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The Brezis–Nirenberg problem near criticality in dimension 3

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Abstract

We consider the problem of finding positive solutions of $\Delta u + \lambda u + u^q = 0$ in a bounded, smooth domain Ω in $\mathbb{R}^3$, under zero Dirichlet boundary conditions. Here q is a number close to the critical exponent 5 and $0 < \lambda < \lambda_1$. We analyze the role of Green's function of $\Delta + \lambda$ in the presence of solutions exhibiting single and multiple bubbling behavior at one point of the domain when either q or λ are regarded as parameters. As a special case of our results, we find that if $\lambda^* < \lambda < \lambda_1$, where λ^* is the *Brezis–Nirenberg number*, i.e., the smallest value of λ for which least energy solutions for $q = 5$ exist, then this problem is solvable if $q > 5$ and $q - 5$ is sufficiently small.

Résumé

Nous considérons le problème de l'existence de solutions positives de $\Delta u + \lambda u + u^q = 0$ dans un domaine borné, régulier Ω de $\mathbb{R}^3$, avec conditions de Dirichlet nulles au bord. Ici q est un nombre proche de l'exposant critique 5 et $0 < \lambda < \lambda_1$. Nous analysons le rôle de la fonction de Green $\Delta + \lambda$ en présence de solutions qui mettent en évidence un comportement de type simple bulle ou bulle multiple quand soit q, soit λ sont considérés comme paramètres. Comme cas particulier de nos résultats, nous trouvons que pour $\lambda^* < \lambda < \lambda_1$, où λ^* est le *nombre de Brezis–Nirenberg*, i.e., la

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plus petite valeur de λ pour laquelle des solutions d'énergie minimale pour $q = 5$ existent, alors le problème possède des solutions si $q > 5$ et si $q - 5$ est suffisamment petit.

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1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. This paper deals with construction of solutions to the boundary value problem:

$$\begin{cases} \Delta u + \lambda u + u^q = 0 & \text{in } \Omega. \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Integrating the equation against a first eigenfunction of the Laplacian yields that a necessary condition for solvability of (1.1) is $\lambda < \lambda_1$. On the other hand, if $1 < q < 5$ and $0 < \lambda < \lambda_1$, a solution may be found as follows. Let us consider the Rayleigh quotient:

$$Q_\lambda(u) \equiv \frac{\int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2}{(\int_\Omega |u|^{q+1})^{2/(q+1)}}, \quad u \in H_0^1(\Omega) \setminus \{0\}, \tag{1.2}$$

and set:

$$S_\lambda \equiv \inf_{u \in H_0^1(\Omega)\setminus\{0\}} Q_\lambda(u).$$

The constant S_λ is achieved thanks to compactness of Sobolev embedding if $q < 5$, and a suitable scalar multiple of it turns out to be a solution of (1.1). The case $q \geqslant 5$ is considerably more delicate: for $q = 5$ compactness of the embedding is lost while for $q > 5$ there is no such embedding. This obstruction is not just technical for the solvability question, but essential. Pohozaev [19] showed that if Ω is strictly star-shaped then no solution of (1.1) exists if $\lambda \leqslant 0$ and $q \geqslant 5$. Let S_0 be the best constant in the critical Sobolev embedding,

$$S_0 = \inf_{u \in C_0^1(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2}{(\int_{\mathbb{R}^3} |u|^6)^{1/3}}.$$

Let us consider $q = 5$ in (1.2) and the number:

$$\lambda^* \equiv \inf\{\lambda > 0:\ S_\lambda < S_0\}. \tag{1.3}$$

In the well-known paper [5], Brezis and Nirenberg established that $0 < \lambda^* < \lambda_1$ and, as a consequence, that S_λ is achieved for $\lambda^* < \lambda < \lambda_1$, hence (1.1) is solvable in this range. When Ω is a ball they find that $\lambda^* = \lambda_1/4$ and that no solution exists for $\lambda \leqslant \lambda^*$.

Let us assume now that $q > 5$. In this case Sobolev embedding fails and the quantity S_λ may only be interpreted as zero. Thus, no direct variational approach applies to find existence of solutions. Consequences of the analysis of this paper are the following existence and multiplicity results for Problem (1.1) in the super-critical regime when q is sufficiently close to 5.

Theorem 1. (a) *Assume that $\lambda^* < \lambda < \lambda_1$, where λ^* is the number given by* (1.3). *Then there exists a number $q_1 > 5$ such that Problem* (1.1) *is solvable for any $q \in (5, q_1)$.*

(b) *Assume that Ω is a ball and that $\lambda^* = \lambda_1/4 < \lambda < \lambda_1$. Then, given $k \geqslant 1$ there exists a number $q_k > 5$ such that Problem* (1.1) *has at least k radial solutions for any $q \in (5, q_k)$.*

While the result of Part (a) resembles that by Brezis and Nirenberg when $q = 5$, in reality the solution we find has a very different nature: it blows up as $q \downarrow 5$ developing a *single bubble* around a certain point inside the domain. The other solutions predicted by Part (b) blow-up only at the origin but exhibit *multiple bubbling*. Let us make this terminology somewhat more precise. By a *blowing-up solution* for (1.1) near the critical exponent we mean an unbounded sequence of solutions u_n of (1.1) for $\lambda = \lambda_n$ bounded, and $q = q_n \to 5$. Setting:

$$M_n \equiv \alpha^{-1} \max_{\Omega} u_n = \alpha^{-1} u_n(x_n) \to +\infty$$

with $\alpha > 0$ to be chosen, we see then that the scaled function

$$v_n(y) \equiv M_n^{-1} u_n\big(x_n + M_n^{-(q_n-1)/2} y\big),$$

satisfies

$$\Delta v_n + v_n^{q_n} + M_n^{-(q_n-1)} \lambda_n v_n = 0$$

in the expanding domain $\Omega_n = M_n^{(q_n-1)/2}(\Omega - x_n)$. Assuming for instance that x_n stays away from the boundary of Ω, elliptic regularity implies that locally over compacts around the origin, v_n converges up to subsequences to a positive solution of

$$\Delta w + w^5 = 0$$

in entire space, with $w(0) = \max w = \alpha$. It is known, see [8], that for the convenient choice $\alpha = 3^{1/4}$, this solution is explicitly given by:

$$w(z) = \frac{3^{1/4}}{\sqrt{1 + |z|^2}}.$$

which corresponds precisely to an extremal of the Sobolev constant S_0, see [2,24]. Coming back to the original variable, we expect then that "near x_n" the behavior of $u_n(y)$ can be approximated as

$$u_n(y) = \frac{3^{1/4} M_n}{\sqrt{1 + M_n^4 |x - x_n|^2}} \big(1 + \mathrm{o}(1)\big). \tag{1.4}$$

Since the convergence in expanded variables is only local over compacts, it is not clear how far from x_n the approximation (1.4) holds true, even if only one maximum point x_n exists. We say that the solution $u_n(x)$ is a *single bubble* if (1.4) holds with $\mathrm{o}(1) \to 0$ uniformly in Ω.

The solution predicted by Part (a) of Theorem 1 has this property around a point of the domain that will be precised below, while those of Part (b) have the form of a "tower" of single bubbles centered at the origin. As we shall see, radial symmetry is not needed for the presence of these solutions: just symmetry with respect to the three coordinate planes around one point of the domain suffices.

The results of [6] concerning asymptotic analysis of radial solutions in a ball when the exponent approaches critical from below, suggest that the object ruling the location of blowing-up in single-bubble solutions of (1.1) is *Robin's function* g_λ defined as follows. Let $\lambda < \lambda_1$ and consider Green's function $G_\lambda(x, y)$, solution for any given $x \in \Omega$ of

$$\begin{cases} -\Delta_y G_\lambda - \lambda G_\lambda = \delta_x, & y \in \Omega, \\ G_\lambda(x, y) = 0, & y \in \partial\Omega. \end{cases}$$

Let $H_\lambda(x, y) = \Gamma(y - x) - G_\lambda(x, y)$ with $\Gamma(z) = 1/(4\pi|z|)$, be its regular part. In other words, $H_\lambda(x, y)$ can be defined as the unique solution of the problem:

$$\begin{cases} \Delta_y H_\lambda + \lambda H_\lambda = \lambda \Gamma(x - y), & y \in \Omega, \\ H_\lambda = \Gamma(x - y), & y \in \partial\Omega. \end{cases}$$

Let us consider Robin's function of G_λ, defined as

$$g_\lambda(x) \equiv H_\lambda(x, x).$$

It turns out that $g_\lambda(x)$ is a smooth function (we provide a proof of this fact in Appendix A) which goes to $+\infty$ as x approaches $\partial\Omega$. Its minimum value is not necessarily positive. In fact this number is strictly decreasing in λ. It is strictly positive when λ is close to 0 and approaches $-\infty$ as $\lambda \uparrow \lambda_1$. It is suggested in [6] and recently proved by Druet in [14] that the number λ^* given by (1.3) can be characterized as

$$\lambda^* = \sup\Big\{\lambda > 0\colon \min_{\Omega} g_\lambda > 0\Big\}. \tag{1.5}$$

Besides, it is shown in [14] that least energy solutions u_λ for $\lambda \downarrow \lambda^*$ constitute a single-bubble with blowing-up near the set where g_{λ_*} attains its minimum value zero.

We consider here the role of *nontrivial critical values* of g_λ in existence of solutions of (1.1). In fact their role is intimate, not only in the critical case $q = 5$ and in the sub-critical $q = 5 - \varepsilon$. More interesting, their connection with solvability of (1.1) for powers above critical is found. In fact a phenomenon apparently unknown even in the case of the ball is established, which puts in evidence an amusing duality between the sub- and super-critical cases.

The meaning we give of a nontrivial critical value of g_λ is as follows: let $\mathcal{D}$ be an open subset of Ω with smooth boundary. We recall that g_λ *links nontrivially in* $\mathcal{D}$ *at critical level* $\mathcal{G}_\lambda$ *relative to* B *and* B_0 if B and B_0 are closed subsets of $\overline{\mathcal{D}}$ with B connected and $B_0 \subset B$ such that the following conditions hold: if we set $\Gamma \equiv \{\Phi \in C(B, \mathcal{D})\colon \Phi|_{B_0} = \mathrm{Id}\}$, then

$$\sup_{y \in B_0} g_\lambda(y) < \mathcal{G}_\lambda \equiv \inf_{\Phi \in \Gamma} \sup_{y \in B} g_\lambda\bigl(\Phi(y)\bigr), \tag{1.6}$$

and for all $y \in \partial\mathcal{D}$ such that $g_\lambda(y) = \mathcal{G}_\lambda$, there exists a vector τ_y tangent to $\partial\mathcal{D}$ at y such that

$$\nabla g_\lambda(y) \cdot \tau_y \neq 0. \tag{1.7}$$

Under these conditions a critical point $\bar{y} \in \mathcal{D}$ of g_λ with $g_\lambda(\bar{y}) = \mathcal{G}_\lambda$ exists, as a standard deformation argument involving the negative gradient flow of g_λ shows. Condition (1.6) is a general way of describing a change of topology in the level sets $\{g_\lambda \leqslant c\}$ in $\mathcal{D}$ taking place at $c = \mathcal{G}_\lambda$, while (1.7) prevents criticality at this level collapsing into the boundary. It is easy to check that the above conditions hold if:

$$\inf_{x \in \mathcal{D}} g_\lambda(x) < \inf_{x \in \partial\mathcal{D}} g_\lambda(x), \quad \text{or} \quad \sup_{x \in \mathcal{D}} g_\lambda(x) > \sup_{x \in \partial\mathcal{D}} g_\lambda(x),$$

namely the case of (possibly degenerate) local minimum or maximum points of g_λ. The level $\mathcal{G}_\lambda$ may be taken in these cases respectively as that of the minimum and the maximum of g_λ in $\mathcal{D}$. These hold also if g_λ is C^1-close to a function with a nondegenerate critical point in $\mathcal{D}$. We call $\mathcal{G}_\lambda$ a nontrivial critical level of g_λ in $\mathcal{D}$.

Theorem 2. *Let us assume that there is a set* $\mathcal{D}$ *where* g_λ *has a nontrivial critical level* $\mathcal{G}_\lambda$.

(a) *Assume that* $\mathcal{G}_\lambda < 0$, $q = 5 + \varepsilon$. *Then Problem* (1.1) *is solvable for all sufficiently small* $\varepsilon > 0$. *More precisely, there exists a solution* u_ε *of* (1.1) *of the form,*

$$u_\varepsilon(y) = \frac{3^{1/4} M_\varepsilon}{\sqrt{1 + M_\varepsilon^4 |y - \zeta_\varepsilon|^2}}\bigl(1 + \mathrm{o}(1)\bigr), \tag{1.8}$$

where $\mathrm{o}(1) \to 0$ *uniformly in* $\overline{\Omega}$ *as* $\varepsilon \to 0$,

$$M_\varepsilon = 8\sqrt{2(-\mathcal{G}_\lambda)\varepsilon^{-1}}, \tag{1.9}$$

and $\zeta_\varepsilon \in \mathcal{D}$ *is such that* $g_\lambda(\zeta_\varepsilon) \to \mathcal{G}_\lambda$, $\nabla g_\lambda(\zeta_\varepsilon) \to 0$, *as* $\varepsilon \to 0$.

(b) *Assume that* $\mathcal{G}_\lambda > 0$, $q = 5 - \varepsilon$. *Then Problem* (1.1) *has a solution* u_ε *of* (1.1) *exactly as in Part* (a) *but with* $M_\varepsilon = 8\sqrt{2\mathcal{G}_\lambda \varepsilon^{-1}}$.

We observe that Theorem 1, Part (a) follows from Part (a) of the above result making use of the characterization (1.5) of the number λ^*. The result of Part (b) recovers the asymptotics found for the radial solution of (1.1) when Ω is a ball and $0 < \lambda < \lambda_1/4$ in Theorem 1 of [6].

Our next result shows in particular that solutions with multiple bubbling from above the critical exponent in a domain exhibiting symmetries exist. We say that $\Omega \subset \mathbb{R}^3$ is *symmetric with respect to the coordinate planes* if for all $(y_1, y_2, y_3) \in \Omega$ we have that

$$(-y_1, y_2, y_3), (y_1, -y_2, y_3), (y_1, y_2, -y_3) \in \Omega.$$

If $0 \in \Omega$, one defines:

$$\tilde{\lambda}^* \equiv \inf\bigl\{\lambda > 0\colon\ g_\lambda(0) < 0\bigr\}.$$

Theorem 3. *Assume that* $0 \in \Omega$, *and that* Ω *is symmetric with respect to the coordinate planes.*

(a) *Assume that* $\tilde{\lambda}^* < \lambda < \lambda_1$ *and let* $q = 5 + \varepsilon$. *Then, given* $k \geqslant 1$, *there exists for all sufficiently small* $\varepsilon > 0$ *a solution* u_ε *of Problem* (1.1) *of the form,*

$$u_\varepsilon(x) = \sum_{j=1}^{k} \frac{3^{1/4} M_{j\varepsilon}}{\sqrt{1 + M_{j\varepsilon}^4 |x|^2}} \bigl(1 + \mathrm{o}(1)\bigr),$$

where $\mathrm{o}(1) \to 0$ *uniformly in* $\overline{\Omega}$ *and for* $j = 1, \ldots, k$,

$$M_{j\varepsilon} \equiv 8\sqrt{2\bigl(-g_\lambda(0)\bigr)k^{-1}} \left(\frac{32\sqrt{2}}{\pi}\right)^{j-1} \frac{(k-j)!}{(k-1)!} \varepsilon^{1/2-j}.$$

(b) *Assume that* $q = 5$. *Then for all* $\lambda > \tilde{\lambda}^*$ *sufficiently close to* $\tilde{\lambda}^*$ *there exists a solution* u_λ *of Problem* (1.1) *of the form,*

$$u_\lambda(x) = \frac{3^{1/4} M_\lambda}{\sqrt{1 + M_\lambda^4 |x|^2}} \bigl(1 + \mathrm{o}(1)\bigr),$$

where $M_\lambda \equiv \frac{1}{2}\sqrt{\tilde{\lambda}^*(-g_\lambda(0))^{-1}}$.

The solution predicted by Part (a) is a superposition of k bubbles with respective blow-up orders $\varepsilon^{1/2-j}$, $j = 1, \ldots, k$. We observe that in the case of a ball $\tilde{\lambda}^* = \lambda^* = \lambda_1/4$ and Theorem 2, Part (b) thus follows.

Part (b) shows that a domain may possess *Brezis–Nirenberg numbers* other than λ^*, where a bubbling branch of solutions at the critical exponent stems to the right. We remark that $\frac{\partial g_\lambda}{\partial \lambda}(0) < 0$ (see Appendix A), so that actually $M_\lambda \sim (\lambda - \tilde{\lambda}^*)^{-1/2}$. Without any symmetry assumption, our next result states that a similar phenomenon holds true at any number $\lambda = \lambda_0$ for which g_{λ_0} has either a local minimizer, or a nondegenerate critical point with value zero.

Theorem 4. *Assume that $q = 5$ and that for a number $\lambda = \lambda_0$, one of the two situations holds*:

(a) *Either there is an open, bounded set $\mathcal{D}$ of Ω such that*

$$0 = \inf_{\mathcal{D}} g_{\lambda_0} < \inf_{\partial\mathcal{D}} g_{\lambda_0}.$$

(b) *Or there is a $\zeta_0 \in \Omega$ such that*

$$g_{\lambda_0}(\zeta_0) = 0, \quad \nabla g_{\lambda_0}(\zeta_0) = 0,$$

and $D^2 g_{\lambda_0}(\zeta_0)$ is nonsingular.

Then for all $\lambda > \lambda_0$ sufficiently close to λ_0 there exists a solution u_λ of Problem (1.1) *of the form,*

$$u_\lambda(x) = \frac{3^{1/4} M_\lambda}{\sqrt{1 + M_\lambda^4 |x - \zeta_\lambda|^2}} \big(1 + \mathrm{o}(1)\big), \tag{1.10}$$

where $\mathrm{o}(1) \to 0$ *uniformly in $\overline{\Omega}$ as $\lambda \downarrow \lambda_0$. Here $\zeta_\lambda \in \mathcal{D}$ in case* (a) *and $\zeta_\lambda \to \zeta_0$ in case* (b). *Besides,*

$$M_\lambda = \beta \sqrt{\frac{\lambda_0}{-g_\lambda(\zeta_\lambda)}},$$

with

$$A_-(\lambda - \lambda_0) \leqslant -g_\lambda(\zeta_\lambda) \leqslant A_+(\lambda - \lambda_0)$$

for certain positive constants $A_\pm$.

The rest of this paper will be devoted to the proofs of Theorems 2–4. The proofs actually provide more accurate information on the solutions found, in particular about the asymptotics for the solutions in Theorem 2 when we allow for instance $q > 5$ and λ moves left toward λ^*, case in which two single-bubble solutions are observed with blow-up orders $\sim \varepsilon^{-1/4}$.

Bubbling at or near the critical exponent in its relation with Green's function of the domain has been broadly considered in the literature. In particular, we refer the reader to [3,4,15,17,20–22], and also to [1,18,23] and their references for related results under Neumann boundary conditions. Conditions (1.6), (1.7) were used in [13] in the construction of spike-patterns in nonlinear Shrödinger equations.

