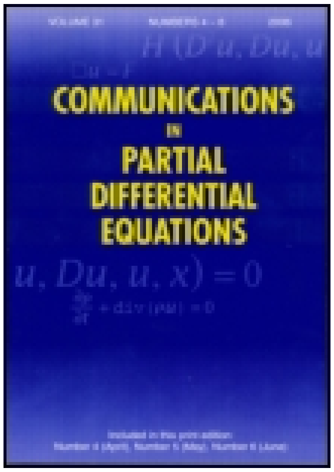

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Curve-Like Concentration Layers for a Singularly Perturbed Nonlinear Problem with Critical Exponents

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In this paper we consider the following problem

$$\triangle u - u + u^{p-\varepsilon} = 0 \quad in\ \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad on\ \partial\Omega,$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$ and p is the critical Sobolev exponent in dimension $n-1$, namely $p = (n+1)/(n-3)$, $\varepsilon > 0$. We show that, if $n \geq 8$, then for a sequence of the small positive parameter ε, the problem admits a positive solution concentrating along a nondegenerate segment connecting two points of the boundary of Ω.

Keywords Concentration phenomena; Critical exponents; Intersection with boundary.

2010 Mathematics Subject Classfication 35J60, 58J05, 58J37, 53C21, 53C22.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. The boundary value problem

$$\begin{cases} -d^2\Delta u + u = u^q & \text{in } \Omega, \quad u > 0 \text{ in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $q > 1$ and $d > 0$, is a model for different problems in applied sciences which exhibit *concentration phenomena* in their solutions. It arises for instance as the *shadow system* associated to activator-inhibitor systems in mathematical theory of biological pattern formation such as the Gierer-Meinhardt model and in certain

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models of chemotaxis, see references in [27]. In such models, and related ones, it is particularly meaningful the presence of solutions exhibiting peaks of concentration, namely one or several local maxima around which the solution remains strictly positive, while being very small away from them.

1.1. *Concentration Phenomena for Subcritical Cases: Perturbation of the Coefficient d*

The works [27, 35, 36] had dealt with precise analysis of least energy solutions to this problem in the subcritical case, $1 < q < \frac{n+2}{n-2}$ when $n > 2$, and $q > 1$ when $n = 2$, namely solutions which minimize the Rayleigh quotient

$$Q(u) = \frac{d^2 \int_\Omega |\nabla u|^2 + \int_\Omega |u|^2}{\left(\int_\Omega |u|^{q+1}\right)^{\frac{2}{q+1}}}, \quad u \in H^1(\Omega)\backslash\{0\}, \tag{1.2}$$

for small d. From those works, it became known that for d sufficiently small, a minimizer u_d of Q has a unique local maximum point x_d which is located on the boundary. Besides, $H(x_d) \to \max_{x\in\partial\Omega} H(x)$ where H denotes the mean curvature of $\partial\Omega$ and

$$u_d(x) \sim w\left(\frac{x - x_d}{d}\right), \tag{1.3}$$

where w is the (unique) radially symmetric solution of

$$\begin{aligned} &\Delta w - w + w^q = 0 \quad \text{in } \mathbb{R}^N, \\ &w > 0, \quad \lim_{|x|\to+\infty} w(x) = 0. \end{aligned} \tag{1.4}$$

This solution w decays exponentially at infinity which implies indeed the presence of a very sharp, bounded spike for the solution u_d around x_d. Solutions other than least energy with similar qualitative behavior around one or several points of the boundary or inside the domain have been found by several authors, see [7–9, 17, 18, 21, 23, 25, 43] and their references.

It is natural to look for solutions to problem (1.1) that exhibit concentration phenomena as $d \to 0$ not just at points but on higher dimensional subsets of $\overline{\Omega}$, see the conjecture by Ni in [32] or [33].

Given a k-dimensional submanifold Γ of $\partial\Omega$ and assuming that either $k \geq n-2$ or $q < \frac{n-k+2}{n-k-2}$, the question is whether there exists a solution u_d which near Γ looks like

$$u_d(x) \approx w\left(\frac{\text{dist}(x, \Gamma)}{d}\right) \tag{1.5}$$

where now $w(|y|)$ denotes the unique positive, radially symmetric solution to the problem

$$\Delta w - w + w^q = 0 \quad \text{in } \mathbb{R}^{n-k}, \quad \lim_{|y|\to\infty} w(|y|) = 0.$$

In [28–31], the authors have established the existence of a solution with the profile (1.5) when either $\Gamma = \partial\Omega$ or Γ is an *embedded closed minimal submanifold* of $\partial\Omega$, which is in addition *non-degenerate* in the sense that its Jacobi operator is non-singular. A difference with point concentration is that existence can only be achieved along a sequence of values $d \to 0$: d must actually remain suitably away from certain values of d where resonance occurs, and the topological type of the solution changes. Unlike the point concentration case, the Morse index of these solutions is very large and grows as $d \to 0$.

