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Ground states of a prescribed mean curvature equation

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Abstract

We study the existence of radial ground state solutions for the problem

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=u^q,\qquad u>0\quad \text{in } \mathbb{R}^N,$$
$$u(x)\to 0\quad \text{as } |x|\to\infty,$$

$N\geqslant 3$, $q>1$. It is known that this problem has infinitely many ground states when $q\geqslant \frac{N+2}{N-2}$, while no solutions exist if $q\leqslant \frac{N}{N-2}$. A question raised by Ni and Serrin in [W.-M. Ni, J. Serrin, Existence and non-existence theorems for ground states for quasilinear partial differential equations, Atti Convegni Lincei 77 (1985) 231–257] is whether or not ground state solutions exist for $\frac{N}{N-2}<q<\frac{N+2}{N-2}$. In this paper we prove the existence of a large, finite number of ground states with fast decay $O(|x|^{2-N})$ as $|x|\to+\infty$ provided that q lies below but close enough to the critical exponent $\frac{N+2}{N-2}$. These solutions develop a *bubble-tower* profile as q approaches the critical exponent.

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1. Introduction

This paper deals with the question of finding radially symmetric solutions $u = u(|x|)$ of the following prescribed mean curvature equation:

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = u^q & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to+\infty} u(x) = 0. \end{cases} \tag{1}$$

We refer to these solutions as ground states. Radial (singular) solutions of (1) when u^q is replaced by κu have been studied in the context of the analysis of capillary surfaces [2,3,6,11,14,18,24,27]. Existence and nonexistence of radial ground states for problem (1) when $q > 1$ has been considered by several authors. Ni and Serrin in [20,21] established that if $1 < q \leqslant N/(N-2)$ no positive solutions exist. On the contrary, if $q \geqslant (N+2)/(N-2)$ there is a continuum of solutions [20]. For any q, radial ground states must satisfy the upper bound $u(0) < (4Nq)^{1/(q+1)}$, see [1,26].

Whether or not ground states of problem (1) exist in the range

$$\frac{N}{N-2} < q < \frac{N+2}{N-2} \tag{2}$$

was left as an open question in [21]. Partial progress was achieved by Clément, Mitidieri and Manásevich in [5] who proved a Liouville type theorem for problem (4), with q in the range (2): there is an explicit positive constant $C(N,q)$ such that no radial ground state u exists with $u(0) < C(N,q)$. In addition, let us recall that for the standard Lane–Emden–Fowler problem

$$\begin{cases} -\Delta u = u^q & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to+\infty} u(x) = 0 \end{cases} \tag{3}$$

ground states do not exist if $q < \frac{N+2}{N-2}$ as established by Gidas and Spruck in [16]. Finally note that replacing u^q by $-au + u^q$ there exists exponentially decaying ground states for sufficiently small $a > 0$ and q in the range (2), see [7,25].

In this paper we will prove that, in striking opposition to the above facts, many ground states of problem (1) *do exist* if q is less than but sufficiently close to $(N+2)/(N-2)$. Thus we assume in what follows that $N \geqslant 3$ and consider the problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = u^{\frac{N+2}{N-2}-\varepsilon} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to+\infty} u(x) = 0, \end{cases} \tag{4}$$

where $\varepsilon > 0$ is a small parameter. We have the validity of the following result which in particular states that an increasingly large number of solutions exist as ε gets smaller and smaller.

Theorem 1. *Given $k \geqslant 1$ there exists a number $\varepsilon_k > 0$ such that for all $0 < \varepsilon < \varepsilon_k$ there exists a radially symmetric solution u_ε to problem* (4) *which asymptotically takes the form*

$$u_\varepsilon(x) = \gamma \sum_{j=1}^{k} \left(\frac{1}{(1 + (\alpha_j \varepsilon^{j - \frac{N+2}{2N}})^{\frac{4}{N-2}} |x|^2)} \right)^{\frac{N-2}{2}} \varepsilon^{j - \frac{N+2}{2N}} \alpha_j \bigl(1 + o(1)\bigr), \tag{5}$$

where $o(1) \to 0$ uniformly in $\mathbb{R}^N$ as $\varepsilon \to 0$. Here the α_j's are (*explicit*) *positive constants and $\gamma = (N(N-2))^{\frac{N-2}{4}}$.*

We recall that the "bubbles"

$$w_m(x) = \gamma \left(\frac{1}{(1 + m^{\frac{4}{N-2}} |x|^2)} \right)^{\frac{N-2}{2}} m, \quad m > 0,$$

constitute all radial positive solutions of the equation $\Delta w + w^{\frac{N+2}{N-2}} = 0$. The solutions found in Theorem 1 constitute at main order superposition of k "flat" bubbles with small maximum values which approach zero uniformly as $\varepsilon \to 0$. The bubble-tower phenomenon, with tall elements, has been detected in slightly supercritical problems for the Laplacian operator in [9,10,13,15].

The question of existence of ground states for (4) remains open for the full range (2). We could mention the possible analogy existing between this problem and

$$\begin{cases} -\Delta u = u^q + u^s \quad \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{in } \mathbb{R}^N, \\ \lim\limits_{|x| \to +\infty} u(x) = 0 \end{cases} \tag{6}$$

with q lying in the range (2) and $s > (N+2)/(N-2)$. It was proven in [4] the existence of a number $\bar{q} \in (N/(N-2), (N+2)/(N-2))$, such that no ground states of problem (6) exist if $q \in (N/(N-2), \bar{q})$, while they do if $q \in [\bar{q}, (N+2)/(N-2))$, with increasing number of them as q approaches $(N+2)/(N-2)$. We conjecture that a similar fact holds for problem (4).

