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Two-bubble solutions in the super-critical Bahri-Coron's problem

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1 Introduction

In this paper we study the existence of positive solutions to the nonlinear elliptic problems

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}+\varepsilon} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $N \geq 3$, and ε is a small positive parameter.

It is well know that problem

$$\begin{cases} -\Delta u = u^{q} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

has at least one solution when $q < \frac{N+2}{N-2}$ for any smooth bounded domain Ω.

On the contrary, when q is critical or supercritical the existence of solutions to problem (1.2) depends strongly on the shape of the domain Ω. Indeed, if $q \geq \frac{N+2}{N-2}$ Pohozaev's identity [18] gives that problem (1.2) has no solution if Ω is star-shaped. On the other hand, if $q = \frac{N+2}{N-2}$, problem (1.2) has at least one solution when Ω is a symmetric annulus, see Kazdan-Warner [15], or when Ω has a "small hole", see Coron [9].

In a remarkable work [3], Bahri and Coron generalize the previous results, by proving that if $q = \frac{N+2}{N-2}$ and if some homology group of Ω with coefficients in $\mathbf{Z}_2$ is nontrivial, then problem (1.2) has a solution.

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As reported by Brezis in [5], Rabinowitz poses the question whether the nontriviality of the topology of Ω in the sense of Bahri-Coron is a sufficient condition for existence of solutions to (1.2) when $q > \frac{N+2}{N-2}$.
In [16, 17] Passaseo constructs examples that show that the answer is in general negative. Among other results, he finds that for $N \geq 4$ there is a topologically nontrivial domain, for which no solution of (1.2) exists if $q > \frac{N+1}{N-3}$. This of course does not rule out the possibility that solutions exist in (1.1) provided that ε is sufficiently small.

Before stating our result, we need to introduce some notation. Let us denote by $G(x,y)$ the Green's function of the domain, namely G satisfies

$$\Delta_x G(x,y) = \delta(x-y), \quad x \in \Omega,$$

$$G(x,y) = 0, \quad x \in \partial\Omega,$$

where $\delta(x)$ denotes the Dirac mass at the origin. We denote by $H(x,y)$ its regular part, namely

$$H(x,y) = \Gamma(x-y) - G(x,y)$$

where Γ denotes the fundamental solution of the Laplacian,

$$\Gamma(x) = b_N |x|^{2-N},$$

so that H satisfies

$$\Delta_x H(x,y) = 0, \quad x \in \Omega,$$

$$H(x,y) = \Gamma(x-y), \quad x \in \partial\Omega.$$

Its *diagonal* $H(x,x)$ is usually called the Robin's function of the domain.

The following function will play a crucial role in our analysis:

$$\varphi(\xi_1,\xi_2) = H^{\frac{1}{2}}(\xi_1,\xi_1) H^{\frac{1}{2}}(\xi_2,\xi_2) - G(\xi_1,\xi_2). \tag{1.3}$$

We will construct solutions of (1.1) which as $\varepsilon \to 0$ develop a spike-shape, blowing-up at exactly two distict points ξ_1, ξ_2 while approaching zero elsewhere, provided that the set where $\varphi < 0$ is topologically nontrivial in a sense to be specified below. The pair (ξ_1, ξ_2) will be a critical point of φ with $\varphi(\xi_1,\xi_2) < 0$.

For a subspace B of Ω we will designate by $H^d(B)$ its d-th cohomology group with integral coefficients. We will consider the homomorphism $\iota^* : H^*(\Omega) \to H^*(B)$, induced by the inclusion $\iota : B \to \Omega$.

Theorem 1.1 *Assume $N \geq 3$ and let Ω be a bounded domain with smooth boundary in $\mathbb{R}^N$, with the following property: There exists a compact manifold $\mathcal{M} \subset \Omega$ and an integer $d \geq 1$ such that, $\varphi < 0$ on $\mathcal{M} \times \mathcal{M}$, $\iota^* : H^d(\Omega) \to H^d(\mathcal{M})$ is nontrivial and either d is odd or $H^{2d}(\Omega) = 0$.*
Then there exists $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$, problem (1.1) has at least one solution u_ε. Moreover, let $\mathcal{C}$ be the component of the set where $\varphi < 0$ which contains $\mathcal{M} \times \mathcal{M}$. Then, given any sequence $\varepsilon = \varepsilon_n \to 0$, there is a subsequence, which we denote in the same way, and a critical point $(\xi_1, \xi_2) \in \mathcal{C}$ of the function

φ such that $u_\varepsilon(x) \to 0$ on compact subsets of $\Omega \setminus \{\xi_1, \xi_2\}$ and such that for any $\delta > 0$

$$\sup_{|x-\xi_i|<\delta} u_\varepsilon(x) \to +\infty, \quad i = 1, 2,$$

as $\varepsilon \to 0$.

Actually, the proof will provide much finer information on the asymptotic profile of the blow-up of these solutions as $\varepsilon \to 0$: after scaling and translation one sees around each ξ_i a "bubble", namely a solution in entire $I\!R^N$ of the equation at the critical exponent. More precisely, we will find,

$$u_\varepsilon(x) = \alpha_N \sum_{i=1}^{2} \left(\frac{\lambda_{i\varepsilon} \varepsilon^{\frac{1}{N-2}}}{\varepsilon^{\frac{2}{N-2}} \lambda_{i\varepsilon}^2 + |x - \xi_{i\varepsilon}|^2} \right)^{\frac{N-2}{2}} + \theta_\varepsilon(x), \tag{1.4}$$

where $\theta_\varepsilon(x) \to 0$ uniformly as $\varepsilon \to 0$, $\xi_{i\varepsilon} \to \xi_i$ up to subsequences, where (ξ_1, ξ_2) is a critical point of φ with negative critical value. Besides, one can identify the limits λ_i of $\lambda_{i\varepsilon}$ as

$$\lambda_i^{N-2} = c_N \frac{H(\xi_j, \xi_j)}{H(\xi_i, \xi_i)|\varphi(\xi_1, \xi_2)|^2}, \quad j \neq i,\ i, j = 1, 2.$$

In the next section we will present two examples to clarify the meaning of Theorem 1.1. In fact, its assumptions are satisfied, hence yielding these two-bubble solutions if for instance to a fixed domain $\mathcal{D}$ one excises a subdomain ω contained in a ball of sufficiently small radius. The other example consists of an arbitrary domain in $I\!R^3$ from which one takes away a solid torus with sufficiently small cross-section.

It is rather intriguing that the former situation is precisely that considered by Coron, who finds existence when $p = \frac{N+2}{N-2}$. The solutions here found of course do not correspond in the limit to those found by Coron or Bahri-Coron since they disappear as $\varepsilon \to 0$. The persistence of this solution for small ε has been conjectured by Dancer, see [10], [11], [12].

The role of Green's and Robin's functions in the concentration phenomena associated to the critical exponent has already been considered in several works, when the exponent q approaches critical *from below*, namely $q = \frac{N+2}{N-2} - \varepsilon$. See Brezis and Peletier [7], Rey [19], [20], [21], Han [14] and Bahri, Li and Rey [4]. In the latter reference, multi-bubble solutions are found for $N \geq 4$ and $q = \frac{N+2}{N-2} - \varepsilon$, concentrating around nondegenerate critical points of certain object which for *two*-spikes corresponds to the function φ (in their case with *positive* critical value.). This construction was improved to dimension $N = 3$ in [21].

Our proof borrows ideas of the above mentioned works. One obvious difficulty one has to circumvent is the fact that Sobolev's embedding is no longer valid in our situation. We are able however to work out in "well-chosen" spaces a somewhat novel reduction to a finite dimensional problem, which we treat with a variational-topological approach. We remark that our method does not use any a priori knowledge of non-degeneracy, an assumption perhaps generic, but hard to check in examples. The main point is then to recognize that critical points of the underlying finite dimesional energy correspond to critical points of the full energy functional, hence to solutions of our problem.

2 Examples and scheme of proof

Let $\mathcal{D}$ be a bounded domain with smooth boundary in $\mathbf{R}^N$, $N \geq 3$, which contains the origin 0. We shall emphasize the dependence of the Green's function on the domain by writing it as $G_{\mathcal{D}}(x, y)$, and similarly for its regular part $H_{\mathcal{D}}(x, y)$. Let us consider a number $\delta > 0$ and the domain

$$\mathcal{D}_\delta = \mathcal{D} \setminus \bar{B}(0, \delta).$$

We denote by G_δ, H_δ respectively its Green's function and regular part.

Lemma 2.1 *The following result holds*

$$\lim_{\delta \to 0} H_\delta(x, y) = H_{\mathcal{D}}(x, y),$$

uniformly on x, y in compact subsets of $\bar{\mathcal{D}} \setminus \{0\}$.

Proof. The maximum principle yields

$$H_\delta(x, y) \leq H_{\mathcal{D}}(x, y),$$

hence the family of functions $H_\delta(x, y)$ is uniformly bounded as $\delta \to 0$ on each compact subset of $\bar{\mathcal{D}} \setminus \{0\} \times \bar{\mathcal{D}} \setminus \{0\}$, and strictly increasing in δ. By harmonicity, its pointwise limit as $\delta \to 0$ is therefore uniform on compacts of $\mathcal{D} \setminus \{0\}$. Since the resulting limit $H(x, y)$ is harmonic in x and bounded, it extends smoothly as a harmonic function in all of $\mathcal{D}$. H therefore satisfies equation

$$\Delta_x H(x, y) = 0, \quad x \in \mathcal{D},$$

$$H(x, y) = \Gamma(x - y), \quad x \in \partial\mathcal{D}$$

and is thus equal to $H_{\mathcal{D}}$. □

Consider now a smooth domain ω such that $\omega \subset \bar{B}(0, \delta) \subset \mathcal{D}$ and the domain

$$\Omega = \mathcal{D} \setminus \omega. \tag{2.1}$$

Denote by G and H its Green's function and regular part, and consider the function $\varphi(\xi_1, \xi_2)$ defined on $\Omega \times \Omega \setminus \{\xi_1 = \xi_2\}$ as in (1.3).

Corollary 2.1 *For any (fixed) sufficiently small number $\rho > 0$ there is a $\delta_0 > 0$ such that if ω is any domain with $\omega \subset \bar{B}(0, \delta)$ and $\delta < \delta_0$, then*

$$\sup_{|\xi_1| = |\xi_2| = \rho} \varphi(\xi_1, \xi_2) < 0.$$

Hence, Theorem 1.1 applies to Ω given by (2.1), with

$$\mathcal{M} = \rho\, S^{N-1}.$$

Proof. We have that $H_{\mathcal{D}}$ is smooth near $(0,0)$ while $G_{\mathcal{D}}$ becomes unbounded, hence for any $\rho > 0$

$$\sup_{|\xi_1|=|\xi_2|=\rho} \tilde{\varphi}(\xi_1,\xi_2) < 0$$

where $\tilde{\varphi}$ is defined by

$$\tilde{\varphi}(\xi_1,\xi_2) = H_{\mathcal{D}}^{\frac{1}{2}}(\xi_1,\xi_1) H_{\mathcal{D}}^{\frac{1}{2}}(\xi_2,\xi_2) - G_{\mathcal{D}}(\xi_1,\xi_2).$$

On the other hand, for this ρ, it follows from the previous lemma that H and hence G become uniformly close to $H_{\mathcal{D}}$ and $G_{\mathcal{D}}$ on $|\xi_1| = |\xi_2| = \rho$ as δ gets smaller. The desired conclusion then readily follows. □

A second example we consider is the following. Let $N = 3$ and $\mathcal{D}$ be as above. Consider now a solid torus in $\mathbb{R}^3$ given by $T(l,r)$, where l is the radius of the axis circle, which we assume centered at 0, and r that of a cross-section. Assume now that there is an $r_0 > 0$ such that $T(l,r_0) \subset \mathcal{D}$. Consider now $\mathcal{D}_\delta$ defined as

$$\mathcal{D}_\delta = \mathcal{D} \setminus T(l,\delta).$$

Similarly as in the previous example the Green's and Robin functions of $\mathcal{D}_\delta$ will approach that of $\mathcal{D}$. Then, fixing now a sufficiently small $\rho > 0$ and considering the boundary of a fixed section $S^1(\rho)$ of $T(l,\rho)$, we will have now that if $\Omega = \mathcal{D}_\delta$ with δ sufficiently small, then

$$\sup_{\xi_1,\xi_2 \in S^1(\rho)} \varphi(\xi_1,\xi_2) < 0.$$

It follows that Theorem 1.1 applies now with

$$\mathcal{M} = S^1(\rho).$$

It is perhaps clear from the above argument that it suffices that for a torus not necessarily symmetric taken away, the same would be true, provided that it is "narrow" only in certain region.

Now we proceed into the proof of Theorem 1.1. As we have mentioned, our approach consists of a combination of a finite dimensional reduction implicit-function like, in suitable spaces, and a variational approach for the finite dimensional resulting problem. In §3, we work out an asymptotic expansion for a finite-dimensional functional which will be, up to lower order terms, that we want to get critical points for. §4 is devoted to a linear problem which plays a crucial role in the finite-dimensional reduction, which is carried out in certain weighted L^∞ spaces in §5. The reduced functional is analyzed asymptotically in §6, and its relation with the expansion in §3 is found. In §7 we set up a min-max scheme to find a critical point for the reduced functional. Here is where the topological assumption of Theorem 1.1 is used in order to prove that a crucial intersection property is accomplished.

