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Beyond the Trudinger–Moser supremum

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Abstract Let Ω be a bounded, smooth domain in $\mathbb{R}^2$. We consider the functional

$$I(u) = \int_\Omega e^{u^2}\, dx$$

in the supercritical Trudinger–Moser regime, i.e. for $\int_\Omega |\nabla u|^2 dx > 4\pi$. More precisely, we are looking for critical points of $I(u)$ in the class of functions $u \in H_0^1(\Omega)$ such that $\int_\Omega |\nabla u|^2\, dx = 4\,\pi\, k\,(1+\alpha)$, for small $\alpha > 0$. In particular, we prove the existence of 1-peak critical points of $I(u)$ with $\int_\Omega |\nabla u|^2 dx = 4\pi(1+\alpha)$ for any bounded domain Ω, 2-peak critical points with $\int_\Omega |\nabla u|^2 dx = 8\pi(1+\alpha)$ for non-simply connected domains Ω, and k-peak critical points with $\int_\Omega |\nabla u|^2 dx = 4k\pi(1+\alpha)$ if Ω is an annulus.

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1 Introduction

The Trudinger–Moser inequality concerns the limiting case $p = N$ of the Sobolev embeddings $W^{1,p}(\Omega) \subset L^{\frac{Np}{N-p}}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is an open domain. If $p = N$, the Sobolev

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space $W^{1,N}(\Omega)$ embeds into any $L^q(\Omega)$, but easy examples show that $W^{1,N}(\Omega) \not\subset L^\infty(\Omega)$. The maximal growth of integrability for functions $u \in W^{1,N}(\Omega)$ has been determined by Pohozaev [27] and Trudinger [32]: it is of exponential type, more precisely, for a bounded domain in $\Omega \subset \mathbb{R}^N$ one has

$$u \in W_0^{1,N}(\Omega) \quad \Rightarrow \quad \int_\Omega e^{u^2} dx < \infty$$

This inequality was sharpened by Moser [26] as follows:

$$\sup_{\|\nabla u\|_N \le 1} \int_\Omega e^{\mu |u|^{N'}} dx \quad \begin{cases} \le C\,|\Omega|, & \text{if } \mu \le \mu_N \\ = +\infty & \text{if } \mu > \mu_N \end{cases} \tag{1.1}$$

where $N' = \frac{N}{N-1}$ and $\mu_N = N\omega_{N-1}^{1/(N-1)}$ with ω_{N-1} the measure of the unit sphere in $\mathbb{R}^N$.

We recall that in the Sobolev case there is a loss of compactness at the limiting Sobolev exponent p^*, and the supremum

$$\sup_{\|\nabla u\|_p \le 1} \int_\Omega |u|^{p^*} dx$$

is not attained for any $\Omega \neq \mathbb{R}^N$. Lions [24] showed that also for (1.1) there is a loss of compactness at the limiting exponent $\mu = \mu_N$. But, despite the loss of compactness and in contrast to the Sobolev case, Carleson–Chang proved in [10] that the supremum

$$\sup_{\|\nabla u\|_N \le 1} \int_\Omega e^{\mu_N |u|^{N'}} dx$$

is attained for the ball $\Omega = B_1(0)$. In [12] an alternative proof of this result was given, showing that there is a close parallel to the famous result of Brezis–Nirenberg [9] on perturbations of the Sobolev embedding. Struwe showed in [29] that the Carleson–Chang result continues to hold if Ω is a small perturbation of the ball; Flucher [20] then proved that the result is true on general bounded domains $\Omega \subset \mathbb{R}^2$, and Lin [23] extended the result to general bounded domains in $\mathbb{R}^N$.

Numerical evidence given by Monahan [25] suggested that in the case $N = 2$ (for Ω being the ball) the Trudinger–Moser functional

$$I(u) = \int_\Omega e^{|u|^2} dx, \ \ \|\nabla u\|_2^2 = \mu \tag{1.2}$$

admits a local maximum and a mountain-pass critical value in the *supercritical regime*, i.e. for $\mu > \mu_2 = 4\pi$ and near 4π. In the cited paper, Struwe was able to prove that this is indeed the case for *almost every* μ in some interval $(4\pi, \mu_0)$. The situation may be visualized as follows, where $u(p)$ and $u(\mu)$ denote the respective critical points (Fig. 1).

Recently, Lamm–Robert–Struwe [22] have studied the heat-flow associated to the Trudinger–Moser functional, and they proved that for $t \to \infty$ the solutions either converge strongly to a solution of the associated stationary Trudinger–Moser equation, or there is the formation of "standard bubbles". This quantization is in correspondence to the results of Adimurthi–Struwe [5], Adimurthi–Druet [4] and Druet [18] concerning the blow-up behavior of sequences of solutions of the associated elliptic equations. Using this characterization of the solutions of the heat-flow, Lamm–Robert–Struwe prove the existence of local maxima and

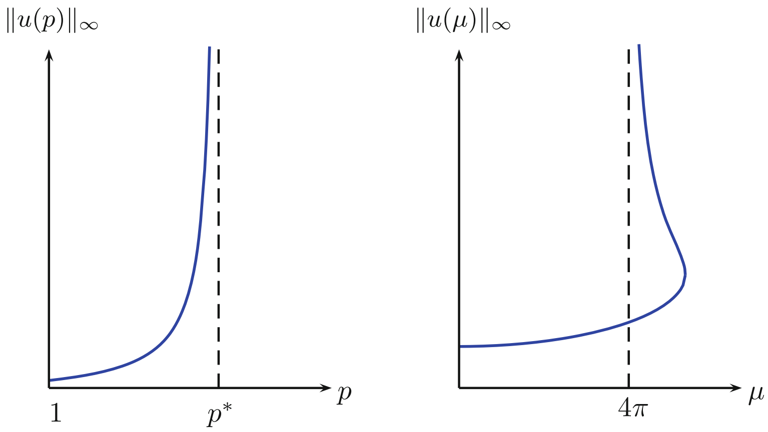

Fig. 1 Approaching criticality in the Sobolev and the Moser case

saddle point solutions in the supercritical regime $\mu \in (4\pi, \mu_0)$ for the functional (1.2), thus completing the result of Struwe in [29].

In recent years a very successful method has been developed for studying elliptic equations in critical or supercritical regimes, see e.g. the survey [16]. The main idea is to try to guess the form of the solution (using the shape of the "standard bubble"), then linearize the equation at this approximate solution and use a Lyapunov–Schmidt reduction to arrive at a reduced finite dimensional variational problem, whose critical points yield actual solutions of the equation. In this paper we use this method to study the functional (1.2) in the supercritical regime. To state the results, we define the Green's function $G(x, y)$ of the problem

$$\begin{aligned} -\Delta_x G &= 8\pi\delta_y(x), \quad x \in \Omega, \\ G(x, y) &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.3}$$

and H its regular part defined as

$$H(x, y) = 4 \log \frac{1}{|x-y|} - G(x, y). \tag{1.4}$$

Then from [4], it follows that the Robin function $x \mapsto H(x, x)$ has a strict minimum $\xi_0 \in \Omega$.

We prove the following results.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain.*

There exists $\mu_1 > 4\pi$ such that for $\mu \in (4\pi, \mu_1)$ the functional $I(u)$ restricted to the sphere $S_\mu = \{u \in H_0^1(\Omega) : \int_\Omega |\nabla u|^2 dx = \mu\}$ has a critical value with corresponding positive solution u_μ.

Furthermore, there exist a point $\xi_\mu \in \Omega$, with $\xi_\mu \to \xi_0$, the minimum of Robin's function, as $\mu \to 4\pi$, and a positive number m_μ with $m_\mu \to m \in (0, \infty)$, as $\mu \to 4\pi$, such that

$$u_\mu(x) = (\mu - 4\pi)^{1/4} \left[m_\mu G(x, \xi_\mu) + o(1) \right],$$

where $o(1) \to 0$ as $\mu \to 4\pi$ uniformly on compact sets of $\Omega \setminus \{\xi_0\}$.

In the next theorem we give a result near and above the critical energy level 8π; again, there exist critical levels with corresponding positive solutions with two blow-up points, provided the domain has non-trivial topology:

Theorem 1.2 *Assume that $\Omega \subset \mathbb{R}^2$ is bounded and not simply connected. Then there exists $\mu_2 > 8\pi$ such that for $\mu \in (8\pi, \mu_2)$ the functional $I(u)$ restricted to S_μ has a critical value with a corresponding positive solution u_μ which blows up around two points $\xi_1, \xi_2 \in \Omega$, as $\mu \to 8\pi$. More precisely, there exist $m_{i,\mu} > 0$ and $\xi_{i\mu} \in \Omega$, for $i = 1, 2$, such that*

$$m_{i,\mu} \to m_i \in (0, \infty) \quad as \quad \mu \to 8\pi,$$

$$\xi_{i\mu} \to \xi_i \quad as \quad \mu \to 8\pi, \quad with \quad \xi_1 \neq \xi_2,$$

and

$$u_\mu = (\mu - 8\pi)^{\frac{1}{4}} \left[m_{1\mu} G(x, \xi_{1\mu}) + m_{2\mu} G(x, \xi_{2\mu}) + o(1) \right]$$

where $o(1) \to 0$ uniformly on compact sets of $\Omega \setminus \{\xi_1, \xi_2\}$, as $\mu \to 8\pi$.

If Ω is an annulus we can look for solutions with symmetry, with several points of blow-up.

Theorem 1.3 *Let $0 < a < b$ and $\Omega = B(0, b) \setminus \overline{B}(0, a)$. Fix a positive integer k. Then there exists $\mu_k > 4\pi k$ such that for $\mu \in (4\pi k, \mu_k)$ the functional $I(u)$ restricted to S_μ has a critical value with corresponding positive solution u_μ which is invariant under rotations of angle $\frac{2\pi}{k}$ and which blows-up around k points arranged on the vertices of a regular polygon as $\mu \to 4\pi k$. More precisely, there exist points $\xi_{j\mu}$, for $j = 1, \ldots, k$, such that*

$$\xi_{j\mu} = r_\mu \hat{\xi}_j \qquad \text{for all } j = 1, \ldots, k, \quad with \quad r_\mu \to r_0 \quad as\ \mu \to 4\pi k$$

where

$$\hat{\xi}_j = e^{\frac{2\pi(j-1)}{k} i} \qquad j = 1, \ldots, k,$$

and r_0 is the minimum of the function

$$r \in (a, b) \to H(r\hat{\xi}_1, r\hat{\xi}_1) - \sum_{i>1} G(r\hat{\xi}_1, r\hat{\xi}_i).$$

There exists a positive number m_μ, with $m_\mu \to m \in (0, \infty)$ as $\mu \to 4\pi k$, such that the solution u_μ satisfies

$$u_\mu(x) = (\mu - 4\pi k)^{\frac{1}{4}} \left[m_\mu \sum_{j=1}^{k} G(x, \xi_{j\mu}) + o(1) \right]$$

where $o(1) \to 0$ uniformly on compact sets of $\Omega \setminus \bigcup_{j=1}^k \{r_0 e^{\frac{2\pi(j-1)}{k} i}\}$, as $\mu \to 4\pi k$.

The critical points u_μ found in Theorems 1.1–1.3 correspond to solutions of the equation

$$-\Delta u = \mu \frac{u e^{u^2}}{\int_\Omega u^2 e^{u^2}} \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{1.5}$$

This equation is related, but not equivalent, to the equation

$$-\Delta u = \lambda u e^{u^2} \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{1.6}$$

Critical growth equations of form (1.6) (and generalizations) have been extensively studied in recent years. The free functional associated with (1.6) is

$$J_\lambda(u) = \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega e^{u^2} dx.$$

In [1,13] existence results for Eq. 1.6 were given using variational methods (in the spirit of Brezis–Nirenberg), working with the functional $J_\lambda(u)$. On the other hand, in [4,5,18] the asymptotic behavior of (certain) sequences of solutions u_λ of (1.6) was studied, and it was found that $J_\lambda(u_\lambda) \to 4k\pi$ as $\lambda \to 0$, for some $k \geq 0$, while the question whether such families of solutions exist for $k \geq 1$ was left open. A positive answer was given in [17] in the case $k = 1, 2$, showing that there exists a family of solutions u_λ to problem (1.6) which blows up near a minimizer of the Robin's function $H(\xi, \xi)$ in the case $k = 1$, and with two bubbles in the case $k = 2$ provided Ω is not simply connected.

The proofs of Theorems 1.1–1.3 are related to the proofs in [17], and for some details we will refer to that paper.

2 A first approximation and outline of the argument

Let us fix an integer $k \geq 1$. For a small number $\alpha > 0$ we consider the set

$$M_\alpha = \{u \in H_0^1(\Omega) \ : \int\limits_\Omega |\nabla u|^2 = 4\pi\, k\, (1+\alpha)\,\}. \tag{2.1}$$

Our problem is to find critical points of the functional

$$I(u) = \int\limits_\Omega e^{u^2}\, dx \tag{2.2}$$

constrained to the manifold M_α. This is equivalent to finding a solution $u \in H_0^1(\Omega)$ to the nonlocal problem

$$\Delta\, u + \frac{4\pi\, k\,(1+\alpha)}{\int_\Omega u^2 e^{u^2}\, dx}\, u\, e^{u^2} = 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega. \tag{2.3}$$

Or equivalently, this reduces to find solutions u to the semilinear elliptic problem

$$\Delta\, u + \lambda\, u\, e^{u^2} = 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \tag{2.4}$$

which furthermore satisfy the constraint

$$\lambda = \frac{4\pi\, k\,(1+\alpha)}{\int_\Omega u^2 e^{u^2}\, dx}. \tag{2.5}$$

Let us consider k distinct points $\xi_1, \xi_2, \ldots, \xi_k$ in Ω and k positive numbers $m_1, m_2, \ldots, m_k$, such that, for a certain given $\delta > 0$ small, we have

$$\operatorname{dist}(\xi_j, \partial\Omega) > \delta, \quad |\xi_i - \xi_j| > \delta, \quad \delta < m_j < \delta^{-1}. \tag{2.6}$$

For points ξ_j and parameters m_j satisfying (2.6), we define new parameters μ_j as follows

$$\log 8\mu_j^2 := -2\log 2m_j^2 - H(\xi_j, \xi_j) + \sum_{i \neq j} m_i m_j^{-1} G(\xi_i, \xi_j), \tag{2.7}$$

where G denotes the Green's function for the Laplace operator with zero Dirichlet boundary conditions in Ω, and H its regular part (1.4). Observe that relation (2.7) defines a diffeomorphism between μ_j and m_j (for a proof of this fact we refer to the end of Sect. 4). For λ small and positive and for parameters m_j as above, we introduce another parameter, denoted by ε_j and defined by

$$\log \varepsilon_j^{-4} := \frac{1}{2m_j^2\lambda} - 2\log 2m_j^2. \tag{2.8}$$

Observe that $\varepsilon_j \to 0$ as $\lambda \to 0$. With these definitions, we introduce the function

$$U(x) := \sqrt{\lambda}\sum_{j=1}^{k} m_j \left[\log \frac{1}{(\mu_j^2\varepsilon_j^2 + |x-\xi_j|^2)^2} \; - \; H_j(x)\right]. \tag{2.9}$$

In (2.9) the function H_j is defined as follows

$$\begin{cases} \Delta H_j = 0, \quad \text{in } \Omega, \\ H_j(x) = \log \frac{1}{(\mu_j^2\varepsilon_j^2+|x-\xi_j|^2)^2}, \quad \text{for } x \in \partial\Omega. \end{cases}$$

This gives in particular that $U = 0$ on $\partial\Omega$. Let us observe that from elliptic estimates,

$$H_j(x) = H(x, \xi_j) + O(\varepsilon_j^2\mu_j^2),$$

uniformly in Ω, as $\varepsilon_j \to 0$, where H is defined in (1.4). Hence, far from the points ξ_j,

$$\log \frac{1}{(\mu_j^2\varepsilon_j^2 + |x-\xi_j|^2)^2} \; - \; H_j(x) \; = G(x, \xi_j) + O(\varepsilon_j^2\mu_j^2), \tag{2.10}$$

so that, far from the points ξ_j,

$$U(x) = \sqrt{\lambda}\left(\sum_{j=1}^{k} m_j\, G(x, \xi_j) \; + \; o(1)\right) \quad \text{as } \lambda \to 0, \tag{2.11}$$

where $o(1) \to 0$ as $\lambda \to 0$.

