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TWO-DIMENSIONAL EULER FLOWS WITH CONCENTRATED VORTICITIES

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ABSTRACT. For a planar model of Euler flows proposed by Tur and Yanovsky (2004), we construct a family of velocity fields $\mathbf{w}_\varepsilon$ for a fluid in a bounded region Ω, with concentrated vorticities ω_ε for $\varepsilon > 0$ small. More precisely, given a positive integer α and a sufficiently small complex number a, we find a family of stream functions ψ_ε which solve the Liouville equation with Dirac mass source,

$$\Delta\psi_\varepsilon + \varepsilon^2 e^{\psi_\varepsilon} = 4\pi\alpha\delta_{p_{a,\varepsilon}} \quad \text{in } \Omega, \quad \psi_\varepsilon = 0 \quad \text{on } \partial\Omega,$$

for a suitable point $p = p_{a,\varepsilon} \in \Omega$. The vorticities $\omega_\varepsilon := -\Delta\psi_\varepsilon$ concentrate in the sense that

$$\omega_\varepsilon + 4\pi\alpha\delta_{p_{a,\varepsilon}} - 8\pi\sum_{j=1}^{\alpha+1}\delta_{p_{a,\varepsilon}+a_j} \rightharpoonup 0 \quad \text{as } \varepsilon \to 0,$$

where the *satellites* $a_1, \dots, a_{\alpha+1}$ denote the complex $(\alpha+1)$-roots of a. The point $p_{a,\varepsilon}$ lies close to a zero point of a vector field explicitly built upon derivatives of order $\leq \alpha + 1$ of the regular part of Green's function of the domain.

1. Introduction and statement of main results

We are concerned with stationary Euler equations for an incompressible and homogeneous fluid,

$$\begin{cases} (\mathbf{w}\cdot\nabla)\,\mathbf{w} + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div}\,\mathbf{w} = 0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $\mathbf{w}$ is the velocity field and p is the pressure. The domain Ω is either the whole $\mathbb{R}^n$, $n = 2, 3$, or a smooth, bounded domain Ω. In the latter situation the velocity field $\mathbf{w}$ is naturally required to be tangent on $\partial\Omega$, that is,

$$\mathbf{w}\cdot\nu = 0 \qquad \text{on } \partial\Omega, \tag{2}$$

ν being a unit normal vector to $\partial\Omega$. We shall restrict our investigation to the planar case $n = 2$ when Ω is bounded and introduce the vorticity $\omega = \operatorname{curl}\,\mathbf{w}$. By applying

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the curl operator to the first equation in (1), we are reduced to the Euler equations in vorticity form,

$$\begin{cases} \mathbf{w} \cdot \nabla \omega = 0 & \text{in } \Omega, \\ \operatorname{div} \mathbf{w} = 0 & \text{in } \Omega, \end{cases} \tag{3}$$

supplemented by (2). We refer the reader to the books [25] and [23] for an exhaustive treatment of fluid mechanics models.

Let us further rewrite Problem (3). The second equation in (3) is equivalent to rewriting the velocity field $\mathbf{w}$ as

$$\mathbf{w} = (\partial_{x_2}\psi, -\partial_{x_1}\psi).$$

In turn, the vorticity ω expresses as $\omega = -\Delta\psi$ in terms of ψ, the so-called stream function.

Now the *ansatz* $\omega = f(\psi)$ guarantees that the first equation in (3) is also automatically satisfied, and then the Euler equations reduce to solving the elliptic problem

$$\Delta\psi + f(\psi) = 0, \tag{4}$$

with Dirichlet boundary condition $\psi = 0$ on $\partial\Omega$ on a bounded domain Ω to account for (2).

Many choices of f are physically meaningful and lead to vortex-type configurations. The Stuart vortex pattern in [30] corresponds to $f(\psi) = \varepsilon^2 e^{\psi}$. Tur and Yanovsky have recently proposed in [35] a singular ansatz

$$f(\psi) = \varepsilon^2 e^{\psi} - 4\pi\alpha\delta_p, \quad \alpha \in \mathbb{N},$$

to describe vortex patterns of necklace type with $(\alpha+1)$-fold symmetry in rotational shear flow. Both papers [30, 35] consider Problem (4) in the whole $\mathbb{R}^2$ and explicit solutions are easily built according to Liouville's formula below. On a bounded domain Ω, a statistical mechanics approach in [5, 6, 20] has provided a rigorous derivation of *Stuart's ansatz*.

In this paper we consider the Tur-Yanovsky problem on a bounded domain Ω, a much harder issue to pursue. In terms of the stream function ψ we are thus led to the singular Liouville equation

$$\begin{cases} \Delta\psi + \varepsilon^2 e^{\psi} = 4\pi\alpha\delta_p & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

The parameter $\varepsilon > 0$ is small and, as we will see, its smallness will yield to flows having vorticities ω concentrated on small "blobs".

Liouville-type equations arise also in several superconductivity theories in the self-dual regime, as for the abelian Maxwell-Higgs and Chern-Simons-Higgs theories. In the latter model, a mean field form of Problem (5) on the torus arises as a limiting equation for nontopological condensates as the Chern-Simons parameter tends to zero as shown in [28, 32]. Problem (5) is a limiting model equation in this context and explains why it has attracted a lot of attention, as we describe precisely below.

In a superconducting sample Ω a Dirichlet boundary condition φ can be imposed and the homogeneous case $\varphi = 0$, discussed in [18], is especially interesting since it describes a perfectly superconductive regime on $\partial\Omega$.