3 Basic estimates in the reduced energy

Let

$$\bar{U}(x) = \alpha_N \left(\frac{1}{1+|x|^2} \right)^{\frac{N-2}{2}}$$

where $\alpha_N = (N(N-2))^{\frac{N-2}{4}}$. Then $\bar{U}$ satisfies the equation

$$-\Delta \bar{U} = \bar{U}^p \quad \text{in } I\!R^N.$$

Here and in what follows $N \geq 3$ and $p = \frac{N+2}{N-2}$. We also denote

$$\bar{U}_{\lambda,\xi}(x) = \alpha_N \left(\frac{\lambda}{\lambda^2 + |x-\xi|^2} \right)^{\frac{N-2}{2}}$$

which also satisfies

$$-\Delta \bar{U}_{\lambda,\xi} = \bar{U}^p_{\lambda,\xi} \quad \text{in } I\!R^N,$$

which constitute the extremals for Sobolev's critical embedding and are actually all positive solutions of the elliptic equation, see [1,22,6,8].
Let Ω be a bounded domain with smooth boundary in $I\!R^N$. We denote by $U_{\lambda,\xi}$ the $H^1_0(\Omega)$-projection of $\bar{U}_{\lambda,\xi}$, namely the unique solution of the equation

$$-\Delta U_{\lambda,\xi} = \bar{U}^p_{\lambda,\xi} \quad \text{in } \Omega$$

$$U_{\lambda,\xi} = 0 \quad \text{on } \partial\Omega.$$

In other words $U_{\lambda,\xi} = \bar{U}_{\lambda,\xi} - \phi_{\lambda,\xi}$ where $\phi_{\lambda,\xi}$ solves

$$-\Delta \phi_{\lambda,\xi} = 0 \quad \text{in } \Omega$$

$$\phi_{\lambda,\xi} = \bar{U}_{\lambda,\xi} \quad \text{on } \partial\Omega.$$

Then the following estimates hold

$$\phi_{\lambda,\xi}(x) = H(x,\xi)\lambda^{\frac{N-2}{2}} \int_{I\!R^N} \bar{U}^p + o(\lambda^{\frac{N-2}{2}}), \tag{3.1}$$

and, away from $x = \xi$,

$$U_{\lambda,\xi}(x) = G(x,\xi)\lambda^{\frac{N-2}{2}} \int_{I\!R^N} \bar{U}^p + o(\lambda^{\frac{N-2}{2}}), \tag{3.2}$$

uniformly for ξ on each compact subset of Ω. Here G and H are respectively the Green function of the Laplacian with Dirichlet boundary condition on Ω and its regular part.

We consider now two points $\xi_1, \xi_2 \in \Omega$, small numbers $\lambda_1, \lambda_2 > 0$ and the functions

$$\bar{U}_i = \bar{U}_{\lambda_i,\xi_i}, \quad U_i = U_{\lambda_i,\xi_i}, \quad i = 1,2.$$

Our purpose is to estimate the following quantity

$$J_0(U_1+U_2)=\frac{1}{2}\int_{\Omega}|\nabla(U_1+U_2)|^2-\frac{1}{p+1}\int_{\Omega}(U_1+U_2)^{p+1}.$$

Let us set

$$C_N=\frac{1}{2}\int_{\mathbb{R}^N}|D\bar{U}|^2-\frac{1}{p+1}\int_{\mathbb{R}^N}\bar{U}^{p+1}$$

and

$$\mathcal{O}_\delta(\Omega)=\{\,(\xi_1,\xi_2)\in\Omega\times\Omega\,:\,|\xi_1-\xi_2|>\delta,\;\; dist(\xi_i,\partial\Omega)>\delta,\\ i=1,\,2\,\}. \tag{3.3}$$

Then the following estimate holds

Lemma 3.1 *Given* $\delta>0$ *we have the validity of the expansion*

$$\begin{aligned}J_0(U_1+U_2)=2C_N&+\frac{1}{2}\left(\int_{\mathbb{R}^N}\bar{U}^p\right)^2\Big\{H(\xi_1,\xi_1)\lambda_1^{N-2}\\&+H(\xi_2,\xi_2)\lambda_2^{N-2}-2G(\xi_1,\xi_2)\lambda_1^{\frac{N-2}{2}}\lambda_2^{\frac{N-2}{2}}\Big\}\\&+o(\max\{\lambda_1,\lambda_2\}^{N-2})\end{aligned}$$

uniformly with respect to $(\xi_1,\xi_2)\in\mathcal{O}_\delta(\Omega)$.

Proof. The basic estimates leading to the above expansion are basically contained in [2], [4]. We recall them in the following:

$$\int_{\Omega}|DU_i|^2=\int_{\mathbb{R}^N}|D\bar{U}|^2-\left(\int_{\mathbb{R}^N}\bar{U}^p\right)^2H(\xi_i,\xi_i)\lambda_i^{N-2}+o(\lambda_i^{N-2}), \tag{3.4}$$

$$\int_{\Omega}\nabla U_1\nabla U_2=\left(\int_{\mathbb{R}^N}\bar{U}^p\right)^2G(\xi_1,\xi_2)\lambda_1^{\frac{N-2}{2}}\lambda_2^{\frac{N-2}{2}}+o(\max\{\lambda_1,\lambda_2\}^{N-2}), \tag{3.5}$$

$$\begin{aligned}\frac{1}{p+1}\int_{\Omega}(U_1+U_2)^{p+1}-U_1^{p+1}-U_2^{p+1}=&\,2(\int_{\mathbb{R}^N}\bar{U}^p)^2G(\xi_1,\xi_2)\lambda_1^{\frac{N-2}{2}}\lambda_2^{\frac{N-2}{2}}\\&+o(\max\{\lambda_1,\lambda_2\}^{N-2})\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\frac{1}{p+1}\int_{\Omega}U_i^{p+1}=&\,\frac{1}{p+1}\int_{\mathbb{R}^N}\bar{U}^{p+1}\\&-\left(\int_{\mathbb{R}^N}\bar{U}^p\right)^2H(\xi_i,\xi_i)\lambda_i^{N-2}+o(\lambda_i^{N-2})\,.\end{aligned} \tag{3.7}$$

Finally we decompose

$$J_0(U_1+U_2)=\sum_{i=1,2}\frac{1}{2}\int_{\Omega}|\nabla U_i|^2-\frac{1}{p+1}\int_{\Omega}U_i^{p+1}+$$

$$\int_\Omega \nabla U_1 \nabla U_2 - \frac{1}{p+1}\int_\Omega (U_1+U_2)^{p+1} - U_1^{p+1} - U_2^{p+1};$$

substituting estimates (3.4), (3.5), (3.6) and (3.7) in this relation we obtain the thesis. □

In what follows of this section we will make a choice of the numbers λ_i in terms of ε: we will assume

$$\lambda_i^{N-2} = c_N \Lambda_i^2 \varepsilon \tag{3.8}$$

where c_N is a constant we will choose later and Λ_i is only allowed to range on a bounded interval of the form $0 < \delta < \Lambda_i < \delta^{-1}$.

Let us consider the energy functional, associated to problem (1.1),

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega |Du|^2 - \frac{1}{p+1+\varepsilon}\int_\Omega u^{p+1+\varepsilon}.$$

We consider next the problem of estimating the quantity $J_\varepsilon(U_1+U_2)$. First we see that

$$J_\varepsilon(U_1+U_2) = J_0(U_1+U_2)+$$

$$\frac{\varepsilon}{(p+1)^2}\int_\Omega (U_1+U_2)^{p+1} - \frac{\varepsilon}{p+1}\int_\Omega (U_1+U_2)^{p+1}\log(U_1+U_2) + o(\varepsilon).$$

As we have seen, we have

$$\int_\Omega (U_1+U_2)^{p+1} = 2\int_{\mathbb{R}^N} \bar{U}^{p+1} + o(1).$$

On the other hand, for a small number ρ we can decompose

$$\int_\Omega (U_1+U_2)^{p+1}\log(U_1+U_2) = \int_{|x-\xi_1|<\rho} (U_1+U_2)^{p+1}\log(U_1+U_2) +$$

$$\int_{|x-\xi_2|<\rho} (U_1+U_2)^{p+1}\log(U_1+U_2) + o(\varepsilon).$$

Now, we have

$$\int_{|x-\xi_1|<\rho} (U_1+U_2)^{p+1}\log(U_1+U_2) = -\frac{N-2}{2}\log\lambda_1 \int_{|x-\xi_1|<\rho} (U_1+U_2)^{p+1} +$$

$$\int_{|x-\xi_1|<\rho} (U_1+U_2)^{p+1}\log(\lambda_1^{\frac{N-2}{2}} U_1 + \lambda_1^{\frac{N-2}{2}} U_2) =$$

$$-\frac{N-2}{2}\log\lambda_1 (\int_{\mathbb{R}^N} \bar{U}^{p+1} + O(\lambda_1^N))) + \int_{\mathbb{R}^N} \bar{U}^{p+1}\log\bar{U} + o(1).$$

We conclude that

$$\int_\Omega (U_1+U_2)^{p+1}\log(U_1+U_2) = -\frac{N-2}{2}\log(\lambda_1\lambda_2)\int_{\mathbb{R}^N} \bar{U}^{p+1}$$

$$+2\int_{\mathbb{R}^N} \bar{U}^{p+1}\log\bar{U} + o(1).$$

Hence

$$
\begin{aligned}
J_\varepsilon(U_1+U_2) = {} & J_0(U_1+U_2) \\
& +2\varepsilon\left\{\frac{1}{(p+1)^2}\int_{I\!R^N}\bar U^{p+1} - \frac{1}{p+1}\int_{I\!R^N}\bar U^{p+1}\log\bar U\right\} \\
& +\frac{N-2}{2(p+1)}\varepsilon\log(\lambda_1\lambda_2)\int_{I\!R^N}\bar U^{p+1}+o(\varepsilon).
\end{aligned}
$$

Combining this estimate with the previous lemma, and our choice (3.8) for λ_1, λ_2 with

$$
c_N = \frac{1}{p+1}\frac{\int_{I\!R^N}\bar U^{p+1}}{(\int_{I\!R^N}\bar U^{p})^2}. \tag{3.9}
$$

we get the following result.

Lemma 3.2 *Given $\delta > 0$ and and choosing $\lambda_i^{N-2} = c_N \Lambda_i^2 \varepsilon$ with c_N given by (3.9), then we have*

$$
J_\varepsilon(U_1+U_2) =
$$

$$
2C_N + w_N\varepsilon\log\varepsilon + \gamma_N\varepsilon + w_N\varepsilon\Psi(\xi_1,\xi_2,\Lambda_1,\Lambda_2)+o(\varepsilon),
$$

uniformly with respect to $(\xi_1,\xi_2,\Lambda_1,\Lambda_2) \in \mathcal{O}_\delta(\Omega)\times(]\delta,\delta^{-1}[)^2$. Here

$$
\begin{aligned}
\Psi(\xi_1,\xi_2,\Lambda_1,\Lambda_2) = {} & \frac{1}{2}\{H(\xi_1,\xi_1)\Lambda_1^2 + H(\xi_2,\xi_2)\Lambda_2^2 - 2G(\xi_1,\xi_2)\Lambda_1\Lambda_2\} \\
& +\log\Lambda_1\Lambda_2,
\end{aligned} \tag{3.10}
$$

$$
\gamma_N = 2\{\frac{1}{(p+1)^2}\int_{I\!R^N}\bar U^{p+1} - \frac{1}{p+1}\int_{I\!R^N}\bar U^{p+1}\log\bar U\} + w_N\log c_N
$$

and $w_N = \frac{1}{p+1}\int_{I\!R^N}\bar U^{p+1}$.

Remark. The quantity $o(\varepsilon)$ in the expansion above is actually also of that size in the C^1-norm as a function of ξ and Λ in the considered region.