The analogues of the results of this paper for dimension $N \geqslant 4$ in the super-critical case are somewhat different and we will treat them in a separate work. It should be remarked that when $N \geqslant 4$ we have that $\lambda^* = 0$, and single-bubbling as $\lambda \downarrow 0$ analogous to Theorem 5 around a nondegenerate critical point of the function g_0 was established by Rey in [20]. The phenomenon of multi-bubbling in the radial case in higher dimensions was described in [11], and, with purely ODE methods, in [12]. Also through an ODE approach, multi-bubbling in the radial case was described in [9] in an equation at the critical exponent with a weight which was taken as the parameter. Bubbling from above the critical exponent when $\lambda = 0$ in domains exhibiting small holes was found in [10].

2. Energy expansion of single bubbling

Given a point $\zeta \in \mathbb{R}^3$ and a positive number μ, we denote in what follows:

$$w_{\mu,\zeta}(y) \equiv \frac{3^{1/4}}{\sqrt{1+\mu^{-2}|y-\zeta|^2}}\mu^{-1/2},$$

which correspond to all positive solutions of the problem:

$$\Delta w + w^5 = 0 \quad \text{in } \mathbb{R}^3.$$

The solutions we look for in Theorems 2, 3, Part (b), and 4 have the form $u(y) \sim w_{\mu,\zeta}(y)$ where $\zeta \in \Omega$ and μ is a very small number. It is natural to correct this initial approximation by a term that provides Dirichlet boundary conditions. We assume in all what follows that $0 < \lambda < \lambda_1$. We define $\pi_{\mu,\zeta}(y)$ to be the unique solution of the problem:

$$\Delta \pi_{\mu,\zeta} + \lambda \pi_{\mu,\zeta} = -\lambda w_{\mu,\zeta} \quad \text{in } \Omega \qquad \text{with } \pi_{\mu,\zeta} = -w_{\mu,\zeta} \quad \text{on } \partial\Omega. \tag{2.1}$$

Fix a small positive number μ and a point $\zeta \in \Omega$. We consider as a first approximation of the solution one of the form:

$$U_{\mu,\zeta}(y) = w_{\mu,\zeta} + \pi_{\mu,\zeta}. \tag{2.2}$$

Observe that $U = U_{\mu,\zeta}$ satisfies then the equation:

$$\Delta U + \lambda U + w_{\mu,\zeta}^5 = 0 \quad \text{in } \Omega, \qquad U = 0 \quad \text{on } \partial\Omega.$$

Classical solutions to (1.1) correspond to critical points of the energy functional,

$$E_{q,\lambda}(u) \equiv \frac{1}{2}\int_\Omega |Du|^2 - \frac{\lambda}{2}\int_\Omega |u|^2 - \frac{1}{q+1}\int_\Omega |u|^{q+1}. \tag{2.3}$$

If there was a solution very close to U_{μ^*,ζ^*} for a certain pair (μ^*,ζ^*), then we would formally expect $E_{q,\lambda}$ to be nearly stationary with respect to variations of (μ,ζ) on $U_{\mu,\zeta}$ around this point. Under this intuitive basis, it seems important to understand critical points of the functional $(\mu,\zeta)\mapsto E_{q,\lambda}(U_{\mu,\zeta})$. Next we will find explicit asymptotic expressions for this functional. For $q=5$ we have the following result.

Lemma 2.1. *For any $\sigma>0$, as $\mu\to 0$, the following expansion holds*:

$$E_{5,\lambda}(U_{\mu,\zeta}) = a_0 + a_1\mu g_\lambda(\zeta) + a_2\mu^2\lambda - a_3\mu^2 g_\lambda^2(\zeta) + \mu^{3-\sigma}\theta(\mu,\zeta), \tag{2.4}$$

where for $j=0,1,2$, $i=0,1$, $i+j\leqslant 2$, the function $\mu^j\frac{\partial^{i+j}}{\partial\zeta^i\partial\mu^j}\theta(\mu,\zeta)$ is bounded uniformly on all small μ and ζ in compact subsets of Ω. The a_j's are explicit constants, given by relations (2.11) *below.*

The proof of this expansion makes use of the following lemma which establishes the relationship between the functions $\pi_{\mu,\zeta}(y)$ and the regular part of Green's function, $H_\lambda(\zeta,y)$. Let us consider the (unique) radial solution $\mathcal{D}_0(z)$ of the problem in entire space,

$$\begin{cases} \Delta\mathcal{D}_0 = -\lambda 3^{1/4}[1/\sqrt{1+|z|^2} - 1/|z|] & \text{in } \mathbb{R}^3,\\ \mathcal{D}_0\to 0 & \text{as } |z|\to\infty. \end{cases}$$

Then $\mathcal{D}_0(z)$ is a $C^{0,1}$ function with $\mathcal{D}_0(z)\sim |z|^{-1}\log|z|$ as $|z|\to+\infty$.

Lemma 2.2. *For any $\sigma>0$ we have the validity of the following expansion as $\mu\to 0$*:

$$\mu^{-1/2}\pi_{\mu,\zeta}(y) = -4\pi 3^{1/4}H_\lambda(\zeta,y) + \mu\mathcal{D}_0\left(\frac{y-\zeta}{\mu}\right) + \mu^{2-\sigma}\theta(\mu,y,\zeta),$$

where for $j=0,1,2$, $i=0,1$, $i+j\leqslant 2$, the function $\mu^j\frac{\partial^{i+j}}{\partial\zeta^i\partial\mu^j}\theta(\mu,y,\zeta)$ is bounded uniformly on $y\in\Omega$, all small μ and ζ in compact subsets of Ω.

Proof. We recall that $H_\lambda(y,\zeta)$ satisfies the equation:

$$\begin{cases} \Delta_y H_\lambda + \lambda H_\lambda = \lambda\Gamma(y-\zeta), & y\in\Omega,\\ H_\lambda(y,\zeta) = \Gamma(y-\zeta), & y\in\partial\Omega, \end{cases}$$

where $\Gamma(z)\equiv\frac{1}{4\pi|z|}$, while $\pi_{\mu,\zeta}$ satisfies:

$$\begin{cases} \Delta\pi + \lambda\pi = -\lambda w_{\mu,\zeta} & \text{in } \Omega,\\ \pi = -w_{\mu,\zeta} & \text{on } \partial\Omega. \end{cases}$$

Let us set $\mathcal{D}_1(y)\equiv\mu\mathcal{D}_0(\mu^{-1}(y-\zeta))$ so that $\mathcal{D}_1$ satisfies:

$$-\Delta\mathcal{D}_1 = \lambda\big[\mu^{-1/2} w_{\mu,\zeta}(y) - 4\pi 3^{1/4}\Gamma(y-\zeta)\big] \quad \text{in } \Omega,$$
$$\mathcal{D}_1 \sim \mu^2 \log\mu \quad \text{on } \partial\Omega \text{ as } \mu \to 0.$$

Let us write:

$$S_1(y) \equiv \mu^{-1/2}\pi_{\mu,\zeta}(y) + 4\pi 3^{1/4} H_\lambda(\zeta, y) - \mathcal{D}_1(y).$$

With the notations of Lemma 2.2, this means

$$S_1(y) = \mu^{2-\sigma}\theta(\mu, y, \zeta).$$

Observe that for $y \in \partial\Omega$, as $\mu \to 0$,

$$\mu^{-1/2}\pi_{\mu,\zeta}(y) + 4\pi 3^{1/4} H_\lambda(\zeta, y) = 3^{1/4}\left[\frac{1}{\sqrt{\mu^2 + |y-\zeta|^2}} - \frac{1}{|y-\zeta|}\right] \sim \mu^2 |y-\zeta|^{-3}.$$

Using the above equations we find that S_1 satisfies:

$$\begin{cases} \Delta S_1 + \lambda S_1 = -\lambda\mathcal{D}_1 & \text{in } \Omega, \\ S_1 = \mathrm{O}(\mu^2\log\mu) \text{ as } \mu \to 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Let us observe that, for any $p > 3$,

$$\int_\Omega \big|\mathcal{D}_1(y)\big|^p \, \mathrm{d}y \leqslant \mu^{p+3} \int_{\mathbb{R}^3} \big|\mathcal{D}_0(z)\big|^p \, \mathrm{d}z,$$

so that $\|\mathcal{D}_1\|_{L^p} \leqslant C_p \mu^{1+3/p}$. Since $0 < \lambda < \lambda_1$, elliptic estimates applied to Eq. (2.5) yield that, for any $\sigma > 0$, $\|S_1\|_\infty = \mathrm{O}(\mu^{2-\sigma})$ uniformly on ζ in compact subsets of Ω. This yields the assertion of the lemma for $i, j = 0$.

Let us consider now the quantity $S_2 = \partial_\zeta S_1$. Then we have:

$$\begin{cases} \Delta S_2 + \lambda S_2 = -\lambda\partial_\zeta\mathcal{D}_1 & \text{in } \Omega, \\ S_2 = \mathrm{O}(\mu^2\log\mu) \text{ as } \mu \to 0 & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

Now, $\partial_\zeta\mathcal{D}_1(y) = -\nabla\mathcal{D}_0((y-\zeta)/\mu)$, so that for any $p > 3/2$,

$$\int_\Omega \big|\partial_\zeta\mathcal{D}_1(y)\big|^p \, \mathrm{d}y \leqslant \mu^{3+p} \int_{\mathbb{R}^3} \big|\nabla\mathcal{D}_0(z)\big|^p \, \mathrm{d}z.$$

We conclude from these facts that $\|S_2\|_\infty = \mathrm{O}(\mu^{2-\sigma})$ for any $\sigma > 0$. This gives the proof of the lemma for $i = 1,\ j = 0$. Let us set now $S_3 \equiv \mu\partial_\mu S_1$. Then

$$\begin{cases} \Delta S_3 + \lambda S_3 = -\lambda\mu\partial_\mu D & \text{in } \Omega, \\ S_3 = \mathrm{O}(\mu^2\log\mu) \text{ as } \mu \to 0 & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

Now,

$$\mu\partial_\mu \mathcal{D}_1(y) = \mu\partial_\mu\left[\mu\mathcal{D}_0\left(\frac{y-\zeta}{\mu}\right)\right] = \mu\big(\mathcal{D}_0 + \widetilde{\mathcal{D}}_0\big)\left(\frac{y-\zeta}{\mu}\right),$$

where $\widetilde{\mathcal{D}}_0(z) = z\cdot\nabla\mathcal{D}_0(z)$. Thus, similarly as the estimate for S_1 itself we obtain again $\|S_3\|_\infty = \mathrm{O}(\mu^{2-\sigma})$ for any $\sigma > 0$. The proof of the remaining estimates comes after applying again $\mu\partial_\mu$ to the equations obtained for S_2 and S_3 above, and the desired result comes after exactly the same arguments. This completes the proof. □

Proof of Lemma 2.1. Let us decompose:

$$\begin{aligned}
&E_{5,\lambda}(U_{\mu,\zeta}) = I + II + III + IV + V + VI,\\
&I = \int_\Omega \left[\frac{1}{2}|Dw_{\mu,\zeta}|^2 - \frac{1}{6}w_{\mu,\zeta}^6\right],\\
&II = \int_\Omega \left[Dw_{\mu,\zeta}\,D\pi_{\mu,\zeta} - w_{\mu,\zeta}^5\pi_{\mu,\zeta}\right],\\
&III = \frac{1}{2}\int_\Omega \left[|D\pi_{\mu,\zeta}|^2 - \lambda(w_{\mu,\zeta}+\pi_{\mu,\zeta})\pi_{\mu,\zeta}\right],\\
&IV = -\frac{\lambda}{2}\int_\Omega (w_{\mu,\zeta}+\pi_{\mu,\zeta})w_{\mu,\zeta},\\
&V = -\frac{5}{2}\int_\Omega w_{\mu,\zeta}^4\pi_{\mu,\zeta}^2,\\
&VI = -\frac{1}{6}\int_\Omega \left[(w_{\mu,\zeta}+\pi_{\mu,\zeta})^6 - w_{\mu,\zeta}^6 - 6w_{\mu,\zeta}^5\pi_{\mu,\zeta} - 15w_{\mu,\zeta}^4\pi_{\mu,\zeta}^2\right].
\end{aligned}$$

Multiplying equation $\Delta w_{\mu,\zeta} + w_{\mu,\zeta}^5 = 0$ by $w_{\mu,\zeta}$ and integrating by parts in Ω we get:

$$\begin{aligned}
I &= \frac{1}{2}\int_{\partial\Omega}\frac{\partial w_{\mu,\zeta}}{\partial\nu}w_{\mu,\zeta} + \frac{1}{3}\int_\Omega w_{\mu,\zeta}^6\\
&= \frac{1}{2}\int_{\partial\Omega}\frac{\partial w_{\mu,\zeta}}{\partial\nu}w_{\mu,\zeta} + \frac{1}{3}\int_{\mathbb{R}^3} w_{\mu,\zeta}^6 - \frac{1}{3}\int_{\mathbb{R}^3\setminus\Omega} w_{\mu,\zeta}^6.
\end{aligned}$$

Here $\partial/\partial\nu$ denotes the derivative along the unit outgoing normal at a point of $\partial\Omega$. Testing equation $\Delta w_{\mu,\zeta} + w_{\mu,\zeta}^5 = 0$ now against $\pi_{\mu,\zeta}$, we find:

$$II = \int_{\partial\Omega} \frac{\partial w_{\mu,\zeta}}{\partial\nu}\pi_{\mu,\zeta} = -\int_{\partial\Omega} \frac{\partial w_{\mu,\zeta}}{\partial\nu} w_{\mu,\zeta},$$

where we have used the fact that $\pi_{\mu,\zeta}$ solves Eq. (2.1). Testing (2.1) against $\pi_{\mu,\zeta}$ and integrating by parts, we get:

$$III = \frac{1}{2}\int_{\partial\Omega} \frac{\partial\pi_{\mu,\zeta}}{\partial\nu}\pi_{\mu,\zeta} = -\frac{1}{2}\int_{\partial\Omega} \frac{\partial\pi_{\mu,\zeta}}{\partial\nu} w_{\mu,\zeta}.$$

Recalling that $U = w_{\mu,\zeta} + \pi_{\mu,\zeta}$ solves:

$$-(\Delta U + \lambda U) = w_{\mu,\zeta}^5 \quad \text{in } \Omega, \qquad U = 0 \quad \text{on } \partial\Omega,$$

by multiplying this equation by $\pi_{\mu,\zeta}$, we get:

$$IV = \frac{1}{2}\int_{\partial\Omega} \frac{\partial U}{\partial\nu} w_{\mu,\zeta} - \frac{1}{2}\int_{\Omega} w_{\mu,\zeta}^5 \pi_{\mu,\zeta}.$$

Now, as for VI, we see from the mean value formula that

$$VI = -10\int_0^1 \mathrm{d}s(1-s)^2 \int_{\Omega} (w_{\mu,\zeta} + s\pi_{\mu,\zeta})^3 \pi_{\mu,\zeta}^3.$$

Adding up the expressions obtained above I–VI we get so far

$$E_{5,\lambda}(U) = \frac{1}{3}\int_{\mathbb{R}^3} w_{\mu,\zeta}^6 - \frac{1}{2}\int_{\Omega} w_{\mu,\zeta}^5 \pi_{\mu,\zeta} - \frac{5}{2}\int_{\Omega} w_{\mu,\zeta}^4 \pi_{\mu,\zeta}^2 + \mathcal{R}_1, \tag{2.8}$$

where

$$\mathcal{R}_1 = -\frac{1}{3}\int_{\mathbb{R}^3\setminus\Omega} w_{\mu,\zeta}^6 - 10\int_0^1 \mathrm{d}s(1-s)^2 \int_{\Omega} (w_{\mu,\zeta} + s\pi_{\mu,\zeta})^3 \pi_{\mu,\zeta}^3. \tag{2.9}$$

We will expand further the second and third integrals in the right-hand side of (2.8).

(1) Using the change of variable $y = \zeta + \mu z$ and calling $\Omega_\mu \equiv \mu^{-1}(\Omega - \zeta)$ we find that

$$A \equiv \int_{\Omega} w_{\mu,\zeta}^5 \pi_{\mu,\zeta}\, \mathrm{d}y = \mu \int_{\Omega_\mu} w_{1,0}^5(z)\mu^{-1/2}\pi_{\mu,\zeta}(\zeta + \mu z)\, \mathrm{d}z.$$

From Lemma 2.2, we have the expansion:

$$\mu^{-1/2}\pi_{\mu,\zeta}(\zeta + \mu z) = -4\pi 3^{1/4} H_\lambda(\zeta + \mu z, \zeta) + \mu\mathcal{D}_0(z) + \mu^{2-\sigma}\theta(\mu, \zeta + \mu z, \zeta).$$

We also have, according to Appendix A,

$$H_\lambda(\zeta+\mu z,\zeta)=g_\lambda(\zeta)+\frac{\lambda}{8\pi}\mu|z|+\theta_1(\zeta,\zeta+\mu z),$$

where θ_1 is a function of class C^2 with $\theta_1(\zeta,\zeta)=0$. Using these facts we obtain then that

$$A\equiv -4\pi 3^{1/4}\mu g_\lambda(\zeta)\int_{\mathbb{R}^3} w_{1,0}^5(z)\,\mathrm{d}z+\mu^2\int_{\mathbb{R}^3} w_{1,0}^5(z)\left[\mathcal{D}_0(z)-\frac{3^{1/4}}{2}\lambda|z|\right]\mathrm{d}z+\mathcal{R}_2$$

with

$$\begin{aligned}\mathcal{R}_2=\mu\int_{\Omega_\mu} w_{1,0}^5(z)\big[\theta_1(\zeta,\zeta+\mu z)+\mu^{2-\sigma}\theta(\mu,\zeta,\zeta+\mu z)\big]\,\mathrm{d}z\\ -\int_{\mathbb{R}^3\setminus\Omega_\mu} w_{1,0}^5(z)\left[\mathcal{D}_0(z)-\frac{3^{1/4}}{2}\lambda|z|\right]\mathrm{d}z\\ +4\pi 3^{1/4}\mu g_\lambda(\zeta)\int_{\mathbb{R}^3\setminus\Omega_\mu} w_{1,0}^5(z)\,\mathrm{d}z.\end{aligned}\tag{2.10}$$

To clean up the above expression for A a bit further, let us recall that

$$-\Delta\mathcal{D}_0=3^{1/4}\lambda\left[\frac{1}{\sqrt{1+|z|^2}}-\frac{1}{|z|}\right],$$

so that,

$$\begin{aligned}\int_{\mathbb{R}^3} w_{1,0}^5\mathcal{D}_0(z)&=-\int_{\mathbb{R}^3}\Delta w_{1,0}\mathcal{D}_0(z)\\ &=-\int_{\mathbb{R}^3} w_{1,0}\Delta\mathcal{D}_0(z)=-3^{1/4}\lambda\int_{\mathbb{R}^3} w_{1,0}\left[\frac{1}{|z|}-\frac{1}{\sqrt{1+|z|^2}}\right].\end{aligned}$$

Combining these relations we get:

$$\begin{aligned}A=&-4\pi 3^{1/4}\mu g_\lambda(\zeta)\int_{\mathbb{R}^3} w_{1,0}^5(z)\,\mathrm{d}z\\ &-\mu^2\lambda 3^{1/4}\int_{\mathbb{R}^3}\left[w_{1,0}(z)\left(\frac{1}{|z|}-\frac{1}{\sqrt{1+|z|^2}}\right)+\frac{1}{2}w_{1,0}^5(z)|z|\right]\mathrm{d}z+\mathcal{R}_2.\end{aligned}$$

(2) Let us consider now:

$$
\begin{aligned}
C &\equiv \int_{\Omega} w_{\mu,\zeta}^4 \pi_{\mu,\zeta}^2 \\
&= \mu \int_{\Omega_\mu} w_{1,0}^4 \pi_{\mu,\zeta}^2 (\zeta + \mu z)\, \mathrm{d}z \\
&= \mu^2 \int_{\Omega_\mu} w_{1,0}^4 \big[-4\pi 3^{1/4} H_\lambda(\zeta, \zeta + \mu z) + \mu \mathcal{D}_0 + \mu^{2-\sigma} \theta(\mu, \zeta, \zeta + \mu z)\big]^2,
\end{aligned}
$$

which we expand as

$$
C = \mu^2 g_\lambda^2(\zeta) 16\pi^2 3^{1/2} \int_{\mathbb{R}^3} w_{1,0}^4 + \mathcal{R}_3.
$$

Combining relation (2.8) and the above expressions we then get:

$$
E_{5,\lambda}(U_{5,\lambda}) = a_0 + a_1 \mu g_\lambda(\zeta) + a_2 \lambda \mu^2 - a_3 \mu^2 g_\lambda^2(\zeta) + \mathcal{R}_1 - \frac{1}{2}\mathcal{R}_2 - \frac{5}{2}\mathcal{R}_3,
$$

where

$$
\begin{aligned}
a_0 &= \frac{1}{3} \int_{\mathbb{R}^3} w_{1,0}^6, \\
a_1 &= 2\pi 3^{1/4} \int_{\mathbb{R}^3} w_{1,0}^5, \\
a_2 &= \frac{3^{1/4}}{2} \int_{\mathbb{R}^3} \left[w_{1,0} \left(\frac{1}{|z|} - \frac{1}{\sqrt{1 + |z|^2}} \right) + \frac{1}{2} w_{1,0}^5 |z| \right] \mathrm{d}z, \\
a_3 &= 40\pi^2 3^{1/2} \int_{\mathbb{R}^3} w_{1,0}^4.
\end{aligned}
$$

(3) We need to analyze the size of the remainders $\mathcal{R}_i$. More precisely we want to establish the estimate,

$$
\mu^j \frac{\partial^{i+j}}{\partial \zeta^i \partial \mu^j} \mathcal{R}_l = \mathrm{O}\big(\mu^{3-\sigma}\big),
$$

for each $j = 0, 1, 2$, $i = 0, 1$, $i + j \leqslant 2$, $l = 1, 2, 3$, uniformly on all small μ and ζ in compact subsets of Ω. This needs a corresponding analysis for each of the individual terms arising in the expressions for $\mathcal{R}_l$. Since several of these computations are similar, we shall only carry in detail those that appear as most representative.

