BUBBLING ON BOUNDARY SUBMANIFOLDS FOR A SEMILINEAR NEUMANN PROBLEM NEAR HIGH CRITICAL EXPONENTS

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Abstract. In this paper we consider the following problem

$$\begin{cases} -\Delta u+u=u^{\frac{n-k+2}{n-k-2}\pm\varepsilon} & \text{in } \Omega \\ u>0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu}=0 & \text{on } \partial\Omega \end{cases} \tag{0.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n\geq 7$, k is an integer with $k\geq 1$, and $\varepsilon>0$ is a small parameter. Assume there exists a k-dimensional closed, embedded, non degenerate minimal submanifold K in $\partial\Omega$. Under a sign condition on a certain weighted avarage of sectional curvatures of $\partial\Omega$ along K, we prove the existence of a sequence $\varepsilon=\varepsilon_j\to 0$ and of solutions u_ε to (0.1) such that

$$|\nabla u_\varepsilon|^2 \rightharpoonup S\delta_K, \quad \text{as} \quad \varepsilon\to 0$$

in the sense of measure, where δ_K denotes a Dirac delta along K and S is a universal positive constant.

Keywords: critical Sobolev exponent, blowing-up solution, non degenerate minimal submanifolds.

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Contents

1. Introduction and statement of main results

Let Ω be a bounded domain in $\mathbb{R}^n$, $n\geq 3$ with smooth boundary $\partial\Omega$. Let ν denote the unitary normal vector of $\partial\Omega$. The boundary value problem

$$\begin{cases} -d^2\Delta u+u=u^q & \text{in } \Omega \\ u>0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu}=0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $q>1$ and $d>0$, is a model for different problems in applied science which exhibit *concentration phenomena* in their solutions. It arises for instance as the *shadow system* associated to activator-inhibitor systems in mathematical theory of biological pattern formation such as the Gierer-Meinhardt model and in certain models of chemotaxis, see references in [26]. In such models, and related ones, it is particularly meaningful the presence of solutions exhibiting peaks of concentration, namely one or several local maxima around which the solution remains strictly positive, while being very small away from them.

If $1<q<\frac{n+2}{n-2}$ a precise analysis of least energy solutions to this problem, namely solutions which minimize the Rayleigh quotient

$$Q(u)=\frac{d^2\int_\Omega|\nabla u|^2+\int_\Omega|u|^2}{\left(\int_\Omega|u|^{q+1}\right)^{\frac{2}{q+1}}}, \quad u\in H^1(\Omega)\setminus\{0\}, \tag{1.2}$$

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for small d, is already known. Indeed, in [12, 26, 35, 36] it is proven that for d sufficiently small, a minimizer u_d of Q has a unique local maximum point x_d which is located on the boundary. Besides, $H(x_d) \to \max_{x\in\partial\Omega} H(x)$ where H denotes the mean curvature of $\partial\Omega$ and u_d behaves qualitatively like

$$u_d(x) \sim W(\frac{x - x_d}{d}), \tag{1.3}$$

where W is the (unique) positive radially symmetric solution of

$$\Delta W - W + W^q = 0 \quad \text{in } \mathbb{R}^N, \qquad \lim_{|x|\to+\infty} W(x) = 0. \tag{1.4}$$

The function W decays exponentially which gives that the solution has a very sharp, bounded spike around x_d. Construction of single and multiple *spike-layer patterns* for this problem in the subcritical case has been the object of many studies, see for instance [5, 6, 8, 10, 11, 12, 13, 19, 20, 22, 23, 25, 45, 33, 34]. In particular, in [45] it was found that whenever one has a non-degenerate critical point x_0 of the mean curvature $H(x)$, a solution with a profile of the form (1.3) can be found with $x_d \to x_0$.

If $q = \frac{n+2}{n-2}$, namely if we consider the problem

$$d^2\Delta u - u + u^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad \Omega, \quad u > 0 \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \tag{1.5}$$

the lack of compactness of Sobolev's embedding makes it harder to apply variational arguments. Nevertheless, in [1, 44] it was proven that a non-constant least energy solution u_d of (1.5) exists, if d is sufficiently small. As in the subcritical case, u_d concentrates, having a unique maximum point x_d which lies on $\partial\Omega$ with

$$H(x_d) \to \max_{x\in\partial\Omega} H(x).$$

See [3, 24, 37, 39]. But a key difference with the subcritical case is that no positive solutions to Problem (1.4) when $q = \frac{n+2}{n-2}$ exists, as consequence of Pohozaev's identity [38], and thus the concentration phenomenon must necessarily be different from the one previously described for the subcritical case. Indeed, unlike the subcritical case, $u_d(x_d) \to +\infty$ the profile of u_d near x_d is given, for suitable $\mu_d \to 0$, is given by

$$u_d(x) \approx d^{\frac{n-2}{2}} w_{\mu_d}(|x - x_d|) \tag{1.6}$$

where $w_\mu(|x|)$ corresponds to the family of radial positive solutions of

$$\Delta w + w^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad \mathbb{R}^n \tag{1.7}$$

namely

$$w_\mu(|x|) = \alpha_n \left(\frac{\mu}{\mu^2 + |x|^2}\right)^{\frac{n-2}{2}}, \quad \alpha_n = (n(n-2))^{\frac{n-2}{4}}, \tag{1.8}$$

which up to translations, correspond to all positive solutions of (1.7), see [9, 41]. In particular $\mu_d \sim d^2$ for $n \ge 5$, so that $u_d(p_d) \sim d^{-\frac{n-2}{2}}$ (see [4, 21, 39] for the concentration rate in lower dimensions). As in the subcritical case, construction and estimates for bubbling solutions to Problem (1.5) have been subjects broadly treated [2, 15, 17, 18, 31, 40, 42, 43]. In particular, in [2] it was found that for $n \ge 6$ and a non-degenerate critical point x_0 of the mean curvature with $H(x_0) > 0$, there exists a solution with its profile near x_0 given by (1.6) with $x_d \to x_0$, and $\mu_d \approx d^2$, as $d \to 0$. The condition of critical point for H with $H(x_0) > 0$ turns out to be necessary for the boundary bubbling phenomenon to take place, see [4, 21].

If q is supercritical in Problem (1.1), namely $q > \frac{n+2}{n-2}$, Sobolev embedding no longer holds, so that variational construction of solutions becomes difficult. In [16] the authors investigate this case for powers close to critical, where now the parameter d is fixed, say $d = 1$. The problem becomes

$$\begin{cases} -\Delta u + u = u^{\frac{n+2}{n-2}+\varepsilon} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.9}$$

where $\varepsilon > 0$. Given a non-degenerate critical point x_0 of the mean curvature (or, more generally, a situation of topologically non-trivial critical point) with *positive critical value*, $H(x_0) > 0$, a solution u_ε exhibiting boundary bubbling around such a point as $\varepsilon \to 0$ exists

$$u_\varepsilon(x) \approx w_{\mu_\varepsilon}(|x - x_\varepsilon|), \quad \text{with} \quad \mu_\varepsilon \approx \varepsilon, \quad x_\varepsilon \to x_0.$$

In [10] it has been found that if $n \ge 4$, and one considers the exponent $\frac{n+2}{n-2} - \varepsilon$ in (1.9), namely an exponent approaching the critical exponent *from below*, then single-bubbling solutions exist with maximum points located on the boundary, near critical points of the mean curvature with *negative value.*

All the results described up to now concern solutions which presents concentration on *one or more points.* It is thus natural to look for solutions to Problem (1.1), or to Problem (1.9), that exhibit concentration phenomena not just at points but on higher dimensional sets.

Given a k-dimensional submanifold K of $\partial\Omega$ and assuming that either $k \geq n-2$ or $q < \frac{n-k+2}{n-k-2}$, the question is whether there exists a solution u_d to Problem (1.1) which near K looks like

$$u_d(x) \approx W\left(\frac{dist(x,K)}{d}\right) \tag{1.10}$$

where now $W(|y|)$ denotes the unique positive, radially symmetric solution to the problem

$$\Delta W - W + W^q = 0 \quad \text{in} \quad \mathbb{R}^{n-k}, \quad \lim_{|y|\to\infty} W(|y|) = 0.$$

In [27, 28, 29, 30], the authors have established the existence of a solution with the profile (1.10) when either $K = \partial\Omega$ or K is an *embedded closed minimal submanifold* of $\partial\Omega$, which is in addition *non-degenerate* in the sense that its Jacobi operator is non-singular. This phenomenon is actually quite subtle compared with concentration at points: existence can only be achieved along a sequence of values $d \to 0$. d must actually remain suitably away from certain values of d where resonance occurs, and the topological type of the solution changes: unlike the point concentration case, the Morse index of these solutions is very large and grows as $d \to 0$.

In the case of the *k-th critical exponent* $q = \frac{n-k+2}{n-k-2}$, namely for the problem

$$d^2\Delta u - u + u^{\frac{n-k+2}{n-k-2}} = 0 \quad \text{in} \quad \Omega, \quad u > 0 \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \tag{1.11}$$

the problem of concentration on a k-dimensional submanifold K of the boundary is treated in [14]. To state the result, we need to introduce a certain notion of weighted avarage of sectional curvatures of $\partial\Omega$ along K. Denote by $T_p\partial\Omega$ the tangent space to $\partial\Omega$ at the point p and consider the *shape operator* $\Lambda : T_p\partial\Omega \to T_p\partial\Omega$ defined as

$$\Lambda[e] := -\nabla_e \nu(p)$$

where $\nabla_e\nu(p)$ is the directional derivative of the vector field ν in the direction e. Let us consider the orthogonal decomposition

$$T_p\partial\Omega = T_pK \oplus N_pK$$

where N_pK stands for the normal bundle of K. We choose orthonormal bases $(e_a)_{a=1,\dots,k}$ of T_pK and $(e_i)_{i=k+1,\dots,n-1}$ of N_pK and we define the $(n-1)\times(n-1)$ matrix $H(p)$ by

$$H_{\alpha\beta}(p) = -e_\alpha \cdot \Lambda[e_\beta].$$

This matrix represents the second fundamental form of $\partial\Omega$ at p in this basis. $H_{\alpha\alpha}(p)$ corresponds to the curvature of $\partial\Omega$ in the direction e_α. By definition, the mean curvature of $\partial\Omega$ at p is given by the trace of this matrix, namely

$$H(p) = \sum_{\alpha=1}^{n-1} H_{\alpha\alpha}(p).$$

We need to consider the mean of the curvatures in the directions of T_pK and N_pK, namely the numbers $\sum_{i=1}^{k} H_{ii}(p)$ and $\sum_{j=k+1}^{n-1} H_{jj}(p)$.

In [14] it is proven that, if $\partial\Omega$ contains a closed embedded, non-degenerate minimal submanifold K of dimension $k \geq 1$ with $n-k \geq 7$, such that

$$\bar{H}(p) := 2\sum_{i=1}^{k} H_{ii}(p) + \sum_{j=k+1}^{n-1} H_{jj}(p) > 0 \text{ for all } p \in K, \tag{1.12}$$

then, for a sequence $d = d_j \longrightarrow 0$, Problem (1.11) has a positive solution u_d concentrating along K.

In the present paper we study concentration phenomena on high dimensional set for the problem

$$\begin{cases} -\Delta u + u = u^{\frac{n-k+2}{n-k-2}\pm\varepsilon} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.13}$$

where $\varepsilon > 0$ is small.

Given a smooth embedded non-degenerate minimal submanifold K of $\partial\Omega$, with dimension $1 \leq k < n-2$, we prove existence of solutions u_ε of (1.13) concentrating along K as the parameter $\varepsilon \to 0$ in the following sense: let

$$x = p + z, \quad p \in K, \quad |z| = dist(x,K),$$

we have

$$u_\varepsilon(x) \approx w_{\mu_\varepsilon}(|z|), \quad \text{where} \quad w_\mu(|z|) = \alpha_{n-k}\left(\frac{\mu}{\mu^2 + |z|^2}\right)^{\frac{n-k-2}{2}} \tag{1.14}$$

and

$$\mu_\varepsilon(p) = \begin{cases} a_{n-k}\, \bar{H}(p)^{-1}\, \varepsilon^2 & \text{if we consider the problem (1.13) with} \quad p+\varepsilon \\ -\, a_{n-k}\, \bar{H}(p)^{-1}\, \varepsilon^2 & \text{if we consider the problem (1.13) with} \quad p-\varepsilon \end{cases}$$

with a_{n-k} a universal positive constant.

We have the validity of the following

Theorem 1.1. *Assume that $\partial\Omega$ contains a closed embedded, non-degenerate minimal submanifold K of dimension $k \geq 1$, with $n-k \geq 7$. If*

$$2\sum_{i=1}^{k} H_{ii}(p) + \sum_{i=k+1}^{n-1} H_{ii}(p) > 0 \quad \textit{for all} \quad p \in K. \tag{1.15}$$

Then there exist a sequence $\varepsilon = \varepsilon_j$ and a sequence of solutions u_ε to Problem

$$\begin{cases} -\Delta u + u = u^{\frac{n-k+2}{n-k-2}+\varepsilon} & \textit{in } \Omega \\ u > 0 & \textit{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \textit{on } \partial\Omega \end{cases}$$

with the shape (1.14) and such that

$$|\nabla u_\varepsilon|^2 \rightharpoonup S\delta_K, \quad \textit{as} \quad \varepsilon \to 0,$$

where δ_K stands for the Dirac measure supported on K and S for a universal constant. If

$$2\sum_{i=1}^{k} H_{ii}(p) + \sum_{i=k+1}^{n-1} H_{ii}(p) < 0 \quad \textit{for all} \quad p \in K. \tag{1.16}$$

Then there exist a sequence $\varepsilon = \varepsilon_j$ and a sequence of solutions u_ε to Problem

$$\begin{cases} -\Delta u + u = u^{\frac{n-k+2}{n-k-2}-\varepsilon} & \textit{in } \Omega \\ u > 0 & \textit{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \textit{on } \partial\Omega \end{cases}$$

with the shape (1.14) and such that

$$|\nabla u_\varepsilon|^2 \rightharpoonup S\delta_K, \quad \textit{as} \quad \varepsilon \to 0,$$

where δ_K stands for the Dirac measure supported on K and S for a universal constant.

Observe that

$$\bar{H}(p) = 2H(p) - \sum_{j=k+1}^{n-1} H_{jj}(p) \quad \text{for all } p \in K,$$

where H denotes the mean curvature of $\partial\Omega$. Formally in the case of point concentration, namely $k=0$, condition (1.15) reduces precisely to $H(p) > 0$, that is exactly the condition known to be necessary for point concentration in the slightly super critical regime. On the other hand, condition (1.16) reduces precisely to $H(p) < 0$, that is exactly the condition known to be necessary for point concentration in the slightly subcritical critical regime. We suspect that this condition is essential for the phenomenon to take place.

Let us also mention that, while the high codimension assumption $n-k \geq 7$ is important in our proof, we expect that a similar phenomenon holds just provided that $n-k \geq 5$, and with a suitable change in the bubbling scales for $n-k \geq 3$ (the difference of rates is formally due to the fact that $\int_{\mathbb{R}^n} w_\mu^2$ is finite if and only if $n \geq 5$).

A final remark: the problem of establishing existence or non existence of solutions to (1.13) which concentrate on high dimensional sets which do not belong to the boundary of Ω is still wide open. The only result available is contained in [32] where it is proved that if $n \geq 8$, $k=1$ then for a sequence of the small positive parameter ε, Problem (1.13) (with $-\varepsilon$ at the exponent) admits a positive solution concentrating along a nondegenerate segment connecting two points of the boundary of Ω. This construction is not known to exist neither for the case $+\varepsilon$, nor for higher dimensional sets of concentration.

It will be convenient to rewrite Problem (1.13) in an equivalent form: Set $N = n-k$ and define

$$u(x) = \varepsilon^{-\frac{2(N-2)}{4\pm(N-2)\varepsilon}}\, \tilde{u}(\varepsilon^{-1}x).$$

Then, setting $\Omega_\varepsilon := \varepsilon^{-1}\Omega$, Problem (1.13) becomes

$$\begin{cases} -\Delta \tilde{u} + \varepsilon^2 \tilde{u} = \tilde{u}^{\frac{N+2}{N-2}\pm\varepsilon} & \text{in } \Omega_\varepsilon \\ \tilde{u} > 0 & \text{in } \Omega_\varepsilon \\ \frac{\partial \tilde{u}}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{1.17}$$

The proof of the theorem has as a main ingredient the construction of an approximate solution with arbitrary degree of accuracy in powers of ε, in a neighborhood of the manifold $K_\varepsilon = \varepsilon^{-1}K$. Later we built the desired solution by linearizing the equation (1.17) around this approximation. The associated linear operator turns out to be invertible with inverse controlled in a suitable norm by certain large negative power of ε, provided that ε remains away from certain critical values where resonance occurs. The interplay of the size of the error and that of the inverse of the linearization then makes it possible a fixed point scheme.

The rest of the paper is organized as follows.

We first introduce some notations and conventions. Next, in Section 2 we collect some notions in differential geometry, like the Fermi coordinates (geodesic normal coordinates) near a minimal submanifold, we expand the coefficients of the metric near these Fermi coordinates and the Laplace-Beltrami operator. Section 4 will be mainly devoted to the construction of the approximate solution to our problem using the local coordinates around the submanifold K introduced before. To perform this construction we need a solvability theory and a-priori estimates for a certain linear operator, which is developed in Section 3. In Section 5 we define globally the approximation and we write the solution to our problem as the sum of the global approximation plus a remaining term. Thus we express our original problem as a non linear problem in the remaining term and we prove our Theorem. To solve such problem, we need to understand the invertibility properties of another linear operator. To do so we start expanding a quadratic functional associated to the linear problem. In Section 6 we develop a linear theory to study our problem. Section 7 is an Appendix, where we postponed the proof of some technical facts to facilitate the reading of the paper.

Notation and conventions

Dealing with coordinates, Greek letters like $\alpha, \beta, \dots,$ will denote indices varying between 1 and $n-1$, while capital letters like $A, B, \dots$ will vary between 1 and n; Roman letters like a or b will run from 1 to k, while indices like $i, j, \dots$ will run between 1 and $N-1 := n-k-1$.

$\xi_1, \dots, \xi_{N-1}, \xi_N$ will denote coordinates in $\mathbb{R}^N = \mathbb{R}^{n-k}$, and they will also be written as $\bar\xi = (\xi_1, \dots, \xi_{N-1})$, $\xi = (\bar\xi, \xi_N)$.

The manifold K will be parameterized with coordinates $y = (y_1, \dots, y_k)$. Its dilation $K_\varepsilon := \frac{1}{\varepsilon}K$ will be parameterized by coordinates $(z_1, \dots, z_k)$ related to the y's simply by $y = \varepsilon z$.

Derivatives with respect to the variables y, z or ξ will be denoted by ∂_y, ∂_z, ∂_ξ, and for brevity sometimes we might use the symbols ∂_a, $\partial_{\overline{a}}$ and ∂_i for ∂_{y_a}, ∂_{z_a} and ∂_{ξ_i} respectively.

In a local system of coordinates, $(\overline{g}_{\alpha\beta})_{\alpha\beta}$ are the components of the metric on $\partial\Omega$ naturally induced by $\mathbb{R}^n$. Similarly, $(\overline{g}_{AB})_{AB}$ are the entries of the metric on Ω in a neighborhood of the boundary. $(H_{\alpha\beta})_{\alpha\beta}$ will denote the components of the mean curvature operator of $\partial\Omega$ into $\mathbb{R}^n$.
Below, for simplicity, the constant C is allowed to vary from one formula to another, also within the same line, and will assume larger and lager values. It is always understood that C depends on Ω and the dimension n, but it is independent of ε.

2. Local coordinates and Expansion of the Laplace Beltrami operator

The solution we are looking at have at main order the shape of w_0, the so called *standard bubble* defined by

$$w_0(\xi) = \frac{\alpha_N}{(1+|\xi|^2)^{\frac{N-2}{2}}}, \quad \alpha_N = (N(N-2))^{\frac{N-2}{4}} \quad \text{for all} \quad \xi \in \mathbb{R}^N \tag{2.1}$$

which solves

$$\Delta w_0 + w_0^{\frac{N+2}{N-2}} = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{2.2}$$

Indeed our solution looks like the function w_0, translated along a small deviation from the k-dimensional submanifold K of $\partial\Omega$ and scaled by a small factor, represented by a small smooth function defined on K. In this way, the solution turns out to be very concentrated around K.

In order to describe this construction, we introduce some local coordinates close to K in $\partial\Omega$, called Fermi coordinates, and then local coordinates in a neighborhood of K in Ω. We then introduce a proper change of variables and we describe the Laplace operator with respect to this new set of coordinates. This is the content of the present Section.

We denote by $\bar g$ the metric induced by the Euclidean metric on $\partial\Omega$ and by $\bar\nabla$ the associated connection. Thus K is a k-dimensional submanifold of $(\partial\Omega, \bar g)$. Along K we choose a local oriented and orthonormal frame field $((E_a)_{a=1,\cdots k}, (E_i)_{i=1,\cdots,N-1})$. At points of K, the tangent bundle $T\partial\Omega$ splits naturally as $TK \oplus NK$, where TK is the tangent bundle to K with orthonormal basis $(E_a)_a$ and NK is the normal bundle, which is spanned by the orthonormal basis $(E_j)_j$. Given a point $q \in K$, we assume that at q the normal vectors $(E_i)_i$, $i = 1, \dots, N-1$, are parallel transported along K, namely

$$\bar g\left(\bar\nabla_{E_a} E_j\, , E_i\right) = 0 \quad \text{at } q, \qquad i, j = 1, \dots, N-1, a = 1, \dots, k. \tag{2.3}$$

In a neighborhood of p in K, we consider normal geodesic coordinates

$$f(y) := \exp_p^K(y_a\, E_a), \qquad y := (y_1, \dots, y_k),$$

where $\exp^K$ is the exponential map on K and summation over repeated indices is understood. This yields the coordinate vector fields $X_a := f_*(\partial_{y_a})$. We extend the E_i along each $\gamma_E(s)$ so that they are parallel with respect to the induced connection on the normal bundle NK. This yields an orthonormal frame field X_i for NK in a neighborhood of p in K which satisfies $\nabla_{X_a} X_i|_p \in T_pK$. A coordinate system in a neighborhood of p in $\partial\Omega$ is now defined by

$$F(y,\bar{x}) := \exp^{\partial\Omega}_{f(y)}(x_i\, X_i), \qquad (y,\bar{x}) := (y_1,\dots,y_k,x_1,\dots,x_{N-1}), \tag{2.4}$$

with corresponding coordinate vector fields $X_i := F_*(\partial_{x_i})$ and $X_a := F_*(\partial_{y_a})$.