4 A linear problem

In this section we introduce a linear problem defined in a suitable functional-analytic setting which is the basis for the reduction of problem (1.1) to the study of a finite dimensional problem. It seems useful to consider the problem in proper streched variables. For this purpose, let us consider the domain $\Omega_\varepsilon = \varepsilon^{-\frac{1}{N-2}}\Omega$. For functions u and v defined on Ω_ε we shall denote in what follows

$$
< u\,,\, v >= \int_{\Omega_\varepsilon} uv.
$$

Consider then a fixed number $\delta > 0$, and points $\xi_i' \in \Omega_\varepsilon$, numbers $\Lambda_i > 0\ i = 1, 2$, with

$$
|\xi_1' - \xi_2'| > \delta\varepsilon^{-\frac{1}{N-2}}, \quad \mathrm{dist}\,(\xi_i', \partial\Omega_\varepsilon) > \delta\varepsilon^{-\frac{1}{N-2}}, \delta < \Lambda_i < \delta^{-1}, \tag{4.1}
$$

and the functions

$$\bar V_i(x) = \bar U_{\Lambda_i^*,\xi_i'}(x) = \alpha_N \left(\frac{\Lambda_i^*}{(\Lambda_i^*)^2 + |x-\xi_i'|^2} \right)^{\frac{N-2}{2}}$$

where $\Lambda_i^* = (c_N \Lambda_i^2)^{\frac{1}{N-2}}$. As before, we take the projections onto $H_0^1(\Omega_\varepsilon)$ of these functions, namely the functions V_i given as the unique solutions of

$$-\Delta V_i = \bar V_i^p \quad \text{in } \Omega_\varepsilon$$

$$V_i = 0 \quad \text{on } \partial\Omega_\varepsilon.$$

Consider further the following functions

$$\bar Z_{ij} = \frac{\partial \bar V_i}{\partial \xi_{ij}},\ j = 1,\ldots,N, \quad \bar Z_{iN+1} = \frac{\partial \bar V_i}{\partial \Lambda_i^*} = (x-\xi_i)\cdot \nabla \bar V_i + (N-2)\bar V_i,$$

and their respective $H_0^1(\Omega_\varepsilon)$-projections Z_{ij}, namely the unique solutions of

$$\Delta Z_{ij} = \Delta \bar Z_{ij} \quad \text{in } \Omega_\varepsilon$$

$$Z_{ij} = 0 \quad \text{on } \partial\Omega_\varepsilon.$$

For further notational simplicity, we we will denote

$$V = V_1 + V_2 \qquad \text{and} \qquad \bar V = \bar V_1 + \bar V_2.$$

Consider now the following problem. Given $h \in C^\alpha(\bar\Omega_\varepsilon)$, find a function ϕ such that for certain constants c_{ij}, $i = 1,2$, $j = 1,\ldots,N+1$ one has

$$\begin{cases} \Delta\phi + (p+\varepsilon)V^{p+\varepsilon-1}\phi = h + \sum_{i,j} c_{ij} V_i^{p-1} Z_{ij} & \text{in } \Omega_\varepsilon \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon \\ < V_i^{p-1} Z_{ij}, \phi >= 0 & \text{for all } i,j. \end{cases} \tag{4.2}$$

We want to show that this problem is uniquely solvable with uniform bounds in certain appropriate norms. To this end, we consider the following weighted L^∞-norms. For a function ψ defined on Ω_ε, we define

$$\|\psi\|_* = \sup_{x\in\Omega_\varepsilon} | \left((1+|x-\xi_1'|^2)^{-\frac{N-2}{2}} + (1+|x-\xi_2'|^2)^{-\frac{N-2}{2}} \right)^{-\beta} \psi(x)|,$$

where $\beta = 1$ if $N = 3$ and $\beta = \frac{2}{N-2}$ if $N \geq 4$. Similarly we define, for any dimension $N \geq 3$,

$$\|\psi\|_{**} = \sup_{x\in\Omega_\varepsilon} | \left((1+|x-\xi_1'|^2)^{-\frac{N-2}{2}} + (1+|x-\xi_2'|^2)^{-\frac{N-2}{2}} \right)^{-\frac{4}{N-2}} \psi(x)|.$$

These norms are easily seen to be equivalent respectively to $\|(\bar V)^{-\beta}\psi\|_\infty$ and $\|(\bar V)^{-\frac{4}{N-2}}\psi\|_\infty$, uniformly in points and numbers satisfying (4.1). It should be noticed that in a related problem in entire space with "almost critical" nonlinearity, Wang and Wei [23] have used instead a weighted Sobolev spaces approach to carry out a finite dimensional reduction in searching for one-spike solutions.

Our purpose in what follows is to prove the following result.

Proposition 4.1 *Assume constraints (4.1) hold. Then there are numbers* $\varepsilon_0 > 0$, $C > 0$, *such that for all* $0 < \varepsilon < \varepsilon_0$ *and all* $h \in C^\alpha(\bar{\Omega}_\varepsilon)$, *problem (4.2) admits a unique solution* $\phi \equiv L_\varepsilon(h)$. *Besides,*

$$\|L_\varepsilon(h)\|_* \leq C\|h\|_{**} \tag{4.3}$$

and

$$|c_{ij}| \leq C\|h\|_{**}. \tag{4.4}$$

Here and in the rest of this paper, we denote by C a generic constant which is independent of ε and of the particular $\xi_i^{'}$, Λ_i chosen satisfying (4.1).

Lemma 4.1 *Under the conditions of Proposition 4.1, assume the existence of a sequence* $\varepsilon = \varepsilon_n \to 0$ *such that there are functions* ϕ_ε *and* h_ε *with* $\|h_\varepsilon\|_{**} = o(1)$ *if* $N \neq 4$, $|\log \varepsilon|\ \|h_\varepsilon\|_{**} = 0(1)$ *if* $N = 4$, *such that*

$$\Delta\phi_\varepsilon + (p+\varepsilon)V^{p-1+\varepsilon}\phi_\varepsilon = h_\varepsilon + \sum_{i,j} c_{ij} V_i^{p-1} Z_{ij} \quad \textit{in } \Omega_\varepsilon$$

$$\phi_\varepsilon = 0 \quad \textit{on } \partial\Omega_\varepsilon,$$
$$< V_i^{p-1} Z_{ij}, \phi_\varepsilon >= 0 \, \textit{for all } i, j,$$

for certain constants c_{ij}, *depending on* ε. *Then*

$$\|\phi_\varepsilon\|_* \to 0.$$

Proof. We shall establish first the slightly weaker assertion that

$$\begin{aligned} &\|\phi_\varepsilon\|_\rho \\ &\quad = \sup_{x\in\Omega_\varepsilon} |\left((1+|x-\xi_1'|^2)^{-\frac{N-2}{2}} + (1+|x-\xi_2'|^2)^{-\frac{N-2}{2}}\right)^{-(\beta-\rho)} \phi_\varepsilon(x)| \to 0 \end{aligned}$$

with $\rho > 0$ a small fixed number. To do this, we assume the opposite, so that with no loss of generality we may take $\|\phi_\varepsilon\|_\rho = 1$. Testing the above equation against Z_{lk}, integrating by parts twice we get that

$$\sum c_{ij} < V_i^{p-1} Z_{ij}, Z_{lk} >=< \Delta Z_{lk} + (p+\varepsilon)V^{p-1+\varepsilon} Z_{lk}, \phi > - < h_\varepsilon, Z_{lk} > . \tag{4.5}$$

This defines a linear system in the c_{ij} which is "almost diagonal" as ε approaches zero, since we have for $k = 1, \ldots, N$

$$< V_i^{p-1} Z_{ij}, Z_{lk} >= \delta_{i,l}\delta_{j,k} \int_{I\!R^N} \bar{U}_{\Lambda_i}^{p-1} \left(\frac{\partial \bar{U}_{\Lambda_i,0}}{\partial x_k}\right)^2 + o(1) \tag{4.6}$$

and for $k = N+1$

$$< V_i^{p-1} Z_{ij}, Z_{l(N+1)} >= \delta_{i,l}\delta_{j,N+1} \int_{I\!R^N} \bar{U}_{\Lambda_i}^{p-1} \left(x \cdot \bar{U}_{\Lambda_i} + (N-2)\bar{U}_{\Lambda_i}\right)^2 + o(1) \tag{4.7}$$

for suitable $\Lambda_i > 0$. On the other hand, it is easy to see that we have, for $l = 1, 2$,

$$< \Delta Z_{lk} + (p+\varepsilon)V^{p+\varepsilon-1} Z_{lk}, \phi >= o(1)\|\phi\|_\rho, \tag{4.8}$$

after noticing that $\Delta Z_{lk} + p\bar{V}_l^{p-1} Z_{lk} = 0$ and an application of dominated convergence. Finally we have

$$| < h_\varepsilon, Z_{ij} > | \le C\|h_\varepsilon\|_{**}.$$

Thus, we conclude that

$$|c_{ij}| \le C\|h_\varepsilon\|_{**} + o(1)\|\phi_\varepsilon\|_\rho \tag{4.9}$$

so that $c_{ij} = o(1)$. Rewrite now the equation in the following form

$$\phi_\varepsilon(x) - (p+\varepsilon)\int_{\Omega_\varepsilon} G_\varepsilon(x,y)V^{p+\varepsilon-1}\phi_\varepsilon dy =$$

$$-\int_{\Omega_\varepsilon} G_\varepsilon(x,y)h_\varepsilon\, dy - \sum c_{ij}\int_{\Omega_\varepsilon} V_i^{p-1} Z_{ij} G_\varepsilon(x,y)\, dy \quad x \in \Omega_\varepsilon, \tag{4.10}$$

where G_ε denotes the Green's function of Ω_ε. We make now the following observation:

$$\int_{\Omega_\varepsilon} G_\varepsilon(x,y)|h_\varepsilon|\, dy \le$$

$$\|h_\varepsilon\|_{**} C\int_{I\!R^N} \Gamma(x-y)\left((1+|y-\xi_1'|^2)^{-2} + (1+|y-\xi_2'|^2)^{-2}\right)\, dy \le$$

$$C\|h_\varepsilon\|_{**}|\log\varepsilon|^m\left((1+|x-\xi_1'|^2)^{-\frac{N-2}{2}} + (1+|x-\xi_2'|^2)^{-\frac{N-2}{2}}\right)^\beta$$

$$\text{with} \quad m = \begin{cases} 1 \text{ if } N = 4 \\ 0 \text{ if } N \ne 4 \end{cases}.$$

On the other hand, we have

$$|\sum c_{ij}\int_{\Omega_\varepsilon} V_i^{p-1} Z_{ij} G_\varepsilon(x,y)\, dy| \le$$

$$C(\|\phi_\varepsilon\|_\rho + \|h_\varepsilon\|_{**})\sum\int_{I\!R^N} \Gamma(x-y)\left((1+|y-\xi_i'|^2)^{-\frac{N+3}{2}}\right) \le$$

$$C(\|\phi_\varepsilon\|_\rho + \|h_\varepsilon\|_{**})\left((1+|x-\xi_1'|^2)^{-\frac{N-2}{2}} + (1+|x-\xi_2'|^2)^{-\frac{N-2}{2}}\right)$$

Similarly, we obtain

$$\int_{\Omega_\varepsilon} G_\varepsilon(x,y)V^{p+\varepsilon-1}|\phi_\varepsilon| dy \le$$

$$C\|\phi_\varepsilon\|_\rho\left((1+|x-\xi_1'|^2)^{-\frac{N-2}{2}} + (1+|x-\xi_2'|^2)^{-\frac{N-2}{2}}\right)^\beta.$$

Equation (4.10) and the above estimates imply that

$$|\phi_\varepsilon(x)| \le C(\|\phi_\varepsilon\|_\rho + \|h_\varepsilon\|_{**}) \ \Big((1+|x-\xi_1'|^2)^{-\frac{N-2}{2}} + (1+|x-\xi_2'|^2)^{-\frac{N-2}{2}}\Big)^{\beta}, \quad (4.11)$$

hence that

$$\Big((1+|x-\xi_1'|^2)^{-\frac{N-2}{2}} + (1+|x-\xi_2'|^2)^{-\frac{N-2}{2}}\Big)^{-(\beta-\rho)} |\phi_\varepsilon(x)| \le C\Big((1+|x-\xi_1'|^2)^{-\frac{N-2}{2}} + (1+|x-\xi_2'|^2)^{-\frac{N-2}{2}}\Big)^{\rho}.$$

Since $\|\phi_\varepsilon\|_\rho = 1$, it follows the existence of a radius $R>0$ and a number $\gamma>0$, both independent of ε such that $\|\phi_\varepsilon\|_{L^\infty(B_R(\xi_i'))} > \gamma$ for either $i=1$ or $i=2$. Assume this happens for $i=1$. Then local elliptic estimates and the bound (4.11) yield that, up to a subsequence, $\tilde\phi_\varepsilon(x) = \phi_\varepsilon(x-\xi_1')$ converges uniformly over compacts of $I\!R^N$ to a nontrivial solution $\tilde\phi$ of

$$\Delta\tilde\phi + p\bar U_{\Lambda,0}^{p-1}\tilde\phi = 0, \quad (4.12)$$

for some $\Lambda>0$, which besides satisfies

$$|\tilde\phi(x)| \le C|x|^{(2-N)\beta}. \quad (4.13)$$

Hence, for $N=3$ we have

$$|\tilde\phi(x)| \le C|x|^{2-N}.$$

Now, since $\tilde\phi$ satisfies (4.12) and estimate (4.13) holds, a bootstrap argument leads to

$$|\tilde\phi(x)| \le C|x|^{2-N} \quad \text{for any } N>3.$$

It is well known that this implies that $\tilde\phi$ is a linear combination of the functions $\frac{\partial \bar U_{\Lambda,0}}{\partial x_j}$, $x\cdot\nabla\bar U_{\Lambda,0} + (N-2)\bar U_{\Lambda,0}$, see for instance [19]. On the other hand, we recall that

$$\int_{\Omega_\varepsilon} \phi_\varepsilon V_i^{p-1} Z_{ij} = 0 \quad \text{for all } i,j.$$

By dominated convergence, this relation is easily seen to be preserved up to the limit, hence

$$\int_{I\!R^N} \tilde\phi \bar U_{\Lambda,0}^{p-1}\frac{\partial \bar U_{\Lambda,0}}{\partial x_j} = \int_{I\!R^N} \tilde\phi\bar U_{\Lambda,0}^{p-1}(x\cdot\nabla\bar U_{\Lambda,0} + (N-2)\bar U_{\Lambda,0}) = 0,$$

for all j. Hence the only possibility is that $\tilde\phi\equiv 0$, which is a contradiction which yields the proof of $\|\phi_\varepsilon\|_\rho \to 0$. Finally, from estimate (4.11), we observe that

$$\|\phi_\varepsilon\|_* \le C(\|h_\varepsilon\|_{**} + \|\phi_\varepsilon\|_\rho),$$

hence $\|\phi_\varepsilon\|_* \to 0$, and the proof is thus complete. □

Now we are in a position to prove Proposition 4.1. To do this, let us consider the space

$$H = \{\phi \in H_0^1(\Omega_\varepsilon) \mid\ < V_i^{p-1} Z_{ij}, \phi >= 0\, \forall\, i, j\ \}$$

endowed with the usual inner product $[\phi, \psi] = \int_{\Omega_\varepsilon} \nabla\phi\nabla\psi$. Problem (4.2) expressed in weak form is equivalent to that of finding a $\phi \in H$ such that

$$[\phi, \psi] =< \big((p+\varepsilon)V^{p+\varepsilon-1}\phi - h\big)\,,\, \psi > \qquad \forall\psi \quad \in H.$$