To describe the function U in a neighborhood of ξ_j, we introduce the functions

$$w_j(x) := w_{\mu_j}\left(\frac{x-\xi_j}{\varepsilon_j}\right) \tag{2.12}$$

with

$$w_\mu(y) := \log \frac{8\mu^2}{(\mu^2+|y|^2)^2}. \tag{2.13}$$

The functions w_μ, $\mu > 0$, are the radially symmetric solutions of the Liouville equation

$$\Delta w + e^w = 0 \quad \text{in } \mathbb{R}^2. \tag{2.14}$$

Thus in a small neighborhood of a given ξ_j, say in $|x-\xi_j| < \delta$, as a consequence of the definition of the parameters μ_j and ε_j given respectively in (2.7) and (2.8), we see that

$$\begin{aligned} U(x) &= \sqrt{\lambda}\Big\{m_j[w_j(x) + \log\varepsilon_j^{-4} - \log 8\mu_j^4 - H_j(x)] + \sum_{i\neq j}[m_i G(x,\xi_i) + O(\varepsilon_i^2)]\Big\} \\ &= \sqrt{\lambda}\Big\{m_j[w_j(x) + \log\varepsilon_j^{-4} - \log 8\mu_j^4 - H(\xi_j,\xi_j)] \\ &\quad + \sum_{i\neq j} m_i G(\xi_j,\xi_i) + O(|x-\xi_j|) + \sum_i O(\varepsilon_i^2)]\Big\} \\ &= m_j\sqrt{\lambda}\left[w_j(x) + \log\varepsilon_j^{-4} + 2\log 2m_j^2 + \theta(x)\right], \end{aligned} \tag{2.15}$$

where θ is a smooth function such that

$$\theta(x) = O(|x-\xi_j|) + \sum_j O(\varepsilon_j^2).$$

The aim of this paper is to construct a solution to Problem (2.3), or equivalently to Problem (2.4)–(2.5), of the form

$$u = U + \phi,$$

where ϕ is a lower order correction. Observe that by construction $U = 0$ on $\partial\Omega$, thus $\phi = 0$ on $\partial\Omega$. We will do this when $k = 1$ (see Theorem 1.1), when $k = 2$ and Ω is not simply connected (see Theorem 1.2), and for arbitrary k when Ω is an annulus (Theorem 1.3).

The first part of our argument is the construction of the function ϕ. For any $\lambda > 0$ small, points ξ_j and parameters m_j satisfying (2.6) we find ϕ as solution to the nonlinear problem (2.4) when projected on a proper subspace of $H_0^1(\Omega)$. Let us be more precise.

The linearized equation at w_j (see (2.12)) of the Liouville equation (2.14) is given by

$$\Delta\psi + e^{w_j}\psi = 0 \quad \text{in} \quad \mathbb{R}^2. \tag{2.16}$$

It is well known (see [8]) that all bounded solutions to (2.16) are given by

$$z_{0j}(y) = \partial_\mu w_{\mu_j}(y), \quad z_{lj}(y) = \partial_{y_l} w_{\mu_j}(y), \quad l = 1, 2.$$

Hence, expanding variables, the functions

$$Z_{ij}(x) := z_{ij}\left(\frac{x-\xi_j}{\varepsilon_j}\right), \quad i = 0, 1, 2 \tag{2.17}$$

are all the admissible solutions to

$$L_j(\phi) = \Delta\phi + \varepsilon_j^{-2} e^{w_j}\phi = 0.$$

Let us further introduce a large but fixed number $R_0 > 0$ and a non-negative function $\zeta(\rho)$ with $\zeta(\rho) = 1$ if $\rho < R_0$ and $\chi(\rho) = 0$ if $\rho > R_0 + 1$. We denote

$$\zeta_j(x) = \varepsilon_j^{-2}\zeta\left(\left|\frac{x-\xi_j}{\varepsilon_j}\right|\right). \tag{2.18}$$

The function ϕ will be a solution to problem (2.4) projected on the subspace of the functions in $H_0^1(\Omega)$ which are L^2-orthogonal to all $\zeta_j Z_{ij}$, for $j = 1, \ldots, k$, $i = 0, 1, 2$.

Using the notations: $m := (m_1, \ldots, m_k)$ and $\xi := (\xi_1, \ldots, \xi_k)$, we have the validity of the following.

Proposition 2.1 *Let $\delta > 0$ be fixed. There exist $\lambda_0 > 0$ and $C > 0$ such that, for any $0 < \lambda < \lambda_0$, for any points $\xi_1, \ldots, \xi_k$ and parameters $m_1, \ldots, m_k$ satisfying (2.6), and for parameters μ_j and ε_j defined by (2.7) and (2.8), there exists a solution $\phi = \phi(\lambda, \xi, m)$ to the Nonlinear Projected Problem*

$$\Delta(U+\phi) + \lambda(U+\phi)e^{(U+\phi)^2} = \sqrt{\lambda}\sum_{i=0}^{2}\sum_{j=1}^{k} c_{ij} Z_{ij}\zeta_j, \quad \text{in } \Omega, \tag{2.19}$$

$$\phi = 0, \quad \text{on } \partial\Omega, \tag{2.20}$$

$$\int_\Omega Z_{ij}\zeta_j\phi = 0, \quad \text{for all } i, j, \tag{2.21}$$

for certain constants $c_{ij} = c_{ij}(\lambda, \xi, m)$, that depend on λ, ξ and m. Furthermore, the function which to each (ξ, m) associates $\phi \in C(\Omega)$ solution to (2.19)–(2.21) is of class C^1 and we have the validity of the following estimates

$$\|\phi\|_\infty \le C\lambda^{\frac{3}{2}} \tag{2.22}$$

and

$$\|D_\xi\phi\|_\infty \le C\lambda^{\frac{3}{2}}, \quad \|D_m\phi\|_\infty \le C\lambda^{\frac{3}{2}}. \tag{2.23}$$

Moreover the function which to each (ξ, m) associates c_{ij} is of class C^1 and we have the validity of the following estimates

$$|c_{ij}| \le C\lambda, \quad |D_\xi c_{ij}| \le C\lambda, \quad |D_m c_{ij}| \le C\lambda. \tag{2.24}$$

We will prove Proposition 2.1 in Sect. 3.

Given the result in Proposition 2.1 we thus observe that the function $U + \phi$, where U is given by (2.9) and ϕ is given by Proposition 2.1, is a solution to our problem (2.3), or equivalently to (2.4)–(2.5), if there exists a proper choice of λ, of the points ξ_j and of the parameters m_j, $j = 1, \ldots, k$, such that

$$\lambda = \frac{4\pi k(1+\alpha)}{\int_\Omega (U+\phi)^2 e^{(U+\phi)^2}} \quad \text{and} \quad c_{ij} = 0 \qquad \text{for all } i, j, \tag{2.25}$$

or equivalently

$$\int_\Omega |\nabla(U+\phi)|^2\, dx = 4\pi k(1+\alpha) \quad \text{and} \quad c_{ij} = 0 \qquad \text{for all } i, j. \tag{2.26}$$

In Sect. 4 we will show the validity of

Proposition 2.2 *Let R be the set of points and parameters (ξ, m) satisfying (2.6). Under the same assumptions of Proposition 2.1, there exists $\alpha_0 > 0$ and a subregion R' of R such that for all $0 < \alpha < \alpha_0$ and for all $(\xi, m) \in R'$ there exists a function $\lambda = \lambda(\alpha, \xi, m)$ such that*

$$\int_\Omega |\nabla(U+\phi)|^2\, dx = 4\pi k(1+\alpha) \quad \text{for all } \alpha > 0, \quad \alpha \to 0 \tag{2.27}$$

Furthermore, λ is a smooth function of the free parameter α, of the points $\xi_1, \ldots, \xi_k$, and of the parameters $m_1, \ldots, m_k$. Moreover $\lambda \to 0$ as $\alpha \to 0$ for points $\xi_1, \ldots, \xi_k$ and parameters $m_1, \ldots, m_k$ belonging to R'.

With this definition of λ, we have that

$$D_{\xi,m} I(U+\phi) = 0 \quad \Longrightarrow \quad c_{ij} = 0 \qquad \text{for all} \quad i, j. \tag{2.28}$$

In (2.27)–(2.28) the function U is defined in (2.9) and ϕ is given by Proposition 2.1.

Given the choice of λ satisfying (2.27), for all $\alpha > 0$ small, Proposition 2.2 gives that $U + \phi$ is a solution to our problem if we can find (ξ, m) to be a critical point of the function

$$g(\xi, m) := I(U+\phi). \tag{2.29}$$

This will be done in Sect. 5 in the case $k = 1$ for any domain Ω, in Sect. 6 in the case $k = 2$ and Ω a not simply connected domain, and in Sect. 7 in the case of arbitrary $k \ge 1$ and Ω an annulus.

3 Proof of Proposition 2.1

The results in Proposition 2.1 are contained in our previous work [17], where we construct solutions u_λ to

$$\Delta u + \lambda u e^{u^2} = 0 \quad \text{in} \quad \Omega, \quad u > 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \tag{3.1}$$

for any $\lambda > 0$ small. Given a certain configuration of points $\xi_1, \ldots, \xi_k$ in Ω and a certain choice of the parameters $m_1, \ldots, m_k$, these solutions u_λ behave as

$$u_\lambda(x) \;=\; \sqrt{\lambda}\, \left(\sum_{j=1}^{k} m_j G(x, \xi_j) \;+\; o(1) \right)$$

where $o(1) \to 0$ on each compact subset of $\bar{\Omega} \setminus \{\xi_1, \ldots, \xi_k\}$.

For completeness, we will sketch the principal steps of the proof of Proposition 2.1 in this Section.

It is convenient for our purpose to rewrite Problem (2.19)–(2.21) using the substitution

$$U + \phi = \sqrt{\lambda}(\tilde{U} + \tilde{\phi}) \;\; \text{and} \;\; f(\tilde{U}) = \lambda \tilde{U} e^{\lambda \tilde{U}^2}$$

so that we get

$$L(\tilde{\phi}) \;=\; -E - N(\tilde{\phi}) + \sum_{i=0}^{2} \sum_{j=1}^{k} c_{ij} Z_{ij} \zeta_j, \quad \text{in } \Omega, \tag{3.2}$$

$$\tilde{\phi} = 0, \quad \text{on } \partial\Omega, \tag{3.3}$$

$$\int_\Omega Z_{ij} \zeta_j \tilde{\phi} = 0, \quad \text{for all } \; i, \, j. \tag{3.4}$$

Here L is the linear operator defined as

$$L(\tilde{\phi}) = \Delta\tilde{\phi} + \left[\sum_{j=1}^{k} \varepsilon_j^{-2} e^{w_j(x)} \right] \tilde{\phi}, \tag{3.5}$$

with ε_j defined in (2.8) and w_j in (2.12), and E and $N(\tilde{\phi})$ are given respectively by

$$E \;=\; \Delta\tilde{U} + f(\tilde{U}), \tag{3.6}$$

and

$$N(\tilde{\phi}) = [f(\tilde{U} + \tilde{\phi}) - f(\tilde{U}) - f'(\tilde{U})\tilde{\phi}] + \left[f'(\tilde{U}) - \sum_{j=1}^{k} \varepsilon_j^{-2} e^{\omega_j(x)} \right] \tilde{\phi}. \tag{3.7}$$

We want to find a small solution $\tilde{\phi}$ to the nonlinear problem (3.2)–(3.4). To do so two key steps are needed: to establish an invertibility theory for the linear operator L in suitable spaces of functions and to determine the size of the error term E in the corresponding norm.

We start by analyzing the error term E. To estimate the size of the right hand side of Eq. 3.2, and in particular of the error term E, we introduce the following L^∞-weighted norm for bounded functions defined in Ω. Let us set

$$\rho(x) := \sum_{j=1}^{k} \rho_i \; \chi_{B_\delta(\xi_j)}(x) + 1,$$

where χ denotes the characteristic function and

$$\rho_j(x) = c\gamma_j \left\{ \left(1 + \frac{1}{\gamma_j}(|w_j| + 1)\right) \left(1 + \frac{1}{\gamma_j}(1 + |w_j|)\right) e^{\frac{w_j^2}{2\gamma_j}} - 1 \right\} \varepsilon_j^{-2} e^{w_j}, \quad (3.8)$$

where $\gamma_j = \log \varepsilon_j^{-4}$. Observe that if $|x - \xi_j| > \delta$ for any j, then $\rho(x) = 1$, while if $|x - \xi_j| = O(\varepsilon_j)$, then $\rho(x) = \varepsilon_j^{-2} e^{w_j}$. Define

$$\|h\|_* = \sup_{x \in \Omega} \rho(x)^{-1} |h(x)|. \quad (3.9)$$

We claim that

$$\|E\|_* \leq C\lambda. \quad (3.10)$$

Indeed, we observe first that in the region $|x - \xi_j| < \delta$, for some fixed j, we have

$$- \Delta \tilde{U} = m_j \varepsilon_j^{-2} e^{w_j} + \sum_{i=1}^{k} O(\varepsilon_i^2). \quad (3.11)$$

On the other hand, thanks to the definition of the parameters μ_j and ε_j given by (2.7) and (2.8), and thanks to the expansion of U given by (2.15) in the region we are considering, we get

$$f(\tilde{U}) = m_j (1 + 2\lambda m_j^2 w_j + O(\lambda)) e^{w_j} \varepsilon_j^{-2} e^{\lambda m_j^2 w_j^2} (1 + \Theta_\lambda(x)),$$

where

$$1 + \Theta_\lambda(x) \leq (1 + C\lambda |w_j|)$$

for some constant $C > 0$. Hence, the error of approximation E near ξ_j is given by

$$E(x) = m_j \varepsilon_j^{-2} e^{w_j} \left\{ 1 - (1 + 2m_j^2 \lambda w_j + O(\lambda)) e^{m_j^2 \lambda w_j^2} \left(1 + O\left(\lambda w_j\right)\right) \right\} + \sum_i O(\varepsilon_i^2).$$

In particular, observe that for $|x - \xi_j| = O(\varepsilon)$ we have that $E(x) \sim \lambda \varepsilon_j^{-2} e^{w_j}$. On the other hand, for $|x - \xi_j| > \delta$ for all j we clearly have that $|E(x)| \leq C\lambda$, as a direct consequence of (2.11). These facts give (3.10); for more details on these estimates, cf. [17, Sect. 2].