In (2.9) let us consider for instance the integral:

$$\int_{\mathbb{R}^3\setminus\Omega} w_{\mu,\zeta}^6 = 3^{3/2}\mu^3 \int_{\mathbb{R}^3\setminus\Omega} \frac{1}{(\mu^2+|y-\zeta|^2)^3}.$$

From this expression it easily follows that

$$\mu^j \frac{\partial^{i+j}}{\partial\zeta^i\partial\mu^j} \int_{\mathbb{R}^3\setminus\Omega} w_{\mu,\zeta}^6 = \mathrm{O}(\mu^3),$$

uniformly in ζ in compact subsets of Ω.

In (2.10), let us consider the term:

$$B \equiv \mu \int_{\Omega_\mu} w_{1,0}^5(z)\big[\theta_1(\zeta,\zeta+\mu z) + \mu^{2-\sigma}\theta(\mu,\zeta,\zeta+\mu z)\big]\,\mathrm{d}z = B_1 + B_2.$$

Let us observe that

$$B_2 \equiv \mu \int_{\Omega_\mu} w_{1,0}^5(z)\mu^{2-\sigma}\theta(\mu,\zeta,\zeta+\mu z)\,\mathrm{d}z = \mu^{-\sigma}\int_{\Omega} w_{1,0}^5\left(\frac{y-\zeta}{\mu}\right)\theta(\mu,\zeta,y)\,\mathrm{d}y.$$

The size of this quantity in absolute value is clearly $\mathrm{O}(\mu^{3-\sigma})$. We have then that

$$\partial_\zeta B_2 = I_{21} + I_{22},$$
$$I_{21} = -\mu^{-\sigma}\int_{\Omega} \mu^{-1} D\big(w_{1,0}^5\big)\left(\frac{y-\zeta}{\mu}\right)\theta(\mu,\zeta,y)\,\mathrm{d}y,$$
$$I_{22} = \mu^{-\sigma}\int_{\Omega} w_{1,0}^5\left(\frac{y-\zeta}{\mu}\right)\partial_\zeta\theta(\mu,\zeta,y)\,\mathrm{d}y.$$

Since $\partial_\zeta\theta(\mu,\zeta,y)$ is uniformly bounded for ζ ranging on compact subsets of Ω, B_{22} is of size $\mathrm{O}(\mu^{3-\sigma})$. Now, using symmetry,

$$I_{22} = \mu^{2-\sigma}\int_{\Omega_\mu} D\big(w_{1,0}^5\big)\big[\theta(\mu,\zeta,\zeta+\mu z) - \theta(\mu,\zeta,\zeta)\big] - \mu^{2-\sigma}\theta(\mu,\zeta,\zeta)\int_{\mathbb{R}^3\setminus\Omega_\mu} D\big(w_{1,0}^5\big)$$
$$= \mu^{2-\sigma}\int_{\Omega_\mu} D\big(w_{1,0}^5\big)\big[\theta(\mu,\zeta,\zeta+\mu z) - \theta(\mu,\zeta,\zeta)\big] + \mathrm{o}\big(\mu^3\big).$$

Now, θ is symmetric in ζ and y, hence has bounded derivative over compacts with respect to each of its arguments. Thus

$$\left|\mu^{2-\sigma}\int_{\Omega_\mu} D\big(w_{1,0}^5\big)(z)\big[\theta(\mu,\zeta,\zeta+\mu z)-\theta(\mu,\zeta,\zeta)\big]\,\mathrm{d}z\right|$$
$$\leqslant C\mu^{2-\sigma}\int_{\mu|z|\leqslant\delta}\mu\big|D\big(w_{1,0}^5\big)(z)\big||z|\,\mathrm{d}z+C\mu^{2-\sigma}\int_{\mu|z|>\delta}|z|^{-6}\,\mathrm{d}z=\mathrm{O}\big(\mu^{3-\sigma}\big).$$

Let us consider now B_1. We can expand,

$$\theta_1(\zeta,\zeta+\mu z)=\mu\mathbf{c}\cdot z+\theta_2(\zeta,\zeta+\mu z),$$

for a constant vector $\mathbf{c}$, where θ_2 is a C^2 function with $|\theta_2(\zeta,y)|\leqslant C|\zeta-y|^2$. Observe that by symmetry,

$$\mu^2\int_{\Omega_\mu} w_{1,0}^5(z)\mathbf{c}\cdot z\,\mathrm{d}z=-\mu^2\int_{\mathbb{R}^3\setminus\Omega_\mu} w_{1,0}^5(z)\mathbf{c}\cdot z\,\mathrm{d}z=\mathrm{O}\big(\mu^3\big).$$

From here it easily follows that $B_1=\mathrm{O}(\mu^3\log\mu)$. Let us decompose it as

$$B_1=B_{11}+B_{12},$$
$$B_{11}\equiv 3^{5/2}\mu^{-2}\int_{\Omega}\big(1+\mu^{-2}|y-\zeta|^2\big)^{-5/2}\theta_2(\zeta,y)\,\mathrm{d}y,$$
$$B_{12}\equiv -3^{5/2}\mu^3\int_{\mathbb{R}^3\setminus\Omega}\big(\mu^2+|y-\zeta|^2\big)^{-5/2}(y-\zeta)\cdot\mathbf{c}\,\mathrm{d}y;$$

B_{12} has derivatives with respect to ζ uniformly bounded by $\mathrm{O}(\mu^3)$. As for the first integral,

$$B_{11}=\mu^{-2}\int_{\Omega} w_{1,0}^5\bigg(\frac{y-\zeta}{\mu}\bigg)\theta_2(\zeta,y)\,\mathrm{d}y,$$

we obtain that $\partial_\zeta B_{11}$ can be written as $I_{111}+I_{112}$ with

$$I_{111}=\mu^{-3}\int_{\Omega} D\big(w_{1,0}^5\big)\bigg(\frac{y-\zeta}{\mu}\bigg)\theta_2(\zeta,y)\,\mathrm{d}y,$$
$$I_{112}=\mu^{-2}\int_{\Omega} w_{1,0}^5\bigg(\frac{y-\zeta}{\mu}\bigg)\partial_\zeta\theta_2(\zeta,y)\,\mathrm{d}y.$$

Let us estimate the second integral:

$$I_{112} = \mu^{-2} \int_{\Omega} w_{1,0}^5 \left(\frac{y-\zeta}{\mu}\right) \partial_\zeta \theta_2(\zeta, y)\,\mathrm{d}y = \mu \int_{\Omega} w_{1,0}^5(z) \partial_\zeta \theta_2(\zeta, \zeta + \mu z)\,\mathrm{d}z.$$

We have that

$$\partial_\zeta \theta_2(\zeta, \zeta + \mu z) = \mu \mathbf{A} z + \mathrm{O}\big(|\mu z|^2\big),$$

where $\mathbf{A} = D_2^2 \theta_2(\zeta, \zeta)$, where we have used the expansion for H_λ made in Appendix A. Replacing the above expression and making use of symmetry we get that $I_{112} = \mathrm{O}(\mu^3 \log \mu)$. As for the integral B_{11}, we observe that after an integration by parts,

$$I_{111} = \mathrm{O}\big(\mu^3\big) - \mu^{-2} \int_{\Omega} w_{1,0}^5 \left(\frac{y-\zeta}{\mu}\right) \partial_y \theta_2(\zeta, y)\,\mathrm{d}y.$$

The integral in the above expression can be treated in exactly the same way as B_{12}, and we thus find $\partial_\zeta B = \mathrm{O}(\mu^{3-\sigma})$ uniformly over compacts of Ω in the variable ζ variable. In analogous way, we find similar bounds for $\mu \partial_\mu B$. The same type of estimate does hold for second derivatives $\mu^2 \partial_\mu^2 B$ and $\mu^2 \partial_{\mu\zeta}^2 B$. As an example, let us estimate, as a part of the latter, the quantity $\mu \partial_\mu I_{21}$. We have:

$$\begin{aligned}
\mu \partial_\mu I_{21} &= -\mu \partial_\mu \left[\mu^{-1-\sigma} \int_{\Omega} D\big(w_{1,0}^5\big)\left(\frac{y-\zeta}{\mu}\right) \theta(\mu, \zeta, y)\,\mathrm{d}y\right] \\
&= (1+\sigma) I_{21} + \mu^{-\sigma} \int_{\Omega} \mu^{-1} D^2\big(w_{1,0}^5\big)\left(\frac{y-\zeta}{\mu}\right) \cdot \left(\frac{y-\zeta}{\mu}\right) \theta(\mu, \zeta, y)\,\mathrm{d}y \\
&\quad - \mu^{-1-\sigma} \int_{\Omega} D\big(w_{1,0}^5\big)\left(\frac{y-\zeta}{\mu}\right) \mu \partial_\mu \theta(\mu, \zeta, y)\,\mathrm{d}y.
\end{aligned}$$

Let us consider the term,

$$\mu^{-\sigma} \int_{\Omega} \mu^{-1} D^2\big(w_{1,0}^5\big)\left(\frac{y-\zeta}{\mu}\right) \cdot \left(\frac{y-\zeta}{\mu}\right) \theta(\mu, \zeta, y)\,\mathrm{d}y,$$

the others being estimated in exactly the same way as before. The observation is that the estimate of this integral by $\mathrm{O}(\mu^{3-\sigma})$ goes over exactly as that one before for I_{21}, where we simply need to replace the function $D(w_{1,0}^5)(z)$ by $D^2(w_{1,0}^5) z \cdot z$ which enjoys the same properties used in the former computation. Corresponding estimates for the remaining terms in $\mathcal{R}_2$ and $\mathcal{R}_3$ are obtained with similar computations, so that we omit them.

Summarizing, we have the validity of the desired expansion (2.4), which with the aid of the formula

$$\int_0^\infty \left(\frac{r}{1+r^2}\right)^q \frac{\mathrm{d}r}{r^{\alpha+1}} = \frac{\Gamma(\frac{q-\alpha}{2})\Gamma(\frac{q+\alpha}{2})}{2\Gamma(q)},$$

has constants a_i given by:

$$a_0 = \frac{1}{4}\sqrt{3}\pi^2, \quad a_1 = 8\sqrt{3}\pi^2, \quad a_2 = \sqrt{3}\pi^2, \quad a_3 = 120\sqrt{3}\pi^4. \quad \square \tag{2.11}$$

Our second result complements the estimate above, now allowing q to be very close to 5 from above or from below.

Lemma 2.3. *Consider $U_{\zeta,\mu}$ and $E_{q,\lambda}$ defined respectively by* (2.2) *and* (2.3). *Then, as $\mu \to 0$,*

$$\begin{aligned} E_{q,\lambda}(U_{\mu,\zeta}) = a_0 + a_1\mu g_\lambda(\zeta) + a_2\mu^2\lambda - a_3\mu^2 g_\lambda^2(\zeta) + (q-5)[a_4\log\mu + a_5] \\ + (q-5)^2\theta_1(\zeta,\mu,q) + \mu^{3-\sigma}\theta_2(\zeta,\mu,q), \end{aligned} \tag{2.12}$$

where for $j = 0, 1, 2$, $i = 0, 1$, $i + j \leqslant 2$, $l = 1, 2$,

$$\mu^j \frac{\partial^{i+j}}{\partial\zeta^i \partial\mu^j}\theta_l(\zeta,\mu,q)$$

is bounded uniformly on all small μ, $|q-5|$ small and ζ in compact subsets of Ω. Here a_0, a_1, a_2, a_3 are given by (2.11), *$a_4 = \sqrt{3}\pi^2/16$ and a_5 is another constant, whose expression is given below in the proof.*

Proof. Observe that

$$E_{q,\lambda}(U_{\mu,\zeta}) - E_{5,\lambda}(U_{\mu,\zeta}) = \frac{1}{6}\int_\Omega U_{\mu,\zeta}^6 - \frac{1}{q+1}\int_\Omega U_{\mu,\zeta}^{q+1}.$$

The desired estimate follows form (2.4), after Taylor expanding in q and estimating the remaining terms similarly to the proof of the previous lemma. More precisely, we estimate:

$$E_{q,\lambda}(U_{\mu,\zeta}) - E_{5,\lambda}(U_{\mu,\zeta}) = (q-5)\left[\frac{1}{36}\int_\Omega U_{\mu,\zeta}^6 - \frac{1}{6}\int_\Omega U_{\mu,\zeta}^6 \log U_{\mu,\zeta}\right] + \mathrm{o}(q-5),$$

which after lengthy computations gives (2.12) with

$$a_4 = \frac{1}{12}\int_{\mathbb{R}^3} w_{1,0}^6(z)\,\mathrm{d}z = \frac{\sqrt{3}\pi^2}{16},$$

$$a_5 = \frac{1}{36}\int_{\mathbb{R}^3} w_{1,0}^6(z)[6\log w_{1,0} - 1]\,\mathrm{d}z. \quad \Box \tag{2.13}$$

The above established expansions provide the presence of critical points for the functional $(\mu,\zeta) \mapsto E_{q,\lambda}(U_{\mu,\zeta})$ under the assumptions of the theorems. These critical points are still present for suitable small perturbations of the functional. We discuss these issues in the next section.

3. Critical single-bubbling

The purpose of this section is to establish that in the situations of Theorems 2 and 4 there are critical points of $E_{q,\lambda}(U_{\mu,\zeta})$ as computed in (2.12) which persist under properly small perturbations of the functional. As we shall rigorously establish later, this analysis does provide critical points of the full functional $E_{q,\lambda}$, namely solutions of (1.1), close to a single bubble of the form $U_{\mu,\zeta}$.

First case. Let us consider first the situation present in Theorem 2, Part (a). We let then $q = 5 + \varepsilon$. Let $\mathcal{D}$ be the set where g_λ is assumed to have nontrivial linking with negative critical value $\mathcal{G}_\lambda$. It is not hard to check, by redefining the sets involved that we may actually assume $g_\lambda(\zeta) < -\delta < 0$ on $\mathcal{D}$. It is convenient to consider Λ defined by:

$$\mu \equiv -\varepsilon \frac{a_4}{a_1}\frac{1}{g_\lambda(\zeta)}\Lambda, \tag{3.1}$$

where a_4 and a_1 are the constants in the expansion (2.12).

Lemma 3.1. *In the situation of Theorem* 2*, Part* (a)*, for μ given by* (3.1)*, consider a functional of the form*:

$$\psi_\varepsilon(\Lambda,\zeta) \equiv E_{5+\varepsilon,\mu}(U_{\mu,\zeta}) + \varepsilon\theta_\varepsilon(\Lambda,\zeta),$$

for $\Lambda > 0$ and $\zeta \in \mathcal{D}$. Denote $\nabla = (\partial_\Lambda, \partial_\zeta)$ and assume that

$$|\theta_\varepsilon| + |\nabla\theta_\varepsilon| + |\nabla\partial_\Lambda\theta_\varepsilon| \to 0 \tag{3.2}$$

uniformly on (ζ,Λ) as $\varepsilon \to 0$, with

$$\delta < \Lambda < \delta^{-1}, \quad g_\lambda(\zeta) < -\delta$$

for any given δ. Then ψ_ε has a critical point $(\Lambda_\varepsilon,\zeta_\varepsilon)$ with $\zeta_\varepsilon \in \mathcal{D}$,

$$\Lambda_\varepsilon \to 1, \quad g_\lambda(\zeta_\varepsilon) \to \mathcal{G}_\lambda, \quad \nabla g_\lambda(\zeta_\varepsilon) \to 0.$$

Proof. The expansion given in Lemma 2.3 implies:

$$\psi_\varepsilon(\Lambda,\zeta) \equiv a_0 + a_4\varepsilon\left[-\Lambda + \log\Lambda + \log\left(-\frac{1}{g_\lambda(\zeta)}\right)\right] + a_4\varepsilon\left[\log\left(\frac{a_4}{a_1}\right) + \log\varepsilon + \frac{a_5}{a_4}\right] + \varepsilon\theta_\varepsilon(\Lambda,\zeta),$$

where θ_ε still satisfies (3.2). The main term in the above expansion is the functional,

$$\psi_0(\Lambda,\zeta) = -\Lambda + \log\Lambda + \log\left(-\frac{1}{g_\lambda(\zeta)}\right),$$

which obviously has a critical point since it has a nondegenerate maximum in Λ at $\Lambda = 1$ and g_λ nontrivially links in $\mathcal{D}$. Consider the equation:

$$\partial_\Lambda \psi_\varepsilon(\Lambda,\zeta) = 0,$$

which has the form

$$\Lambda = 1 + \mathrm{o}(1)\theta_\varepsilon(\Lambda,\zeta),$$

where the function θ_ε has a continuous, uniformly bounded derivative in (Λ,ζ) in the considered region. It then follows that for each $\zeta \in \mathcal{D}$ there exists a unique $\Lambda = \Lambda_\varepsilon(\zeta)$, function of class C^1 satisfying the above equation which has the form:

$$\Lambda_\varepsilon(\zeta) = 1 + \mathrm{o}(1)\beta_\varepsilon(\zeta),$$

where β_ε and its derivative are uniformly bounded in the considered region. Clearly we get a critical point of ψ_ε if we have one of the functional $\zeta \mapsto \psi_\varepsilon(\Lambda_\varepsilon(\zeta),\zeta)$. Observe that on $\mathcal{D}$,

$$\psi_\varepsilon\big(\Lambda_\varepsilon(\zeta),\zeta\big) = c_\varepsilon + a_4\varepsilon\left[\log\left(-\frac{1}{g_\lambda(\zeta)}\right) + \mathrm{o}(1)\right],$$

where $\mathrm{o}(\varepsilon)$ is small uniformly on $\mathcal{D}$ in the C^1-sense and c_ε is a constant. The linking structure is thus preserved, and a critical point $\zeta_\varepsilon \in \mathcal{D}$ of the above functional with the desired properties thus exists. □

We observe that the associated bubble $U_{\mu,\zeta}$, where μ is given by (3.1) and with $\zeta = \zeta_\varepsilon$, has then precisely the form of that in (1.8)–(1.9) in Theorem 1, Part (a).