With the aid of Riesz's representation theorem, this equation gets rewritten in H in the operational form

$$\phi = T_\varepsilon(\phi) + \tilde{h} \tag{4.14}$$

with certain $\tilde{h} \in H$ which depends linearly in h and where T_ε is a compact operator in H. Fredholm's alternative guarantees unique solvability of this problem for any h provided that the homogeneous equation

$$\phi = T_\varepsilon(\phi)$$

has only the zero solution in H. Let us observe that this last equation is equivalent to

$$\Delta\phi + (p+\varepsilon)V^{p-1+\varepsilon}\phi = \sum_{i,j} c_{ij} V_i^{p-1} Z_{ij} \quad \text{in } \Omega_\varepsilon \tag{4.15}$$

$$\phi = 0 \quad \text{on } \partial\Omega_\varepsilon,$$

$$< \phi, V_i^{p-1} Z_{ij} >= 0$$

for certain constants c_{ij}. Assume it has a nontrivial solution $\phi = \phi_\varepsilon$, which with no loss of generality may be taken so that $\|\phi_\varepsilon\|_* = 1$. But this makes the previous lemma applicable, so that necessarily $\|\phi_\varepsilon\|_* \to 0$. This is certainly a contradiction that proves that this equation only has the trivial solution in H. We conclude then that for each h, problem (4.2) admits a unique solution. We check that

$$\|\phi\|_* \le C\|h\|_{**}.$$

We assume again the opposite. In doing so, we find a sequence h_ε with $\|h_\varepsilon\|_{**} = o(1)$ and solutions $\phi_\varepsilon \in H$ of problem (4.2) with $\|\phi_\varepsilon\|_* = 1$. Again this makes the previous lemma applicable, and a contradiction has been found. This proves estimate (4.3). Estimate (4.4) follows from this and relation (4.9). This concludes the proof of the proposition. □

It is important for later purposes to understand the differentiability of the operator L_ε on the variables

$$\xi' = (\xi_1', \xi_2') \in \Omega_\varepsilon^2, \quad (\Lambda_1, \Lambda_2) \in \mathbb{R}_+^2$$

which satisfy constraints (4.1). Consider the L_*^∞ (resp. L_{**}^∞) of functions defined on Ω_ε with finite $\|\|_*$ norm (resp. $\|\|_{**}$ norm). We consider the map

$$(\xi', \Lambda, h) \mapsto S(\xi', \Lambda, h) \equiv L_\varepsilon(h), \tag{4.16}$$

as a map with values in $L_*^\infty \cap H_0^1(\Omega_\varepsilon)$. We have the following result:

Proposition 4.2 *Under the conditions of Proposition 4.1, the map S is of class C^1. Besides, we have*

$$\|\nabla_{\xi',\Lambda} S(\xi',\Lambda,h)\|_* \le C\|h\|_{**}$$

Proof. Let us consider differentiation with respect to the variable ξ'_{kl}, $k=1,2$, $l=1,\dots,N$. For notational simplicity we write $\frac{\partial}{\partial \xi'_{ij}} = \partial_{\xi'}$. Let us set, $\phi = S(\xi',\Lambda,h)$ and, still formally, $Z = \partial_{\xi'}\phi$. We seek for an expression for Z. Then Z satisfies the following equation:

$$\Delta Z + (p+\varepsilon)V^{p+\varepsilon-1}Z = -(p+\varepsilon)\partial_{\xi'}(V^{p-1+\varepsilon})\phi +$$

$$\sum_{i,j} d_{ij}V_i^{p-1}Z_{ij} + c_{ij}\partial_{\xi'}(V_i^{p-1}Z_{ij}) \quad \text{in } \Omega_\varepsilon.$$

Here $d_{ij} = \partial_{\xi'}c_{ij}$. Besides, from differentiating the orthogonality condition $<\phi, V_i^{p-1}Z_{ij}>=0$ we further obtain the relations

$$<\phi, \partial_{\xi'}(V_i^{p-1}Z_{ij})> + <Z, V_i^{p-1}Z_{ij}>=0.$$

Let us consider constants b_{ij} such that

$$<Z - \sum_{l,k} b_{lk}Z_{lk}, V_i^{p-1}Z_{ij}>=0.$$

These relations amount to

$$\sum_{l,k} b_{lk} <Z_{lk}, V_i^{p-1}Z_{ij}>=<\phi, \partial_{\xi'}V_i^{p-1}Z_{ij}> \tag{4.17}$$

Since this system is diagonal dominant with uniformly bounded coefficients, we see that it is uniquely solvable and that

$$b_{lk} = O(\|\phi\|_*)$$

uniformly on ξ', Λ in the considered region. Now, we easily see that

$$\|\phi\partial_{\xi'}(V^{p-1+\varepsilon})\|_{**} \le C\|\phi\|_*.$$

Recall now that from Proposition 4.2 $c_{ij} = O(\|h\|_{**})$. On the other hand

$$|\partial_{\xi'}(V_i^{p-1}Z_{ij}(x))| \le C|x-\xi'_i|^{-N-4},$$

hence

$$\|c_{ij}\partial_{\xi'}V_i^{p-1}Z_{ij}\|_{**} \le C\|h\|_{**}.$$

Let us now set $\eta = Z - \sum_{i,j} b_{ij}Z_{ij}$. Then, summing up the estimates above, and using that $\|\phi\|_* \le C\|h\|_{**}$, we get that η satisfies the relation

$$\Delta\eta + (p+\varepsilon)V^{p-1+\varepsilon}\eta = f + \sum_{i,j} d_{ij}V_i^{p-1}Z_{ij} \quad \text{in } \Omega_\varepsilon, \tag{4.18}$$

where

$$f = \sum_{i,j} b_{ij}(-(\Delta + (p+\varepsilon)V^{p-1+\varepsilon})Z_{ij} + c_{ij}\partial_{\xi'}(V_i^{p-1}Z_{ij}) -$$

$$(p+\varepsilon)\partial_{\xi'}(V^{p-1+\varepsilon})\phi, \tag{4.19}$$

so that

$$\|f\|_{**} \le C\|h\|_{**}.$$

Since besides $\eta \in H_0^1(\Omega_\varepsilon)$ and

$$< \eta, V_i^{p-1}Z_{ij} >= 0 \quad \text{for all } i,j, \tag{4.20}$$

we have that $\eta = L_\varepsilon(f)$. Reciprocally, if we now *define*

$$Z = L_\varepsilon(f) + \sum_{i,j} b_{ij}Z_{ij},$$

with b_{ij} given by relations (4.17) and f by (4.19), then it is a matter of routine to check that indeed $Z = \partial_{\xi'}\phi$. In fact Z depends continuously on the parameters ξ', Λ and h for the norm $\| \ \|_*$, and $\|Z\|_* \le C\|h\|_{**}$ for points in the considered region. The corresponding result for differentiation with respect to the Λ_i's follow similarly. This concludes the proof. □

Remark 4.1 We can also state the above result by saying that the map $(\xi', \lambda) \mapsto L_\varepsilon$ is of class C^1 in $\mathcal{L}(L^\infty_{**}, L^\infty_*)$ and, for instance

$$(D_{\xi'}L_\varepsilon)(h) = L_\varepsilon(f) + \sum_{i,j} b_{ij}Z_{ij}, \tag{4.21}$$

where f is given by (4.19) and b_{ij} by (4.17) .

5 The finite-dimensional reduction

At this point we are ready to start the finite dimensional reduction. Again for notational brevity, we write $V = V_1 + V_2$ and $\bar V = \bar V_1 + \bar V_2$. We consider now the nonlinear problem of finding a function ϕ such that for some constants c_{ij} the following equation holds

$$\begin{cases} \Delta(V+\psi+\phi) + (V+\psi+\phi)_+^{p+\varepsilon} = \sum_{i,j} c_{ij}V_i^{p-1}Z_{ij} & \text{in } \Omega_\varepsilon, \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \phi V_i^{p-1}Z_{ij} = 0 & \text{for all } i,j, \end{cases} \tag{5.1}$$

where the function ψ will be chosen below. Let us rewrite the first equation in (5.1) in the following form

$$\Delta\phi + (p+\varepsilon)V^{p+\varepsilon-1}\phi =$$

$$-N_\varepsilon(\psi+\phi) - (\Delta\psi + (p+\varepsilon)V^{p+\varepsilon-1}\psi + R^\varepsilon) + \sum_{i,j} c_{ij}V_i^{p-1}Z_{ij} \quad \text{in } \Omega_\varepsilon,$$

where

$$N_\varepsilon(\eta) = (V+\eta)_+^{p+\varepsilon} - V^{p+\varepsilon} - (p+\varepsilon)V^{p+\varepsilon-1}\eta,$$
$$R^\varepsilon = V^{p+\varepsilon} - \bar V_1^p - \bar V_2^p. \tag{5.2}$$

We choose in what follows, ψ as

$$\psi = -L_\varepsilon(R^\varepsilon), \tag{5.3}$$

where L_ε is the operator defined in Proposition 4.1. We will estimate separately each term in (5.2) in the $\| \ \|_{**}$-norm. To estimate $N_\varepsilon(\eta)$, it is convenient, and sufficient for our purposes, to assume $\|\eta\|_* < 1$. Note that

$$N_\varepsilon(\eta) = \frac{(p+\varepsilon)(p-1+\varepsilon)}{2}(V_1+V_2+t\eta)^{p-2+\varepsilon}\eta^2 \tag{5.4}$$

with $t \in (0,1)$. If $N \le 6$, then $p \ge 2$ and we can estimate

$$|\bar V^{-\frac{4}{N-2}} N_\varepsilon(\eta)| \le C\bar V^{(p-2)\beta-\frac{4}{N-2}+2\beta}\|\eta\|_*^2,$$

hence

$$\|N_\varepsilon(\eta)\|_{**} \le C\|\eta\|_*^2.$$

Assume now that $N > 6$. If $|\eta| \le \frac{1}{2}\bar V$ then relation (5.4) yields that

$$|\bar V^{-\frac{4}{N-2}} N_\varepsilon(\eta)| \le C\bar V^{2\beta-1}\|\eta\|_*^2 \le C\varepsilon^{2\beta-1}\|\eta\|_*^2.$$

In the other case, we see directly from (5.2) that $|N_\varepsilon(\eta)| \le C|\eta|^p$ and hence

$$|\bar V^{-\frac{4}{N-2}} N_\varepsilon(\eta)| \le \bar V^{p\beta-\frac{4}{N-2}}\|\eta\|_*^p \le C\varepsilon^{-(2-p)\beta}\|\eta\|_*^p.$$

Combining these relations we get

$$\|N_\varepsilon(\eta)\|_{**} \le \begin{cases} C\|\eta\|_*^2 & \text{if } N \le 6 \\ C(\varepsilon^{2\beta-1}\|\eta\|_*^2 + \varepsilon^{-(2-p)\beta}\|\eta\|_*^p) & \text{if } N > 6. \end{cases} \tag{5.5}$$

Next we estimate the term R^ε. We have

$$|R^\varepsilon| \le |\bar V_i^{p+\varepsilon} - \bar V_i^p| + o(\varepsilon^{\frac{N+2}{N-2}}) \le \varepsilon C\bar V_i^p|\log \bar V_i|(x) + o(\varepsilon^{\frac{N+2}{N-2}})$$

in the regions where $|x-\xi_i'| \le \bar\delta\varepsilon^{-\frac{1}{N-2}}$, for small $\bar\delta > 0$. Taking into account that $|R^\varepsilon| \le C\varepsilon^{\frac{N+2}{N-2}}$ in the complement of these two regions, we get

$$\|R^\varepsilon\|_{**} \le C\varepsilon.$$

Combining this with (5.3) and (5.5), we obtain then the following estimate.

Lemma 5.1 *Assume that the conditions of Proposition 4.1 are satisfied. Then there is a positive constant C such that, for any sufficiently small ε and $\|\phi\|_* \le 1$,*

$$\|N_\varepsilon(\psi+\phi)\|_{**} \le \begin{cases} C(\|\phi\|_*^2 + \varepsilon^2) & \text{if } N \le 6 \\ C(\varepsilon^{2\beta-1}\|\phi\|_*^2 + \varepsilon^{-(2-p)\beta}\|\phi\|_*^p + \varepsilon^{p\beta+1}) & \text{if } N > 6. \end{cases} \tag{5.6}$$

Proposition 5.1 *Assume the conditions of Proposition 4.1 are satisfied. Then there is a $C > 0$, such that for all small ε there exists a unique solution $\phi = \phi(\xi', \Lambda)$ with*

$$||\phi||_* \leq C\varepsilon$$

to the problem

$$\begin{cases} \Delta\phi + (p+\varepsilon)V^{p+\varepsilon-1}\phi = -N_\varepsilon(\psi+\phi) + \sum_{i,j} c_{ij} V_i^{p-1} Z_{ij} & \text{in } \Omega_\varepsilon \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon \\ \int_{\Omega_\varepsilon} \phi V_i^{p-1} Z_{ij} = 0 & \text{for all } i, j, \end{cases} \tag{5.7}$$

where ψ is the function defined in (5.3).

