Large energy entire solutions for the Yamabe equation

Manuel del Pino [a], Monica Musso [b,*], Frank Pacard [c,d], Angela Pistoia [e]

[a] *Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático (UMI 2807 CNRS), Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile*
[b] *Departamento de Matemática, Pontificia Universidad Católica de Chile, Avda. Vicuña Mackenna 4860, Macul, Chile*
[c] *Centre de Mathématiques Laurent Schwartz, École Polytechnique, 91128 Palaiseau, France*
[d] *Institut Universitaire de France, France*
[e] *Dipartimento SBAI, Palazzina B, Via Scarpa 16, Roma, Italy*

A R T I C L E   I N F O

A B S T R A C T

We consider the Yamabe equation $\Delta u + \frac{n(n-2)}{4}|u|^{\frac{4}{n-2}}u = 0$ in $\mathbb{R}^n$, $n \geqslant 3$. Let $k \geqslant 1$ and $\xi_j^k = (e^{\frac{2j\pi i}{k}}, 0) \in \mathbb{R}^n = \mathbb{C} \times \mathbb{R}^{n-2}$. For all large k we find a solution of the form $u_k(x) = U(x) - \sum_{j=1}^{k} \mu_k^{-\frac{n-2}{2}} U \times (\mu_k^{-1}(x - \xi_j)) + o(1)$, where $U(x) = (\frac{2}{1+|x|^2})^{\frac{n-2}{2}}$, $\mu_k = \frac{c_n}{k^2}$ for $n \geqslant 4$, $\mu_k = \frac{c}{k^2(\log k)^2}$ for $n = 3$ and $o(1) \to 0$ uniformly as $k \to +\infty$.

© 2011 Elsevier Inc. All rights reserved.

Contents

*   Corresponding author.
    E-mail addresses: delpino@dim.uchile.cl (M. del Pino), mmusso@mat.puc.cl (M. Musso), frank.pacard@math.polytechnique.fr (F. Pacard), pistoia@dmmm.uniroma1.it (A. Pistoia).

1. Introduction and statement of main results

This paper deals with the construction of finite energy solutions to the Yamabe equation in $\mathbb{R}^n$,

$$\Delta u + \gamma |u|^{p-1} u = 0 \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $n \geqslant 3$ and p is the critical Sobolev exponent $p = \frac{n+2}{n-2}$, and the constant $\gamma > 0$ is chosen (for normalization purposes) as

$$\gamma = \frac{n(n-2)}{4}.$$

By finite energy solutions of problem (1.1) we mean critical points of the functional

$$J(u) = \frac{1}{2} \int\limits_{\mathbb{R}^n} |\nabla u|^2 - \frac{\gamma}{p+1} \int\limits_{\mathbb{R}^n} |u|^{p+1}, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^n),$$

where $\mathcal{D}^{1,2}(\mathbb{R}^N) = \{u \in L^{p+1}(\mathbb{R}^N) / \nabla u \in L^2(\mathbb{R}^N)\}$.

It has been known after the work by Obata [14] that the only finite energy *positive* solutions to (1.1) are given by the family of functions

$$\mu^{-\frac{n-2}{2}} U\big(\mu^{-1}(x-\xi)\big), \qquad U(x) = \left(\frac{2}{1+|x|^2}\right)^{\frac{N-2}{2}}, \quad \xi \in \mathbb{R}^n, \ \mu > 0 \tag{1.2}$$

corresponding to the extremals for the critical Sobolev embedding [1,17]. These functions are indeed *all* positive solutions of (1.1), even without the finite energy requirement, see Caffarelli, Gidas and Spruck [6]. Finite energy sign-changing solutions to (1.1) are only partly understood. Radially symmetric ones for instance do not exist as it readily follows from Pohozaev's identity [15]. On the other hand, via stereographic projection to S^n Eq. (1.1), which is *conformally invariant* becomes

$$\Delta_{S^n} v + \gamma \big(|v|^{\frac{4}{n-2}} v - v\big) = 0 \quad \text{in } S^n, \tag{1.3}$$

see for instance [16,9]. Ding [8] found that compactness of critical Sobolev's embedding holds within functions of the form

$$v(x) = v\big(|x_1|, |x_2|\big), \quad x = (x_1, x_2) \in S^n \subset \mathbb{R}^{n+1} = \mathbb{R}^k \times \mathbb{R}^{n+1-k}, \ k \geqslant 2,$$

so that infinitely many solutions of this form exist, for any $n \geqslant 3$, thanks to Ljusternik–Schnirelmann theory. No qualitative features of these solutions other than their radial symmetry are known. See also [10].

Using a different method, we have built in [7] sequences of sign changing solutions for problem (1.3) for $n \geqslant 4$ without radial symmetry. These solutions have large energy and exhibit concentration patterns of their energy densities along special submanifolds of S^n. They can be visualized as a large number of bubbles of the form (1.2) with small scaling parameters μ. Among the possible concentration sets included are great circles in S^n and higher-dimensional sets such as the Clifford torus $\frac{1}{\sqrt{2}} S^1 \times \frac{1}{\sqrt{2}} S^1 \times \{0\} \subset S^n \subset \mathbb{R}^{n+1}$ for $n \geqslant 5$. The construction in [7] does not include the 3-dimensional case.

The purpose of this paper is to devise an approach which provides examples of non-radial solutions in all dimensions $n \geqslant 3$, at the same time providing fine knowledge on the core asymptotic behavior. We will concentrate on the simplest case considered in [7]: solutions of (1.3) concentrating on a great circle. We construct for any $n \geqslant 3$ a solution to Eq. (1.1) which looks like *the soliton U*

crowned with k negative spikes arranged on a regular polygon with radius 1, *and precisely described in our main result:*

Theorem 1. *Let $n \geqslant 3$ and write $\mathbb{R}^n = \mathbb{C} \times \mathbb{R}^{n-2}$ and let $\xi_j^k = (e^{\frac{2j\pi i}{k}}, 0)$, $j = 1, \dots, k$. Then for any sufficiently large k there is a finite energy solution of the form*

$$u_k(x) = U(x) - \sum_{j=1}^{k} \mu_k^{-\frac{n-2}{2}} U\left(\mu_k^{-1}(x - \xi_j)\right) + o(1),$$

where

$$\mu_k = \frac{c_n}{k^2} \quad \text{for } n \geqslant 4, \qquad \mu_k = \frac{c}{k^2 (\log k)^2} \quad \text{for } n = 3.$$

Moreover,

$$J(u_k) = (k+1) J(U) + O(1). \tag{1.4}$$

Here $O(1)$ remains bounded and $o(1) \to 0$ uniformly as $k \to +\infty$.

The proof of this result consists of linearizing the equation around a first approximation and devising an invertibility theory for the linearized operator which takes advantage of the symmetry of the configuration, and reduces the problem to just slightly adjusting the scaling parameter μ_k. The basic outline is similar to that in [7], but with the main ingredient worked out in a different way: in order to cover the lower-dimensional case, the invertibility theory for the linearization needs to involve norms which describe in more accurate way the behavior of the error of approximation and the corresponding remainders. In this way, together with covering the case $n = 3$, the above result describes a more precise asymptotic behavior for $n \geqslant 4$ than that in the parallel construction [7].

We believe that the approach devised in this paper may also be applicable to cover lower dimensions in higher-dimensional lattices, for instance for $n = 4$ in the Clifford torus. We will not treat these issues in this paper but just concentrate in the simplest case of the *crown solution*. We point out that the idea of using the (large) number of bubbles as a parameter of the problem has been previously developed by Wei and Yan in [18] for critical problems with the presence of weights. The possibility of concentration of positive solutions on lattices is discussed in [13]. Finally, we point out that sign-changing, non-radial solutions were found in [4,5] in the subcritical range while in the critical exponent case and $n = 3$ the topology of lower energy level sets was analyzed in [2,3]. The result described here is the first semi-explicit construction, with an approach which naturally yields spectral information on the linearization. Such properties for finite-energy solutions may be important for instance in understanding the long-term dynamics in the corresponding Schrödinger equations, a topic of recent high interest, see [11,12].

The rest of the paper will be devoted to the proof of Theorem 1.

2. First approximation and the error

In this section we construct a first approximation to find a solution to our problem (1.1). As mentioned in the introduction, it is known that all positive solutions of (1.1) are given by the family

$$w_\mu(y - \xi), \quad \xi \in \mathbb{R}^n, \ \mu > 0, \ \text{where } w_\mu(y) := \mu^{-\frac{n-2}{2}} U\left(\mu^{-1} y\right), \tag{2.1}$$

with U defined in (1.2). Eq. (1.1) is invariant under Kelvin's transform. This means that if $u(y)$ solves (1.1) in $\mathbb{R}^n$, then so does

$$\hat{u}(y) := |y|^{2-n} u\big(|y|^{-2} y\big)$$

in $\mathbb{R}^n \setminus \{0\}$.

The solution U in (1.2) has the characteristic of being invariant under this transformation. We may wonder, more generally when the function $w_\mu(y - \xi)$ satisfies this property. As simple algebra shows, this happens if and only if

$$|\xi|^2 + \mu^2 = 1.$$

We build an approximation U_* to a solution of (1.1) as follows. Let k be a large positive integer and let us select k points $\xi_1, \ldots, \xi_k$ with

$$|\xi_j|^2 = 1 - \mu^2$$

where $\mu > 0$ is a small number which we write in the form

$$\mu = \frac{\delta^{\frac{2}{n-2}}}{k^2} \quad \text{for } n \geqslant 4, \qquad \mu = \frac{\delta^2}{k^2 (\log k)^2} \quad \text{for } n = 3. \tag{2.2}$$

In what follows we assume that δ is a parameter with uniform lower and upper bounds δ_0, δ_1,

$$0 < \delta_0 \leqslant \delta \leqslant \delta_1 \tag{2.3}$$

for k large. Moreover, we assume that the points ξ_j are arranged symmetrically, as the vertices of a planar regular polygon.

We denote points $y \in \mathbb{R}^n$, $n \geqslant 3$, as

$$y = \big(\bar{y}, y'\big), \qquad \bar{y} = (y_1, y_2), \qquad y' = (y_3, \ldots, y_n).$$

Using complex notation for $\bar{y}$ variables, we then assume

$$\xi_j = \sqrt{1 - \mu^2} \big(e^{\frac{2\pi(j-1)}{k} i}, 0, \ldots, 0\big), \quad j = 1, \ldots, k.$$

We write

$$U_j(y) := w_\mu(y - \xi_j), \quad j = 1, \ldots, k,$$

and consider the function

$$U_*(y) := U(y) - \sum_{j=1}^{k} U_j(y). \tag{2.4}$$

For a large number k, which at the same time makes the concentration parameter μ very small, we have that U_* defines a rather good approximation to a solution of (1.1), which is in addition invariant under Kelvin's transform:

$$U_*(y) = |y|^{2-n} U_*\big(|y|^{-2} y\big).$$

Next we derive some estimates on the error of approximation, defined as

$$E := \Delta U_* + \gamma |U_*|^{\frac{4}{n-2}} U_*. \tag{2.5}$$

A basic issue is to measure the size of the error near and far from the concentration points ξ_j.