The regular case $\alpha = 0$ in Problem (5), sometimes referred to as the Gelfand problem [16],

$$\begin{cases} \Delta\psi + \varepsilon^2 e^{\psi} = 0 & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

has been broadly considered in the literature. When $\varepsilon > 0$ is sufficiently small, the existence of both a small and a large solution as $\varepsilon \to 0$ has long been known as first found in [11, 19]. In the language of the calculus of variations, applied to the functional

$$J(\psi) = \frac{1}{2}\int_\Omega |\nabla\psi|^2 - \varepsilon^2 \int_\Omega e^{\psi}, \quad \psi \in H_0^1(\Omega),$$

the small solution corresponds to a local minimizer, while the large solution is a mountain pass critical point. The blowing-up behavior of a large solution was first described in [36] when Ω is simply connected. In the general case, the analysis in the works [4, 22, 26, 27, 31] yields that, if ψ_ε is a family of blowing-up solutions of (6) with $\varepsilon^2 \int_\Omega e^{\psi_\varepsilon}$ uniformly bounded, then, up to subsequences, there is an integer $k \geq 1$ such that $\varepsilon^2 \int_\Omega e^{\psi_\varepsilon} \to 8\pi k$. Moreover, there are points $\xi_1^\varepsilon, \ldots, \xi_k^\varepsilon$ in Ω, which remain away one from each other and away from $\partial\Omega$, such that

$$\varepsilon^2 e^{\psi_\varepsilon} - \sum_{i=1}^{k} 8\pi\delta_{\xi_i^\varepsilon} \rightharpoonup 0 \quad \text{as } \varepsilon \to 0 \tag{7}$$

in the measure sense. Besides,

$$\nabla\varphi_k(\xi_1^\varepsilon, \ldots, \xi_k^\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0, \tag{8}$$

where

$$\varphi_k(\xi_1, \ldots, \xi_k) := \begin{cases} H(\xi_1, \xi_1) & \text{if } k = 1, \\ \sum_{i=1}^k H(\xi_i, \xi_i) + \sum_{i \neq j} G(\xi_i, \xi_j) & \text{if } k \neq 1. \end{cases}$$

Here $G(z, \xi)$ denotes the Dirichlet Green's function in Ω, namely

$$G(z, \xi) = \Gamma(z - \xi) + H(z, \xi), \quad \Gamma(z) := \frac{1}{2\pi}\log\frac{1}{|z|},$$

where $H(z, \xi)$, its regular part, satisfies

$$\begin{cases} \Delta_z H(\cdot, \xi) = 0 & \text{in } \Omega, \\ H(\cdot, \xi) = -\Gamma(\cdot - \xi) & \text{on } \partial\Omega. \end{cases} \tag{9}$$

Moreover, refined asymptotic estimates hold, as established in a general setting in [7, 21], which lead to the computation of the Leray-Schauder degree of an associated nonlinear operator in [8]. In particular, ψ_ε satisfies

$$\psi_\varepsilon(z) = 8\pi \sum_{i=1}^{k} G(z, \xi_i^\varepsilon) + o(1), \quad \psi_\varepsilon(z) = \log \frac{8\mu_i^2}{(\mu_i^2\varepsilon^2 + |z - \xi_i^\varepsilon|^2)^2} + o(1), \tag{10}$$

respectively, away from all points ξ_i^ε, and around each of them, for some $\mu_i > 0$. The family of functions

$$w_{\varepsilon,\mu,\xi}(z) := \log \frac{8\mu^2}{(\mu^2\varepsilon^2 + |z - \xi|^2)^2} \tag{11}$$

corresponds to all solutions of the equation

$$\Delta w + \varepsilon^2 e^w = 0 \quad \text{in } \mathbb{R}^2,$$

such that $\int_{\mathbb{R}^2} e^w < +\infty$; see [9]. Besides,

$$\varepsilon^2 \int_{\mathbb{R}^2} e^{w_{\varepsilon,\mu,\xi}} = 8\pi, \qquad \varepsilon^2 e^{w_{\varepsilon,\mu,\xi}} \rightharpoonup 8\pi\delta_\xi \quad \text{as } \varepsilon \to 0.$$

The reciprocal issue, namely the existence of solutions with the above properties, has been addressed in [1], near nondegenerate critical points of φ_k, and in [12, 15] associated to topologically nontrivial critical point situations for φ_k. In particular, it is found in [12] that ψ_ε as above exists for any $k \geq 1$ provided that the domain is not simply connected.

Important progress in the understanding of blowing-up solutions of Problem (5) with $\alpha > 0$ has been achieved in the local analysis in the works [2, 3, 34] from the point of view of quantization of blow-up levels and Harnack-type estimates. See [33] for a complete account on the topic.

Concerning construction of solutions to (5) with $\alpha > 0$, only a few results are available. The first important remark is that the functions

$$w_{\varepsilon,\mu}(z) = -4\pi\alpha\Gamma(z-p) + \log \frac{8\mu^2(1+\alpha)^2}{(\mu^2\varepsilon^2 + |z-p|^{2+2\alpha})^2} \tag{12}$$

satisfy precisely

(13)

$$\Delta w + \varepsilon^2 e^w = 4\pi\alpha\delta_p, \quad \varepsilon^2 \int_{\mathbb{R}^2} e^{w_{\varepsilon,\mu}} = 8\pi(1+\alpha), \quad \varepsilon^2 e^{w_{\varepsilon,\mu}} \rightharpoonup 8\pi(1+\alpha)\delta_p \text{ as } \varepsilon \to 0.$$

When $\alpha \in \mathbb{N}$, an integer, the family above extends to one carrying an extra parameter a which plays a similar role as ξ in (11): in complex notation, all functions

$$w_{\varepsilon,\mu,a}(z) = -4\pi\alpha\Gamma(z-p) + \log \frac{8\mu^2(1+\alpha)^2}{(\mu^2\varepsilon^2 + |(z-p)^{\alpha+1} - a|^2)^2} \tag{14}$$

satisfy (13) with the third property replaced by

$$\varepsilon^2 e^{w_{\varepsilon,\mu,a}} \rightharpoonup 8\pi \sum_{j=1}^{\alpha+1} \delta_{p+a_j} \quad \text{as } \varepsilon \to 0, \tag{15}$$

where the a_j's are the complex $(\alpha+1)$-roots of a.