The assumption that K is a minimal submanifold in $\partial\Omega$ translated into the condition

$$\Gamma^a_a(E_i) = 0 \quad \text{ for all } i = 1,\dots N-1, \tag{2.5}$$

where here we have denoted by $\Gamma^b_a(\cdot)$ the 1-forms defined on the normal bundle, NK, of K by the formula

$$\bar{g}_{bc}\Gamma^c_{ai} := \bar{g}_{bc}\Gamma^c_a(X_i) = \bar{g}(\nabla_{X_a} X_b, X_i) \quad \text{at } q = f(y). \tag{2.6}$$

To parameterize a neighborhood of q in $\partial\Omega$, we introduce

$$\Upsilon_0(y,\bar{x}) = \exp^{\partial\Omega}_y\Big(\sum_{i=1}^{N-1} x_i\, E_i\Big); \qquad (y,\bar{x}) = ((y_a)_a, (x_i)_i)\,, \tag{2.7}$$

where $\exp^{\partial\Omega}_y$ is the exponential map at y in $\partial\Omega$. The coordinates $(y,\bar{x})$ are called *Fermi coordinates* on $\partial\Omega$. A parametrization of a neighborhood of $q \in \partial\Omega$ in Ω is given by the map Υ_1 defined by

$$\Upsilon_1(y,x) = \Upsilon_0(y,\bar{x}) + x_N \nu(y,\bar{x}), \qquad x = (\bar{x}, x_N) \in \mathbb{R}^{N-1}\times\mathbb{R}, \tag{2.8}$$

where Υ_0 is the parametrization introduced in (2.7) and $\nu(y,\bar{x})$ is the outward unit normal to $\partial\Omega$ at $\Upsilon_0(y,\bar{x})$. Let us define the tensor matrix $\mathbf{H}$ to be given by

$$d\nu_x[v] = -\mathbf{H}(x)[v]. \tag{2.9}$$

We thus find that

$$\frac{\partial\Upsilon_1}{\partial y_a} = [Id - x_N\mathbf{H}(y,\bar{x})]\,\frac{\partial\Upsilon_0}{\partial y_a}(y,\bar{x}); \qquad \frac{\partial\Upsilon_1}{\partial x_i} = [Id - x_N\mathbf{H}(y,\bar{x})]\,\frac{\partial\Upsilon_0}{\partial x_i}(y,\bar{x}). \tag{2.10}$$

Differentiating Υ_1 with respect to x_N we also get

$$\frac{\partial\Upsilon_1}{\partial x_N} = \nu(y,\bar{x}). \tag{2.11}$$

Given a positive smooth function $\mu_\varepsilon = \mu_\varepsilon(y)$ defined on K and a smooth normal section (in $\partial\Omega$) $\Phi_\varepsilon : K \longrightarrow NK^{\partial\Omega}$ defined by $\Phi_\varepsilon(y) = \Phi^j_\varepsilon(y)E_j$, $y \in K$, $j = 1,\cdots,N-1$, we introduce the following change of variables

$$\tilde{u}(\Upsilon_1(y,\overline{x},x_N)) = \mu_\varepsilon^{-\frac{N-2}{2}}(y)\, v\Big(\frac{y}{\varepsilon}, \frac{\overline{x}-\Phi_\varepsilon(y)}{\mu_\varepsilon(y)}, \frac{x_N}{\mu_\varepsilon(y)}\Big), \tag{2.12}$$

where

$$v = v(z,\xi), \quad z = \frac{y}{\varepsilon}, \quad \overline{\xi} = \frac{\overline{x}-\Phi_\varepsilon}{\mu_\varepsilon}, \quad \xi_N = \frac{x_N}{\mu_\varepsilon}. \tag{2.13}$$

To emphasize the dependence of the above change of variables on μ_ε and Φ_ε, we will use the notation

$$\tilde{u} = \mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(v) \quad \Longleftrightarrow \tilde{u} \quad \text{and} \quad v \quad \text{satisfy (2.12)}. \tag{2.14}$$

We assume for the moment that the function μ_ε and the normal section Φ_ε are smooth functions on K, uniformly bounded as $\varepsilon \to 0$. Since the original variables $(y,x) \in \mathbb{R}^{k+N}$ are local coordinates along K, we let the variables (z,ξ) vary in the set $\mathcal{D}$ defined by

$$\mathcal{D} = \Big\{(z,\bar{\xi},\xi_N) \,:\, \varepsilon z \in K, \quad |\bar{\xi}| < \frac{\delta}{\varepsilon}, \quad 0 < \xi_N < \frac{\delta}{\varepsilon}\Big\} \tag{2.15}$$

for some small and positive number δ, independent of ε, that we choose properly later on. We will also use the notation $\mathcal{D} = K_\varepsilon \times \hat{\mathcal{D}}$, where $K_\varepsilon = \frac{K}{\varepsilon}$ and

$$\hat{\mathcal{D}} = \Big\{(\bar{\xi},\xi_N) \,:\, |\bar{\xi}| < \frac{\delta}{\varepsilon}, \quad 0 < \xi_N < \frac{\delta}{\varepsilon}\Big\}.$$

We note that $\partial\hat{\mathcal{D}} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ with

$$\Gamma_1 = \Big\{(\bar{\xi},\xi_N) \in \hat{\mathcal{D}} \,:\, |\bar{\xi}| < \frac{\delta}{\varepsilon}, \quad \xi_N = 0\Big\}, \tag{2.16}$$

$$\Gamma_2 = \Big\{(\bar{\xi},\xi_N) \in \hat{\mathcal{D}} \,:\, |\bar{\xi}| < \frac{\delta}{\varepsilon}, \quad \xi_N = \frac{\delta}{\varepsilon}\Big\}, \tag{2.17}$$

and

$$\Gamma_3 = \Big\{(\bar{\xi},\xi_N) \in \hat{\mathcal{D}} \,:\, |\bar{\xi}| = \frac{\delta}{\varepsilon}, \quad 0 < \xi_N < \frac{\delta}{\varepsilon}\Big\}. \tag{2.18}$$

We now need to compute the Laplace operator in the new variables (z,ξ), taking into account the change of variables (2.12). For this purpose, we introduce $R_{\alpha\beta\gamma\delta}$ the components of the curvature tensor with lowered indices, which are obtained by means of the usual ones $R^{\sigma}_{\beta\gamma\delta}$ by

$$R_{\alpha\beta\gamma\delta} = \overline{g}_{\alpha\sigma}\, R^{\sigma}_{\beta\gamma\delta}. \tag{2.19}$$

We have the validity of the following

Lemma 2.1. *Given the change of variables defined in* (2.12), *the following expansion for the Laplace Beltrami operator holds true*

$$\mu_\varepsilon^{\frac{N+2}{2}} \Delta\tilde{u} = \mathcal{A}_{\mu_\varepsilon,\Phi_\varepsilon}(v) := \mu_\varepsilon^2 \Delta_{K_\varepsilon} v + \Delta_\xi v + \sum_{\ell=0}^{5} \mathcal{A}_\ell v + B(v). \tag{2.20}$$

Above, the expression $\mathcal{A}_k$ denotes the following differential operators

$$\begin{aligned} \mathcal{A}_0 v \;&=\; -\varepsilon^2\mu_\varepsilon D_{\overline{\xi}}\, v\,[\Delta_K\Phi_\varepsilon] - \varepsilon^2\,\mu_\varepsilon\,\Delta_K\mu_\varepsilon\,\left(\gamma v + D_\xi v\,[\xi]\right) \\ &+\; \varepsilon^2\,|\nabla_K\mu_\varepsilon|^2\left[D_{\xi\xi}v\,[\xi]^2 + 2(1+\gamma)D_\xi v[\xi] + \gamma(1+\gamma)v\right] \\ &+\; \varepsilon^2\nabla_K\mu_\varepsilon\cdot\left\{2D_{\overline{\xi}\,\overline{\xi}}v[\overline{\xi}] + ND_{\overline{\xi}}v\right\}\,[\nabla_K\Phi_\varepsilon] + \varepsilon^2 D_{\overline{\xi}\,\overline{\xi}}v\,[\nabla_K\Phi_\varepsilon]^2 \\ &-\; 2\,\varepsilon\mu_\varepsilon\,\tilde{g}^{ab}\left[D_\xi(\partial_{\overline{a}}v)[\partial_b\mu_\varepsilon\xi] + D_{\overline{\xi}}(\partial_{\overline{a}}v)[\partial_b\Phi_\varepsilon] + \gamma\partial_a\mu_\varepsilon\,\partial_{\overline{b}}v\right], \end{aligned} \tag{2.21}$$

where we have set $\gamma = \frac{N-2}{2}$,

$$\begin{aligned} \mathcal{A}_1\, v \;&=\; \sum_{i,j}\Big[2\mu_\varepsilon\varepsilon H_{ij}\xi_N - \frac{\varepsilon^2}{3}\sum_{m,l} R_{mijl}(\mu_\varepsilon\xi_m + \Phi_\varepsilon^m)(\mu_\varepsilon\xi_l + \Phi_\varepsilon^l) \\ &+\; \mu_\varepsilon^2\varepsilon^2\xi_N^2\,Q(H)_{ij} + \mu_\varepsilon\varepsilon^2\xi_N\sum_l \mathfrak{D}^{ij}_{Nl}\,(\mu_\varepsilon\xi_l + \Phi_\varepsilon^l)\Big]\partial^2_{ij}v, \end{aligned} \tag{2.22}$$

where the functions $\mathfrak{D}^{ij}_{Nk}$ are smooth functions of the variable $y = \varepsilon z$, which are uniformly bounded, as $\varepsilon \to 0$. Furthermore,

$$\mathcal{A}_2 v = \varepsilon^2\mu_\varepsilon\sum_j\left[\sum_s\frac{2}{3}R_{mssj} + \sum_{m,a,b}\left(\tilde{g}^{ab}_\varepsilon\,R_{mabj} - \Gamma^b_{am}\Gamma^a_{bj}\right)\right](\mu_\varepsilon\xi_m + \Phi_\varepsilon^m)\partial_j v \tag{2.23}$$

and

$$\mathcal{A}_3 v \;=\; \Big[-\varepsilon\mathrm{tr}(H) - 2\,\mu_\varepsilon\,\varepsilon^2\mathrm{tr}(H^2)\,\xi_N - 2\varepsilon^2\sum_{i,a,b}(\mu_\varepsilon\xi_i + \Phi_\varepsilon^i)H_{ab}\Gamma^a_{bi}\Big]\mu_\varepsilon\partial_N v. \tag{2.24}$$

Moreover

$$\mathcal{A}_4 v = 4\,\varepsilon\,\mu_\varepsilon\xi_N\sum_{a,j}H_{aj}\left(-\varepsilon D_y(\partial_j v)[\partial_a\Phi_\varepsilon] + \mu_\varepsilon\partial^2_{\overline{a}j}v - \varepsilon\partial_a\mu_\varepsilon(\gamma\partial_j v + D_\xi(\partial_j v)[\xi])\right) \tag{2.25}$$

and

$$\begin{aligned} \mathcal{A}_5 v \;&=\; \left(\sum_{a,j}\mathfrak{D}^a_j\varepsilon^2[\mu_\varepsilon\xi_j + \Phi_\varepsilon^j] + \varepsilon^2\mu_\varepsilon\,\mathfrak{D}^a_N\,\xi_N\right)\times \\ &\left\{\mu_\varepsilon\left[-\varepsilon D_{\overline{\xi}}v\,[\partial_a\Phi_\varepsilon] + \mu_\varepsilon\partial_{\overline{a}}v - \varepsilon\partial_a\mu_\varepsilon(\gamma v + D_\xi v\,[\xi])\right]\right\} \end{aligned} \tag{2.26}$$

where $\mathfrak{D}^a_j$ and $\mathfrak{D}^a_N$ are smooth functions of $z = \frac{y}{\varepsilon}$. Finally, the operator $B(v)$ is defined by

$$\begin{aligned} \mathcal{B}(v) \;&=\; O\left(\varepsilon^2(\mu_\varepsilon\overline{y} + \Phi)^2 + \varepsilon^2\mu_\varepsilon\xi_N(\mu_\varepsilon\overline{\xi} + \Phi) + \varepsilon^2\mu_\varepsilon^2\xi_N^2\right)\left(\partial_l v + \partial^2_{\overline{a}l}v\right) \\ &+\; O\left(\varepsilon^3|\mu_\varepsilon\overline{y} + \Phi|^3 + \varepsilon^3\mu_\varepsilon\xi_N\,|\mu_\varepsilon\overline{y} + \Phi|^2 + \varepsilon^3\mu_\varepsilon^2\xi_N^2|\mu_\varepsilon\overline{y} + \Phi| + \varepsilon^3\mu_\varepsilon^3\xi_N^3\right)\partial^2_{ij}v. \end{aligned}$$

We recall that the symbols ∂_a, $\partial_{\overline{a}}$ and ∂_i denote the derivatives with respect to ∂_{y_a}, ∂_{z_a} and ∂_{ξ_i} respectively.

The first part of the Appendix, Section 7, is devoted to recall some results which are the basic tools to prove the above result. But we also refer the reader to the proof of Lemma 3.3 in [14] for the detailed proof of this Lemma 2.1.

3. A linear theory

Using the local coordinates along the submanifold K introduced in Section 2, after performing the change of variables in (2.12), the original equation (1.17) in $\tilde{u}$ reduces locally close to $K_\varepsilon = \frac{1}{\varepsilon}K$ to the following equation in v

$$-\mathcal{A}_{\mu_\varepsilon,\Phi_\varepsilon} v + \varepsilon^2 \mu_\varepsilon^2 v - \mu_\varepsilon^{\mp\varepsilon\frac{N-2}{2}} v^{p\pm\varepsilon} = 0, \tag{3.1}$$

where $\mathcal{A}_{\mu_\varepsilon,\Phi_\varepsilon}$ is defined in (2.20) and $p = \frac{N+2}{N-2}$. Let us denote by $\mathcal{S}_\varepsilon$ the operator given by

$$\mathcal{S}_\varepsilon(v) := -\mathcal{A}_{\mu_\varepsilon,\Phi_\varepsilon} v + \varepsilon^2 \mu_\varepsilon^2 v - \mu_\varepsilon^{\mp\varepsilon\frac{N-2}{2}} v^{p\pm\varepsilon}. \tag{3.2}$$

Our aim is to construct a function v so that $\mathcal{S}_\varepsilon(v)$ is small (in a proper sense we clarify later) in the set $\mathcal{D} = K_\varepsilon \times \hat{\mathcal{D}}$ (see (2.15)).

The basic tool for this construction is a linear theory we describe below.

Let $a : K \to \mathbb{R}$ be a smooth function with $a(y) \geq \lambda > 0$ for all $y \in K$ and consider an operator of the form

$$L(\phi) := -\Delta\phi - p w_0^{p-1}\phi + \varepsilon^2\, a(\varepsilon z)\phi + b_{ij}(\varepsilon z, \xi)\partial_{ij}\phi + b_i(\varepsilon z, \xi)\partial_i\phi, \tag{3.3}$$

for functions ϕ defined on $\mathcal{D}$. In (3.3), b_{ij} and b_i are functions defined in $\mathcal{D}$, which depend smoothly on $y \in K$. Recall that a variable $z \in K_\varepsilon$ has the form $\varepsilon z = y \in K$.

We want to find a linear theory for the following linear problem

$$\begin{cases} L(\phi) = h, & \text{in } \mathcal{D} \\ \frac{\partial \phi}{\partial \xi_N} = 0, & \text{on } \Gamma_1 \\ \phi = 0, & \text{on } \Gamma_2 \cup \Gamma_3 \\ \int_{\hat{\mathcal{D}}} \phi(\varepsilon z, \xi) Z_j(\xi)\, d\xi = 0 & \text{for all } z \in K_\varepsilon, \quad j = 0, \dots N-1, \end{cases} \tag{3.4}$$

for a given function $h : \mathcal{D} \to \mathbb{R}$, which depends smoothly on the variable $y \in K$. We recall that Γ_1, Γ_2 and Γ_3 are defined respectively in (2.16), (2.17) and (2.18). The functions $Z_j(\xi)$, $j = 0, \dots, N-1$, are

$$Z_j(\xi) = \frac{\partial w_0}{\partial \xi_j}, \quad j = 1, \dots, N-1, \quad Z_0(\xi) = \xi \cdot \nabla w_0(\xi) + \frac{N-2}{2} w_0(\xi). \tag{3.5}$$

It is well known (see for instance [7]) that these functions are the only bounded solutions to the linearized equation around w_0 of problem (2.2)

$$-\Delta\phi - p w_0^{p-1}\phi = 0 \quad \text{in} \quad \mathbb{R}^{N-1} \times \mathbb{R}^+, \quad \frac{\partial \phi}{\partial \xi_N} = 0 \quad \text{on } \xi_N = 0.$$

In order to solve the above linear problem, we define the following norms. Let $\delta > 0$ be a positive, small fixed number. Let r be an integer. For a function w defined in $\mathcal{D} = K_\varepsilon \times \hat{\mathcal{D}}$, we define

$$\|w\|_{\varepsilon,r} := \sup_{(z,\xi)\in K_\varepsilon \times \hat{\mathcal{D}}} \left((1+|\xi|^2)^{\frac{r}{2}} |w(z,\xi)| \right). \tag{3.6}$$

Let $\sigma \in (0,1)$. We define

$$\|w\|_{\varepsilon,r,\sigma} := \|w\|_{\varepsilon,r} + \sup_{(z,\xi)\in K_\varepsilon \times \hat{\mathcal{D}}} \left((1+|\xi|^2)^{\frac{r+\sigma}{2}} [w]_{\sigma,B(\xi,1)} \right) \tag{3.7}$$

where we have denoted

$$[w]_{\sigma,B(\xi,1)} := \sup_{\xi_1,\xi_2 \in B(\xi,1)} \frac{|w(z,\xi_2) - w(z,\xi_1)|}{|\xi_1 - \xi_2|^\sigma}. \tag{3.8}$$

We will established existence and uniform a priori estimates for problem (3.4) in the above norms, provided that appropriate bounds for coefficients hold. We have the validity of the following result.

Proposition 3.1. *Let r be an integer such that $4 < r < N$. Let $a : K \to \mathbb{R}$ be a smooth function, such that $a(y) \geq \lambda > 0$ for all $y \in K$. Assume that there exists a positive number η, such that for all i, j,*

$$\|b_{ij}\|_\infty + \|Db_{ij}\|_\infty + \|(1+|\xi|)b_i\|_\infty < \eta. \tag{3.9}$$

Let $h : K \times \hat{\mathcal{D}} \to \mathbb{R}$ be a function that depends smoothly on the variable $y \in K$, such that $\|h\|_{\varepsilon,r}$ is bounded, uniformly in ε, and

$$\int_{\hat{\mathcal{D}}} h(\varepsilon z, \xi) Z_j(\xi) d\xi = 0 \quad \text{for all } z \in K_\varepsilon, \quad j = 0, 1, \dots, N-1.$$

Then there exists a solution ϕ of problem (3.4) and a constant $C > 0$ such that

$$\|D_\xi^2 \phi\|_{\varepsilon,r,\sigma} + \|D_\xi \phi\|_{\varepsilon,r-1,\sigma} + \|\phi\|_{\varepsilon,r-2,\sigma} \leq C\|h\|_{\varepsilon,r,\sigma}. \tag{3.10}$$

Furthermore, the function ϕ depends smoothly on the variable εz, and the following estimates hold true: for any integer l there exists a positive constant C_l such that

$$\|D_y^l \phi\|_{\varepsilon,r-2,\sigma} \le C_l \left(\sum_{k\le l} \|D_y^k h\|_{\varepsilon,r,\sigma} \right). \tag{3.11}$$

What is left of this section is devoted to the proof of the above result.

Proof. The proof of this Proposition will be divided into several steps.

Step 1. Let us assume that in problem (3.4) the coefficients b_{ij}, b_i are identically zero. Thus assume that ϕ is a solution to

$$\begin{cases} -\Delta\phi - pw_0^{p-1}\phi + \varepsilon^2\, a(\varepsilon z)\phi = h & \text{in } \mathcal{D} \\ \frac{\partial \phi}{\partial \xi_N} = 0, & \text{on } \Gamma_1 \\ \phi = 0, & \text{on } \Gamma_2 \cup \Gamma_3 \\ \int_{\hat{\mathcal{D}}} \phi(\varepsilon z, \xi) Z_j(\xi)\, d\xi = 0 & \text{for all } z \in K_\varepsilon, \quad j = 0, \dots N-1. \end{cases} \tag{3.12}$$

We claim that there exists $C > 0$ such that

$$\|\phi\|_{\varepsilon,r-2} \le C\|h\|_{\varepsilon,r}. \tag{3.13}$$

By contradiction, assume that there exist sequences $\varepsilon_n \to 0$, h_n with $\|h_n\|_{\varepsilon_n,r} \to 0$ and solutions ϕ_n to (3.12) with $\|\phi_n\|_{\varepsilon_n,r-2} = 1$.

Let $z_n \in K_{\varepsilon_n}$ and ξ_n be such that

$$|\phi_n(\varepsilon_n z_n, \xi_n)| = \sup |\phi_n(y, \xi)|.$$

We may assume that, up to subsequences, $(\varepsilon_n z_n) \to \bar{y}$ in K. In particular one gets that $|\xi_n| \le C\varepsilon_n^{-1}$ for some positive constant C independent of ε_n.

Let us now assume that there exists a positive constant M such that $|\xi_n| \le M$. In this case, up to subsequences, one gets that $\xi_n \to \xi_0$. Consider the functions

$$\tilde{\phi}_n(z, \xi) = \phi_n(z, \xi + \xi_n).$$

This is a sequence of uniformly bounded functions, that converges uniformly over compact sets of $K \times \hat{\mathcal{D}}$ to a function $\tilde{\phi}$ solution to

$$\begin{cases} -\Delta\tilde{\phi} - pw_0^{p-1}\tilde{\phi} = 0 & \text{in } \mathbb{R}^{N-1} \times (-M, +\infty) \\ \frac{\partial \tilde{\phi}}{\partial \xi_N} = 0, & \text{on } \{\xi_N = -M\}. \end{cases}$$

Since the orthogonality conditions pass to the limit, we get that furthermore

$$\int_{\mathbb{R}^{N-1}\times[-M,\infty)} \tilde{\phi}(y, \xi) Z_j(\xi)\, d\xi = 0 \quad \text{ for all } y \in K, \quad \text{ for all } j = 0, \dots N-1.$$

These facts imply that $\tilde{\phi} \equiv 0$, that is a contradiction.