The proof of Theorem 1 follows a scheme close in spirit to that in [9]. It is based on a transformation of the ODE equivalent to the problem of finding radially symmetric solutions, via an Emden–Fowler type transformation, after which the problem of finding the desired solution becomes equivalent to that of finding a multibump solution in which the centers are located toward $+\infty$, at the same time far away one to each other. The ε-dependent location of the bumps is then derived as stationary configurations for critical points of the energy functional along a suitable manifold of approximate solutions. This procedure gets rigorously carried out via a Lyapunov–Schmidt procedure broadly used in elliptic singular perturbation problems starting with [12], after which actual critical points of the full energy are found close to those on the approximate manifold. The phenomenon here described has resemblance with *spike clustering* as found for instance in [8,19,22,23].

We carry out this program in what remains of this paper.

2. The set up and energy computations

2.1. Scaling

For the analysis we shall study the problem

$$-\left(\eta^{N-1}\frac{f_\eta}{\sqrt{1+\varepsilon f_\eta^2}}\right)_\eta=\eta^{N-1}f^q\quad \text{for } \eta>0, \tag{7}$$

$$f_\eta(0)=0,\qquad f(\eta)\to 0\quad \text{as } \eta\to\infty, \tag{8}$$

which is the radial form of problem (1) under the scaling

$$f(\eta):=\varepsilon^{-\frac{1}{q+1}}u(r)\quad\text{and}\quad \eta=\varepsilon^{\frac{1}{2}\frac{q-1}{q+1}}r.$$

In particular we study f_ε solution of

$$-\left(\eta^{N-1}\frac{f_\eta}{\sqrt{1+\varepsilon f_\eta^2}}\right)_\eta=\eta^{N-1}f^{\frac{N+2}{N-2}-\varepsilon}\quad \text{for } \eta>0. \tag{9}$$

In terms of f_ε, the expansion (5) takes the form

$$f_\varepsilon(\eta)=\gamma\sum_{j=1}^{k}\left(\frac{1}{(1+(\alpha_j\varepsilon^{j-1})^{\frac{4}{N-2}}|\eta|^2)}\right)^{\frac{N-2}{2}}\varepsilon^{j-1}\alpha_j\big(1+o(1)\big), \tag{10}$$

where $o(1)\to 0$ uniformly in $\mathbb{R}^N$ as $\varepsilon\to 0$. Here $\alpha_1=\Lambda_1^{-1}$ and $\alpha_i=\Lambda_1^{-1}\prod_{j=2}^{i}\Lambda_j$ for $i=2,\dots,k$, where Λ_i are explicitly given at the end of Section 3. See Fig. 1 for numerical solutions of (9)–(8) of the form (10) with $k=1,2,3$.

Note that this scaling is invariant for the Emden–Fowler variable, so $\eta^{\frac{2}{q-1}}f(\eta)=r^{\frac{2}{q-1}}u(r)$. For $q\geqslant (N+2)/(N-2)$ by the Ni and Serrin result [20], we have existence for (7) with the initial boundary values

$$f(0)=\gamma>0\quad\text{and}\quad f_\eta(0)=0 \tag{11}$$

provided that ε is small. Because (7) is a regular perturbation of Eq. (7) with $\varepsilon=0$, the solution f of (7) with initial values (11) has the property that $f\to u_\gamma$ as $\varepsilon\to 0$, uniformly in compact sets $[0,R]$, where u_γ solves (7) with $\varepsilon=0$ and initial values (11). It was shown in [17] that when $q>(N+2)/(N-2)$ the solution u_γ has infinitely many oscillations in the Emden–Fowler variable for q close to $(N+2)/(N-2)$, so the behavior of f will be close to that as we see in Fig. 2. The case $q=(N+2)/(N-2)$ is more delicate. By the previous argument the solution f converges uniformly, as $\varepsilon\to 0$, to the unique solution of (3) in $[0,R]$. But we can say more, we conjecture that the structure of f is given by (10) with $k=\infty$, as we see numerically in Fig. 2. Consequently as $\varepsilon\to 0$ the first bubble will remain in the same position and the others will move towards infinity in slow manner.

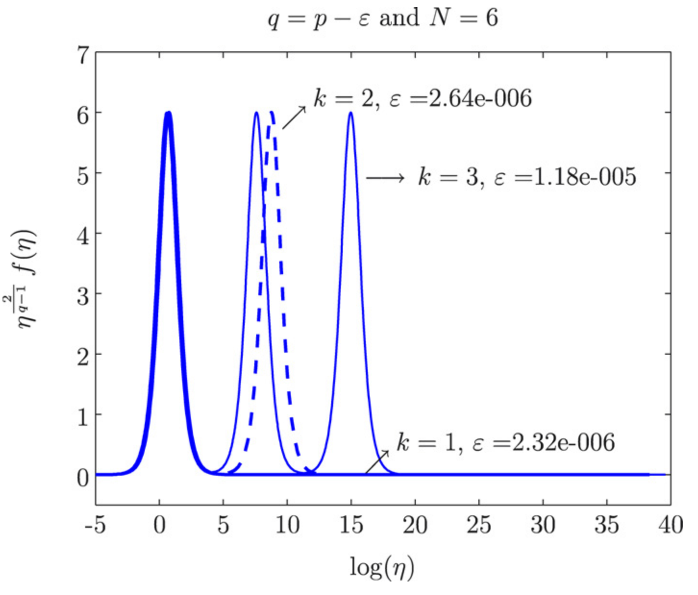

Fig. 1. Ground state solutions of Eq. (9) in the Emden–Fowler variable for $k = 1, 2, 3,$ for ε small. It is not seen here, but the first bumps are moving to the left as k increase, as shown by the formula of t_k.