Next we will establish an invertibility theory for the linear operator L. Given $h \in L^\infty(\Omega)$, we consider the linear problem of finding a function ϕ such that for certain scalars c_{ij}, $i = 0, 1, 2$, $j = 1, \ldots, k$, it satisfies

$$L(\phi) = h + \sum_{i=0}^{2} \sum_{j=1}^{k} c_{ij} Z_{ij} \zeta_j, \quad \text{in } \Omega, \quad (3.12)$$

$$\phi = 0, \quad \text{on } \partial\Omega, \quad (3.13)$$

$$\int_\Omega Z_{ij} \zeta_j \phi = 0, \quad \text{for all } i, j. \quad (3.14)$$

Consider the norm

$$\|\phi\|_\infty = \sup_{x \in \Omega} |\phi(x)|.$$

Proposition 3.1 *Let $\delta > 0$ be fixed. Under the same assumptions of Proposition 2.1, for any $h \in L^\infty(\Omega)$, there is a unique solution $\phi =: T_\lambda(h)$ to problem (3.12)–(3.14) for all λ positive and sufficiently small. Moreover*

$$\|\phi\|_\infty \le C\|h\|_*. \tag{3.15}$$

We omit the proof of Proposition 3.1, since it can be obtained with minor changes from the proof of Proposition 2.1 in [17].

We proceed then to prove Proposition 2.1.

Proof of Proposition 2.1 To solve problem (3.2)–(3.4) in $L^\infty(\Omega)$, we recast it in fixed point form

$$\tilde\phi = T_\lambda(-E - N(\tilde\phi)) := A(\tilde\phi), \tag{3.16}$$

where T_λ is the operator in Proposition 3.1. We will show that A has a fixed point in the set $\mathcal{B} = \{\tilde\phi \,/\, \|\tilde\phi\|_\infty \le M\lambda\}$ for a sufficiently large and fixed M and all small λ.

We recall the definition of $N(\phi)$ in (3.7),

$$N(\phi) = f(\tilde U + \phi) - f(\tilde U)\phi - f'(\tilde U) + \left[f'(\tilde U) - \sum_j \varepsilon^{-2} e^{w_j}\right]\phi^2.$$

Since $f'(\tilde U) = \lambda(2\lambda\tilde U^2 + 1)e^{\lambda\tilde U^2}$, we have that $f'(\tilde U) = O(\lambda)$ away from the points ξ_j. Repeating the corresponding computations to obtain the estimate of $\|E\|_*$ in (3.10), we readily get that

$$\left\| f'(\tilde U) - \sum_j \varepsilon^{-2} e^{w_j} \right\|_* \le C\lambda. \tag{3.17}$$

Furthermore, for any $\phi \in \mathcal{B}$ we write $f(\tilde U + \phi) - f(\tilde U) - f'(\tilde U)\phi = \frac{f''(\tilde U + t\phi)}{2}\phi^2$ for some $0 \le t \le 1$. Using again the computations to obtain the estimate of $\|E\|_*$ in (3.10), we readily get that $\|f''(\tilde U)\|_* \le C$. Using a Taylor expansion we find that

$$\|N(\tilde\phi)\|_* \le C\|\tilde\phi\|_\infty^2 + C\lambda\|\tilde\phi\|_\infty. \tag{3.18}$$

This estimate, Proposition 2.3, and estimate (3.10) imply that $A(\mathcal{B}) \subset \mathcal{B}$, for a sufficiently large and fixed M and all small λ. Indeed, if $\phi \in \mathcal{B}$, then

$$\|A(\tilde\phi)\|_\infty = \|T_\lambda(-E - N(\tilde\phi))\|_\infty \le c\left(\|E\|_* + \|N(\tilde\phi)\|_*\right) \le C\lambda(1+\lambda)$$

for some proper positive constants c and C, independent of λ. Thus, choosing M in the definition of $\mathcal{B}$ large, but independent of λ, we get that if $\tilde\phi \in \mathcal{B}$ then $A(\tilde\phi) \in \mathcal{B}$. Besides, a direct computation gives that, for any $\tilde\phi_1, \tilde\phi_2$ in $\mathcal{B}$, we have

$$\begin{aligned} N(\tilde\phi_1) - N(\tilde\phi_2) &= \left[f(\tilde U + \tilde\phi_1) - f(\tilde U + \tilde\phi_2) - f'(\tilde U)(\tilde\phi_1 - \tilde\phi_2)\right] \\ &\quad + \left[f'(\tilde U) - \sum_j \varepsilon^{-2} e^{w_j}\right](\tilde\phi_1 - \tilde\phi_2). \end{aligned}$$

Estimate (3.17) readily gives

$$\left\| \left[f'(\tilde U) - \sum_j \varepsilon^{-2} e^{w_j}\right](\tilde\phi_1 - \tilde\phi_2) \right\|_* \le C\lambda\|\tilde\phi_1 - \tilde\phi_2\|_\infty$$

for some constant $C > 0$. On the other hand, for some $0 \le t \le 1$, we have that

$$\left[f(\tilde U + \tilde\phi_1) - f(\tilde U + \tilde\phi_2) - f'(\tilde U)(\tilde\phi_1 - \tilde\phi_2)\right] = f''(\tilde U + t(\tilde\phi_1 - \tilde\phi_2))[\tilde\phi_1 - \tilde\phi_2]^2.$$

Since $\|f''(\tilde U + t(\tilde\phi_1 - \tilde\phi_2))\|_* \le C$, for some positive constant C, we get readily

$$\left\|\left[f(\tilde U + \tilde\phi_1) - f(\tilde U + \tilde\phi_2) - f'(\tilde U)(\tilde\phi_1 - \tilde\phi_2)\right]\right\|_* \le C\|\tilde\phi_1 - \tilde\phi_2\|_\infty^2.$$

Thus it is directly checked that the operator A has a small Lipschitz constant in $\mathcal{B}$ for all small λ. Indeed, for any $\tilde\phi_1$ and $\tilde\phi_2$ in $\mathcal{B}$,

$$\|A(\tilde\phi_1) - A(\tilde\phi_2)\|_\infty \le c\|N(\tilde\phi_1) - N(\tilde\phi_2)\|_*$$

$$\le c\lambda\left(1 + \|\tilde\phi_1 - \tilde\phi_2\|_\infty\right)\|\tilde\phi_1 - \tilde\phi_2\|_\infty \le C\lambda\|\tilde\phi_1 - \tilde\phi_2\|_\infty$$

for constants c, C positive and independent of λ.

Thus, the contraction mapping principle leads to a solution $\tilde\phi$ of (3.2)–(3.4) with $\|\tilde\phi\|_\infty \le C\lambda$. This gives (2.22). The C^1-dependence of the function $\tilde\phi$ on the points ξ, the parameter m and the corresponding estimates (2.23) can be obtained as in [17, Sect. 4].

We will next show the validity of (2.24). Let us fix $h \in \{0, 1, 2\}$ and $l \in \{1, \dots, k\}$. Multiply Eq. 3.3 against $Z_{hl}\zeta_l$ and integrate all over Ω. Since $\int_\Omega Z_{ij}\zeta_j Z_{hl}\zeta_l = C\varepsilon_l^{-2}\delta_{jl}(1+o(1))$, with C a positive constant, and $o(1) \to 0$ as $\lambda \to 0$, we have that

$$c_{hl}\varepsilon_l^{-2}C(1 + o(1)) = \int_\Omega L(\tilde\phi)Z_{hl}\zeta_k + \int_\Omega EZ_{hl}\zeta_k + \int_\Omega N(\tilde\phi)Z_{hl}\zeta_k. \tag{3.19}$$

We will estimate each of the terms in the right hand side of the above formula. Using the definition of the cut off functions ζ_l in (2.18), the definition of the $\|\cdot\|_*$-norm in (3.9) and the estimate (3.10), we first observe that

$$\left|\int_\Omega EZ_{hl}\zeta_k\right| \le C\|E\|_* \int_{B(\xi_l,\sigma\varepsilon_l)} \varepsilon_l^{-2}e^{w_l}|Z_{hl}|\zeta_l \le C\lambda\varepsilon_l^{-2},$$

for some $\sigma > 0$ and $C > 0$, independent of λ. On the other hand, we have

$$\left|\int_\Omega N(\tilde\phi)Z_{hl}\zeta_k\right| \le C\|N(\tilde\phi)\|_* \int_{B(\xi_l,\sigma\varepsilon_l)} \varepsilon_l^{-2}e^{w_l}|Z_{hl}|\zeta_l \le C\lambda^2\varepsilon_l^{-2},$$

since (3.18) holds and $\|\tilde\phi\|_\infty \le C\lambda$. Since $\Delta Z_{hl} + \varepsilon_l^{-2}e^{w_l}Z_{hl} = 0$, a double integration by parts in the remaining term in (3.19) yields to

$$\int_\Omega L(\tilde\phi)Z_{hl}\zeta_k = \left[2\int_\Omega \nabla Z_{hl}\nabla\zeta_l + \int_\Omega Z_{hl}\Delta\zeta_l\right](1 + o(1)).$$

A direct computation shows that $|\int_\Omega \nabla Z_{hl}\nabla\zeta_l|, |\int_\Omega Z_{hl}\Delta\zeta_l| \le C\lambda\varepsilon_l^{-2}$. Thus, combining all the above estimates in (3.19), yields the validity of the first one of the estimates in (2.24). Next we will estimate $D_\xi c_{ij}$. To fix the ideas, we will consider $D_{\xi_{11}}c_{ij}$. Define $Z = \partial_{\xi_{11}}\phi$, $\tilde Z = Z + \sum_{a,b}\alpha_{a,b}Z_{ab}\zeta_b$, where

$$\alpha_{ab} = \frac{\int_\Omega \phi\partial_{\xi_{11}}(Z_{ab}\zeta_b)}{\int_\Omega (Z_{ab}\zeta_b)^2}.$$

As shown in [17, Sect. 4], we have that $\int_\Omega \tilde{Z} Z_{ij}\zeta_j = 0$ for all i, j, $\tilde{Z} = 0$ on $\partial\Omega$ and

$$L(\tilde{Z}) = -\partial_{\xi_{11}}\left(\sum_{j=1}^{k} \varepsilon_j^{-2} e^{w_j}\right)\tilde{\phi} + \sum_{i,j} c_{ij}\partial_{\xi_{11}}(Z_{ij}\zeta_j) + \sum \alpha_{ab} L(Z_{ab}\zeta_b) + \sum_{i,j} \partial_{\xi_{11}}(c_{ij}) Z_{ij}\zeta_j.$$

Arguing as before, we get that

$$|\partial_{\xi_{11}} c_{ij}| \le C\|\tilde{Z}\|_\infty \le C\|\partial_{\xi_{11}}\tilde{\phi}\|_\infty,$$

which, together with (2.23), readily gives $|\partial_{\xi_{11}} c_{ij}| \le C\lambda$. The remaining estimates in (2.24) on $|D_m c_{ij}|$ can be obtained in a similar way, so we omit the details. This concludes the proof of the Proposition. □

4 Proof of Proposition 2.2

We start with the following.

Proposition 4.1 *Let $\delta > 0$ be a fixed small number, let U be the function defined in* (2.9) *and ϕ the function given by Proposition* 2.1. *Assume the parameters μ_j and ε_j are defined as in* (2.7) *and* (2.8)*, and assume that λ is a free parameter. Then, as $\lambda \to 0$, the following expansions hold true*

$$\int_\Omega |\nabla(U+\phi)|^2\,dx = 4\pi k\left\{1 + \lambda f_k(\xi, m) + \lambda^2 \Theta_\lambda(\xi, m)\right\}, \tag{4.1}$$

where

$$f_k(\xi, m) = 2\left[\sum_{j=1}^{k} m_j^2\left(b + 2\log 2m_j^2 + H(\xi_j, \xi_j)\right) - \sum_{i\neq j} m_i m_j G(\xi_i, \xi_j)\right] \tag{4.2}$$

with $b = 2\log 8 - 2 > 0$. Furthermore, as $\lambda \to 0$,

$$\int_\Omega e^{(U+\phi)^2}\,dx = |\Omega| + 16\pi\sum_{j=1}^{k} m_j^2 + \lambda\sum_{j=1}^{k} m_j^2 \int_\Omega G^2(x, \xi_j)\,dx + \lambda^2\Theta_\lambda(\xi, m). \tag{4.3}$$

In (4.1) *and* (4.3)*, $\Theta_\lambda(\xi, m)$ denotes a smooth function, uniformly bounded together with its derivatives, as $\lambda \to 0$, for (ξ, m) satisfying constraint* (2.6).

Proof We write

$$\int_\Omega |\nabla(U+\phi)|^2\,dx = \int_\Omega |\nabla U|^2 + 2\int_\Omega \nabla U\nabla\phi\,dx + \int_\Omega |\nabla\phi|^2.$$

In [17] Lemma 6.1, formula (6.11), we showed that

$$\int_\Omega |\nabla U|^2 = 4\pi k\left\{1 + \lambda f_k(\xi, m) + \sum_{j=1}^{k}\varepsilon_j^2 \log\frac{1}{\varepsilon_j}\theta_\lambda(\xi, m)\right\}, \tag{4.4}$$

where f_k is the function defined in (4.2) and θ_λ is a smooth function, uniformly bounded, as $\lambda \to 0$, in the region for (ξ, m) satisfying (2.6). In [17, Lemma 6.1], we also showed that

$$D_\xi \left(\int_\Omega |\nabla U|^2 \right) = 4\pi k \lambda D_\xi \left(f_k(\xi, m) \right) + \sum_{j=1}^k \varepsilon_j^2 \log \frac{1}{\varepsilon_j} \; \theta_\lambda(\xi, m), \tag{4.5}$$

$$D_m \left(\int_\Omega |\nabla U|^2 \right) = 4\pi k \lambda D_m \left(f_k(\xi, m) \right) + \sum_{j=1}^k \varepsilon_j^2 \log \frac{1}{\varepsilon_j} \; \theta_\lambda(\xi, m), \tag{4.6}$$

where again θ_λ denotes a smooth function, uniformly bounded, as $\lambda \to 0$, in the region for (ξ, m) satisfying (2.6). Observe that, given the definition of the parameters ε_j in (2.8), in (4.4), (4.5) and (4.6) we get that $\varepsilon_j^2 \log \frac{1}{\varepsilon_j} = o(\lambda^3)$. In order to get (4.1), we are left with the estimate of $2\int_\Omega \nabla U \nabla \phi \, dx + \int_\Omega |\nabla \phi|^2$. Observe first that $2\int_\Omega \nabla U \nabla \phi \, dx + \int_\Omega |\nabla \phi|^2 \le 2\left[\int_\Omega \nabla U \nabla \phi \, dx + \int_\Omega |\nabla \phi|^2 \right]$.