Assume now that $\lim_{n\to\infty} |\xi_n| = \infty$. Consider the scaled function

$$\tilde{\phi}_n(z, \xi) = \phi_n(z, |\xi_n|\xi + \xi_n)$$

defined on the set

$$\bar{\mathcal{D}} = \left\{ (z, \bar{\xi}, \xi_N) : \ |\bar{\xi}| < \frac{\delta}{\varepsilon_n|\xi_n|} - \frac{\xi_n}{|\xi_n|},\ -\frac{\xi_n}{|\xi_n|} < \xi_N < \frac{\delta}{\varepsilon_n|\xi_n|} - \frac{\xi_n}{|\xi_n|} \right\}.$$

Thus $\tilde{\phi}_n$ satisfies the equation

$$\Delta\tilde{\phi}_n + p\, C_N \frac{|\xi_n|^2}{(1 + |\,|\xi_n|\xi + \xi_n|^2)^2} \tilde{\phi}_n - |\xi_n|^2 \varepsilon_n^2 a \tilde{\phi}_n = |\xi_n|^2 h(z, |\xi_n|\xi + \xi_n) \ \text{ in } \bar{\mathcal{D}}.$$

Consider first the case in which $\lim_{n\to\infty} \varepsilon_n^2 |\xi_n|^2 = 0$. Under our assumptions, we have that $\tilde{\phi}_n$ is uniformly bounded and it converges locally over compact sets to $\tilde{\phi}$ solution to

$$\Delta\tilde{\phi} = 0, \quad |\tilde{\phi}| \le C|\xi|^{2-r} \quad \text{in} \quad \mathbb{R}^{N-1} \times [-l, +\infty),$$

and

$$\frac{\partial \tilde{\phi}}{\partial \nu} = 0 \quad \text{ on } \{\xi_N = -l\}.$$

Since $4 < r < N$, we conclude that $\tilde{\phi} \equiv 0$, which is a contradiction.

Consider now the other possible case, namely that

$$\lim_{n\to\infty} \varepsilon_n^2 |\xi_n|^2 = \beta > 0.$$

Then,

$$\bar{\mathcal{D}} \to \mathcal{S} := \{(\bar{\xi}, \xi_N) \in (-L, L) \times (-l, L)\} \quad \text{as } n \to \infty$$

where l and L are some positive constant. And up to subsequences we get that $\tilde{\phi}_n$ converges uniformly over compact sets to $\tilde{\phi}$ solution to

$$\Delta\tilde{\phi} - \beta a\tilde{\phi} = 0, \quad |\tilde{\phi}| \le C|\xi|^{2-r} \quad \text{in } \mathcal{S}$$

with

$$\frac{\partial\tilde{\phi}}{\partial\nu} = 0 \ \text{ on } \{\xi_N = -l\}, \quad \tilde{\phi} = 0 \ \text{ on } \ \{|\bar{\xi}| = L\} \cup \{\xi_N = L\}.$$

Multiplying equation by $\tilde{\phi}$, and integrating it over $\mathcal{S}$ only in ξ, we get

$$\int_{\mathcal{S}} (|\nabla\tilde{\phi}|^2 + \beta a\tilde{\phi}^2) d\xi = 0.$$

Thus we conclude that $\tilde{\phi} \equiv 0$, which is a contradiction. The proof of (3.13) is completed.

Step 2. We shall now show that there exists $C > 0$ such that, if ϕ is a solution to (3.12), then

$$\|D^2_\xi\phi\|_{\varepsilon,r} + \|D_\xi\phi\|_{\varepsilon,r-1} + \|\phi\|_{\varepsilon,r-2} \le C\|h\|_{\varepsilon,r}. \tag{3.14}$$

For $z \in K_\varepsilon$, we have that ϕ solves $-\Delta\phi = h + pw_0^{p-1}\phi - \varepsilon^2\, a(\varepsilon z)\phi := \tilde{h}$ in $|\xi| < \delta\varepsilon^{-1}$. From Step 1, we have that $|\tilde{h}| \le \frac{\|h\|_{\varepsilon,r}}{(1+|\xi|^r)}$.. Elliptic estimates give that $|\phi| \le \frac{C}{(1+|\xi|^{r-2})}$.

Let us now fix a point $e \in \mathbb{R}^N$ and a positive number $R > 0$. Perform the change of variables $\tilde{\phi}(z,t) = R^{r-2}\,\phi(z, Rt+3Re)$, so that

$$\Delta\tilde{\phi} = R^r\tilde{h}(z, Rt + 3Re) \quad \text{in} \quad |t| \le 1.$$

Elliptic estimates give then that

$$\|D^2\tilde{\phi}\|_{L^\infty(B(0,1))} + \|D\tilde{\phi}\|_{L^\infty(B(0,1))} \le C\|R^r\tilde{h}(z, Rt + 3Re)\|_{L^\infty(B(0,2))}.$$

It then follows that

$$\|(1 + |\xi|)^r D^2\phi\|_{L^\infty(|\xi|\le\delta\varepsilon^{-1})} \le C\|(1 + |\xi|)^r h\|_{L^\infty(|\xi|\le\delta\varepsilon^{-1})}.$$

Arguing in a similar way, one gets the internal weighted estimate for the first derivative of ϕ

$$\|(1 + |\xi|)^{r-1} D\phi\|_{L^\infty(|\xi|\le\delta\varepsilon^{-1})} \le C\|(1 + |\xi|)^r h\|_{L^\infty(|\xi|\le\delta\varepsilon^{-1})}.$$

By using the representation formula for solution ϕ to the above equation, we see that $|\phi| \le C\varepsilon^{\frac{r-2}{2}}$ in $|\xi| < \delta\varepsilon^{-1}$. Furthermore, elliptic estimates give that in this region $|D\phi| \le C\varepsilon^{\frac{r-1}{2}}$ and $|D^2\phi| \le C\varepsilon^{\frac{r}{2}}$. This concludes the proof of (3.14).

Step 3. We shall now show that there exists $C > 0$ such that, if ϕ is a solution to (3.12), then

$$\|D^2_\xi\phi\|_{\varepsilon,r,\sigma} + \|D_\xi\phi\|_{\varepsilon,r-1,\sigma} + \|\phi\|_{\varepsilon,r-2,\sigma} \le C\|h\|_{\varepsilon,r,\sigma}. \tag{3.15}$$

Let us first assume we are in the region $|\xi| < \delta\varepsilon^{-1}$, and $z \in K_\varepsilon$. We first claim that from elliptic regularity, we have that if $\|h\|_{\varepsilon,r,\sigma} \le C$ then $\|\phi\|_{\varepsilon,r-2,\sigma} \le C$. Thus, we write that ϕ solves $-\Delta\phi = \tilde{h}$ in $|\xi| < \delta\varepsilon^{-1}$ where $\|\tilde{h}\|_{\varepsilon,r,\sigma} \le C$.

Arguing as in the previous step, we fix a point $e \in \mathbb{R}^N$ and a positive number $R > 0$. Perform the change of variables $\tilde{\phi}(z,t) = \phi(z, Rt + 3Re)$, so that

$$\Delta\tilde{\phi} = \frac{1}{R^{r-2}}\tilde{h} \quad \text{in} \quad |t| \le 1$$

where $|\tilde{h}| \le \frac{c}{|t+3e|^r}$. Elliptic estimates give then that $\|R^{r-2}D^2\tilde{\phi}\|_{C^{0,\sigma}(B(0,1))} \le C\|\tilde{h}\|_{L^\infty(B(0,2))}$. This implies that

$$R^{r-2}\|D^2_\xi\tilde{\phi}\|_{L^\infty(B_1)} + R^{r-2}[D^2\tilde{\phi}]_{\sigma,B(0,1)} \le C.$$

In particular, we have for any $z \in K_\varepsilon$, that

$$R^{r-2} \sup_{y_1,y_2\in B(0,1)} \frac{|D^2\tilde{\phi}(z,y_1) - D^2\tilde{\phi}(z,y_2)|}{|y_1 - y_2|^\sigma} \le C.$$

This inequality gets translated in term of ϕ as

$$R^{r+\sigma} \sup_{\xi_1,\xi_2\in B(\xi,1)} \frac{|D^2\phi(z,\xi_1) - D^2\phi(z,\xi_2)|}{|\xi_1 - \xi_2|^\sigma} \le C.$$

In a very similar way, one gets the estimate on $D\phi$. This concludes the proof of (3.15).

Step 4. Differentiating equation (3.12) with respect to the z variable l times and using elliptic regularity estimates, one proves that

$$\|D^l_y\phi\|_{\varepsilon,r-2,\sigma} \le C_l \left(\sum_{k\le l} \|D^k_y h\|_{\varepsilon,r,\sigma} \right) \tag{3.16}$$

for any given integer l.

Step 5. Assume now that the function b_{ij} and b_i in (3.4) are not zero, and assume that ϕ is a solution of problem (3.4), then by (3.15) we obtain

$$\|D^2_\xi\phi\|_{\varepsilon,r,\sigma} + \|D_\xi\phi\|_{\varepsilon,r-1,\sigma} + \|\phi\|_{\varepsilon,r-2,\sigma}$$

$$\leq \quad C\|h\|_{\varepsilon,r,\sigma} + C\|b_{ij}\partial_{ij}\phi\|_{\varepsilon,r,\sigma} + C\|b_i\partial_i\phi\|_{\varepsilon,r,\sigma}.$$

By definition of the norms and from (3.9), we have

$$\|b_{ij}\partial_{ij}\phi\|_{\varepsilon,r,\sigma} + \|b_i\partial_i\phi\|_{\varepsilon,r,\sigma} \leq C\eta\left(\|D^2_\xi\phi\|_{\varepsilon,r,\sigma} + \|D_\xi\phi\|_{\varepsilon,r-1,\sigma} + \|\phi\|_{\varepsilon,r-2,\sigma}\right).$$

Therefore, taking $\eta > 0$ small enough, we get (3.10). Also we get (3.11) as a consequence of (3.16).

Step 6. Now we shall prove the existence of the solution ϕ to problem (3.12). We consider the Hilbert space $\mathcal{H}$ defined as the subspace of functions ψ which are in $H^1(\mathcal{D})$ such that $\frac{\partial\psi}{\partial\nu} = 0$ on Γ_1 and $\psi = 0$ on $\Gamma_2 \cup \Gamma_3$, and

$$\int_{\hat{\mathcal{D}}} \psi(\varepsilon z, \xi) Z_j(\xi)\, d\xi = 0 \quad \text{for all } z \in K_\varepsilon, \quad j = 0, \dots N-1.$$

Define a bilinear form in $\mathcal{H}$ by

$$B(\phi,\psi) := \int_{\hat{\mathcal{D}}} \psi L\phi.$$

Then problem (3.4) gets weakly formulated as that of finding $\phi \in \mathcal{H}$ such that

$$B(\phi,\psi) = \int_{\hat{\mathcal{D}}} h\psi \quad \forall\, \psi \in \mathcal{H}.$$

By the Riesz representation theorem, this is equivalent to solve

$$\phi = T(\phi) + \tilde{h}$$

with $\tilde{h} \in \mathcal{H}$ depending linearly on h, and $T : \mathcal{H} \to \mathcal{H}$ being a compact operator. Fredholm's alternative guarantees that there is a unique solution to problem (3.4) for any h provided that

$$\phi = T(\phi) \tag{3.17}$$

has only the zero solution in $\mathcal{H}$. Equation (3.17) is equivalent to problem (3.4) with $h = 0$. If $h = 0$, the estimate in (3.10) implies that $\phi = 0$.

This concludes the proof of Proposition 3.1. □

4. Construction of a local approximation

This section is devoted to build an approximate solution to problem (3.1), which can be made arbitrarily accurate. In other words, we construct a function so that $\|\mathcal{S}_\varepsilon(v)\|_{\varepsilon,N-2,\sigma}$ (see (3.2)) is arbitrarily small in the set $\mathcal{D} = K_\varepsilon \times \hat{\mathcal{D}}$ (see (2.15)).

We have the validity of the following

Lemma 4.1. *For any integer $I \in \mathbb{N}$ there exist a constant $C > 0$, a smooth function $\mu_\varepsilon : K \to \mathbb{R}$, a smooth normal section $\Phi_\varepsilon : K \to NK^{\partial\Omega}$, of the form $\Phi_\varepsilon(y) = \Phi^j_\varepsilon(y)E_j$ for $y \in K$ and $j = 1, \dots, N-1$ such that*

$$\|\mu_\varepsilon\|_{L^\infty(K)} + \|\partial_a\mu_\varepsilon\|_{L^\infty(K)} + \|\partial^2_a\mu_\varepsilon\|_{L^\infty(K)} \leq C \tag{4.1}$$

$$\|\Phi_\varepsilon\|_{L^\infty(K)} + \|\partial_a\Phi_\varepsilon\|_{L^\infty(K)} + \|\partial^2_a\Phi_\varepsilon\|_{L^\infty(K)} \leq C \tag{4.2}$$

and a positive function $v_{I+1,\varepsilon} : K_\varepsilon \times \hat{\mathcal{D}} \to \mathbb{R}$ such that

$$-\mathcal{A}_{\mu_\varepsilon,\Phi_\varepsilon}(v_{I+1,\varepsilon}) + \varepsilon^2\mu_\varepsilon^2 v_{I+1,\varepsilon} - \mu_\varepsilon^{\mp\frac{N-2}{2}\varepsilon} v_{I+1,\varepsilon}^{p\pm\varepsilon} = \mathcal{E}_{I+1,\varepsilon} \quad \text{in} \quad K_\varepsilon \times \hat{\mathcal{D}}$$

$$\frac{\partial v_{I+1,\varepsilon}}{\partial\nu} = 0 \quad \text{on} \quad \Gamma_1, \qquad v_{I+1,\varepsilon} = 0 \quad \text{on} \quad \Gamma_2 \cup \Gamma_3$$

with

$$\|v_{I+1,\varepsilon} - v_{I,\varepsilon}\|_{\varepsilon,N-4,\sigma} \leq C\varepsilon^{1+I} \tag{4.3}$$

and

$$\|\mathcal{E}_{I+1,\varepsilon}\|_{\varepsilon,N-2,\sigma} \leq C\varepsilon^{1+I}. \tag{4.4}$$

We refer to (2.20) for the definition of $\mathcal{A}_{\mu_\varepsilon,\Phi_\varepsilon}$, to (2.15) for $K_\varepsilon \times \hat{\mathcal{D}}$ and to (2.16), (2.17) and (2.18) respectively for Γ_1, Γ_2 and Γ_3.

The idea of the construction of $v_{I+1,\varepsilon}$ is the following. Let I be an integer. By an iterative scheme we define

$$\mu_\varepsilon = \mu_0 + \mu_{1,\varepsilon} + \cdots + \mu_{I,\varepsilon}, \tag{4.5}$$

where μ_0, $\mu_{1,\varepsilon}, \dots, \mu_{I,\varepsilon}$ are smooth positive functions on K,

$$\Phi_\varepsilon = \Phi_{1,\varepsilon} + \cdots + \Phi_{I,\varepsilon}, \tag{4.6}$$

where $\Phi_{1,\varepsilon}$, ..., $\Phi_{I,\varepsilon}$ are smooth normal sections on Ω defined along K, and

$$v_{I+1,\varepsilon}(z, \bar{\xi}, \xi_N) \quad = \quad w_0(\xi) + w_{1,\varepsilon}(z,\xi) + \cdots + w_{I+1,\varepsilon}(z,\xi), \tag{4.7}$$

where w_0 is defined by (2.1) and the functions $w_{j,\varepsilon}$'s for $j \geq 1$ are to be determined accordingly so that the above function $v_{I+1,\varepsilon}$ satisfies "*formally*"

$$-\mathcal{A}_{\mu_\varepsilon,\Phi_\varepsilon} v_{I+1,\varepsilon} + \varepsilon^2\mu_\varepsilon^2 v_{I+1,\varepsilon} - \mu_\varepsilon^{\mp\varepsilon\frac{N-2}{2}} v_{I+1,\varepsilon}^{\frac{N+2}{N-2}\pm\varepsilon} = \mathcal{O}(\varepsilon^{1+I}) \quad \text{in} \quad K_\varepsilon \times \hat{\mathcal{D}}.$$

We start with $I = 0$.

• **Construction of** $w_{1,\varepsilon}$**,** μ_0 **:** Let $\mu_\varepsilon = \mu_0$, $\Phi_\varepsilon = 0$ and $v_{1,\varepsilon} = w_0 + w_{1,\varepsilon}$. Using the expansion of Laplace-Beltrami operator given in Lemma 2.1, and the fact that w_0 solves (2.2) we have

$$\begin{aligned}
& -\mathcal{A}_{\mu_\varepsilon,\Phi_\varepsilon} v_{1,\varepsilon} + \varepsilon^2\mu_\varepsilon^2 v_{1,\varepsilon} - \mu_\varepsilon^{\mp\varepsilon\frac{N-2}{2}} v_{1,\varepsilon}^{p\pm\varepsilon} \\
= \; & -\mathcal{A}_{\mu_0}\left(w_0(\xi) + w_{1,\varepsilon}(z,\xi)\right) + \varepsilon^2\mu_0^2\left(w_0(\xi) + w_{1,\varepsilon}(z,\xi)\right) \\
& -\mu_0^{\mp\varepsilon\frac{N-2}{2}}\left(w_0(\xi) + w_{1,\varepsilon}(z,\xi)\right)^{p\pm\varepsilon} \\
= \; & \left(-\Delta_{\mathbb{R}^N} w_{1,\varepsilon} - p w_0^{p-1} w_{1,\varepsilon} + \varepsilon^2\mu_0^2 w_{1,\varepsilon} + \varepsilon\mu_0 H_\alpha^\alpha \partial_{\xi_N} w_{1,\varepsilon} - 2\varepsilon\mu_0\xi_N H_{ij}\partial_{ij}^2 w_{1,\varepsilon}\right) \\
& -\varepsilon\left[\pm\left(-\frac{N-2}{2} w_0^p \log\mu_0 + \log(w_0) w_0^p\right) - \mu_0(H_\alpha^\alpha \partial_{\xi_N} w_0 - 2\xi_N H_{ij}\partial_{ij}^2 w_0)\right] \\
& +\mathcal{E}_{1,\varepsilon} + \mathcal{L}_\varepsilon(w_{1,\varepsilon}) + Q_\varepsilon(w_{1,\varepsilon}).
\end{aligned}$$

The function $\mathcal{E}_{1,\varepsilon}$ is a function which is a sum of functions of the form

$$\varepsilon\mu_0\left(\varepsilon\mu_0 + \varepsilon\partial_a\mu_0 + \varepsilon\partial_a^2\mu_0\right) a(\varepsilon z) b(\xi)$$

where $a(\varepsilon z)$ is a smooth function uniformly bounded, together with its derivatives, in ε as $\varepsilon \to 0$, while the function b is such that

$$\sup_\xi (1 + |\xi|^{N-2})|b(\xi)| < \infty.$$

The term $\mathcal{L}_\varepsilon(w_{1,\varepsilon})$ has the form

$$\mathcal{L}_\varepsilon(w_{1,\varepsilon}) = -\mu_0^2 \Delta_{K_\varepsilon} w_{1,\varepsilon} + \varepsilon a_1(\varepsilon z)\partial_{z_a} w_{1,\varepsilon} + \varepsilon a_2(\varepsilon z)\partial_{z_a}\nabla w_{1,\varepsilon} \tag{4.8}$$

where a_1 and a_2 are smooth and uniformly bounded functions on K. The term $Q_\varepsilon(w_{1,\varepsilon})$ is quadratic in $w_{1,\varepsilon}$, in fact it is explicitly given by

$$\mu_0^{\mp\frac{N-2}{2}\varepsilon}\left[(w_0 + w_{1\varepsilon})^{p\pm\varepsilon} - w_0^{p\pm\varepsilon} - p w_0^{p-1\pm\varepsilon} w_{1,\varepsilon}\right].$$

Let

$$h_{1,\varepsilon}(\varepsilon z,\xi) = \pm\left(-\frac{N-2}{2} w_0^p \log\mu_0 + \log(w_0) w_0^p\right) - \mu_0\left(H_\alpha^\alpha \partial_{\xi_N} w_0 - 2\xi_N H_{ij}\partial_{ij}^2 w_0\right) \tag{4.9}$$

and

$$\begin{aligned}
L_{1,\varepsilon} w_{1,\varepsilon} \quad := \quad & -\Delta_{\mathbb{R}^N} w_{1,\varepsilon} - p w_0^{p-1} w_{1,\varepsilon} + \varepsilon^2\mu_0^2 w_{1,\varepsilon} \\
+ \quad & \varepsilon\mu_0 H_\alpha^\alpha \partial_{\xi_N} w_{1,\varepsilon} - 2\varepsilon\mu_0\xi_N H_{ij}\partial_{ij}^2 w_{1,\varepsilon}.
\end{aligned} \tag{4.10}$$

We define $w_{1,\varepsilon}$ to be solution of the Problem

$$\begin{cases}
L_{1,\varepsilon} w_{1,\varepsilon} = \varepsilon h_{1,\varepsilon}(\varepsilon z,\xi) & \text{in } \mathcal{D} \\
\frac{\partial w_{1,\varepsilon}}{\partial \xi_N} = 0, & \text{on } \Gamma_1 \\
w_{1,\varepsilon} = 0, & \text{on } \Gamma_2 \cup \Gamma_3. \\
\int_D w_{1,\varepsilon}(\varepsilon z,\xi) Z_j(\xi)\, d\xi = 0 & \text{for all } z \in K_\varepsilon, \quad j = 0,\dots N-1.
\end{cases} \tag{4.11}$$

Observe that, given the result of Proposition 3.1, (4.11) is solvable if the L^2 product of the right-hand side with Z_i for $i = 0, 1, \dots, N-1$ vanishes. We note that w_0^p is orthogonal to all Z_j's, for $j = 0, \dots, N-1$, and that the projection onto Z_i for $i = 1, \dots, N-1$ is clearly satisfied since both $\partial_N w_0$ and $\partial_{ij}^2 w_0$ are even in $\bar{\xi}$, while the Z_i's are odd in $\bar{\xi}$ for every i. It remains to compute the L^2 product of the right-hand side with Z_0. We claim that

$$\int_{\hat{\mathcal{D}}} \left(H_{\alpha\alpha}\partial_N w_0 - 2H_{ij}\xi_N\,\partial_{ij}^2 w_0\right) Z_0\, d\xi = \left[\mathfrak{A}_0\, H_{\alpha\alpha} - \mathfrak{A}_1 H_{ii}\right] + O(\varepsilon^{N-3}) \tag{4.12}$$

where $\mathfrak{A}_0$ and $\mathfrak{A}_1$ are positive constants defined by

$$\mathfrak{A}_0 = \int_{\mathbb{R}^N_+} \partial_N w_0 Z_0, \quad \mathfrak{A}_1 = \int_{\mathbb{R}^N_+} \xi_N |\partial_1 w_0|^2 > 0. \tag{4.13}$$