Second case. Let us consider the situation in Part (b). Let $q = 5 - \varepsilon$ and assume now that g_λ has nontrivial linking in a set $\mathcal{D}$ with critical value $\mathcal{G}_\lambda > 0$. Again, we may assume $g_\lambda > \delta > 0$ on $\mathcal{D}$ and set the change of variables,

$$\mu \equiv \varepsilon\frac{a_4}{a_1}\frac{1}{g_\lambda(\zeta)}\Lambda. \tag{3.3}$$

In this case we get the following result:

Lemma 3.2. *In the situation above of Theorem* 2*, Part* (b)*, for* μ *given by* (3.3)*, consider a functional of the form*:

$$\psi_\varepsilon(\Lambda,\zeta) = E_{5-\varepsilon,\lambda}(U_{\mu,\zeta}) + \varepsilon\theta_\varepsilon(\Lambda,\zeta)$$

for $\Lambda > 0$ *and* $\zeta \in \mathcal{D}$*. Assume that*

$$|\theta_\varepsilon| + |\nabla\theta_\varepsilon| + |\nabla\partial_\Lambda\theta_\varepsilon| \to 0$$

uniformly on (Λ,ζ) *as* $\varepsilon \to 0$*, with*

$$\delta < \Lambda < \delta^{-1}, \quad g_\lambda(\zeta) > \delta.$$

Then ψ_ε *has a critical point* $(\Lambda_\varepsilon,\zeta_\varepsilon)$ *with* $\zeta_\varepsilon \in \mathcal{D}$*,*

$$\Lambda_\varepsilon \to 1, \quad g_\lambda(\zeta_\varepsilon) \to \mathcal{G}_\lambda, \quad \nabla g_\lambda(\zeta_\varepsilon) \to 0.$$

Proof. For $\Lambda > 0$ and $\zeta \in \mathcal{D}$ now we find the expansion:

$$\begin{aligned}\psi_\varepsilon(\Lambda,\zeta) &\equiv a_0 + a_4\varepsilon\bigl[\Lambda - \log\Lambda + \log\bigl(g_\lambda(\zeta)\bigr)\bigr]\\ &\quad - a_4\varepsilon\left[\log\left(\frac{a_4}{a_1}\right) + \log\varepsilon + \frac{a_5}{a_4}\right] + \varepsilon\theta_\varepsilon(\Lambda,\zeta),\end{aligned}$$

where θ_ε satisfies (3.2). The main term in the expansion has now a nondegenerate minimum at $\Lambda = 1$. The rest of the proof is identical to that of Lemma 3.1. □

Third case. Let us consider the situation in Theorem 4 where now $q = 5$. Let us assume the situation (a) of local minimizer:

$$0 = \inf_{x\in\mathcal{D}} g_{\lambda_0}(x) < \inf_{x\in\partial\mathcal{D}} g_{\lambda_0}(x).$$

Then for λ close to λ_0, $\lambda > \lambda_0$, we will have:

$$\inf_{x\in\mathcal{D}} g_\lambda(x) < -A(\lambda-\lambda_0).$$

Let us consider the shrinking set:

$$\mathcal{D}_\lambda = \left\{y \in \mathcal{D}\colon\ g_\lambda(x) < -\frac{A}{2}(\lambda-\lambda_0)\right\}.$$

Assume $\lambda > \lambda_0$ is sufficiently close to λ_0 so that $g_\lambda = -\frac{A}{2}(\lambda-\lambda_0)$ on $\partial\mathcal{D}_\lambda$.

Now, let us consider the situation of Part (b). Since $g_\lambda(\zeta)$ has a nondegenerate critical point at $\lambda = \lambda_0$ and $\zeta = \zeta_0$, this is also the case at a certain critical point ζ_λ for all λ close to λ_0 where $|\zeta_\lambda - \zeta_0| = \mathrm{O}(\lambda - \lambda_0)$.

Besides, for some intermediate point $\tilde{\zeta}_\lambda$,

$$g_\lambda(\zeta_\lambda) = g_\lambda(\zeta_0) + Dg_\lambda\big(\tilde{\zeta}_\lambda\big)(\zeta_\lambda - \zeta_0) \geqslant A(\lambda - \lambda_0) + \mathrm{o}(\lambda - \lambda_0)$$

for a certain $A > 0$. Let us consider the ball B_ρ^λ with center ζ_λ and radius $\rho(\lambda - \lambda_0)$ for fixed and small $\rho > 0$. Then we have that $g_\lambda(\zeta) > \frac{A}{2}(\lambda - \lambda_0)$ for all $\zeta \in B_\rho^\lambda$. In this situation we set $\mathcal{D}_\lambda = B_\rho^\lambda$.

It is convenient to make the following relabeling of the parameter μ. Let us set:

$$\mu \equiv -\frac{a_1}{2a_2}\frac{g_\lambda(\zeta)}{\lambda}\Lambda, \tag{3.4}$$

where $\zeta \in \mathcal{D}_\lambda$. The result we have now is the following:

Lemma 3.3. *Assume the validity of one of the conditions* (a) *or* (b) *of Theorem* 4*, and consider a functional of the form,*

$$\psi_\lambda(\Lambda, \zeta) = E_{5,\lambda}(U_{\mu,\zeta}) + g_\lambda(\zeta)^2\theta_\lambda(\Lambda, \zeta),$$

where μ is given by (3.4) *and*

$$|\theta_\lambda| + |\nabla\theta_\lambda| + |\nabla\partial_\Lambda\theta_\lambda| \to 0 \tag{3.5}$$

uniformly on $\zeta \in \mathcal{D}_\lambda$ and $\Lambda \in (\delta, \delta^{-1})$. Then ψ_λ has a critical point $(\Lambda_\lambda, \zeta_\lambda)$ with $\zeta_\lambda \in \mathcal{D}_\lambda$, $\Lambda_\lambda \to 1$.

Proof. Using the expansion for the energy with μ given by (3.4) we find now that

$$\psi_\lambda(\Lambda, \zeta) \equiv E_{5,\lambda}(U_{\zeta,\mu}) = a_0 + \frac{a_1^2}{4a_2}\frac{g_\lambda(\zeta)^2}{\lambda}\big[-2\Lambda + \Lambda^2\big] + g_\lambda(\zeta)^2\theta_\lambda(\Lambda, \zeta),$$

where θ_λ satisfies property (3.5). Observe then that $\partial_\Lambda\psi_\lambda = 0$ if and only if

$$\Lambda = 1 + \mathrm{o}(1)\theta_\lambda(\Lambda, \zeta),$$

where θ_λ is bounded in C^1-sense. This implies the existence of a unique solution close to 1 of this equation, $\Lambda = \Lambda_\lambda(\zeta) = 1 + \mathrm{o}(1)$ with $\mathrm{o}(1)$ small in C^1 sense. Thus we get a critical point of ψ_λ if we have one of

$$p_\lambda(\zeta) \equiv \psi_\lambda\big(\Lambda_\lambda(\zeta), \zeta\big) = a_0 + cg_\lambda(\zeta)^2\big[1 + \mathrm{o}(1)\big]$$

with $\mathrm{o}(1)$ uniformly small in C^1-sense and $c > 0$. In the case of Part (a), i.e., of the minimizer, it is clear that we get a local maximum in the region $\mathcal{D}_\lambda$ and therefore a critical point.

Let us consider the case (b). With the same definition for p_λ as above, we have:

$$\nabla p_\lambda(\zeta) = g_\lambda(\zeta)\big[\nabla g_\lambda + \mathrm{o}(1) g_\lambda\big].$$

Consider a point $\zeta \in \partial\mathcal{D}_\lambda = \partial B_\rho^\lambda$. Then $|\nabla g_\lambda(\zeta)| = |D^2 g_\lambda(\tilde{x})(\zeta - \zeta_\lambda)| \geqslant \alpha\rho(\lambda - \lambda_0)$, for some $\alpha > 0$. We also have $g_\lambda(\zeta) = \mathrm{O}(\lambda - \lambda_0)$. We conclude that for all $t \in (0,1)$, the function $\nabla g_\lambda + t\mathrm{o}(1) g_\lambda$ does not have zeros on the boundary of this ball, provided that $\lambda - \lambda_0$ is small. In conclusion, its degree on the ball is constant along t. Since for $t = 0$ is not zero, thanks to nondegeneracy of the critical point ζ_λ, we conclude the existence of a zero of $\nabla p_\lambda(\zeta)$ inside $\mathcal{D}_\lambda$. $\square$

4. The method

Our purpose in what follows is to find in each of the situations stated in the theorems, solutions with single or multiple bubbling for some well chosen $\zeta \in \Omega$, which at main order look like:

$$U = \sum_{i=1}^{k} (w_{\mu_i,\zeta} + \pi_{\mu_i,\zeta}), \tag{4.1}$$

with μ_1 small and, in case $k \geqslant 1$, also with $\mu_{i+1} \ll \mu_i$. This requires the understanding of the linearization of the equation around this initial approximation. It is convenient and natural, especially in what concerns multiple bubbling to recast the problem using spherical coordinates around the point ζ and a transformation which takes into account the natural dilation invariance of the equation at the critical exponent. This transformation is a variation of the so-called Emden–Fowler transformation, see [16].

Let ζ be a point in Ω. We consider spherical coordinates $y = y(\rho, \Theta)$ centered at ζ given by:

$$\rho = |y - \zeta| \quad \text{and} \quad \Theta = \frac{y - \zeta}{|y - \zeta|},$$

and the transformation $\mathcal{T}$ defined by:

$$v(x, \Theta) = \mathcal{T}(u)(x, \Theta) \equiv 2^{1/2} \mathrm{e}^{-x} u\big(\zeta + \mathrm{e}^{-2x}\Theta\big). \tag{4.2}$$

Denote by D the ζ-dependent subset of $S = \mathbb{R} \times S^2$ where the variables (x, Θ) vary. After these changes of variables, Problem (1.1) becomes:

$$\begin{aligned} &4\Delta_{S^2} v + v'' - v + 4\lambda\, \mathrm{e}^{-4x} v + c_q\, \mathrm{e}^{(q-5)x} v^q = 0 \quad \text{in } D,\\ &v > 0 \quad \text{in } D, \qquad v = 0 \quad \text{on } \partial D \end{aligned} \tag{4.3}$$

with

$$c_q \equiv 2^{-(q-5)/2}.$$

Here and in what follows, $' = \frac{\partial}{\partial x}$. We observe then that

$$\mathcal{T}(w_{\mu,\zeta})(x,\Theta) = W(x-\xi),$$

where

$$W(x) \equiv (12)^{1/4} \mathrm{e}^{-x}\big(1+\mathrm{e}^{-4x}\big)^{-1/2} = 3^{1/4}\big[\cosh(2x)\big]^{-1/2}$$

and $\mu = \mathrm{e}^{-2\xi}$. The function W is the unique solution of the problem:

$$\begin{cases} W'' - W + W^5 = 0 & \text{on } (-\infty,\infty), \\ W'(0) = 0, \\ W > 0, \quad W(x) \to 0 & \text{as } x \to \pm\infty. \end{cases}$$

We see also that setting:

$$\Pi_{\xi,\zeta} \equiv \mathcal{T}(\pi_{\mu,\zeta}) \quad \text{with } \mu = \mathrm{e}^{-2\xi},$$

then $\Pi = \Pi_{\xi,\zeta}$ solves the boundary value problem:

$$\begin{cases} -(4\Delta_{S^2}\Pi + \Pi'' - \Pi + 4\lambda\,\mathrm{e}^{-4x}\Pi) = 4\lambda\,\mathrm{e}^{-4x}W(x-\xi) & \text{in } D, \\ \Pi = -W(x-\xi) & \text{on } \partial D. \end{cases}$$

An observation useful to fix ideas is that this transformation leaves the energy functional associated invariant. In fact associated to (4.3) is the energy:

$$J_{q,\lambda}(v) \equiv 2\int_D |\nabla_\Theta v|^2 + \frac{1}{2}\int_D \big[|v'|^2 + |v|^2\big] - 2\lambda\int_D \mathrm{e}^{-4x}v^2 - \frac{c_q}{q+1}\int_D \mathrm{e}^{(q-5)x}|v|^{q+1}. \tag{4.4}$$

If $v = \mathcal{T}(u)$ we have the identity:

$$4E_{q,\lambda}(u) = J_{q,\lambda}(v).$$

Let $\zeta \in \Omega$ and consider the numbers $0 < \xi_1 < \xi_2 < \cdots < \xi_k$. Set:

$$W_i(x) = W(x-\xi_i), \quad \Pi_i = \Pi_{\xi_i,\zeta}, \quad V_i = W_i + \Pi_i, \quad V = \sum_{i=1}^k V_i.$$

We observe then that $V = \mathcal{T}(U)$ where U is given by (4.1) and $\mu_i = \mathrm{e}^{-2\xi_i}$. Thus finding a solution of (1.1) which is a small perturbation of U is equivalent to finding a solution of (4.3) of the form $v = V + \phi$ where ϕ is small in some appropriate sense. Then solving (4.3) is equivalent to finding ϕ such that,

$$\begin{cases} L(\phi) = -N(\phi) - R, \\ \phi = 0 \quad \text{on } \partial D, \end{cases}$$

where

$$L(\phi) \equiv 4\Delta_{S^2}\phi + \phi'' - \phi + 4\lambda\,\mathrm{e}^{-4x}\phi + qc_q\,\mathrm{e}^{(q-5)x}V^{q-1}\phi,$$
$$N(\phi) \equiv c_q\,\mathrm{e}^{(q-5)x}\big[(V+\phi)_+^q - V^q - qV^{q-1}\phi\big],$$

and

$$R \equiv c_q \mathrm{e}^{(q-5)x}V^q - \sum_{i=1}^k W_i^5. \tag{4.5}$$

Rather than solving (4.3) directly, we consider first the following intermediate problem: Given points $\xi = (\xi_1, \dots, \xi_k) \in \mathbb{R}^k$ and a point $\zeta \in \Omega$, find a function ϕ such that for certain constants c_{ij},

$$\begin{cases} L(\phi) = -N(\phi) - R + \sum_{i,j} c_{ij} Z_{ij} & \text{in } D, \\ \phi = 0 & \text{on } \partial D, \\ \int_D Z_{ij}\phi\,\mathrm{d}x\,\mathrm{d}\Theta = 0 & \text{for all } i, j, \end{cases} \tag{4.6}$$

where the Z_{ij} span an "approximate kernel" for L. They are defined as follows:

Let $\mathbf{z}_{ij}$ be given by $\mathbf{z}_{ij}(x,\Theta) = \mathcal{T}(z_{ij})$, $i = 1, \dots, k$, $j = 1, \dots, 4$, where z_{ij} are respectively given by:

$$z_{ij}(y) = \frac{\partial}{\partial \zeta_j} w_{\mu_i,\zeta}(y), \quad j = 1, \dots, 3,$$
$$z_{i4}(y) = \mu_i \frac{\partial}{\partial \mu_i} w_{\mu_i,\zeta}(y), \quad i = 1, \dots, k,$$

with $\mu_i = \mathrm{e}^{-2\xi_i}$. We recall that for each i, the functions z_{ij} for $j = 1, \dots, 4$, span the space of all bounded solutions of the linearized problem:

$$\Delta z + 5w_{\mu_i,\zeta}^4 z = 0 \quad \text{in } \mathbb{R}^3.$$

A proof of this fact can be found for instance in [20]. This implies that the $\mathbf{z}_{ij}$'s satisfy:

$$4\Delta_{S^2}\mathbf{z}_{ij} + \mathbf{z}_{ij}'' - \mathbf{z}_{ij} + 5W_i^4\mathbf{z}_{ij} = 0.$$

Explicitly, we find that setting:

$$Z(x) = (12)^{1/4} \mathrm{e}^{-3x} \big(1 + \mathrm{e}^{-4x}\big)^{-3/2} = 3^{1/4} 2^{-1} \big[\cosh(2x)\big]^{-3/2},$$

we get:

$$\mathbf{z}_{ij} = Z(x - \xi_i)\Theta_j, \quad j = 1, 2, 3, \qquad \mathbf{z}_{i4} = W'(x - \xi_i).$$

Observe that

$$\int\limits_{\mathbb{R}\times S^2} \mathbf{z}_{ij}\mathbf{z}_{il} = 0 \quad \text{for } l \neq j.$$

The Z_{ij} are corrections of $\mathbf{z}_{ij}$ which vanish for very large x. Let $\eta_M(s)$ be a smooth cut-off function with

$$\eta_M(s) = 0 \text{ for } s < M, \quad \eta_M(s) = 1 \text{ for } s > M + 1.$$

We define:

$$Z_{ij} = \big(1 - \eta_M(x - \xi_i)\big)\mathbf{z}_{ij},$$

where $M > 0$ is a large fixed number. We will see that with these definitions, Problem (4.6) is uniquely solvable if the points ξ_i, ζ satisfy appropriate constrains and q is close enough to 5. After this is done, the remaining task is to adjust the parameters ζ and ξ_i in such a way that all constants $c_{ij} = 0$. We will see that this is indeed possible under the different assumptions of the theorems.

5. The linear problem

In order to solve Problem (4.6) it is necessary to understand first its linear part. Given a function h, we consider the problem of finding ϕ such that for certain real numbers c_{ij} the following is satisfied:

$$\begin{cases} L(\phi) = h + \sum_{i,j} c_{ij} Z_{ij} & \text{in } D, \\ \phi = 0 & \text{on } \partial D, \\ \int_D Z_{ij}\phi = 0 & \text{for all } i, j. \end{cases} \tag{5.1}$$

Recall that L defined by (4.5) takes the expression:

$$L(\phi) = 4\Delta_{S^2}\phi + \phi'' - \phi + 4\lambda\, \mathrm{e}^{-4x}\phi + q c_q\, \mathrm{e}^{(q-5)x} V^{q-1}\phi.$$

We need uniformly bounded solvability in proper functional spaces for Problem (5.1), for a proper range of the ξ_i's and ζ. To this end, it is convenient to introduce the following norm. Given an arbitrarily small but fixed number $\sigma > 0$, we define:

$$\|f\|_* = \sup_{(x,\Theta)\in D} \omega(x)^{-1}\big|f(x,\Theta)\big| \quad \text{with } \omega(x) = \sum_{i=1}^{k} \mathrm{e}^{-(1-\sigma)|x-\xi_i|}.$$

We shall denote by $\mathcal{C}_*$ the set of continuous functions f on $\overline{D}$ such that $\|f\|_*$ is finite.