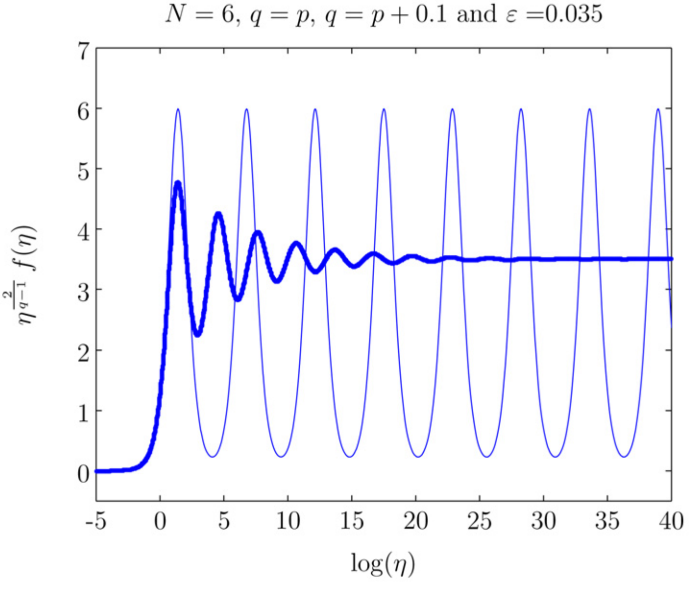

Fig. 2. Ground states of (7) with $f(0) = 0.9$ and ε small, plotted in the Emden–Fowler variable. For $q = (N+2)/(N-2)$ (regular line), the solution is conjectured to be a superposition of infinitely many bumps. For $q > (N+2)/(N-2)$ (thick line), the solution behaves as Eq. (3), here we draw the case q near $(N+2)/(N-2)$.

2.2. Emden–Fowler transformation

For the analysis we make the following change of variables

$$v(s)=\tau^{-2/(p-1-\varepsilon)}\eta^{\tau}f(\eta),\qquad \eta=e^{s/\tau},\qquad \tau=\frac{2}{p-1},$$

where $p=(N+2)/(N-2)$. This gives

$$f_\eta(\eta)=\tau^{1+2/(p-1-\varepsilon)}\eta^{-\tau-1}(v'-v)=\sqrt{\ell e^{-(p+1)s}}(v'-v),$$

where $\ell=\tau^{2\frac{p+1-\varepsilon}{p-1-\varepsilon}}$. Note that $\ell\to\tau^N$ as $\varepsilon\to 0$. Using this, problem (9) becomes

$$\begin{cases}\left(\dfrac{v'-v}{\sqrt{1+\varepsilon\ell e^{-(p+1)s}(v'-v)^2}}\right)'+\dfrac{v'-v}{\sqrt{1+\varepsilon\ell e^{-(p+1)s}(v'-v)^2}}+v^{p-\varepsilon}e^{s\varepsilon}=0\quad\text{on }\mathbb{R},\\ v>0,\quad v(s)\to 0\quad\text{as } s\to\pm\infty.\end{cases}\tag{12}$$

The functional associated to problem (12) is given by

$$E_\varepsilon(w)=\int_{-\infty}^{\infty}\frac{1}{\varepsilon\ell}\left(\sqrt{1+\varepsilon\ell e^{-(p+1)s}(w'-w)^2}-1\right)e^{(p+1)s}\,ds$$
$$-\frac{1}{p+1-\varepsilon}\int_{-\infty}^{\infty}w^{p+1-\varepsilon}e^{\varepsilon s}\,ds$$

but we have the Taylor's expansion

$$\begin{aligned}&\frac{1}{\varepsilon\ell}\left(\sqrt{1+\varepsilon\ell e^{-(p+1)s}(w'-w)^2}-1\right)e^{(p+1)s}\\ &\quad=\frac{1}{2}(w'-w)^2-\frac{\varepsilon}{8}\ell e^{-(p+1)s}(w'-w)^4\\ &\qquad+\varepsilon^2e^{-2(p+1)s}\sum_{m=3}^{\infty}D_\varepsilon(m,N)\varepsilon^{m-3}e^{-(m-3)(p+1)s}\frac{(w'-w)^{2m}}{(2m)!},\end{aligned}$$

where $D_\varepsilon(m,N)=(-\ell)^{m-1}(2m-1)\prod_{j=1}^{m-2}(2j+1)^2$. This yields

$$E_\varepsilon(w)=I_\varepsilon(w)-\frac{\varepsilon}{8}\ell\int_{-\infty}^{\infty}e^{-(p+1)s}(w'-w)^4\,ds+\varepsilon^2J_\varepsilon(w)\tag{13}$$

with

$$I_\varepsilon(w) = \frac{1}{2}\int_{-\infty}^{\infty} (w' - w)^2\, ds - \frac{1}{p+1-\varepsilon}\int_{-\infty}^{\infty} e^{\varepsilon s}|w|^{p-\varepsilon+1}\, ds, \tag{14}$$

$$J_\varepsilon(w) = \sum_{m=3}^{\infty} \varepsilon^{m-3} D_\varepsilon(m,N)\int_{-\infty}^{\infty} e^{-(m-1)(p+1)s}\frac{(w'-w)^{2m}}{(2m)!}\, ds. \tag{15}$$