Moreover,

$$\int_{\hat{\mathcal{D}}} \left(-\frac{N-2}{2} \log \mu_0 + \log(w_0) \right) w_0^p \, Z_0 = \frac{(N-2)^2}{4N} \, C_0 + O(\varepsilon^{N-2}) \tag{4.14}$$

where C_0 is a constant given by

$$C_0 := \int_{\mathbb{R}^N_+} w_0^{\frac{2N}{N-2}} = C_N^{\frac{2N}{N-2}} \int_{\mathbb{R}^N_+} \frac{d\xi}{(1+|\xi|^2)^N}.$$

We postpone the proofs of (4.12) and (4.14) to the Appendix, Section 7. We turn now to the solvability in $w_{1,\varepsilon}$. If in (1.17), or in (3.1), we consider $p+\varepsilon$, we define

$$\begin{aligned} \mu_0(y) &:= \frac{\frac{(N-2)^2}{4N} \, C_0}{[\mathfrak{A}_0 \, H_{\alpha\alpha} - \mathfrak{A}_1 H_{ii}]} (1 + O(\varepsilon^{N-3})) \\ &= \frac{\frac{(N-2)^2}{4N} \, C_0}{\mathfrak{A}_1 \, [2 \, H_{\alpha\alpha} - H_{ii}]} (1 + O(\varepsilon^{N-3})), \end{aligned} \tag{4.15}$$

where we use the relation $\mathfrak{A}_0 = 2\mathfrak{A}_1$, which implies

$$\mathfrak{A}_0 \, H_{\alpha\alpha} - \mathfrak{A}_1 H_{ii} = \mathfrak{A}_1 \, [2 \, H_{\alpha\alpha} - H_{ii}] \, ,$$

whose proof can be found in [14]. On the other hand, if we consider $p-\varepsilon$, we define

$$\begin{aligned} \mu_0(y) &:= \frac{\frac{(N-2)^2}{4N} \, C_0}{-\,[\mathfrak{A}_0 \, H_{\alpha\alpha} - \mathfrak{A}_1 H_{ii}]} (1 + O(\varepsilon^{N-3})) \\ &= \frac{\frac{(N-2)^2}{4N} \, C_0}{-\mathfrak{A}_1 \, [2 \, H_{\alpha\alpha} - H_{ii}]} (1 + O(\varepsilon^{N-3})). \end{aligned} \tag{4.16}$$

In both cases, under the assumptions of Theorem 1.1 the above definitions define a function μ_0 which is strictly positive along K. Furthermore, μ_0 is a smooth function on K, uniformly bounded, together with its first and second derivatives, with respect to ε, as $\varepsilon \to 0$. With this choice for μ_0, the integral of the right hand side in (4.11) against Z_0 also vanishes on K. Furthermore, the operator $L_{1,\varepsilon}$ has the form (3.3). Thus choosing δ in the definition of $\mathcal{D}$ in (2.15) sufficiently small, we have that condition (3.9) is satisfied. Moreover, it is straightforward to check that

$$\|h_{1,\varepsilon}\|_{\varepsilon,N-1,\sigma} \le C$$

for some $\sigma \in (0,1)$. Thus Proposition 3.1 guarantees the existence of $w_{1,\varepsilon}$ solution of (4.11). Using again Proposition 3.1 we get that

$$\|D^2_\xi w_{1,\varepsilon}\|_{\varepsilon,N-1,\sigma} + \|D_\xi w_{1,\varepsilon}\|_{\varepsilon,N-2,\sigma} + \|w_{1,\varepsilon}\|_{\varepsilon,N-3,\sigma} \le C\varepsilon \tag{4.17}$$

and that there exists a positive constant λ (depending only on Ω, K and N) such that for any integer ℓ there holds

$$\|\nabla^{(\ell)}_y w_{1,\varepsilon}(z,\cdot)\|_{\varepsilon,N-3,\sigma} \le \lambda C_l \varepsilon \qquad z \in K_\varepsilon \tag{4.18}$$

where C_ℓ depends only on ℓ, p, K and Ω.

With this definition of μ_0 and $w_{1,\varepsilon}$, we have in particular that

$$\| -\mathcal{A}_{\mu_\varepsilon,\Phi_\varepsilon} v_{1,\varepsilon} + \varepsilon^2 \mu_\varepsilon^2 v_{1,\varepsilon} - \mu_\varepsilon^{\mp\varepsilon \frac{N-2}{2}} \, v_{1,\varepsilon}^{p\pm\varepsilon} \|_{\varepsilon,N-2,\sigma} \le C\varepsilon^2.$$

• **Construction of** $w_{2,\varepsilon}$**,** $\mu_{1,\varepsilon}$**,** $\Phi_{1,\varepsilon}$ **:** Let $\mu_\varepsilon = \mu_0 + \mu_{1,\varepsilon}$, $\Phi = \Phi_{1,\varepsilon}$ and $v_{2,\varepsilon}(z,\xi) = w_0(\xi) + w_{1,\varepsilon}(z,\xi) + w_{2,\varepsilon}(z,\xi)$, where μ_0 and $w_{1,\varepsilon}$ have already been constructed in the previous step. Computing $\mathcal{S}_\varepsilon(v_{2,\varepsilon})$ (see (3.2)) we get

$$\begin{aligned} -\mathcal{A}_{\mu_\varepsilon,\Phi_\varepsilon} v_{2,\varepsilon} + \varepsilon^2 \mu_\varepsilon^2 v_{2,\varepsilon} - \mu_\varepsilon^{\mp\varepsilon \frac{N-2}{2}} \, v_{2,\varepsilon}^{p\pm\varepsilon} &= L_{1,\varepsilon} w_{2,\varepsilon} - \varepsilon h_{2,\varepsilon} + \varepsilon \mathcal{E}_{2,\varepsilon} \\ &+ \mathcal{L}_\varepsilon(w_{2,\varepsilon}) + Q_\varepsilon(w_{2,\varepsilon}) \end{aligned} \tag{4.19}$$

where $L_{1,\varepsilon}$ is defined in (4.10), the operator $\mathcal{L}_\varepsilon$ is defined in (4.8) and the function $h_{2,\varepsilon}$ is given by

$$\begin{aligned} h_{2,\varepsilon} &= -\mu_{1,\varepsilon}(y) \Big[H^\alpha_\alpha \partial_{\xi_N} w_0 - 2\xi_N H_{ij} \partial^2_{ij} w_0 \Big] - \varepsilon \mu_0^2 w_0 \qquad (4.20) \\ &\quad -\varepsilon \, \mu_0 \, \Delta_K \Phi^j_{1,\varepsilon} \, \partial_j w_0 - \varepsilon \, \frac{1}{3} \, \mu_0 \, R_{mijl} (\xi_m \Phi^l_{1,\varepsilon} + \xi_l \Phi^m_{1,\varepsilon}) \, \partial^2_{ij} w_0 \\ &\quad +\varepsilon \, \frac{2}{3} \, \mu_0 \, R_{mssj} \, \Phi^m_{1,\varepsilon} \, \partial_j w_0 + \varepsilon \, \mu_0 \left(\tilde{g}^{ab}_\varepsilon \, R_{maaj} - \Gamma^c_a(E_m) \Gamma^a_c(E_j) \right) \Phi^m_{1,\varepsilon} \, \partial_j w_0 \\ &\quad +\varepsilon \mathfrak{G}_{2,\varepsilon}(\xi, z, w_0, w_{1,\varepsilon}, \mu_0). \qquad (4.21) \end{aligned}$$

In (4.20) $\mathfrak{G}_{2,\varepsilon}(\xi, z, w_0, w_{1,\varepsilon}, \mu_0)$ is the sum of functions of the form

$$\mathcal{Q}(\mu_0, \partial_a \mu_0, \partial_a^2 \mu_0) a(\varepsilon z) b(\xi)$$

where $\mathcal{Q}$ denotes a quadratic function of its arguments, $a(\varepsilon z)$ is a smooth function uniformly bounded, together with its derivatives, in ε as $\varepsilon \to 0$, while the function b is such that

$$\sup_{\xi}(1 + |\xi|^{N-2})|b(\xi)| < \infty.$$

In (4.19) the term $\mathcal{E}_{2,\varepsilon}$ can be described as the sum of functions of the form

$$\left(\varepsilon \mathcal{L}(\mu_1, \varepsilon\Phi_1) + \mathcal{Q}(\mu_1, \varepsilon\Phi_1)\right) a(\varepsilon z) b(\xi)$$

where $(\mu_1, \varepsilon\Phi_1) = (\mu_{1,\varepsilon}, \partial_a \mu_{1,\varepsilon}, \partial_a^2 \mu_{1,\varepsilon}, \varepsilon\Phi_{1,\varepsilon}, \varepsilon\partial_a \Phi_{1,\varepsilon}, \varepsilon\partial_a^2 \Phi_{1,\varepsilon})$, $\mathcal{L}$ denotes a linear function of its arguments, $\mathcal{Q}$ denotes a quadratic function of its arguments, $a(\varepsilon z)$ is a smooth function uniformly bounded, together with its derivatives, in ε as $\varepsilon \to 0$, while the function b is such that

$$\sup_{\xi}(1 + |\xi|^{N-2})|b(\xi)| < \infty.$$

Finally the term $Q_\varepsilon(w_{2,\varepsilon})$ in (4.19) is a sum of quadratic terms in $w_{2,\varepsilon}$ like

$$(\mu_0 + \mu_1)^{\mp\frac{N-2}{2}\varepsilon} \left[(w_0 + w_{1,\varepsilon} + w_{2,\varepsilon})^{p\pm\varepsilon} - (w_0 + w_{1,\varepsilon})^{p\pm\varepsilon} - (p \pm \varepsilon)(w_0 + w_{1,\varepsilon})^{p-1\pm\varepsilon} w_{2,\varepsilon}\right]$$

and linear terms in $w_{2,\varepsilon}$ multiplied by a term of order ε^2, like

$$p\left((w_0 + w_{1,\varepsilon})^{p-1\pm\varepsilon} - w_0^{p-1\pm\varepsilon}\right) w_{2,\varepsilon}.$$

Consider now the following equation

$$\begin{cases} L_{1,\varepsilon} w_{2,\varepsilon} = \varepsilon h_{2,\varepsilon}(\varepsilon z, \xi) & \text{in } \mathcal{D} \\ \frac{\partial w_{2,\varepsilon}}{\partial \xi_N} = 0, & \text{on } \Gamma_1 \\ w_{2,\varepsilon} = 0, & \text{on } \Gamma_2 \cup \Gamma_3 \\ \int_D w_{2,\varepsilon}(\varepsilon z, \xi) Z_j(\xi)\, d\xi = 0 & \text{for all } z \in K_\varepsilon, \quad j = 0, \dots N-1. \end{cases} \tag{4.22}$$

Again by Proposition 3.1, to guarantee solvability of (4.22) , we need that $h_{2,\varepsilon}$ is L^2-orthogonal to Z_j, $j = 0, 1, \cdots, N-1$. These orthogonality conditions will define the parameters $\mu_{1,\varepsilon}$ and the normal section $\Phi_{1,\varepsilon}$.

Projection onto Z_0 and choice of $\mu_{1,\varepsilon}$: We define $\mu_{1,\varepsilon}$ to make the following quantity zero.

$$\int_{\hat{\mathcal{D}}} h_{2,\varepsilon} Z_0 = 0.$$

The above relation defines $\mu_{1,\varepsilon}$ as a smooth function of εz in K. From estimates (4.17) for $w_{1,\varepsilon}$ we get that

$$\|\mu_{1,\varepsilon}\|_{L^\infty(K)} + \|\partial_a \mu_{1,\varepsilon}\|_{L^\infty(K)} + \|\partial_a^2 \mu_{1,\varepsilon}\|_{L^\infty(K)} \le C\varepsilon. \tag{4.23}$$

Projection onto Z_l and choice of $\Phi_{1,\varepsilon}$: Multiplying $h_{2,\varepsilon}$ with $Z_l = \partial_l w_0$, integrating over $\hat{\mathcal{D}}$ and using the fact w_0 is even in the variable $\bar{\xi}$, one obtains

$$\begin{aligned} \int_{\hat{\mathcal{D}}} h_{2,\varepsilon}\, \partial_l w_0 \quad &= \quad -\varepsilon\, \mu_0\, \Delta_K \Phi_{1,\varepsilon}^j \int_{\hat{\mathcal{D}}} \partial_j w_0 \partial_l w_0 + \varepsilon \int_{\hat{\mathcal{D}}} \mathfrak{G}_{2,\varepsilon}\, \partial_l w_0 \\ &- \quad \varepsilon\, \frac{1}{3}\, \mu_0\, R_{mijs} \int_{\hat{\mathcal{D}}} (\xi_m \Phi_{1,\varepsilon}^s + \xi_s \Phi_{1,\varepsilon}^m)\, \partial_{ij}^2 w_0 \partial_l w_0 \\ &+ \quad \varepsilon\, \mu_0 \left[\frac{2}{3} R_{mssj}\, \Phi_{1,\varepsilon}^m + \left(\tilde{g}_\varepsilon^{ab}\, R_{maaj} - \Gamma_a^c(E_m)\Gamma_c^a(E_j)\right) \Phi_{1,\varepsilon}^m\right] \int_{\hat{\mathcal{D}}} \partial_j w_0\, \partial_l w_0. \end{aligned} \tag{4.24}$$

First of all, observe that by oddness in $\bar{\xi}$ we have that

$$\int_{\hat{\mathcal{D}}} \partial_j w_0 \partial_l w_0 = \delta_{lj} \left(\int_{\mathbb{R}_+^N} |\partial_l w_0|^2 + O(\varepsilon^{N-2})\right) = \delta_{lj}\, C_0 + O(\varepsilon^{N-2})$$

with $C_0 := \int_{\mathbb{R}_+^N} |\partial_l w_0|^2$. On the other hand the integral $\int_{\hat{\mathcal{D}}} \xi_m\, \partial_{ij}^2 w_0 \partial_l w_0$ is non-zero only if, either $i = j$ and $m = l$, or $i = l$ and $j = m$, or $i = m$ and $j = l$. In the latter case we have $R_{mijs} = 0$ (by the antisymmetry of the curvature tensor in the first two indices). Therefore, the first term of the second line of the above formula becomes simply

$$\begin{aligned} R_{mijs} \int_{\hat{\mathcal{D}}} \xi_m \Phi_{1,\varepsilon}^s\, \partial_{ij}^2 w_0 \partial_l w_0 = \sum_i R_{liis} \Phi_{1,\varepsilon}^s \int_{\hat{\mathcal{D}}} \xi_l \partial_l w_0 \partial_{ii}^2 w_0 d\xi \\ + \sum_i R_{jijs} \Phi_{1,\varepsilon}^s \int_{\hat{\mathcal{D}}} \xi_j \partial_i w_0 \partial_{ij}^2 w_0 d\xi = \sum_i R_{liis} \Phi_{1,\varepsilon}^s \int_{\mathbb{R}_+^N} \xi_l \partial_l w_0 \partial_{ii}^2 w_0 d\xi \end{aligned}$$

$$+\sum_i R_{jijs}\Phi^s_{1,\varepsilon}\int_{\mathbb{R}^N_+}\xi_j\partial_i w_0\partial^2_{ij}w_0 d\xi+O(\varepsilon^{N-2}).$$

Observe that, integrating by parts, when $l\neq i$ (otherwise $R_{liis}=0$) there holds

$$\begin{aligned}\int_{\hat{\mathcal{D}}}\xi_l\partial_l w_0\partial^2_{ii}w_0 d\xi &= \int_{\mathbb{R}^N_+}\xi_l\partial_l w_0\partial^2_{ii}w_0 d\xi+O(\varepsilon^{N-2})\\ &= -\int_{\mathbb{R}^N_+}\xi_l\partial_i w_0\partial^2_{li}w_0 d\xi+O(\varepsilon^{N-2}).\end{aligned}$$

On the other hand, we have

$$-\frac{2\mu_0}{3}\sum_i R_{ijjs}\Phi^s_{1,\varepsilon}\int_{\hat{\mathcal{D}}}\xi_j\partial_i w_0\partial^2_{ij}w_0 d\xi=-\frac{2\mu_0}{3}\sum_i R_{ijjs}\Phi^s_{1,\varepsilon}\int_{\mathbb{R}^N_+}\xi_j\partial_i w_0\partial^2_{ij}w_0 d\xi+O(\varepsilon^{N-2}).$$

The last integral can be computed with a further integration by parts and is equal to $-\frac{1}{2}C_0$, so we get

$$-\frac{2\mu_0}{3}\sum_i R_{ijjs}\Phi^s_{1,\varepsilon}\int_{\hat{\mathcal{D}}}\xi_j\partial_i w_0\partial^2_{ij}w_0 d\xi=\frac{\mu_0}{3}\,C_0\sum_i R_{ijjs}\Phi^s_{1,\varepsilon}+O(\varepsilon^{N-2}).$$

In a similar way (permuting the indices s and m in the above argument), one obtains

$$\frac{1}{3}\,\mu_0\,R_{sijm}\int_{\hat{\mathcal{D}}}\xi_s\Phi^m_{1,\varepsilon}\,\partial^2_{ij}w_0\partial_l w_0=\frac{\mu_0}{3}\,C_0\,\sum_i R_{ijjm}\Phi^m_{1,\varepsilon}+O(\varepsilon^{N-2}).$$

Collecting the above computations, we conclude that

$$-\frac{1}{3}\,\mu_0\,R_{mijs}\int_{\hat{\mathcal{D}}}(\xi_m\Phi^s_{1,\varepsilon}+\xi_s\Phi^m_{1,\varepsilon})\,\partial^2_{ij}w_0\partial_l w_0+\frac{2}{3}\,\mu_0\,R_{mssj}\,\Phi^m_{1,\varepsilon}\int_{\hat{\mathcal{D}}}\partial_j w_0\,\partial_l w_0=O(\varepsilon^{N-2}).$$

Hence formula (4.24) becomes simply

$$\begin{aligned}\int_{\hat{\mathcal{D}}}h_{2,\varepsilon}\,\partial_l w_0 &= -\varepsilon\,\mu_0\,C_0\,\Delta_K\,\Phi^l_{1,\varepsilon}+\varepsilon\,\mu_0\,C_0\left(\tilde g^{ab}_\varepsilon\,R_{maal}-\Gamma^c_a(E_m)\Gamma^a_c(E_l)\right)\Phi^m_{1,\varepsilon}\\ &\quad+\mu_0C_0O(\varepsilon^{N-1})\,\Phi^m_{1,\varepsilon}+\varepsilon\int_{\hat{\mathcal{D}}}\mathfrak{G}_{2,\varepsilon}\,\partial_l w_0.\end{aligned}$$

We thus obtain that $h_{2,\varepsilon}(z,\xi,w_0,\dots,w_{1,\varepsilon})$, the right-hand side of (4.22), is L^2-orthogonal to Z_l ($l=1,\cdots,N-1$) if and only if $\Phi_{1,\varepsilon}$ satisfies an equation of the form

$$\Delta_K\,\Phi^l_{1,\varepsilon}-\left(\tilde g^{ab}_\varepsilon\,R_{maal}-\Gamma^c_a(E_m)\Gamma^a_c(E_l)+O(\varepsilon^{N-2})\right)\Phi^m_{1,\varepsilon}=G_{2,\varepsilon}(\varepsilon z), \tag{4.25}$$

for some expression $G_{2,\varepsilon}$ smooth on its argument. Observe that the operator acting on $\Phi_{1,\varepsilon}$ in the left hand side is nothing but the Jacobi operator, which is invertible by the non-degeneracy condition on K. This implies the solvability of the above equation in $\Phi_{1,\varepsilon}$. Furthermore, equation (4.41) defines $\Phi_{1,\varepsilon}$ as a smooth function on K, of order ε, more precisely we have

$$\|\Phi_{1,\varepsilon}\|_{L^\infty(K)}+\|\partial_a\Phi_{1,\varepsilon}\|_{L^\infty(K)}+\|\partial^2_a\Phi_{1,\varepsilon}\|_{L^\infty(K)}\le C. \tag{4.26}$$

By our choice of $\mu_{1,\varepsilon}$ and $\Phi_{1,\varepsilon}$ we have solvability of equation (4.22) in $w_{2,\varepsilon}$. Moreover, it is straightforward to check that

$$|\varepsilon h_{2,\varepsilon}(\varepsilon z,\xi)|\le C\varepsilon|\partial_{\xi_N}w_{1,\varepsilon}|\le C\frac{\varepsilon^2}{(1+|\xi|)^{N-2}}.$$

Furthermore, for a given $\sigma\in(0,1)$ we have

$$\|\varepsilon h_{2,\varepsilon}\|_{\varepsilon,N-2,\sigma}\le C\varepsilon^2.$$

Proposition 3.1 thus gives then that

$$\|D^2_\xi w_{2,\varepsilon}\|_{\varepsilon,N-2,\sigma}+\|D_\xi w_{2,\varepsilon}\|_{\varepsilon,N-3,\sigma}+\|w_{2,\varepsilon}\|_{\varepsilon,N-4,\sigma}\le C\varepsilon^2 \tag{4.27}$$

and that there exists a positive constant β (depending only on Ω, K and n) such that for any integer ℓ there holds

$$\|\nabla^{(\ell)}_y w_{2,\varepsilon}(z,\cdot)\|_{\varepsilon,N-2,\sigma}\le\beta C_\ell\,\varepsilon^2\qquad\varepsilon y=z\in K_\varepsilon \tag{4.28}$$

where C_ℓ depends only on ℓ, p, K and Ω.

With this choice of $\mu_{1,\varepsilon}$, $\Phi_{1,\varepsilon}$ and $w_{2,\varepsilon}$ we get that

$$\|-\mathcal{A}_{\mu_\varepsilon,\Phi_\varepsilon}v_{2,\varepsilon}+\varepsilon^2\mu^2_\varepsilon v_{2,\varepsilon}-\mu_\varepsilon^{\mp\varepsilon\frac{N-2}{2}}\,v_{2,\varepsilon}^{p\pm\varepsilon}\|_{\varepsilon,N-2,\sigma}\le C\varepsilon^3.$$

• **Expansion at an arbitrary order:** We take now an arbitrary integer I. Let

$$\mu_\varepsilon:=\mu_0+\mu_{1,\varepsilon}+\cdots+\mu_{I-1,\varepsilon}+\mu_{I,\varepsilon}, \tag{4.29}$$

$$\Phi=\Phi_{1,\varepsilon}+\cdots+\Phi_{I-1,\varepsilon}+\Phi_{I,\varepsilon} \tag{4.30}$$

and

$$v_{I+1,\varepsilon} = w_0(\xi) + w_{1,\varepsilon}(z,\xi) + \ldots + w_{I,\varepsilon}(z,\xi) + w_{I+1,\varepsilon}(z,\xi) \tag{4.31}$$

where $\mu_0, \mu_{1,\varepsilon}, \cdots, \mu_{I-1,\varepsilon}$, $\Phi_{1,\varepsilon}, \cdots, \Phi_{I-1,\varepsilon}$ and $w_{1,\varepsilon}$, .. , $w_{I,\varepsilon}$ have already been constructed following an iterative scheme, as described in the previous steps of the construction.