The difference between the cases $\alpha \in \mathbb{N}$ and $\alpha \notin \mathbb{N}$ is not just cosmetics but analytically essential. In the latter case, a suitable form of nondegeneracy up to dilations holds for the solutions (12), which allows in [13, 14] the construction of solutions to (5) so that away from p,

(16)

$$\psi_\varepsilon(z) = -4\pi\alpha G(z,p) + 8\pi(1+\alpha)G(z,p) + o(1) \text{ and } \varepsilon^2 e^{\psi_\varepsilon} \rightharpoonup 8\pi(1+\alpha)\delta_p \text{ as } \varepsilon \to 0.$$

On the other hand, solutions with concentration points away from p have been built in [12], regardless whether or not α is an integer. Whenever $k < 1+\alpha$, there is a solution ψ_ε of (5) and, up to a subsequence, k points $\xi_j^\varepsilon \in \Omega \setminus \{p\}$ such that away from them,

(17)

$$\psi_\varepsilon(z) = -4\pi\alpha G(z,p) + 8\pi \sum_{j=1}^{k} G(z,\xi_j^\varepsilon) + o(1) \text{ and } \varepsilon^2 e^{\psi_\varepsilon} - 8\pi \sum_{j=1}^{k} \delta_{\xi_j^\varepsilon} \rightharpoonup 0 \text{ as } \varepsilon \to 0.$$

A natural, unsettled question for which the methods in [12, 14] fail is whether in case $\alpha \in \mathbb{N}$, solutions ψ_ε with property (16), or for $k = \alpha + 1$ with property (17) exist. Expressions (14) and (15) suggest that both scenarios may be possible, using these functions, suitably corrected, as approximations of a solution of (5).

The case $\alpha \in \mathbb{N}$ in Problem (5) is more difficult, and at the same time the most relevant to physical applications. Our main result states that both of the above situations do take place; however, the location of the vortex p does not seem possible to be prescribed in an arbitrary way as in [13, 14] in case $\alpha \notin \mathbb{N}$ but rather, as in the situation of standard type II superconductivity, it is determined by the geometry of the region and boundary conditions in an ε-dependent way.

Let us assume that Ω is simply connected and consider a holomorphic extension

$$h_p(z) = H(z,p) - H(p,p) + i\tilde{H}(z,p) \quad \text{in } \Omega$$

of the harmonic function $H(z,p) - H(p,p)$ so that $h_p(p) = 0$. This means $\tilde{H}(p,p) = 0$ and

$$\tilde{H}_{z_1}(z,p) = H_{z_2}(z,p), \quad \tilde{H}_{z_2}(z,p) = -H_{z_1}(z,p) \quad \text{for all } z \in \Omega.$$

Then we let

$$\Psi(p) = \frac{d^{\alpha+1}}{dz^{\alpha+1}} \left(e^{2\pi(\alpha+2)h_p(z)} \right)(p).$$

Our main result for problem (5) is stated as follows.

Theorem 1.1. *Assume that $\alpha \in \mathbb{N}$ and that Ω is simply connected. Then there exists $\delta > 0$ such that for each a with $|a| \le \delta$, there is a point $p_{a,\varepsilon} \in \Omega$ away from $\partial\Omega$ and a flow described by ψ_ε, the solution of*

$$\begin{cases} \Delta\psi_\varepsilon + \varepsilon^2 e^{\psi_\varepsilon} = 4\pi\alpha\delta_{p_{a\varepsilon}} & \text{in } \Omega, \\ \psi_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

for which the associated vorticity $\omega_\varepsilon := \varepsilon^2 e^{\psi_\varepsilon} - 4\pi\alpha\delta_{p_{a,\varepsilon}}$ is concentrated in the sense that

(18)
$$\omega_\varepsilon + 4\pi\alpha\delta_{p_{a,\varepsilon}} - 8\pi \sum_{j=1}^{\alpha+1} \delta_{p_{a,\varepsilon}+a_j} \rightharpoonup 0 \quad \text{as } \varepsilon \to 0,$$

where the a_j's are the complex $(\alpha+1)$-roots of a. Besides we have

$$\Psi(p_{a,\varepsilon}) \to 0 \quad \text{as } |a| + \varepsilon \to 0.$$

For $\alpha \ge 1$, we observe that $\alpha + 1$ vertices of any sufficiently tiny regular polygon can be allocated with a suitable center to yield these vertices as an asymptotic concentration set. The solution predicted satisfies

$$\psi_\varepsilon(z) = -4\pi\alpha G(z, p_{a,\varepsilon}) + 8\pi \sum_{j=1}^{\alpha+1} G(z, p_{a,\varepsilon} + a_j) + o(1) \quad \text{as } \varepsilon \to 0$$

away from the concentration points. Let us observe that this result recovers the one known for $\alpha = 0$, where $p_{a,\varepsilon} = \xi^\varepsilon - a$ and where ξ^ε, the point of concentration, approaches a critical point of the Robin's function. Let us notice that indeed, for $\alpha = 0$,

$$\Psi(p) = 0 \iff \nabla_z H(p,p) = 0.$$

In general, the condition $\Psi(p) = 0$ involves derivatives of orders up to $\alpha + 1$ of the function $H(z,p)$ at $z = p$. Notice also that the function $\Psi(p)$ is well-defined even if Ω is not simply connected. We suspect that this requirement may be lifted but our proof uses this fact. Theorem 1.1 will be a consequence of a more general result, which states that, associated to any region $\Lambda \subset \Omega$ for which $\deg(\Psi, \Lambda, 0) \neq 0$, a solution as in Theorem 1.1 exists with concentration points inside Λ. The proof applies with just slight changes to the case of nonzero boundary data φ.