For reasons that will become apparent later, it is convenient to do this measurement using the following norm: Let us fix a number $q > \frac{n}{2}$ and consider the weighted L^q norm

$$\|h\|_{**} = \left\| (1 + |y|)^{n+2-\frac{2n}{q}} h \right\|_{L^q(\mathbb{R}^n)}. \tag{2.6}$$

To be more precise, we will estimate the $\|\cdot\|_{**}$-norm of the error term E first in the *exterior region* $\bigcap_{j=1}^{k} \{|y - \xi_j| > \frac{\eta}{k}\}$, then in the *interior regions* $\{|y - \xi_j| < \frac{\eta}{k}\}$, for any $j = 1, \ldots, k$. Here $\eta > 0$ is a positive and small constant, independent of k. We will do it in what is left of this section.

In the exterior region. Observe first that

$$\gamma^{-1} E = \left| U - \sum_{j=1}^{k} U_j \right|^{p-1} \left(U - \sum_{j=1}^{k} U_j \right) - U^p - \sum_{j=1}^{k} U_j^p.$$

For $y \in \bigcap_{j=1}^{k} \{|y - \xi_j| > \frac{\eta}{k}\}$ we can estimate

$$|E(y)| \leqslant C \left[\frac{1}{(1+|y|^2)^2} + \left| \sum_{j=1}^{k} \frac{\mu^{\frac{n-2}{2}}}{|y-\xi_j|^{n-2}} \right|^{\frac{4}{n-2}} \right] \left(\sum_{j=1}^{k} \frac{\mu^{\frac{n-2}{2}}}{|y-\xi_j|^{n-2}} \right)$$

$$\leqslant C \frac{\mu^{\frac{n-2}{2}}}{(1+|y|^2)^2} \sum_{j=1}^{k} \frac{1}{|y-\xi_j|^{n-2}}.$$

Now we compute the weighted L^q norm above for this quantity. We get, for $n \geqslant 4$,

$$\left\| (1+|y|)^{n+2-\frac{2n}{q}} E_1 \right\|_{L^q(\bigcap_{j=1}^{k} \{|y-\xi_j| > \frac{\eta}{k}\})}$$

$$\leqslant C \mu^{\frac{n-2}{2}} \sum_{j=1}^{k} \left(\int\limits_{|y-\xi_j| > \frac{\eta}{k}} \frac{(1+|y|)^{(n+2)q-2n}}{(1+|y|^2)^{2q}} \frac{1}{|y-\xi_j|^{(n-2)q}} \, dy \right)^{\frac{1}{q}}$$

$$\leqslant C \mu^{\frac{n-2}{2}} k \left(\int\limits_{\frac{\eta}{k}}^{1} \frac{t^{n-1}}{t^{(n-2)q}} \, dt \right)^{\frac{1}{q}} \leqslant C \frac{\mu^{\frac{n-2}{2}} k^{n-2}}{k^{\frac{n}{q}-1}} \leqslant C k^{1-\frac{n}{q}}. \tag{2.7}$$

On the other hand, if $n = 3$, we get

$$\left\| (1+|y|)^{n+2-\frac{2n}{q}} E \right\|_{L^q(\bigcap_{j=1}^{k} |x-\xi_j| > \frac{\eta}{k})} \leqslant \frac{C}{\log k}, \tag{2.8}$$

where C depends on η and on positive upper and lower bounds for the parameter δ.

In the interior regions. Now, if $|y - \xi_j| < \frac{\eta}{k}$ for some $j \in \{1, \ldots, k\}$ fixed, we have

$$\gamma^{-1} E = p \left(U_j + s \left(-\sum_{i \neq j} U_i + U \right) \right)^{p-1} \left(-\sum_{i \neq j} U_i + U \right) - U^p + \sum_{i \neq j} U_i^p. \tag{2.9}$$

Note that very close to ξ_j, $U_j = O(\mu^{-\frac{n-2}{2}})$. More in general taking η small, we have that U_j dominates globally the other terms. Note that in particular

$$\sum_{i \neq j} \frac{\mu^{\frac{n+2}{2}}}{|\xi_j - \xi_i|^{n+2}} \sim k^{n+2} \mu^{\frac{n+2}{2}} = O(1) \quad \text{for any } n \geqslant 3.$$

It is convenient to measure the error after a change of scale. Define

$$\tilde{E}_j(y) := \mu^{\frac{n+2}{2}} E(\xi_j + \mu y), \quad |y| < \frac{\eta}{\mu k}.$$

We observe that

$$\mu^{\frac{n-2}{2}} U_j(\xi_j + \mu y) = U(y) \quad \text{and} \quad U_i(y) = \mu^{-\frac{n-2}{2}} U\big(\mu^{-1}(y - \xi_i)\big).$$

Thus

$$\mu^{\frac{n-2}{2}} U_i(\xi_j + \mu y) = U\big(y - \mu^{-1}(\xi_i - \xi_j)\big).$$

Notice also that

$$\mu^{-1} |\xi_i - \xi_j| \sim \frac{\mu^{-1}}{k} |j - i|$$

hence for $i \neq j$ and $|y| < \frac{\eta}{\mu k}$ we estimate

$$U\big(y - \mu^{-1}(\xi_i - \xi_j)\big) \leqslant \frac{c \mu^{n-2} k^{n-2}}{|j - i|^{n-2}}.$$

Therefore we obtain that for some $s \in (0, 1)$

$$\gamma^{-1} \tilde{E}_j(y) = p \left(U(y) + s \left(-\sum_{i \neq j} U\big(y - \mu^{-1}(\xi_i - \xi_j)\big) + \mu^{\frac{n-2}{2}} U(\xi_j + \mu y) \right) \right)^{p-1}$$

$$\times \left(-\sum_{i \neq j} U\big(y - \mu^{-1}(\xi_i - \xi_j)\big) + \mu^{\frac{n-2}{2}} U(\xi_j + \mu y) \right)$$

$$+ \sum_{i \neq j} U^p\big(y - \mu^{-1}(\xi_i - \xi_j)\big) - \mu^{\frac{n+2}{2}} U^p(\xi_j + \mu y). \tag{2.10}$$

Hence we can estimate for $|y| < \frac{\eta}{\mu k}$ and when $n \geqslant 4$

$$\left| \tilde{E}_j(y) \right| \leqslant C \left[\frac{\mu^{\frac{n-2}{2}}}{1 + |y|^4} + \mu^{\frac{n+2}{2}} \right]. \tag{2.11}$$

We compute now

$$\left\| (1 + |y|)^{n+2-\frac{2n}{q}} \tilde{E}_j(y) \right\|_{L^q(|y|<\eta\mu^{-\frac{1}{2}})} \leqslant C \mu^{\frac{n-2}{2}} \left\| (1 + |y|)^{n-2-\frac{2n}{q}} \right\|_{L^q(|y|<\eta\mu^{-\frac{1}{2}})}.$$

Since

$$\left\| (1 + |y|)^{n-2-\frac{2n}{q}} \right\|_{L^q(|y|<\eta\mu^{-\frac{1}{2}})}^q \sim \int\limits_0^{\eta\mu^{-\frac{1}{2}}} (1+r)^{(n-2)q-n-1}\, dr \leqslant C\mu^{-\frac{(n-2)q-n}{2}}$$

it follows that, for $n \geqslant 4$,

$$\left\| (1 + |y|)^{n+2-\frac{2n}{q}} \tilde{E}_j(y) \right\|_{L^q(|y|<\eta\mu^{-\frac{1}{2}})} \leqslant C \frac{1}{k^{\frac{n}{q}}}. \tag{2.12}$$

When $n = 3$, one gets the estimate

$$\left\| (1 + |y|)^{n+2-\frac{2n}{q}} \tilde{E}_j(y) \right\|_{L^q(|y|<\frac{\eta}{\mu k})} \leqslant C \frac{1}{k \log k}. \tag{2.13}$$

3. A linear result

We consider the operator L_0 defined as

$$L_0(\phi) := \Delta\phi + p\gamma U^{p-1}\phi. \tag{3.1}$$

It is well known that the set of bounded solutions of the homogeneous equation $L_0(\phi) = 0$ is spanned by the $n + 1$ functions

$$Z_\ell = \partial_{y_\ell} U, \quad \ell = 1, \ldots, n \quad \text{and} \quad Z_{n+1} = y \cdot \nabla U + \frac{n-2}{2} U.$$

This section is devoted to establish an invertibility theory for

$$L_0(\phi) = h(y) \quad \text{in } \mathbb{R}^n. \tag{3.2}$$

To do so, let us introduce the norm

$$\|\phi\|_* := \left\| (1 + |y|^{n-2})\phi \right\|_{L^\infty(\mathbb{R}^n)}. \tag{3.3}$$

We have the following result.

Lemma 3.1. *Assume that $\frac{n}{2} < q < n$ in the definition of the norm $\| \cdot \|_{**}$ in (2.6). Let $h(y)$ be a function such that $\|h\|_{**} < +\infty$, and*

$$\int_{\mathbb{R}^n} Z_\ell h = 0 \quad \text{for all } \ell = 1, \ldots, n+1. \tag{3.4}$$

Then Eq. (3.2) has a unique solution ϕ with $\|\phi\|_* < +\infty$, such that

$$\int_{\mathbb{R}^n} U^{p-1} Z_\ell \phi = 0 \quad \text{for all } \ell = 1, \ldots, n+1.$$

Moreover, there is a constant C depending only on q and n such that

$$\|\phi\|_* \leqslant C \|h\|_{**}. \tag{3.5}$$

Proof. Let us consider the subspace

$$H = \left\{ \phi \in \mathcal{D}^{1,2}(\mathbb{R}^n) \Big/ \int_{\mathbb{R}^n} U^{p-1} Z_\ell \phi = 0 \text{ for all } \ell = 1, \ldots, n+1 \right\}.$$

Observe that for h as in the statement of the lemma,

$$\|h\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leqslant C \left\| \left(1 + |y|\right)^{n+2-\frac{2n}{q}} h \right\|_{L^q(\mathbb{R}^n)}, \tag{3.6}$$

as a direct consequence of Holder inequality

$$\int_{\mathbb{R}^n} |h|^r \leqslant \left(\int_{\mathbb{R}^n} |h|^q \left(1 + |y|\right)^{(n+2)q-2n} \right)^{\frac{r}{q}} \left(\int_{\mathbb{R}^n} \left(1 + |y|\right)^{-2n} \right)^{\frac{q-r}{q}},$$

with $r = \frac{2n}{n+2}$. Let us consider the problem of finding a function $\phi \in H$ such that

$$\int_{\mathbb{R}^n} \nabla \phi \nabla \psi - p \int_{\mathbb{R}^n} U^{p-1} \phi \psi + \int_{\mathbb{R}^n} h \psi = 0 \quad \text{for all } \psi \in H, \tag{3.7}$$

which makes sense because of (3.6) and Sobolev's embedding. Since the orthogonality conditions (3.4) hold, we easily check that a solution of problem (3.7) produces a weak solution of (3.2).