In particular one has, for any $i = 1, \ldots, I-1$

$$\|\mu_{i,\varepsilon}\|_{L^\infty(K)} + \|\partial_a \mu_{i,\varepsilon}\|_{L^\infty(K)} + \|\partial_a^2 \mu_{i,\varepsilon}\|_{L^\infty(K)} \leq C\varepsilon^i \tag{4.32}$$

$$\|\Phi_{i,\varepsilon}\|_{L^\infty(K)} + \|\partial_a \Phi_{i,\varepsilon}\|_{L^\infty(K)} + \|\partial_a^2 \Phi_{i,\varepsilon}\|_{L^\infty(K)} \leq C\varepsilon^{i-1} \tag{4.33}$$

and, for now $i = 0, \ldots, I-1$,

$$\|D_\xi^2 w_{i+1,\varepsilon}\|_{\varepsilon,N-2,\sigma} + \|D_\xi w_{i+1,\varepsilon}\|_{\varepsilon,N-3,\sigma} + \|w_{i+1,\varepsilon}\|_{\varepsilon,N-4,\sigma} \leq C\varepsilon^{1+i} \tag{4.34}$$

and, for any integer ℓ

$$\|\nabla_z^{(\ell)} w_{i+1,\varepsilon}(z,\cdot)\|_{\varepsilon,N-2,\sigma} \leq \beta C_l \varepsilon^{1+i}. \qquad z \in K_\varepsilon \tag{4.35}$$

The new triplet $(\mu_{I,\varepsilon}, \Phi_{I,\varepsilon}, w_{I+1,\varepsilon})$ will be found reasoning as in the construction of $(\mu_{1,\varepsilon}, \Phi_{1,\varepsilon}, w_{2,\varepsilon})$. Computing $\mathcal{S}_\varepsilon(v_{I+1,\varepsilon})$ (see (3.2)) we get

$$\begin{aligned} & -\mathcal{A}_{\mu_\varepsilon,\Phi_\varepsilon} v_{I+1,\varepsilon} + \varepsilon^2 \mu_\varepsilon^2 v_{I+1,\varepsilon} - \mu_\varepsilon^{\mp\varepsilon\frac{N-2}{2}} v_{I+1,\varepsilon}^{p\pm\varepsilon} \\ = \;& L_{1,\varepsilon} w_{I+1,\varepsilon} - \varepsilon h_{I+1,\varepsilon} + \varepsilon \mathcal{E}_{I+1,\varepsilon} + \mathcal{L}_\varepsilon(w_{I+1,\varepsilon}) + Q_\varepsilon(w_{I+1,\varepsilon}) \end{aligned} \tag{4.36}$$

where $L_{1,\varepsilon}$ is defined in (4.10), $\mathcal{L}_\varepsilon$ is defined by (4.8) and the function $h_{I+1,\varepsilon}$ is given by

$$\begin{aligned} h_{I+1,\varepsilon} \;=\; & -\mu_{I,\varepsilon}(y)\Big[H_\alpha^\alpha \partial_{\xi_N} w_0 - 2\xi_N H_{ij} \partial_{ij}^2 w_0\Big] \\ & -\varepsilon\,\mu_0\,\Delta_K \Phi_{I,\varepsilon}^j\,\partial_j w_0 - \varepsilon\,\frac{1}{3}\,\mu_0\,R_{mijl}(\xi_m \Phi_{I,\varepsilon}^l + \xi_l \Phi_{I,\varepsilon}^m)\,\partial_{ij}^2 w_0 \\ & +\varepsilon\,\frac{2}{3}\,\mu_0\,R_{mssj}\,\Phi_{I,\varepsilon}^m\,\partial_j w_0 + \varepsilon\,\mu_0\Big(\tilde g_\varepsilon^{ab}\,R_{maaj} - \Gamma_a^c(E_m)\Gamma_c^a(E_j)\Big)\,\Phi_{1,\varepsilon}^m\,\partial_j w_0 \\ & +\varepsilon \mathfrak{G}_{I+1,\varepsilon}(\xi, z, w_0, \cdots, w_{I,\varepsilon}, \mu_0, \cdots, \mu_{I-1,\varepsilon}). \end{aligned} \tag{4.37}$$

In (4.37) $\mathfrak{G}_{I+1,\varepsilon}(\xi, z,)$ is a smooth function with

$$\|\mathfrak{G}_{I+1,\varepsilon}\|_{\varepsilon,N-2,\sigma} \leq C\varepsilon^{I-1}. \tag{4.38}$$

In (4.36) the term $\mathcal{E}_{I+1,\varepsilon}$ can be described as the sum of functions of the form

$$(\varepsilon\mathcal{L}(\mu_I, \varepsilon\Phi_I) + \mathcal{Q}(\mu_I, \Phi_I))\,a(\varepsilon z) b(\xi)$$

where $(\mu_I, \varepsilon\Phi_I) = (\mu_{I,\varepsilon}, \partial_a \mu_{I,\varepsilon}, \partial_a^2 \mu_{I,\varepsilon}, \varepsilon\Phi_{I,\varepsilon}, \varepsilon\partial_a \Phi_{I,\varepsilon}, \varepsilon\partial_a^2 \Phi_{I,\varepsilon})$, $\mathcal{L}$ denotes a linear function of its arguments, $\mathcal{Q}$ denotes a quadratic function of its arguments, $a(\varepsilon z)$ is a smooth function uniformly bounded, together with its derivatives, in ε as $\varepsilon \to 0$, while the function b is such that

$$\sup_\xi (1 + |\xi|^{N-2})|b(\xi)| < \infty.$$

Finally the term $Q_\varepsilon(w_{I+1,\varepsilon})$ in (4.36) is a sum of quadratic terms in $w_{I+1,\varepsilon}$ like

$$\begin{aligned} & (\mu_0 + \mu_{1,\varepsilon} + \cdots + \mu_{I-1,\varepsilon} + \mu_{I,\varepsilon})^{\mp\frac{N-2}{2}\varepsilon} \big[(w_0 + w_{1,\varepsilon} + \ldots + w_{I+1,\varepsilon})^{p\pm\varepsilon} - \\ & (w_0 + w_{1,\varepsilon} + \ldots w_{I+1,\varepsilon})^{p\pm\varepsilon} - (p\pm\varepsilon)(w_0 + w_{1,\varepsilon} + \ldots |w_{I,\varepsilon})^{p-1\pm\varepsilon} w_{I+1,\varepsilon}\big] \end{aligned}$$

and linear terms in $w_{I+1,\varepsilon}$ multiplied by a term of order ε^2, like

$$p\left((w_0 + w_{1,\varepsilon} + \cdots + w_{I-1,\varepsilon})^{p-1} - (w_0 + \cdots + w_{I,\varepsilon})^{p-1\pm\varepsilon}\right) w_{I+1,\varepsilon}.$$

Consider the following problem

$$\begin{cases} L_{I+1,\varepsilon} w_{I+1,\varepsilon} = \varepsilon h_{I+1,\varepsilon}(\varepsilon z, \xi) & \text{in } \mathcal{D} \\ \frac{\partial w_{I+1,\varepsilon}}{\partial \xi_N} = 0, & \text{on } \Gamma_1 \\ w_{I+1,\varepsilon} = 0, & \text{on } \Gamma_2 \cup \Gamma_3 \\ \int_D w_{I+1,\varepsilon}(\varepsilon z, \xi) Z_j(\xi)\,d\xi = 0 & \text{for all } z \in K_\varepsilon, \quad j = 0, \ldots N-1. \end{cases} \tag{4.39}$$

Again by Proposition 3.1, the above problem is solvable in $w_{I+1,\varepsilon}$ if $h_{I+1,\varepsilon}$ is L^2-orthogonal to Z_j, $j = 0, 1, \cdots, N-1$. These orthogonality conditions will define the parameters $\mu_{I,\varepsilon}$ and the normal section $\Phi_{I,\varepsilon}$.

Projection onto Z_0 and choice of $\mu_{I,\varepsilon}$: We define $\mu_{I,\varepsilon}$ to make

$$\int_{\hat{\mathcal{D}}} h_{I+1,\varepsilon} Z_0 = 0.$$

The above relation defines $\mu_{I,\varepsilon}$ as a smooth function of εz in K. From estimates (4.34) for $w_{I,\varepsilon}$ we get that

$$\|\mu_{I,\varepsilon}\|_{L^\infty(K)} + \|\partial_a \mu_{I,\varepsilon}\|_{L^\infty(K)} + \|\partial_a^2 \mu_{I,\varepsilon}\|_{L^\infty(K)} \le C\varepsilon^I. \tag{4.40}$$

Projection onto Z_l and choice of $\Phi_{I,\varepsilon}$: Multiplying $h_{I+1,\varepsilon}$ with $\partial_l w_0$, integrating over $\hat{\mathcal{D}}$ and arguing as in the construction of $\Phi_{1,\varepsilon}$, we get

$$\int_{\hat{\mathcal{D}}} h_{I+1,\varepsilon}\, \partial_l w_0 = -\,\varepsilon\, \Delta_K\, \Phi^l_{I,\varepsilon}$$
$$+\quad \varepsilon\, \left(\tilde{g}^{ab}_\varepsilon\, R_{maal} - \Gamma^c_a(E_m)\Gamma^a_c(E_l) + O(\varepsilon^{N-2})\right) \Phi^m_{I,\varepsilon} + \varepsilon \int_{\hat{\mathcal{D}}} \mathfrak{G}_{I+1,\varepsilon}\, \partial_l w_0.$$

We then conclude that $h_{I+1,\varepsilon}(z, \xi, w_0, \dots, w_{I,\varepsilon})$, the right-hand side of (4.39), is L^2-orthogonal to Z_l $(l = 1, \cdots, N-1)$ if and only if $\Phi_{I,\varepsilon}$ satisfies an equation of the form

$$\Delta_K\, \Phi^l_{I,\varepsilon} - \left(\tilde{g}^{ab}_\varepsilon\, R_{maal} - \Gamma^c_a(E_m)\Gamma^a_c(E_l) + O(\varepsilon^{N-2})\right) \Phi^m_{I,\varepsilon} = \varepsilon^{I-1} G_{I+1,\varepsilon}(\varepsilon z), \tag{4.41}$$

where $G_{I+1,\varepsilon}$ is a smooth function on K, uniformly bounded as $\varepsilon \to 0$. Using again the non-degeneracy condition on K we have solvability of the above equation in $\Phi_{I,\varepsilon}$. Furthermore, taking into account (4.38), we get

$$\|\Phi_{I,\varepsilon}\|_{L^\infty(K)} + \|\partial_a \Phi_{I,\varepsilon}\|_{L^\infty(K)} + \|\partial_a^2 \Phi_{I,\varepsilon}\|_{L^\infty(K)} \le C\varepsilon^{I-1}. \tag{4.42}$$

By our choice of $\mu_{I+1,\varepsilon}$ and $\Phi_{I+1,\varepsilon}$ we have solvability of equation (4.39) in $w_{I+1,\varepsilon}$. Moreover, it is straightforward to check that

$$|\varepsilon h_{I+1,\varepsilon}(\varepsilon z, \xi)| \le C \frac{\varepsilon^{1+I}}{(1+|\xi|)^{N-2}}.$$

Furthermore, for a given $\sigma \in (0,1)$ we have

$$\|\varepsilon h_{I+1,\varepsilon}\|_{\varepsilon, N-2,\sigma} \le C\varepsilon^{1+I}.$$

Proposition 3.1 gives then that

$$\|D^2_\xi w_{I+1,\varepsilon}\|_{\varepsilon,N-2,\sigma} + \|D_\xi w_{I+1,\varepsilon}\|_{\varepsilon,N-3,\sigma} + \|w_{I+1,\varepsilon}\|_{\varepsilon,N-4,\sigma} \le C\varepsilon^{1+I} \tag{4.43}$$

and that there exists a positive constant β (depending only on Ω, K and N) such that for any integer ℓ there holds

$$\|\nabla^{(\ell)}_y w_{I+1,\varepsilon}(z,\cdot)\|_{\varepsilon,N-2,\sigma} \le \beta C_l \varepsilon^{1+I} \qquad \varepsilon y = z \in K_\varepsilon. \tag{4.44}$$

With this choice of $\mu_{I,\varepsilon}$, $\Phi_{I,\varepsilon}$ and $w_{I+1,\varepsilon}$ we obtain that

$$\| -\mathcal{A}_{\mu_\varepsilon,\Phi_\varepsilon} v_{I+1,\varepsilon} + \varepsilon^2 \mu_\varepsilon^2 v_{I+1,\varepsilon} - \mu_\varepsilon^{\mp\varepsilon\frac{N-2}{2}}\, v^{p\pm\varepsilon}_{I+1,\varepsilon}\|_{\varepsilon,N-2,\sigma} \le C\varepsilon^{I+1}.$$

This concludes our construction.

5. A global approximation and proof of the result

Let $\mu_\varepsilon(y)$, $\Phi_\varepsilon(y)$ and $v_{I+1,\varepsilon}$ be the functions whose existence and properties have been established in Lemma 4.1. We define locally around $K_\varepsilon := \frac{K}{\varepsilon} \subset \partial\Omega_\varepsilon$ in Ω_ε the function

$$\tilde{U}_\varepsilon(z, X) := \mu_\varepsilon^{-\frac{N-2}{2}}(\varepsilon z)\, v_{I+1,\varepsilon}\left(z,\, \mu_\varepsilon^{-1}(\varepsilon z)(\bar{X} - \Phi_\varepsilon(\varepsilon z)),\, \mu_\varepsilon^{-1}(\varepsilon z) X_N\right) \times$$
$$\chi_\varepsilon(|(\bar{X} - \Phi_\varepsilon(\varepsilon z), X_N)|) \tag{5.1}$$

where $z \in K_\varepsilon$. In (5.1) the function χ_ε is a smooth cut-off function with

$$\chi_\varepsilon(r) = \begin{cases} 1, & \text{for} \quad r \in [0, 2\varepsilon^{-\gamma}] \\ 0, & \text{for} \quad r \in [3\varepsilon^{-\gamma}, 4\varepsilon^{-\gamma}], \end{cases} \tag{5.2}$$

and

$$|\chi^{(l)}_\varepsilon(r)| \le C_l \varepsilon^{l\gamma}, \qquad \text{for all } l \ge 1,$$

for some $\gamma \in (\frac{1}{2}, 1)$ to be fixed later.

The function $\tilde{U}_\varepsilon$ is well defined in a small neighborhood of K_ε inside Ω_ε. We will look at a solution to (1.17) of the form

$$\tilde{u}_\varepsilon = \tilde{U}_\varepsilon + \phi.$$

This translates into the fact that ϕ has to satisfy the non linear problem

$$\begin{cases} -\Delta\phi + \varepsilon^2\phi - (p\pm\varepsilon)\tilde{U}^{p\pm\varepsilon-1}_\varepsilon \phi = S_\varepsilon(\tilde{U}_\varepsilon) + N_\varepsilon(\phi) & \text{in } \Omega_\varepsilon, \\ \frac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{5.3}$$

where

$$S_\varepsilon(\tilde{U}_\varepsilon) = \Delta\tilde{U}_\varepsilon - \varepsilon^2 \tilde{U}_\varepsilon + \tilde{U}^{p\pm\varepsilon}_\varepsilon \tag{5.4}$$

and

$$N_\varepsilon(\phi) = (\tilde U_\varepsilon + \phi)^{p\pm\varepsilon} - \tilde U_\varepsilon^{p\pm\varepsilon} - (p\pm\varepsilon)\tilde U_\varepsilon^{p\pm\varepsilon-1}\phi. \tag{5.5}$$

Define

$$L_\varepsilon(\phi) = -\Delta\phi + \varepsilon^2\phi - (p\pm\varepsilon)\tilde U_\varepsilon^{p\pm\varepsilon-1}\phi.$$

Our strategy consists in solving the Non-Linear Problem (5.3) using a fixed point argument based on the contraction Mapping Principle. To do so, we need to establish some invertibility properties of the linear problem

$$L_\varepsilon(\phi) = f \quad \text{in} \quad \Omega_\varepsilon, \quad \frac{\partial\phi}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega_\varepsilon,$$

with $f \in L^2(\Omega_\varepsilon)$. We do this in two steps. First we study the above problem in a strip close to the scaled manifold $K_\varepsilon = \frac{K}{\varepsilon}$ in $\partial\Omega_\varepsilon$. Let $\gamma \in (\frac12, 1)$ be the number fixed before in (5.2) and consider

$$\Omega_{\varepsilon,\gamma} := \{x \in \Omega_\varepsilon \,:\, \mathrm{dist}(x, K_\varepsilon) < 2\varepsilon^{-\gamma}\}. \tag{5.6}$$

We are first interested in solving the following problem: given $f \in L^2(\Omega_{\varepsilon,\gamma})$

$$\begin{cases} -\Delta\phi + \varepsilon^2\phi - (p\pm\varepsilon)\tilde U_\varepsilon^{p\pm\varepsilon-1}\phi = f & \text{in } \Omega_{\varepsilon,\gamma}, \\ \frac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial\Omega_\varepsilon \bigcap \bar\Omega_{\varepsilon,\gamma}, \\ \phi = 0 & \text{in } \partial\Omega_{\varepsilon,\gamma} \setminus \partial\Omega_\varepsilon. \end{cases} \tag{5.7}$$

Define

$$H^1_\varepsilon = \Big\{ u \in H^1(\Omega_{\varepsilon,\gamma}) \,:\, \frac{\partial u}{\partial\nu} = 0 \ \text{ for } x \in \partial\Omega_\varepsilon \cap \bar\Omega_{\varepsilon,\gamma}, \quad u(x) = 0 \quad \text{for } x \in \partial\Omega_{\varepsilon,\gamma} \setminus \partial\Omega_\varepsilon \Big\}. \tag{5.8}$$

We have the validity of the following result.

Proposition 5.1. *There exist a constant $C > 0$ and a sequence $\varepsilon_l = \varepsilon \to 0$ such that, for any $f \in L^2(\Omega_{\varepsilon,\gamma})$ there exists a solution $\phi \in H^1_\varepsilon$ to Problem (5.7) such that*

$$\|\phi\|_{H^1_\varepsilon} \le C\varepsilon^{-\max\{2,k\}} \|f\|_{L^2(\Omega_{\varepsilon,\gamma})}. \tag{5.9}$$

We postpone the proof of Proposition 5.1 to Section 6.

Now we establish the existence of solutions to the linear problem in the whole domain Ω_ε.

Proposition 5.2. *There exist a sequence $\varepsilon_l \to 0$ and a positive constant $C > 0$, such that, for any $f \in L^2(\Omega_{\varepsilon_l})$, there exists a solution $\phi \in H^1(\Omega_{\varepsilon_l})$ to the equation*

$$L_{\varepsilon_l}\phi = f \quad \textit{in } \Omega_{\varepsilon_l}, \qquad \frac{\partial\phi}{\partial\nu} = 0 \quad \textit{on} \quad \partial\Omega_{\varepsilon_l}.$$

Furthermore,

$$\|\phi\|_{H^1(\Omega_{\varepsilon_l})} \le C\,\varepsilon_l^{-\max\{2,k\}} \|f\|_{L^2(\Omega_{\varepsilon_l})}. \tag{5.10}$$

Proof. By contradiction, assume that for all $\varepsilon \to 0$ there exists a solution $(\phi_\varepsilon, \lambda_\varepsilon)$, $\phi_\varepsilon \ne 0$, to

$$L_\varepsilon(\phi_\varepsilon) = \lambda_\varepsilon\phi_\varepsilon \quad \text{in} \quad \Omega_\varepsilon, \quad \frac{\partial\phi_\varepsilon}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega_\varepsilon \tag{5.11}$$

with

$$|\lambda_\varepsilon|\varepsilon^{-\max\{2,k\}} \to 0, \quad \text{as} \quad \varepsilon \to 0. \tag{5.12}$$

Let η_ε be a smooth cut off function (like the one defined in (5.2)) so that

$$\eta_\varepsilon = 1 \quad \text{if dist } (y, K_\varepsilon) < \tfrac{\varepsilon^{-\gamma}}{2} \qquad \text{and} \quad \eta_\varepsilon = 0 \quad \text{if dist } (y, K_\varepsilon) > \varepsilon^{-\gamma}.$$

In particular one has that

$$|\nabla\eta_\varepsilon| \le c\varepsilon^\gamma \qquad \text{and} \qquad |\Delta\eta_\varepsilon| \le c\varepsilon^{2\gamma}$$

in the whole domain.