2. A more general result

In what follows we assume that Ω is a simply connected bounded domain. Theorem 1.1 will be a consequence of a more general result concerning the Dirichlet problem

$$\begin{cases} \Delta u + \varepsilon^2 e^u = 4\pi N \delta_p & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{19}$$

where φ is a smooth function and $N \geq 1$ is an integer. Let Φ be the harmonic extension of φ to all of Ω. The substitution

$$v(z) := u(z) + 4\pi N G(z,p) - \Phi(z)$$

transforms Problem (19) into the (regular) boundary value problem

$$\begin{cases} \Delta v + \varepsilon^2 |z-p|^{2N} e^{2K(z)} e^v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{20}$$

where

$$2K(z) = \Phi(z) - 4\pi N H(z,p).$$

The homogeneous case $\varphi = 0$ corresponds to simply having $K(z) = -2\pi N H(z,p)$.

In what follows, we identify $(z_1, z_2) \in \mathbb{R}^2$ and $z = z_1 + iz_2 \in \mathbb{C}$. We can associate to (20) a limiting problem of Liouville type, which will play a crucial role in the construction of solutions blowing up at p as $\varepsilon \to 0^+$:

$$\begin{cases} \Delta V + \varepsilon^2 |z-p|^{2N} e^V = 0 \text{ in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} |z-p|^{2N} e^V < \infty. \end{cases} \tag{21}$$

Let us recall Liouville's formula [24]. Given a holomorphic function f on $\mathbb{C}$, the function

$$\ln \frac{8|f'(z)|^2}{(1+|f(z)|^2)^2} - 2\ln\varepsilon \tag{22}$$

solves the equation $\Delta V + \varepsilon^2 e^V = 0$ in the set $\{z \in \mathbb{C} \mid f'(z) \neq 0\}$. We can allow $f(z)$ to have simple poles since the Liouville formula (22) still makes sense.

If now f' vanishes only at the point p with multiplicity N, the function

$$\ln \frac{8|f'(z)|^2}{(1+|f(z)|^2)^2} - 2\ln\varepsilon - \ln|z-p|^{2N} \tag{23}$$

solves the equation in (21) but possibly does not have energy $\int_{\mathbb{R}^2} |z-p|^{2N} e^V$ finite. The finite energy condition in the entire space forces the choice

$$f(z) = \frac{(z-p)^{N+1} - a}{\mu\varepsilon}, \quad \mu > 0,\ a \in \mathbb{C},$$

which leads exactly to the following three-parameter family of solutions to (21):

$$\tilde{V}_a(z) = \ln \frac{8(N+1)^2\mu^2}{(\mu^2\varepsilon^2 + |(z-p)^{N+1} - a|^2)^2}\,, \quad z \in \mathbb{C}; \tag{24}$$

see [10, 29]. Let us note that $\tilde{V}_a(z)$ is the regular part of $w_{\varepsilon,\mu,a}$ as defined in (14). For $a = |a|e^{i\theta} \in \mathbb{C}$, we consider the $(N+1)$-roots of a,

$$a_j = |a|^{\frac{1}{N+1}} e^{i\frac{\theta}{N+1} + \frac{2\pi i}{N+1}(j-1)}, \quad j = 1, \dots, N+1,$$

where i is the imaginary unit in $\mathbb{C}$.

The function $\tilde{V}_a(z)$ solves the PDE in (20) with $K=0$ but does not have the right boundary condition. By the Liouville formula, we will choose more carefully the function $f(z)$ to achieve the Dirichlet boundary condition and include the potential $e^{2K(z)}$ in the equation. Our approach is related to that by Weston in [36] for $N=0$. Nonetheless, the assumptions in [36] on the Riemann conformal map of Ω onto the unit disc are shown here to be unnecessary, and our improved approach is also essential to deal with a nonzero $N \in \mathbb{N}$.

We explain below how to choose $f(z)$, and the simply connectedness of Ω will be crucial. The details of our construction will be presented in Section 3.

Let $Q \in C(\bar{\Omega}, \mathbb{C})$ be a holomorphic function in Ω so that

$$Q(p) = 1, \qquad Q(z) - \frac{z-p}{N+1} Q'(z) \neq 0 \quad \text{in } \Omega.$$

Since, as already observed, f can have simple poles, let us take

$$f(z) = \frac{(z-p)^{N+1} Q^{-1}(z) - a}{\mu\varepsilon}$$

in Liouville's formula (23) in order to get a solution

$$V_a(z) = \ln \frac{8(N+1)^2\mu^2 |Q(z) - \frac{z-p}{N+1} Q'(z)|^2}{(\mu^2\varepsilon^2 |Q(z)|^2 + |(z-p)^{N+1} - aQ(z)|^2)^2}$$

of the equation

$$-\Delta v = \varepsilon^2 |z-p|^{2N} e^v \quad \text{in } \Omega.$$

Given the projection operator $P : H^1(\Omega) \to H_0^1(\Omega)$, we will search for Q and μ (depending on a and $\varepsilon > 0$ small) such that $PV_a = V_a - 2K$. The function PV_a will then be a solution of (20).