If we multiply Eq. (2.19) against ϕ and then integrate on Ω, we get through an integration by parts

$$\int_\Omega \nabla U \nabla \phi \, dx + \int_\Omega |\nabla \phi|^2 = \lambda \int_\Omega (U + \phi) e^{(U+\phi)^2} \phi, \tag{4.7}$$

since we recall that $\phi = 0$ on $\partial\Omega$ and the orthogonality conditions (2.21) hold. By (2.22) we have $\|\phi\|_\infty \le C\lambda^{\frac{3}{2}}$, for some fixed constant C independent of λ, and hence

$$\left| \int_\Omega (U + \phi) e^{(U+\phi)^2} \phi \right| \le C\lambda^{\frac{3}{2}} \left| \int_\Omega (U + \phi) e^{(U+\phi)^2} \right|.$$

Now, a Taylor expansion on the last integral also gives,

$$\left| \int_\Omega (U + \phi) e^{(U+\phi)^2} \phi \right| \le C\lambda^{|frac32} \left| \int_\Omega U e^{U^2} \right| + \lambda^3 C. \tag{4.8}$$

We now write, for some $\delta > 0$ small,

$$\int_\Omega U e^{U^2} \, dx = \int_{\Omega \setminus \cup_j B(\xi_j, \delta\sqrt{\varepsilon_j})} U e^{U^2} + \sum_{j=1}^k \int_{B(\xi_j, \delta\sqrt{\varepsilon_j})} U e^{U^2}.$$

Since in the region $\Omega \setminus \cup_j B(\xi_j, \delta\sqrt{\varepsilon_j})$ the function $U(x)$ satisfies $U(x) = \sqrt{\lambda}(\sum_j m_j G(x, \xi_j) + o(1))$, with $o(1) \to 0$ as $\lambda \to 0$, a Taylor expansion gives

$$\begin{aligned} \int_{\Omega \setminus \cup_j B(\xi_j, \delta\sqrt{\varepsilon_j})} U e^{U^2} &= \sqrt{\lambda} \sum_{j=1}^k m_j \int_\Omega G(x, \xi_j) \left[1 + \lambda \left(\sum_{j=1}^k m_j G(x, \xi_j) \right)^2 \right] (1 + o(1)) \\ &= \sqrt{\lambda} \sum_{j=1}^k m_j \int_\Omega G(x, \xi_j) \, dx (1 + o(1)). \end{aligned}$$

Let us now fix an index j. We write

$$\int_{B(\xi_j,\delta\sqrt{\varepsilon_j})} Ue^{U^2} = \int_{B(\xi_j,\delta\varepsilon_j|\log\varepsilon_j|)} Ue^{U^2} + \int_{B(\xi_j,\delta\sqrt{\varepsilon_j})\setminus B(\xi_j,\delta\varepsilon_j|\log\varepsilon_j|)} Ue^{U^2} = I_1 + I_2.$$

We estimate first I_1. Performing the change of variables $y = \frac{x-\xi_j}{\varepsilon_j}$ and using the notations $V_j(y) = 2\gamma_j U(\varepsilon_j y + \xi_j) - 2\gamma_j^2$ and $\gamma_j = \log\varepsilon_j^{-4}$, we have

$$\begin{aligned} I_1 &= \frac{\varepsilon_j^2}{\gamma_j} e^{\gamma_j^2} \int_{B(0,\delta|\log\varepsilon_j|)} (V_j(y) + 2\gamma_j^2) e^{V_j(y) + \frac{V_j^2(y)}{4\gamma_j^2}}\, dy \\ &= 2\varepsilon_j^2 e^{\gamma_j^2} \sqrt{\lambda} m_j \int_{B(0,\delta|\log\varepsilon_j|)} \omega_j(y) e^{\omega_j(y)} (1 + o(1)) \\ &= 2m_j^3 \sqrt{\lambda} \int_{\mathbb{R}^2} \omega(y) \frac{8}{(1+|y|^2)^2}\, dy (1 + o(1)). \end{aligned}$$

On the other hand

$$|I_2| \le C\sqrt{\lambda} \int_{\delta|\log\varepsilon_j|}^{\delta\varepsilon_j^{-\frac{1}{2}}} \frac{1}{r^4} e^{\frac{\log^2 r}{\gamma_j^2}}\, r\, dr$$

$(t = \log r)$

$$= C\sqrt{\lambda} \int_{\log(\frac{\delta}{4})+\log\gamma_j}^{\log\delta + \frac{\gamma_j}{8}} e^{-2t + \frac{4t^2}{\gamma_j^2}}\, dt \le C\sqrt{\lambda} \int_{\log(\frac{\delta}{4})+\log\gamma_j}^{\log\delta + \frac{\gamma_j}{8}} e^{-t}\, dt \le C\lambda^{\frac{3}{2}}. \tag{4.9}$$

We thus conclude that

$$\int_\Omega Ue^{U^2}\, dx = \sqrt{\lambda}\Theta_\lambda(\xi, m)(1 + o(1)) \tag{4.10}$$

where $\Theta_\lambda(\xi, m)$ is bounded together with its derivatives in the considered region, as $\lambda \to 0$. This fact, together with (4.7) and (4.8), gives

$$\int_\Omega \nabla U \nabla\phi\, dx + \int_\Omega |\nabla\phi|^2 = \lambda^3 \Theta_\lambda(\xi, m).$$

We conclude that estimate (4.1) holds in the C^0-sense. Next we show the C^1-closeness in estimate (4.1), namely we prove that

$$D_\xi \int_\Omega |\nabla(U + \phi)|^2\, dx = 4\pi k\lambda D_\xi f_k(\xi, m) + \lambda^2 \Theta_\lambda(\xi, m) \tag{4.11}$$

and

$$D_m \int_\Omega |\nabla(U + \phi)|^2\, dx = 4\pi k\lambda D_m f_k(\xi, m) + \lambda^2 \Theta_\lambda(\xi, m), \tag{4.12}$$

for a function $\Theta_\lambda(\xi, m)$ which is bounded in the considered region, as $\lambda \to 0$. The proof of (4.12) follows the lines of the proof of (4.11). For this reason, we will focus only on the proof of (4.11).

An integration by parts gives that

$$D_\xi \int_\Omega |\nabla(U+\phi)|^2\, dx = -2\int_\Omega (U+\phi)\Delta U_\xi - 2\int_\Omega (U+\phi)\Delta\phi_\xi.$$

By definition, far from the points ξ_j, say in $|x-\xi_j| > \delta$ for any j, the function U writes as $U(x) = \sqrt{\lambda}\left(\sum_{j=1}^k m_j G(x,\xi_j) + o(1)\right)$, while in each set $|x-\xi_j| \le \delta$, $j = 1, \ldots, k$, one has that $U(x) \ge a\sqrt{\lambda}$ for some positive number a. We refer to (2.9), (2.11), (2.15). On the other hand, Proposition 2.1 gives that $\|\phi\|_\infty \le C\lambda^{\frac{3}{2}}$ for some positive constant C. Thus we get that

$$D_\xi \int_\Omega |\nabla(U+\phi)|^2\, dx = -2\left(\int_\Omega U\Delta U_\xi)(1+\lambda O(1)\right) - 2\left(\int_\Omega U\Delta\phi_\xi)(1+\lambda O(1)\right),$$

where $O(1)$ is a continuous function of ξ and m, which is uniformly bounded, in the considered region, as $\lambda \to 0$.

Observe now that, using a further integration by parts, we have

$$\left|\int_\Omega U\Delta\phi_\xi\right| = \left|\int_\Omega \Delta U\phi_\xi\right| \le C\|\phi_\xi\|_\infty \int_\Omega |\Delta U|.$$

Now taking into account (3.11) and (2.11), it is straightforward to show that $\int_\Omega |\Delta U| \le C\sqrt{\lambda}$ for some positive constant C. We thus conclude that

$$\left|\int_\Omega U\Delta\phi_\xi\right| \le C\lambda^2,$$

and thus, recollecting the previous estimates, we conclude that

$$\begin{aligned} D_\xi \int_\Omega |\nabla(U+\phi)|^2\, dx &= 2\int_\Omega \nabla U\nabla D_\xi U + \lambda^2\Theta_\lambda(\xi,m) \\ &= D_\xi \int_\Omega |\nabla U|^2 + \lambda^2\Theta_\lambda(\xi,m) \end{aligned} \tag{4.13}$$

where again $\Theta(\xi, m)$ is uniformly bounded, for ξ and m varying in the corresponding region, as $\lambda \to 0$. From (4.13) and (4.5) we deduce the validity of (4.11) and henceforth the validity of (4.1).

Let us now evaluate $\int_\Omega e^{(U+\phi)^2}$. A Taylor expansion, together with estimate (4.10), gives that

$$\int_\Omega e^{(U+\phi)^2} = \int_\Omega e^{U^2} + \lambda^2\Theta_\lambda(\xi,m). \tag{4.14}$$

Now we write

$$\int_\Omega e^{U^2}\,dx = \left[\sum_{j=1}^k \int_{B(\xi_j,\delta\sqrt{\varepsilon_j})} e^{U^2}\,dx\right] + A_\lambda. \tag{4.15}$$

Since in the region $\Omega \setminus \cup_j B(\xi_j, \delta\sqrt{\varepsilon_j})$ we have

$$U(x) = \sqrt{\lambda}\left(\sum_j m_j G(x,\xi_j) + o(1)\right),$$

with $o(1) \to 0$ as $\lambda \to 0$, a Taylor expansion gives

$$\begin{aligned}A_\lambda &= \int_{\Omega\setminus\cup_j B(\xi_j,\delta\sqrt{\varepsilon_j})} \left[1+\lambda\sum_{j=1}^k m_j^2 G^2(x,\xi_j)\right](1+o(1))\\ &= |\Omega| + \lambda\sum_{j=1}^k m_j^2\int_\Omega G^2(x,\xi_j)\,dx + \lambda^2\Theta_\lambda(\xi,m).\end{aligned}$$

Now, we write

$$\begin{aligned}\int_{B(\xi_j,\delta\sqrt{\varepsilon_j})} e^{U^2}\,dx &= \int_{B(\xi_j,\delta\varepsilon_j|\log\varepsilon_j|)} e^{U^2}\,dx + \int_{B(\xi_j,\delta\sqrt{\varepsilon_j})\setminus B(\xi_j,\delta\varepsilon_j|\log\varepsilon_j|)} e^{U^2}\,dx\\ &= I_1 + I_2.\end{aligned}$$

We will show next that

$$I_1 = 16\pi m_j^2 + \lambda\Theta_\lambda(\xi,m), \quad I_2 = \lambda\Theta_\lambda(\xi,m) \tag{4.16}$$

for some function Θ_λ, uniformly bounded together with its derivatives, as $\lambda \to 0$. Indeed, performing the change of variables $y = \frac{x-\xi_j}{\varepsilon_j}$ and using the notations $V_j(y) = 2\gamma_j U(\varepsilon_j y + \xi_j) - 2\gamma_j^2$ and $\gamma_j = \log\varepsilon_j^{-4}$, we have

$$\begin{aligned}I_1 &= \varepsilon_j^2 e^{\gamma_j^2}\int_{B(0,\delta|\log\varepsilon_j|)} e^{V_j(y)+\frac{V_j^2(y)}{4\gamma_j^2}}\,dy\\ &= 2m_j^2\int_{\mathbb{R}^2}\frac{8}{(1+|y|^2)^2}\,dy + \lambda\Theta_\lambda(\xi,m) = 16\pi m_j^2\left[1+\lambda\Theta_\lambda(\xi,m)\right].\end{aligned}$$

On the other hand

$$|I_2| \le C\int_{\delta|\log\varepsilon_j|}^{\delta\varepsilon_j^{-\frac{1}{2}}} \frac{1}{r^4} e^{\frac{\log^2 r}{\gamma_j^2}}\, r\,dr = O(\lambda),$$

by the calculations in (4.9).

We can thus conclude that estimate (4.3) holds true in C^0-sense. The C^1-closeness in (4.3) can be obtained proceeding as in the proof of the C^1-closeness in (4.1). We leave the details to the reader. □

We are now ready to give the proof of Proposition 2.2.

Proof of Proposition 2.2. Let R' be the subset of R defined as follows

$$R' = \{(\xi, m) \in R \, : \, f_k(\xi, m) \neq 0\}.$$

Replacing expansion (4.1) into (2.27), we see that (2.27) gives

$$\lambda f_k(\xi, m) + \lambda^2 \Theta_\lambda(\xi, \lambda) = \alpha. \tag{4.17}$$

In R', the relation (4.17) defines λ as a function of α, ξ and m, which is smooth in (ξ, m) in the region R'. Furthermore, as $\alpha \to 0$, we have

$$\lambda = \frac{\alpha}{f_k(\xi, m)} + \frac{\alpha^2}{f_k^3(\xi, m)} \Theta_\alpha(\xi, m),$$

where Θ_α is a smooth function of (ξ, m), which is uniformly bounded together with its derivatives, in the region R', as $\alpha \to 0$.

Assume now (2.27). We shall prove (2.28). Let us define the functional

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{\lambda}{2} \int_\Omega e^{u^2}$$

for functions $u \in H_0^1(\Omega)$. Under our assumptions, we have that

$$J'(U + \phi)[\partial(U + \phi)] = 0, \tag{4.18}$$

where with ∂ we either denote the partial derivative with respect to m_j for any $j = 1, \ldots, k$, or the partial derivative with respect to ξ_{ji} for $j = 1, \ldots, k$, $i = 1, 2$.