Let us consider the unique solution $U(s)$ to the problem

$$\begin{cases} U'' - U + U^p = 0 \quad \text{on } (-\infty,\infty), \\ U'(0) = 0, \\ U > 0, \quad U(s) \to 0 \quad \text{as } s \to \pm\infty, \end{cases} \tag{16}$$

which is the well-known function,

$$U(s) = C_N \cosh\left(\frac{2s}{N-2}\right)^{\frac{2-N}{2}} \quad \text{with } C_N = \left(\frac{N}{N-2}\right)^{(N-2)/4}. \tag{17}$$

Let us consider points $-\infty < \xi_1 < \xi_2 < \cdots < \xi_k$. We look for a solution of (12) of the form

$$v(s) = \sum_{i=1}^{k} U(s-\xi_i) + \phi,$$

where ϕ is small. Note that $v(s) \sim \sum_{i=1}^{k} U(s-\xi_i)$ solves (12) if and only if (going back in the change of variables)

$$f(\eta) \sim \gamma \sum_{i=1}^{k}\left(\frac{1}{1+e^{-\frac{4\xi_i}{N-2}}\eta^2}\right)^{\frac{N-2}{2}} e^{-\xi_i}$$

solves (9).

Let us write

$$U_i(s) = U(s-\xi_i), \qquad \mathbf{w} = \sum_{i=1}^{k} U_i. \tag{18}$$

We shall work out asymptotics for the energy functional associated at the function $\mathbf{w}$, assuming that the numbers ξ_i are very far apart but at comparable distances from each other.

We make the following choices for the points ξ_i:

$$\begin{aligned} &\xi_1 = \log \Lambda_1, \\ &\xi_{i+1} - \xi_i = -\log\varepsilon - \log \Lambda_{i+1}, \quad i = 1,\dots,k-1, \end{aligned} \tag{19}$$

where the Λ_i's are positive parameters. For notational convenience, we also set

$$\Lambda = (\Lambda_1, \Lambda_2, \dots, \Lambda_k).$$

The advantage of the above choice is the validity of the expansion of the energy E_ε defined by (13) given as follows.

Lemma 1. *Fix a small number $\delta > 0$ and assume that*

$$\delta < \Lambda_i < \delta^{-1} \quad \text{for all } i = 1, \dots, k. \tag{20}$$

Let $\mathbf{w}$ *be given by* (18). *Then, with the choice* (19) *of the points ξ_i, there are positive numbers a_i, $i = 0, \dots, 4$, depending only on N* (*which have the explicit expressions* (29)) *such that the following expansion holds*:

$$E_\varepsilon(\mathbf{w}) = ka_0 + \varepsilon \Psi_k(\Lambda) - ka_4\varepsilon + \varepsilon\theta_\varepsilon(\Lambda), \tag{21}$$

$$\Psi_k(\Lambda) = -ka_2 \log \Lambda_1 - a_3 \Lambda_1^{-(p+1)} + \sum_{i=2}^{k} \big[(k-i+1)a_2 \log \Lambda_i - a_1 \Lambda_i\big], \tag{22}$$

and as $\varepsilon \to 0$, the term $\theta_\varepsilon(\Lambda)$ converges to 0 *uniformly and in the C^1-sense on the set of Λ_i's satisfying constraints* (20).

Proof. We will estimate the different terms in the expansion of $E_\varepsilon(\mathbf{w})$ with V defined by (18), for the ξ_i's given by (19)–(20). Let I_ε be the functional in (14). We may write

$$I_\varepsilon(\mathbf{w}) = I_0(\mathbf{w}) - \frac{1}{p+1} \int_{-\infty}^{\infty} \big(e^{\varepsilon s} - 1\big)|\mathbf{w}|^{p+1}\, ds + A_\varepsilon,$$

$$A_\varepsilon = \left(\frac{1}{p+1} - \frac{1}{p-\varepsilon+1}\right) \int_{-\infty}^{\infty} e^{\varepsilon s} |\mathbf{w}|^{p+\varepsilon+1}\, ds + \frac{1}{p+1} \int_{-\infty}^{\infty} e^{\varepsilon s}\big(|\mathbf{w}|^{p+1} - |\mathbf{w}|^{p-\varepsilon+1}\big)\, ds.$$

Then, we find that

$$A_\varepsilon = -k\varepsilon\left(\frac{1}{p+1}\int_{-\infty}^{\infty} U^{p+1}\log U\, ds + \frac{1}{(p+1)^2}\int_{-\infty}^{\infty} U^{p+1}\, ds\right) + o(\varepsilon). \tag{23}$$

On the other hand, for the same reason, we have

$$\begin{aligned} \int_{-\infty}^{\infty} \big(e^{\varepsilon s} - 1\big)\mathbf{w}^{p+1}\, ds &= \varepsilon \int_{-\infty}^{\infty} s\mathbf{w}^{p+1}\, ds + o(\varepsilon) \\ &= \varepsilon\left(\sum_{i=1}^{k} \xi_i\right) \int_{-\infty}^{\infty} U^{p+1}\, ds + o(\varepsilon). \end{aligned} \tag{24}$$

Now, we have the validity of the identity

$$I_0(\mathbf{w}) = \sum_{i=1}^{k} I_0(U_i) + \frac{1}{p+1} B, \tag{25}$$

where

$$B = \int_{-\infty}^{\infty} \left[\sum_{i=1}^{k} U_i^{p+1} - \left(\sum_{i=1}^{k} U_i \right)^{p+1} + (p+1) \sum_{i<j} U_i^p U_j \right] ds.$$