Proof. Let us set

$$\mathcal{F} = \{\phi \in H_0^1 \, : \, ||\phi||_* \leq \varepsilon\}.$$

Define now the map $A_\varepsilon : \mathcal{F} \to H_0^1$ as

$$A_\varepsilon(\phi) = -L_\varepsilon(N_\varepsilon(\phi + \psi)),$$

where L_ε is the linear operator defined in Proposition 4.1. Since $\psi = -L_\varepsilon(R^\varepsilon)$ and since L_ε is a linear operator, solving (5.7) is equivalent to finding a fixed point ϕ for A_ε. From Proposition 4.1 and Lemma 5.1 we conclude that, for ε sufficiently small and any $\phi \in \mathcal{F}_r$ we have

$$\|A_\varepsilon(\phi)\|_* = \|L_\varepsilon(N_\varepsilon(\phi+\psi))\|_* \leq C\|N_\varepsilon(\phi+\psi)\|_{**} \leq$$

$$\begin{cases} C\varepsilon^2 \leq \varepsilon & \text{if } N \leq 6 \\ C(\varepsilon^{2\beta+1} + \varepsilon^{p\beta+1}) \leq \varepsilon & \text{if } N > 6, \end{cases}$$

where the last inequality holds provided that ε is sufficiently small. Now we will show that the map A_ε is a contraction, for any ε small enough. That will imply that A_ε has a unique fixed point in $\mathcal{F}$ and hence problem (5.7) has a unique solution.

For any ϕ_1, ϕ_2 in $\mathcal{F}_r$ we have

$$\|A_\varepsilon(\phi_1) - A_\varepsilon(\phi_2)\|_* \leq C\|N_\varepsilon(\psi+\phi_1) - N_\varepsilon(\psi+\phi_2)\|_{**},$$

hence we just need to check that N_ε is a contraction in its corresponding norms. By definition of N_ε

$$D_{\bar\phi} N_\varepsilon(\bar\phi) = (p+\varepsilon)[(V+\bar\phi)_+^{p+\varepsilon-1} - V^{p+\varepsilon-1}].$$

Hence we get

$$|N_\varepsilon(\psi+\phi_1) - N_\varepsilon(\psi+\phi_2)| \leq C\bar V^{p-2}|\bar\phi||\phi_1 - \phi_2|.$$

for some $\bar\phi$ in the segment joining $\psi + \phi_1$ and $\psi + \phi_2$. Hence, we get for small enough $\|\bar\phi\|_*$,

$$\bar V^{-\frac{4}{N-2}}|N_\varepsilon(\psi+\phi_1) - N_\varepsilon(\psi+\phi_2)| \leq C\bar V^{2\beta-1}\|\bar\phi\|_*\|\phi_1 - \phi_2\|_*.$$

We conclude

$$\|N_\varepsilon(\psi+\phi_1)-N_\varepsilon(\psi+\phi_2)\|_{**} \le \bar V^{2\beta-1}(\|\phi_1\|_*+\|\phi_2\|_*+\|\psi\|_*)\|\phi_1-\phi_2\|_*$$
$$\le \varepsilon^{\min\{2\beta,1\}}\|\phi_1-\phi_2\|_*$$

and hence A_ε is a contraction mapping for the $\|\ \|_*$-norm inside $\mathcal{F}_r$. □

Our purpose in what remains of this section is to analyze the differentiability properties of the function $\phi(\xi',\Lambda)$ defined in Proposition 5.1

Proposition 5.2 *The function $(\xi',\Lambda)\mapsto\phi(\xi',\Lambda)$ provided by Proposition 5.1 is of class C^1 for the norm $\|\ \|_*$. Moreover,*

$$\|\nabla_{(\xi',\Lambda)}\phi\|_* \le C\varepsilon.$$

Proof. We recall that ϕ is defined through the relation

$$B(\xi',\Lambda,\phi)\equiv\phi+L_\varepsilon(N_\varepsilon(\phi+\psi))=0.$$

Write $N(\xi',\Lambda,\bar\phi)=N_\varepsilon(\bar\phi)$, namely

$$N(\xi',\Lambda,\bar\phi)=(V+\bar\phi)_+^{p+\varepsilon}-V^{p+\varepsilon}-(p+\varepsilon)V^{p+\varepsilon-1}\bar\phi.$$

Then

$$D_{\bar\phi}N(\xi',\Lambda,\bar\phi)=(p+\varepsilon)[(V+\bar\phi)_+^{p+\varepsilon-1}-V^{p+\varepsilon-1}]$$

and

$$D_{\xi'}N(\xi',\Lambda,\bar\phi)=(p+\varepsilon)[(V+\bar\phi)_+^{p+\varepsilon-1} -V^{p+\varepsilon-1}-(p+\varepsilon-1)V^{p+\varepsilon-2}\bar\phi]D_{\xi'}V, \tag{5.8}$$

similarly for $D_\Lambda N(\xi',\Lambda,\bar\phi)$. We have that

$$D_\phi B(\xi',\Lambda,\phi)[\theta]=\theta+L_\varepsilon(\theta D_{\bar\phi}N_\varepsilon(\phi+\psi))\equiv\theta+M(\theta).$$

Now,

$$\|M(\theta)\|_*\le C\|(\theta D_{\bar\phi}N_\varepsilon(\phi+\psi))\|_{**}\le C\|V^{-\frac{4}{N-2}+\beta}D_{\bar\phi}N_\varepsilon(\phi+\psi))\|_\infty\|\theta\|_*.$$

Now,

$$\bar V^{-\frac{4}{N-2}+\beta}|D_{\bar\phi}N_\varepsilon(\phi+\psi))|\le\bar V^{2\beta-1}\|\phi+\psi\|_*\le C\varepsilon^{\min\{2\beta,1\}}.$$

It follows that for small ε, the linear operator $D_\phi B(\xi',\Lambda,\phi)$ is invertible in L^∞_*, with uniformly bounded inverse. It also depends continuously on its parameters.

Now, let us consider differentiability with respect to the (ξ',Λ) variables. We have

$$D_{\xi'}B(\xi',\Lambda,\phi)=(D_{\xi'}L_\varepsilon)(N_\varepsilon(\phi+\psi))+$$
$$\big[L_\varepsilon((D_{\xi'}N)(\xi',\Lambda,\phi+\psi))+L_\varepsilon((D_{\bar\phi}N)(\xi',\Lambda,\phi+\psi)D_{\xi'}\psi)\big].$$

Here $D_{\xi'}L_\varepsilon$ is the operator defined by the expression (4.21) and the second quantity by (5.8). Observe also that

$$D_{\xi'}\psi = (D_{\xi'}L_\varepsilon)(R^\varepsilon) + L_\varepsilon(D_{\xi'}R^\varepsilon). \tag{5.9}$$

Also,

$$D_{\xi'_1}R^\varepsilon = (p+\varepsilon)V^{p+\varepsilon-1}D_{\xi'_1}V_1 - p\bar{V}_1^{p-1}D_{\xi'_1}\bar{V}_1. \tag{5.10}$$

These expressions also depend continuously on their parameters. We have a similar expression for the derivative with respect to Λ.

The implicit function theorem then applies to yield that $\phi(\xi', \Lambda)$ indeed defines a C^1 function into L^∞_*. Moreover, we have for instance

$$D_{\xi'}\phi = -(D_\phi B(\xi, \Lambda, \phi))^{-1}\big[(D_{\xi'}L_\varepsilon)(N_\varepsilon(\phi+\psi)) + [L_\varepsilon(D_{\xi'}[N(\xi', \Lambda, \phi+\psi)]) +$$

$$L_\varepsilon((D_{\bar{\phi}}N)(\xi', \Lambda, \phi+\psi)D_{\xi'}\psi)]\big].$$

Hence,

$$\|D_{\xi'}\phi\|_* \le C(\|N_\varepsilon(\phi+\psi)\|_{**} +$$

$$\|D_{\xi'}N(\xi', \Lambda, \phi+\psi)\|_{**} + \|D_{\bar{\phi}}N(\xi', \Lambda, \psi+\phi)D_{\xi'}\psi\|_{**}),$$

where we have used Remark 4.1. From Lemma 5.1, we get

$$\|N_\varepsilon(\phi+\psi)\|_{**} \le \begin{cases} C\varepsilon^2 & \text{if } N \le 6 \\ C\varepsilon^{p\beta+1} & \text{if } N > 6. \end{cases} \tag{5.11}$$

On the other hand, from (5.8) we have

$$|(D_{\xi'}N)(\xi', \Lambda, \bar{\phi})| \le C\bar{V}^{\frac{N-1}{N-2}}|(V+\bar{\phi})_+^{p+\varepsilon-1} - V^{p+\varepsilon-1} - (p+\varepsilon-1)V^{p+\varepsilon-2}\bar{\phi}| \le$$

$$C\bar{V}^{\frac{5}{N-2}+\varepsilon+\beta}\|\bar{\phi}\|_*,$$

hence

$$\|(D_{\xi'}N)(\xi', \Lambda, \psi+\phi)\|_{**} \le C\|\phi+\psi\|_* \le C\varepsilon.$$

In similar way we get that

$$\|D_{\bar{\phi}}N(\xi', \Lambda, \psi+\phi)D_\xi\psi\|_{**} \le C\varepsilon.$$

Hence, we finally get

$$\|D'_\xi\phi\|_* \le C\varepsilon,$$

as desired. A similar estimate holds for differentiation with respect to the Λ_i's. This concludes the proof. □

6 The reduced functional

Now we have all elements at hand for the resolution of the full problem. In what follows we consider points $(\xi_1', \xi_2', \Lambda_1, \Lambda_2) = (\xi', \Lambda)$ with, for $i = 1, 2$,

$$|\xi_1' - \xi_2'| \geq \varepsilon^{-\frac{1}{N-2}}\delta, \quad \operatorname{dist}(\xi_i', \partial\Omega_\varepsilon) \geq \varepsilon^{-\frac{1}{N-2}}\delta, \quad \delta < \Lambda_i < \delta^{-1}. \tag{6.1}$$

The estimates obtained below will be uniform on points satisfying these constraints. Let $\phi(x) = \phi(\xi', \Lambda)(x)$ be the unique solution of problem

$$\begin{cases} \Delta(V + \psi + \phi) + (V + \psi + \phi)_+^{p+\varepsilon} = \sum_{i,j} c_{i,j} V_i^{p-1} Z_{ij} & \text{in } \Omega_\varepsilon \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon \\ \int_{\Omega_\varepsilon} \phi V_i^{p-1} Z_{ij} = 0 & \text{for all } i, j \end{cases} \tag{6.2}$$

as predicted by Proposition 5.1. We observe that if $\xi' = \varepsilon^{-\frac{1}{N-2}}\xi$ with $\xi \in \Omega \times \Omega$, and Λ are so that $c_{ij} = 0$ for all i, j, we obtain a solution of our original problem, by means of the scaling

$$u(x) = \varepsilon^{-\zeta} v(x\varepsilon^{-\frac{1}{N-2}}), \tag{6.3}$$

where

$$v = V + \psi + \phi(\varepsilon^{-\frac{1}{N-2}}\xi, \Lambda) \quad \text{and} \quad \zeta = \frac{1}{2 + \varepsilon\frac{N-2}{2}}. \tag{6.4}$$

v will be a critical point of the functional

$$\mathcal{I}_\varepsilon(v) = \frac{1}{2}\int_{\Omega_\varepsilon} |Dv|^2 - \frac{1}{p + \varepsilon + 1}\int_{\Omega_\varepsilon} v^{p+\varepsilon+1},$$

while u one of

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega |Du|^2 - \frac{1}{p + 1 + \varepsilon}\int_\Omega u^{p+1+\varepsilon}.$$

It seems reasonable therefore to consider the functions defined in Ω

$$\hat{\phi}(\xi, \Lambda)(x) = \varepsilon^{-\zeta}\phi(\varepsilon^{-\frac{1}{N-2}}\xi, \Lambda)(\varepsilon^{-\frac{1}{N-2}}x),$$

$$\hat{\psi}(x) = \varepsilon^{-\zeta}\psi(\varepsilon^{-\frac{1}{N-2}}x) \quad \text{and} \quad \hat{U}_i(x) = \varepsilon^{-\zeta}V_i(\varepsilon^{-\frac{1}{N-2}}x) = \varepsilon^{(\frac{1}{2}-\zeta)}U_{\lambda_i^\varepsilon, \xi_i}(x),$$

where

$$\lambda_i^\varepsilon = (c_N \Lambda_i^2 \varepsilon)^{\frac{1}{N-2}} \quad \text{and} \quad \xi_i = \varepsilon^{\frac{1}{N-2}}\xi_i';$$

in particular,

$$(\xi_1, \xi_2) \in \mathcal{O}_\delta(\Omega)$$

(see (3.3)) since (6.1) holds. Let us set $\hat{U} = \hat{U}_1 + \hat{U}_2$. Consider now the functional

$$I(\xi, \Lambda) \equiv J_\varepsilon(\hat{U} + \hat{\psi} + \hat{\phi}(\xi, \Lambda)). \tag{6.5}$$

We see that

$$I(\xi, \Lambda) = \varepsilon^{1-2\zeta}\mathcal{I}_\varepsilon(V + \psi + \phi)$$

We start with the following basic claim:

Lemma 6.1 *$u = \hat{U} + \hat{\psi} + \hat{\phi}$ is a solution of problem (1.1) if and only if (ξ, Λ) is a critical point of I.*

Proof. We will have that the numbers c_{ij} in (5.7) are all zero if and only if

$$D\mathcal{I}_\varepsilon(V + \bar{\phi})[Z_{ij}] = 0 \quad \text{for all } i, j, \tag{6.6}$$

where, for now, we write $\bar{\phi} = \psi + \phi$. On the other hand,

$$\frac{\partial}{\partial \xi_{lk}} I(\xi, \Lambda) = 0$$

for all l, k if and only if

$$\frac{\partial}{\partial \xi'_{lk}} \mathcal{I}_\varepsilon(V + \bar{\phi}) = 0$$

where $\xi' = \varepsilon^{-1/(N-2)}\xi$, namely if and only if

$$D\mathcal{I}_\varepsilon(V + \bar{\phi}) \left[\frac{\partial V}{\partial \xi'_{lk}} + \frac{\partial \bar{\phi}}{\partial \xi'_{lk}} \right] = 0.$$