Indeed, a direct computation gives

$$\begin{aligned} J'(U + \phi)[\partial(U + \phi)] &= \int_\Omega \nabla(U + \phi) \nabla(\partial(U + \phi)) - \lambda \int_\Omega (U + \phi) e^{(U+\phi)^2} \partial(U + \phi) \\ &= \frac{1}{2} \partial \left(\int_\Omega |\nabla(U + \phi)|^2 \right) - \frac{\lambda}{2} \partial \left(\int_\Omega e^{(U+\phi)^2} \right) = 0, \end{aligned}$$

where we used the fact that $\partial \left(\int_\Omega |\nabla(U + \phi)|^2 \right) = 0$ since (2.27) holds, and the assumption that $\partial \left(\int_\Omega e^{(U+\phi)^2} \right) = 0$.

Let us now consider the following change of variables

$$\frac{1}{\sqrt{\lambda}} (U + \phi)(x) = m_l v_l \left(\frac{x - \xi_l}{\varepsilon_l} \right) + \frac{1}{2 m_l \lambda}$$

for some $l = 1, \ldots, k$. A direct consequence of (2.15) is

$$v_l(y) = w_{\mu_l}(y) + \sum_j \left(O(|\varepsilon_l y + \xi_l - \xi_j|) + O(\varepsilon_j^2) \right), \quad \text{for } |y| \leq \frac{\delta}{\varepsilon_l}. \tag{4.19}$$

A straightforward computation gives that

$$J(U + \phi) = I_l(v_l),$$

where

$$I_l(v_l) = \frac{m_l^2}{2} \int_{\Omega_l} |\nabla v_l|^2 \, dx - m_l^2 \int_{\Omega_l} e^{v_l} e^{\lambda m_l^2 v^2},$$

with $\Omega_l = \frac{\Omega - \xi_l}{\varepsilon_l}$. Furthermore,

$$J'(U+\phi)[\partial(U+\phi)] = I_l'(v_l)[\partial v_l]. \tag{4.20}$$

Now, since $(U+\phi)$ solves (2.19) in Ω, we see that $v_l(y)$ solves in Ω_l

$$m_l \varepsilon_l^{-2}\left[\Delta\tilde v_l + e^{v_l}(1+2\lambda m_l^2 v_l) e^{\lambda m_l^2 v_l^2}\right] = \sum_{ij} c_{ij} \zeta\left(\frac{\varepsilon_l y + \xi_l - \xi_j}{\varepsilon_j}\right) \varepsilon_j^{-2} z_{ij}\left(\frac{\varepsilon_l y + \xi_l - \xi_j}{\varepsilon_j}\right).$$

Thus from (4.18) and (4.20) we get

$$\begin{aligned} 0 &= I_l'(\tilde v_l)[\partial v_l] \\ &= \sum_{ij} \left(\int_{\Omega_l} \zeta(\frac{\varepsilon_l y + \xi_l - \xi_j}{\varepsilon_j}) \varepsilon_j^{-2} z_{ij}(\frac{\varepsilon_l y + \xi_l - \xi_j}{\varepsilon_j}) \partial v_l dy \right) c_{ij}. \end{aligned}$$

Now assume that $\partial = \partial_{m_1}$. Fix two indices i and j. To compute the coefficient in front of c_{ij} in the above expression, we choose $l = j$ and obtain

$$\int_{\Omega_l} \zeta\left(\frac{\varepsilon_l y + \xi_l - \xi_j}{\varepsilon_j}\right) \varepsilon_j^{-2} z_{ij}\left(\frac{\varepsilon_l y + \xi_l - \xi_j}{\varepsilon_j}\right) D_{m_1} v_l dy = \frac{\partial \mu_j}{\partial m_1} \int_{\mathbb{R}^2} z_{0j}^2(y)\, dy\, (1 + o(1)).$$

Thus we conclude that, for any $h = 1, \ldots, k$,

$$0 = \sum_j \left(\frac{\partial \mu_j}{\partial m_h} \int_{\mathbb{R}^2} z_{0j}^2(y)\, dy \right) c_{0j}\, (1 + o(1)).$$

If we now assume that $\partial = \partial_{\xi_{ab}}$ for $a = 1, 2$, $b = 1, \ldots, k$, a direct argument shows on the other hand that

$$0 = \sum_j \left(\frac{\partial \mu_j}{\partial \xi_{ab}} \int_{\mathbb{R}^2} z_{0j}^2(y)\, dy \right) c_{0j} + \left(\int_{\mathbb{R}^2} z_{1j}^2(y)\, dy \right) c_{ab}\, (1 + o(1)).$$

We can conclude that $D_{\xi,m} I(\xi, m) = 0$ implies the validity of a system of equations of the form

$$\left[\sum_j \frac{\partial \mu_j}{\partial m_h} c_{0j} \right] (1 + o(1)) = 0, \quad j = 1, \ldots, k, \tag{4.21}$$

$$\left[A \sum_j \frac{\partial \mu_j}{\partial \xi_{ab}} c_{0j} + c_{ab} \right] (1 + o(1)) = 0, \quad a = 1, 2, \quad b = 1, \ldots, k, \tag{4.22}$$

for some fixed constant A, with $o(1)$ small in the sense of the L^∞ norm as $\lambda \to 0$. The conclusion of the Lemma follows if we show that the matrix $(D_m \mu_j)_j$ of dimension $k \times k$ is invertible in the range for the points ξ_j and parameters m_j we are considering. Indeed, this fact implies unique solvability of (4.21). Inserting this in (4.22) we get unique solvability of (4.22).

We will now show that $(D_m \mu_j)$ is invertible. Consider the definition of the μ_j's, in terms of m_j's and points ξ_j given in (2.7). These relations correspond to the gradient $D_m F(m, \xi)$ of the function F defined as follows

$$F(m, \xi) = \frac{1}{2} \sum_{j=1}^{k} m_j^2 \left[\log 2m_j^2 - \log 8\mu_j^2 - 1 - H(\xi_j, \xi_j) \right] + \sum_{i \neq j} G(\xi_i, \xi_j) m_i m_j.$$

It is natural to perform the change of variable $s_j = m_j^2$. With abuse of notation, the above function now reads as follows

$$F(s, \xi) = \frac{1}{2} \sum_{j=1}^{k} s_j \left[\log 2s_j - \log 8\mu_j^2 - 1 - H(\xi_j, \xi_j) \right] + \sum_{i \neq j} G(\xi_i, \xi_j) \sqrt{s_i s_j}.$$

This is a strictly convex function of the parameters s_j, for parameters s_j uniformly bounded and uniformly bounded away from 0 and for points ξ_j in Ω uniformly far away from each other and from the boundary. For this reason, the matrix $\left(\frac{\partial^2 F}{\partial s_i \partial s_j} \right)$ is invertible in the range of parameters and points we are considering. Thus, by the implicit function theorem, relation (2.7) defines a diffeomorphism between μ_j and m_j. This fact gives the invertibility of $\left(\frac{\partial \mu_j}{\partial m_l} \right)$ we were aiming at.

This concludes the proof of Proposition 2.2. □

5 Proof of Theorem 1.1

In this section we assume that Ω is a bounded domain with smooth boundary and we let $k = 1$. Thus in this case the point $\xi \in \Omega$ and the parameter $m \in \mathbb{R}_+$ satisfy the constraints

$$\operatorname{dist}(\xi, \partial\Omega) > \delta, \quad \delta < m < \delta^{-1}. \tag{5.1}$$

Condition (2.27) reduces to

$$\lambda f_1(\xi, m) + \lambda^2 \Theta_\lambda(\xi, m) = \alpha \tag{5.2}$$

where

$$f_1(\xi, m) = 2m^2(b + 2\log 2m^2 + H(\xi, \xi)). \tag{5.3}$$

In (5.2), $\Theta_\lambda(\xi, m)$ is a smooth function, uniformly bounded together with its derivatives, as $\lambda \to 0$, for (ξ, m) satisfying constraint (2.6). For simplicity we introduce the change of variables $s = m^2$. Observe that, for any $\xi \in \Omega$, the function $s \to b + 2\log 2s + H(\xi, \xi)$ is strictly monotone in $(0, \infty)$, it has a unique zero $\bar{s} = \bar{s}(\xi) = \frac{1}{2} e^{\frac{-b + H(\xi,\xi)}{2}}$, it is negative for $0 < s < \bar{s}$ and strictly positive for $s > \bar{s}$. We thus conclude that in the region

$$(\xi, s) \in R' = \{(\xi, s) \, : \, \xi \in \Omega, \; s > \bar{s}(\xi)\},$$

where $f_1(\xi, s)$ is strictly positive, relation (5.2) defines λ as a smooth function of α and (ξ, s). More precisely,

$$\lambda = \frac{\alpha}{f_1(\xi, s)} + \alpha^2 f_1(\xi, s)^3 \Theta_\alpha(\xi, s)$$

where Θ_α is a smooth function of (ξ, s), uniformly bounded together with its derivatives, as $\alpha \to 0$.

Replacing (5.3) and (5.2) in (4.3) for $k = 1$, we get that

$$g(\xi, s) := \int_\Omega e^{(U+\phi)^2}\, dx = |\Omega| + 16\pi s + \frac{\alpha \int_\Omega G^2(x,\xi)\, dx}{2(b + 2\log 2s + H(\xi,\xi))} + \left(\frac{\alpha}{2s(b + 2\log 2s + H(\xi,\xi))}\right)^2 \Theta_\alpha(\xi, s), \tag{5.4}$$

where Θ_α is a smooth function which is uniformly bounded together with its derivatives in the region R'.

As a consequence of (5.2) and of Proposition 2.2, we have that the function $u = U + \phi$, where U is defined in (2.9) and ϕ is given by Proposition 2.1, is a solution to our problem (2.3) if we establish the existence of a critical point (ξ, s) for the function g given by (5.4).

We claim that, given $\delta > 0$, for all $\alpha > 0$ small enough, the function

$$\varphi_\alpha(\xi, s) := |\Omega| + 16\pi s + \frac{\alpha \int_\Omega G^2(x,\xi)\, dx}{2(b + 2\log 2s + H(\xi,\xi))} \tag{5.5}$$

has a critical point in the region $\mathrm{dist}(\xi, \partial\Omega) > \delta$ and $\bar{s}(\xi) + \delta\sqrt{\alpha} < s < \bar{s}(\xi) + \delta^{-1}\sqrt{\alpha}$, with value $\left(|\Omega| + 8\pi e^{-\frac{b+H(\xi,\xi)}{2}}\right)(1 + O(\sqrt{\alpha}))$, as $\alpha \to 0$, in the region considered. Furthermore, we show that this critical point situation is stable under proper small C^1 perturbation of φ_α: to be more precise, any function ψ such that $\|\psi - \varphi_\alpha\|_\infty + \|\nabla\psi - \nabla\varphi_\alpha\|_\infty \le C\alpha$ in the region considered, also has a critical point. This fact will conclude the proof of Theorem 1.1 since, in the above region, the function $g(\xi, s)$ given by (5.4) is a proper small C^1 perturbation of φ_α as $\alpha \to 0$ in the sense described above.

Thus what is left of this section is devoted to prove the existence of a C^1-stable critical point situation for φ_α (see Proposition 4.1).

Consider the new change of variable $t = s - \bar{s}$, so that $t > 0$ in the region we are considering. In this new variable, the function φ_α takes the form

$$\varphi_\alpha(\xi, t) = |\Omega| + 16\pi(\bar{s} + t) + \frac{\alpha \int_\Omega G^2(x,\xi)\, dx}{2\log(1 + \frac{t}{\bar{s}})}. \tag{5.6}$$

Let $H_0 > 0$ be the minimum value of the function H inside Ω. Let $\delta > 0$ be such that the set $\Omega_\delta := \{\xi \in \Omega : H(\xi,\xi) < H_0 + 10\delta\}$ is strictly contained in Ω.

Let

$$D = \Omega_\delta \times \mathbb{R}_+$$

and let $B = \bar{\Omega}_{\frac{\delta}{2}} \times [\delta\sqrt{\alpha}, \delta^{-1}\sqrt{\alpha}]$ and $B_0 = \bar{\Omega}_{\frac{\delta}{2}} \times \{\delta\sqrt{\alpha}, \delta^{-1}\sqrt{\alpha}\}$. Consider the set Γ of all functions $\phi \in C(B, D)$ such that there exists $\psi \in C([0,1] \times B; D)$ with

$$\psi(0, \cdot) = Id_B, \quad \psi(1, \cdot) = \phi, \quad \psi(t, \cdot)_{|B_0} = Id_{B_0}.$$

Define

$$\sup_{\phi\in\Gamma} \inf_{(\xi,t)\in B} \varphi_\alpha(\xi, t) = c.$$

Observe that the value c is strictly positive, as a direct consequence of the definition of φ_α in the region considered. We will show that, given $\delta > 0$ small, for all $\alpha > 0$ small enough,

(a)

$$\inf_{(\xi,t)\in B_0} \varphi_\alpha(\xi,t) > c$$

(b) there exists $K > 0$ independent of $\delta > 0$ such that

$$c < K$$

(c) if $\delta > 0$ is small enough, then for any $(\xi, t) \in \partial D$ such that $\varphi_\alpha(\xi, t) = c$ there exists a tangent vector τ to ∂D such that

$$\nabla \varphi_\alpha(\xi, t) \cdot \tau \neq 0.$$

Under the conditions (a), (b), (c), a critical point $(\bar{\xi}, \bar{t})$ for φ_α with $\varphi_\alpha(\bar{\xi}, \bar{t}) = c$ exists, as a standard deformation argument involving the negative gradient flow of φ_α shows. This structure is clearly preserved for small $C^1(\bar{D})$-perturbations of φ_α and hence a stable critical point situation for the functional g, which is C^1-close to φ_α, is established.