Indeed we have

$$\int_{-\infty}^{\infty} (W' - W)^2 \, ds = \int_{-\infty}^{\infty} |W'|^2 \, ds + \int_{-\infty}^{\infty} |W|^2 \, ds$$

for $W = \mathbf{w}$, $U_1 \dots U_k$ and so

$$\begin{aligned} &\frac{1}{p+1} B - \int_{-\infty}^{\infty} \left[\sum_{i=1}^{k} U_i^{p+1} - \left(\sum_{i=1}^{k} U_i \right)^{p+1} \right] ds \\ &= \sum_{i<j} \int_{-\infty}^{\infty} \left(U_i' U_j' + U_i U_j \right) ds \\ &= \sum_{i<j} \int_{-\infty}^{\infty} \left(-U_i'' + U_i \right) U_j \, ds = \sum_{i<j} \int_{-\infty}^{\infty} U_i^p . U_j \, ds. \end{aligned}$$

To estimate this latter quantity, we consider the numbers

$$\mu_1 = -\infty, \qquad \mu_l = \frac{1}{2}(\xi_{l-1} + \xi_l), \quad l = 2, \dots, k, \qquad \mu_{k+1} = +\infty,$$

and decompose B as $B = -C_0 + C_1$ where

$$C_0 = (p+1) \sum_{\substack{1 \leqslant l \leqslant k \\ j > l}} \int_{\mu_l}^{\mu_{l+1}} U_l^p U_j \, dx.$$

We follow the argument in [9,10] and find that $C_1 = o(\varepsilon)$. Let us now estimate C_0. We have for $l = 1, \dots, k$ that

$$\int_{\mu_l}^{\mu_{l+1}} U_l^p U_{l+1} \, ds = \int_{\mu_l - \xi_l}^{\mu_{l+1} - \xi_l} U^p(s) U\big(s - (\xi_{l+1} - \xi_l)\big) \, ds.$$

On the other hand, according to (17), it is directly checked that

$$\left|U(s-\xi)-2^{(N-2)/2}C_N e^{-|\xi-s|}\right|=O\left(e^{-p|\xi-s|}\right)$$

as $\xi\to+\infty$. We conclude then that

$$C_0=(p+1)\sum_{l=1}^{k-1}e^{-|\xi_{l+1}-\xi_l|}2^{(N-2)/2}C_N\int_{-\infty}^{\infty}e^s U(s)^p\,ds+o(\varepsilon).$$

This yields

$$B=-a_1\sum_{l=1}^{k-1}e^{-|\xi_{l+1}-\xi_l|}+o(\varepsilon) \tag{26}$$

with $a_1=2^{(N-2)/2}C_N\int_{-\infty}^{\infty}e^sU(s)^p\,ds$.

Continuing our estimate of $I_\varepsilon(\mathbf{w})$, we have now to consider $I_0(U_i)$ for $i=1,\dots,k$. We find

$$I_0(U_i)=a_0=\frac{1}{2}\int_{-\infty}^{\infty}\left(|U'|^2+U^2\right)dx-\frac{1}{p+1}\int_{-\infty}^{\infty}U^{p+1}\,dx\quad\text{for all } i\geqslant 2. \tag{27}$$

Finally, as for the last term in the decomposition (13), we easily check that

$$\int_{-\infty}^{\infty}e^{-(p+1)s}(\mathbf{w}'-\mathbf{w})^4\,ds=e^{-(p+1)\xi_1}\int_{-\infty}^{\infty}e^{-(p+1)s}\left|U'(s)-U(s)\right|^4ds+o(\varepsilon). \tag{28}$$

Summarizing, we obtain from estimates (23)–(28) the validity of the following expansion:

$$\begin{aligned}E_\varepsilon(\mathbf{w})=ka_0-a_1\sum_{l=1}^{k}e^{-|\xi_{l+1}-\xi_l|}-a_2\varepsilon\left(\sum_{i=1}^{k}\xi_i\right)\\-\varepsilon a_3e^{-(p+1)\xi_1}-ka_4\varepsilon+o(\varepsilon).\end{aligned}$$

Here the constants a_i, $i=0,\dots,4$, depend only on N and can be expressed as follows:

$$\begin{cases}a_0=\frac{1}{2}\int_{-\infty}^{\infty}(|U'|^2+U^2)\,dx-\frac{1}{p+1}\int_{-\infty}^{\infty}U^{p+1}\,ds,\\ a_1=2^{(N-2)/2}C_N\int_{-\infty}^{\infty}e^sU^p\,ds,\\ a_2=\frac{1}{p+1}\int_{-\infty}^{\infty}U^{p+1}\,ds,\\ a_3=\frac{1}{8}\left(\frac{N-2}{2}\right)^N\int_{-\infty}^{\infty}e^{-(p+1)s}(U'-U)^4\,ds,\\ a_4=\frac{1}{p+1}\int_{-\infty}^{\infty}U^{p+1}\log U\,ds+\frac{1}{(p+1)^2}\int_{-\infty}^{\infty}U^{p+1}\,ds.\end{cases} \tag{29}$$

These constants can be explicitly computed using the explicit expression for U given by (17) and the identity

$$\int_{-\infty}^{\infty} \cosh\left(\frac{2}{N-2}s\right)^{-q} e^{-\frac{2}{N-2}\alpha s}\, ds = 2^{q-2}(N-2)\frac{\Gamma(\frac{q-\alpha}{2})\Gamma(\frac{q+\alpha}{2})}{\Gamma(q)}$$

for all $q > \max\{\alpha, -\alpha\}$. The above decomposition of E_ε finally reads

$$E_\varepsilon(\mathbf{w}) = ka_0 + \varepsilon\Psi_k(\Lambda) - ka_4\varepsilon + o(\varepsilon),$$

with Ψ_k given by (22). In fact, the term $o(\varepsilon)$ is uniform on the Λ_i's satisfying (20). A further computation along the same lines shows that differentiation with respect to the Λ_i's leaves the term $o(\varepsilon)$ of the same order in the C^1-sense. This concludes the proof of Lemma 1. □