Define $\tilde\phi_\varepsilon = \phi_\varepsilon\eta_\varepsilon$. Then $\tilde\phi_\varepsilon$ solves

$$\begin{cases} L_\varepsilon(\tilde\phi_\varepsilon) = \lambda_\varepsilon\tilde\phi_\varepsilon - \nabla\eta_\varepsilon\nabla\phi_\varepsilon - \Delta\eta_\varepsilon\phi_\varepsilon & \text{in } \Omega_{\varepsilon,\gamma} \\ \frac{\partial\phi_\varepsilon}{\partial\nu} = 0 & \text{on } \partial\Omega_\varepsilon \setminus \overline\Omega_{\varepsilon,\gamma}, \\ \phi_\varepsilon = 0 & \text{in } \partial\Omega_\varepsilon \bigcap \partial\Omega_{\varepsilon,\gamma}, \end{cases} \tag{5.13}$$

where $\Omega_{\varepsilon,\gamma}$ is the set defined in (5.6). We now apply Proposition 5.1, that guarantees the existence of a sequence $\varepsilon_l \to 0$ and a constant c such that

$$\|\tilde\phi_{\varepsilon_l}\|_{H^1_{\varepsilon_l}} \le c\varepsilon_l^{-\max\{2,k\}} \Big[\lambda_{\varepsilon_l}\|\tilde\phi_{\varepsilon_l}\|_{L^2} + \|\nabla\eta_{\varepsilon_l}\nabla\phi_{\varepsilon_l}\|_{L^2} + \|\Delta\eta_{\varepsilon_l}\phi_{\varepsilon_l}\|_{L^2}\Big]. \tag{5.14}$$

Observe now that, in the region where $\nabla\eta_{\varepsilon_l} \neq 0$ and $\Delta\eta_{\varepsilon_l} \neq 0$, the function $\tilde{U}_{\varepsilon_l}$ can be uniformly bounded $|\tilde{U}_\varepsilon(y)| \leq c\varepsilon$, with a positive constant c, fact that follows directly from (5.1) and (4.3). Furthermore, since we are assuming (5.12), we see that in the region we are considering, namely where $\nabla\eta_{\varepsilon_l} \neq 0$ and $\Delta\eta_{\varepsilon_l} \neq 0$, the function ϕ_{ε_l} satisfies the equation

$$-\Delta\phi_{\varepsilon_l} + \varepsilon_l^2 a_{\varepsilon_l}(y)\phi_{\varepsilon_l} = 0$$

for a certain smooth function a_{ε_l}, which is uniformly positive and bounded as $\varepsilon_l \to 0$. Elliptic estimates give that, in this region, $|\phi_{\varepsilon_l}| \leq ce^{-\varepsilon_l^{\gamma'}}$, and $|\nabla\phi_{\varepsilon_l}| \leq ce^{-\varepsilon_l^{\gamma'}}$ for some $\gamma' > 0$ and $c > 0$. Inserting this information in (5.14), it is easy to see that

$$\|\tilde{\phi}_{\varepsilon_l}\|_{H^1_{\varepsilon_l}} \leq c\varepsilon_l^{-\max\{2,k\}}\lambda_{\varepsilon_l}\|\tilde{\phi}_{\varepsilon_l}\|_{H^1_{\varepsilon_l}}(1 + o(1))$$

where $o(1) \to 0$ as $\varepsilon_l \to 0$. Taking into account (5.12) the above inequality gives a contradiction with the fact that, for all ε, the function ϕ_ε is not identically zero. This concludes the proof. □

Proof of Theorem 1.1. By Proposition 5.2, $\phi \in H^1(\Omega_\varepsilon)$ is a solution to (5.3) if and only if

$$\phi = L_\varepsilon^{-1}\left(S_\varepsilon(\tilde{U}_\varepsilon) + N_\varepsilon(\phi)\right).$$

Notice that

$$\|N_\varepsilon(\phi)\|_{L^2(\Omega_\varepsilon)} \leq C\begin{cases}\|\phi\|^p_{H^1(\Omega_\varepsilon)} & \text{for } p \leq 2,\\ \|\phi\|^2_{H^1(\Omega_\varepsilon)} & \text{for } p > 2\end{cases} \qquad \|\phi\|_{H^1(\Omega_\varepsilon)} \leq 1 \tag{5.15}$$

and

$$\begin{aligned}&\|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)\|_{L^2(\Omega_\varepsilon)}\\ &\leq C\begin{cases}\left(\|\phi_1\|^{p-1}_{H^1(\Omega_\varepsilon)} + \|\phi_2\|^{p-1}_{H^1(\Omega_\varepsilon)}\right)\|\phi_1 - \phi_2\|_{H^1(\Omega_\varepsilon)} & \text{for } p \leq 2,\\ \left(\|\phi_1\|_{H^1(\Omega_\varepsilon)} + \|\phi_2\|_{H^1(\Omega_\varepsilon)}\right)\|\phi_1 - \phi_2\|_{H^1(\Omega_\varepsilon)} & \text{for } p > 2\end{cases},\end{aligned} \tag{5.16}$$

for any ϕ_1, ϕ_2 in $H^1(\Omega_\varepsilon)$ with $\|\phi_1\|_{H^1(\Omega_\varepsilon)}$, $\|\phi_2\|_{H^1(\Omega_\varepsilon)} \leq 1$.

Defining $T_\varepsilon : H^1(\Omega_\varepsilon) \to H^1(\Omega_\varepsilon)$ as

$$T_\varepsilon(\phi) = L_\varepsilon^{-1}\left(S_\varepsilon(\tilde{U}_\varepsilon) + N_\varepsilon(\phi)\right)$$

we will show that T_ε is a contraction in some small ball in $H^1(\Omega_\varepsilon)$. A direct consequence of (4.4), we have

$$\|S_\varepsilon(\tilde{U}_\varepsilon)\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{1+I}.$$

Using this inequality and by (5.15), (5.16) and (5.10), we obtain

$$\|T_\varepsilon(\phi)\|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^{-\max\{2,k\}}\begin{cases}\left(\varepsilon^{1+I} + \|\phi\|^p_{H^1(\Omega_\varepsilon)}\right) & \text{for } p \leq 2,\\ \left(\varepsilon^{1+I} + \|\phi\|^2_{H^1(\Omega_\varepsilon)}\right) & \text{for } p > 2.\end{cases}$$

Now we choose integers d and I so that

$$d > \begin{cases}\frac{\max\{2,k\}}{p-1} & \text{for } p \leq 2,\\ \max\{2,k\} & \text{for } p > 2\end{cases} \qquad I > d - 1 + \max\{2,k\}.$$

Thus one easily gets that T_ε has a unique fixed point in the set

$$\mathcal{B} = \{\phi \in H^1(\Omega_\varepsilon) \,:\, \|\phi\|_{H^1(\Omega_\varepsilon)} \leq \varepsilon^d\},$$

as a direct application of the contraction mapping Theorem. This concludes the proof of the main Theorem 1.1. □

6. Proof of Proposition 5.1

In this section, we will establish a solvability theory for Problem (5.7). The quadratic functional of problem ((5.7)) given by

$$E(\phi) = \frac{1}{2}\int_{\Omega_{\varepsilon,\gamma}}(|\nabla\phi|^2 + \varepsilon^2\phi^2 - (p \pm \varepsilon)\tilde{U}_\varepsilon^{p\pm\varepsilon-1}\phi^2) \tag{6.1}$$

for functions $\phi \in H^1_\varepsilon$ (see (5.8)).

Let $(z, X) \in \mathbb{R}^{k+N}$ be the local coordinates along K_ε introduced in (2.8). With abuse of notation we will denote

$$\phi(\Upsilon_\varepsilon(z, X)) = \phi(z, X). \tag{6.2}$$

Since the original variable $(z, X) \in \mathbb{R}^{k+N}$ are only local coordinates along K_ε we let the variable (z, X) vary in the set $\mathcal{C}_\varepsilon$ defined by

$$\mathcal{C}_\varepsilon = \{(z, \bar{X}, X_N) \;/\; \varepsilon z \in\ K, \quad 0 < X_N < \varepsilon^{-\gamma}, \quad |\bar{X}| < \varepsilon^{-\gamma}\}. \tag{6.3}$$

We write $\mathcal{C}_\varepsilon = \frac{1}{\varepsilon}K \times \hat{\mathcal{C}}_\varepsilon$ where

$$\hat{\mathcal{C}}_\varepsilon = \{(\bar{X}, X_N) \ / \ 0 < X_N < \varepsilon^{-\gamma}, \quad |\bar{X}| < \varepsilon^{-\gamma}\}. \tag{6.4}$$

Observe that $\hat{\mathcal{C}}_\varepsilon$ approaches, as $\varepsilon \to 0$, the half space $\mathbb{R}^N_+$.

In these new local coordinates, the energy density associated to the energy E in (6.1) is given by

$$\left[\frac{1}{2}\left(|\nabla_{g^\varepsilon}\phi|^2 + \varepsilon^2\phi^2 - (p\pm\varepsilon)\tilde{U}_\varepsilon^{p\pm\varepsilon-1}\phi^2\right)\right]\sqrt{\det(g^\varepsilon)}, \tag{6.5}$$

where ∇_{g^ε} denotes the gradient in the new variables and where g^ε is the flat metric in $\mathbb{R}^{N+k}$ in the coordinates (z, X). After a careful expansion of the metric g_ε, see Lemma 7.2, we get that, if (z, X) vary in $\mathcal{C}_\varepsilon$, then, the energy functional (6.1) in the new variables (6.2) is given by

$$\begin{aligned} E(\phi) \quad = \quad & \int_{K_\varepsilon \times \hat{\mathcal{C}}_\varepsilon} \left(\frac{1}{2}(|\nabla_X\phi|^2 + \varepsilon^2\phi^2 - (p\pm\varepsilon)\tilde{U}_\varepsilon^{p\pm\varepsilon-1}\phi^2)\right)\sqrt{\det(g^\varepsilon)}\, dz\, dX \\ & + \int_{K_\varepsilon \times \hat{\mathcal{C}}_\varepsilon} \frac{1}{2}\,\Xi_{ij}(\varepsilon z, X)\,\partial_i\phi\partial_j\phi\,\sqrt{\det(g^\varepsilon)}\, dz\, dX \\ & + \frac{1}{2}\int_{K_\varepsilon \times \hat{\mathcal{C}}_\varepsilon} |\nabla_{K_\varepsilon}\phi|^2\sqrt{\det(g^\varepsilon)}\, dz\, dX + \int_{K_\varepsilon \times \hat{\mathcal{C}}_\varepsilon} B(\phi,\phi)\sqrt{\det(g^\varepsilon)}\, dz\, dX. \end{aligned} \tag{6.6}$$

In the above expression, we have

$$\Xi_{ij}(\varepsilon z, X) = 2\varepsilon H_{ij}X_N - \frac{\varepsilon^2}{3}\,R_{islj}X_lX_s, \tag{6.7}$$

we denoted by $B(\phi,\phi)$ a quadratic term in ϕ that can be expressed in the following form

$$\begin{aligned} B(\phi,\phi) \quad = \quad & O\left(\varepsilon^2X_N^2 + \varepsilon^3|\bar{X}|^3 + \varepsilon^3X_N|\bar{X}|^2 + \varepsilon^3X_N^2|\bar{X}|\right)\partial_i\phi\partial_j\phi \\ & + \varepsilon^2\,|\nabla_{K_\varepsilon}\phi|^2\,O(\varepsilon|X|) + \partial_j\phi\partial_{\bar{a}}\phi\left(\mathcal{O}(\varepsilon|\bar{X}| + \varepsilon^2X_N^2)\right) \end{aligned} \tag{6.8}$$

and we used the Einstein convention over repeated indices. Furthermore we use the notation $\partial_a = \partial_{y_a}$ and $\partial_{\bar{a}} = \partial_{z_a}$. A detailed proof of expansion (6.6) can be found in [14].

Given a function $\phi \in H^1_\varepsilon$ (see (5.8)), we decompose it as

$$\phi = \left[\frac{\delta}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_0) + \sum_{j=1}^{N-1}\frac{d^j}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_j) + \frac{e}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z)\right]\bar{\chi}_\varepsilon + \phi^\perp \tag{6.9}$$

where the expression $\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(v)$ is defined in (2.14), the functions Z_0 and Z_j are already defined in (3.5) and where Z is the eigenfunction, with $\int_{\mathbb{R}^N} Z^2 = 1$, corresponding to the unique positive eigenvalue λ_0 in $L^2(\mathbb{R}^N)$ of the problem

$$\Delta_{\mathbb{R}^N}\phi + pw_0^{p-1}\phi = \lambda_0\phi \quad \text{in} \quad \mathbb{R}^N. \tag{6.10}$$

It is worth mentioning that $Z(\xi)$ is even and it has exponential decay of order $O(e^{-\sqrt{\lambda_0}|\xi|})$ at infinity. The function $\bar{\chi}_\varepsilon$ is a smooth cut off function defined by

$$\bar{\chi}_\varepsilon(X) = \hat{\chi}_\varepsilon\left(\left|\left(\frac{\bar{X}-\Phi_\varepsilon}{\mu_\varepsilon}, \frac{X_N}{\mu_\varepsilon}\right)\right|\right), \tag{6.11}$$

with $\hat{\chi}(r) = 1$ for $r \in (0, \frac{3}{2}\varepsilon^{-\gamma})$, and $\chi(r) = 0$ for $r > 2\varepsilon^{-\gamma}$. Finally, in (6.9) we have that $\delta = \delta(\varepsilon z)$, $d^j = d^j(\varepsilon z)$ and $e = e(\varepsilon z)$ are function defined in K such that for all $z \in K_\varepsilon$

$$\int_{\hat{\mathcal{C}}_\varepsilon}\phi^\perp\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_0)\bar{\chi}_\varepsilon dX = \int_{\hat{\mathcal{C}}_\varepsilon}\phi^\perp\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_j)\bar{\chi}_\varepsilon = \int_{\hat{\mathcal{C}}_\varepsilon}\phi^\perp\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z)\bar{\chi}_\varepsilon = 0. \tag{6.12}$$

We will denote by $(H^1_\varepsilon)^\perp$ the subspace of the functions in H^1_ε that satisfy the orthogonality conditions (6.12).

A direct computation shows that

$$\delta(\varepsilon z) = \frac{\int\phi\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_0)}{\mu_\varepsilon\int Z_0^2}(1 + O(\varepsilon^2)) + O(\varepsilon^2)(\sum_j d^j(\varepsilon z) + e(\varepsilon z)),$$

$$d^j(\varepsilon z) = \frac{\int\phi\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_j)}{\mu_\varepsilon\int Z_j^2}(1 + O(\varepsilon^2)) + O(\varepsilon^2)(\delta(\varepsilon z) + \sum_{i\neq j} d^i(\varepsilon z) + e(\varepsilon z)),$$

and

$$e(\varepsilon z) = \frac{\int\phi\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z)}{\mu_\varepsilon\int Z^2}(1 + O(\varepsilon^2)) + O(\varepsilon^2)(\delta(\varepsilon z) + \sum_j d^j(\varepsilon z)).$$

Observe that, since $\phi \in H^1_\varepsilon$, one easily get that the functions δ, d^j and e belong to the Hilbert space

$$\mathcal{H}^1(K) = \{\zeta \in \mathcal{L}^2(K) : \partial_a\zeta \in \mathcal{L}^2(K), \quad a = 1, \cdots, k\}. \tag{6.13}$$

Thanks to the above decomposition (6.9), we have the validity of the following expansion for $E(\phi)$.

Theorem 6.1. *Let $\gamma = 1-\sigma$, for some $\sigma > 0$ and small. Assume we write $\phi \in H^1_\varepsilon$ as in (6.9) and let $d = (d^1, \dots, d^{N-1})$. Then, there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$, the following expansion holds true*

$$E(\phi) = E(\phi^\perp) + \varepsilon^{-k}\left[P_\varepsilon(\delta) + Q_\varepsilon(d) + R_\varepsilon(e)\right] + \mathcal{M}(\phi^\perp, \delta, d, e). \tag{6.14}$$

In (6.14)

$$P_\varepsilon(\delta) = P(\delta) + P_1(\delta) \tag{6.15}$$

with

$$P(\delta) = \left[\frac{A_\varepsilon}{2}\int_K \varepsilon^2 |\partial_a(\delta(1+o(\varepsilon^2)\beta_1^\varepsilon(y)))|^2 + \varepsilon\frac{B}{2}\int_K \delta^2\right] \tag{6.16}$$

where

$$B = \begin{cases} -\frac{(N-2)^2}{4N} C_0 & \text{if we consider the problem (1.13) with } \quad p+\varepsilon \\ \frac{(N-2)^2}{4N} C_0 & \text{if we consider the problem (1.13) with } \quad p-\varepsilon \end{cases}, \tag{6.17}$$

A_ε a real number such that $\lim_{\varepsilon\to 0} A_\varepsilon = A := \int_{\mathbb{R}^N_+} Z_0^2$, and β_1^ε is an explicit smooth function defined on K which is uniformly bounded as $\varepsilon \to 0$; furthermore, $P_1(\delta)$ is a small compact perturbation in $\mathcal{H}^1(K)$ whose shape is a sum of quadratic functional in δ of the form

$$\varepsilon^2 \int_K b(y)|\delta|^2$$

where $b(y)$ denotes a generic explicit function, smooth and uniformly bounded, as $\varepsilon \to 0$, in K. In (6.14),

$$Q_\varepsilon(d) = Q(d) + Q_1(d) \tag{6.18}$$

with

$$Q(d) = \frac{\varepsilon^2}{2} C_\varepsilon \left(\int_K |\partial_a(d(1+o(\varepsilon^2)\beta_2^\varepsilon(y)))|^2 + \int_K ((\tilde{g}^\varepsilon)^{ab} R_{mabl} - \Gamma_a^c(E_m)\Gamma_c^a(E_l))d^m d^l\right) \tag{6.19}$$

where C_ε is a real number such that $\lim_{\varepsilon\to 0} C_\varepsilon = C := \int_{\mathbb{R}^N_+} Z_1^2$, β_2^ε is an explicit smooth function defined on K which is uniformly bounded as $\varepsilon \to 0$ and the terms R_{maal} and $\Gamma_a^c(E_m)$ are smooth functions on K defined respectively in (2.19) and (2.6). Furthermore, $Q_1(d)$ is a small compact perturbation in $\mathcal{H}^1(K)$ whose shape is a sum of quadratic functional in d of the form

$$\varepsilon^3 \int_K b(y) d^i d^j$$

where again $b(y)$ is a generic explicit function, smooth and uniformly bounded, as $\varepsilon \to 0$, in K. In (6.14),

$$R_\varepsilon(e) = R(e) + R_1(e) \tag{6.20}$$

$$R(e) = \varepsilon^{-k}\left[\frac{D_\varepsilon}{2}\left(\varepsilon^2\int_K |\partial_a(e(1+e^{-\frac{\lambda_0}{2}\varepsilon^{-\gamma}}\beta_3^\varepsilon(y)))|^2 - \lambda_0\int_K e^2\right)\right] \tag{6.21}$$

with D_ε a real number so that $\lim_{\varepsilon\to 0} D_\varepsilon = D := \int_{\mathbb{R}^N_+} Z^2$, β_3^ε an explicit smooth function in K, which is uniformly bounded as $\varepsilon \to 0$, and λ_0 the positive number defined in (6.10). Furthermore, R_1 is a small compact perturbation in $\mathcal{H}^1(K)$ whose shape is a sum of quadratic functional in e of the form

$$\varepsilon^3 \int_K b(y) e^2$$

where again $b(y)$ is a generic explicit function, smooth and uniformly bounded, as $\varepsilon \to 0$, in K. Finally in (6.14)

$$\mathcal{M} : (H^1_\varepsilon)^\perp \times (\mathcal{H}^1(K))^{N+1} \to \mathbb{R}$$

is a continuous and differentiable functional with respect to the natural topologies, homogeneous of degree 2

$$\mathcal{M}(t\phi^\perp, t\delta, td, te) = t^2 \mathcal{M}(\phi^\perp, \delta, d, e) \quad \text{for all } t.$$

The derivative of $\mathcal{M}$ with respect to each one of its variable is given by a small multiple of a linear operator in $(\phi^\perp, \delta, d, e)$ and it satisfies

$$\|D_{(\phi^\perp,\delta,d)}\mathcal{M}(\phi_1^\perp, \delta_1, d_1, e_1) - D_{(\phi^\perp,\delta,d)}\mathcal{M}(\phi_2^\perp, \delta_2, d_2, e_2)\| \le C\varepsilon^{\gamma(N-3)} \times$$
$$\left[\|\phi_1^\perp - \phi_2^\perp\| + \varepsilon^{-k}\|\delta_1 - \delta_2\|_{\mathcal{H}^1(K)} + \varepsilon^{-k}\|d_1 - d_2\|_{(\mathcal{H}^1(K))^{N-1}} + \varepsilon^{-k}\|e_1 - e_2\|_{\mathcal{H}^1(K)}\right]. \tag{6.22}$$

Furthermore, there exists a constant $C > 0$ such that

$$\left|\mathcal{M}(\phi^\perp, \delta, d, e)\right| \le C\varepsilon^2\left[\|\phi^\perp\|^2 + \varepsilon^{-2k}\left(\|\delta\|^2_{\mathcal{H}^1(K)} + \|d\|^2_{\mathcal{H}^1(K)} + \|e\|^2_{\mathcal{H}^1(K)}\right)\right]. \tag{6.23}$$

We refer the reader to the proof of Theorem 2, Section 5, in [14] for the proof of Theorem 6.1.

The rest of this section is devoted to prove Proposition 5.1.

Let $\phi \in H^1_\varepsilon(\Omega_{\varepsilon,\gamma})$. As in (6.9), we have the following decomposition of ϕ

$$\phi = \left[\frac{\delta}{\mu_\varepsilon} \mathcal{T}_{\mu_\varepsilon, \Phi_\varepsilon}(Z_0) + \sum_{j=1}^{N-1} \frac{d^j}{\mu_\varepsilon} \mathcal{T}_{\mu_\varepsilon, \Phi_\varepsilon}(Z_j) + \frac{e}{\mu_\varepsilon} \mathcal{T}_{\mu_\varepsilon, \Phi_\varepsilon}(Z) \right] \bar{\chi}_\varepsilon + \phi^\perp.$$

We then define the energy functional associated to Problem (5.7)

$$\mathcal{E} : (H^1_\varepsilon)^\perp \times (\mathcal{H}^1(K))^{N+1} \to \mathbb{R}$$

by

$$\mathcal{E}(\phi^\perp, \delta, d, e) = E(\phi) - \mathcal{L}_f(\phi) \tag{6.24}$$

where E is the functional in (6.1) and $\mathcal{L}_f(\phi)$ is the linear operator given by

$$\mathcal{L}_f(\phi) = \int_{\Omega_{\varepsilon,\gamma}} f\phi.$$

Observe that

$$\mathcal{L}_f(\phi) = \mathcal{L}^1_f(\phi^\perp) + \varepsilon^{-k}\left[\mathcal{L}^2_f(\delta) + \mathcal{L}^3_f(d) + \mathcal{L}^4_f(e)\right]$$

where $\mathcal{L}^1_f : H^1_\varepsilon \to \mathbb{R}$, $\mathcal{L}^2_f, \mathcal{L}^4_f : \mathcal{H}^1(K) \to \mathbb{R}$ and $\mathcal{L}^3_f : (\mathcal{H}^1(K))^{N-1} \to \mathbb{R}$ with

$$\mathcal{L}^1_f(\phi^\perp) = \int_{\Omega_{\varepsilon,\gamma}} f\phi^\perp, \quad \varepsilon^{-k}\mathcal{L}^2_f(\delta) = \int_{\Omega_{\varepsilon,\gamma}} f\frac{\delta}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon, \Phi_\varepsilon}(Z_0)\bar{\chi}_\varepsilon$$

$$\varepsilon^{-k}\mathcal{L}^3_f(d) = \sum_{j=1}^{N-1}\int_{\Omega_{\varepsilon,\gamma}} f\frac{d^j}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon, \bar{\Phi}_\varepsilon}(Z_j)\bar{\chi}_\varepsilon \quad \text{and} \quad \varepsilon^{-k}\mathcal{L}^4_f(e) = \int_{\Omega_{\varepsilon,\gamma}} f\frac{e}{\bar{\mu}_\varepsilon}\mathcal{T}_{\mu_\varepsilon, \Phi_\varepsilon}(Z)\bar{\chi}_\varepsilon.$$

Finding a solution $\phi \in H^1_\varepsilon$ to Problem (5.7) reduces to finding a critical point $(\phi^\perp, \delta, d, e)$ for $\mathcal{E}$. This will be done in several steps.