Now, for $f \in L^{\frac{2n}{n+2}}(\mathbb{R}^n)$, let us denote by $\phi = A(f) \in H$ the unique solution of the problem

$$\int_{\mathbb{R}^n} \nabla \phi \nabla \psi + \int_{\mathbb{R}^n} f \psi = 0 \quad \text{for all } \psi \in H, \tag{3.8}$$

given by Riesz's theorem. Then A defines a continuous linear map between $L^{\frac{2n}{n+2}}(\mathbb{R}^n)$ and H. Problem (3.7) can be formulated as

$$\phi - A\left(p\gamma U^{p-1}\phi\right) = A(h), \quad \phi \in H. \tag{3.9}$$

The map $\phi \in H \mapsto U^{p-1}\phi \in L^{\frac{n}{2}}(\mathbb{R}^n)$ is easily seen to be compact, thanks to local compactness of Sobolev's embeddings and the fact that $U^{p-1} = O(|y|^{-4})$.

Hence, Fredholm's alternative applies to problem (3.9): for $f = 0$, (3.9) reduces to $L_0(\phi) = 0$ with $\phi \in H$. Elliptic regularity yields that ϕ is also bounded, and hence it is a linear combination of the functions Z_ℓ. Then, the definition of H implies that necessarily $\phi = 0$. We conclude that problem (3.9) is uniquely solvable in H for any h. Besides,

$$\|\nabla\phi\|_{L^2(\mathbb{R}^n)} + \|\phi\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leqslant C\left\|\left(1+|y|\right)^{n+2-\frac{2n}{q}}h\right\|_{L^q(\mathbb{R}^n)}.$$

It remains to prove that ϕ satisfies the estimate (3.14). Being ϕ solution to (3.9), local elliptic estimates yield

$$\left\|D^2\phi\right\|_{L^q(B_1)} + \|D\phi\|_{L^q(B_1)} + \|\phi\|_{L^\infty(B_1)} \leqslant C\left\|\left(1+|y|\right)^{n+2-\frac{2n}{q}}h\right\|_{L^q(\mathbb{R}^n)}.$$

Now, let us consider Kelvin's transform of ϕ,

$$\tilde{\phi}(y) = |y|^{2-n}\phi(|y|^{-2}y).$$

Then we check that $\tilde{\phi}$ satisfies the equation

$$\Delta\tilde{\phi} + p\gamma U^{p-1}(y)\tilde{\phi} = \tilde{h} \quad \text{in } \mathbb{R}^n \setminus \{0\}, \tag{3.10}$$

where $\tilde{h}(y) = |y|^{-n-2}h(|y|^{-2}y)$. We observe that

$$\|\tilde{h}\|_{L^q(|y|<2)} = \left\||y|^{n+2-\frac{2n}{q}}h\right\|_{L^q(|y|>\frac{1}{2})} \leqslant C\left\|\left(1+|y|\right)^{n+2-\frac{2n}{q}}h\right\|_{L^q(\mathbb{R}^n)},$$

and

$$\|\nabla\tilde{\phi}\|_{L^2(\mathbb{R}^n)} + \|\tilde{\phi}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} = \|\nabla\phi\|_{L^2(\mathbb{R}^n)} + \|\phi\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}.$$

Then we get, from elliptic estimates applied to Eq. (3.10),

$$\left\|D^2\tilde{\phi}\right\|_{L^q(B_1)} + \|D\tilde{\phi}\|_{L^q(B_1)} + \|\tilde{\phi}\|_{L^\infty(B_1)} \leqslant C\|\tilde{h}\|_{L^q(B_2)} \leqslant C\left\|\left(1+|y|\right)^{n+2-\frac{2n}{q}}h\right\|_{L^q(\mathbb{R}^n)}.$$

But

$$\|\tilde{\phi}\|_{L^\infty(B_1)} = \left\||y|^{n-2}\phi\right\|_{L^\infty(\mathbb{R}^n\setminus B_1)},$$

and we also check that

$$\left\||y|^{n+1-\frac{2n}{q}}D\phi\right\|_{L^q(\mathbb{R}^n\setminus B_1)} \leqslant C\left[\left\|D^2\tilde{\phi}\right\|_{L^q(B_1)} + \|D\tilde{\phi}\|_{L^q(B_1)}\right].$$

Combining the above estimates, relation (3.14) follows. The proof is concluded. $\square$

For later purpose we consider now the following perturbation of problem (3.2):

$$L_0(\phi) + a(y)\phi = g(y) + \sum_{\ell=1}^{n+1} c_\ell U^{p-1}Z_\ell \quad \text{in } \mathbb{R}^n \tag{3.11}$$

where

$$\int_{\mathbb{R}^n} U^{p-1} Z_\ell \phi = 0 \quad \text{for all } \ell = 1, \dots, n+1 \tag{3.12}$$

and

$$c_\ell \int_{\mathbb{R}^n} U^{p-1} Z_\ell^2 = \int_{\mathbb{R}^n} \big(a(y)\phi - g(y) \big) Z_\ell \phi = 0 \quad \text{for all } \ell = 1, \dots, n+1. \tag{3.13}$$

Lemma 3.2. *Let $2 < \nu < n$. There exist numbers $\delta, C > 0$ depending on ν, n such that the following holds: If g and ϕ are such that $\|(1 + |y|^\nu)g\|_\infty < +\infty$, $\|(1 + |y|^{\nu-2})\phi\|_\infty < +\infty$, and $\|(1 + |y|^2)a\|_\infty < \delta$, and (3.11)–(3.13) are satisfied, then*

$$\big\| \big(1 + |y|^{\nu-2}\big)\phi \big\|_\infty \leqslant C \big\| \big(1 + |y|^\nu\big)g \big\|_\infty. \tag{3.14}$$

Proof. By contradiction, let us assume the existence of functions ϕ_n, a_n, g_n and constants c_ℓ^n such that (3.11)–(3.13) hold, and

$$\big\| \big(1 + |y|^\nu\big)g_n \big\|_\infty \to 0, \qquad \big\| \big(1 + |y|^{\nu-2}\big)\phi_n \big\|_\infty = 1, \qquad \big\| \big(1 + |y|^2\big)a_n \big\|_\infty \to 0.$$

Clearly we have that $\|(1 + |y|^\nu)a_n\phi_n\|_\infty \to 0$, and also that by their definition that $c_\ell^n \to 0$, so that with no loss of generality we may assume that $a_\ell^n \equiv 0$, and $c_\ell^n = 0$. We claim first that

$$\|\phi\|_\infty \to 0.$$

Indeed, otherwise there are numbers $\gamma, R > 0$ and points x_n such that

$$\big| \phi_n(x_n) \big| \geqslant \gamma, \quad |x_n| \leqslant R.$$

Passing to a subsequence, and using local elliptic estimates, we find that ϕ_n converges locally uniformly over compact sets to a bounded function $\phi_0 \neq 0$ with

$$L_0(\phi_0) = 0 \quad \text{and} \quad \int_{\mathbb{R}^n} U^{p-1}\phi Z_\ell = 0 \quad \text{for all } \ell$$

which gives $\phi_0 = 0$, a contradiction.

Since $\nu < N$ we have that for some constants d_0, R_0 we have

$$-L_0\big(|y|^{2-\nu}\big) > d_0 |y|^{-\nu} \quad \text{for all } |y| > R_0.$$

Therefore if we let

$$h_n(y) = \big(d_0^{-1} \big\| |y|^\nu g_n \big\|_\infty + \|\phi_n\|_\infty R_0^{\nu-2} \big) |y|^{2-\nu}$$

maximum principle yields

$$\big| \phi_n(y) \big| \leqslant h_n(y) \quad \text{for all } |y| > R_0.$$

From here we obtain that actually

$$\big\|\big(1 + |y|^{\nu-2}\big)\phi_n\big\|_{\infty} \to 0,$$

a contradiction that finishes the proof. □

4. A gluing procedure

To prove our result Theorem 1 we will show that problem (1.1) admits a solution of the form

$$u(y) = U_*(y) + \phi(y)$$

where ϕ is a function small when compared with U_*. Then Eq. (1.1) gets rewritten in terms of ϕ as

$$\Delta\phi + p\gamma |U_*|^{p-1}\phi + E + \gamma N(\phi) = 0 \qquad (4.1)$$

where E is defined in (2.5) and

$$N(\phi) = |U_* + \phi|^{p-1}(U_* + \phi) - |U_*|^{p-1}U_* - p|U_*|^{p-1}\phi.$$

In this section we prove the existence of a function ϕ solution to (4.1).

Let ζ_j be a cut-off function defined as follows. Let $\zeta(s)$ be a smooth function such that $\zeta(s) = 1$ for $s < 1$ and $\zeta(s) = 0$ for $s > 2$. We also let $\zeta^-(s) = \zeta(2s)$. Then we set

$$\zeta_j(y) = \begin{cases} \zeta(k\eta^{-1}|y|^{-2}|(y - \xi|y|)|) & \text{if } |y| > 1, \\ \zeta(k\eta^{-1}|y - \xi|) & \text{if } |y| \leqslant 1, \end{cases}$$

in such a way that

$$\zeta_j(y) = \zeta_j\big(y/|y|^2\big).$$

We consider in addition the cut-off functions $\zeta_j^-(y)$, defined as above with ζ replaced by ζ^-.

A function ϕ of the form

$$\phi = \sum_{j=1}^{k} \tilde{\phi}_j + \psi$$

is a solution of problem (4.1) if we can solve the following coupled system of elliptic equations in $\tilde{\phi} = (\tilde{\phi}_1, \ldots, \tilde{\phi}_k)$ and ψ:

$$\Delta\tilde{\phi}_j + p\gamma |U_*|^{p-1}\zeta_j\tilde{\phi}_j + \zeta_j\bigg[p\gamma |U_*|^{p-1}\psi + E + \gamma N\bigg(\tilde{\phi}_j + \sum_{i\neq j}\tilde{\phi}_i + \psi\bigg)\bigg] = 0, \quad j = 1, \ldots, k,$$

$$(4.2)$$

$$\Delta \psi + p\gamma U^{p-1}\psi + \left[p\gamma \left(|U_*|^{p-1} - U^{p-1} \right) \left(1 - \sum_{j=1}^{k} \zeta_j \right) + p\gamma U^{p-1} \sum_{j=1}^{k} \zeta_j \right] \psi$$

$$+ p\gamma |U_*|^{p-1} \sum_{j} (1 - \zeta_j) \tilde{\phi}_j + \left(1 - \sum_{j=1}^{k} \zeta_j \right) \left(E + \gamma N \left(\sum_{j=1}^{k} \tilde{\phi}_j + \psi \right) \right) = 0. \qquad (4.3)$$

To solve this system (4.2)–(4.3) we will solve first problem (4.3) for given $\tilde{\phi}_j$'s of a special form that we describe next. We assume that

$$\tilde{\phi}_j(\bar{y}, y') = \tilde{\phi}_1 \left(e^{\frac{2\pi j}{k} i} \bar{y}, y' \right), \quad j = 1, \dots, k-1. \qquad (4.4)$$

On the other hand, we assume that $\tilde{\phi}_1$ is even in the variables $y_2, \dots, y_n$, namely

$$\tilde{\phi}_1(y_1, \dots, y_j, \dots, y_n) = \tilde{\phi}_1(y_1, \dots, -y_j, \dots, y_n), \quad j = 2, \dots, n, \qquad (4.5)$$

and invariant under Kelvin's transform:

$$\tilde{\phi}_1(y) = |y|^{2-n} \tilde{\phi}_1 \left(|y|^{-2} y \right). \qquad (4.6)$$

We also assume that

$$\|\phi_1\|_* \leqslant \rho \quad \text{where } \phi_1(y) := \mu^{\frac{n-2}{2}} \tilde{\phi}_1(\xi_1 + \mu y), \qquad (4.7)$$

for a ρ fixed, but sufficiently small. The following result holds.