Before proving existence, let us analyze the critical points of Ψ_k:

$$\Psi_k(\Lambda) = \varphi_1^k(\Lambda_1) + \sum_{i=2}^{k} \varphi_i(\Lambda_i),$$

$$\varphi_1^k(t) = -ka_2 \log t - a_3 t^{-(p+1)} \quad \text{and} \quad \varphi_i(t) = (k-i+1)a_2 \log t - a_1 t.$$

Now the equation $\varphi_1^k(t)' = 0$ yields

$$t = t_k := \left(\frac{(p+1)b_1}{k}\right)^{\frac{1}{p+1}} \quad \text{with } b_1 = \frac{a_3}{a_2},$$

which is a maximum for φ_1^k.

On the other hand, each of the functions φ_j has exactly one nondegenerate critical point, a maximum,

$$t = (k-j+1)b_2 \quad \text{for each } j = 2, \dots, k,$$

with $b_2 = a_2/a_1$. Now we compute b_1 and b_2. We find that

$$a_1 = 2^{N-2}(N-2)C_N^{\frac{2N}{N-2}} \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{N+2}{2})}, \qquad a_2 = \frac{(N-2)^2}{N} C_N^{\frac{2N}{N-2}} 2^{N-3} \frac{\Gamma(\frac{N}{2})^2}{\Gamma(N)},$$

$$a_3 = 2^{N-5}(N-2)^{N+1} C_N^4 \frac{\Gamma(\frac{N+4}{2})\Gamma(\frac{3N-4}{2})}{\Gamma(2N)}$$

and we obtain

$$b_1 = \frac{a_3}{a_2} = \frac{1}{4}(N-2)^{\frac{N}{2}+1} N^{\frac{N}{2}-1} \frac{\Gamma(N)\Gamma(\frac{N+4}{2})\Gamma(\frac{3N-4}{2})}{\Gamma(\frac{N}{2})^2 \Gamma(2N)}$$

and

$$b_2 = \frac{a_2}{a_1} = \frac{N-2}{2} \frac{\Gamma(\frac{N}{2})\Gamma(\frac{N+2}{2})}{N\Gamma(N)}.$$

Lemma 2. *The function $\Psi_k(\Lambda)$ has exactly one critical point, given by*

$$\Lambda^* = \big(t_k, (k-1)b_2, (k-2)b_2, \ldots, b_2\big).$$

This critical point is nondegenerate.

3. Linear theory

Let us consider points $-\infty < \xi_1 < \xi_2 < \cdots < \xi_k$ and a large number $R > 0$, which is for now arbitrary, such that

$$\xi_{i+1} - \xi_i > R \quad \text{for all } i = 1, \ldots, k-1. \tag{30}$$

Associated to these points we consider the functions

$$U_i(x) = U(x - \xi_i), \qquad Z_i(x) = U'(x - \xi_i)$$

and, for a number $0 < \sigma < 1$ the norms

$$\|h\|_{**} = \sup_{x\in\mathbb{R}} \left(\sum_{j=1}^{k} e^{-\sigma|x-\xi_i|} \right)^{-1} \big|h(x)\big|, \qquad \|\phi\|_* = \|\phi\|_{**} + \|\phi'\|_{**} + \|\phi''\|_{**}.$$

In this notation the dependence of the norms on the points ξ_i and the number σ is understood but will not be made explicit as long as it does not create confusion. Let us denote

$$\mathbf{w} = \sum_{i=1}^{k} U_i. \tag{31}$$

Given a function h for which $\|h\|_{**} < +\infty$ we consider the problem of finding a function ϕ such that for certain constants $c_1, \ldots, c_k$ the following equation holds

$$\begin{cases} \phi'' + p\mathbf{w}^{p-1}\phi - \phi = h + \displaystyle\sum_{i=1}^{k} c_i Z_i \chi_i & \text{in } \mathbb{R}, \\ \displaystyle\lim_{|x|\to+\infty} \phi(x) = 0, \end{cases} \tag{32}$$

where $\chi_i(x) = 1$ if $|x - \xi_i| < \frac{R}{2}$ and $= 0$ otherwise.

Our main result in this section is the following.

Proposition 1. *There exist positive numbers R and C such that if the points ξ_i satisfy constraints* (30)*, then for all h with $\|h\|_{**} < +\infty$, problem* (32) *has a solution $\phi =: T(h)$, which defines a linear operator of h and satisfies*

$$\big\|T(h)\big\|_* \leqslant C\|h\|_{**} \quad \text{and} \quad |c_i| \leqslant C\|h\|_*.$$

Proof. For the proof of this result we will consider first the basic case $k = 1$, $\xi_1 = 0$, namely the problem

$$\begin{cases} \phi'' + pU^{p-1}\phi - \phi = h + cZ\chi & \text{in } \mathbb{R}, \\ \lim\limits_{|x|\to+\infty} \phi(x) = 0, \end{cases} \tag{33}$$

where $Z = U'$, $\chi(x) = 1$ if $|x| < \frac{R}{2}$ and $= 0$ otherwise. We will find a solution of this problem by means of an explicit formula.