Proof of (a) We start with the following observations. For any given ξ, since $\alpha > 0$ the function $t \to \varphi_\alpha(\xi, t)$ is strictly convex and positive in $(0, \infty)$, thus it has a unique minimum which we denote by $\bar{t} = \bar{t}(\xi)$. This minimum is non degenerate, namely $\frac{\partial^2}{\partial^2 t}\varphi_\alpha(\xi, \bar{t}) \neq 0$, as a direct computation shows. From $\partial_t \varphi_\alpha(\xi, \bar{t}) = 0$ we get that

$$\log^2\left(1 + \frac{\bar{t}}{\bar{s}}\right)\left(1 + \frac{\bar{t}}{\bar{s}}\right) = \frac{\alpha \int_\Omega G^2(x, \xi)\, dx}{32\pi \bar{s}}. \tag{5.7}$$

The function $\tilde{h}(r) := \log^2(1+r)\,(1+r)$ is strictly monotone in $(0, \infty)$ and a Taylor expansion gives

$$\tilde{h}(r) = r^2 + o(r^2) \quad \text{as} \quad r \to 0.$$

In particular, as $\alpha \to 0$, from (5.7) we obtain

$$\bar{t} = \frac{\sqrt{\alpha \bar{s}}\sqrt{\int_\Omega G^2(x, \xi)\, dx}}{\sqrt{32\pi}} + o(\sqrt{\alpha \bar{s}}). \tag{5.8}$$

Choosing, if necessary, a smaller δ, we may assume that, for any $\xi \in \Omega_\delta$, we have that

$$\frac{\delta}{2}\sqrt{\alpha} \le \bar{t}(\xi) \le \frac{\delta^{-1}}{2}\sqrt{\alpha}.$$

In particular, this fact gives the validity of (a). □

Proof of (b) Inserting (5.8) in (5.5), we get

$$\varphi_\alpha(\xi, \bar{t}) = |\Omega| + 16\pi \bar{s}(\xi) + \tilde{\varphi}_\alpha(\xi), \tag{5.9}$$

where

$$\tilde{\varphi}_\alpha(\xi) = \sqrt{\alpha \bar{s}}\left[4\sqrt{\frac{\pi}{2}\int_\Omega G^2(x, \xi)} + \frac{\sqrt{\alpha}\int_\Omega G^2(x, \xi)}{2\sqrt{\bar{s}}\log\left(1 + \frac{\sqrt{\alpha \int_\Omega G^2(x,\xi)}}{\sqrt{32\pi \bar{s}}}\right)}\right] + o(\sqrt{\alpha \bar{s}}). \tag{5.10}$$

From (5.9) we thus get the following estimates for $\varphi_\alpha(\xi, \bar t(\xi))$

$$\varphi_\alpha(\xi, \bar t(\xi)) := |\Omega| + 16\pi e^{-\frac{b+H(\xi,\xi)}{2}} + O(\sqrt{\alpha}). \tag{5.11}$$

From this we conclude that there exists a constant M independent of δ such that

$$c \le \sup_{\xi \in \Omega} \left(|\Omega| + 16\pi e^{-\frac{b+H(\xi,\xi)}{2}} + M \right).$$

Thus (b) is proven. □

Proof of (c) We argue by contradiction: assume that for a sequence $\delta \to 0$, points (ξ, t) with $\varphi_\alpha(\xi, t) = c$ and $(\xi, t) \in \partial D$ we have that $\nabla \varphi_\alpha(\xi, t) \cdot \tau = 0$ for any tangent vector τ to ∂D. Since c is uniformly bounded above and below away from zero, we have that t is uniformly bounded above and away from 0. Thus we assume that $\partial_t \varphi_\alpha(\xi, \bar t) = 0$ (otherwise we would get a contradiction right away). But if this is the case, our function $\varphi_\alpha(\xi, \bar t)$ is given by (5.9), or equivalently (5.11). Now, since $\xi \to e^{-\frac{b+H(\xi,\xi)}{2}}$ has a maximum in the region Ω_δ, we have the existence of a tangent vector τ to $\partial\Omega_\delta$ such that $\nabla\varphi_\alpha(\xi, \bar t) \cdot \tau \neq 0$. This concludes our argument, and also the proof of Theorem 1.1. □

6 Proof of Theorem 1.2

In this section we assume that Ω is not simply connected and that $k = 2$. In this case $\xi = (\xi_1, \xi_2) \in \Omega^2$ and $m = (m_1, m_2) \in \mathbb{R}^2_+$ satisfy

$$\operatorname{dist}(\xi_i, \partial\Omega) > \delta, \quad |\xi_1 - \xi_2| > \delta, \quad \delta < m_i < \delta^{-1} \tag{6.1}$$

for some given $\delta > 0$ small, but independent of α. We denote

$$\Omega^2_\delta = \{\xi = (\xi_1, \xi_2) \in \Omega^2 : \operatorname{dist}(\xi_i, \partial\Omega) > \delta, |\xi_1 - \xi_2| > \delta\}. \tag{6.2}$$

Condition (2.27) reduces to

$$\lambda f_2(\xi, m) + \lambda^2 \Theta(\xi, m) = \alpha \tag{6.3}$$

where

$$f_2(\xi, m) = 2\left[\sum_{j=1}^{2} m_j^2 (b + 2\log 2m_j^2 + H(\xi_j, \xi_j)) - m_1 m_2 G(\xi_1, \xi_2)\right]. \tag{6.4}$$

From now on, we will use the change of variables $m_j^2 = s_j$, $j = 1, 2$.

We will next describe the set

$$Z = \{(\xi, s) \in \Omega^2 \times \mathbb{R}^2_+ : \xi_1 \neq \xi_2,\ s_1 s_2 \neq 0,\ f_2(\xi, s) = 0\}. \tag{6.5}$$

For any fixed $\xi = (\xi_1, \xi_2) \in \Omega^2$, with $\xi_1 \neq \xi_2$, there exists a unique intersection in $\mathbb{R}^2_+$ between the set

$$Z_\xi = \{s \in \mathbb{R}^2_+ : s_1 s_2 \neq 0,\ f_2(\xi, s) = 0\} \tag{6.6}$$

and the lines $s_2 = t s_1$, for any $0 < t < \infty$. This gives a parametrization of the curve described by Z_ξ. Namely

$$Z_\xi = \{s = (s_1, s_2) \in \mathbb{R}^2_+ : s_1 = \frac{1}{2} e^{\phi(\xi,t)},\ s_2 = \frac{t}{2} e^{\phi(\xi,t)},\ 0 < t < \infty\}$$

where

$$\phi(\xi,t)=\frac{\sqrt{t}G(\xi_1,\xi_2)-(1+t)b-H(\xi_1,\xi_1)-tH(\xi_2,\xi_2)-2t\log t}{2(1+t)}. \tag{6.7}$$

In particular, the set Z_ξ is a smooth curve in $\mathbb{R}^2_+$, which is diffeomorphic to $\{s_1^2+s_2^2=1\}\cap\mathbb{R}^2_+$, and it divides $\mathbb{R}^2_+$ into two connected components, one bounded and the other unbounded. In the unbounded component f_2 is positive. We call this region Z_ξ^+ and we denote by Z_δ^+ the region given by

$$Z_\delta^+=\{(\xi,s)\in\Omega_\delta^2\times\mathbb{R}^2_+ : s_1s_2\neq 0,\ f_2(\xi,s)>0\}.$$

In Z_δ^+ relation (6.3) defines λ as a smooth function of the free parameter α and of the points (ξ,s). More precisely,

$$\lambda=\frac{\alpha}{f_2(\xi,s)}+\frac{\alpha^2}{f_2(\xi,s)^3}\Theta_\alpha(\xi,s)$$

where Θ_α is a smooth function of (ξ,s), uniformly bounded together with its derivatives, as $\alpha\to 0$.

Replacing (6.4) and (6.3) in (4.3) for $k=2$, we get that

$$g(\xi,s):=\int_\Omega e^{(U+\phi)^2}\,dx=|\Omega|+16\pi(s_1+s_2)+\frac{\alpha h(\xi,s)}{f_2(\xi,s)} + \left(\frac{\alpha}{f_2(\xi,s)}\right)^2\Theta_\alpha(\xi,s), \tag{6.8}$$

where

$$h(\xi,s)=s_1\int_\Omega G^2(x,\xi_1)\,dx+s_2\int_\Omega G^2(x,\xi_2)\,dx, \tag{6.9}$$

f_2 is defined in (6.4) and Θ_α is a smooth function which is uniformly bounded together with its derivatives in Z_δ^+, as $\alpha\to 0$.

As a consequence of Proposition 2.2, we have that the function $u=U+\phi$, where U is defined in (2.9) and ϕ is given by Proposition 2.1, is a solution to our problem (2.3) if we establish the existence of a critical point (ξ,s) for the function g given by (6.8). We will devote the rest of this Section to prove the above assertion.

We first introduce a further change of variables. For any fixed $\xi\in\Omega_\delta^2$, we consider the connected region Z_ξ^+ in $\mathbb{R}^2_+$ where $f_2(\xi,s)>0$. We perform the following change of variable: for any $\xi\in\Omega_\delta^2$ and $t\in(0,\infty)$, we define $s=(1+r)\bar{s}$, where

$$\bar{s}=(\bar{s}_1(\xi,t),\bar{s}_2(\xi,t))=\left(\frac{e^{\phi(\xi,t)}}{2},\frac{t\,e^{\phi(\xi,t)}}{2}\right) \tag{6.10}$$

with ϕ given by (6.7), and $r\in\mathbb{R}$. In particular, $\bar{s}\in Z_\xi$ and if $r>0$ then $(1+r)\bar{s}\in Z_\xi^+$.

In the new variables (ξ,t,r) the function g takes the form

$$\begin{aligned} g(\xi,t,r)=&\ |\Omega|+16\pi(1+r)(\bar{s}_1+\bar{s}_2)\\ &+\frac{\alpha h(\xi,(1+r)\bar{s})}{f_2(\xi,(1+r)\bar{s})}+\left(\frac{\alpha}{f_2(\xi,(1+r)\bar{s})}\right)^2\Theta_\alpha(\xi,(1+r)\bar{s}). \end{aligned}\tag{6.11}$$

Finding critical points for $g(\xi, s)$, for $\xi \in \Omega^2_\delta$ and $s \in Z^+_\xi$, is equivalent to finding critical points for $g(\xi, t, r)$ for $\xi \in \Omega^2_\delta$ and $(t, r) \in \mathbb{R}^2_+$.

We claim that, given $\delta > 0$, for all $\alpha > 0$ small enough, the function

$$\varphi_\alpha(\xi, t, r) := |\Omega| + 16\pi(1 + r)(\bar{s}_1 + \bar{s}_2) + \frac{\alpha h(\xi, (1 + r)\bar{s})}{f_2(\xi, (1 + r)\bar{s})} \tag{6.12}$$

has a critical point in the region dist $(\xi_i, \partial\Omega) > \delta$, $|\xi_1 - \xi_2| > \delta$, $\delta < t < \delta^{-1}$ and $\delta\sqrt{\alpha} < r < \delta^{-1}\sqrt{\alpha}$, with value $(|\Omega| + O(1))\,(1 + O(\sqrt{\alpha}))$, as $\alpha \to 0$, in the region considered. Furthermore, we show that this critical point situation is stable under proper small C^1 perturbation of φ_α: to be more precise, any function ψ such that $\|\psi - \varphi_\alpha\|_\infty + \|\nabla\psi - \nabla\varphi_\alpha\|_\infty \le C\alpha$ in the region considered, also has a critical point. Since g is properly C^1-close to φ_α (see Proposition 4.1) this fact will conclude the proof of Theorem 1.2.

Let us fix a small number $\delta > 0$ to be chosen later. We define $\mathcal{D}$ to be

$$\mathcal{D} = \hat{\Omega}^2_\delta \times \mathbb{R}^2_+, \quad \text{where} \quad \hat{\Omega}^2_\delta = \{y \in \Omega^2 / \text{dist}\,(y, \partial\Omega^2) > \delta\}. \tag{6.13}$$

Denote by Ω_1 a bounded nonempty component of $\mathbb{R}^2 \setminus \bar{\Omega}$ and assume that $0 \in \Omega_1$. Consider a closed, smooth Jordan curve γ contained in Ω which encloses Ω_1. We let S be the image of γ and $B = S \times S \times [\delta, \delta^{-1}] \times [\delta\sqrt{\alpha}, \delta^{-1}\sqrt{\alpha}]$. Thus B is a closed and connected subset of $\mathcal{D}$. Define $B_0 = S \times S \times [\delta, \delta^{-1}] \times \{\delta\sqrt{\alpha}, \delta^{-1}\sqrt{\alpha}\}$.$B_0$ is a closed subset of B.

Let Γ be the class of all maps $\Phi \in C(B, \mathcal{D})$ with the property that there exists a function $\Psi \in C([0, 1] \times B, \mathcal{D})$ such that

$$\Psi(0, \cdot) = \mathrm{Id}_B, \quad \Psi(1, \cdot) = \Phi, \quad \Psi(t, \cdot)_{|B_0} = \mathrm{Id}_{|B_0}. \tag{6.14}$$

Then we define

$$\mathcal{C} = \sup_{\Phi \in \Gamma} \inf_{z \in B} \varphi_\alpha(\Phi(z)). \tag{6.15}$$

We will show that, given $\delta > 0$ small, for all $\alpha > 0$ small enough,

(a)

$$\inf_{B_0} \varphi_\alpha(\xi, t, r) > \mathcal{C}$$

(b) there exists $K > 0$ independent of $\delta > 0$ such that

$$\mathcal{C} < K$$

(c) if $\delta > 0$ is small enough, then for any $(\xi, t, r) \in \partial\mathcal{D}$ such that $\varphi_\alpha(\xi, t, r) = \mathcal{C}$ there exists a tangent vector τ to $\partial\mathcal{D}$ such that

$$\nabla\varphi_\alpha(\xi, t, r) \cdot \tau \neq 0.$$

As in Sect. 5, the conditions (a), (b) and (c) yield a critical point $(\bar{\xi}, \bar{t}, \bar{r})$ for φ_α with $\varphi_\alpha(\bar{\xi}, \bar{t}, \bar{r}) = \mathcal{C}$, by a standard deformation argument involving the negative gradient flow of φ_α. This structure is preserved for small $C^1(\bar{D})$-perturbations of φ_α and hence we have a stable critical point situation for the functional g, which is C^1-close to φ_α.