Lemma 4.1. *There exist constants k_0, C, ρ_0, such that for all $k \geqslant k_0$ the following holds: Let $\tilde{\phi}_j$, $j = 1, \dots, k$ satisfy conditions (4.4)–(4.7), with $\rho < \rho_0$. Then there exists a unique solution $\psi = \Psi(\phi_1)$ to Eq. (4.3), that satisfies the symmetries*

$$\psi(\bar{y}, y_3, \dots, y_\ell, \dots, y_n) = \psi(\bar{y}, y_3, \dots, -y_\ell, \dots, y_n), \quad \ell = 3, \dots, n, \qquad (4.8)$$

$$\psi(\bar{y}, y') = \psi \left(e^{\frac{2\pi j}{k} i} \bar{y}, y' \right), \quad j = 1, \dots, k-1, \qquad (4.9)$$

$$\psi(y) = |y|^{2-n} \tilde{\psi} \left(|y|^{-2} y \right), \qquad (4.10)$$

and such that

$$\|\psi\|_* \leqslant \frac{C}{k^{\frac{n}{q}-1}} + C\|\phi_1\|_*^2 \quad \text{if } n \geqslant 4 \qquad (4.11)$$

and

$$\|\psi\|_* \leqslant \frac{C}{\log k} + C\|\phi_1\|_*^2 \quad \text{if } n = 3. \qquad (4.12)$$

Moreover, the operator Ψ satisfies the Lipschitz condition

$$\left\| \Psi\left(\phi_1^1\right) - \Psi\left(\phi_1^2\right) \right\|_* \leqslant C \left\| \phi_1^1 - \phi_1^2 \right\|_*.$$

Proof. Let us write Eq. (4.3) in the form

$$\Delta \psi + p\gamma U^{p-1}(y)\psi + \gamma V(y)\psi + p\gamma |U_*|^{p-1}\sum_j (1 - \zeta_j)\tilde{\phi}_j + M(\psi) = 0 \qquad (4.13)$$

where

$$V(y) := \underbrace{p\big(|U_*|^{p-1} - U^{p-1}\big)\left(1 - \sum_{j=1}^{k}\zeta_j\right)}_{V_1(y)} + \underbrace{pU^{p-1}\sum_{j=1}^{k}\zeta_j}_{V_2(y)}, \qquad (4.14)$$

and

$$M(\psi) := \left(1 - \sum_{j=1}^{k}\zeta_j\right)\left(E + \gamma N\left(\sum_{j=1}^{k}\tilde{\phi}_j + \psi\right)\right). \qquad (4.15)$$

The desired result will be a consequence of a corresponding linear result and an application of the contraction mapping principle. Thus we consider first the linear problem

$$\Delta \psi + p\gamma U^{p-1}(y)\psi = h \quad \text{in } \mathbb{R}^n, \qquad (4.16)$$

where h is a function that satisfies symmetries (4.8), (4.9), and

$$h(y) = |y|^{-n-2}h\big(|y|^{-2}y\big), \qquad (4.17)$$

and in addition such that $\|h\|_{**} < +\infty$.

Claim. *Eq. (4.16) has a unique bounded solution $\psi = T(h)$ that satisfies symmetries (4.8), (4.9), (4.10). Moreover, there is a constant C, dependent only on q and n such that*

$$\|\psi\|_* \leqslant C\|h\|_{**}. \qquad (4.18)$$

To prove this claim we will apply Lemma 3.1. We will check that under the symmetries assumed we have that

$$\int_{\mathbb{R}^n} Z_\ell h = 0 \quad \text{for all } \ell = 1, \ldots, n+1. \qquad (4.19)$$

For $l = 3, \ldots, n$, this is a consequence of the oddness of Z_l and assumption (4.5) on h. Now, we consider the vector integral

$$I = \int_{\mathbb{R}^n} h \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} dy = c_n \int_{\mathbb{R}^n} \frac{h(y)}{(1 + |y|^2)^{\frac{n}{2}}} \bar{y}\, dy.$$

Changing the variables $\bar{y}$ into $e^{\frac{2\pi}{k}i}\bar{z}$ and using the symmetry (4.9) we get the identity

$$e^{\frac{2\pi}{k}i}I = I,$$

which yields $I = 0$, since $k \geqslant 2$.

Define now for $\lambda > 0$

$$I(\lambda) = \lambda^{\frac{n-2}{2}} \int\limits_{\mathbb{R}^n} U(\lambda y) h(y) \, dy.$$

Changing variables y into $y|y|^{-2}$ yields $I(\lambda) = I(\lambda^{-1})$, and hence $\int_{\mathbb{R}^n} h Z_{n+1} = \partial_\lambda I(\lambda)|_{\lambda=1} = 0$. Given the orthogonality conditions checked above, Lemma 3.1 then yields the existence of a unique solution to Eq. (4.16) such that

$$\int\limits_{\mathbb{R}^n} U^{p-1} Z_\ell \psi = 0 \quad \text{for all } \ell = 1, \ldots, n+1, \tag{4.20}$$

and in addition, estimate (4.18) holds. Now, the functions

$$\psi_\ell(y) := \psi(\bar{y}, y_3, \ldots, -y_\ell, \ldots, y_n), \quad \ell = 3, \ldots, n$$

also solve (4.16) and satisfy relations (4.20). Hence $\psi = \psi_\ell$, and thus ψ satisfies symmetries (4.8). The same argument applies to the functions

$$\psi_{12}(y) := \psi\left(e^{\frac{2\pi}{k} i} \bar{y}, y_3, \ldots, -y_\ell, \ldots, y_n\right)$$

and

$$\psi_{n+1}(y) = |y|^{2-n} \psi\left(|y|^{-2} y\right),$$

therefore giving the symmetries (4.9) and (4.10). The proof of the claim is complete.

Let us go back to problem (4.13). Let T be the linear operator defined in the claim. Then we write our problem in fixed point form as

$$\psi = -T\left(V\psi + p\gamma |U_*|^{p-1} \sum_j (1 - \zeta_j)\tilde{\phi}_j + M(\psi)\right) =: \mathcal{M}(\psi), \quad \psi \in X, \tag{4.21}$$

where X is the space of continuous functions ψ with $\|\psi\|_* < +\infty$ that satisfy symmetries (4.8), (4.9) and (4.10). Indeed, we readily check that if ψ satisfies those properties then $V\psi + M(\psi)$ inherits symmetries (4.8), (4.9), and (4.10), so that the operator $\mathcal{M}$ is well defined.

We will see that the operator $\mathcal{M}$ is a contraction mapping in the $\|\ \|_*$ norm, in a small ball centered at the origin in X.

With reference to (4.14), we check that

$$\left|V_1(y)\right| \leqslant p(p-1)\left|U - s\sum_{i=1}^k U_i\right|^{p-2} \left(\sum_{i=1}^k |U_i|\right) \leqslant C U^{p-2} \sum_{i=1}^k \frac{\mu^{\frac{n-2}{2}}}{|y - \xi_i|^{n-2}}.$$

Thus, if $|y - \xi_j| > \frac{\eta}{k}$ for all j, we get

$$\left|V_1\psi(y)\right| \leqslant C\|\psi\|_* U^{p-1}(y) \sum_{j=1}^k \frac{\mu^{\frac{n-2}{2}}}{|y - \xi_j|^{n-2}},$$

and, as in the computation in (2.7), we obtain

$$\|V_1\psi\|_{**} = \left\|\left(1+|y|\right)^{(n+2)-\frac{2n}{q}} V_1\psi\right\|_{L^q(\mathbb{R}^n)} \leqslant \frac{c}{k^{\frac{n}{q}-1}}\|\psi\|_* \quad \text{for } n \geqslant 4$$

and

$$\|V_1\psi\|_{**} \leqslant \frac{c}{\log k}\|\psi\|_* \quad \text{for } n = 3.$$

Now,

$$\|V_2\psi\|_{**} \leqslant C\big|\big\{\big||y|-1\big| < c\mu k\big\}\big|\|\psi\|_* \leqslant \frac{c}{k}\|\psi\|_*,$$

hence, since $q > \frac{n}{2}$,

$$\|V\psi\|_{**} \leqslant \frac{c}{k^{\frac{n}{q}-1}}\|\psi\|_* \quad \text{for all } n \geqslant 4, \qquad \|V\psi\|_{**} \leqslant \frac{c}{\log k}\|\psi\|_* \quad \text{if } n = 3. \tag{4.22}$$

We observe that

$$\big|\tilde{\phi}_j(y)\big| \leqslant CU(y)\|\phi_1\|_* \frac{\mu^{\frac{n-2}{2}}}{|y-\xi_j|^{n-2}}.$$

For the moment we shall only assume that

$$\|\psi\|_* + \|\phi_1\|_* \leqslant 2\rho$$

for a sufficiently small ρ.