The function Z solves the homogeneous equation

$$\phi'' + pU^{p-1}\phi - \phi = 0, \quad x \in \mathbb{R}.$$

Its asymptotic behavior is given by

$$Z(x) \sim e^{-|x|} \quad \text{as } |x| \to +\infty.$$

One can find a second, linearly independent solution $\tilde{Z}(x)$ of this equation normalized such that the (constant) Wronskian $\tilde{Z}Z' - Z\tilde{Z}'$ is identically equal to 1. Its asymptotic behavior is then given by

$$\tilde{Z}(x) \sim e^{|x|} \quad \text{as } |x| \to +\infty.$$

For a bounded function h with $\|h\|_{**} < +\infty$ let us choose the constant c as

$$c = -\frac{\int_{-\infty}^{\infty} h(s)Z(s)\,ds}{\int_{-\infty}^{\infty} Z(s)^2\chi\,ds}.$$

The formula of variation of parameters then gives us a solution ϕ of problem (33) as

$$\phi(x) := T_0(h) = -Z(x)\int_0^x \tilde{h}(s)\tilde{Z}(s)\,ds + \tilde{Z}(x)\int_{-\infty}^x \tilde{h}(s)Z(s)\,ds, \tag{34}$$

where $\tilde{h} = h + cZ\chi$. Observe that with this choice of c we have $\int_{-\infty}^{\infty} \tilde{h}(s)Z(s)\,ds = 0$ and thus, if we just allow $R > 1$, we find a constant C independent of R such that

$$\|T_0(h)\|_* \leqslant C\|h\|_{**}.$$

T_0 of course defines a linear operator in h. We want to use this operator in order to construct, by linear perturbations, an inverse with similar properties for the full equation (32). First, we observe that the problem

$$\begin{cases} \phi'' + pU_j^{p-1}\phi - \phi = h + cZ_j\chi_j & \text{in } \mathbb{R}, \\ \lim\limits_{|x|\to+\infty} \phi(x) = 0 \end{cases} \tag{35}$$

has a solution given by

$$\phi := T_j(h) = \tau_{\xi_j} T(\tau_{-\xi_j} h),$$

where $\tau_\xi h(x) = h(x+\xi_j)$, which satisfies a similar bound, provided that $R > 1$. In order to solve problem (32) we consider now a smooth cut-off function $\eta(s)$ with $\eta(s) = 1$ if $s < 1$ and $= 0$ if $s > 2$. We also set

$$\eta_j(x) = \eta\big(|x-\xi_j|/R\big).$$

We look for a solution ϕ of the form

$$\phi = \sum \eta_j \phi_j + \psi, \tag{36}$$

where ϕ_j's and ψ solve the following coupled linear system:

$$-\psi'' + \left(1 - f(\mathbf{w})\left(1 - \sum_{j=1}^{k} \chi_j\right)\right)\psi = g(\phi_1, \dots, \phi_k, h), \tag{37}$$

with

$$g(\phi_1, \dots, \phi_k, h) = \sum_{j=1}^{k} \big(2\eta_j'\phi_j' + \eta_j''\phi_j + \eta_j\big(f(\mathbf{w}) - f(w_j)\big)\big)\phi_j - \left(1 - \sum_{j=1}^{k} \chi_j\right)h, \tag{38}$$

and

$$\phi_j'' + \big(f(w_j) - 1\big)\phi_j = -f(\mathbf{w})\chi_j\psi + \chi_j h + c_j Z_j \chi_j, \quad j = 1, \dots, k. \tag{39}$$

Here we have denoted $f(w) = pw^{p-1}$. Let us assume that $\|\phi_j\|_*$ is finite for all j. Then Eq. (37) has a unique bounded solution ψ for R large enough. More precisely, we see that if $\sigma' < \sigma$ then

$$\big|g(\phi_1, \dots, \phi_k, h)\big| \leqslant e^{-\alpha R}\left[\|h\|_{**} + \sum_{j=1}^{k} \|\phi_j\|_* e^{-\sigma'|x-\xi_j|}\right],$$

for some $\alpha > 0$ depending on σ and σ'. Since for all R large enough we have that $f(\mathbf{w})(1 - \sum_j \chi_j))$ becomes as small as we wish, then by the use of a suitable barrier we get as well that this solution ψ satisfies

$$\big|\psi(x)\big| \leqslant Ce^{-\alpha R} \sum_{j=1}^{k} e^{-\sigma'|x-\xi_j|}.$$

ψ defines a linear operator of the $k+1$-tuple $(\phi_1, \dots, \phi_k, h)$. Thus, in order to get a solution of problem (32) we just need to solve the linear system

$$\phi_j = T_j\big(-f(\mathbf{w})\chi_j\psi(\phi_1,\dots,\phi_k,h) + h\chi_j\big), \quad j = 1,\dots,k. \tag{40}$$

Let us observe that, by construction, if $\sigma - \sigma' < p - 1$, then

$$\big\| f(\mathbf{w})\chi_j\psi(\phi_1,\dots,\phi_k,0)\big\|_{**} \leqslant Ce^{-\alpha R}\sum_{j=1}^{k}\|\phi_j\|_*.$$

Using this, the boundedness of the operator T_j and Banach fixed point theorem, the existence of a unique solution $(\phi_1,\dots,\phi_k)$ follows, which besides satisfies

$$\sum_{j=1}^{k}\|\phi_j\|_* \leqslant C\|h\|_{**}.$$

This k-tuple determines ϕ given by (36), as a linear operator in h with the desired bounds. This concludes the proof of the proposition. □

4. Proof of Theorem 1

We start by restating problem (12) in the following way.