Step 1. We claim that there exist $\sigma > 0$ and ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $\phi^\perp \in (H^1_\varepsilon)^\perp$ then

$$E(\phi^\perp) \geq \sigma\|\phi^\perp\|^2_{L^2}. \tag{6.25}$$

A direct consequence of (6.6) is that, for sufficiently small $\varepsilon > 0$

$$E(\phi^\perp) \geq \frac{1}{4}E_0(\phi^\perp),$$

$$\text{with} \quad E_0(\phi^\perp) = \int_{K_\varepsilon \times \hat{\mathcal{C}}_\varepsilon}\left[|\nabla_X\phi^\perp|^2 - (p\pm\varepsilon)\tilde{U}^{p\pm\varepsilon-1}_\varepsilon\phi^\perp\right]\sqrt{\det(g^\varepsilon)}$$

for any $\phi^\perp = \phi^\perp(\varepsilon z, X)$, with $z \in K_\varepsilon = \frac{1}{\varepsilon}K$. The set $\hat{\mathcal{C}}_\varepsilon$ is defined in (6.4) and $\hat{\mathcal{C}}_\varepsilon \to \mathbb{R}^N_+$ as $\varepsilon \to 0$. We will establish (6.25) showing that

$$E_0(\phi^\perp) \geq \sigma\|\phi^\perp\|^2_{L^2} \quad \text{for all } \phi^\perp. \tag{6.26}$$

To do so, we first observe that if we scale in the z-variable, defining $\varphi^\perp(y, X) = \phi^\perp(\frac{y}{\varepsilon}, X)$, the relation (6.26) becomes

$$E_0(\varphi^\perp) \geq \sigma\|\varphi^\perp\|^2_{L^2}. \tag{6.27}$$

Thus we are led to show the validity of (6.27). We argue by contradiction, for any $n \in \mathbb{N}^*$, there exist $\varepsilon_n \to 0$ and $\varphi^\perp_n \in (H^1_{\varepsilon_n})^\perp$ such that

$$E_0(\varphi^\perp_n) \leq \frac{1}{n}\|\varphi^\perp_n\|^2_{L^2}. \tag{6.28}$$

Without loss of generality we can assume that the sequence $(\|\varphi^\perp_n\|)_n$ is bounded, as $n \to \infty$. Hence, up to subsequences, we have that

$$\varphi^\perp_n \rightharpoonup \varphi^\perp \quad \text{in} \quad H^1(K \times \mathbb{R}^N_+) \qquad \text{and} \qquad \varphi^\perp_n \to \varphi^\perp \quad \text{in} \quad L^2(K \times \mathbb{R}^N_+).$$

Furthermore, using the estimate in (4.3) we get that

$$\sup_{y\in K, X\in\mathbb{R}^N_+}\left|(1+|X|)^{N-4}\left[\tilde{U}_\varepsilon(\frac{y}{\varepsilon}, X) - \mu_0^{-\frac{N-2}{2}}(y)w_0(\frac{\bar{X}-\Phi_0(y)}{\mu_0(y)}, \frac{X_N}{\mu_0(y)})\right]\right| \to 0,$$

as $\varepsilon \to 0$, where μ_0 and w_0 are the smooth explicit function defined in (4.15), (4.16) and (2.1).

Passing to the limit as $n \to \infty$ in (6.28) and applying dominated convergence Theorem, we get

$$\int_{K\times\mathbb{R}^N_+}\left[|\nabla_X\varphi^\perp|^2 - p\left(\mu_0^{-\frac{N-2}{2}}(y)w_0(\frac{\bar{X}-\Phi_0(y)}{\mu_0(y)}, \frac{X_N}{\mu_0(y)})\right)^{p-1}(\varphi^\perp)^2\right]dydX \leq 0. \tag{6.29}$$

Furthermore, passing to the limit in the orthogonality condition we get, for any $y \in K$

$$\int_{\mathbb{R}^N_+} \varphi^\perp(y,X) Z_0(\frac{\bar X - \Phi_0(y)}{\mu_0(y)}, \frac{X_N}{\mu_0(y)}) dX = 0, \tag{6.30}$$

$$\int_{\mathbb{R}^N_+} \varphi^\perp(y,X) Z_j(\frac{\bar X - \Phi_0(y)}{\mu_0(y)}, \frac{X_N}{\mu_0(y)}) dX = 0, \quad j = 1, \dots N-1 \tag{6.31}$$

and

$$\int_{\mathbb{R}^N_+} \varphi^\perp(y,X) Z(\frac{\bar X - \Phi_0(y)}{\mu_0(y)}, \frac{X_N}{\mu_0(y)}) dX = 0. \tag{6.32}$$

We thus get a contradiction with (6.29), since for any function $\varphi^\perp$ satisfying the orthogonality conditions (6.30)–(6.32) for any $y \in K$ one has

$$\int_{K\times\mathbb{R}^N_+} \left[|\nabla_X \varphi^\perp|^2 - p \left(\mu_0^{-\frac{N-2}{2}}(y) w_0(\frac{\bar X - \Phi_0(y)}{\mu_0(y)}, \frac{X_N}{\mu_0(y)}) \right)^{p-1} (\varphi^\perp)^2 \right] dy dX > 0$$

(see for instance [15]).

Step 2. For all $\varepsilon > 0$ small, the functional $P_\varepsilon(\delta)$ defined in (6.15) is continuous and differentiable in $\mathcal{H}^1(K)$. If we consider Problem (1.17), or (3.1), with $p - \varepsilon$, we see that $P_\varepsilon(\delta)$ is strictly convex and bounded from below since

$$P_\varepsilon(\delta) \geq \frac{1}{4} \left[\frac{A}{2} \varepsilon^2 \int_K |\partial_a \delta|^2 + \frac{B}{2} \varepsilon \int_K \delta^2 \right] \geq \sigma \varepsilon \|\delta\|^2 \tag{6.33}$$

for some small but fixed $\sigma > 0$. A direct consequence of these properties is that

$$\delta \in \mathcal{H}^1(K) \longmapsto P_\varepsilon(\delta) - \mathcal{L}^2_f(\delta)$$

has a unique minimum δ, and furthermore

$$\varepsilon^{-\frac{k}{2}} \|\delta\|_{\mathcal{H}^1(K)} \leq \mathcal{C} \varepsilon^{-2} \|f\|_{L^2(\Omega_{\varepsilon,\gamma})}$$

for a given positive constant $\mathcal{C}$.

If we consider Problem (1.17), or (3.1), with $p + \varepsilon$, the functional $P_\varepsilon(\delta)$ has a mountain pass structure. Thanks to the compactness of the problem, also in this case the functional $\delta \mapsto P_\varepsilon(\delta) - \mathcal{L}^2_f(\delta)$ has a critical point. Furthermore

$$\varepsilon^{-\frac{k}{2}} \|\delta\|_{\mathcal{H}^1(K)} \leq \mathcal{C} \varepsilon^{-2} \|f\|_{L^2(\Omega_{\varepsilon,\gamma})}$$

for some positive number $\mathcal{C}$.

Step 3. For all $\varepsilon > 0$ small, the functional Q_ε defined in (6.18) is a small perturbation in $(\mathcal{H}^1(K))^{N-1}$ of the quadratic form $\varepsilon^2 Q_0(d)$, defined by

$$\varepsilon^2 Q_0(d) = \frac{\varepsilon^2}{2} C \left[\int_K |\partial_a d|^2 + \int_K ((\tilde g_\varepsilon)^{ab} R_{mabl} - \Gamma^c_a(E_m)\Gamma^a_c(E_l)) d^m d^l \right]$$

with $C := \int_{\mathbb{R}^N_+} Z_1^2$ and the terms R_{maal} and $\Gamma^c_a(E_m)$ are smooth functions on K defined respectively in (2.19) and (2.6). Recall that the non-degeneracy assumption on the minimal submanifold K is equivalent to the invertibility of the operator $Q_0(d)$. A consequence, for each $f \in L^2(\Omega_{\varepsilon,\gamma})$,

$$d \in (\mathcal{H}^1(K))^{N-1} \longrightarrow \mathbb{R}, \quad d \longmapsto Q_\varepsilon(d) - \mathcal{L}^3_f(d)$$

has a unique critical point d, which satisfies

$$\varepsilon^{-\frac{k}{2}} \|d\|_{(\mathcal{H}^1(K))^{N-1}} \leq \tilde\sigma \varepsilon^{-2} \|f\|_{L^2(\Omega_{\varepsilon,\gamma})}$$

for some proper $\tilde\sigma > 0$.

Step 4. Let $f \in L^2(\Omega_{\varepsilon,\gamma})$ and assume that e is a given (fixed) function in $\mathcal{H}^1(K)$. We claim that for all $\varepsilon > 0$ small enough, the functional $\mathcal{G} : (H^1_\varepsilon)^\perp \times (\mathcal{H}^1(K))^N \to \mathbb{R}$

$$(\phi^\perp, \delta, d) \to \mathcal{E}(\phi^\perp, \delta, d, e)$$

has a critical point $(\phi^\perp, \delta, d)$. Furthermore there exists a positive constant C, independent of ε, such that

$$\|\phi^\perp\| + \varepsilon^{-\frac{k}{2}} \left[\|\delta\|_{\mathcal{H}^1(K)} + \|d\|_{(\mathcal{H}^1(K))^{N-1}} \right] \leq C \varepsilon^{-2} \left[\|f\|_{L^2(\Omega_{\varepsilon,\gamma})} + \varepsilon^{-\frac{k}{2}} \varepsilon^2 \|e\|_{\mathcal{H}^1(K)} \right]. \tag{6.34}$$

To prove the above assertion, we first consider the functional

$$\mathcal{G}_0(\phi^\perp, \delta, d) = \mathcal{G}(\phi^\perp, \delta, d, e) - \mathcal{M}(\phi^\perp, \delta, d, e)$$

where $\mathcal{M}$ is the functional that recollects all mixed terms, as defined in (6.14). A direct consequence of Step 1, Step 2 and Step 3 is that $\mathcal{G}_0$ has a critical point $(\phi^\perp = \phi^\perp(f), \delta = \delta(f), d = d(f))$, namely the system

$$D_{\phi^\perp} E(\phi^\perp) = D_{\phi^\perp} \mathcal{L}^1_f(\phi^\perp), \quad \varepsilon^{-\frac{k}{2}} D_\delta P_\varepsilon(\delta) = D_\delta \mathcal{L}^2_f(\delta), \quad \varepsilon^{-\frac{k}{2}} D_d Q_\varepsilon(d) = D_d \mathcal{L}^3_f(d)$$

is uniquely solvable in $(H^1_\varepsilon)^\perp \times (\mathcal{H}^1(K))^N$ and furthermore

$$\|\phi^\perp\|_{H^1_\varepsilon} + \varepsilon^{-\frac{k}{2}}\|\delta\|_{\mathcal{H}^1(K)} + \varepsilon^{-\frac{k}{2}}\|d\|_{(\mathcal{H}^1(K))^{N-1}} \leq C\varepsilon^{-2}\|f\|_{L^2(\Omega_{\varepsilon,\gamma})}$$

for some constant $C > 0$, independent of ε.

If we now consider the complete functional $\mathcal{G}$, a critical point of $\mathcal{G}$ shall satisfy the system

$$\begin{cases} D_{\phi^\perp} E(\phi^\perp) = D_{\phi^\perp}\mathcal{L}^1_f(\phi^\perp) + D_{\phi^\perp}\mathcal{M}(\phi^\perp, \delta, d, e) \\ D_\delta P_\varepsilon(\delta) = D_\delta \mathcal{L}^2_f(\delta) + D_\delta\mathcal{M}(\phi^\perp, \delta, d, e) \\ D_d Q_\varepsilon(d) = D_d\mathcal{L}^3_f(d) + D_d\mathcal{M}(\phi^\perp, \delta, d, e). \end{cases} \tag{6.35}$$

On the other hand, as we have already observed in Theorem 6.1, we have

$$\|D_{(\phi^\perp,\delta,d)}\mathcal{M}(\phi^\perp_1, \delta_1, d_1, e_1) - D_{(\phi^\perp,\delta,d)}\mathcal{M}(\phi^\perp_2, \delta_2, d_2, e_2)\| \leq C\varepsilon^2 \times$$
$$\left[\|\phi^\perp_1 - \phi^\perp_2\| + \varepsilon^{-\frac{k}{2}}\|\delta_1 - \delta_2\|_{\mathcal{H}^1(K)} + \varepsilon^{-\frac{k}{2}}\|d_1 - d_2\|_{(\mathcal{H}^1(K))^{N-1}} + \varepsilon^{-\frac{k}{2}}\|e_1 - e_2\|_{\mathcal{H}^1(K)}\right].$$

Thus the contraction mapping Theorem guarantees the existence of a unique solution $(\bar\phi^\perp, \bar\delta, \bar d)$ to (6.35) in the set

$$\|\phi^\perp\|_{H^1_\varepsilon} + \varepsilon^{-\frac{k}{2}}\|\delta\|_{\mathcal{H}^1(K)} + \varepsilon^{-\frac{k}{2}}\|d\|_{(\mathcal{H}^1(K))^{N-1}} \leq C\left[\varepsilon^{-2}\|f\|_{L^2(\Omega_{\varepsilon,\gamma})} + \varepsilon^2\varepsilon^{-\frac{k}{2}}\|e\|_{\mathcal{H}^1(K)}\right].$$

Furthermore, the solution $\bar\phi^\perp = \bar\phi^\perp(f, e)$, $\bar\delta = \bar\delta(f, e)$ and $\bar d = \bar d(f, e)$ depend on e in a smooth and non-local way.

Step 5. Given $f \in L^2(\Omega_{\varepsilon,\gamma})$, we replace the critical point $(\bar\phi^\perp = \bar\phi^\perp(f, e), \bar\delta = \bar\delta(f, e), \bar d = \bar d(f, e))$ of $\mathcal{G}$ obtained in the previous step into the functional $\mathcal{E}(\phi^\perp, \delta, d, e)$ thus getting a new functional depending only on $e \in \mathcal{H}^1(K)$, that we denote by $\mathcal{F}_\varepsilon(e)$, given by

$$\begin{aligned} \mathcal{F}_\varepsilon(e) &= \varepsilon^{-k}[R_\varepsilon(e) - \mathcal{L}^4_f(e)] + E(\bar\phi^\perp(e)) - \varepsilon^{-k}\mathcal{L}^1_f(\bar\phi^\perp(e)) + \varepsilon^{-k}[P_\varepsilon(\bar\delta(e)) - \mathcal{L}^2_f(\bar\delta(e))] \\ &+ \varepsilon^{-k}[Q_\varepsilon(\bar d(e)) - \mathcal{L}^3_f(\bar d(e))] + \mathcal{M}(\bar\phi^\perp(e), \bar\delta(e), \bar d(e), e). \end{aligned}$$

The rest of the proof is devoted to show that there exists a sequence $\varepsilon = \varepsilon_l \to 0$ such that

$$D_e\mathcal{F}_\varepsilon(e) = 0 \tag{6.36}$$

is solvable. Using the fact that $(\bar\phi^\perp, \bar\delta, \bar d)$ is a critical point for $\mathcal{G}$ (see Step 4 for the definition), we have that

$$D_e\mathcal{F}_\varepsilon(e) = \varepsilon^{-k}D_e[R_\varepsilon(e) - \mathcal{L}^4_f(e)] + D_e\mathcal{M}(\bar\phi^\perp(e), \bar\delta(e), \bar d(e), e). \tag{6.37}$$

Define

$$\mathcal{L}_\varepsilon := \varepsilon^{-k}D_eR_\varepsilon(e) + D_e\mathcal{M}(\bar\phi^\perp(e), \bar\delta(e), \bar d(e), e), \tag{6.38}$$

regarded as self adjoint in $\mathcal{L}^2(K)$. The work to solve the equation $D_e\mathcal{F}_\varepsilon(e) = 0$ consists in showing the existence of a sequence $\varepsilon_l \to 0$ such that 0 lies suitably far away from the spectrum of $\mathcal{L}_{\varepsilon_l}$.

We recall now that the map

$$(\phi^\perp, \delta, d, e) \to D_e\mathcal{M}(\phi^\perp, \delta, d, e)$$

is a linear operator in the variables $\phi^\perp, \delta, d$, while it is constant in e. This is contained in the result of Theorem 6.1. If we furthermore take into account that the terms $\bar\phi^\perp$, $\bar\delta$ and $\bar d$ depend smoothly and in a non-local way through e, we conclude that, for any $e \in \mathcal{H}^1(K)$,

$$D_e\mathcal{M}(\bar\phi^\perp(e), \bar\delta(e), \bar d(e), e)[e] = \varepsilon^{\gamma(N-3)}\varepsilon^{-k}\int_K (\varepsilon\eta_1(e)\partial_a e + \eta_2(e)e)^2 \tag{6.39}$$

where η_1 and η_2 are non local operators in e, that are bounded, as $\varepsilon \to 0$, on bounded sets of $\mathcal{L}^2(K)$. Thanks to the result contained in Theorem 6.1 and the above observation, we conclude that the quadratic from

$$\Upsilon_\varepsilon(e) := \varepsilon^{-k}D_eR_\varepsilon(e)[e] + D_e\mathcal{M}(\bar\phi^\perp(e), \bar\delta(e), \bar d(e), e)[e]$$

can be described as follows

$$\tilde\Upsilon_\varepsilon(e) = \varepsilon^k\Upsilon_\varepsilon(e) = \Upsilon^0_\varepsilon(e) - \bar\lambda_0\int_K e^2 + \varepsilon\Upsilon^1_\varepsilon(e) \tag{6.40}$$

where

$$\Upsilon^0_\varepsilon(e) = \varepsilon^2\int_K (1 + \varepsilon^{\gamma(N-3)}\eta_1(e))\left|\partial_a\left(e(1 + e^{-\varepsilon^{-\lambda'}}\beta^\varepsilon_3(y))\right)\right|^2. \tag{6.41}$$

In the above expression $\bar\lambda_0$ is the positive number defined by

$$\bar\lambda_0 = (\int_{\mathbb{R}^N_+} Z_1^2)\,\lambda_0,$$

$\Upsilon^1_e(e)$ is a compact quadratic form in $\mathcal{H}^1(K)$, β^ε_3 is a smooth and bounded (as $\varepsilon \to 0$) function on K, given by (6.21). Finally, η_1 is a non local operator in e, which is uniformly bounded, as $\varepsilon \to 0$ on bounded sets of $\mathcal{L}^2(K)$.

Thus, for any $\varepsilon > 0$, the eigenvalues of

$$\mathcal{L}_\varepsilon e = \lambda e, \quad e \in \mathcal{H}^1(K)$$

are given by a sequence $\lambda_j(\varepsilon)$, characterized by the Courant-Fisher formulas

$$\lambda_j(\varepsilon) = \sup_{dim(M)=j-1} \inf_{e \in M^\perp \setminus \{0\}} \frac{\tilde{\Upsilon}_\varepsilon(e)}{\int_K e^2} = \inf_{dim(M)=j} \sup_{e \in M \setminus \{0\}} \frac{\tilde{\Upsilon}_\varepsilon(e)}{\int_K e^2}. \tag{6.42}$$

The proof of Theorem 5.1 and of the inequality (5.9) will follow then from Step 4 and formula (6.34), together with the validity of the following

Lemma 6.1. *There exist a sequence $\varepsilon_l \to 0$ and a constant $c > 0$ such that, for all j, we have*

$$|\lambda_j(\varepsilon_l)| \geq c\varepsilon_l^k. \tag{6.43}$$

The proof of this Lemma can be found in [14].

7. Appendix

The proof Lemma 2.1 is simply based on a Taylor expansion of the metric coefficients in terms of the geometric properties of $\partial\Omega$ and K. Indeed recall that the Laplace-Beltrami operator is given by

$$\Delta_{g^\varepsilon} = \frac{1}{\sqrt{\det g^\varepsilon}}\, \partial_A(\sqrt{\det g^\varepsilon}\, (g^\varepsilon)^{AB}\, \partial_B\,), \tag{7.44}$$

where indices A and B run between 1 and $n = N + k$. Here g^ε is the scaled metric whose coefficients re defined by

$$g^\varepsilon_{\alpha,\beta}(z, x) = g_{\alpha,\beta}(\varepsilon z, \varepsilon x)$$

where $g_{\alpha,\beta}$ are the coefficients of the flat metric g of $\mathbb{R}^{N+k}$ in the coordinates $(y, \bar{x}, x_N)$. To compute first $g_{\alpha,\beta}$, recall that, by choosing Fermi coordinates, on K the metric $\overline{g}$ (see Section 2) splits in the following way

$$\overline{g}(q) = \overline{g}_{ab}(q)\, dy_a \otimes dy_b + \overline{g}_{ij}(q)\, dx_i \otimes dx_j, \qquad q \in K. \tag{7.45}$$

Letting $g_{\alpha\beta}$ be the coefficients of the flat metric g of $\mathbb{R}^{N+k}$ in the coordinates $(y, \bar{x}, x_N)$, with easy computations we deduce for $\tilde{y} = (y, \bar{x})$ that

$$g_{\alpha\beta}(\tilde{y}, x_N) = \overline{g}_{\alpha\beta}(\tilde{y}) - x_N \left(H_{\alpha\delta}\overline{g}_{\delta\beta} + H_{\beta\delta}\overline{g}_{\delta\alpha} \right)(\tilde{y}) + x_N^2 H_{\alpha\delta} H_{\sigma\beta} \overline{g}_{\delta\sigma}(\tilde{y}); \tag{7.46}$$

$$g_{\alpha N} \equiv 0; \qquad g_{NN} \equiv 1. \tag{7.47}$$

In the above expressions, with α and β we denote any index of the form $a = 1, \ldots, k$ or $i = 1, \ldots, N-1$. We first provide a Taylor expansion of the coefficients of the metric g.