Let us assume that $|y - \xi_j| > \frac{\eta}{2k}$ for all j. Then we find in this region

$$\left|N\left(\sum_{j=1}^{k}\tilde{\phi}_j + \psi\right)\right| \leqslant CU^{p-2}\left(\left|\sum_{j=1}^{k}\tilde{\phi}_j\right|^2 + |\psi|^2\right).$$

But

$$U^{p-2}\left|\sum_{j=1}^{k}\tilde{\phi}_j\right|^2 \leqslant C\|\phi_1\|_*^2 U^p \sum_{j=1}^{k} \frac{\mu^{n-2}}{|y-\xi_j|^{2(n-2)}}, \qquad U^{p-2}|\psi|^2 \leqslant U^p\|\psi\|_*^2.$$

As a conclusion, using again the computation in (2.7), we find that

$$\big\|M(\psi)\big\|_{**} \leqslant \frac{c}{k^{\frac{n}{q}-1}} + \frac{c}{k^{\frac{n}{q}-1}}\|\phi_1\|_*^2 + c\|\psi\|_*^2 \quad \text{for all } n \geqslant 4 \tag{4.23}$$

and, if $n = 3$,

$$\big\|M(\psi)\big\|_{**} \leqslant \frac{c}{\log k} + \frac{c}{\log k}\|\phi_1\|_*^2 + c\|\psi\|_*^2. \tag{4.24}$$

Given ψ_1, ψ_2 satisfying similar constraints, we also find,

$$\left\| M(\psi_1) - M(\psi_2) \right\|_{**} \leqslant C\rho \|\psi_1 - \psi_2\|_*. \tag{4.25}$$

Using the above estimates, we readily see that if ρ is fixed sufficiently small, but k-independent, then the operator $\mathcal{M}$ in (4.21) defines a contraction map in the set of functions $\psi \in X$ with

$$\|\psi\| \leqslant C\left[\|\phi_1\|^2 + k^{1-\frac{n}{q}} \right],$$

in dimension 4 or higher, and

$$\|\psi\| \leqslant C\left[\|\phi_1\|^2 + (\log k)^{-1} \right],$$

if $n = 3$. The existence result of the lemma thus follows. The Lipschitz condition is straightforwardly checked. $\quad\square$

Let us consider the operator $\Psi(\phi_1)$ defined in the above lemma. Then all Eqs. (4.2) reduce to just one, say that for $\tilde{\phi}_1$. Then we will find a solution to our problem if we solve

$$\Delta\tilde{\phi}_1 + p\gamma |U_*|^{p-1}\zeta_1\tilde{\phi}_1 + \zeta_1\left[p\gamma |U_*|^{p-1}\Psi(\phi_1) + E + \gamma N\left(\tilde{\phi}_1 + \sum_{i\neq 1}\tilde{\phi}_i + \Psi(\phi_1) \right) \right] = 0 \quad \text{in } \mathbb{R}^n. \tag{4.26}$$

We write this equation in the form

$$\Delta\tilde{\phi}_1 + p\gamma |U_1|^{p-1}\tilde{\phi}_1 + \zeta_1 E + \gamma\mathcal{N}(\phi_1) = 0 \quad \text{in } \mathbb{R}^n. \tag{4.27}$$

A key observation we make is the following: if the symmetry assumptions (4.8), (4.9), (4.10) hold, then thanks to the properties of $\Psi(\phi_1)$ we find that the function

$$\tilde{h}(y) = \zeta_1 E + \mathcal{N}(\phi_1)$$

satisfies the symmetries (4.8), (4.9) and the invariance

$$\tilde{h}(y) = |y|^{-n-2}h\left(|y|^{-2}y \right). \tag{4.28}$$

For a general function $\tilde{h}(y)$ satisfying the above properties, we consider first the linear problem

$$\Delta\tilde{\phi} + p\gamma U_1^{p-1}\tilde{\phi} + \tilde{h}(y) = c_{n+1}U_1^{p-1}\tilde{Z}_{n+1} \quad \text{in } \mathbb{R}^n. \tag{4.29}$$

where

$$\tilde{Z}_{n+1}(y) := \mu^{-\frac{n-2}{2}}Z_{n+1}\left(\mu^{-1}(y - \xi_1) \right), \qquad c_{n+1} := \frac{\int_{\mathbb{R}^n}\tilde{h}\tilde{Z}_{n+1}}{\int_{\mathbb{R}^n}U_1^{p-1}\tilde{Z}_{n+1}^2}.$$

We have the following result.

Lemma 4.2. *Let us assume that $\tilde{h}$ is even with respect to each of the variables $y_2, \ldots, y_n$ and that it satisfies the invariance* (4.28). *We assume in addition that*

$$h(y) := \mu^{\frac{n+2}{2}} \tilde{h}(\xi_1 + \mu y) \tag{4.30}$$

satisfies $\|h\|_{**} < +\infty$. Then problem (4.29) *has a unique solution $\tilde{\phi} := \tilde{T}(\tilde{h})$ that is even with respect to each of the variables $y_2, \ldots, y_n$, invariant under Kelvin's transform*

$$\tilde{\phi}(y) = |y|^{2-n} \tilde{\phi}(|y|^{-2} y), \tag{4.31}$$

and with

$$\phi(y) := \mu^{\frac{n-2}{2}} \tilde{\phi}(\xi_1 + \mu y) \tag{4.32}$$

satisfying

$$\int_{\mathbb{R}^n} \phi U^{p-1} Z_{n+1} = 0, \quad \|\phi\|_* \leqslant C \|h\|_{**}.$$

Proof. With no loss of generality we may assume that

$$\int_{\mathbb{R}^n} \tilde{h} \tilde{Z}_{n+1} = 0. \tag{4.33}$$

Let us scale out μ and consider the equivalent problem, for ϕ and h given by (4.32) and (4.30),

$$\Delta \phi + p\gamma |U|^{p-1} \phi = h(y) \quad \text{in } \mathbb{R}^n. \tag{4.34}$$

The evenness of h in the $n-1$ last coordinates guarantees that we have

$$\int_{\mathbb{R}^n} h Z_\ell = 0, \quad \ell = 2, \ldots, n+1.$$

We need to show that condition (4.28) implies the solvability condition

$$\int_{\mathbb{R}^n} h Z_1 = 0.$$

We have that

$$U_1(y) = w_\mu(y - \xi_1) \quad \text{where } w_\mu(y) = \mu^{-\frac{n-2}{2}} U(\mu^{-1} y),$$

$\xi_1 = (\xi^1, 0, \ldots, 0)$. We let

$$I(t) = \int_{\mathbb{R}^n} w_\mu(y - t\xi_1) \tilde{h}(y) \, dy$$

and observe that

$$\xi^1 \int_{\mathbb{R}^n} h Z_1 = \partial_t I(t)\Big|_{t=1} = -\xi^1 \int_{\mathbb{R}^n} \partial_{y_1} w_\mu(y - \xi_1) h(y)\, dy. \tag{4.35}$$

After changing variables we obtain

$$I(t) = \int_{\mathbb{R}^n} w_\mu\big(|y|^{-2} y - t\xi_1\big) \tilde{h}\big(|y|^{-2} y\big) |y|^{-2n}\, dy = \int_{\mathbb{R}^n} w_{\mu(t)}\big(y - s(t)\xi_1\big) \tilde{h}(y)\, dy$$

where

$$\mu(t) = \frac{\mu t}{\mu^2 + |\xi_1|^2 t^2}, \qquad s(t) = \frac{t}{\mu^2 + |\xi_1|^2 t^2}.$$

Hence

$$\partial_t I(t)\big|_{t=1} = \dot{\mu}(1) \int_{\mathbb{R}^n} \partial_\mu w_\mu(y - \xi_1)\Big|_{\mu=1} \tilde{h}(y)\, dy - \dot{s}(1)\xi_1 \int_{\mathbb{R}^n} \partial_{y_1} w_\mu(t)(y - \xi) \tilde{h}(y)\, dy. \tag{4.36}$$

We readily check that

$$\int_{\mathbb{R}^n} \partial_\mu w_\mu(y - \xi_1)\Big|_{\mu=\mu} \tilde{h}(y)\, dy = \int_{\mathbb{R}^n} Z_{n+1}(y) h(y)\, dy = 0$$

and $\dot{s}(1) = 1 - 2|\xi_1|^2$. Hence, using (4.35) and (4.36) we obtain

$$\xi_1 \int_{\mathbb{R}^n} h Z_1 = \xi^1\big(1 - 2|\xi_1|^2\big) \int_{\mathbb{R}^n} h Z_1$$

hence $\int_{\mathbb{R}^n} h Z_1 = 0$, as desired.

It follows from Lemma 3.1 that Eq. (4.34) possesses a unique solution ϕ_1 with

$$\int \phi Z_\ell = 0 \quad \text{for all } \ell = 1, \dots, n+1,$$

and $\|\phi\|_* \leqslant C \|h\|_{**}$. Arguing by uniqueness, as in the proof of Lemma 4.1, we find that $\tilde{\phi}$ satisfies the corresponding symmetries. The proof is complete. $\quad\square$

We use the above lemma to solve the corresponding projected version of (4.27),

$$\Delta \tilde{\phi}_1 + p\gamma |U_1|^{p-1} \tilde{\phi}_1 + \zeta_1 E + \gamma \mathcal{N}(\phi_1) = c_{n+1} U_1^{p-1} \tilde{Z}_{n+1} \quad \text{in } \mathbb{R}^n, \tag{4.37}$$

$$c_{n+1} := \frac{\int_{\mathbb{R}^n} (\zeta_1 E + \gamma \mathcal{N}(\phi_1)) \tilde{Z}_{n+1}}{\int_{\mathbb{R}^n} U_1^{p-1} \tilde{Z}_{n+1}^2}. \tag{4.38}$$

Let $\tilde{T}$ be the linear operator predicted by Lemma 4.2. Then we can set up problem (4.37) as that of solving the fixed point problem

$$\tilde{\phi}_1 = \tilde{T}\big(\zeta_1 E + \gamma \mathcal{N}(\phi_1)\big) =: \mathcal{M}(\phi_1). \tag{4.39}$$

We will solve this problem by means of contraction mapping principle. We recall that

$$\mathcal{N}(\phi_1) := p\big(|U_*|^{p-1}\zeta_1 - |U_1|^{p-1}\big)\tilde{\phi}_1$$
$$+ \zeta_1\left[p|U_*|^{p-1}\Psi(\phi_1) + N\left(\tilde{\phi}_1 + \sum_{i\neq 1}\tilde{\phi}_i + \Psi(\phi_1)\right)\right]. \tag{4.40}$$

In general, we denote

$$\bar{f}(y) = \mu^{\frac{n+2}{2}} f(\xi_1 + \mu y).$$

Assume that $n \geqslant 4$. Let us consider first the linear term

$$f_1(y) = p\zeta_1\big(|U_*|^{p-1} - |U_1|^{p-1}\big)\tilde{\phi}_1.$$

Then we have that for $|y| < \frac{\eta}{\mu k}$,

$$\big|\bar{f}_1(y)\big| = \left|p\left(\left(U(y) + \sum_{j=2}^{k} U\big(y + \mu^{-1}(\xi_1 - \xi_j)\big) - \mu^{\frac{n-2}{2}} U(\xi_1 + \mu y)\right)^{p-1} - U^{p-1}(y)\right)\phi_1(y)\right|.$$

We notice that, independently of small η, we have

$$\sum_{j=2}^{k} U\big(y + \mu^{-1}(\xi_1 - \xi_j)\big) \leqslant C\mu^{n-2}k^{n-2}\sum_{j=1}^{k}\frac{1}{j^{n-2}},$$

and hence we find

$$\big|\bar{f}_1(y)\big| \leqslant C\mu^{\frac{n-2}{2}} U(y)^{p-1}\|\phi_1\|_*.$$

Just like in the computation dealing with estimate (2.11), we then find that

$$\big\|\bar{f}_1(y)\big\|_{**} \leqslant C\mu^{\frac{n}{2q}}\|\phi_1\|_*. \tag{4.41}$$