Since $Q - \frac{z-p}{N+1} Q' \neq 0$ in Ω, the harmonic function

$$R_{\mu,Q,a}(z) = \frac{1}{2}\left[PV_a - V_a + \ln\left(8(N+1)^2\mu^2\right) + \ln|Q(z) - \frac{z-p}{N+1}Q'(z)|^2 - 8\pi H_{Q,a}(z)\right]$$

satisfies by the Maximum Principle $R_{\mu,Q,a} = O(\varepsilon^2)$ uniformly in Ω, where $H_{Q,a}(z)$ is the solution of

$$\begin{cases} \Delta H = 0 & \text{in } \Omega, \\ H(z) = \frac{1}{2\pi} \ln|(z-p)^{N+1} - aQ(z)| & \text{on } \partial\Omega. \end{cases}$$

Observe that

$$H_{Q,0}(z) = (N+1)H(z,p),$$

where $H(z,p)$ is the regular part of the Green's function at p. Since Ω is a simply connected domain, let $h_{Q,a}(z)$, $r_{\mu,Q,a}(z)$ and $k(z)$ be the holomorphic extensions in Ω of the harmonic functions $H_{Q,a}(z) - H_{Q,a}(p)$, $R_{\mu,Q,a}(z) - R_{\mu,Q,a}(p)$ and $K(z) - K(p)$, respectively, so that $h_{Q,a}(p) = r_{\mu,Q,a}(p) = k(p) = 0$. Set

$$c_{\mu,Q,a} = \frac{1}{(N+1)!} \frac{d^{N+1}}{dz^{N+1}} \left(e^{4\pi h_{Q,a}(z) + r_{\mu,Q,a}(z) + k(z)} \right)(p).$$

For a and $\varepsilon > 0$ small, in Section 3 we will find a solution $Q_{\mu,a}$ of the equation:

$$\begin{aligned} &-\ln |Q(z) - \frac{z-p}{N+1} Q'(z)| + 4\pi H_{Q,a}(z) + R_{\mu,Q,a}(z) + K(z) \\ &\quad = 4\pi H_{Q,a}(p) + R_{\mu,Q,a}(p) + K(p) + \text{Re } \left(c_{\mu,Q,a}(z-p)^{N+1} \right) \end{aligned} \tag{25}$$

in Ω, where Re stands for the real part of a complex number. Next, we will solve

$$\mu = \frac{e^{4\pi H_{Q,a}(p) + R_{\mu,Q,a}(p) + K(p)}}{\sqrt{8}(N+1)} \Big|_{Q=Q_{\mu,a}} \tag{26}$$

for some μ_a, where a and $\varepsilon > 0$ are small parameters. Setting $Q_a = Q_{\mu_a,a}$, the function

$$V_a(z) = \ln \frac{8(N+1)^2 \mu_a^2 |Q_a(z) - \frac{z-p}{N+1} Q_a'(z)|^2}{(\mu_a^2 \varepsilon^2 |Q_a(z)|^2 + |(z-p)^{N+1} - aQ_a(z)|^2)^2}$$

satisfies:

$$PV_a = V_a - 2K(z) + 2\text{Re } \left(c_a (z-p)^{N+1} \right) \qquad \text{in } \Omega, \tag{27}$$

where $c_a := c_{\mu_a,Q_a,a}$.

As a conclusion, by means of (27), Problem (20) reduces to solving the equation

$$c_a(p) = \frac{1}{(N+1)!} \frac{d^{N+1}}{dz^{N+1}} \left(e^{4\pi h_{Q_a,a}(z) + r_{\mu_a,Q_a,a}(z) + k(z)} \right)(p) = 0.$$

This implies that every "stable" isolated zero point $p_0 \in \Omega$ for the map

$$\Psi(p) := \frac{d^{N+1}}{dz^{N+1}} \left(e^{4\pi(N+1)h_p(z) + k(z)} \right)(p),$$

in the sense that Ψ has a nonzero local index at p_0, will provide us with points $p_{a,\varepsilon}$, for a and $\varepsilon > 0$ small, so that $c_a = 0$. More generally, we have the validity of the following result:

Theorem 2.1. *Let $\Lambda \Subset \Omega$ be a region such that $\deg(\Psi, \Lambda, 0) \neq 0$. Then there exists $\delta > 0$ such that for each a with $|a| \leq \delta$, there is a point $p_{a,\varepsilon} \in \Lambda$ so that Problem (20) for $p = p_{a,\varepsilon}$ has a solution v_ε with*

$$\varepsilon^2 |z - p_{a,\varepsilon}|^{2N} e^{2K(z)} e^{v_\varepsilon} - 8\pi \sum_{j=1}^{N+1} \delta_{p_{a,\varepsilon} + a_j} \rightharpoonup 0 \quad \text{as } \varepsilon \to 0, \tag{28}$$

where the a_j's are the complex $(N+1)$-roots of a. Besides we have

$$\Psi(p_{a,\varepsilon}) \to 0 \quad \text{as } |a| + \varepsilon \to 0.$$

Let us stress that if p_0 is a stable isolated zero of Ψ, then we can obtain solutions as above with $p_{a,\varepsilon} \to p_0$ as $|a| + \varepsilon \to 0$.

In what remains of this paper we will prove Theorem 2.1 with the scheme described above and deduce Theorem 1.1 as a corollary.