Now,

$$\frac{\partial V}{\partial \xi'_{lk}} = Z_{lk} + o(1).$$

Hence

$$D\mathcal{I}_\varepsilon(V + \bar{\phi})[Z_{lk} + o(1)] = 0$$

with $o(1) \to 0$ in, say, $\| \|_*$-norm, since we have also seen that $\|\frac{\partial \bar{\phi}}{\partial \xi'_{ij}}\|_* = o(1)$. Now, by definition of ϕ we have that

$$D\mathcal{I}_\varepsilon(V + \bar{\phi})[\varphi] = 0$$

for all φ in H_0^1 with $< \varphi, V_i^{p-1} Z_{ij} >= 0$ for all i, j. For a given function θ we can find constants b_{ij} such that $\theta - \sum b_{ij} Z_{ij}$ satisfies

$$< \theta - \sum_{i,j} b_{ij} Z_{ij}, V_l^{p-1} Z_{lk} >= 0$$

for all l, k. In fact this amounts to the system

$$< \theta, V_l^{p-1} Z_{lk} >= \sum_{i,j} b_{ij} < Z_{ij}, V_l^{p-1} Z_{lk} >$$

which has a uniformly invertible associated matrix. We see in particular that $b_{ij} = O(\|\theta\|_*)$. Then the above estimate implies that

$$\frac{\partial I}{\partial \xi_{ij}} = 0$$

if and only if

$$D\mathcal{I}_\varepsilon(V + \bar{\phi})[Z_{ij} + o(1)\theta] = 0$$

where θ is a uniformly bounded element of the space spanned by the Z_{lk}'s. Thus the above relation for all i, j is equivalent to

$$D\mathcal{I}_\varepsilon(V + \bar{\phi})[Z_{ij}] = 0$$

for all i, j. By definition of the c_{ij}, it is easily seen that this is indeed equivalent to $c_{ij} = 0$ for all i, j. Therefore finding (ξ', Λ) in such a way that the numbers c_{ij} which appear in problem (6.2) are zero is equivalent to finding critical points of the function $I(\xi, \Lambda)$. □

Our next purpose is to establish an asymptotic estimate for the functional $I(\xi, \Lambda)$. We prove

Proposition 6.1 *Let ζ be given by (6.4). Then we have the expansion,*

$$\varepsilon^{2\zeta-1} I(\xi, \Lambda) = 2C_N + \gamma_N \varepsilon + w_N \varepsilon \Psi(\xi, \Lambda) + o(\varepsilon)\theta(\xi, \Lambda), \tag{6.7}$$

uniformly with respect to $(\xi, \Lambda) \in \mathcal{O}_\delta(\Omega) \times (]\delta, \delta^{-1}[)^2$, where θ and $\nabla_{\xi,\Lambda}\theta$ are uniformly bounded, independently of ε. Here, we recall

$$\Psi(\xi, \Lambda) = \frac{1}{2}\{H(\xi_1, \xi_1)\Lambda_1^2 + H(\xi_2, \xi_2)\Lambda_2^2 - 2G(\xi_1, \xi_2)\Lambda_1\Lambda_2\} + \log \Lambda_1\Lambda_2,$$

and the constants in (6.7) are those in Lemma 3.2.

Proof. To begin with, we will prove that the following relations hold

$$I(\xi, \Lambda) - J_\varepsilon(\hat{U}) = o(\varepsilon) \tag{6.8}$$

and

$$\nabla_{\xi,\Lambda}[I(\xi, \Lambda) - J_\varepsilon(\hat{U})] = o(\varepsilon). \tag{6.9}$$

First, a Taylor expansion gives that

$$J_\varepsilon(\hat{U} + \hat{\psi}) - I(\xi, \Lambda) = J_\varepsilon(\hat{U} + \hat{\psi}) - J_\varepsilon(\hat{U} + \hat{\psi} + \hat{\phi}) =$$

$$\int_0^1 tdtD^2 J_\varepsilon(\hat{U} + \hat{\psi} + t\hat{\phi})[\hat{\phi}, \hat{\phi}], \tag{6.10}$$

since $0 = D\mathcal{I}_\varepsilon(V + \psi + \phi)[\phi] = \varepsilon^{2\zeta-1} DJ_\varepsilon(\hat{U} + \hat{\psi} + \hat{\phi})[\hat{\phi}]$. Now, from the definition of ϕ, we see that

$$\begin{aligned}
&\int_0^1 tdtD^2 J_\varepsilon(\hat{U} + \hat{\psi} + t\hat{\phi})[\hat{\phi}, \hat{\phi}] = \varepsilon^{1-2\zeta} \int_0^1 tdtD^2\mathcal{I}_\varepsilon(V + \psi + t\phi)[\phi, \phi] \\
&= \varepsilon^{1-2\zeta} \int_0^1 tdt \left[\int_{\Omega_\varepsilon} |\nabla\phi|^2 - (p+\varepsilon)(V + \psi + t\phi)^{p+\varepsilon-1}\phi^2\right] \\
&= \varepsilon^{1-2\zeta} \int_0^1 tdt \left(\int_{\Omega_\varepsilon} N_\varepsilon(\phi + \psi)\phi \right.\\
&\quad \left. + \int_{\Omega_\varepsilon} (p+\varepsilon)[V^{p+\varepsilon-1} - (V + \psi + t\phi)^{p+\varepsilon-1}]\phi^2\right).
\end{aligned} \tag{6.11}$$

Since, we recall $\|\phi\|_* + \|\psi\|_* = O(\varepsilon)$, the above relation together with (5.11) yield in particular,

$$I(\xi,\Lambda) - J_\varepsilon(\hat U + \hat\psi) = \begin{cases} O(\varepsilon^2) & \text{if } N < 6 \\ O(\varepsilon^2|\log\varepsilon|) & \text{if } N = 6 \\ O(\varepsilon^{1+\frac{4}{N-2}}) & \text{if } N \geq 7, \end{cases} \tag{6.12}$$

uniformly on ξ, Λ in the considered region. Let us estimate now difference in derivatives. Differentiating with respect to ξ variables we get from (6.11) that

$$\begin{aligned} &D_\xi[I(\xi,\Lambda) - J_\varepsilon(\hat U + \hat\psi)] \\ &\quad = \varepsilon^{1-2\zeta-\frac{1}{N-2}} \int_0^1 t dt \left(\int_{\Omega_\varepsilon} D_{\xi'}[(N_\varepsilon(\phi+\psi))\phi] \right. \\ &\qquad \left. +(p+\varepsilon) \int_{\Omega_\varepsilon} \nabla_{\xi'}[((V+\psi+t\phi)^{p+\varepsilon-1} - (V)^{p+\varepsilon-1})\phi^2] \right). \end{aligned} \tag{6.13}$$

Here $\xi_i' = \varepsilon^{-\frac{1}{N-2}}\xi_i$. Using the computations in the proof of Proposition 5.2 we get that

$$D_\xi[I(\xi,\Lambda) - J_\varepsilon(\hat U + \hat\psi)] = o(\varepsilon)$$

Now,

$$\begin{aligned} J_\varepsilon(\hat U + \hat\psi) - J_\varepsilon(\hat U) &= \varepsilon^{1-2\zeta}[\mathcal{I}_\varepsilon(V+\psi) - \mathcal{I}_\varepsilon(V)] \\ &= \varepsilon^{1-2\zeta}\left\{ \int_0^1 (1-t)dt[(p+\varepsilon)\int_{\Omega_\varepsilon}((V+t\psi)^{p+\varepsilon-1} \right. \\ &\qquad \left. -V^{p+\varepsilon-1})\psi^2] - 2\int_{\Omega_\varepsilon} R^\varepsilon\psi \right\} \end{aligned} \tag{6.14}$$

where we have used that

$$D\mathcal{I}_\varepsilon(V)[\psi] = -\int_{\Omega_\varepsilon} R^\varepsilon \psi.$$

Arguing as before and taking into account that (6.12) holds, we get (6.8). On the other hand, using (6.14), we see that

$$\begin{aligned} &D_\xi[J_\varepsilon(\hat U + \hat\psi) - J(\hat U)] \\ &\quad = \varepsilon^{1-2\zeta-\frac{1}{N-2}} D_{\xi'}\left\{ \int_0^1 (1-t)dt[(p+\varepsilon)\int_{\Omega_\varepsilon}((V+t\psi)^{p+\varepsilon-1} \right. \\ &\qquad \left. -V^{p+\varepsilon-1})\psi^2] - 2\int_{\Omega_\varepsilon} R^\varepsilon\psi \right\}. \end{aligned}$$

A computation similar to those already carried out yields then that

$$D_\xi[J_\varepsilon(\hat U + \hat\psi) - J(\hat U)] = o(\varepsilon) - 2\varepsilon^{-\frac{1}{N-2}} D_{\xi'}(\int_{\Omega_\varepsilon} R^\varepsilon\psi).$$

The desired result will follow if we prove that

$$\varepsilon^{-\frac{1}{N-2}} D_{\xi'}(\int_{\Omega_\varepsilon} R^\varepsilon \psi) = o(\varepsilon). \tag{6.15}$$

First, if $N > 3$, Proposition 5.2 yields

$$\varepsilon^{-\frac{1}{N-2}} D_{\xi'}(\int_{\Omega_\varepsilon} R^\varepsilon \psi) = \begin{cases} O(\varepsilon^{2-\frac{1}{N-2}}) & \text{if } N = 4, 5 \\ O(\varepsilon^{7/4} |\log \varepsilon|) & \text{if } N = 6 \\ O(\varepsilon^{\frac{N+1}{N-2}}) & \text{if } N \geq 7. \end{cases}$$

Let us consider now the case $N = 3$. We have that

$$D_{\xi_1'}\left(\int_{\Omega_\varepsilon} R^\varepsilon \psi\right) = \int_{\Omega_\varepsilon} (D_{\xi_1'} R^\varepsilon)\psi + \int_{\Omega_\varepsilon} (D_{\xi_1'}\psi) R^\varepsilon = \varepsilon^2 (I + II).$$

Let us estimate first II. Our first observation is that, locally, around ξ_i',

$$\varepsilon^{-1} R^\varepsilon(\xi_1 + x) \to V_0^5 \log V_0 + c V_0^4$$

uniformly over compacts, for certain constant c. Here $V_0(|x|) = \bar{U}_{\lambda,0}$ for some $\lambda > 0$. We also set $Z_0 = x \cdot \nabla V_0 + V_0$. Hence, $\varepsilon^{-1}\psi(x + \xi_1) \to w(|x|)$ where w is the unique radial solution of

$$\Delta w + p V_0^4 w = V_0^5 \log V_0 + c V_0^4 + b V_0^4 Z_0$$

which goes to zero at ∞, and is such that

$$\int_{I\!R^N} V_0^4 Z_0 w = 0.$$

The constant b is precisely that making the integral of the right hand side of the above equation against Z_0 equal to zero. In a similar way,

$$\varepsilon^{-1} D_{\xi_1} \psi(x + \xi_1) \to w'(|x|)\frac{x}{|x|}.$$

After a suitable application of dominated convergence, we get that After a suitable application of dominated convergence, we get that

$$\begin{aligned} II &= \varepsilon^{-2} \int_{\Omega_\varepsilon} (D_{\xi_1}\psi) R^\varepsilon = \varepsilon^{-2} \left\{ \sum_{j=1}^{2} \int_{B(\xi_j')} (D_{\xi_1}\psi) R^\varepsilon \right. \\ &\quad \left. + \int_{\Omega_\varepsilon \setminus \cap_{j+1}^2 B(\xi_{\prime}^j)} (D_{\xi_1}\psi) R^\varepsilon \right\} \\ &\to \int_{I\!R^N} (V_0^5 \log V_0 + c V_0^4 + b V_0^4 Z_0)(|x|) w'(|x|) \frac{x}{|x|} + O\left(\frac{1}{R}\right) = O\left(\frac{1}{R}\right), \end{aligned}$$

by symmetry. The term I can we dealt with in a similar manner. We conclude that $I + II \to 0$ since R can be chosen as large as desired. Hence relation (6.15) has been established, and this proves the result in what concerns to derivatives with respect to ξ. Derivatives with respect to Λ are actually dealt with in fact in simpler way, since the term $\varepsilon^{-\frac{1}{N-2}}$ does not appear in the differentiation. The validity of (6.15) thus follows.