Now, we consider the term

$$f_2 = (\zeta_1 - 1)U_1^{p-1}\tilde{\phi}_1.$$

Now we have, for $|y| > c\mu^{-\frac{1}{2}}$,

$$\big|\bar{f}_2(y)\big| \leqslant U^p(y)\|\phi_1\|_*.$$

Combining these estimates we find then that

$$\|\bar{f}_2\|_{**} \leqslant C\mu^{\frac{n}{2q}}\|\phi_1\|_*. \tag{4.42}$$

Now, let

$$f_3 = \zeta_1 p |U_*|^{p-1} \Psi(\phi_1).$$

Then, for $|y| < \eta\mu^{-\frac{1}{2}}$ we obtain

$$\left|\bar{f}_3(y)\right| \leqslant CU^{p-1}\mu^{\frac{n-2}{2}}\|\Psi(\phi_1)\|_\infty \leqslant CU^{p-1}\mu^{\frac{n-2}{2}}\left(\|\phi_1\|_*^2 + k^{1-\frac{n}{q}}\right).$$

Thus

$$\|\bar{f}_3\|_{**} \leqslant C\mu^{\frac{n}{2q}}\left(\|\phi_1\|_*^2 + k^{1-\frac{n}{q}}\right). \tag{4.43}$$

Now, for

$$f_4 = \zeta_1 N\left(\tilde{\phi}_1 + \sum_{i\neq 1}\tilde{\phi}_i + \Psi(\phi_1)\right).$$

Let us notice that

$$\bar{N}(\phi) = (V_* + \hat{\phi})^p - V_*^p - pV_*^{p-1}\hat{\phi}$$

where $\hat{\phi}(y) := \mu^{\frac{n-2}{2}}\phi(\xi_1 + \mu y)$, and

$$V_*(y) = U(y) + \sum_{j=2}^{k} U\left(y + \mu^{-1}(\xi_1 - \xi_j)\right) - \mu^{\frac{n-2}{2}}U(\xi_1 + \mu y).$$

Thus for

$$\phi = \tilde{\phi}_1 + \sum_{i\neq 1}\tilde{\phi}_i + \Psi(\phi_1)$$

we get

$$\left|\bar{f}_4(y)\right| \leqslant C\left[U^{p-1}\mu^{\frac{n-2}{2}}\|\phi_1\|_* + U^{p-1}\mu^{\frac{n-2}{2}}\left(\|\phi_1\|_*^2 + k^{1-\frac{n}{q}}\right)\right]$$

thus

$$\|\bar{f}_4\|_{**} \leqslant C\left[\mu^{\frac{n}{2q}}\|\phi_1\|_* + \mu^{\frac{n}{2q}}\left[\|\phi_*\| + k^{1-\frac{n}{q}}\right]^2\right]. \tag{4.44}$$

We recall that for the error $f_5 = \zeta_1 E$ we have already determined in (2.12) the estimate

$$\|\bar{f}_5\|_{**} \leqslant C\mu^{\frac{n}{2q}}. \tag{4.45}$$

A direct consequence of the fact that $q < n$ and the validity of estimates (4.41)–(4.45) is that $\mathcal{M}$ defined in (4.39) maps functions ϕ_1 with $\|\phi_1\|_* \leqslant c\mu^{\frac{n}{2q}}$ in the same class of functions. Furthermore, in a very similar way one proves the small Lipschitz character of the operators involved.

When $n = 3$, estimates (4.41)–(4.45) read respectively as follows

$$\|\bar{f}_1\|_{**}, \|\bar{f}_2\|_{**} \leqslant C\frac{1}{k(\log k)}\|\phi_1\|_*,$$

$$\|\bar{f}_3\|_{**} \leqslant C\frac{1}{k\log k}\left(\|\phi_1\|_*^2 + \frac{1}{k\log k}\right),$$

$$\|\bar{f}_4\|_{**} \leqslant C\left[\frac{1}{k\log k}\|\phi_1\|_* + \frac{1}{\log k}\left[\|\phi\|_*^2 + \frac{1}{k\log k}\right]\right]$$

and from (2.13)

$$\|\bar{f}_5\|_{**} \leqslant C\frac{1}{k\log k}.$$

Besides, one can prove that the map $\mathcal{M}$ is a contraction in the set of functions ϕ_1 with $\|\phi_1\|_* \leqslant c\frac{1}{k\log k}$. Thus we conclude

Proposition 4.1. *There exists a unique small solution $\phi_1 = \Phi(\delta)$ to problem (4.37)–(4.38). This solution satisfies*

$$\|\Phi\|_* \leqslant Ck^{-\frac{n}{q}}, \quad \forall n \geqslant 4$$

and

$$\|\Phi\|_* \leqslant Ck^{-1}(\log k)^{-1} \quad \text{if } n = 3.$$

We also have

$$\|\mathcal{N}(\Phi)\|_{**} \leqslant Ck^{-\frac{2n}{q}} \quad \forall n \geqslant 4$$

and

$$\|\mathcal{N}(\Phi)\|_* \leqslant Ck^{-2}(\log k)^{-2} \quad \text{if } n = 3.$$

Furthermore, there is a continuous dependence on δ on these operators.

5. Conclusion: Proof of Theorem 1

In this section we will adjust the parameter δ defined in (2.2), which we have so far left free between two uniform bounds (2.3), in such a way the constant $c_{n+1} = c_{n+1}(\delta)$ in problem (4.37)–(4.38) be equal to zero. Indeed this fact gives that the function

$$u(y) = U_*(y) + \phi(y)$$

is the solution to problem (1.1) predicted by Theorem 1. In the above formula we recall that U_* is the function defined in (2.4) and ϕ is the function defined by

$$\phi = \sum_{j=1}^{k} \phi_j + \psi$$

with ϕ_1 given by Proposition 4.1,

$$\phi_j(\bar{y}, y_3, \ldots, y_n) = \phi_1\big(e^{\frac{2\pi j}{k}i}\bar{y}, y_3, \ldots, y_n\big), \quad j = 2, \ldots, k$$

and ψ is given by Lemma 4.1.

Thus we need to choose δ such that

$$c_{n+1}(\delta) := \frac{\int_{\mathbb{R}^n}(\zeta_1 E + \gamma \mathcal{N}(\phi_1))\tilde{Z}_{n+1}}{\int_{\mathbb{R}^n} U_1^{p-1}\tilde{Z}_{n+1}^2} = 0.$$

This is equivalent to find δ such that

$$\hat{c}_{n+1}(\delta) := \int\limits_{\mathbb{R}^n} \big(\zeta_1 E + \gamma \mathcal{N}(\phi_1)\big)\tilde{Z}_{n+1} = 0.$$

In the rest of this section, with $\Theta_k(\delta)$ we will denote a generic continuous function of the variable δ, which is uniformly bounded as $k \to \infty$.

Let $n \geqslant 4$. We claim that

$$\hat{c}_{n+1}(\delta) = -A_n \frac{\delta}{k^{n-2}}[\delta a_n - 1] + \frac{1}{k^{n-1}}\Theta_k(\delta) \tag{5.1}$$

where

$$A_n = p\gamma \int\limits_{\mathbb{R}^N} U^{p-1}Z_{n+1}\,dx \quad \text{and} \quad a_n = 2^{\frac{n-2}{2}}\tilde{a}_n$$

with $\tilde{a}_n$ the positive number defined as

$$\lim_{k \to \infty} \frac{1}{k^{n-2}} \sum_{j=2}^{k} \frac{1}{|\hat{\xi}_1 - \hat{\xi}_j|^{n-2}} = \tilde{a}_n,$$

where $\hat{\xi}_1 = (1, 0, \ldots, 0)$ and $\hat{\xi}_j = e^{\frac{2\pi(j-1)}{k}i}\hat{\xi}_1$.

Let now $n = 3$. We claim that

$$\hat{c}_{n+1}(\delta) = -A_3 \frac{\delta}{k \log k}[\delta a_3 - 1] + \frac{1}{k^2(\log k)^2}\Theta_k(\delta) \tag{5.2}$$

where

$$A_3 = p\gamma \int\limits_{\mathbb{R}^N} U^{p-1}Z_4\,dx \quad \text{and} \quad a_3 = \sqrt{2}\tilde{a}_3$$

with

$$\lim_{k\to\infty} \frac{1}{k\log k} \sum_{j=2}^{k} \frac{1}{|\hat{\xi}_1 - \hat{\xi}_j|} = \tilde{a}_3 > 0.$$

We assume for the moment the validity of (5.1) and (5.2). By continuity, we have the existence of a positive δ solution to

$$\hat{c}_{n+1}(\delta) = 0.$$

Furthermore, $\delta = \frac{1}{a_n} + O(\frac{1}{k})$ if the dimensions are greater or equal to 4 and $\delta = \frac{1}{a_3} + O(\frac{1}{k\log k})$ if $n = 3$. This proves the existence of the solutions predicted by Theorem 1.

In order to prove the key estimates (5.1) and (5.2) we write

$$\hat{c}_{n+1}(\delta) = \int_{\mathbb{R}^n} E\tilde{Z}_{n+1} + \int_{\mathbb{R}^n} (\zeta_1 - 1)E\tilde{Z}_{n+1} + \int_{\mathbb{R}^n} \gamma \mathcal{N}(\Phi)\tilde{Z}_{n+1}.$$

Taking into account the definition of μ given in (2.2), it is easy to check that estimates (5.1) and (5.2) are direct consequences of the following claims, to be established below.

Claim 1.

$$\int_{\mathbb{R}^n} E\tilde{Z}_{n+1} = \begin{cases} -A_n \frac{\delta}{k^{n-2}}[\delta a_n - 1] + \frac{1}{k^{n-1}}\Theta_k(\delta) & \text{if } n \geqslant 4, \\ -A_3 \frac{\delta}{k\log k}[\delta a_3 - 1] + \frac{1}{k^2(\log k)^2}\Theta_k(\delta) & \text{if } n = 3, \end{cases} \tag{5.3}$$

where A_n and a_n are the positive constants defined in (5.1)–(5.2) and $\Theta_k(\delta)$ is a continuous function of δ, which is uniformly bounded as $k \to \infty$.

Claim 2. For k large

$$\int_{\mathbb{R}^n} (\zeta_1 - 1)E\tilde{Z}_{n+1} = \begin{cases} k^{1-n}\Theta_k(\delta) & \text{if } n \geqslant 4, \\ k^{-2}(\log k)^{-2}\Theta_k(\delta) & \text{if } n = 3, \end{cases} \tag{5.4}$$

where $\Theta_k(\delta)$ is a continuous function of the variable δ, which is uniformly bounded as $k \to \infty$.

Claim 3. For k large

$$\int_{\mathbb{R}^n} \mathcal{N}(\Phi)\tilde{Z}_{n+1} = \begin{cases} k^{-\frac{2n}{q}}\Theta_k(\delta) & \text{if } n \geqslant 4, \\ k^{-2}\log^{-2}k\Theta_k(\delta) & \text{if } n = 3, \end{cases} \tag{5.5}$$

where $\Theta_k(\delta)$ is a continuous function of δ, which is uniformly bounded as $k \to \infty$.

Claim 4. Assume $n \geqslant 5$. Estimate (5.5) can be improved as follows: for k large

$$\int_{\mathbb{R}^n} \mathcal{N}(\Phi)\tilde{Z}_{n+1} = k^{3-n-\frac{n}{q}}\Theta_k(\delta) \tag{5.6}$$

where $\Theta_k(\delta)$ is a continuous function of δ, which is uniformly bounded as $k \to \infty$.