Proposition 5.1. *Fix a small number $\delta > 0$ and take the cut-off parameter $M > 0$ of Section 4 large enough. Then there exist positive numbers ε_0, δ_0, R_0, and a constant $C > 0$ such that if $|q-5| < \varepsilon_0$,*

$$0 \leqslant \lambda \leqslant \lambda_1 - \delta, \quad \operatorname{dist}(\zeta, \partial\Omega) > \delta_0, \tag{5.2}$$

and the numbers $0 < \xi_1 < \xi_2 < \cdots < \xi_k$ satisfy:

$$R_0 < \xi_1, \quad R_0 < \min_{1\leqslant i<k} (\xi_{i+1} - \xi_i) \tag{5.3}$$

with $\xi_k < \delta_0/|q-5|$ if $q \neq 5$, then for any $h \in C^{\alpha}(D)$ with $\|h\|_* < +\infty$, Problem (5.1) admits a unique solution $\phi \equiv T(h)$. Besides,

$$\big\|T(h)\big\|_* \leqslant C\|h\|_* \quad \textit{and} \quad |c_{ij}| \leqslant C\|h\|_*.$$

For the proof we need the following result:

Lemma 5.1. *Assume the existence of sequences $(\varepsilon_n)_{n\in\mathbb{N}}$, $(\lambda_n)_{n\in\mathbb{N}}$, $(\zeta_n)_{n\in\mathbb{N}}$, $(\xi_i^n)_{n\in\mathbb{N}}$, $1 \leqslant i \leqslant k$, such that $\varepsilon_n \to 0$, $\lambda_n \in (0, \lambda_1 - \delta)$, $\operatorname{dist}(\zeta_n, \partial\Omega) > \delta_1$ and $0 < \xi_1^n < \xi_2^n < \cdots < \xi_k^n$ with*

$$\xi_1^n \to +\infty, \quad \min_{1\leqslant i<k}\big(\xi_{i+1}^n - \xi_i^n\big) \to +\infty, \quad \xi_k^n = \mathrm{o}\big(\varepsilon_n^{-1}\big),$$

such that for certain q_n with $|q_n - 5| < \varepsilon_n$, certain functions ϕ_n and h_n with $\|h_n\|_* \to 0$, and scalars c_{ij}^n, one has:

$$\begin{cases} L(\phi_n) = h_n + \sum_{i,j} c_{ij}^n Z_{ij}^n, & \\ \phi_n = 0 & \textit{on } \partial D, \\ \int_D Z_{ij}\phi_n \,\mathrm{d}x = 0 & \textit{for all } i, j. \end{cases} \tag{5.4}$$

Here the functions Z_{ij}^n are given in terms of $\mathbf{z}_{ij}$ as in Section 4 and the cut-off parameter $M > 0$ is chosen large enough. Then

$$\lim_{n\to\infty} \|\phi_n\|_* = 0.$$

Proof. We will establish first the weaker assertion that

$$\lim_{n\to\infty} \|\phi_n\|_\infty = 0.$$

By contradiction, we may assume that $\|\phi_n\|_\infty = 1$. Recall that $D \ni (x,\theta)$ is a subset of $\mathbb{R}^+ \times S^2$. We will establish first that $\lim_{n\to\infty} c_{ij}^n = 0$. Fix a number $M > 0$ such that the region $\{x > M\}$ is contained in D and consider $\eta(x)$, a smooth cut-off function with $\eta(x) = 0$ if $x < M$ and $\eta(x) = 1$ for $x > M + 1$ as in the previous section. Testing the above equation against $\eta \mathbf{z}_{lm}^n$, and integrating by parts twice we get the following relation:

$$\begin{aligned}
&\int_D \left(4\Delta_{S^2}\mathbf{z}_{lm}^n + {\mathbf{z}_{lm}^n}'' - \mathbf{z}_{lm}^n + 5W_l^5\mathbf{z}_{lm}^n\right)\eta\phi_n + \int_D \left[2\left(\mathbf{z}_{lm}^n\right)'\eta' + \mathbf{z}_{lm}^n\eta''\right]\phi_n \\
&\quad + \int_D \left[4\lambda_n \mathrm{e}^{-4x}\mathbf{z}_{lm}^n + \left(c_{q_n}q_n \mathrm{e}^{(q_n-5)x}V_n^{q_n-1} - 5W_l^5\right)\mathbf{z}_{lm}^n\right]\eta\phi_n \\
&\quad = \int_D h_n Z_{lm}^n + \sum_{i,j} c_{ij}^n \int_D \eta Z_{ij}^n \mathbf{z}_{lm}^n.
\end{aligned}$$

The first integral on the left-hand side of the above equality is zero, while the other three can be bounded by $\mathrm{o}(1)\|\phi_n\|_\infty$ and therefore go to 0 as $n \to \infty$. The same is true for the first integral in the right-hand side. The definition of the Z_{ij}^n's makes this linear system in the c_{ij}'s "almost diagonal" as $n \to \infty$. We conclude then that $\lim_{n\to\infty} c_{ij}^n = 0$ as desired.

Now let $(x_n, \Theta_n) \in D$ be such that $\phi_n(x_n, \Theta_n) = 1$, so that ϕ_n maximizes at this point. We claim that, for n large enough, there exist $R > 0$ and $i \in \{1,\ldots,k\}$ such that $|x_n - \xi_i^n| < R$. We argue by contradiction and suppose that $|x_n - \xi_i^n| \to +\infty$ as $n \to +\infty$ for any $i = 1,\ldots,k$. Then either $|x_n| \to +\infty$ or $|x_n|$ remains bounded. Assume first that $|x_n| \to +\infty$. Let us define:

$$\tilde\phi_n(x,\Theta) = \phi_n(x + x_n, \Theta).$$

Then, from elliptic estimates, $\tilde\phi_n$ converges uniformly over compacts to a nontrivial solution $\tilde\phi$ of

$$\begin{cases} 4\Delta_{S^2}\tilde\phi + \tilde\phi'' - \tilde\phi = 0 & \text{in } \mathbb{R} \times S^2, \\ \tilde\phi \to 0 & \text{as } |(x,\Theta)| \to \infty. \end{cases}$$

For a function $g(y)$ defined in $\mathbb{R}^3 \setminus \{0\}$, let us denote:

$$\mathcal{T}_0(g)(x,\Theta) \equiv \sqrt{2}\,\mathrm{e}^{-x} g\left(\mathrm{e}^{-2x}\Theta\right).$$

Then the function $\tilde\psi$ defined by the relation $\tilde\phi = \mathcal{T}_0(\tilde\psi)$ satisfies:

$$\Delta\tilde\psi = 0 \quad \text{in } \mathbb{R}^3 \setminus \{0\}.$$

Moreover, $\|\tilde{\phi}\|_\infty = 1$, translates into $|\tilde{\psi}(y)| \leqslant |y|^{-1/2}$. It follows that $\tilde{\psi}$ extends smoothly to 0, to a harmonic function in $\mathbb{R}^3$ with this decay condition, hence $\tilde{\psi} \equiv 0$, yielding a contradiction.

Assume now that $|x_n|$ is bounded. Hence, up to subsequence, the function ϕ_n converges uniformly over compacts to a nontrivial solution of

$$\begin{cases} 4\Delta_{S^2}\phi + \phi'' - \phi + 4\lambda\, \mathrm{e}^{-2x}\phi = 0 & \text{in } D, \\ \phi = 0 & \text{on } \partial D, \end{cases}$$

for some $\lambda \in [0, \lambda_1)$. But this implies that $\phi \equiv 0$, since the function $\psi = \mathcal{T}^{-1}(\phi)$ is identically 0 in Ω because it solves:

$$\begin{cases} \Delta\psi + \lambda\psi = 0 & \text{in } \Omega \setminus \{\zeta\}, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases}$$

with the additional condition $|\psi(y)| \leqslant C|y|^{-1/2}$ for some λ such that $0 \leqslant \lambda < \lambda_1$. We again reach a contradiction, and the claim is thus proved. Hence, there exists an integer $l \in \{1, \ldots, k\}$ and a positive number $R > 0$ such that, for n sufficiently large, $|x_n - \xi_l^n| \leqslant R$. Let again $\tilde{\phi}_n(x, \Theta) \equiv \phi_n(x + \xi_l^n, \Theta)$. This relation implies that $\tilde{\phi}_n$ converges uniformly over compacts to $\tilde{\phi}$ which is a nontrivial, bounded solution of the problem:

$$\Delta_{S^2}\tilde{\phi} + \tilde{\phi}'' - \tilde{\phi} + 5W^4\tilde{\phi} = 0 \quad \text{in } \mathbb{R} \times S^2,$$

and also satisfies:

$$\int\limits_{\mathbb{R}\times S^2} \tilde{\phi}\mathbf{z}_m\big(1 - \eta_M(x)\big)\,\mathrm{d}x\,\mathrm{d}\Theta = 0, \tag{5.5}$$

where $\mathbf{z}_m(x, \Theta) = \mathcal{T}_0(z_m)$ with

$$z_m(y) = \partial_{y_m} w_{1,0}(y), \quad m = 1, 2, 3, \qquad z_4(y) = \frac{1}{2} w_{1,0}(y) + y \cdot \nabla w_{1,0}(y).$$

This means that the function $\tilde{\psi} = \mathcal{T}_0^{-1}(\tilde{\phi})$ is a nontrivial solution of

$$\Delta\tilde{\psi} + 5w_{1,0}^4\tilde{\psi} = 0 \quad \text{in } \mathbb{R}^3 \setminus \{0\}$$

with $|\tilde{\psi}(y)| \leqslant C|y|^{-1/2}$ for all y. Thus we get a classical solution in $\mathbb{R}^3 \setminus \{0\}$ which decays at infinity and hence equals a linear combination of the z_m's. It follows that ϕ is a linear combination of the Z_m's. But then the orthogonality relations (5.5) imply $\tilde{\phi} = 0$, at least for $M > 0$ large enough, again a contradiction. We have thus proved $\|\phi_n\|_\infty \to 0$ as $n \to \infty$.

Next we shall establish that $\|\phi_n\|_* \to 0$. Let us write:

$$\bar{L}(\phi) = 4\Delta_{S^2}\phi + \phi'' - \phi + 4\lambda\,\mathrm{e}^{-4x}\phi. \tag{5.6}$$

Let us observe that Eq. (5.4) takes the form:

$$\bar{L}(\phi_n) = g_n$$

with $g_n(x,\Theta) \equiv -c_{q_n} q_n \, \mathrm{e}^{(q_n-5)x} V_n^{q_n-1} \phi_n + h_n + \sum_{i,j} c_{ij}^n Z_{ij}$. Hence

$$\left|g_n(x,\Theta)\right| \leqslant \mathcal{G}_n \equiv \eta_n \sum_{i=1}^{k} \mathrm{e}^{-(1-\sigma)|x-\xi_i|}$$

with $\eta_n \to 0$. We claim that the operator $\bar{L}$ satisfies the Maximum Principle in the following sense:

If ϕ is bounded, continuous in $\bar{D}$, $\phi \in H^1(D \cap \{x < R\})$ for any $R > 0$ and satisfies $\bar{L}(\phi) \leqslant 0$ in the weak sense in D and $\phi \geqslant 0$ on ∂D, then $\phi \geqslant 0$.

To see this, let us observe that if $\phi = \mathcal{T}(\psi)$ then ψ satisfies:

$$\Delta\psi + \lambda\psi \leqslant 0 \quad \text{in } \Omega \setminus \{\zeta\}$$

in the weak sense, and $|\psi(y)| \leqslant C|y-\zeta|^{-1/2}$. Fix a small number $\nu > 0$. Then $\psi(y) \geqslant -\nu G_\lambda(\zeta, y)$ if $|y-\zeta| < C\nu^2$, for some eventually larger constant C. Let $\widetilde{\Omega} = \Omega \setminus B(\zeta, C\nu^2)$. If ν is small enough, then we have $\lambda < \lambda_1(\widetilde{\Omega})$. Thus $\bar{L}$ satisfies maximum principle in $\widetilde{\Omega}$ and therefore $\psi(y) \geqslant -\nu G_\lambda(\zeta, y)$ for all $y \in \Omega \setminus \{\zeta\}$. Letting ν go to zero, the desired assertion follows.

Since $\lambda < \lambda_1$, there is a unique bounded solution $\bar{\phi}$ of

$$\begin{cases} 4\Delta_{S^2}\bar{\phi} + \bar{\phi}'' - \bar{\phi} + 4\lambda\, \mathrm{e}^{-4x}\bar{\phi} = -\mathrm{e}^{-x} & \text{in } D,\\ \bar{\phi} = 0 & \text{on } \partial D,\end{cases}$$

and it satisfies $\bar{\phi} \leqslant C(1+|x|)\,\mathrm{e}^{-x}$. Indeed, $\bar{\phi} = \mathcal{T}(\bar{\psi})$, where $\bar{\psi}$ solves:

$$\begin{cases} \Delta\bar{\psi} + \lambda\bar{\psi} = -\frac{1}{2^{5/2}|y-\zeta|^2} & \text{in } \Omega,\\ \bar{\psi} = 0 & \text{on } \partial\Omega.\end{cases}$$

Observe that $Z = \bar{\psi} + 2^{3/2}\log|x-\zeta|$ satisfies:

$$\begin{cases} \Delta Z + \lambda Z = 2^{3/2}\lambda\log|y-\zeta| & \text{in } \Omega,\\ Z = -\log|y-\zeta| & \text{on } \partial\Omega,\end{cases}$$

so that Z is at least of class $C^{1,\alpha}(\overline{\Omega})$. This gives the required assertion for $\bar{\phi}$.

Let us consider now the quantity:

$$s_n = K\eta_n\left(\sum_{i=1}^{k} \mathrm{e}^{-(1-\sigma)|x-\xi_i|} + K\,\mathrm{e}^{-(1-\sigma)\xi_1}\bar{\phi}\right).$$

Direct substitution shows that $\bar{L}(s_n) \leqslant -\mathcal{G}_n$ in weak sense, provided that K is chosen large enough but independent of n. From Maximum Principle, we obtain then that $\phi_n \leqslant s_n$. Similarly we obtain $\phi_n \geqslant -s_n$. Since, as well $s_n \leqslant C\mathcal{G}_n$, this shows that $\|\phi_n\|_* \to 0$, and the proof of Lemma 5.1 is completed. □

Proof of Proposition 5.1. Let us consider the space:

$$H = \left\{ \phi \in H_0^1(D) \colon \int_D Z_{ij}\phi \,\mathrm{d}x = 0 \text{ for all } i, j \right\}$$

endowed with the usual inner product:

$$[\phi, \psi] = 2\int_D \nabla_\Theta \phi \cdot \nabla_\Theta \psi + \frac{1}{2}\int_D (\phi'\psi' + \phi\psi).$$

Problem (5.1) expressed in weak form is equivalent to that of finding a $\phi \in H$ such that

$$[\phi, \psi] = \int \left[c_q q\, \mathrm{e}^{(q-5)x} V^{q-1}\phi + 4\lambda\, \mathrm{e}^{-4x}\phi + h\right]\psi \,\mathrm{d}x \quad \text{for all } \psi \in H.$$

With the aid of Riesz's representation theorem, this equation gets, rewritten in H in the operational form $\phi = K(\phi) + \tilde{h}$, for certain $\tilde{h} \in H$, where K is a compact operator in H. Fredholm's alternative guarantees unique solvability of this problem for any h provided that the homogeneous equation $\phi = K(\phi)$ has only the zero solution in H. Let us observe that this last equation is precisely equivalent to (5.1) with $h \equiv 0$. Thus existence of a unique solution follows. The bounded solvability in the sense of the $\|\ \|_*$-norm follows after an indirect argument from the previous lemma. □

Before proceeding, let us see how this result translates in terms of the original variables in Ω. Consider the function $\psi(y)$ defined in Ω for which $\mathcal{T}(\phi) = \psi$, where $\mathcal{T}$ is given by (4.2). Then ϕ satisfies Problem (5.1) if and only if ψ satisfies:

$$\begin{cases} \Delta\psi + qU_{\mu,\zeta}^{q-1}\psi + \lambda\psi = g + \sum_{i,j} \frac{c_{ij}}{|y-\zeta|^2}(1-\eta^\mu)z_{ij} & \text{in } \Omega \setminus \{\zeta\}, \\ \psi = 0 & \text{on } \partial\Omega, \\ \int_\Omega \psi z_{ij} \frac{1}{|y-\zeta|^2}\,\mathrm{d}y = 0, \end{cases} \tag{5.7}$$

where $\mathcal{T}(|y-\zeta|^2 g) = h$ and $\eta^\mu(y) = \eta^1(|y-\zeta|/\mu)$ is a family of smooth cut-off functions with

$$\eta^1(s) = 1 \text{ for } s < \delta, \quad \eta^1(s) = 0 \text{ for } s > 2\delta. \tag{5.8}$$

The size of δ is determined by M in the definition of Z_{ij}. Observe that

$$\left|g(y)\right| \leqslant \|h\|_* |y-\zeta|^{-2-\sigma/2} \sum_{i=1}^k w_{\mu_i,\zeta}^{1-\sigma}(y).$$

Thus what we have proved in Proposition 5.1 can be restated like this:

If $\|h\|_* < +\infty$ then (5.7) has a unique solution ψ which satisfies,

$$\bigl|\psi(y)\bigr| \leqslant C\|h\|_*|y-\zeta|^{-\sigma/2}\sum_{i=1}^k w^{1-\sigma}_{\mu_i,\zeta}(y),$$

and given any $\delta_0 > 0$, there exists a constant C such that

$$|y-\zeta| > \delta_0 \quad \Longrightarrow \quad \bigl|\psi(y)\bigr| \leqslant C\|h\|_*\mu_1^{(1-\sigma)/2}.$$

Hence as well, from the equation satisfied in this region and elliptic estimates,

$$\bigl|\nabla\psi(y)\bigr| \leqslant C\|h\|_*\mu_1^{(1-\sigma)/2}. \tag{5.9}$$

It is important, for later purposes, to understand the differentiability of the operator $T: h \mapsto \phi$, with respect to the variables ξ_i and ζ. Let us assume that conditions (5.2), (5.3) hold. Fix $h \in \mathcal{C}_*$ and let $\phi = T(h)$. Let us recall that ϕ satisfies the equation:

$$L(\phi) = h + \sum_{i,j} c_{ij} Z_{ij},$$

and the vanishing and orthogonality conditions, for some (uniquely determined) constants c_{ij}. We want to compute derivatives of ϕ with respect to the parameters ζ and ξ. Let us begin with differentiation with respect to ζ. A main observation is that the functions Z_{ij} do not exhibit explicit dependence on ζ. Formal differentiation then yields that $X = \partial_{\zeta_l}\phi$ should satisfy:

$$\begin{cases} L(X) = \sum_{i,j} \tilde{c}_{ij} Z_{ij} - q(q-1)c_q\,\mathrm{e}^{(q-5)x} V^{q-2}[\partial_{\zeta_l} V]\phi & \text{in } D,\\ X = B & \text{on } \partial D,\\ \int_D Z_{ij} X = 0 & \text{for all } i, j, \end{cases}$$

where

$$B \equiv \mathcal{T}(\partial_{y_l}\psi) = \sqrt{2}\,\mathrm{e}^{-x}\partial_{y_l}\psi\bigl(\zeta + \mathrm{e}^{-2x}\Theta\bigr),$$

with $\phi \equiv \mathcal{T}(\psi)$ and (formally) $\tilde{c}_{ij} \equiv \partial_{\zeta_l} c_{ij}$. Let us consider the equation:

$$\bar{L}(Y) = 0 \quad \text{in } D, \qquad Y = B \quad \text{on } \partial D,$$

where $\bar{L}$ is given by (5.6). This problem for $Y = \mathcal{T}(\overline{Y})$ is equivalent to:

$$\Delta\overline{Y} + \lambda\overline{Y} = 0 \quad \text{in } D, \qquad \overline{Y} = \partial y_l \psi \quad \text{on } \partial D$$

and, according to estimate (5.9), has a unique solution with

$$\|\overline{Y}\|_\infty \leqslant C\|h\|_*\mu_1^{(1-\sigma)/2},$$

so that $|Y(x,\Theta)| \leqslant C\|h\|_* \,\mathrm{e}^{-x}\,\mathrm{e}^{-(1-\sigma)\xi_1}$, and in particular, $\|Y\|_* \leqslant C\|h\|_*$.