3. The reduction to $c_a = 0$

Let $Q \in C(\bar{\Omega}, \mathbb{C})$ be a holomorphic function in Ω so that $Q(p) = 1$ and

$$Q(z) - \frac{z-p}{N+1}Q'(z) \neq 0 \quad \text{for all } z \in \Omega.$$

Let

$$V_a(z) = \ln \frac{8(N+1)^2\mu^2|Q(z) - \frac{z-p}{N+1}Q'(z)|^2}{(\mu^2\varepsilon^2|Q(z)|^2 + |(z-p)^{N+1} - aQ(z)|^2)^2}, \tag{29}$$

and denote by PV_a the projection of V_a onto the space $H_0^1(\Omega)$. Namely PV_a is the unique solution of

$$\begin{cases} -\Delta PV_a = -\Delta V_a = \varepsilon^2|z-p|^{2N}e^{V_a} & \text{in } \Omega, \\ PV_a = 0 & \text{on } \partial\Omega. \end{cases}$$

For a small, let $H_{Q,a}(z)$ be the solution of

$$\begin{cases} \Delta H = 0 & \text{in } \Omega, \\ H(z) = \frac{1}{2\pi}\ln|(z-p)^{N+1} - aQ(z)| & \text{on } \partial\Omega. \end{cases} \tag{30}$$

Since we assume $Q - \frac{z-p}{N+1}Q' \neq 0$ in Ω, the function $\ln|Q(z) - \frac{z-p}{N+1}Q'(z)|$ is harmonic in Ω. Since the harmonic function

$$\begin{aligned} R_{\mu,Q,a}(z) =& \frac{1}{2}\Big[PV_a(z) - V_a(z) + \ln(8(N+1)^2\mu^2) + \ln|Q(z) \\ &- \frac{z-p}{N+1}Q'(z)|^2 - 8\pi H_{Q,a}(z)\Big] \end{aligned}$$

satisfies $R_{\mu,Q,a} = O(\varepsilon^2)$ uniformly on $\partial\Omega$ as $\varepsilon \to 0$ (together with any boundary derivatives), by elliptic estimates (see [17]) we get that

(31)

$$PV_a = V_a - \ln\left(8(N+1)^2\mu^2\right) - \ln|Q(z) - \frac{z-p}{N+1}Q'(z)|^2 + 8\pi H_{Q,a}(z) + 2R_{\mu,Q,a}(z),$$
$$R_{\mu,Q,a} = O(\varepsilon^2)$$

in $C^1(\bar{\Omega})$. Let $h_{Q,a}(z)$, $r_{\mu,Q,a}(z)$ and $k(z)$ be the holomorphic extensions in Ω of the harmonic functions $H_{Q,a}(z) - H_{Q,a}(p)$, $R_{\mu,Q,a}(z) - R_{\mu,Q,a}(p)$ and $K(z) - K(p)$ so that $h_{Q,a}(p) = r_{\mu,Q,a}(p) = k(p) = 0$. Set

$$c_{\mu,Q,a} = \frac{1}{(N+1)!}\frac{d^{N+1}}{dz^{N+1}}\left(e^{4\pi h_{Q,a}(z) + r_{\mu,Q,a}(z) + k(z)}\right)(p).$$

It suffices for equation (25) to hold true that $Q(z)$ satisfies the equation

$$Q(z) - \frac{z-p}{N+1}Q'(z) = \exp\left(4\pi h_{Q,a}(z) + r_{\mu,Q,a}(z) + k(z) - c_{\mu,Q,a}(z-p)^{N+1}\right),$$

or equivalently

(32)

$$\begin{aligned} Q(z) = -(N+1)(z-p)^{N+1}\int \frac{dz}{(z-p)^{N+2}} \exp\ &\big(4\pi h_{Q,a}(z) + r_{\mu,Q,a}(z) \\ &+ k(z) - c_{\mu,Q,a}(z-p)^{N+1}\big), \end{aligned}$$

where the symbol $\int$ designates a primitive in Ω of the argument function. The choice of the complex constant $c = c_{\mu,Q,a}$ guarantees that the right hand side in expression (32) is a well-defined single-valued function. Indeed, let us set

$$W(z) := \exp\ (4\pi h_{Q,a}(z) + r_{\mu,Q,a}(z) + k(z))$$

and

$$\begin{aligned}\Sigma(z) &:= \exp\ \left(4\pi h_{Q,a}(z) + r_{\mu,Q,a}(z) + k(z) - c(z-p)^{N+1}\right)\\ &= W(z)\exp(-c(z-p)^{N+1}).\end{aligned}$$

We expand near $z = p$, using $W(p) = 1$,

$$\begin{aligned}\Sigma(z) = &\left(1 + W'(p)(z-p) + \cdots + \frac{1}{(N+1)!}\frac{d^{N+1}W}{dz^{N+1}}(p)(z-p)^{N+1} + \cdots\right)\\ &\times\left(1 - c(z-p)^{N+1} + \frac{c^2}{2}(z-p)^{2N+2} + \cdots\right),\end{aligned}$$

so that our choice of $c = c_{\mu,Q,a} = \frac{1}{(N+1)!}\frac{d^{N+1}W}{dz^{N+1}}(p)$ guarantees that the Taylor expansion of $\Sigma(z)$ does not contain a term of order $(z-p)^{N+1}$, say

$$\Sigma(z) = \sum_{j=0}^{\infty} b_j (z-p)^j$$

with $b_{N+1} = 0$. Thus we make sense of the whole R.H.S. in expression (32) as

$$\begin{aligned}&-(N+1)(z-p)^{N+1}\int \frac{\Sigma(z)}{(z-p)^{N+2}}\\ &= -(N+1)(z-p)^{N+1}\sum_{j=0}^{N} b_j \int \frac{dz}{(z-p)^{N+2-j}}\\ &-(N+1)(z-p)^{N+1}\int \frac{dz}{(z-p)^{N+2}}\left(\Sigma(z) - \sum_{j=0}^{N+1} b_j (z-p)^j\right)\\ &= \sum_{j=0}^{N} b_j \frac{N+1}{N+1-j}(z-p)^j\\ &\quad -(N+1)(z-p)^{N+1}\int_p^z \frac{dw}{(w-p)^{N+2}}\left(\Sigma(w) - \sum_{j=0}^{N+1} b_j (w-p)^j\right).\end{aligned}$$

The latter integral is well-defined since its argument function is holomorphic and Ω is simply connected. Let us note that for $b_{N+1} \neq 0$, an additional term would arise of the form $-(N+1)b_{N+1}(z-p)^{N+1}\ln(z-p)$ making the R.H.S. in (32) a multi-valued function.