We consider next the problem of estimating the quantity

$$J_\varepsilon(\hat{U}_1 + \hat{U}_2) = J_\varepsilon(\varepsilon^{\bar{\zeta}}(U_1 + U_2)).$$

where $\bar{\zeta} = \frac{1}{2} - \zeta$. We have the expansion

$$\varepsilon^{-2\bar{\zeta}} J_\varepsilon(\varepsilon^{\bar{\zeta}}(U_1 + U_2)) =$$

$$J_\varepsilon(U_1 + U_2) + \frac{1 - \varepsilon^{\frac{\varepsilon}{2}}}{p + 1 + \varepsilon} \int_\Omega (U_1 + U_2)^{p+1+\varepsilon}. \tag{6.16}$$

Let us now consider the second term in (6.16). From estimates already carried out in §3, we see that

$$\frac{1 - \varepsilon^{\frac{\varepsilon}{2}}}{p + 1 + \varepsilon} \int_\Omega (U_1 + U_2)^{p+1+\varepsilon} = (-\frac{\varepsilon}{2} \log \varepsilon + o(\varepsilon))\Big[\frac{1}{p+1} \int_\Omega (U_1 + U_2)^{p+1} +$$

$$\frac{\varepsilon}{(p+1)^2} \int_\Omega (U_1 + U_2)^{p+1} - \frac{\varepsilon}{p+1} \int_\Omega (U_1 + U_2)^{p+1} \log(U_1 + U_2) + o(\varepsilon)\Big] =$$

$$-\frac{1}{p+1} (\int_{I\!R^N} \bar{U}^{p+1}) \varepsilon \log \varepsilon + o(\varepsilon). \tag{6.17}$$

Combining (6.16), (6.17), (6.8) and (6.9) and the results of Lemma 3.2, (6.7) finally follows. $\square$

Lemma 3.2 and its remark, together with (6.9), yield

$$I(\xi, \Lambda) = 2C_N + \varepsilon\gamma_N + \varepsilon \frac{1}{p+1} (\int_{I\!R^N} U^{p+1}) \Psi(\xi, \Lambda) + o(\varepsilon), \tag{6.18}$$

and

$$\nabla I(\xi, \Lambda) = \varepsilon \frac{1}{p+1} (\int_{I\!R^N} U^{p+1}) (\nabla \Psi(\xi, \Lambda) + o(1)), \tag{6.19}$$

estimates that will be crucial later for our purposes

7 The min-max

In this section we set up a min-max scheme to find a critical point of the function Ψ. This scheme can be also used to find a critical point for the reduced functional I. We recall that the function Ψ is well defined in $(\Omega \times \Omega \setminus \Delta) \times \mathbb{R}^2_+$, where Δ is the diagonal $\Delta = \{(\xi_1, \xi_2) \in \Omega \times \Omega \,/\, \xi_1 = \xi_2\}$. In order to avoid the singularity of Ψ over Δ we consider $M > 0$ and define

$$G_M(\xi) = \begin{cases} G(\xi) & \text{if } G(\xi) \leq M \\ M & \text{if } G(\xi) > M, \end{cases} \tag{7.1}$$

and we consider $\Psi_{M,\rho} : \Omega_\rho \times \Omega_\rho \times \mathbb{R}^2_+ \to \mathbb{R}$ defined by

$$\Psi_{M,\rho}(\xi, \Lambda) = \Psi(\xi, \Lambda) - G_M(\xi)\Lambda_1\Lambda_2 + G(\xi)\Lambda_1\Lambda_2, \tag{7.2}$$

where $\rho > 0$ and $\Omega_\rho = \{\xi_1 \in \Omega \,/\, \mathrm{dist}(\xi_1, \partial\Omega) > \rho\}$. We will specify ρ and M later, and for notational convenience we will simply write $\Psi_{M,\rho} = \Psi$ and $D = \Omega_\rho \times \Omega_\rho \times \mathbb{R}^2_+$. We consider a further restriction $D_\varphi = \{(\xi, \Lambda) \in D \,/\, \varphi(\xi) < -\rho_0\}$, where $\rho_0 = \min\{\frac{1}{2}\exp(-2C_0 - 1), -\frac{1}{2}\max\{\varphi \,/\, \text{in } \mathcal{M}^2\}\}$, with

$$C_0 = \sup_{(\xi,\sigma)\in\mathcal{M}^2\times I_0} \Psi(\xi, \sigma).$$

With this choice certainly $\mathcal{M}^2 \times \mathbb{R}^2_+ \subset D_\varphi$.

Aiming to define the min-max class, for every $\xi \in \mathcal{M}^2$ we let $d(\xi) = (d_1(\xi), d_2(\xi)) \in \mathbb{R}^2$ be the negative direction of the quadratic form defining Ψ. Such a direction exists since, by hypothesis of Theorem 1.1, the function φ is negative over $\mathcal{M}^2$. We easily see that there is a constant $c > 0$ so that $c < d_1(\xi)d_2(\xi) < c^{-1}$ for all $\xi \in \mathcal{M}^2$.

Next we let Γ be the class of continuous functions $\gamma : \mathcal{M}^2 \times I_0 \times [0,1] \to D_\varphi$, such that

1. $\gamma(\xi, \sigma_0, t) = (\xi, \sigma_0 d(\xi))$, and $\gamma(\xi, \sigma_0^{-1}, t) = (\xi, \sigma_0^{-1}d(\xi))$ for all $\xi \in \mathcal{M}^2$, $t \in [0,1]$, and
2. $\gamma(\xi, \sigma, 0) = (\xi, \sigma d(\xi))$ for all $(\xi, \sigma) \in \mathcal{M}^2 \times I_0$,

where $I_0 = [\sigma_0, \sigma_0^{-1}]$ with σ_0 is a small number to be chosen later. Then we define the min-max value

$$c(\Omega) = \inf_{\gamma\in\Gamma} \sup_{(\xi,\sigma)\in\mathcal{M}^2\times I_0} \Psi(\gamma(\xi, \sigma, 1)) \tag{7.3}$$

and we will prove in what follows that $c(\Omega)$ is a critical value of Ψ. The first step in this direction is an intersection lemma. The idea behind this result is the topological continuation of the set of solution of an equation, and is based on the work of Fitzpatrick, Massabó and Pejsachowicz [13]. For every $(\xi, \sigma, t) \in \mathcal{M}^2 \times I_0 \times [0,1]$ we denote $\gamma(\xi, \sigma, t) = (\tilde{\xi}(\xi, \sigma, t), \tilde{\Lambda}(\xi, \sigma, t)) \in D_\varphi$, and we define the set

$$\mathcal{S} = \{(\xi, \sigma) \in \mathcal{M}^2 \times I_0 \ / \ \tilde{\Lambda}_1(\xi, \sigma, 1) \cdot \tilde{\Lambda}_2(\xi, \sigma, 1) = 1\}.$$

Lemma 7.1 *For every open neighborhood V of $\mathcal{S}$ in $\mathcal{M}^2 \times I_0$, the projection $g : V \to \mathcal{M}^2$ induces a monomorphism in cohomology, that is*

$$g^* : H^*(\mathcal{M}^2) \longrightarrow H^*(V)$$

is a monomorphism.

Proof. Let us define the set

$$Z([0,1]) = \{(\xi,\sigma) \in \mathcal{M}^2 \times I_0 \ / \ f(\xi,\sigma,t) \neq 1, \text{ for all } t \in [0,1]\},$$

where $f(\xi,\sigma,t) = \tilde{\Lambda}_1(\xi,\sigma,t)\cdot\tilde{\Lambda}_2(\xi,\sigma,t)$. Then the function h defined by $h(\xi,\sigma,t) = (g(\xi,\sigma), f(\xi,\sigma,t))$ is a homotopy of pairs

$$h : (\mathcal{M}^2 \times I_0, Z([0,1])) \times [0,1] \longrightarrow (\mathcal{M}^2 \times I\!R^+, \mathcal{M}^2 \times (I\!R^+ \setminus \{1\}).$$

By choosing σ_0 small enough we have that the following inclusion is well defined:

$$j : (\mathcal{M}^2 \times I_0, \mathcal{M}^2 \times \partial I_0) \longrightarrow (\mathcal{M}^2 \times I_0, Z([0,1])).$$

If i is also an inclusion map and $h_0(\cdot) = h(\cdot,0)$, then we have the following commutative diagram in cohomology

$$\begin{array}{ccc} H^*(\mathcal{M}^2 \times I_0, Z([0,1])) & \overset{h_0^*}{\longleftarrow} & H^*(\mathcal{M}^2 \times I\!R^+, \mathcal{M}^2 \times (I\!R^+ \setminus \{1\})) \\ {\scriptstyle j^*}\searrow & & \swarrow{\scriptstyle i^*} \\ & H^*(\mathcal{M}^2 \times I_0, \mathcal{M}^2 \times \partial I_0) & \end{array}$$

Since i^* is an isomorphism we conclude that h_0^* is a monomorphism and then from the homotopy axiom, we find that

$$h_1 = (g, f_1) : (\mathcal{M}^2 \times I_0, Z([0,1])) \to (\mathcal{M}^2 \times I\!R^+, \mathcal{M}^2 \times (I\!R^+ \setminus \{1\}))$$

induces a monomorphism in cohomology, where $h_1(\cdot) = h(\cdot,1)$. Next, defining

$$Z(1) = \{(\xi,\sigma) \in \mathcal{M}^2 \times I_0 \ / \ f(\xi,\sigma,1) \neq 1\}$$

and noting that $Z([0,1]) \subset Z(1)$ we also find that

$$h_1 : (\mathcal{M}^2 \times I_0, Z(1)) \to (\mathcal{M}^2 \times I\!R^+, \mathcal{M}^2 \times (I\!R^+ \setminus \{1\})$$

induces a monomorphism in cohomology. Since V and $Z(1)$ are open, and $V^c \subset Z(1)$, defining $Z = Z(1) \cap V$ and using the excision axiom, we conclude that

$$h_1^* : H^*(\mathcal{M}^2 \times I\!R^+, \mathcal{M}^2 \times (I\!R^+ \setminus \{1\})) \to H^*(V, Z)$$

is a monomorphism. Let e be a generator of $H^1(I\!R^+, I\!R^+ \setminus \{1\})$ and $u \in H^i(\mathcal{M}^2)$, with $i \geq 0$, then following from the basic relation between cross product and cup product in cohomology, we have

$$h_1^*(u \times e) = d^*(g^*(u) \times f_1^*(e)) = g^*(u) \smile f_1^*(e).$$

Since h_1^* is a monomorphism, it follows that g^* is also a monomorphism. $\square$

Corollary 7.1 *There is a constant K, independent of σ_0, so that*

$$\sup_{(\xi,\sigma)\in\mathcal{M}^2\times I_0} \Psi(\gamma(\xi,\sigma,1)) \geq -K \quad \text{for all } \gamma \in \Gamma.$$

Proof. Since Ω is smooth, there is $\delta_0 > 0$ such that if $\xi_1, \xi_2 \in \Omega_\rho$ and $|\xi_1 - \xi_2| < \delta_0$ then the line segment $[\xi_1, \xi_2] \subset \Omega$. Then we let $K > 0$ so that $G(\xi_1, \xi_2) \geq K$ implies $|\xi_1 - \xi_2| < \delta_0$. We observe that, if we assume that M is chosen such that $M \geq 2K$, then the implication remains valid both for G and G_M.

Assume, for contradiction, that for certain $\gamma \in \Gamma$

$$\Psi(\gamma(\xi,\sigma,1)) \leq -K \qquad \text{for all} \quad (\xi,\sigma) \in \mathcal{M}^2 \times I_0.$$

This implies that, for a small neighborhood V of $\mathcal{S}$ in $\mathcal{M}^2 \times I_0$, we have

$$G(\tilde{\xi}(\xi,\sigma,1)) \geq K \quad \text{for all} \quad (\xi,\sigma) \in V. \tag{7.4}$$

Let $D_0 = \Omega \times \Omega \times I\!R^2_+$ and $\gamma_1 = \gamma(\cdot, 1)$. Consider the inclusion $i_2 : \gamma_1(V) \to D_0$ and the maps $p : \gamma_1(V) \to \Omega \times I\!R^2_+$ and $\delta : \Omega \times I\!R^2_+ \to D_0$ defined as $p(\xi_1, \xi_2, \Lambda) = (\xi_1, \Lambda)$ and $\delta(\xi_1, \Lambda) = (\xi_1, \xi_1, \Lambda)$. From (7.4) we find that the function $h : \gamma_1(V) \times [0,1] \to D_0$ defined as $h(\xi_1, \xi_2, \Lambda, t) = (\xi_1, \xi_2 + t(\xi_1 - \xi_2), \Lambda)$ is a homotopy between i_2 and $\delta \circ p$. Let d be the integer given in Theorem 1.2 and consider the following commutative diagram

$$\begin{array}{ccc} H^{2d}(\mathcal{M}^2 \times I_0) & \overset{\gamma_1{}^*}{\longleftarrow} & H^{2d}(D_0) \\ i_1^*\downarrow & & i_2^*\downarrow \\ H^{2d}(V) & \overset{\gamma_2{}^*}{\longleftarrow} & H^{2d}(\gamma_1(V)), \end{array}$$

where i_1 is inclusion map and $\gamma_2 = \gamma_1|_V$. From the hypothesis of Theorem 1.2 we find $u \in H^d(\mathcal{M})$ and $v \in H^d(\Omega)$ nontrivial elements such that $\iota^*(v) = u$. If $\hat{v} \times \hat{v} \in H^{2d}(D_0)$ is the corresponding element, then by homotopy axiom and Lemma 4.1 we have $i_1^* \circ \gamma_1^*(\hat{v} \times \hat{v}) \neq 0$. On the other hand we see that $\delta^*(\hat{v} \times \hat{v}) = \hat{v} \smile \hat{v} \in H^{2d}(\Omega \times I\!R^2_+)$ is zero, either because d is odd or because $H^{2d}(\Omega) = 0$. In both cases we have then $\gamma_2^* \circ i_2^*(\hat{v} \times \hat{v}) = 0$, providing a contradiction. □

In view of Corollary 7.1, in order to prove that the min-max number (7.3) is a critical value, we need to care about the fact that the domain in which Ψ is defined is not necessarily closed for the gradient flow of Ψ. The following lemma, involving the original φ, is a step in this direction

Lemma 7.2 *Given $c < 0$ there exists a sufficiently small number $\rho > 0$ with the following property: If $(\bar{\xi}_1, \bar{\xi}_2) \in \partial(\Omega_\rho \times \Omega_\rho)$ is such that $\varphi(\bar{\xi}_1, \bar{\xi}_2) = c$, then there is a vector τ, tangent to $\partial(\Omega_\rho \times \Omega_\rho)$ at the point $(\bar{\xi}_1, \bar{\xi}_2)$, so that*

$$\nabla\varphi(\bar{\xi}_1, \bar{\xi}_2) \cdot \tau \neq 0. \tag{7.5}$$

The number ρ does not depend on c.