Let us look closer into $\partial_{\zeta_l} V$. Since W_i does not exhibit dependence on ζ, we get that

$$\partial_\zeta V = \sum_{i=1}^{k} \partial_\zeta \Pi_i.$$

By definition of Π_i,

$$\partial_\zeta \Pi_i(x,\Theta) = \partial_\zeta \mathcal{T}[\pi_{\mu_i,\zeta}] = \mathrm{e}^{-x}\partial_\zeta\big[\pi_{\mu_i,\zeta}\big(\zeta + \mathrm{e}^{-2x}\Theta\big)\big],$$

where $\mu_i = \mathrm{e}^{-2\xi_i}$. Let us recall the expansion we found for $\pi_{\mu_i,\zeta}$ in Lemma 2.2:

$$\begin{aligned}\pi_{\mu_i,\zeta}\big(\zeta + \mathrm{e}^{-2x}\Theta\big) = \mu_i^{1/2}\big[&-4\pi 3^{1/4} H_\lambda\big(\zeta, \zeta + \mathrm{e}^{-2x}\Theta\big)\\ &+ \mu_i \mathcal{D}_0\big(\mathrm{e}^{-2(x-\xi_i)}\Theta\big) + \mu_i^{2-\sigma}\mathcal{R}\big(\zeta + \mathrm{e}^{-2x}\Theta, \mu_i, \zeta\big)\big].\end{aligned}$$

In particular we see that $|\partial_\zeta \Pi_i(x,\Theta)| \leqslant C\,\mathrm{e}^{-\xi_i}\,\mathrm{e}^{-x}$ and conclude that

$$\big\|\mathrm{e}^{(q-5)x} V^{q-2}[\partial_{\zeta_l} V]\phi\big\|_* \leqslant C\|\phi\|_* \leqslant C\|h\|_*.$$

We observe, incidentally, that in the same way we get:

$$\big|\partial_{\xi_i}\partial_\zeta \Pi_i(x,\Theta)\big| \leqslant C\,\mathrm{e}^{-\xi_i}\,\mathrm{e}^{-x}.$$

We shall use this below for the computation of derivatives with respect to ξ_i.

Let us fix a number $M > 0$ such that the region $\{x > M\}$ is contained in D and consider $\eta_M(x)$, a smooth cut-off function with $\eta_M(x) = 0$ if $x < M$ and $\eta_M(x) = 1$ for $x > M + 1$. Let us consider the constants d_{ij} defined as

$$\sum_{i,j} d_{ij} \int_D \eta_M Z_{ij} Z_{lk} = -\int_D Z_{lk} Y.$$

This linear system has a unique solution since it is almost diagonal. We also have: $|d_{ij}| \leqslant C\|h\|_*$. Consider $H = X - Y - \sum_{i,j} d_{ij}\eta_M Z_{ij}$. Then

$$\begin{cases} L(H) = \sum_{i,j} \tilde{c}_{ij} Z_{ij} + f & \text{in } D,\\ H = 0 & \text{on } \partial D,\\ \int_D Z_{ij} H = 0 & \text{for all } i, j \end{cases} \tag{5.10}$$

with

$$f = -\sum_{ij} d_{ij} L(\eta_M Z_{ij}) + q c_q\,\mathrm{e}^{(q-5)x}\big[-V^{q-1}Y + (q-1)V^{q-2}\partial_{\zeta_l} V\big]\phi.$$

The above equation has indeed a unique solution H for certain constants $\tilde{c}_{ij}$ provided that the assumptions of Proposition 5.1 are fulfilled. This computation is not just formal. Indeed one gets, as arguing directly by definition shows,

$$\partial_{\zeta_l}\phi = Y + \sum_{i,j} d_{ij}\eta_M Z_{ij} + T(f),$$

so that $\|\partial_{\zeta_l}\phi\|_* \leqslant C\|h\|_*$.

Let us now differentiate with respect to ξ_m. Let us consider $\eta_M(x)$, a smooth cut-off function as above. For a given $l \in \{1,\dots,k\}$, we consider the constant b_{lm} defined as

$$b_{lm}\int_D |Z_{lm}|^2\eta_M \equiv \int_D \phi\,\partial_{\xi_l} Z_{lm}$$

and the function:

$$f \equiv -\sum_{m=1}^{4}\bigl[b_{lm}L(\eta_M Z_{lm}) + c_{lm}\partial_{\xi_l} Z_{lm}\bigr] + q c_q\,\mathrm{e}^{(q-5)x}\partial_{\xi_l}\bigl(V^{q-1}\bigr)\phi,$$

one can then directly check that $\partial_{\xi_l}\phi$ is given by:

$$\partial_{\xi_l}\phi = T(f) + \sum_{m=1}^{4} b_{lm}\eta_M Z_{lm},$$

and that $\|\partial_{\xi_l}\phi\|_* \leqslant C\|h\|_*$. Let us denote $\nabla = [\partial_\xi, \partial_\zeta]$. Then we have proved that $\|\nabla\phi\|_* \leqslant C\|h\|_*$. Examining the above differentiation with respect to ξ, we see that we may also apply it to $\nabla\phi$, so that $\|\partial_\xi\nabla\phi\|_* \leqslant C\|h\|_*$. Actually, elaborating a bit more we get as well continuity of these derivatives in the $*$-norm.

On the Banach space $\mathcal{C}_*$ of all functions ψ in $C(\overline{D})$ for which $\|\psi\|_* < \infty$, T defines a continuous linear map of $\mathcal{C}_*$. It is easily checked that the map $(\xi,\zeta) \mapsto T$ is continuous into $\mathcal{L}(\mathcal{C}_*)$. Moreover, we have the validity of the following result:

Proposition 5.2. *Under the assumptions of Proposition* 5.1*, the derivatives* ∇T *and* $\partial_\xi\nabla T$ *exist and define continuous functions of the pair* (ξ,ζ)*. In particular, there is a constant* $C_0 > 0$*, uniform in points* (ξ,ζ)*, satisfying the constraints in Proposition* 5.1*, such that*

$$\|\nabla T\|_* + \|\partial_\xi\nabla T\|_* \leqslant C_0.$$

6. Solving the nonlinear problem

In this section we will solve Problem (4.6). We assume that the conditions in Proposition 5.1 hold. We have the following result:

Lemma 6.1. *Under the assumptions of Proposition* 5.1 *there exist numbers* $c_0 > 0$, $C_1 > 0$, *such that if* ξ *and* ζ *are additionally such that* $\|R\|_* < c_0$, *then Problem* (4.6) *has a unique solution* ϕ *which satisfies*:

$$\|\phi\|_* \leqslant C_1 \|R\|_*.$$

Proof. In terms of the operator T defined in Proposition 5.1, Problem (4.6) becomes:

$$\phi = T\big(N(\phi) + R\big) \equiv A(\phi), \tag{6.1}$$

where $N(\phi)$ and R were defined in (4.5) and (4.5). For a given R, let us consider the region:

$$\mathcal{F}_\gamma \equiv \big\{\phi \in C\big(\overline{D}\big): \ \|\phi\|_* \leqslant \gamma \|R\|_*\big\}$$

for some $\gamma > 0$, to be fixed later. From Proposition 5.1, we get:

$$\big\|A(\phi)\big\|_* \leqslant C_0\big[\big\|N(\phi)\big\|_* + \|R\|_*\big].$$

On the other hand we can represent

$$N(\phi) = c_q\, \mathrm{e}^{(q-5)x} q(q-1) \int_0^1 (1-t)\, \mathrm{d}t [V + t\phi]^{q-2} \phi^2,$$

so that (making $q-5$ smaller if necessary) $|N(\phi)| \leqslant C_1 |\phi|^2$, and hence $\|N(\phi)\|_* \leqslant C_1 \|\phi\|_*^2$. It is also easily checked that N satisfies, for $\phi_1, \phi_2 \in \mathcal{F}_\gamma$,

$$\big\|N(\phi_1) - N(\phi_2)\big\|_* \leqslant C_2 \gamma \|R\|_* \|\phi_1 - \phi_2\|_*.$$

Hence for a constant C_3 depending on C_0, C_1, C_2, we get:

$$\big\|A(\phi)\big\|_* \leqslant C_3\big[\gamma^2 \|R\|_* + 1\big] \|R\|_*,$$
$$\big\|A(\phi_1) - A(\phi_2)\big\|_* \leqslant C_3 \gamma \|R\|_* \|\phi_1 - \phi_2\|_*.$$

With the choices:

$$\gamma = 2C_3, \quad \|R\|_* \leqslant c_0 = \frac{1}{4C_3^2},$$

we get that A is a contraction mapping of $\mathcal{F}_\gamma$, and therefore a unique fixed point of A exists in this region. □

Since R depends continuously for the $*$-norm in the pair (ξ, ζ), the fixed point characterization obviously involves the map $(\xi, \zeta) \mapsto \phi$. We shall next analyze the differentiability of this map. Assume for instance that the partial derivative $\partial_{\zeta_l}\phi$ exists. Then, formally, with $c = c_q\,\mathrm{e}^{(q-5)x}q(q-1)$,

$$\partial_{\zeta_l}N(\phi) = c\,\mathrm{e}^{(q-5)x}\int_0^1 (1-t)\,\mathrm{d}t\big[(q-2)[V+t\phi]^{q-3}(V_{\zeta_l}+t\partial_{\zeta_l}\phi)\phi^2 + 2[V+t\phi]^{q-2}\partial_{\zeta_l}\phi\phi\big].$$

As we have seen in the previous section, $V_{\zeta_l} = \sum_{j=1}^k \partial_{\zeta_l}\Pi_{\xi_j,\zeta}$ is uniformly bounded: Hence we conclude:

$$\big\|\partial_{\zeta_l}N(\phi)\big\|_* \leqslant C\big[\|\phi\|_* + \|\partial_{\zeta_l}\phi\|_*\big]\|\phi\|_* \leqslant C\big[\|R\|_* + \|\partial_{\zeta_l}\phi\|_*\big]\|R\|_*.$$

Also observe that we have:

$$\partial_{\zeta_l}\phi = (\partial_{\zeta_l}T)\big(N(\phi)+R\big) + T\big(\partial_{\zeta_l}\big[N(\phi)+R\big]\big)$$

so that, using Proposition 5.2,

$$\|\partial_{\zeta_l}\phi\|_* \leqslant C\big[\big\|N(\phi)+R\big\|_* + \big\|\partial_{\zeta_l}N(\phi)\big\|_* + \|\partial_{\zeta_l}R\|_*\big]$$

for some constant $C>0$. Reducing the constant c_0 for which $\|R\|_* \leqslant c_0$ if necessary, we conclude from the above computation that

$$\|\partial_{\zeta_l}\phi\|_* \leqslant C\big[\|R\|_* + \|\partial_{\zeta_l}R\|_*\big].$$

A similar computation shows that, as well:

$$\|\partial_{\xi_l}\phi\|_* \leqslant C\big[\|R\|_* + \|\partial_{\xi_l}R\|_*\big].$$

The above computation can be made rigorous by using the implicit function theorem in the space $\mathcal{C}_*$ and the fixed point representation (6.1) which guarantees C^1 regularity in (ξ,ζ). This differentiation procedure can be iterated to obtain second derivatives. This can be summarized as follows:

Lemma 6.2. *Under the assumptions of Proposition* 5.1 *and Lemma* 6.1 *consider the map* $(\xi,\zeta)\mapsto\phi$ *into the space* $\mathcal{C}_*$. *The partial derivatives* $\nabla\phi$ *and* $\nabla\partial_\xi\phi$ *exist and define continuous functions of the pair* (ξ,ζ). *Besides, there is a constant* $C>0$, *such that*

$$\|\nabla\phi\|_* + \|\nabla\partial_\zeta\phi\|_* \leqslant C\big(\|R\|_* + \|\nabla R\|_* + \|\nabla\partial_\zeta R\|_*\big).$$

The size of ϕ and of its derivatives is proportional to the corresponding sizes for R.

After Problem (4.6) has been solved, we will find solutions to the full problem (4.3) if we manage to adjust the pair (ξ,ζ) in such a way that $c_{ij}(\xi,\zeta)=0$ for all i, j. This is the *reduced problem*. A nice feature of this system of equations is that it turns out to be equivalent to finding critical points of a functional of the pair (ξ,ζ) which is close, in appropriate sense, to the energy of the single or multiple-bubble V. We make this precise in the next section for the case of single-bubbling, $k=1$.

7. Variational formulation of the reduced problem for $k=1$

In this section we assume $k=1$ in Problem (4.6). We omit the subscript $i=1$ in c_{ij}, Z_{ij} and ξ_i. Then in order to obtain a solution of (4.3) we need to solve the system of equations:

$$c_j(\xi,\zeta)=0 \quad \text{for all } j=1,\dots,4. \tag{7.1}$$

If (7.1) holds, then $v=V+\phi$ will be a solution to (4.3). This system turns out to be equivalent to a variational problem, as we discuss next.

Let us consider the functional $J_{q,\lambda}$ in (4.4), the energy associated to Problem (4.3). Let us define:

$$F(\mu,\zeta)\equiv J_{q,\lambda}(V+\phi), \quad \mu=\mathrm{e}^{-2\xi}, \tag{7.2}$$

where $\phi=\phi(\xi,\zeta)$ is the solution of Problem (4.6) given by Proposition 5.1. Critical points of F correspond to solutions of (7.1) under a mild assumption that will be satisfied in the proofs of the theorems, as we shall see below.

Lemma 7.1. *Under the assumptions of Proposition* 5.1*, the functional* $F(\zeta,\xi)$ *is of class* C^1*. Assume additionally that* R *in* (4.5) *satisfies* $\|R\|_*\leqslant\mu^{8\sigma}$*, where* $\sigma>0$ *is the number in the definition of the* $*$*-norm. Then, for all* $\mu>0$ *sufficiently small, if* $\nabla F(\xi,\zeta)=0$*, then* (ξ,ζ) *satisfies System* (7.1)*.*

Proof. Let us first differentiate with respect to ξ. We can differentiate directly $J_{q,\lambda}(V+\phi)$ under the integral sign, since the domain D depends on ζ but not on ξ. Thus

$$\partial_\xi F(\xi,\zeta)=DJ_{q,\lambda}(V+\phi)[\partial_\xi V+\partial_\xi\phi]=\sum_{j=1}^{4}\int_D c_jZ_j[\partial_\xi V+\partial_\xi\phi].$$

From this expression and the results of the previous section, it is continuous with respect to the pair (ξ,ζ). Let us assume that $\partial_\xi F(\xi,\zeta)=0$. Then

$$\sum_{j=1}^{4}c_j\int_D Z_j[\partial_\xi V+\partial_\xi\phi]=0.$$

We recall that we proved $\|\partial_\xi\phi\|_*\leqslant C\|R\|_*$, thus we directly check that as $\mu\to 0$, we have $\partial_\xi V+\partial_\xi\phi=Z_4+\mathrm{o}(1)$ with $\mathrm{o}(1)$ small in terms of the $*$-norm as $\mu\to 0$.

Let us consider now differentiation with respect to ζ. This is a bit more involved since it is no longer sufficient to differentiate under the integral sign. It is convenient to relate the functional with its expression in terms of the original variable in Ω. Let us observe first that the following identity holds:

$$DJ_{q,\lambda}(v)[f] = 4DE_{q,\lambda}(u)[g] \quad \text{where } v = \mathcal{T}(u),\ f = \mathcal{T}(g).$$

Let us define U and ψ by $V \equiv \mathcal{T}(U)$, $\psi \equiv \mathcal{T}(\phi)$. Let us recall then that $J_{q,\lambda}(V+\phi) = 4E_{q,\lambda}(U+\psi)$. Given l, we compute:

$$\begin{aligned}\partial_{\zeta_l} F &= 4DE_{q,\lambda}(U+\psi)[\partial_{\zeta_l} U + \partial_{\zeta_l}\psi] \\ &= DJ_{q,\lambda}(V+\phi)\big[\mathcal{T}(\partial_{\zeta_l} U) + \mathcal{T}(\partial_{\zeta_l}\psi)\big] = \sum_{j=1}^{4} c_j \int_D Z_j\big[\mathcal{T}(\partial_{\zeta_l} U) + \mathcal{T}(\partial_{\zeta_l}\psi)\big].\end{aligned}$$

This expression depends continuously on (ξ,ζ). Let us consider $\mathcal{T}(\partial_\zeta U)$. We have that

$$\partial_\zeta U = \partial_\zeta w_{\mu,\zeta} + \partial_\zeta \pi_{\mu,\zeta} = c\mu^{-5/2}\left[1 + \frac{r^2}{\mu^2}\right]^{-3/2} r\Theta + \mathrm{O}\big(\mu^{1/2}\big),$$

where $r = |y-\zeta|$. Hence,

$$\mathcal{T}(\partial_\zeta U) = \mu^{-1} Z(x-\xi_1) + \mathrm{e}^{-\xi_1}\,\mathrm{e}^{-x}\,\mathrm{O}(1).$$

Let us consider now the term $\mathcal{T}(\partial_\zeta\psi)$. If $\psi = \psi(y,\zeta)$, we have:

$$(\partial_{\zeta_l}\phi)(x,\Theta,\zeta) = \sqrt{2}\,\mathrm{e}^{-x}\partial_\zeta\big[\psi\big(\zeta + \mathrm{e}^{2x}\Theta,\zeta\big)\big] = \mathcal{T}(\partial_{y_l}\psi) + \mathcal{T}(\partial_{\zeta_l}\psi),$$

so that $\mathcal{T}(\partial_{\zeta_l}\psi) = \partial_{\zeta_l}\phi - \mathcal{T}(\partial_{y_l}\psi)$. We have already established that

$$\|\partial_{\zeta_l}\phi\|_* \leqslant C\big(\|R\|_* + \|\nabla R\|_*\big).$$

Let us recall the equation satisfied by $\partial_{y_l}\psi$. It is convenient to define $\tilde\psi(z) = \mu^{1/2}\psi(\zeta + \mu z)$. Then $\tilde\psi$ satisfies:

$$\Delta\tilde\psi + q\mu^{(5-q)/2}\big[w_{1,0} + \mathrm{O}(\mu)\big]^{q-1}\tilde\psi + \lambda\mu^2\tilde\psi = -E + \sum_{l=1}^{4} c_l \frac{1}{|z|^2}\big(1 - \eta^1(|z|)\big) z_l(z),$$

where $E = \widetilde{N}(\psi) + \widetilde{R}$ with

$$\begin{aligned}\widetilde{R} &= \big(w_{1,0} + \mathrm{O}(\mu)\big)^q - w_{1,0}^5, \\ \widetilde{N}(\tilde\psi) &= \frac{q(q-1)}{2}\int_0^1 (1-t)\,\mathrm{d}t\big[w_{1,0} + \mathrm{O}(\mu) + t\tilde\psi\big]^{q-2}\tilde\psi^2.\end{aligned}$$

Here η^1 is the smooth cut-off function in (5.8). We also know that, globally,

$$\left|\tilde{\psi}(z)\right| \leqslant C\mu^{-\sigma/2}\|R\|_*|x|^{-\sigma/2}w_{1,0}^{1-\sigma}(z).$$

Since σ is small, it follows from elliptic estimates that near the origin actually $|\tilde{\psi}(z)| \leqslant C\mu^{-q\sigma}\|R\|_*$ and

$$\left|D\tilde{\psi}(z)\right| \leqslant \mu^{-q\sigma}\|R\|_* w_{1,0}^{1-\sigma}(z).$$

As a conclusion we get that

$$\left\|\mathcal{T}(D\psi)\right\|_* \leqslant C\mu^{-q\sigma}\|R\|_*.$$

Thus $\partial_\zeta F = 0$ if and only if:

$$0 = \sum_{j=1}^{4} c_j \int\limits_D Z_j\left[Z_l + \mathrm{O}\left(\mu^{-\frac{q}{2}\sigma}\|R\|_*\right)\right]$$

for each $l = 1, 2, 3$. We get then that $\nabla F(\xi, \zeta) = 0$ implies the validity of a system of equations of the form:

$$\sum_{j=1}^{4} c_j \int\limits_D Z_j\left[Z_l + \mathrm{o}(1)\right] = 0, \quad l = 1, \ldots, 4,$$

with $\mathrm{o}(1)$ small in the sense of the $*$-norm as $\mu \to 0$. The above system is diagonal dominant and we thus get $c_j = 0$ for all j. □

In order to solve for critical points of the function F, a key step is its expected closeness to the function $4E_{q,\lambda}(U_{\mu,\zeta}) = J_{q,\lambda}(V)$, which we analyzed in Section 2. From now on we shall use the notation:

$$\nabla \equiv [\partial_\xi, \partial_\zeta].$$

Lemma 7.2. *The following expansion holds*:

$$F(\xi, \zeta) = J_{q,\lambda}(V) + \left[\|R\|_*^2 + \|\nabla R\|_*^2 + \|\nabla\partial_\xi R\|_*^2\right]\theta(\xi, \zeta),$$

where for a certain positive constant C the function θ satisfies:

$$|\theta| + |\nabla\theta| + |\nabla\partial_\xi\theta| \leqslant C,$$

uniformly on points satisfying the constraints in Proposition 5.1.