Let us finally observe that an easy adaptation of the arguments used to prove estimates (5.3)–(5.6) also gives the expansion of the energy contained in (1.4) in the statement of Theorem 1, which we do not include in the paper.

The rest of this section is devoted to prove estimates (5.3)–(5.6).

Proof of Claim 1. Let $\eta > 0$ be a small number, independent of k. We write

$$\int_{\mathbb{R}^n} E \tilde{Z}_{n+1} = \int_{B_1} E \tilde{Z}_{n+1} + \int_{\mathbb{R}^n \setminus \bigcup_j B_j} E \tilde{Z}_{n+1} + \sum_{j \neq 1} \int_{B_j} E \tilde{Z}_{n+1}, \tag{5.7}$$

where $B_j = B(\xi_j, \frac{\eta}{k})$.

The main contribution in the above integral is given by the first term $\int_{B_1} E \tilde{Z}_{n+1}$. After scaling $x = \mu y + \xi_1$ and denoting $\tilde{E}_j(y) = \mu^{\frac{n+2}{2}} E(\xi_1 + \mu y)$, we get

$$\int_{B_1} E \tilde{Z}_{n+1} = \int_{B(0, \frac{\eta}{\mu k})} \tilde{E}_1(y) Z_{n+1}(y) \, dy.$$

In the region $|y| \leqslant \frac{\eta}{\mu k}$ we use the expansion (2.10) and we obtain

$$
\begin{aligned}
\int_{B(0, \frac{\eta}{\mu k})} \tilde{E}_1(y) Z_{n+1}(y) \, dy = {} & -\gamma p \sum_{j \neq 1} \int_{B(0, \frac{\eta}{\mu k})} U^{p-1} U\big(y - \mu^{-1}(\xi_j - \xi_1)\big) Z_{n+1} \\
& + \gamma p \mu^{\frac{n-2}{2}} \int_{B(0, \frac{\eta}{\mu k})} U^{p-1} U(\xi_1 + \mu y) Z_{n+1} \, dy \\
& + \gamma p \int_{B(0, \frac{\eta}{\mu k})} \big[(U(y) + sV)^{p-1} - U^{p-1} \big] V(y) Z_{n+1} \, dy \\
& + \gamma \sum_{i \neq 1} \int_{B(0, \frac{\eta}{\mu k})} U^p\big(y - \mu^{-1}(\xi_j - \xi_1)\big) Z_{n+1} \\
& - \mu^{\frac{n+2}{2}} \gamma \int_{B(0, \frac{\eta}{\mu k})} U^p(\xi_j + \mu y) Z_{n+1} \tag{5.8}
\end{aligned}
$$

where

$$V(y) = \left(-\sum_{j \neq 1} U\big(y - \mu^{-1}(\xi_j - \xi_1)\big) + \mu^{\frac{n-2}{2}} U(\xi_1 + \mu y) \right).$$

We see that, for $j \neq 1$,

$$\int_{B(0, \frac{\eta}{\mu k})} U^{p-1} U\big(y - \mu^{-1}(\xi_j - \xi_1)\big) Z_{n+1} = 2^{\frac{n-2}{2}} C_1 \mu^{n-2} \frac{1}{|\hat{\xi}_j - \hat{\xi}_1|^{n-2}} \big(1 + (\mu k)^2 \Theta_k(\delta)\big)$$

where $C_1 = \int_{\mathbb{R}^n} U^{p-1} Z_{n+1}$ and $\hat{\xi}_1 = (1, 0, \ldots, 0)$ and $\hat{\xi}_j = e^{\frac{2\pi(j-1)}{k}i}\hat{\xi}_1$. Furthermore, a Taylor expansion gives

$$\mu^{\frac{n-2}{2}} \int\limits_{B(0, \frac{\eta}{\mu k})} U^{p-1} U(\xi_1 + \mu y) Z_{n+1} \, dy = C_1 \mu^{\frac{n-2}{2}} \big(1 + (\mu k)^2 \Theta_k(\delta)\big).$$

On the other hand, the remaining terms in (5.8) are higher order. Indeed, we have

$$\left| \int\limits_{B(0, \frac{\eta}{\mu k})} \big[(U(y) + sV)^{p-1} - U^{p-1}\big] V(y) Z_{n+1} \, dy \right| \leqslant \left| \sum_{i \neq 1} \int\limits_{B(0, \frac{\eta}{\mu k})} U^p \big(y - \mu^{-1}(\xi_j - \xi_1)\big) Z_{n+1} \right|$$

$$\leqslant C \sum_{i \neq 1} \frac{\mu^{n+2}}{|\hat{\xi}_1 - \hat{\xi}_i|^{n+2}} \int\limits_{B(0, \frac{\eta}{\mu k})} \frac{1}{(1 + |y|)^{n-2}}$$

$$\leqslant C(\mu k)^{-2} \sum_{i \neq 1} \frac{\mu^{n+2}}{|\hat{\xi}_1 - \hat{\xi}_i|^{n+2}},$$

and

$$\left| \mu^{\frac{n+2}{2}} \gamma \int\limits_{B(0, \frac{\eta}{\mu k})} U^p (\xi_j + \mu y) Z_{n+1} \right| \leqslant C \mu^{\frac{n+2}{2}} \int\limits_{B(0, \frac{\eta}{\mu k})} \frac{1}{(1 + |y|)^{n-2}} \leqslant C \mu^{\frac{n-2}{2}} k^{-2}.$$

In the above estimates, C denotes a positive constant independent of k.

To estimate the second term in (5.7), by Holder inequality, we observe that

$$\left| \int\limits_{\mathbb{R}^n \backslash \bigcup_j B_j} E \tilde{Z}_{n+1} \right| \leqslant C \big\| (1 + |y|)^{n+2-\frac{2n}{q}} E \big\|_{L^q(\mathbb{R}^n \backslash \bigcup B_j)} \big\| (1 + |y|)^{-n-2+\frac{2n}{q}} \tilde{Z}_{n+1} \big\|_{L^{\frac{q}{q-1}}(\mathbb{R}^n \backslash \bigcup B_j)}.$$

From the definition of $\tilde{Z}_{n+1}$ in (4.29)

$$\big\| (1 + |y|)^{-n-2+\frac{2n}{q}} \tilde{Z}_{n+1} \big\|_{L^{\frac{q}{q-1}}(\mathbb{R}^n \backslash \bigcup B_j)} \leqslant C \mu^{\frac{n-2}{2}} \left(\int\limits_{\mathbb{R}^n \backslash \bigcup B_j} \left[\frac{|y - \xi_1|^{2/n}}{(1 + |y|)^{n+2-\frac{2n}{q}}} \right]^{\frac{q}{q-1}} \right)^{\frac{q-1}{q}}$$

$$\leqslant C \mu^{\frac{n-2}{2}} \left(\int\limits_{\frac{\eta}{k}}^1 \frac{t^{n-1}}{t^{(n-2)\frac{q}{q-1}}} \, dt \right)^{\frac{q-1}{q}} \leqslant C \mu^{\frac{n-2}{2}} k^{n-2} k^{-n\frac{q-1}{q}}.$$

Thus we conclude that

$$\left| \int\limits_{\mathbb{R}^n \backslash \bigcup_j B_j} E \tilde{Z}_{n+1} \right| \leqslant C \frac{\mu^{n-2} k^{2(n-2)}}{k^{n-1}} \tag{5.9}$$

since $\|(1 + |y|)^{n+2-\frac{2n}{q}} E\|_{L^q(\mathbb{R}^n \backslash \bigcup B_j)} \leqslant C \mu^{\frac{n-2}{2}} k^{n-2} k^{1-\frac{n}{q}}.$

Let us now fix $j \neq 1$ and denote $\tilde{E}_j(y) = \mu^{\frac{n+2}{2}} E(\xi_j + \mu y)$. Performing the change of variables $x = \mu y - \xi_j$,

$$\left| \int_{B_j} E \tilde{Z}_{n+1} \right| = \left| \mu^{\frac{n-2}{2}} \int_{B(0, \frac{\eta}{\mu k})} \tilde{E}_j(y) \tilde{Z}_{n+1}(\mu y + \xi_j) \, dy \right|$$

$$\leqslant C \mu^{\frac{n-2}{2}} \left\| \left(1 + |y|\right)^{n+2 - \frac{2n}{q}} \tilde{E}_j \right\|_{L^q(B(0, \frac{\eta}{\mu k}))}$$

$$\times \left\| \left(1 + |y|\right)^{-n+2 + \frac{2n}{q}} \mu^{-\frac{n-2}{2}} Z_{n+1}\left(y + \mu^{-1}(\xi_j - \xi_1)\right) \right\|_{L^{\frac{q}{q-1}}(B(0, \frac{\eta}{\mu k}))}.$$

Now

$$\left\| \left(1 + |y|\right)^{-n+2 + \frac{2n}{q}} \mu^{-\frac{n-2}{2}} Z_{n+1}\left(y + \mu^{-1}(\xi_j - \xi_1)\right) \right\|_{L^{\frac{q}{q-1}}(B(0, \frac{\eta}{\mu k}))}$$

$$\leqslant C \frac{\mu^{\frac{n-2}{2}}}{|\xi_j - \xi_1|^{n-2}} \left(\int_1^{\frac{\eta}{\mu k}} \frac{t^{n-1}}{t^{(n+2 - \frac{2n}{q}) \frac{q}{q-1}}} \, dt \right)^{\frac{q-1}{q}} \leqslant C \frac{\mu^{\frac{n-2}{2}}}{|\xi_j - \xi_1|^{n-2}} (\mu k)^{2 - \frac{n}{q}}.$$

Since in this region the error can be estimated as follows $\| (1 + |y|)^{n+2 - \frac{2n}{q}} \tilde{E}_j \|_{L^q(B(0, \frac{\eta}{\mu k}))} \leqslant C(\mu k)^{-n+2 + \frac{n}{q}}$, we conclude that

$$\left| \sum_{j \neq 1} \int_{B_j} E \tilde{Z}_{n+1} \right| \leqslant \frac{\mu^{\frac{n-2}{2}}}{(\mu k)^{n-4}} \left[\mu^{n-2} \sum_{j \neq 1} \frac{1}{|\xi_j - \xi_1|^{n-2}} \right].$$

Thus we have that

$$\int_{\mathbb{R}^n} E \tilde{Z}_{n+1} = -\gamma \, p C_1 \left[2^{\frac{n-2}{2}} \mu^{n-2} \sum_{j \neq 1} \frac{1}{|\hat{\xi}_j - \hat{\xi}_1|^{n-2}} - \mu^{\frac{n-2}{2}} \right]$$

$$+ \left[\frac{\mu^{n-2} k^{2(n-2)}}{k^{n-1}} + k^{-2} \mu^{n-2} \sum_{j \neq 1} \frac{1}{|\hat{\xi}_j - \hat{\xi}_1|^{n-2}} \right] \Theta_k(\delta),$$

where $C_1 = \int_{\mathbb{R}^n} U^{p-1} Z_{n+1}$ and Θ_k is a continuous function of the parameter δ, uniformly bounded as $k \to \infty$. This fact gives the validity of (5.3). $\quad \square$