Since $b_0 = 1$, let us stress that any solution Q of (32) automatically satisfies $Q(p) = 1$ and $Q - \frac{z-p}{N+1}Q' \neq 0$ in Ω, as required. For $a = 0$ and $\varepsilon = 0$, the constant $c_{\mu,Q,a}$ reduces to

$$c_0 = \frac{1}{(N+1)!}\frac{d^{N+1}}{dz^{N+1}}\left(e^{4\pi(N+1)h_p(z)+k(z)}\right)(p),$$

where $h_p(z)$ is the holomorphic extension of $H(z,p) - H(p,p)$ so that $h_p(p) = 0$. Correspondingly, (25) has the solution Q_0 given by:

$$Q_0(z) = -(N+1)(z-p)^{N+1} \int \frac{dz}{(z-p)^{N+2}} \exp \left(4\pi(N+1)h_p(z) + k(z) - c_0(z-p)^{N+1}\right).$$

By the Implicit Function Theorem, we have the following existence result.

Lemma 3.1. *For a and $\varepsilon > 0$ small, in a small neighborhood of Q_0 there exists a unique holomorphic function $Q_{\mu,a} \in C(\bar{\Omega}, \mathbb{C})$ (depending smoothly on μ and a) satisfying* (32)*:*

$$Q(z) = -(N+1)(z-p)^{N+1} \int \frac{dz}{(z-p)^{N+2}} \; exp \; \left(4\pi h_{Q,a}(z) + r_{\mu,Q,a}(z) \right.$$
$$\left. + k(z) - c_{\mu,Q,a}(z-p)^{N+1}\right).$$

Proof. Let $H = \{Q \in C(\bar{\Omega}, \mathbb{C})$: Q is holomorphic in $\Omega\}$. For given $\mu > 0$, define the map

$$\begin{array}{rccl} P: & \mathbb{R}^+ \times H \times \mathbb{C} & \to & H \\ & (\varepsilon, Q, a) & \to & Q(z) + (N+1)(z-p)^{N+1} \int \frac{\Sigma(z)}{(z-p)^{N+2}}, \end{array}$$

where $\Sigma(z) = \exp \left(4\pi h_{Q,a}(z) + r_{\mu,Q,a}(z) + k(z) - c_{\mu,Q,a}(z-p)^{N+1}\right)$. We have that

$$P(0,Q,0) = Q + (N+1)(z-p)^{N+1} \int \frac{1}{(z-p)^{N+2}} \exp \; \left(4\pi(N+1)h_p(z) \right.$$
$$\left. + k(z) - c_0(z-p)^{N+1}\right).$$

Since $\partial_Q P(0, Q_0, 0) = \mathrm{Id}$ and $P(0, Q_0, 0) = 0$, by the Implicit Function Theorem we find $\varepsilon_0 > 0$ small and a smooth map $(\varepsilon, a) \in (0, \varepsilon_0) \times B_{\varepsilon_0}(0) \to Q_{\varepsilon,\mu,a} \in H$ so that $Q_{0,\mu,0} = Q_0$ and $P(\varepsilon, Q_{\varepsilon,\mu,a}, a) = 0$. The required smallness of ε_0 is easily shown to be independent of μ, and $Q_{\varepsilon,\mu,a}$ depends smoothly also on μ, provided μ remains bounded and bounded away from zero. To keep light notation, we will omit the explicit dependence of $Q_{\varepsilon,\mu,a}$ on ε and simply write $Q_{\mu,a}$. □

Next, for $a = 0$ and $\varepsilon = 0$, equation (26) has a unique solution

$$\mu_0 = \frac{e^{4\pi(N+1)H(p,p)+K(p)}}{\sqrt{8}(N+1)}.$$

By perturbation, we get

Lemma 3.2. *For a and $\varepsilon > 0$ small, in a small neighborhood of μ_0 there exists a unique solution μ_a (depending smoothly on a) to* (26)*.*

Proof. Set

$$\begin{array}{rccl} T: & \mathbb{R}^+ \times \mathbb{R} \times \mathbb{C} & \to & \mathbb{R} \\ & (\varepsilon, \mu, a) & \to & \mu - \frac{e^{4\pi H_{Q_{\mu,a},a}(p) + R_{\mu,Q_{\mu,a},a}(p) + K(p)}}{\sqrt{8}(N+1)}. \end{array}$$

Since $Q_{\mu,a}$ depends smoothly on μ and a, the same regularity holds for T. We have that

$$T(0,\mu,0) = \mu - \frac{e^{4\pi(N+1)H(p,p)+K(p)}}{\sqrt{8}(N+1)}.$$

Since $T(0, \mu_0, 0) = 0$ and $\partial_\mu T(0, \mu_0, 0) = 1$, we find $\varepsilon_0 > 0$ small and a smooth map

$$(\varepsilon, a) \in (0, \varepsilon_0) \times B_{\varepsilon_0}(0) \mapsto \mu_{\varepsilon,a}$$

so that $\mu_{0,0} = \mu_0$ and $T(\varepsilon, \mu_{\varepsilon,a}, a) = 0$, by means of the Implicit Function Theorem. As before, we simply write μ_a instead of $\mu_{\varepsilon,a}$, and the proof is complete. □

4. Proofs of the main results

Proof of Theorem 2.1. Let $\Lambda \Subset \Omega$ be a region as in the statement of the theorem and set

$$\Psi(p) := \frac{d^{N+1}}{dz^{N+1}} \left(e^{4\pi(N+1)h_p(z)+k(z)} \right)(p).$$