Proof. Consider, for small ρ, the modified domain

$$\tilde{\Omega} = \rho^{-1}\Omega,$$

and observe that for this domain, its associated Green's function and regular part are given by

$$\tilde{G}(x_1, x_2) = \rho^{N-2} G(\rho x_1, \rho x_2), \quad \tilde{H}(x_1, x_2) = \rho^{N-2} H(\rho x_1, \rho x_2).$$

Then $\varphi(\rho x_1, \rho x_2) = c$ translates into $\tilde{\varphi}(x_1, x_2) = c\rho^{N-2}$ where

$$\tilde{\varphi}(x_1, x_2) = \tilde{H}(x_1, x_1)^{1/2} \tilde{H}(x_2, x_2)^{1/2} - \tilde{G}(x_1, x_2).$$

Assume that $\operatorname{dist}(\rho x_1, \partial\Omega) = \rho$, namely that $\operatorname{dist}(x_1, \partial\tilde{\Omega}) = 1$. After a rotation and a translation, we assume that the closest point of the boundary to x_1 is the origin, that $x_1 = (\mathbf{0}, 1)$, where $\mathbf{0} = 0_{\mathbb{R}^{N-1}}$ and that as $\rho \to 0$ the domain $\tilde{\Omega}$ becomes the half-space $x^N > 0$. In order to make the relation $\tilde{\varphi}(x_1, x_2) = c\rho^{N-2}$ remain, as $\rho \to 0$, we claim that necessarily we must have $d = |x_1 - x_2| = O(1)$ as $\rho \to 0$. In fact, otherwise we will have

$$\tilde{H}(x_1, x_1)^{1/2} \tilde{H}(x_2, x_2)^{1/2} \geq C d^{-\frac{N-2}{2}}$$

while

$$\tilde{G}(x_1, x_2) \leq C d^{-(N-2)}.$$

Hence, for large d

$$C d^{-\frac{N-2}{2}} \leq \tilde{\varphi}(x_1, x_2) = c\rho^{N-2},$$

which is impossible since $c < 0$. We observe that this conclusion does not depend on the value of c, but on the fact c is negative. By assumption, we also have $|x_1 - x_2| \geq 1$. Then we let $\rho \to 0$ and then assume that the point x_2 converges to some $\bar{x}_2 = (\bar{x}_2', \bar{x}_2^N)$, where $\bar{x}_2^N \geq 1$. We also set, consistently $\bar{x}_1 = (\mathbf{0}, 1)$. The functions $\tilde{H}(x, y)$ and $\tilde{G}(x, y)$ converge to the corresponding ones $\hat{H}$ and $\hat{G}$ in the half-space $x_N > 0$, namely to

$$\hat{H}(x, y) = \frac{b_N}{|x - \hat{y}|^{N-2}}$$

and

$$\hat{G}(x, y) = b_N \left(\frac{1}{|x - y|^{N-2}} - \frac{1}{|x - \hat{y}|^{N-2}} \right).$$

Here, for $y = (y', y_N)$ we denote $\hat{y} = (y', -y_N)$. Similarly, $\nabla\tilde{\varphi}$ converges to $\nabla\hat{\varphi}$ where

$$\hat{\varphi}(x_1, x_2) = \hat{H}(x_1, x_1)^{1/2} \hat{H}(x_2, x_2)^{1/2} - \hat{G}(x_1, x_2).$$

We have that

$$\hat{\varphi}(\bar{x}_1, \bar{x}_2) = 0,$$

Assume first that $\bar{x}_2' \neq 0$. Then

$$\begin{aligned} \nabla_{x_2'} \hat{\varphi}(\bar{x}_1, \bar{x}_2) &= -\nabla_{x_2'} \hat{G}(\bar{x}_1, \bar{x}_2) \\ &= -(N-2) b_N \left(\frac{1}{|\bar{x}_2 - \bar{x}_1|^{N-2}} - \frac{1}{|\bar{x}_2 - \hat{\bar{x}}_1|^{N-2}} \right) x_2^{'} \neq 0 \end{aligned}$$

since clearly $|\bar{x}_2 - \bar{x}_1| < |\bar{x}_2 - \hat{\bar{x}}_1|$. The vector of $I\!R^N \times I\!R^N$ $(0', 0, x_2', 0)$ is clearly tangent to the boundary of the restriction $x_1^N > 0$, where we are assuming the considered point lies. Assume now that $x_2' = 0$, case in which otherwise $\bar{x}_2 = (0', a_0)$ with $a_0 \geq 2$, and then

$$b_N^{-1}\hat{\varphi}(\bar{x}_1, \bar{x}_2 + (a - a_0)\bar{x}_1) = \frac{1}{2^{\frac{N-2}{2}}} \frac{1}{(2a)^{\frac{N-2}{2}}} - \left(\frac{1}{(a-1)^{N-2}} - \frac{1}{(1+a)^{N-2}} \right)$$

Differentiating with respect to a we get

$$\begin{aligned} &b_N^{-1}\nabla_{x_2^N}\hat{\varphi}(\bar{x}_1, \bar{x}_2) \\ &\quad = -(N-2)[2^{-(N-1)}a_0^{-N/2} - (a_0 - 1)^{-(N-1)} + (a_0 + 1)^{-(N-1)}] \end{aligned}$$

This combined with the relation $\hat{\varphi}(\bar{x}_1, \bar{x}_2) = 0$ yields

$$\begin{aligned} &b_N^{-1}\nabla_{x_2^N}\hat{\varphi}(\bar{x}_1, \bar{x}_2) \\ &\quad = (N-2)[(a_0 - 1)^{-(N-1)} - 2^{-(N-1)}a_0^{-N/2} - (a_0 + 1)^{-(N-1)}] > 0. \end{aligned}$$

Indeed, since $\hat{\varphi}(\bar{x}_1, \bar{x}_2) = 0$ we have

$$\frac{1}{2^{N-1}a_0^{\frac{N}{2}}} = \frac{1}{2a_0} \frac{1}{2^{N-2}a_0^{\frac{N-2}{2}}} < \left[\frac{1}{(a_0 - 1)^{N-1}} - \frac{1}{(a_0 + 1)^{N-1}} \right].$$

So we can conclude that

$$b_N^{-1}\nabla_{x_2^N}\hat{\varphi}(\bar{x}_1, \bar{x}_2) > 0. \qquad \square$$

We finally can prove

Proposition 7.1 *The number $c(\Omega)$ given in (7.3) is a critical value for Ψ in D.*

Proof. We first prove that for every sequence $\{(\xi_n, \Lambda_n)\} \subset D_\varphi$ such that $(\xi_n, \Lambda_n) \to (\bar{\xi}, \bar{\Lambda}) \in \partial D_\varphi$ and $\Psi(\xi_n, \Lambda_n) \to c(\Omega)$ there is a vector T, tangent to ∂D_φ at $(\bar{\xi}, \bar{\Lambda})$, such that

$$\nabla\Psi(\bar{\xi}, \bar{\Lambda}) \cdot T \neq 0. \tag{7.6}$$

In order to prove (7.6) we first observe that if $\Lambda_n \to \bar{\Lambda} \in \partial I\!R^2_+$ then $\Psi(\xi_n, \Lambda_n) \to -\infty$. Thus we can assume that $\bar{\Lambda} \in I\!R^2_+$, $\bar{\xi} \in \bar{\Omega}_\rho \times \bar{\Omega}_\rho$ and $\varphi(\bar{\xi}) \leq -\rho_0$. Two cases arise, if $\nabla_\Lambda\Psi(\bar{\xi}, \bar{\Lambda}) \neq 0$ then T can be chosen parallel to $\nabla_\Lambda\Psi(\bar{\xi}, \bar{\Lambda})$. Otherwise, when $\nabla_\Lambda\Psi(\bar{\xi}, \bar{\Lambda}) = 0$ we have that $\bar{\Lambda}$ satisfies

$$\bar{\Lambda}_1^2 = -\frac{H(\bar{\xi}_2, \bar{\xi}_2)^{1/2}}{H(\bar{\xi}_1, \bar{\xi}_1)^{1/2}\varphi(\bar{\xi}_1, \bar{\xi}_2)}, \quad \bar{\Lambda}_2^2 = -\frac{H(\bar{\xi}_1, \bar{\xi}_1)^{1/2}}{H(\bar{\xi}_2, \bar{\xi}_2)^{1/2}\varphi(\bar{\xi}_1, \bar{\xi}_2)},$$

and $\bar{\xi}$ satisfies $\varphi(\bar{\xi}) < 0$. Substituting back in Ψ, we get

$$\Psi(\bar{\xi}_1, \bar{\xi}_2, \bar{\Lambda}_1, \bar{\Lambda}_2) = -\frac{1}{2} + \frac{1}{2}\log\frac{1}{|\varphi(\bar{\xi}_1, \bar{\xi}_2)|}$$

and then $\varphi(\bar{\xi}) = -\exp(-2c(\Omega) - 1) \leq -2\rho_0 < -\rho_0$, so that $\bar{\xi} \in \partial(\Omega_\rho \times \Omega_\rho)$. At this point we choose M: We take $\rho > 0$ as in Lemma 7.2, then we let $H_\rho =$

$\max\{H(\xi_1, \xi_1) \,/\, \xi_1 \in \Omega_\rho\}$ and consider $M \geq \exp(2K-1) + H_\rho$. We observe then, that the use of Corollary 7.1 implies $G(\bar{\xi}_1, \bar{\xi}_2) \leq M$. Thus, we can apply (7.5) to complete the proof of (7.6). Now we can define an appropriate negative gradient flow that will remain in D_φ at level $c(\Omega)$.

To finish we only need to prove the Palais Smale condition in D_φ at level $c(\Omega)$, that is, that if $\{(\xi_n, \Lambda_n)\} \subset D_\varphi$ satisfies $\Psi(\xi_n, \Lambda_n) \to c(\Omega)$ and $\nabla\Psi(\xi_n, \Lambda_n) \to 0$ then $\{(\xi_n, \Lambda_n)\}$ has a subsequence converging to some $(\bar{\xi}, \bar{\Lambda}) \in D_\varphi$. In fact, it can be shown that the sequence Λ_n remains bounded. Then we conclude using (7.6). □

Now we are in a position to complete the proof of Theorem 1.1, proving that the reduced functional has a critical point.

Proof of Theorem 1.1 completed. We consider the domain $D_{r,R} = \Omega_\rho \times \Omega_\rho \times [r, R]^2 \cap D_\varphi$, with r, R to be chosen later. The functional I is well defined on $D_{r,R}$ except on the set $\Delta_\rho = \{\xi \in \Omega_\rho \times \Omega_\rho \,/\, |\xi_1 - \xi_2| < \rho\}$. Proceeding as with Ψ, we can extend I to all $D_{r,R}$, keeping the relations (6.18) and (6.19) over $D_{r,R}$.

By the Palais Smale condition for Ψ proved in Proposition 7.1 there are numbers $R > 0$, $c > 0$ and $\alpha_0 > 0$ such that for all $0 < \alpha < \alpha_0$, and $(\xi, \Lambda) \in D_{r,R}$ satisfying $|\Lambda| \geq R$ and $c(\Omega) - 2\alpha \leq \Psi(\xi, \Lambda) \leq c(\Omega) + 2\alpha$ we have $|\nabla\Psi(\xi, \Lambda)| \geq c$.

Next by the min-max characterization of $c(\Omega)$ to choose $\gamma \in \Gamma$ so that

$$c(\Omega) \leq \sup_{(\xi,\sigma) \in \mathcal{M}^2 \times I_0} \Psi(\gamma(\xi, \sigma, 1)) \leq c(\Omega) + \alpha.$$

By making r small and R larger if necessary, we can assume that $\gamma(\xi, \sigma, 1) \in D_{r/2,R/2} \subset D_{r,R}$ for all $(\xi, \sigma) \in \mathcal{M}^2 \times I_0$.

We define a min-max value for the functional I using γ and the negative gradient flow for I. More precisely we consider $\eta : D_{r,R} \times [0, \infty] \to D_{r,R}$ being the solution of the equation $\dot{\eta} = -h(\eta)\nabla I(\eta)$ with initial condition $\eta(\xi, \Lambda, 0) = (\xi, \Lambda)$. Here the function h is defined in $D_{r,R}$ so that $h(\xi, \Lambda) = 0$ for all (ξ, Λ) with $\Psi(\xi, \Lambda) \leq c(\Omega) - 2\alpha$ and $h(\xi, \Lambda) = 1$ if $\Psi(\xi, \Lambda) \geq c(\Omega) - \alpha$, satisfying $0 \leq h \leq 1$.

By the choice of r and R and taking in account (6.18) and (6.19) we have $\eta(\xi, \Lambda, t) \in D_{r,R}$ for all $t \geq 0$. Then the following min-max value

$$C(\Omega) = \inf_{t \geq 0} \sup_{(\xi,\sigma) \in \mathcal{M}^2 \times I_0} I(\eta(\gamma(\xi, \sigma, 1), t))$$

is a critical value for I. In all this reasoning we are assuming that ε is small enough, to make the errors in (6.18) and (6.19) sufficiently small. □

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