$$\begin{cases} S(v) := v'' - v + v^{p-\varepsilon}e^{s\varepsilon} + M_1(v) = 0 \quad \text{on } (-\infty,\infty), \\ v > 0, \quad v(s) \to 0 \quad \text{as } s \to \pm\infty, \end{cases} \tag{41}$$

where

$$\begin{aligned} M_1(v) = {} & -\frac{\varepsilon\ell e^{-(p+1)s}}{2}\frac{v'-v}{(1+\varepsilon\ell e^{-(p+1)s}(v'-v)^2)^{\frac{3}{2}}}\big(-(p+1)(v'-v)^2 + 2(v'-v)v''\big) \\ & + (v''-v)\left(1 - \frac{1}{\sqrt{1+\varepsilon\ell e^{-(p+1)s}(v'-v)^2}}\right). \end{aligned} \tag{42}$$

We consider now points ξ_i chosen according to formula (19)–(20) and the function $\mathbf{w}$ defined by (31) for these ξ_i's. We shall look for a solution to (41) of the form

$$v = \mathbf{w} + \phi$$

for a small $\varepsilon > 0$. We write problem (41) in terms of ϕ as

$$\begin{cases} \phi'' + p\mathbf{w}^{p-1}\phi - \phi = R - N_1(\phi) - N_2(\phi) \quad \text{in } \mathbb{R}, \\ \lim\limits_{|x|\to+\infty}\phi(x) = 0, \end{cases} \tag{43}$$

where

$$R = S(\mathbf{w}), \qquad N_1(\phi) = M_1(\mathbf{w}+\phi) - M_1(\mathbf{w})$$

and

$$N_2(\phi) = \left[(\mathbf{w}+\phi)^{p-\varepsilon} e^{s\varepsilon} - (\mathbf{w}+\phi)^p\right] + (\mathbf{w}+\phi)^p - \mathbf{w}^p - p\mathbf{w}^{p-1}\phi.$$

Instead of dealing directly with (43) we consider the intermediate nonlinear problem

$$\begin{cases} \phi'' + p\mathbf{w}^{p-1}\phi - \phi = R - N_1(\phi) - N_2(\phi) + \displaystyle\sum_{j=0}^{k} c_i Z_i \chi_i \quad \text{in } \mathbb{R}, \\ \displaystyle\lim_{|x|\to+\infty} \phi(x) = 0. \end{cases} \tag{44}$$

Let T be the operator defined in Proposition 1. Then we obtain a solution of (44) if ϕ solves the fixed point problem

$$\phi = T\big(R - N_1(\phi) - N_2(\phi)\big). \tag{45}$$

Let us be more explicit in what regards to the size of the expression for the error of approximation of $\mathbf{w}$. We have that

$$\begin{aligned} R = {} & \mathbf{w}^{p-\varepsilon} e^{s\varepsilon} - \sum_{i=1}^{k} U_i^p \\ & - \frac{\varepsilon\ell e^{-(p+1)s}}{2} \frac{\mathbf{w}' - \mathbf{w}}{(1+\varepsilon\ell e^{-(p+1)s}(\mathbf{w}'-\mathbf{w})^2)^{3/2}} \big(-(p+1)(\mathbf{w}'-\mathbf{w})^2 + 2(\mathbf{w}'-\mathbf{w})\mathbf{w}''\big) \\ & + (\mathbf{w}''-\mathbf{w})\left(1 - \frac{1}{\sqrt{1+\varepsilon\ell e^{-(p+1)s}(\mathbf{w}'-\mathbf{w})^2}}\right). \end{aligned} \tag{46}$$

From this expression we directly check the validity of the following estimate:

$$\|R\|_{**} \leqslant C\varepsilon^{\lambda}$$

with $0 < \lambda < 1$. Let us consider the region of all functions ϕ of class C^2 for which $\|\phi\|_* \leqslant M\varepsilon^{\lambda}$ for a large constant M. We check directly that for ϕ_1, ϕ_2 in this region we have

$$\big\|N_1(\phi_1) - N_1(\phi_2)\big\|_{**} + \big\|N_2(\phi_1) - N_2(\phi_2)\big\|_{**} \leqslant C\varepsilon^{\alpha}\|\phi_1 - \phi_2\|_*.$$

We conclude from these estimates and the boundedness of the operator T that the fixed point problem (45) actually has a unique solution ϕ in the region $\|\phi\|_* \leqslant M\varepsilon^{\lambda}$ for some suitably chosen M. The dependence of ϕ on the points ξ is by construction continuous. It only remains to choose these points in such a way that the constants c_i are all zero.

Testing Eq. (44) against Z_i for $i = 1, \dots, k$, we obtain an almost diagonal system for the relations $c_i = 0$ for all i. In fact we obtain that these relations hold if

$$\int_{-\infty}^{\infty} R Z_i + o(\varepsilon) = 0 \quad \text{for all } i = 1, \dots, k, \tag{47}$$

where the term $o(\varepsilon)$ encodes a continuous function of the parameters Λ_i which approaches zero uniformly in the considered range as $\varepsilon \to 0$. Let us observe that

$$\int_{-\infty}^{\infty} RZ_i = \int_{-\infty}^{\infty} S(\mathbf{w})\partial_{\xi_i}\mathbf{w} = \partial_{\xi_i} E_\varepsilon(\mathbf{w}).$$

According to the expansion in C^1-sense found for $E_\varepsilon(\mathbf{w})$ in Lemma 1, we then have that system (47) takes the form

$$\varepsilon\big(\nabla\Psi_k(\Lambda) + o(1)\big) = 0,$$

where the quantity $o(1)$ goes to zero uniformly on the considered region for the parameters Λ_i and depends continuously on them. We recall that according to Lemma 2 the functional Ψ_k possesses one and only one critical point Λ^*, which is nondegenerate. The above equation thus have a solution which lies close to Λ^*. The proof of the theorem is concluded.

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