Proof. Taking into account that $0 = DJ_{q,\lambda}(V+\phi)[\phi]$, a Taylor expansion gives:

$$J_{q,\lambda}(V+\phi) - J_{q,\lambda}(V) \tag{7.3}$$

$$= \int_0^1 D^2 J_{q,\lambda}(V+t\phi)[\phi]^2(1-t)\,\mathrm{d}t \tag{7.4}$$

$$= \int_0^1 \left(\int_D \big[N(\phi)+R\big]\phi + \int_D q\big[V^{q-1} - (V+t\phi)^{q-1}\big]\phi^2 \right)(1-t)\,\mathrm{d}t.$$

Since $\|\phi\|_* \leqslant C\|R\|_*$, we get:

$$J_{q,\lambda}(V+\phi) - J_{q,\lambda}(V) = \mathrm{O}\big(\|R\|_*^2\big).$$

Let us differentiate now with respect to the pair (ξ,ζ). Since the quantity inside the integral in the representation (7.4) vanishes on ∂D, we may differentiate directly under the integral sign, thus obtaining:

$$\nabla\big[J_{q,\lambda}(V+\phi) - J_{q,\lambda}(V)\big]$$

$$= \int_0^1 \left(\int_D \nabla\big[\big(N(\phi)+R\big)\phi\big] + q\int_D \nabla\big[\big((V+t\phi)^{q-1} - V^{q-1}\big)\phi^2\big] \right)(1-t)\,\mathrm{d}t.$$

Using the fact that $\|\nabla\phi\|_* \leqslant C[\|R\|_* + \|\nabla R\|_*]$ and the computations in the proof of Lemma 6.2 we get that the above integral can be estimated by $\mathrm{O}(\|R\|_*^2 + \|\nabla R\|_*^2)$. Finally, we can also differentiate under the integral sign if we do it first with respect to ξ, and then apply ∇, using the fact that $\partial_\xi\phi = 0$ on ∂D. We obtain then

$$\nabla\partial_\xi\big[J_{q,\lambda}(V+\phi) - J_{q,\lambda}(V)\big] = \mathrm{O}\big(\|R\|_*^2 + \|\nabla R\|_*^2 + |\nabla\partial_\xi R\|_*^2\big).$$

The continuity in (ξ,ζ) of all these expressions is inherited from that of ϕ and its derivatives in (ξ,ζ) in the $*$-norm. □

We have now all the elements for the proof of our main results regarding single bubbling.

8. Existence of single bubbling solutions

In this section we will prove our main results concerning solutions of (4.3) close to $V = W(x-\xi) + \Pi_\xi$ where $\Pi_\xi \equiv \mathrm{e}^{-x}\pi_{\mu,\zeta}(\zeta + \mathrm{e}^{-2x}\Theta)$ with $\mu = \mathrm{e}^{-2\xi}$. Before going into the proofs, we point out properties of this function which essentially translate those in Lemma 2.2. We have:

$$\begin{aligned}\Pi_\xi(x,\Theta) = {} & -4\pi\sqrt{2}\,3^{1/4}\,\mathrm{e}^{-(x+\xi)}H_\lambda\big(\zeta,\zeta+\mathrm{e}^{-2x}\Theta\big)\\ & +\sqrt{2}\,\mathrm{e}^{-(x+3\xi)}\mathcal{D}_0\big(\mathrm{e}^{-2(x-\xi)}\Theta\big)\\ & +\sqrt{2}\,\mathrm{e}^{-x-(5-2\sigma)\xi}\theta\big(\zeta,\zeta+\mathrm{e}^{-2x}\Theta,\xi\big),\end{aligned}$$

where for $j=0,1,2$, $i=0,1$, $i+j\leqslant 2$, the function $\frac{\partial^{i+j}}{\partial\zeta^i\partial\xi^j}\theta(y,\zeta,\xi)$, is bounded uniformly on $y\in\Omega$, all large ξ and ζ in compact subsets of Ω. We recall that as $x\gg\xi$,

$$\mathcal{D}_0\big(\mathrm{e}^{2(x-\xi)}\Theta\big)=\mathrm{e}^{-2(x-\xi)}+\mathcal{D}_1\big(\mathrm{e}^{-2(x-\xi)}\big),$$

and with $\mathcal{D}_1$ smooth, $\mathcal{D}_1'(0)=0$, while $\mathcal{D}_0(r)\sim\frac{\log r}{r}$ as $r\to+\infty$. It follows that

$$\begin{aligned}|\Pi_\xi|+|\partial_\xi\Pi_\xi|+\big|\partial_\xi^2\Pi_\xi\big| & \leqslant C\,\mathrm{e}^{-x}\,\mathrm{e}^{-\xi}\big[\big|H_\lambda\big(\zeta,\zeta+\mathrm{e}^{-2x}\Theta\big)\big|+\mathrm{e}^{-2\xi}\big]\\ & \leqslant C\,\mathrm{e}^{-x}\,\mathrm{e}^{-\xi}\big[\mathrm{e}^{-2x}+\big|g_\lambda(\zeta)\big|+\mathrm{e}^{-2\xi}\big].\end{aligned}\tag{8.1}$$

On the other hand $H_\lambda(\zeta,\zeta+y)=h_0(y)+h_1(\zeta,y)$ where h_1 is smooth, from where it follows that

$$|\partial_\zeta\Pi_\xi|+|\partial_\zeta\partial_\xi\Pi_\xi|\leqslant C\,\mathrm{e}^{-x}\,\mathrm{e}^{-\xi}\big(\big|\partial_\zeta g_\lambda(\zeta)\big|+\mathrm{e}^{-2x}+\mathrm{e}^{-2\xi}\big).\tag{8.2}$$

Proof of Theorem 2. We choose μ as in (3.1),

$$\mu=-\varepsilon\frac{a_4}{a_1}\frac{1}{g_\lambda(\zeta)}\Lambda,$$

where $\varepsilon=q-5$. We have to find a critical point of the functional $F(\mu,\zeta)$ in (7.2) for $q=5+\varepsilon$. Consider:

$$R=c_q\,\mathrm{e}^{(q-5)x}\big(W(x-\xi)+\Pi_\xi(x,\Theta)\big)^{5+\varepsilon}-W(x-\xi)^5,$$

where $\mathrm{e}^{-2\xi}=\mu$. We write as usual $W_1=W(x-\xi)$, $V=W_1+\Pi_\xi$. Then we can decompose $R=R_1+R_2+R_3+R_4$, where

$$\begin{aligned} & R_1\equiv\mathrm{e}^{\varepsilon x}\big(V^{5+\varepsilon}-V^5\big), && R_2\equiv V^5\big(\mathrm{e}^{\varepsilon x}-1\big),\\ & R_3\equiv V^5-W_1^5, && R_4\equiv(c_q-1)\mathrm{e}^{\varepsilon x}V^{5+\varepsilon}.\end{aligned}$$

We have:

$$R_1=\varepsilon\,\mathrm{e}^{\varepsilon x}\int_0^1(1-t)\,\mathrm{d}t\big(V^{5+t\varepsilon}\log V\big),$$

from where it follows that

$$|R_1| \leqslant C\varepsilon\, \mathrm{e}^{\varepsilon\xi}\, \mathrm{e}^{\varepsilon|x-\xi|} V^{4+1/2} \leqslant C\varepsilon V^4.$$

Since $|\Pi_\xi| \leqslant C\,\mathrm{e}^{-(x+\xi)} \leqslant C\,\mathrm{e}^{-|x-\xi|}$, we get $|R_1| \leqslant C\varepsilon\,\mathrm{e}^{-4|x-\xi|}$ and hence $\|R_1\|_* \leqslant C\varepsilon$. Direct differentiation of the above expression, using the bounds for derivatives of Π_ξ yields as well

$$\left\|\partial^2_{\xi^2} R_1\right\|_* + \left\|\partial^2_{\zeta\xi} R_1\right\|_* + \|\partial_\zeta R_1\|_* \leqslant \varepsilon.$$

Let us denote $\nabla = [\partial_\xi, \partial_\zeta]$. Thus we have:

$$\|R_1\|_* + \|\nabla R_1\|_* + \|\nabla\partial_\xi R_1\|_* \leqslant C\varepsilon.$$

Observe that the same estimate is also valid for R_4. On the other hand, we have:

$$R_2 = V^5\bigl(\mathrm{e}^{\varepsilon x} - 1\bigr) = \varepsilon x V^5 \int_0^1 \mathrm{e}^{t\varepsilon x}\,\mathrm{d}t.$$

Since $\xi \sim c\log(1/\varepsilon)$ we obtain for R_2 and derivatives the bounds:

$$\|R_2\|_* + \|\nabla R_2\|_* + \|\nabla\partial_\xi R_2\|_* \leqslant C\varepsilon|\log\varepsilon|.$$

Finally, for

$$R_3 = 5\int_0^1 (1-t)\,\mathrm{d}t\,(W_1 + t\Pi_\xi)^4 \Pi_\xi,$$

we find the bound:

$$|R_3| \leqslant C\,\mathrm{e}^{-\xi-x-4|x-\xi|} \leqslant C\,\mathrm{e}^{-2\xi-|x-\xi|},$$

and similarly for derivatives. We get, recalling that $\mathrm{e}^{-2\xi} \equiv \mu \leqslant C\varepsilon$,

$$\|R_3\|_* + \|\nabla R_3\|_* + \|\nabla\partial_\xi R_3\|_* \leqslant C\,\mathrm{e}^{-2\xi}.$$

Concerning R_4, a direct computation gives $|R_4| \leqslant C\varepsilon\,\mathrm{e}^{-5|x-\xi|}$. Thus for full R we have:

$$\|R\|_* + \|\nabla R\|_* + \|\nabla\partial_\xi R\|_* \leqslant C\varepsilon|\log\varepsilon|.$$

It follows from Lemma 7.2 that for this choice of μ,

$$F(\xi,\zeta) = J_{q,\lambda}(V) + \mu^2|\log\mu|^2\theta(\xi,\zeta)$$

with $|\theta|+|\nabla\partial_\xi\theta|+|\nabla\theta|\leqslant C$. Define $\psi_\varepsilon(\Lambda,\zeta)=F(\frac{1}{2}\log\frac{1}{\mu},\zeta)$ with μ as above. A critical point for ψ_ε is in correspondence with one of F. We conclude that

$$\psi_\varepsilon(\Lambda,\zeta)=4E_{5+\varepsilon,\mu}(U_{\mu,\zeta})+\varepsilon\theta_\varepsilon(\Lambda,\zeta)$$

with θ_ε as in Lemma 3.1. The lemma thus applies to predict a critical point of ψ_ε and the proof of Part (a) is complete. Part (b) is analogous, invoking instead Lemma 3.2. □

Proof of Theorem 4. Let us choose now μ as in (3.4),

$$\mu=-\frac{a_1g_\lambda(\zeta)}{2a_2\lambda}\Lambda,$$

where $\zeta\in\mathcal{D}_\lambda$. Now R is just given by:

$$R=5\int_0^1(1-t)\,\mathrm{d}t\,(W_1+t\Pi_\xi)^4\Pi_\xi.$$

It follows from estimates (8.1) and (8.2) that

$$|R|+|\nabla R|+|\nabla\partial_\xi R|\leqslant C\,\mathrm{e}^{-x-\xi-4|x-\xi|}\big[\big|g_\lambda(\zeta)\big|+\big|\nabla g_\lambda(\zeta)\big|+\mathrm{e}^{-2x}+\mathrm{e}^{-2\xi}\big].$$

Let $\delta_\lambda\equiv\sup_{\mathcal{D}_\lambda}(|g_\lambda|+|\nabla g_\lambda|)$. Then we see that $\delta_\lambda\to0$ as $\lambda\downarrow0$. We conclude that

$$\|R\|_*+\|\nabla R\|_*+\|\nabla\partial_\xi R\|_*\leqslant C\,\mathrm{e}^{-2\xi}\delta_\lambda.$$

We have now

$$F(\xi,\zeta)=4E_{5,\lambda}(U_{\zeta,\mu})+\mu^2\delta_\lambda^2\theta(\xi,\zeta)$$

with $|\theta|+|\nabla\partial_\xi\theta|+|\nabla\theta|\leqslant C$. Define $\psi_\lambda(\Lambda,\zeta)=F(\frac{1}{2}\log\frac{1}{\mu},\zeta)$ with μ as above. Again, a critical point for ψ_ε is in correspondence with one of F. We conclude that

$$\psi_\lambda(\zeta,\Lambda)=4E_{5,\lambda}(U_{\zeta,\mu})+g_\lambda(\zeta)^2\theta_\lambda(\Lambda,\zeta),$$

where μ is given by (3.4) and θ_λ is as in Lemma 3.3. Hence ψ_λ has a critical point as in the statement of Lemma 3.3, and the result of the theorem follows, with the constant β given by $\beta=(2a_2/a_1)^{1/2}$. □

Proof of Theorem 3, Part (b). In this case the consideration we make is slightly different. Observe that if we choose $\zeta=0$, then the assumption of symmetry of the domain, and uniqueness of the solution $\phi_{(\xi,0)}(x,\Theta)$ of Problem (4.6) makes it even in each of the coordinates Θ_i, $i=1,2,3$, since so is V. Moreover, as a by-product, we find that

$c_j = 0$ for $j = 1, 2, 3$. Thus only c_4 survives. As a consequence, we find that $c_4 = 0$ if we have $\partial_\xi F(\xi, 0) = 0$. With the choice,

$$\mu = -\frac{a_1 g_\lambda(0)}{2a_2\lambda}\Lambda,$$

we find that $F_\lambda(\Lambda) = F(\frac{1}{2}\log\frac{1}{\mu}, 0)$ satisfies:

$$F_\lambda(\Lambda) = \frac{a_1^2}{4a_2\lambda} g_\lambda(0)^2\big[-2\Lambda + \Lambda^2\big] + g_\lambda(0)^2\theta_\lambda(\Lambda),$$

where θ_λ and its derivative are small uniformly on Λ in bounded sets. We conclude the existence of a critical point Λ_λ of F_λ close to 1, and the desired result follows. $\square$

9. Multiple bubbling

In this section we will prove Theorem 4, Part (a). Let us consider the solution $\phi(\xi, \zeta)$ of (4.6) given by Proposition 5.1 where $\xi = (\xi_1, \xi_2, \dots, \xi_k)$. Similarly as in the proof of Theorem 3, Part (b), choosing $\zeta = 0$ makes ϕ symmetric in the Θ_i variables, which automatically yields $c_{ij} = 0$ for all $i = 1, \dots, k$ and $j = 1, 2, 3$. Thus we just need to adjust ξ in such a way that $c_{i4} = 0$ for $i = 1, \dots, k$. Arguing exactly as in the proof of Lemma 7.1 we get that this is equivalent to finding a critical point of the functional $F(\xi) = J_{q,\lambda}(V + \phi)$, where ζ has been fixed to be zero. Similarly as before, we find now that

$$F(\xi) = J_{q,\lambda}(V) + \big(\|R\|_*^2 + \|\partial_\xi R\|_*^2\big)\theta(\xi),$$

where θ and its first derivative are continuous and uniformly bounded in large ξ.

In what remains of this section we fix a number $\delta > 0$, set $\varepsilon = q - 5 > 0$ and choose $\mu_i = \mathrm{e}^{-2\xi_i}$ in order that

$$\mu_1 = \varepsilon\Lambda_1, \quad \mu_{j+1} = \mu_j(\Lambda_{j+1}\varepsilon)^2, \quad j = 1, \dots, k-1, \tag{9.1}$$

with

$$\delta < \Lambda_j < \delta^{-1}, \quad j = 1, \dots, k. \tag{9.2}$$

Let us measure the size of $\|R\|_*$ and $\|\partial_\xi R\|_*$ for this Ansatz. We can now decompose $R = R_1 + R_2 + R_3 + R_4 + R_5$, where

$$R_1 \equiv \mathrm{e}^{\varepsilon x}\big(V^{5+\varepsilon} - V^5\big), \qquad R_2 \equiv V^5\big(\mathrm{e}^{\varepsilon x} - 1\big),$$
$$R_3 \equiv V^5 - \Big(\sum_i W_i\Big)^5, \qquad R_4 \equiv \Big(\sum_i W_i\Big)^5 - \sum_i W_i^5,$$
$$R_5 \equiv (c_q - 1)\,\mathrm{e}^{\varepsilon x} V^{5+\varepsilon}.$$

We can estimate,

$$|R_4| \leqslant C \sum_{i=1}^{k-1} \mathrm{e}^{-(\xi_{i+1}-\xi_i)}\, \mathrm{e}^{-3|x-\xi_i|},$$

hence $\|R_4\|_* \leqslant C\varepsilon$, a similar bound being valid for its derivatives in ξ_i's. The quantities R_j for $j = 1, 2, 3, 5$ can be estimated in exactly the same way as in the proof of Theorem 2. Thus $\|R\|_* + \|\partial_\xi R\|_* \leqslant C\varepsilon|\log\varepsilon|$. Let us set $\Lambda = (\Lambda_1, \dots, \Lambda_k)$ and define $\psi_\varepsilon(\Lambda) = F(\xi)$ with ξ given by (9.1). We need to find a critical point of ψ_ε. We have proved that

$$\psi_\varepsilon(\Lambda) = J_{q,\lambda}(V) + \mathrm{O}\big(\varepsilon^2|\log\varepsilon|^2\big)\theta_\varepsilon(\Lambda), \tag{9.3}$$

where θ_ε and its first derivative are uniformly bounded. We have the validity of the following fact, whose proof we postpone for the moment,

$$\frac{1}{4} J_{q,\lambda}(V) = k a_0 + \big[\psi_*(\Lambda) + \mathrm{o}(1)\big]\varepsilon + \frac{1}{2} k(k+1) a_4 \varepsilon|\log\varepsilon|, \tag{9.4}$$

where

$$\psi_*(\Lambda) = a_1 g_\lambda(0)\Lambda_1 + k a_4 \log\Lambda_1 + \sum_{j=2}^{k} \big[(k-j+1) a_4 \log\Lambda_j - a_6 \Lambda_j\big]$$

and the term $\mathrm{o}(1)$ as $\varepsilon \to 0$ is uniformly small in C^1-sense on parameters Λ_j satisfying (9.2). Here the constants a_0, a_1, a_4 are the same as those in Lemma 2.3, while $a_6 = 16\pi\sqrt{3}$. The assumption $g_\lambda(0) < 0$ implies the existence of a unique critical point Λ_* which can easily be solved explicitly. It follows that $\mathrm{o}(1)$ C^1 perturbation of ψ_* will have a critical point located at $\mathrm{o}(1)$ distance of Λ_*. After this observation, the combination of relations (9.4) and (9.3) give the existence of a critical point of ψ_ε close to Λ_* which translates exactly as the result of Theorem 4, Part (a).