Finding a solution of Problem (20) as desired reduces then to finding $p \in \Lambda$ such that

$$\Psi_{a,\varepsilon}(p) = \frac{d^{N+1}}{dz^{N+1}} \left(e^{4\pi h_{Q_a,a}(z)+r_{\mu_a,Q_a,a}(z)+k(z)} \right)(p) = 0,$$

where $Q_a = Q_{\mu_a,a}$, with $Q_{\mu,a}$ and μ_a given by Lemmas 3.1 and 3.2, respectively. Since $\Psi_{a,\varepsilon}(p)$ depends continuously on a, ε, p and $\Psi_{0,0}(p) = \Psi(p)$, we get that the degree of $\Psi_{a,\varepsilon}(p)$ on Λ w.r.t. 0 is well-defined, for $|a| + \varepsilon$ small, and coincides with $\deg(\Psi, \Lambda, 0)$. The latter one being nontrivial by assumption, for a and $\varepsilon > 0$ small we find a solution $p_{a,\varepsilon} \in \Lambda$ of $\Psi_{a,\varepsilon}(p) = 0$.

In this way, the function

$$v_\varepsilon = \ln \frac{8(N+1)^2\mu_a^2 |Q_a(z) - \frac{z-p}{N+1} Q_a'(z)|^2}{(\mu_a^2 \varepsilon^2 |Q_a(z)|^2 + |(z - p_{a,\varepsilon})^{N+1} - aQ_a(z)|^2)^2}$$

is a solution to Problem (20) with $p = p_{a,\varepsilon}$ for which the concentration property (28) easily follows. Since $\Psi_{a,\varepsilon}(p_{a,\varepsilon}) = 0$, we finally have that $\Psi(p_{a,\varepsilon}) \to 0$ as $|a| + \varepsilon \to 0$. □

Proof of Theorem 1.1. To establish Theorem 1.1, we observe that in the homogeneous case $\varphi = 0$, $K(z) = -2\pi N H(z, p)$ and the function Ψ becomes

$$\Psi(p) = \frac{d^{N+1}}{dz^{N+1}} \left(e^{2\pi(N+2)h_p(z)} \right)(p).$$

We will compute the total degree of this map by estimating its behavior near $\partial\Omega$. Let us observe that

$$H(z,p) - \frac{1}{2\pi} \ln |z - \hat{p}| \to 0 \quad \text{in } C^{N+1}(\bar{\Omega}) \text{ as } p \to \partial\Omega, \tag{33}$$

where $\hat{p} \in \mathbb{R}^2 \setminus \bar{\Omega}$ is the reflection of p with respect to $\partial\Omega$. Let us stress that the reflection map is well-defined for points in Ω that are near $\partial\Omega$. The C^0-validity of (33) follows from the Maximum Principle applied to the harmonic function $H(z,p) - \frac{1}{2\pi} \ln |z - \hat{p}|$ by means of the asymptotic behavior

$$H(z,p) - \frac{1}{2\pi} \ln |z - \hat{p}| = \frac{1}{2\pi} \ln \frac{|z - p|}{|z - \hat{p}|} \to 0 \text{ unif. on } \partial\Omega, \text{ as } p \to \partial\Omega.$$

Elliptic regularity (see [17]) then implies the validity of (33).

Let us denote $d = \text{dist}(p, \partial\Omega)$. Then, from (33), we obtain

$$H(p,p) = \frac{1}{2\pi} \ln |p - \hat{p}| + o(1) = \frac{1}{2\pi} \ln(2d) + o(1),$$
$$H(z,p) - H(p,p) = \frac{1}{2\pi} \ln \frac{|z - \hat{p}|}{2d} + o(1) \quad \text{in } C^{N+1}(\bar{\Omega})$$

as $d \to 0$. We extend $H(z,p) - H(p,p)$ holomorphically in Ω by $h_p(z)$ with $h_p(p) = 0$, and as $d \to 0$, the expansion

$$h_p(z) = \frac{1}{2\pi} \ln \frac{z - \hat{p}}{2d} + o(1) \quad \text{in } C^{N+1}(\bar{\Omega})$$

holds. Since as $d \to 0$,

$$e^{2\pi(N+2)h_p(z)} = (\frac{z-\hat{p}}{2d})^{N+2}(1 + o(1))$$

in $C^{N+1}(\bar{\Omega})$, we finally get that in the homogeneous case,

$$\Psi(p) = (N+2)! \, \frac{p - \hat{p}}{2d}(1 + o(1))$$

as $d \to 0$. But $\frac{p-\hat{p}}{2d} = \nu_\Omega(p + \hat{p}/2)$, where $\nu_\Omega(x)$ is the inward unit normal vector at $x \in \partial\Omega$ and $p + \hat{p}/2$ is the projection of p onto the boundary. The winding number of ν_Ω along $\partial\Omega$ is ± 1 and, by stability, we get that

$$\deg\,(\Psi, 0, \Omega_\delta) = \pm 1 \neq 0$$

for $\delta > 0$ small. Here, $\Omega_\delta = \{x \in \Omega : d > \delta\}$.

Theorem 2.1 thus applies with $\Lambda = \Omega_\delta$ to provide, for a and $\varepsilon > 0$ small, $p_{a,\varepsilon} \in \Omega_\delta$ and solutions v_ε to Problem (20) with $p = p_{a,\varepsilon}$ for which the concentration property (28) holds. Setting $\psi_\varepsilon(z) = v_\varepsilon(z) - 4\pi N G(z,p)$, the function ψ_ε is a solution to Problem (5) with $\alpha = N$, $p = p_{a,\varepsilon}$ and (28) rewritten in terms of ω_ε as (18). The proof is concluded. □

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