Proof of (a) We claim that, given $\delta > 0$ small, for any $\xi \in \Omega^2$, with dist$(\xi_i, \partial\Omega) > \delta$, $|\xi_1 - \xi_2| > \delta$, and $t \in \mathbb{R}^+$ the function $r \to \varphi_\alpha(\xi, t, r)$ has a non degenerate minimum in $[\delta\sqrt{\alpha}, \delta^{-1}\sqrt{\alpha}]$. Indeed, observe that in this region, the function φ_α takes the form

$$\varphi_\alpha(\xi, t, r) = \tilde{\varphi}_\alpha(\xi, t, r) + \alpha\Theta_\alpha(\xi, t, r) \tag{6.16}$$

where

$$\tilde{\varphi}_\alpha(\xi, t, r) = |\Omega| + 16\pi(1+r)(\bar{s}_1 + \bar{s}_2) + \frac{\alpha h(\xi, \bar{s})}{[\partial_{s_1} f_2(\xi, \bar{s})\bar{s}_1 + \partial_{s_2} f_2(\xi, \bar{s})\bar{s}_2]r}$$

and $\Theta_\alpha(\xi, t, r)$ is a smooth function, uniformly bounded together with its derivatives, as $\alpha \to 0$. A direct computation gives

$$\partial_{s_1} f_2(\xi, \bar{s}) = 2\left(b + 2\log 2\bar{s}_1 + 2 + H(\xi_1, \xi_1) - \frac{\sqrt{\bar{s}_2}}{2\sqrt{\bar{s}_1}} G(\xi_1, \xi_2)\right),$$

$$\partial_{s_2} f_2(\xi, \bar{s}) = 2\left(b + 2\log 2\bar{s}_2 + 2 + H(\xi_2, \xi_2) - \frac{\sqrt{\bar{s}_1}}{2\sqrt{\bar{s}_2}} G(\xi_1, \xi_2)\right)$$

and, since $f_2(\xi, \bar{s}) = 0$, we get that

$$\partial_{s_1} f_2(\xi, \bar{s})\bar{s}_1 + \partial_{s_2} f_2(\xi, \bar{s})\bar{s}_2 = 4(\bar{s}_1 + \bar{s}_2).$$

Observe that, in the region we are considering, we have that $m < \bar{s}_1 + \bar{s}_2 < m^{-1}$, for a certain positive constant m depending on δ, but independent of α. Inserting the above computation in $\tilde{\varphi}_\alpha$, we get

$$\tilde{\varphi}_\alpha(\xi, t, r) = |\Omega| + 16\pi(1+r)(\bar{s}_1 + \bar{s}_2) + \frac{\alpha h(\xi, \bar{s})}{4(\bar{s}_1 + \bar{s}_2)r}.$$

Since $\alpha > 0$ and $h > 0$, the function $r \to \tilde{\varphi}_\alpha(\xi, t, r)$ has a critical point $\bar{r}$ given by

$$\bar{r} = \frac{\sqrt{h(\xi, \bar{s})}}{8\sqrt{\pi}(\bar{s}_1 + \bar{s}_2)}\sqrt{\alpha},$$

which is a non degenerate minimum, since

$$\partial_r^2 \tilde{\varphi}_\alpha(\xi, t, \bar{r}) = \frac{\alpha h(\xi, \bar{s})}{8(\bar{s}_1 + \bar{s}_2)\bar{r}^3} > 0.$$

Choosing, if necessary, a smaller δ, we may assume that, for any $\xi \in \hat{\Omega}_\delta^2$ and $\delta \leq t \leq \delta^{-1}$, we have that

$$\frac{\delta}{2}\sqrt{\alpha} \leq \bar{r}(\xi, t) \leq \frac{\delta^{-1}}{2}\sqrt{\alpha}.$$

In particular, this fact gives the validity of (a). □

Proof of (b) Inserting the value of $\bar{r}$ in $\tilde{\varphi}_\alpha$, see (6.16), we get

$$\tilde{\varphi}_\alpha(\xi, t, \bar{r}) = |\Omega| + 16\pi(\bar{s}_1 + \bar{s}_2) + \sqrt{\alpha}\, 2\sqrt{\pi}\, \sqrt{h(\xi, t)} + O(\alpha),$$

and thus from (6.16) and (6.10)

$$\begin{aligned} \varphi_\alpha(\xi, t, \bar{r}) &= |\Omega| + 16\pi(\bar{s}_1 + \bar{s}_2) + O(\sqrt{\alpha}) \\ &= |\Omega| + 8\pi(1+t)e^{\phi(\xi,t)} + O(\sqrt{\alpha}). \end{aligned} \tag{6.17}$$

To prove (b), we need to show the existence of $K > 0$ independent of small δ such that if $\Phi \in \Gamma$, then there exists a point $\bar{z} \in B$ for which

$$\varphi_\alpha(\Phi(\bar{z})) \leq K. \tag{6.18}$$

We write

$$z = (z_1, z_2, z_3, z_4), \quad \Phi(z) = (\Phi_1(z), \Phi_2(z), \Phi_3(z), \Phi_4(z)),$$

with

$$(z_1, z_2),\ (\Phi_1(z), \Phi_2(z)) \in \hat{\Omega}^2_\delta, \quad z_3,\ z_4,\ \Phi_3(z), \quad \Phi_4(z) \in \mathbb{R}_+.$$

Since by definition (6.14) the map Ψ keeps B_0 fixed, we have that for any $(z_1, z_2) \in S \times S$ and $z_3 \in [\delta, \delta^{-1}]$, there exists $\hat{z}_4 \in [\delta\sqrt{\alpha}, \delta^{-1}\sqrt{\alpha}]$ so that for this z

$$\varphi_\alpha(\Phi(z)) \le |\Omega| + 8\pi(1 + \Phi_3(z))e^{\phi(\Phi_1(z), \Phi_2(z), \Phi_3(z))} + O(\sqrt{\alpha}).$$

We now claim that for any $z_3 \in \mathbb{R}^2_+$ there exists a $\hat{z} \in S \times S$ such that $\Phi_1(\hat{z}, z_3, \hat{z}_4)$ and $\Phi_2(\hat{z}, z_3, \hat{z}_4)$ have antipodal directions, more precisely

$$\frac{\Phi_1(\hat{z}, z_3, \hat{z}_4)}{|\Phi_1(\hat{z}, z_3, \hat{z}_4)|} = R_\pi \frac{\Phi_2(\hat{z}, z_3, \hat{z}_4)}{|\Phi_2(\hat{z}, z_3, \hat{z}_4)|}, \tag{6.19}$$

where R_π denotes a rotation in the plane of an angle π. This implies the existence of a number $M > 0$, which depends only on Ω, such that $G\left(\Phi_1(\hat{z}, z_3, \hat{z}_4), \Phi_2(\hat{z}, z_3, \hat{z}_4)\right) \le M$. This yields by (6.7)

$$\varphi_\alpha(\Phi(\hat{z}, z_3, \hat{z}_4)) \le |\Omega| + 8\pi e^{-\frac{b}{2}}(1 + t)e^{-\frac{t\log t}{1+t}} \le K$$

for some explicit number K, which depends on M, but is independent of δ. This gives estimate (6.18).

We prove (6.19) by a degree argument. For fixed z_3, consider an orientation-preserving homeomorphism $h : S^1 \to S^1$ and the map $f : S^1 \times S^1 \to S^1 \times S^1$ defined as $f(\zeta) = (f_1(\zeta), f_2(\zeta))$ with

$$f_1(\zeta_1, \zeta_2) = \frac{\Phi_1(h(\zeta_1), h(\zeta_2), z_3, \hat{z}_4)}{|\Phi_1(h(\zeta_1), h(\zeta_2), z_3, \hat{z}_4)|}, \quad f_2(\zeta_1, \zeta_2) = \frac{\Phi_2(h(\zeta_1), h(\zeta_2), z_3, \hat{z}_4)}{|\Phi_2(h(\zeta_1), h(\zeta_2), z_3, \hat{z}_4)|}.$$

If we show that f is onto, we get in particular the validity of (6.19).

By (6.14) there exists a map $\Psi \in \Gamma$ such that $\Psi(1, \cdot) = \Phi$. Let $\Psi_i(t, \cdot)$ denote the components of the map Ψ and set $\tilde{\Psi}_i(t, \xi_1, \xi_2) = \Psi_i(t, \xi_2, \xi_2, z_3, \hat{z}_4)$. We then have $\tilde{\Psi}_i \in C([0,1] \times S^1 \times S^1, \Omega^2_\delta)$, $\tilde{\Psi}_i(0, \cdot) = \mathrm{Id}_{S^1 \times S^1}$ and $\tilde{\Psi}_i(1, \cdot) = \Phi_i$, $i = 1, 2$. We now define a homotopy $F : [0,1] \times S^1 \times S^1 \to S^1 \times S^1$ by

$$F_1(t, \zeta) = \frac{\tilde{\Psi}_1(t, h(\zeta_1), h(\zeta_2))}{|\tilde{\Psi}_1(t, h(\zeta_1), h(\zeta_2))|} \quad \text{and} \quad F_2(t, \zeta) = \frac{\tilde{\Psi}_2(t, h(\zeta_1), h(\zeta_2))}{|\tilde{\Psi}_2(t, h(\zeta_1), h(\zeta_2))|}.$$

Note that $F(1, \zeta) = f(\zeta)$ and

$$F(0, \zeta) = (h(\zeta_1), h(\zeta_2)).$$

This function defines a homeomorphism of $S^1 \times S^1$, which we regard as embedded in $\mathbb{R}^3$, parametrized as follows:

$$\zeta : (\theta_1, \theta_2) \in [0, 2\pi)^2 \mapsto (\rho_1 \cos\theta_1, \rho_1 \sin\theta_1, 0) + (0, \rho_2 \cos\theta_2, \rho_2 \sin\theta_2),$$

for $0 < \rho_2 < \rho_1$. The map f defined above can be read in the introduced variables as

$$f(\zeta) = (\rho_1 f_1(\zeta), 0) + (0, \rho_2 f_2(\zeta)),$$

and it can be extended to a continuous map $\tilde{f} : T \to T$, where T is the solid torus described by

$$(\theta_1, \theta_2, \rho) \in [0, 2\pi)^2 \times [0, \rho_2] \mapsto (\rho_1 \cos\theta_1, \rho_1 \sin\theta_1, 0) + (0, \rho\cos\theta_2, \rho\sin\theta_2),$$

with

$$\tilde{f}(\zeta, \rho) = (\rho_1 f_1(\zeta), 0) + (0, \rho f_2(\zeta)).$$

Note that $\tilde{f}$ is homotopic to a homeomorphism of T, along a deformation which maps $\partial T = S^1 \times S^1$ into itself. This implies that $\deg(\tilde{f}, T, P) \neq 0$ for all P lying in the interior of T. This implies that f is onto: indeed, taking $(\theta_1^*, \theta_2^*) \in [0, 2\pi)^2$ and $\rho^* \in (0, \rho_2)$, there exist $\zeta^{**} \in S^1 \times S^1$ and $\rho^{**} \in (0, \rho_2)$ such that

$$(\rho_1 f_1(\zeta^{**}), 0) + (0, \rho^{**} f_2(\zeta^{**})) = (\rho_1 \cos\theta_1^*, \rho_1 \sin\theta_1^*, 0) + (0, \rho^* \cos\theta_2^*, \rho^* \sin\theta_2^*),$$

which implies that $f_1(\zeta^{**}) = (\cos\theta_1^*, \sin\theta_1^*)$, $f_2(\zeta^{**}) = (\cos\theta_2^*, \sin\theta_2^*)$ and $\rho^* = \rho^{**}$. Thus f is onto, and this concludes the proof of (6.19). □

Proof of (c) Consider $(\xi, t, r) \in \partial\mathcal{D}$ with $\varphi_\alpha(\xi, t, r) = \mathcal{C}$. Since $\mathcal{C}$ is uniformly bounded above and below away from zero, we have that r is uniformly bounded above and away from 0. Thus we may assume that $\partial_r \varphi_\alpha(\xi, t, \bar{r}) = 0$ (otherwise we would get the result right away). But if this is the case, our function $\varphi_\alpha(\xi, t, \bar{r})$ takes the form (6.17). Let us set

$$\psi(\xi, t) = |\Omega| + 8\pi(1 + t)e^{\phi(\xi,t)}. \tag{6.20}$$

We argue by contradiction. Let us assume there exist a sequence $\delta = \delta_n \to 0$ and points $(\xi_n, t_n, \bar{r}(\xi_n, t_n)) \in \partial\mathcal{D}$ such that $\psi(\xi_n, t_n) = \mathcal{C}$ and $\nabla\psi(\xi_n, t_n) \cdot \tau = 0$ for all tangent vector τ to $\partial\mathcal{D}$.

Passing to a subsequence, if necessary, we have that $(\xi_n, t_n) \to (\bar{\xi}, \bar{t}) \in \bar{\Omega}^2 \times \mathbb{R}_+$, with $(\bar{\xi}, \bar{t}) \in \partial\left(\bar{\Omega}^2 \times \bar{\mathbb{R}}_+\right)$ and $\psi(\bar{\xi}, \bar{t}) = \mathcal{C}$. We will show the existence of a vector $\tau \neq 0$, tangent to the boundary of $\bar{\Omega}^2 \times \bar{\mathbb{R}}_+$ such that $\nabla\psi(\bar{\xi}, \bar{t}) \cdot \tau \neq 0$, reaching thus a contradiction.

We start with the following observation: given any $\xi = (\xi_1, \xi_2)$, the function $t \in (0, \infty) \mapsto \psi(\xi, t)$ has a unique non degenerate maximum $\hat{t} = \hat{t}(\xi)$. Indeed, observe that

$$\lim_{t\to 0^+} \psi(\xi, t) = 8\pi e^{-\frac{b+H(\xi_1,\xi_1)}{2}}, \quad \lim_{t\to +\infty} \psi(\xi, t) = 8\pi e^{-\frac{b+H(\xi_2,\xi_2)}{2}},$$
$$\partial_t\psi(\xi, t) = 4\pi e^{\frac{\phi(\xi,t)}{(1+t)}} \left[\ell(\xi, t) + H(\xi_1, \xi_1) - H(\xi_2, \xi_2)\right] \tag{6.21}$$

where

$$\ell(\xi, t) = \frac{1 - t}{2\sqrt{t}} G(\xi_1, \xi_2) - 2\log t.$$

The function $t \to \ell(\xi, t)$ is monotone decreasing in $(0, \infty)$,

$$\lim_{t\to 0+} \ell(\xi, t) = +\infty, \quad \lim_{t\to +\infty} \ell(\xi, t) = -\infty$$

and

$$\partial_t \ell(\xi, t) = -\frac{t + 1}{2t\sqrt{t}} G(\xi_1, \xi_2) - \frac{2}{t} < 0, \quad \text{for all} \quad t > 0.$$

Let $\hat{t} = \hat{t}(\xi)$ be the unique non degenerate maximum point of $\psi(\xi, t)$, defined by the relation

$$\frac{1 - \hat{t}}{2\sqrt{\hat{t}}} G(\xi_1, \xi_2) - 2\log\hat{t} = H(\xi_2, \xi_2) - H(\xi_1, \xi_1). \tag{6.22}$$

A key fact is that there exists a positive constant $c > 0$, independent of δ, such that $c < \hat{t} < c^{-1}$. This fact is a direct consequence of the definition of $\hat{t}$ given by (6.22) and of the following property: there exists a positive number C, independent of δ, such that

$$\left| \frac{H(\xi_2, \xi_2) - H(\xi_1, \xi_1)}{2G(\xi_1, \xi_2)} \right| \leq C. \tag{6.23}$$

We postpone the proof of (6.23).

Assume now first that $(\bar{\xi}, \bar{t}) \in \partial \left(\bar{\Omega}^2 \times \bar{\mathbb{R}}_+ \right)$, with $\bar{\xi} \in \Omega^2$. This implies that $t_n \to 0$ or $t_n \to \infty$. In both cases, thanks to the above discussion and to (6.21), we get that $|\nabla_t \psi(\xi_n, t_n)| \geq M$, for some positive fixed M, as $n \to \infty$. We thus get the result.

Let us consider now the case in which $\mathrm{dist}(\xi_2, \partial\Omega) = \delta$. As $\delta \to 0$, this fact implies that $H(\xi_2, \xi_2) \to \infty$, but then we must also have that $|\xi_1 - \xi_2| \to 0$ to keep the value of ψ bounded. By construction we have $\mathrm{dist}(\xi_1, \partial\Omega) \geq \delta$. Two cases arise: if $\nabla_t \psi(\xi, \bar{t}) \neq 0$, then we can choose τ parallel to $\nabla_t \psi(\xi, \bar{t})$. Otherwise, we are in the case in which $\nabla_t \psi(\xi, \bar{t}) = 0$. This implies that $\bar{t} = \hat{t}$. What is left of this proof concerns the analysis of this case.