Lemma 7.1. *For the (Euclidean) metric g in the above coordinates we have the expansions*

$$g_{ij} = \delta_{ij} - 2x_N H_{ij} + \frac{1}{3}\, R_{istj}\, x_s\, x_t + x_N^2 (H^2)_{ij} \, + \, \mathcal{O}(|x|^3), \quad 1 \leq i, j \leq N-1;$$

$$g_{aj} = -x_N \left(H_{aj} + \tilde{g}_{ac} H_{cj} \right) + \mathcal{O}(|x|^2), \quad 1 \leq a \leq k,\ 1 \leq j \leq N-1;$$

$$\begin{aligned} g_{ab} &= \tilde{g}_{ab} - \left\{ \tilde{g}_{ac}\, \Gamma^c_{bi} + \tilde{g}_{bc}\, \Gamma^c_{ai} \right\} x_i - x_N \left\{ H_{ac}\, \tilde{g}_{bc} + H_{bc}\, \tilde{g}_{ac} \right\} + \left[R_{sabl} + \tilde{g}_{cd} \Gamma^c_{as}\, \Gamma^b_{dl} \right] x_s x_l \\ &+ x_N^2 (H^2)_{ab} + x_N\, x_k \left[H_{ac} \left\{ \tilde{g}_{bf} \Gamma^f_{ck} + \tilde{g}_{cf} \Gamma^f_{bk} \right\} + H_{bc} \left\{ \tilde{g}_{af} \Gamma^f_{ck} + \tilde{g}_{cf} \Gamma^f_{ak} \right\} \right] + \mathcal{O}(|x|^3), \\ &\quad 1 \leq a, b \leq k; \end{aligned}$$

$$g_{aN} \equiv 0, \quad a = 1, \ldots, k; \qquad g_{iN} \equiv 0, \quad i = 1, \ldots, N-1; \qquad g_{NN} \equiv 1.$$

In the above expressions $H_{\alpha\beta}$ denotes the components of the matrix tensor H defined in (2.9), R_{istj} are the components of the curvature tensor as defined in (2.19), $\Gamma^b_a(E_i)$ are defined in (2.6). Here we have set

$$(A^2)_{\alpha\beta} = A_{\alpha i} A_{i\beta} + \tilde{g}_{cd} A_{\alpha c} A_{\beta d}.$$

Furthermore, we have the validity of the following expansion for the log of the determinant of g

$$\begin{aligned} \log\left(\det g\right) \quad &= \quad \log\left(\det \tilde{g}\right) - 2x_N \mathrm{tr}\,(H) - 2\Gamma^b_{bk}\, x_k + \frac{1}{3} R_{miil} x_m x_l \\ &+ \quad \left(\tilde{g}^{ab}\, R_{mabl} - \Gamma^c_{am} \Gamma^a_{cl} \right) x_m x_l - x_N^2\, \mathrm{tr}\,(H^2) + \mathcal{O}(|x|^3). \end{aligned}$$

We refer to Lemma 2.2 in [14] for a detailed proof of this lemma.

Letting now $g^\varepsilon_{\alpha,\beta}$ be the coefficients of the metric g^ε, we have $g^\varepsilon_{\alpha,\beta}(z,x) = g_{\alpha,\beta}(\varepsilon z, \varepsilon x)$. With an easy computation we deduce the following

Lemma 7.2. *For the (Euclidean) metric g^ε in the above coordinates (z, X) we have the expansions*

$$g^\varepsilon_{ij} = \delta_{ij} - 2\varepsilon X_N H_{ij} + \frac{\varepsilon^2}{3} R_{istj}\, X_s\, X_t + \varepsilon^2 X_N^2 (H^2)_{ij} \;+\; \mathcal{O}(\varepsilon^3(|X|^3),$$
$$1 \le i, j \le N-1;$$

$$g^\varepsilon_{aj} = -\varepsilon X_N \Big(H_{aj} + \tilde{g}^\varepsilon_{ac} H_{cj} \Big) + \mathcal{O}(\varepsilon^2 |X|^2)$$
$$1 \le a \le k,\ 1 \le j \le N-1;$$

$$\begin{aligned} g^\varepsilon_{ab} = \tilde{g}^\varepsilon_{ab} &- \varepsilon \Big\{ \tilde{g}^\varepsilon_{ac}\, \Gamma^c_{bi} + \tilde{g}^\varepsilon_{bc}\, \Gamma^c_{ai} \Big\} X_i - \varepsilon X_N \Big\{ H_{ac}\, \tilde{g}^\varepsilon_{bc} + H_{bc}\, \tilde{g}^\varepsilon_{ac} \Big\} \\ &+ \varepsilon^2 \Big[R_{sabl} + \tilde{g}^\varepsilon_{cd} \Gamma^c_{as}\, \Gamma^b_{dl} \Big] X_s X_l + \varepsilon^2 X_N^2 (H^2)_{ab} \\ &+ \varepsilon^2 X_N\, X_k \Big[H_{ac} \big\{ \tilde{g}^\varepsilon_{bf} \Gamma^f_{ck} + \tilde{g}^\varepsilon_{cf} \Gamma^f_{bk} \big\} + H_{bc} \big\{ \tilde{g}^\varepsilon_{af} \Gamma^f_{ck} + \tilde{g}^\varepsilon_{cf} \Gamma^f_{ak} \big\} \Big] + \mathcal{O}(\varepsilon^3 |X|^3), \end{aligned}$$
$$1 \le a, b \le k;$$

$$g^\varepsilon_{aN} \equiv 0, \quad a = 1, \dots, k; \qquad g^\varepsilon_{iN} \equiv 0, \quad i = 1, \dots, N-1; \qquad g^\varepsilon_{NN} \equiv 1.$$

In the above expressions $H_{\alpha\beta}$ denotes the components of the matrix tensor H defined in (2.9), R_{istj} are the components of the curvature tensor as defined in (2.19), Γ^b_{ai} are defined in (2.6) and $\tilde{g}^\varepsilon_{ab}(z) = \tilde{g}_{ab}(\varepsilon z)$.

Furthermore, we have the validity of the following expansions for the square root of the determinant of g^ε and the log of determinant of g^ε

$$\begin{aligned} \sqrt{\det g^\varepsilon} \;&=\; \sqrt{\det \tilde{g}^\varepsilon} \Big\{ 1 - \varepsilon X_N tr(H) + \frac{\varepsilon^2}{6} R_{miil} X_m X_l + \frac{\varepsilon^2}{2} \Big((\tilde{g}^\varepsilon)^{ab}\, R_{mabl} - \Gamma^c_{am} \Gamma^a_{cl} \Big) X_m X_l \\ &+ \; \frac{\varepsilon^2}{2} X_N^2 tr(H)^2 - \varepsilon^2 X_N^2\, tr(H^2) \Big\} + \varepsilon^3 \mathcal{O}(|X|^3), \end{aligned} \tag{7.48}$$

and

$$\begin{aligned} \log\big(\det g^\varepsilon\big) \;&=\; \log\big(\det \tilde{g}^\varepsilon\big) - 2\varepsilon X_N \mathrm{tr}\,(H) + \frac{\varepsilon^2}{3} R_{miil}\, X_m X_l \\ &+ \; \varepsilon^2 \Big((\tilde{g}^\varepsilon)^{ab}\, R_{mabl} - \Gamma^c_{am} \Gamma^a_{cl} \Big) X_m X_l - \varepsilon^2 X_N^2\, \mathrm{tr}\,(H^2) + \mathcal{O}(\varepsilon^3 |X|^3). \end{aligned}$$

We refer to Lemma 3.1 and Lemma 3.2 in [14] for a detailed proof of the above lemma.

The proof of Lemma 2.1 is based on Lemma 7.1 and Lemma 7.2. We refer the reader to Lemma 3.3 in [14] for a detailed proof of Lemma 2.1.

We end up this section with the proofs of (4.12) and (4.14).

Proof of (4.12). We write

$$\begin{aligned} & \int_{\hat{\mathcal{D}}} \big(H_{\alpha\alpha} \partial_N w_0 - 2 H_{ij} \xi_N\, \partial^2_{ij} w_0 \big)\, Z_0 d\xi \\ =\;& H_{\alpha\alpha} \int_{\hat{\mathcal{D}}} \partial_N w_0 Z_0 d\xi - 2 H_{ij} \int_{\hat{\mathcal{D}}} \xi_N\, \partial^2_{ij} w_0 Z_0 d\xi \\ =\;& H_{\alpha\alpha} \int_{\mathbb{R}^N_+} \partial_N w_0 Z_0 d\xi - 2 H_{ij} \int_{\mathbb{R}^N_+} \xi_N\, \partial^2_{ij} w_0 Z_0 d\xi \\ & - \int_{\mathbb{R}^N_+ \setminus \hat{\mathcal{D}}} \big(H_{\alpha\alpha} \partial_N w_0 - 2 H_{ij} \xi_N\, \partial^2_{ij} w_0 \big)\, Z_0 d\xi. \end{aligned} \tag{7.49}$$

We compute $-2 \int_{\mathbb{R}^N_+} \xi_N \partial^2_{ij} w_0 Z_0$. By symmetry we have that $2 \int_{\mathbb{R}^N_+} \xi_N \partial^2_{ij} w_0 Z_0 = 0$ if $i \ne j$. Assume then that $i = j$ is fixed and integrations by parts, a direct differentiation yields

$$-2 \int_{\mathbb{R}^N_+} \xi_N \partial^2_{ii} w_0 Z_0 = 2 \int_{\mathbb{R}^N_+} \xi_N \partial_i w_0 \partial_i Z_0 = -\partial_\lambda \left[\int_{\mathbb{R}^N_+} \xi_N |\partial_1 w_\lambda|^2 \right]_{|_{\lambda=1}}$$

and also

$$\int_{\mathbb{R}^N_+} \xi_N |\partial_1 w_\lambda|^2 = \lambda \int_{\mathbb{R}^N_+} \xi_N |\partial_1 w_0|^2.$$

Then

$$-2 \int_{\mathbb{R}^N_+} \xi_N \partial^2_{ii} w_0 Z_0 = -\mathfrak{A}_1.$$

Therefore

$$H_{\alpha\alpha} \int_{\mathbb{R}^N_+} \partial_N w_0 Z_0 d\xi - 2H_{ij} \int_{\mathbb{R}^N_+} \xi_N \, \partial^2_{ij} w_0 Z_0 d\xi = H_{\alpha\alpha}\mathfrak{A}_0 - H_{ii}\mathfrak{A}_1. \tag{7.50}$$

Last let us estimate the third term in (7.49), using the definition of

$$Z_0 = \gamma w_0 + \xi \cdot \nabla w_0,$$

we easily get

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N_+ \setminus \hat{\mathcal{D}}} \left(H_{\alpha\alpha} \partial_N w_0 - 2H_{ij} \xi_N \, \partial^2_{ij} w_0 \right) Z_0 d\xi \right| \\
= &\left| \int_{\mathbb{R}^N_+ \setminus \hat{\mathcal{D}}} \left(H_{\alpha\alpha} \partial_N w_0 - 2H_{ij} \xi_N \, \partial^2_{ij} w_0 \right) (\gamma w_0 + \xi \cdot \nabla w_0) d\xi \right| \\
\leq & \; C \int_{\mathbb{R}^N_+ \setminus \hat{\mathcal{D}}} \frac{\xi_N \xi_1^2}{(1+|\xi|^2)^N} d\xi + C \int_{\mathbb{R}^N_+ \setminus \hat{\mathcal{D}}} \frac{\xi_N}{(1+|\xi|^2)^{N-1}} d\xi \\
\leq & \; C \int_{\mathbb{R}^N_+ \setminus \hat{\mathcal{D}}} \frac{\xi_N}{(1+|\xi|^2)^{N-1}} d\xi.
\end{aligned} \tag{7.51}$$

It is convinient to rewrite the last integral as

$$\int_{\mathbb{R}^N_+ \setminus \hat{\mathcal{D}}} \frac{\xi_N}{(1+|\xi|^2)^{N-1}} d\xi \quad = \quad \mathcal{A}_1 + \mathcal{A}_2 \tag{7.52}$$

with

$$\begin{aligned}
\mathcal{A}_1 \quad &= \quad \int_{\mathbb{R}^{N-1}} \int_{\frac{\delta}{\varepsilon}}^{+\infty} \frac{\xi_N}{(1+|\bar{\xi}|^2 + \xi_N^2)^{N-1}} d\xi_N \; d\bar{\xi} \\
\mathcal{A}_2 \quad &= \quad \int_{|\bar{\xi}| > \frac{\delta}{\varepsilon}} \int_0^{\frac{\delta}{\varepsilon}} \frac{\xi_N}{(1+|\bar{\xi}|^2 + \xi_N^2)^{N-1}} d\xi_N \; d\bar{\xi}.
\end{aligned}$$

We first estimate $\mathcal{A}_1$. We have

$$\begin{aligned}
\mathcal{A}_1 \quad \leq \quad & C \int_{\mathbb{R}^{N-1}} \frac{1}{(1+|\bar{\xi}|^2)^{N-1}} d\bar{\xi} \int_{\frac{\delta}{\varepsilon}}^{+\infty} \frac{\xi_N}{\left(1 + \left(\frac{\xi_N}{\sqrt{1+|\bar{\xi}|^2}}\right)^2\right)^{N-1}} d\xi_N \\
& \text{set} \quad z_N = \frac{\xi_N}{\sqrt{1+|\bar{\xi}|^2}} \\
= \quad & C \int_{\mathbb{R}^{N-1}} \frac{1}{(1+|\bar{\xi}|^2)^{N-2}} d\bar{\xi} \int_{\frac{\delta}{\varepsilon}\frac{1}{\sqrt{1+|\bar{\xi}|^2}}}^{+\infty} \frac{z_N}{(1+z_N^2)^{N-1x}} dz_N \\
\leq \quad & C \left| \int_{\mathbb{R}^{N-1}} \frac{1}{(1+|\bar{\xi}|^2)^{N-2}} d\bar{\xi} \int_{\frac{\delta}{\varepsilon}\frac{1}{\sqrt{1+|\bar{\xi}|^2}}}^{+\infty} \frac{\partial}{\partial z_N} \frac{1}{(1+z_N^2)^{N-2}} dz_N \right| \\
\leq \quad & C \int_{\mathbb{R}^{N-1}} \frac{1}{(1+|\bar{\xi}|^2)^{N-2}} \frac{1}{\left(1 + \left(\frac{\delta}{\varepsilon}\frac{1}{\sqrt{1+|\bar{\xi}|^2}}\right)^2\right)^{N-2}} d\bar{\xi} \\
= \quad & C \int_{\mathbb{R}^{N-1}} \frac{1}{(1+|\bar{\xi}|^2)^{N-2}} \frac{1}{\frac{1}{(1+|\bar{\xi}|^2)^{N-2}} \left(1 + |\bar{\xi}|^2 + \frac{\delta^2}{\varepsilon^2}\right)^{N-2}} d\bar{\xi} \\
= \quad & C \int_{\mathbb{R}^{N-1}} \frac{1}{\left(1 + |\bar{\xi}|^2 + \frac{\delta^2}{\varepsilon^2}\right)^{N-2}} d\bar{\xi}
\end{aligned}$$

$$= C\frac{1}{\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{N-2}}\int_{\mathbb{R}^{N-1}}\frac{1}{\left(1+\left(\frac{|\bar\xi|}{\sqrt{1+\frac{\delta^2}{\varepsilon^2}}}\right)^2\right)^{N-2}}d\bar\xi. \tag{7.53}$$

We make the following change of variable $\bar z = \frac{|\bar\xi|}{\sqrt{1+\frac{\delta^2}{\varepsilon^2}}}$ we clearly get

$$\begin{aligned}\mathcal{A}_1 &\le C\frac{1}{\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{\frac{N-3}{2}}}\int_{\mathbb{R}^{N-1}}\frac{1}{(1+|\bar z|^2)^{N-2x}}d\bar z\\ &\le C\frac{1}{\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{\frac{N-3}{2}}} = O(\varepsilon^{N-3}).\end{aligned} \tag{7.54}$$

For $\mathcal{A}_2$, we have

$$\begin{aligned}\mathcal{A}_2 &= C\int_{|\bar\xi|>\frac{\delta}{\varepsilon}}\int_0^{\frac{\delta}{\varepsilon}}\frac{\xi_N}{(1+|\bar\xi|^2+\xi_N^2)^N}d\xi_N\ d\bar\xi\\ &\le C\int_{|\bar\xi|>\frac{\delta}{\varepsilon}}\frac{1}{(1+|\bar\xi|^2)^N}\ d\bar\xi\int_0^{\frac{\delta}{\varepsilon}}\xi_N d\xi_N\\ &\le C\frac{\delta^2}{\varepsilon^2}\int_{\frac{\delta}{\varepsilon}}^{+\infty}\frac{r^{N-2}}{(1+r^2)^N}\ dr = O(\varepsilon^{N-1}).\end{aligned} \tag{7.55}$$

Formula (4.12) then follows from (7.50) and (7.51)-(7.55).

Proof of (4.14). To prove Formula (4.14), we first mention the following basic observations

$$\int_{\hat{\mathcal{D}}} w_0^p Z_0 = \int_{\hat{\mathcal{D}}} w_0^{\frac{N+2}{N-2}} Z_0 = -\frac{N-2}{2N}\partial_\lambda\left(\int_{\hat{\mathcal{D}}} w_\lambda^{\frac{2N}{N-2}}\right)_{|\lambda=1} = 0.$$

Here we used the fact that $(\partial_\lambda w_\lambda)_{|\lambda=1} = -Z_0$, and $\int_{\hat{\mathcal{D}}} w_\lambda^{\frac{2N}{N-2}}$ does not depend on λ (by simple change of variables argument). It remains then to prove that

$$\int_{\hat{\mathcal{D}}}\log(w_0)w_0^p\, Z_0 = -\frac{N-2}{4N}\, C_0 + O(\varepsilon^{N-2}).$$

We rewrite

$$\int_{\hat{\mathcal{D}}}\log(w_0)w_0^p\, Z_0 = \int_{\mathbb{R}^N_+}\log(w_0)w_0^p\, Z_0 - \int_{\mathbb{R}^N_+\setminus\hat{\mathcal{D}}}\log(w_0)w_0^p\, Z_0.$$

We clearly have that

$$\begin{aligned}\int_{\mathbb{R}^N_+}\log(w_0)w_0^p\, Z_0 &= -\frac{1}{p+1}\left(\int_{\mathbb{R}^N_+}\log(w_\lambda)\partial_\lambda\,(w_\lambda^{p+1})\right)_{|\lambda=1}\\ &= -\frac{1}{p+1}\left(\partial_\lambda\int_{\mathbb{R}^N_+}\log(w_\lambda)w_\lambda^{p+1} - \int_{\hat{\mathcal{D}}} w_\lambda^{p+1}\frac{\partial_\lambda w_\lambda}{w_\lambda}\right)_{|\lambda=1}\\ &= -\frac{1}{p+1}\left(\partial_\lambda\int_{\mathbb{R}^N_+}\log(w_\lambda)w_\lambda^{p+1} - \frac{1}{p+1}\partial_\lambda\int_{\mathbb{R}^N_+} w_\lambda^{p+1}\right)_{|\lambda=1}\\ &= -\frac{1}{p+1}\,\partial_\lambda\left(\int_{\mathbb{R}^N_+}\log(w_\lambda)w_\lambda^{p+1}\right)_{|\lambda=1}.\end{aligned}$$

Now, recall that w_λ is given by

$$w_\lambda(\xi) = \alpha_N\left(\frac{\lambda}{\lambda^2+|\xi|^2}\right)^{\frac{N-2}{2}}.$$

Hence, performing a change a variables $\xi\mapsto\frac{1}{\lambda}\,\xi$, one gets

$$\begin{aligned}\int_{\mathbb{R}^N_+}\log(w_\lambda)w_\lambda^{p+1}\,d\xi &= \alpha_N^{\frac{2N}{N-2}}\int_{\mathbb{R}^N_+}\left(\frac{\lambda}{\lambda^2+|\xi|^2}\right)^N\log(\alpha_N\,(\frac{\lambda}{\lambda^2+|\xi|^2})^{\frac{N-2}{2}})\,d\xi\\ &= \alpha_N^{\frac{2N}{N-2}}\int_{\mathbb{R}^N_+}\left(\frac{1}{1+|\xi|^2}\right)^N\log(\alpha_N\,\lambda^{\frac{2-N}{2}})\,d\xi\\ &\quad+\alpha_N^{\frac{2N}{N-2}}\int_{\mathbb{R}^N_+}\left(\frac{1}{1+|\xi|^2}\right)^N\log\left(\frac{1}{(1+|\xi|^2)^{\frac{N-2}{2}}}\right)\,d\xi.\end{aligned}$$

Differentiating with respect to λ and taking $\lambda = 1$, we immediately obtain

$$\begin{aligned}\int_{\mathbb{R}^N_+} \log(w_0) w_0^p \, Z_0 &= \frac{1}{p+1}\frac{N-2}{2}\alpha_N^{\frac{2N}{N-2}} \int_{\mathbb{R}^N_+} \left(\frac{1}{1+|\xi|^2}\right)^N \, d\xi \\ &= \frac{(N-2)^2}{4N}\, C_0\end{aligned}$$

with

$$C_0 := \alpha_N^{\frac{2N}{N-2}} \int_{\mathbb{R}^N_+} \frac{d\xi}{(1+|\xi|^2)^N}.$$

On the other hand,

$$\begin{aligned}\left|\int_{\mathbb{R}^N_+\setminus\hat{\mathcal{D}}} \log(w_0) w_0^p \, Z_0\right| &= \left|\int_{\mathbb{R}^N_+\setminus\hat{\mathcal{D}}} \log(w_0) w_0^p \, (\gamma w_0 + \xi\cdot\nabla w_0)\right| \\ &\le C\int_{\mathbb{R}^N_+\setminus\hat{\mathcal{D}}} \frac{1}{(1+|\xi|^2)^{N-1}} d\xi \\ &= C\int_{|\bar\xi|>\frac{\delta}{\varepsilon}}\int_0^{\frac{\delta}{\varepsilon}} \frac{1}{(1+|\bar\xi|^2+\xi_N^2)^{N-1}} d\xi_N d\bar\xi \\ &+ C\int_{\mathbb{R}^{N-1}}\int_{\frac{\delta}{\varepsilon}}^{+\infty} \frac{1}{(1+|\bar\xi|^2+\xi_N^2)^{N-1}} d\xi_N d\bar\xi.\end{aligned}$$

We have clearly

$$\int_{|\bar\xi|>\frac{\delta}{\varepsilon}}\int_0^{\frac{\delta}{\varepsilon}} \frac{1}{(1+|\bar\xi|^2+\xi_N^2)^{N-1}} d\xi_N d\bar\xi \le \frac{\delta}{\varepsilon}\int_{|\bar\xi|>\frac{\delta}{\varepsilon}} \frac{1}{(1+|\bar\xi|^2)^{N-1}} d\bar\xi = O(\varepsilon^{N-2})$$

and

$$\begin{aligned}&\int_{\mathbb{R}^{N-1}}\int_{\frac{\delta}{\varepsilon}}^{+\infty} \frac{1}{(1+|\bar\xi|^2+\xi_N^2)^{N-1}} d\xi_N d\bar\xi \\ =& \int_{\frac{\delta}{\varepsilon}}^{+\infty} \frac{1}{(1+\xi_N^2)^{N-1}} d\xi_N \int_{\mathbb{R}^{N-1}} \frac{1}{\left(1+\left|\frac{\bar\xi}{\sqrt{1+\xi_N^2}}\right|^2\right)^{N-1}} d\bar\xi \\ =& \int_{\frac{\delta}{\varepsilon}}^{+\infty} \frac{1}{(1+\xi_N^2)^{\frac{N-1}{2}}} d\xi_N \int_{\mathbb{R}^{N-1}} \frac{1}{\left(1+|\bar z|^2\right)^{N-1}} d\bar z \\ \le& \int_{\frac{\delta}{\varepsilon}}^{+\infty} \frac{1}{(1+\xi_N^2)^{\frac{N-1}{2}}} d\xi_N = O(\varepsilon^{N-2}).\end{aligned}$$

This ends the proof of Formula (4.14). □

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