Proof of Claim 2. We first observe that

$$\left| \int_{\mathbb{R}^n} (\zeta_1 - 1) E \tilde{Z}_{n+1} \right| \leqslant C \left| \int_{|x - \xi_1| > \frac{\eta}{k}} E \tilde{Z}_{n+1} \right|.$$

We now write

$$
\int\limits_{|x-\xi_1|>\frac{\eta}{k}} E\tilde{Z}_{n+1} = \left(\int\limits_{\bigcap_{j=1}^{k}\{|x-\xi_j|>\frac{\eta}{k}\}} + \sum_{j\neq 1} \int\limits_{|x-\xi_j|<\frac{\eta}{k}} \right) E\tilde{Z}_{n+1}.
$$

In the region $\bigcap_{j=1}^{k}\{|x-\xi_j|>\frac{\eta}{k}\}$, we already observed that

$$
\big|E(x)\big| \leqslant C \frac{\mu^{\frac{n-2}{2}}}{(1+|x|^2)^2} \sum_{j=1}^{k} \frac{1}{|x-\xi_j|^{n-2}},
$$

where C is a positive constant, independent of k. Furthermore, in this region, we have $\tilde{Z}_{n+1}(x) \leqslant C\frac{\mu^{\frac{n-2}{2}}}{|x-\xi_1|^{n-2}}$. Thus we can estimate

$$
\int\limits_{\bigcap_{j=1}^{k}\{|x-\xi_j|>\frac{\eta}{k}\}} E\tilde{Z}_{n+1} \leqslant Ck\mu^{n-2} \int\limits_{\frac{\eta}{k}}^{1} \frac{t^{n-1}}{t^{2n-4}}\,dt \leqslant Ck\mu^{n-2}k^{n-4}
$$

and we conclude that

$$
\int\limits_{\bigcap_{j=1}^{k}\{|x-\xi_j|>\frac{\eta}{k}\}} E\tilde{Z}_{n+1} =
\begin{cases}
\frac{1}{k^{n-1}}\Theta_k(\delta) & \text{if } n \geqslant 4, \\
\frac{1}{k^2(\log k)^2}\Theta_k(\delta) & \text{if } n = 3.
\end{cases}
\tag{5.10}
$$

On the other hand, performing the change of variables $\mu y = x - \xi_j$, we get

$$
\int\limits_{|x-\xi_j|\leqslant\frac{\eta}{k}} E\tilde{Z}_{n+1} = \mu^{\frac{n+2}{2}} \int\limits_{|y|\leqslant\frac{\eta}{k\mu}} E(\xi_j+\mu y)Z_{n+1}\big(y+\mu^{-1}(\xi_j-\xi_1)\big).
$$

We already observed that, for $|y| \leqslant \frac{\eta}{k\mu}$, the error $\mu^{\frac{n+2}{2}}|E(\xi_j+\mu y)| \leqslant C\frac{\mu^{\frac{n-2}{2}}}{(1+|y|^2)^2}$. Furthermore, in the same region, $|Z_{n+1}(y+\mu^{-1}(\xi_j-\xi_1))| \leqslant C\frac{\mu^{n-2}k^{n-2}}{|j-1|^{n-2}}$. Thus we have

$$
\left| \sum_{j\neq 1} \int\limits_{|x-\xi_j|<\frac{\eta}{k}} E\tilde{Z}_{n+1} \right| \leqslant Ck\mu^{\frac{n-2}{2}}(k\mu)^{n-2} \int\limits_{|y|<\frac{\eta}{k\mu}} \frac{1}{(1+|y|^2)^2}\,dy \leqslant C\frac{\mu^{\frac{n-2}{2}}}{k}.
$$

This last estimate, together with (5.10), concludes the proof of (5.4). $\quad\square$

Proof of Claim 3. Using the change of variables $x = \mu y + \xi_1$,

$$
\int\limits_{\mathbb{R}^n} \mathcal{N}(\phi_1)\tilde{Z}_{n+1}\,dx = \int\limits_{\mathbb{R}^n} \mathcal{N}(\phi_1)\mu^{-\frac{n-2}{2}} Z_{n+1}\big(\mu^{-1}(x-\xi_1)\big)\,dx
$$

$$
= \int\limits_{\mathbb{R}^n} \mu^{\frac{n+2}{2}} \mathcal{N}(\phi_1)(\mu y+\xi_1)Z_{n+1}(y)\,dy.
$$

Thus we have

$$\left| \int\limits_{\mathbb{R}^n} \mathcal{N}(\phi_1) \tilde{Z}_{n+1} \, dx \right| \leqslant C \left\| \mu^{\frac{n+2}{2}} \mathcal{N}(\phi_1)(\mu y + \xi_1) \right\|_{**} \left(\int\limits_{\mathbb{R}^n} \frac{1}{(1+|y|)^{2n}} \, dy \right)^{\frac{q-1}{q}}$$

$$\leqslant \begin{cases} Ck^{-\frac{2n}{q}} & \text{if } n \geqslant 4, \\ Ck^{-2} \log^{-2} k & \text{if } n = 3, \end{cases}$$

where C is a positive constant independent of k. □

Proof of Claim 4. Assume $n \geqslant 5$. It is convenient to decompose

$$\mathcal{N}(\phi_1) = \tilde{\mathcal{N}}(\phi_1) + N(\tilde{\phi}_1)$$

where

$$\tilde{\mathcal{N}}(\phi_1) = p \big(|U_*|^{p-1} \zeta_1 - U_1^{p-1} \big) \tilde{\phi}_1 + p \zeta_1 |U_*|^{p-1} \Psi(\phi_1)$$

$$+ N \bigg(\tilde{\phi}_1 + \sum_{j \neq 1} \tilde{\phi}_j + \Psi(\phi_1) \bigg) - N(\tilde{\phi}_1)$$

and

$$N(\tilde{\phi}_1) = |U_* + \tilde{\phi}_1|^{p-1}(U_* + \tilde{\phi}_1) - |U_*|^{p-1} U_* - p |U_*|^{p-1} \tilde{\phi}_1.$$

We have that

$$I := \int\limits_{\mathbb{R}^N} \tilde{\mathcal{N}}(\phi_1) \tilde{Z}_{n+1} = \mu^{\frac{n+2}{2}} \int\limits_{\mathbb{R}^N} \tilde{\mathcal{N}}(\phi_1)(\xi_1 + \mu x) Z_{n+1}(x) \, dx$$

so that, from the estimates found we readily check

$$|I| \leqslant Ck^{2-n} k^{1-\frac{n}{q}} \int\limits_{\mathbb{R}^n} U^{p-1} |Z_{n+1}|. \tag{5.11}$$

On the other hand, if we let

$$II := \int\limits_{\mathbb{R}^N} N(\tilde{\phi}_1) \tilde{Z}_{n+1}$$

we find that

$$|II| \leqslant \|\phi_1\|_* \int\limits_{\mathbb{R}^n} U^{p-1} |\phi_1| |Z_{n+1}|.$$

Now, we notice from Eq. (4.37), that we can write

$$L_0(\phi_1) + a(y)\phi_1 = g + \sum_\ell c_\ell U^{p-1} Z_\ell \quad \text{where } a(y) = \mu^{\frac{n+2}{2}} \gamma N(\tilde{\phi}_1)(\xi_1 + \mu y)$$

so that

$$|a(y)| \leqslant C U^{p-1} \|\phi_1\|_*, \quad \text{and} \quad |g(y)| \leqslant C \mu^{\frac{n-2}{2}} (1 + |y|)^{-4}.$$

Thus, applying Lemma 3.2 with $\nu = 4$, assuming that $n \geqslant 5$, we find

$$|\phi_1(y)| \leqslant C \mu^{\frac{n-2}{2}} (1 + |y|)^{-2}.$$

As a conclusion,

$$|II| \leqslant C \|\phi_1\|_* \mu^{\frac{n-2}{2}} \leqslant C k^{2-n-\frac{n}{q}}.$$

Combining this estimate with (5.11) we then find

$$\left| \int_{\mathbb{R}^N} \mathcal{N}(\phi_1) \tilde{Z}_{n+1} \right| \leqslant C k^{2-n} k^{1-\frac{n}{q}}$$

and relation (5.6) is established. $\quad \Box$

References

[1] T. Aubin, Problèmes isopérimétriques et espaces de Sobolev, J. Differential Geom. 11 (1976) 573–598.
[2] A. Bahri, S. Chanillo, The difference of topology at infinity in changing-sign Yamabe problems on $S^3 S3$ (the case of two masses), Comm. Pure Appl. Math. 54 (4) (2001) 450–478.
[3] A. Bahri, Y. Xu, Recent Progress in Conformal Geometry, ICP Adv. Texts Math., vol. 1, Imperial College Press, London, 2007.
[4] T. Bartsch, M. Willem, Infinitely many nonradial solutions of a Euclidean scalar field equation, J. Funct. Anal. 117 (1993) 447–460.
[5] T. Bartsch, M. Willem, Infinitely many radial solutions of a semilinear elliptic problem on $\mathbb{R}^N$, Arch. Ration. Mech. Anal. 124 (1993) 261–276.
[6] L. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behaviour of semilinear elliptic equations with critical Sobolev growth, Comm. Pure Appl. Math. 42 (1989) 271–297.
[7] M. del Pino, M. Musso, F. Pacard, A. Pistoia, Torus action on S^n and sign changing solutions for conformally invariant equations, preprint, 2010.
[8] W. Ding, On a conformally invariant elliptic equation on $\mathbb{R}^n$, Comm. Math. Phys. 107 (1986) 331–335.
[9] E. Hebey, Introduction à l'analyse non linéaire sur les variétées, Diderot éditeur, 1997.
[10] E. Hebey, M. Vaugon, Existence and multiplicity of nodal solutions for nonlinear elliptic equations with critical Sobolev growth, J. Funct. Anal. 119 (2) (1994) 298–318.
[11] C.E. Kenig, F. Merle, Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case, Invent. Math. 166 (3) (2006) 645–675.
[12] C.E. Kenig, F. Merle, Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation, Acta Math. 201 (2) (2008) 147–212.
[13] Y.-Y. Li, J. Wei, H. Xu, Multibump solutions for $-\Delta u = K(x) u^{\frac{n+2}{n-2}}$ on lattices in $\mathbb{R}^n$, preprint, 2010.

[14] M. Obata, Conformal changes of Riemannian metrics on a Euclidean sphere, in: Differential Geometry (in Honor of Kentaro Yano), Kinokuniya, Tokyo, 1972, pp. 347–353.
[15] S. Pohozaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Soviet. Math. Dokl. 6 (1965) 1408–1411.
[16] R. Schoen, S.T. Yau, Lectures on differential geometry, in: Conference Proceedings and Lecture Notes in Geometry and Topology, I, International Press, Cambridge, MA, 1994.
[17] G. Talenti, Best constants in Sobolev inequality, Ann. Math. 10 (1976) 353–372.
[18] J. Wei, S. Yan, Infinitely many solutions for the prescribed scalar curvature problem on S^N, preprint, 2009.