We recall that the case we are discussing is the following: $\mathrm{dist}(\xi_2, \partial\Omega) = \delta$, $\xi_1 \to \xi_2$, with $\mathrm{dist}(\xi_1, \partial\Omega) \geq \delta$, $\nabla_t \psi(\xi, \bar{t}) = 0$, with $\bar{t} = \hat{t}$. Inserting the value of $\bar{t}$ given by (6.22) into the expression of ψ, we get

$$\psi(\xi, \bar{t}(\xi)) = 8\pi e^{-b}(1 + \bar{t}) e^{\frac{\sqrt{\bar{t}} G(\xi_1, \xi_2) - 2H(\xi_1, \xi_1)}{4}}$$

We assume by contradiction that

$$\nabla_{\xi_2} \psi(\xi, \bar{t}) \cdot \tau = 0 \tag{6.24}$$

for any vector τ tangent to $\partial\Omega_\delta$ at ξ_2, where $\Omega_\delta = \{x \in \Omega \,:\, \mathrm{dist}(x, \partial\Omega) = \delta\}$. Taking into account (6.22) we have that

$$\nabla_\xi \psi(\xi, \bar{t}) = \psi(\xi, \bar{t}) \frac{\sqrt{\bar{t}} \partial_\xi G(\xi_1, \xi_2) - 2\partial_\xi H(\xi_1, \xi_1)}{4}.$$

We denote $\rho = |\xi_1 - \xi_2| \to 0$. Only two cases may occur, namely $\frac{\delta}{\rho} \to \infty$ or $\delta\rho \leq c_0$, for some constant c_0. We shall show that in both cases relation (6.24) is impossible.

Let us assume first that $\frac{\delta}{\rho} \to \infty$ and define

$$x_j = \frac{\xi_2 - \xi_j}{\rho} \quad \text{for} \quad j = 1, 2$$

and $\tilde{x}_j = \lim_{\delta \to 0} x_j$. Let us define

$$\tilde{\varphi}(x_1, x_2) = \varphi_2(\xi_1 + \rho x_1, \xi_2 + \rho x_2, s).$$

Since away from the boundary the function $H(x, x)$ is bounded, we get

$$\lim_{\delta \to 0} \nabla_{\xi_2} \varphi_2(\xi + \rho x) = -C \partial_{x_{lj}} \log \frac{1}{|\tilde{x}_1 - \tilde{x}_2|} \neq 0$$

contradicting (6.24). Thus, we necessarily have that $\frac{\delta}{\rho}$ is bounded. The interesting case is when $\xi_1 \in \partial\Omega_\delta$. If not, we can reproduce the argument above to reach a contradiction. The case $\delta = o(\rho)$ cannot happen because of (6.23). Let us assume then that $\frac{\delta}{\rho} \to c$. We consider the scaled domain $\tilde{\Omega} = \delta^{-1}\Omega$, whose associated Green's function $\tilde{G}$ and regular part $\tilde{H}$ are given by (6.26). In this scaled domain the number t defined by relation (6.22) remains away from 0 and 1, since the quantity

$$\frac{\tilde{H}(\xi_2,\xi_2)-\tilde{H}(\xi_1,\xi_1)}{2\tilde{G}(\xi_1,\xi_2)}$$

remains bounded. Furthermore, after a rotation and translation, we may assume that $\tilde{\xi}_2 := \frac{\xi_2}{\delta} \to (0,1)$, $\tilde{\xi}_1 := \frac{\xi_1}{\delta} \to (a,1)$, for some $a>0$, as $\delta \to 0$ and the domain $\tilde{\Omega}$ becomes the half-plane $x_2>0$. Under this condition, we see that the derivative of φ_2 in the direction $e=(0,1)$ is not 0, reaching again a contradiction with (6.24), and the proof is concluded.

In the rest of this proof, we will show the validity of (6.23). We assume by contradiction that

$$0 \le \frac{H(\xi_2,\xi_2)-H(\xi_1,\xi_1)}{2G(\xi_1,\xi_2)} \to +\infty. \tag{6.25}$$

We have $\delta = \operatorname{dist}(\xi_2,\partial\Omega)$. Let us denote $d_1 = \operatorname{dist}(\xi_1,\partial\Omega)$, and $d=|\xi_1-\xi_2|$. Condition (6.25) implies that d_1 and $d \to 0$, with $\delta = o(d_1)$ and $\delta = o(d)$. Let us consider the expanded domain $\tilde{\Omega} = \delta^{-1}\Omega$ and observe that for this domain its associated Green's function and regular part are given by

$$\tilde{H}(x_1,x_2) = 4\log\delta + H(\delta x_1,\delta x_2),\ \tilde{G}(x_1,x_2) = G(\delta x_1,\delta x_2). \tag{6.26}$$

Furthermore, $\operatorname{dist}(\xi_2,\partial\Omega)=\delta$ implies $\operatorname{dist}\left(\frac{\xi_2}{\delta},\partial\tilde{\Omega}\right)=1$. After a rotation and translation, we assume that $\frac{\xi_2}{\delta}=(0,1)$ and as $\delta\to 0$ the domain $\tilde{\Omega}$ becomes the half-plane $x_2>0$. We denote respectively by G_0 and H_0 Green's function and its regular part, associated to the half plane $x_2>0$. The expressions for G_0 and H_0 are explicit:

$$H_0(x,y) = 4\log\frac{1}{|x-\bar{y}|}, \quad \bar{y}=(y_1,-y_2)$$

where $y=(y_1,y_2)$, and

$$G_0(x,y) = 4\log\frac{1}{|x-y|} - 4\log\frac{1}{|x-\bar{y}|}.$$

We thus compute the expression in (6.25)

$$0 \le \frac{H(\xi_2,\xi_2)-H(\xi_1,\xi_1)}{2G(\xi_1,\xi_2)} = \frac{\tilde{H}(\frac{\xi_2}{\delta},\frac{\xi_2}{\delta})-\tilde{H}(\frac{\xi_1}{\delta},\frac{\xi_1}{\delta})}{2\tilde{G}(\frac{\xi_1}{\delta},\frac{\xi_2}{\delta})}$$

$$= \frac{H_0((0,1),(0,1)) - 4\log\frac{\delta}{|\xi_1-\bar{\xi}_1|} + o(1)}{4\log\frac{\delta}{|\xi_1-\delta(0,1)|}} = O(1),$$

but this is in contradiction with (6.25). □

7 Proof of Theorem 1.3

In this section we assume that $\Omega = B(0,b)\setminus\overline{B}(0,a)$ with $0<a<b$ and we fix an integer k. We look for a solution to Problem (2.4)–(2.5), or equivalently for a critical point of the functional $I(u)$ constrained to S_μ, for $\mu > 4\pi k$ (see (1.2)), in the class of functions that are invariant under rotations of angle $\frac{2\pi}{k}$.

To be more precise, we build a solution of the form

$$u(x) = U(x) + \phi(x)$$

where $U(x)$ is given by (2.9), with the following choice for the points ξ_j and the parameters m_j

$$\xi_j = re^{\frac{2\pi(j-1)}{k}i}, \quad m_j = m \qquad \text{for all} \quad j = 1, \dots, k$$

with $r > 0, m > 0$. Recall that the function ϕ is a solution to the nonlinear Problem (2.19)–(2.21).

Since by construction the function U is invariant under rotations of $\frac{2\pi}{k}$ and it is even in the x_2-direction, a direct consequence of the uniqueness of ϕ guaranteed by Proposition 2.1 is that the function ϕ shares the same symmetries. These facts, together with the uniqueness of ϕ, imply that the constants c_{ij} that appear in Eq. 2.19 satisfy the following conditions

$$c_{ij} = c_{i1} \qquad \text{for all} \quad j = 2, \dots, k, \qquad \text{for all} \quad i = 0, 1, 2$$

and

$$c_{21} = 0.$$

We thus conclude that if we work in the class of functions that are invariant under rotations of $\frac{2\pi}{k}$ and even in the x_2-direction, the function $U + \phi$ is a solution to (2.4)–(2.5) if there exists a proper choice of λ, $r > 0$ and $m > 0$ such that

$$\int_\Omega |\nabla(U + \phi)|^2 \, dx = 4\pi k(1 + \alpha), \quad \text{and} \quad c_{01} = c_{11} = 0.$$

From Proposition 4.1 we deduce that the first of the above conditions reduces to

$$\lambda f_k(r, m) + \lambda^2 \Theta_\lambda(r, m) = \alpha \tag{7.1}$$

where

$$f_k(r, m) = 2\,k\,m^2 \left[b + 2\log(2m^2) + H(\xi_1, \xi_1) - \sum_{i>1} G(\xi_1, \xi_i) \right]$$

and $\Theta_\lambda(r, m)$ is a smooth function of the variables r, m, which is uniformly bounded together with its derivatives, as $\lambda \to 0$, in the region $a + \delta < r < b - \delta$, $\delta < m < \delta^{-1}$ for any given $\delta > 0$.

For simplicity we introduce the change of variables $s = m^2$, and we define $H(r) = H(\xi_1, \xi_1)$ and $G(r) = \sum_{i>1} G(\xi_1, \xi_i)$. The function f_k gets rewritten as

$$f_k(r, s) = 2\,k\,s \left[b + 2\log 2s + H(r) - G(r) \right].$$

For any $r \in (a, b)$, let $\bar{s} = \bar{s}(r) = \frac{1}{2} e^{-\frac{b+H(r)-G(r)}{2}}$. In the region

$$(r, s) \in R' = \{(r, s) : r \in (a, b), s > \bar{s}(r)\},$$

$f_k(r, s)$ is strictly positive and relation (7.1) defines λ as a smooth function of α and (r, s). More precisely,

$$\lambda = \frac{\alpha}{f_k(r, s)} + \frac{\alpha^2}{f_k(r, s)^3} \Theta_\alpha(r, s)$$

where Θ_α is a smooth function of (r, s) uniformly bounded, together with its derivatives, as $\alpha \to 0$.

Inserting this information into (4.3) we get that

$$g(r,s) := \int_\Omega e^{(U+\phi)^2}\,dx = |\Omega| + 16\pi k s + \frac{\alpha \int_\Omega G^2(x,\xi_1)\,dx}{2(b + 2\log 2s + H(r) - G(r))} + \left(\frac{\alpha}{2s(b + 2\log 2s + H(r) - G(r))}\right)^2 \Theta_\alpha(r,s), \tag{7.2}$$

where Θ_α is a smooth function which is uniformly bounded together with its derivatives in the region R'.

As a consequence of Proposition 2.2 and the symmetries of the function $U + \phi$, we have that $u = U + \phi$ is a solution to Problem (2.4)–(2.5) if we establish the existence of a critical point (r, s) for the function g given by (7.2).

Arguing as in Sect. 5 we will prove that, given $\delta > 0$, for all $\alpha > 0$ small enough, the function

$$\varphi_\alpha(r,s) := |\Omega| + 16\pi k s + \frac{\alpha \int_\Omega G^2(x,\xi_1)\,dx}{2(b + 2\log 2s + H(r) - G(r))} \tag{7.3}$$

has a critical point in the region $a+\delta < r < b-\delta$ and $\bar s(r)+\delta\sqrt{\alpha} < s < \bar s(r)+\delta^{-1}\sqrt{\alpha}$, with value $|\Omega| + 8\pi e^{-\frac{b+H(r)-G(r)}{2}} + O(\sqrt{\alpha})$, as $\alpha \to 0$, in the region considered. Furthermore, we show that this critical point situation is stable under proper small C^1 perturbation of φ_α: to be more precise, any function ψ such that $\|\psi - \varphi_\alpha\|_\infty + \|\nabla\psi - \nabla\varphi_\alpha\|_\infty \le C\alpha$ in the region considered, also has a critical point. This fact will conclude the proof of Theorem 1.3.

Thus what is left of this section is devoted to prove the existence of a C^1-stable critical point situation for φ_α.

Consider the new change of variable $t = s - \bar s$, so that $t > 0$ in the region we are considering. In this new variable, the function φ_α takes the form

$$\varphi_\alpha(r,t) = |\Omega| + 16\pi k(\bar s + t) + \frac{\alpha \int_\Omega G^2(x,\xi_1)\,dx}{2\log(1+\frac{t}{\bar s})}. \tag{7.4}$$

Since the points ξ_j are uniformly separated from each other, the function $G(r)$ is bounded from above in the interval (a, b). Since $\lim_{r\to a} H(r) = \lim_{r\to b} H(r) = \infty$, the function $H(r) - G(r)$ has a minimum r_0 in (a, b). Let $\delta > 0$ and define the set $I_\delta := \{r \in (a,b) : h(r) < H(r_0) - G(r_0) + 10\delta\}$. Let

$$D := I_\delta \times \mathbb{R}_+,$$

$B := \bar I_{\frac{\delta}{2}} \times [\delta\sqrt{\alpha}, \delta^{-1}\sqrt{\alpha}]$ and $B_0 := \bar I_{\frac{\delta}{2}} \times \{\delta\sqrt{\alpha}, \delta^{-1}\sqrt{\alpha}\}$. Consider the set Γ of all function $\phi \in C(B, D)$ such that there exists $\psi \in C([0,1]\times B; D)$ with

$$\psi(0,\cdot) = Id_B, \quad \psi(1,\cdot) = \phi, \quad \psi(t,\cdot)_{|B_0} = Id_{B_0}.$$

Define

$$\sup_{\phi\in\Gamma} \inf_{(r,t)\in B} \varphi_\alpha(r,t) = c.$$

The value c is strictly positive, as a consequence of the definition of φ_α in the region considered. As in Sect. 5 one proves that for given $\delta > 0$ small and for all $\alpha > 0$ small enough,

(a) $\inf_{(r,t)\in B_0} \varphi_\alpha(r,t) > c$
(b) there exists $K > 0$ independent of $\delta > 0$ such that $c < K$
(c) if $\delta > 0$ is small enough, then for any $(r,t) \in \partial D$ such that $\varphi_\alpha(r,t) = c$ there exists a tangent vector τ to ∂D such that $\nabla\varphi_\alpha(r,t) \cdot \tau \neq 0$.

From (a), (b) and (c) one obtains as in Sect. 5 a critical point $(\bar{r}, \bar{t})$ for φ_α with $\varphi_\alpha(\bar{r}, \bar{t}) = c$, by a standard deformation argument involving the negative gradient flow of φ_α. This structure is clearly preserved for small $C^1(\bar{D})$-perturbations of φ_α, and hence a stable critical point situation for the functional g, which is C^1-close to φ_α, is established.

This concludes the proof of Theorem 1.3. □

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