It only remains to establish the validity of expansion (9.4). We recall that

$$V = \sum_{i=1}^{k} V_i = \sum_{i=1}^{k} W_i + \Pi_i = \mathcal{T}(U),$$

where $U = \sum_{i=1}^{k} w_i + \pi_i$, and we denote $w_i = w_{\mu_i,0}$, $\pi_i = \pi_{\mu_i,0}$, $U_i = w_i + \pi_i$. We have that $J_{q,\lambda}(V) = 4E_{q,\lambda}(U)$, where $q = 5 + \varepsilon$. Observe that we can write:

$$E_{q,\lambda}(U) = E_{5,\lambda}(U) + \mathcal{R},$$

where

$$\mathcal{R} \equiv -\frac{1}{6} \int_D \big(\mathrm{e}^{(q-5)x} - 1\big)|V|^6 + 4\pi A_q.$$

A direct computation yields:

$$A_q = k(q-5)\left(\frac{1}{6}\int_{-\infty}^{\infty} W^6 \log W \,\mathrm{d}x + \frac{1}{36}\int_{-\infty}^{\infty} W^6 \,\mathrm{d}x\right) + \mathrm{o}(\varepsilon).$$

On the other hand,

$$\begin{aligned}
\mathcal{R} - 4\pi A_q &= -\frac{1}{6}\int_D \left[\mathrm{e}^{(q-5)x} - 1\right] V^6 \,\mathrm{d}x \\
&= -\frac{1}{6}(q-5)4\pi \int_D x V^6 \,\mathrm{d}x + \mathrm{o}(\varepsilon) \\
&= -\frac{1}{6}(q-5)\left(\sum_{i=1}^{k} \xi_i\right)\int_{-\infty}^{\infty} W^6 \,\mathrm{d}x + \mathrm{o}(\varepsilon) \\
&= a_4(q-5)\sum_{j=1}^{k} \log \mu_j + \mathrm{o}(\varepsilon).
\end{aligned}$$

Now we have:

$$E_{5,\lambda}(U) = \sum_{j=1}^{k} E_{5,\lambda}(U_j) + \frac{1}{6}\int_D \left[\sum_{i=1}^{k} V_i^6 - \left(\sum_{i=1}^{k} V_i\right)^6 + 6\sum_{i<j} W_i^5 V_j\right] \tag{9.5}$$

since

$$\begin{aligned}
&E_{5,\lambda}(U) - \sum_{j=1}^{k} E_{5,\lambda}(U_j) - \int_D \left[\sum_{i=1}^{k} V_i^6 - \left(\sum_{i=1}^{k} V_i\right)^6\right] \\
&= \sum_{i<j}\int_D \left(2\nabla_\Theta V_i \nabla_\Theta V_j + V_i' V_j' + V_i V_j - 2\lambda\,\mathrm{e}^{-4x} V_i V_j\right) \\
&= \sum_{i<j}\int_D \left(-4\Delta_{S^2} V_i - V_i'' + V_i - 4\lambda\,\mathrm{e}^{-4x} V_i\right) V_j = \sum_{i<j}\int_D W_i^5 V_j.
\end{aligned}$$

To estimate the quantities in (9.5), we consider the numbers:

$$\chi_1 = 0, \quad \chi_l = \frac{1}{2}(\xi_{l-1} + \xi_l), \quad l = 2, \ldots, k, \quad \chi_{k+1} = +\infty,$$

and decompose

$$E_{5,\lambda}(U) - \sum_{j=1}^{k} E_{5,\lambda}(U_j) = \sum_{\substack{1\leqslant l\leqslant k\\ j>l}} \int\limits_{D\cap\{\chi_l<x<\chi_{l+1}\}} V_l^5 V_j + B.$$

A straightforward computation yields $B = \mathrm{o}(\varepsilon)$. On the other hand,

$$\begin{aligned}\sum_{\substack{1\leqslant l\leqslant k\\ j>l}} \int\limits_{D\cap\{\chi_l<x<\chi_{l+1}\}} V_l^5 V_j &= 4\pi \sum_{l=1}^{k} \int\limits_{\chi_l}^{\chi_{l+1}} W_l^5 W_{l+1} + \mathrm{o}(\varepsilon)\\ &= 4\pi \int\limits_{\chi_l}^{\chi_{l+1}} W_l^5 W_{l+1} + \mathrm{o}(\varepsilon)\\ &= 4\pi \int\limits_{\chi_l-\xi_l}^{\chi_{l+1}-\xi_l} W^5(x) W\big(x - (\xi_{l+1} - \xi_l)\big) + \mathrm{o}(\varepsilon)\\ &= 4\pi \sum_{l=1}^{k-1} \mathrm{e}^{-|\xi_{l+1}-\xi_l|} (12)^{1/4} \int\limits_{-\infty}^{\infty} \mathrm{e}^x W(x)^5 + \mathrm{o}(\varepsilon)\\ &= a_6 \sum_{j=1}^{k-1} \left(\frac{\mu_{j+1}}{\mu_j}\right)^{1/2} + \mathrm{o}(\varepsilon).\end{aligned}$$

Taking into account the estimate given in Lemma (2.1) and the above estimates, we get (9.4) in the uniform sense. Similar arguments yield that the remainder is as well $\mathrm{o}(\varepsilon)$ small after a differentiation with respect to the ξ_i's.

10. Further asymptotics, final comments

Let λ_0 be a number for which a critical value 0 as in Theorem 4 is present. What we want to discuss next is the situation present when λ is close to λ_0 and, at the same time, q is close to 5, both from above and below. We shall do this only in the case of a local minimizer,

$$0 = \inf_{\mathcal{D}} g_{\lambda_0} < \inf_{\partial\mathcal{D}} g_{\lambda_0}.$$

As we have discussed this local minimum situation remains whenever λ is sufficiently close to λ_0. Let us set $m_\lambda \equiv \inf_{\mathcal{D}} g_\lambda$. Then m_λ is strictly decreasing. In fact $m_\lambda \sim -(\lambda - \lambda_0)$. *Dual asymptotics* are found for the sub- and super-critical cases as follows:

Theorem 5. (a) *Assume that* $q = 5 + \varepsilon$. *Let* $\gamma > (8\sqrt{2})^{-1}$ *be fixed and assume that* $\lambda > \lambda_0$ *is the unique number for which*

$$m_\lambda = -\gamma\sqrt{\varepsilon\lambda_0}.$$

Then for all ε *sufficiently small there exist two solutions* $u_\varepsilon^\pm$ *to Problem* (1.1) *of the form,*

$$u_\varepsilon^\pm(x) = \frac{3^{1/4} M_\varepsilon^\pm}{\sqrt{1 + (M_\varepsilon^\pm)^4 |x - \zeta_\varepsilon|^2}}\bigl(1 + \mathrm{o}(1)\bigr), \tag{10.1}$$

where $\mathrm{o}(1) \to 0$ *uniformly in* $\overline{\Omega}$ *as* $\varepsilon \to 0$,

$$M_\varepsilon^\pm = \Lambda_\pm^{-1/2}(\gamma)\varepsilon^{-1/4}.$$

Here $\sqrt{\lambda_0}\Lambda_\pm(\gamma) = x_\pm(\gamma)$ *are the two positive roots of*

$$32x^2 - 128\gamma x + 1 = 0$$

and ζ_ε *is a point in* $\mathcal{D}$ *such that* $g_\lambda(\zeta_\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

(b) *Assume that* $q = 5 - \varepsilon$. *Let* $\gamma \in \mathbb{R}$ *be fixed and assume additionally that* λ (*close to* λ_0) *is the unique number for which*

$$m_\lambda = \gamma\sqrt{\lambda_0}\varepsilon^{1/2}.$$

Then for all ε *sufficiently small there exist a solution* u_ε *to Problem* (1.1) *of the form* (10.1), *with* $M_\varepsilon^\pm$ *replaced by* M_ε, *where* $M_\varepsilon = \Lambda^{-1/2}(\gamma)\varepsilon^{-1/4}$. *Here* $\sqrt{\lambda_0}\Lambda(\gamma) = x(\gamma)$ *is the positive root of*

$$32x^2 + 128\gamma x - 1 = 0$$

and ζ_ε *is a point in* $\mathcal{D}$ *such that* $g_\lambda(\zeta_\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

Proof. Let $q = 5 + \varepsilon$ and Λ be such that

$$\mu = \Lambda\sqrt{\varepsilon}, \quad \Lambda > \delta.$$

This choice allows us to regard the functional $F(\xi, \zeta)$, where $\mu = \mathrm{e}^{-2\xi}$, as a small perturbation of $4E_{q,\lambda}(U)$ after restricting conveniently the range of variation of ζ. We will not carry out all details but just concentrate on the asymptotic expression of $4E_{q,\lambda}(U)$. Using the expansion in Lemma 2.3, we find that $\psi_\varepsilon(\Lambda, \zeta) = 4E_{q,\lambda}(U_{\mu,\zeta})$ can be expanded as

$$\psi_\varepsilon(\Lambda, \zeta) = a_0 + \tilde{\psi}_\varepsilon(\Lambda, \zeta) + \mathrm{o}(\varepsilon),$$

uniformly with respect to $\Lambda > \delta$, with

$$\tilde{\psi}_\varepsilon(\Lambda, \zeta) = a_1 g_\lambda(\zeta)\Lambda\sqrt{\varepsilon} - a_3\big(g_\lambda(\zeta)\big)^2\Lambda^2\varepsilon + a_2\lambda\Lambda^2\varepsilon + a_4\varepsilon\log\Lambda + \frac{a_4}{2}\varepsilon\log\varepsilon.$$

For fixed ζ, it turns out that the equation,

$$\partial_\Lambda\tilde{\psi}_\varepsilon(\Lambda, \zeta) = 0,$$

reduces in Λ at main order to the quadratic equation:

$$\frac{a_1 g_\lambda(\zeta)}{\sqrt{\varepsilon}}\Lambda + 2a_2\lambda_0\Lambda^2 + a_4 = 0$$

which has exactly two positive solutions $\Lambda^\pm(\zeta)$ provided that

$$\frac{-g_\lambda(\zeta)}{\sqrt{\varepsilon}} > \frac{1}{a_1}\sqrt{8a_2a_4\lambda_0} = \frac{\sqrt{\lambda_0}}{8\sqrt{2}}.$$

What we are assuming is that $m_\lambda = -\gamma\sqrt{\varepsilon\lambda_0}$ with $\gamma > (8\sqrt{2})^{-1}$. Let $\gamma' = (\gamma_0 + \gamma)/2$ and call $\mathcal{D}_\varepsilon$ the set of $\zeta \in \mathcal{D}$, where $-g_\lambda(\zeta)/\sqrt{\varepsilon} > \gamma'/\sqrt{\varepsilon}$. Using $\inf_{\mathcal{D}} g_\lambda = m_\lambda = -\gamma\sqrt{\varepsilon\lambda_0}$ and the expressions of a_1, a_2 and a_4 given in (2.11) and (2.13), the equation for Λ reduces to

$$32\big(\sqrt{\lambda_0}\,\Lambda\big)^2 - 128\gamma\big(\sqrt{\lambda_0}\,\Lambda\big) + 1 = 0.$$

The conclusion then holds if we take $M_\varepsilon^\pm = (\mu_\varepsilon^\pm)^{-1/2}$, $\mu_\varepsilon^\pm = \Lambda_\pm(\gamma)\sqrt{\varepsilon}$, where $x_\pm(\gamma) = \sqrt{\lambda_0}\Lambda_\pm(\gamma)$ are the two roots of the above equation. As in the proof of Lemma 3.3, we finally find that $\tilde{\psi}_\varepsilon(\Lambda_\pm(\gamma)(\zeta), \zeta)$ has a critical point in $\mathcal{D}_\varepsilon$ thus giving the two searched bubble-solutions. In fact, after a perturbation argument similar to those in Theorems 2 and 4, we find actual solutions to (1.1) with the form stated in the theorem. The proof of Part (b) is exactly the same, except that in this case the quadratic equation for Λ becomes:

$$\frac{a_1 g_\lambda(\zeta)}{\sqrt{\varepsilon}}\Lambda + 2a_2\lambda_0\Lambda^2 - a_4 = 0$$

which has exactly one positive solution, regardless the sign of $g_\lambda(\zeta)$. $\square$

It is illustrative to describe the results of this paper in terms of the bifurcation branch for the positive solutions of (1.1) in a ball which stems from $\lambda = \lambda_1$, $u = 0$, for any value of $q > 1$. This branch does not have turning points for $q = 5$ (uniqueness of the positive radial solution is known from [25]) and blows-up at $\lambda = \lambda_1/4$. On the other hand, an oscillating behavior has been observed from a variational point of vein in [11] and with ODE tools in [12]. As soon as $\varepsilon > 0$, $q = 5 + \varepsilon$, the branch turns right near the asymptote and then lives until getting close to λ_1. This “upper part” of the branch is the one described in Theorem 3, Part (a). It is of course reasonable to ask how the turning point looks like, in particular showing the presence of two solutions for λ slightly to the right of it. This is the interpretation Theorem 5, Part (a). Formal asymptotics of this first turning point, which are fully recovered by this result, were found by Budd and Norbury in [7]. The behavior of this branch "later" corresponds to the result of Theorems 4 (a): for $\varepsilon > 0$ small, the branch oscillates wildly between $\lambda_1/4$ and λ_1, giving rise for fixed λ between these numbers to an arbitrarily large number of solutions. The towers of Theorem 4 (a) may be interpreted as the solution found on the branch between the kth and $k+1$ turning points. Except in a ball or in a domain with symmetry, we have not found asymptotics of the turning points that lie close to λ_1, nor we know whether multiple bubbling is a generic phenomenon or rather a big coincidence due to symmetry.

Appendix A. Robin's function

In this appendix we prove two facts we have used in the course of the proofs about *Robin's function* g_λ. Recall that $g_\lambda(x) \equiv H_\lambda(x,x)$, where the function $y \mapsto H_\lambda(x,y)$ satisfies the boundary value problem:

$$\begin{cases} \Delta_y H_\lambda + \lambda H_\lambda = \lambda \frac{1}{4\pi|x-y|}, & y \in \Omega, \\ H_\lambda(x,y) = \frac{1}{4\pi|x-y|}, & x \in \partial\Omega. \end{cases}$$

Lemma A.1. *The function g_λ is of class $C^\infty(\Omega)$.*

Proof. We will show that $g_\lambda \in C^k$, for any k. Fix $x \in \Omega$. Let $h_{1,\lambda}$ be the function defined in $\Omega \times \Omega$ by the relation:

$$H_\lambda(x,y) = \beta_1|x-y| + h_{1,\lambda}(x,y),$$

where $\beta_1 = \lambda/(8\pi)$. Then $h_{1,\lambda}$ satisfies the boundary value problem:

$$\begin{cases} \Delta_y h_{1,\lambda} + \lambda h_{1,\lambda} = -\lambda\beta_1|x-y| & \text{in } \Omega, \\ h_{1,\lambda}(x,y) = \frac{1}{4\pi|x-y|} - \beta_1|x-y| & \text{on } \partial\Omega. \end{cases}$$

Elliptic regularity then yields that $h_{1,\lambda}(x,\cdot) \in C^2(\Omega)$. Its derivatives are clearly continuous as functions of the joint variable. Let us observe that the function $H_\lambda(x,y)$ is symmetric, thus so is h_1, and then $h_{1,\lambda}(\cdot,y)$ is also of class C^2 with derivatives jointly continuous. It follows that $h_1(x,y)$ is a function of class $C^2(\Omega\times\Omega)$. Iterating this procedure, we get that, for any k,

$$H_\lambda(x,y) = \sum_{j=1}^{k} \beta_j|x-y|^{2j-1} + h_{k,\lambda}(x,y)$$

with $\beta_{j+1} = -\lambda\beta_j/((2j+1)(2j+2))$ and $h_{k,\lambda}$ solution of the boundary value problem:

$$\begin{cases} \Delta_x h_{k,\lambda} + \lambda h_{k,\lambda} = -\lambda\beta_k |x-y|^{2k-1} & \text{in } \Omega, \\ h_{k,\lambda}(x,y) = \frac{1}{4\pi|x-y|} - \sum_{j=1}^{k} \beta_j |x-y|^{2j-1} & \text{on } \partial\Omega. \end{cases}$$

We may remark that

$$\Delta_y h_{k+1,\lambda} + \lambda h_{k,\lambda} = 0 \quad \text{in } \Omega.$$

Elliptic regularity then yields that $h_{k,\lambda}$, is a function of class $C^{k+1}(\Omega \times \Omega)$. Let us observe now that by definition of g_λ we have $g_\lambda(x) = h_{k,\lambda}(x,x)$, and the conclusion of the lemma follows. □

Lemma A.2. *The function* $\frac{\partial g_\lambda}{\partial \lambda}$ *is well defined, smooth and strictly negative in* Ω*. Its derivatives depend continuously on* λ*.*

Proof. For a fixed given $x \in \Omega$, consider the unique solution $F(y)$ of

$$\begin{cases} \Delta_y F + \lambda F = G(x,y), & y \in \Omega, \\ F = 0, & y \in \partial\Omega. \end{cases}$$

Using elliptic regularity, F is at least of class $C^{0,\alpha}$. Besides a convergence argument using elliptic estimates shows that actually:

$$F(y) = \frac{\partial H_\lambda}{\partial \lambda}(x,y).$$

Since $\lambda < \lambda_1$, the Maximum Principle implies that $F < 0$ in Ω. Hence, in particular,

$$\frac{\partial g_\lambda}{\partial \lambda}(x) = F(x) < 0.$$

Arguing as in the previous lemma, this function turns out to be smooth in x. The resulting expansions easily provide the continuous dependence in λ of its derivatives in x-variable. □

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