In the papers above mentioned, the higher dimensional concentration set lies *on* the boundary. In [45, 46] the question whether there are solutions with high dimensional concentration set *inside* the domain is considered. Indeed, the authors show the existence of solutions of the form (1.5) where now Γ is a *nondegenerate* straight line (hence $k = 1$) inside a two dimensional bounded smooth domain Ω and intersects orthogonally the boundary in two points. The reader can also refer to the review paper[44] for higher dimensional concentration phenomena. In the present paper we will address a similar question for the supercritical case.

1.2. *Bubbling Phenomena for Critical Cases: Perturbation of the Coefficient d*

Phenomena of the types described above occur as well in the critical case $q = \frac{n+2}{n-2}$, and $n \geq 3$, however several important differences are present. For instance, since compactness of the embedding of $H^1(\Omega)$ into $L^{q+1}(\Omega)$ is lost, existence of minimizers of $Q(u)$ becomes non-obvious (and in general not true for large d as established in [26]). Nevertheless it is the case, as shown in [1, 40], that such a minimizer does exist if d is sufficiently small. However the asymptotic profile (1.3) is lost. The profile and asymptotic behavior of this least energy solution has been analyzed in [4, 34, 39]. Again only one local maximum point x_d located around a point of maximum of the mean curvature of $\partial\Omega$ exists. However, unlike the subcritical case now its maximum value $M_d = u_d(x_d) \to +\infty$. Let $w(x)$ be the *standard bubble* in $\mathbb{R}^n$,

$$w(x) = \alpha_n \left(\frac{1}{1+|x|^2}\right)^{\frac{n-2}{2}}, \quad \alpha_n = [n(n-2)]^{\frac{n-2}{4}}, \tag{1.6}$$

which solves

$$\Delta w + w^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n. \tag{1.7}$$

The asymptotic profile of u_d is now, at leading order

$$u_d(x) \sim (M_d/\alpha_n)\, w\Big((M_d/\alpha_n)^{\frac{q-1}{2}}(x - x_d)\Big).$$

Construction of solutions with this type of *bubbling behavior* around one or more critical points of the mean curvature has been achieved for instance in [2, 3, 16, 19, 38, 41, 42]. An important difference with the subcritical case is that now mean curvature is required to be positive at these critical points. In fact, non-negativity of curvature is actually necessary for existence [5, 20, 39].

In [12] the authors study the problem of existence of solutions concentrating along a k dimensional set, with $k \geq 1$, for the critical case of the *k-critical exponent*, namely for the problem

$$-d^2\Delta u + u = u^{\frac{n-k+2}{n-k-2}} \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1.8}$$

for $\Omega \subset \mathbb{R}^n$. They proved the following result: assume the boundary $\partial\Omega$ contains a closed embedded non-degenerate minimal manifold Γ of dimensional $k \geq 1$, with $n - k \geq 7$, such that a certain linear combination of the sectional curvatures along Γ is positive, then, for a sequence $d = d_j \to 0$ there exists a positive solution u_d for (1.8) concentrating along Γ, as $d \to 0$, namely

$$d^2|\nabla u_d|^2 \rightharpoonup S_{n-k}\delta_\Gamma \quad \text{as } d \to 0,$$

in the sense of measure, where δ_Γ stands for the Dirac measure supported on Γ, and S_{n-k} is an explicit positive number. As far as we know, there are no results available in the literature for concentration phenomena for (1.8) on high dimensional sets which are *interior* to Ω.

1.3. *Bubbling Phenomena: Perturbation of the Exponent q*

In [14] the authors concerned the existence of point concentration solutions for problem (1.1) for q supercritical, namely

$$q = \frac{n+2}{n-2} + \varepsilon, \quad n \geq 3, \quad d = 1.$$

They established existence of bubbling solutions concentrating on points of the boundary when q approaches the critical exponent from the super critical side, i.e., $\varepsilon \to 0^+$. More precisely, given a non-degenerate critical point of the mean curvature on the boundary (or, more generally, a situation of topologically non trivial critical point) with *positive critical value* then a positive solution exhibiting boundary bubbling around such a point for problem

$$-\Delta u + u = u^{\frac{n+2}{n-2}+\varepsilon} \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

exists for all $\varepsilon > 0$ small enough. We refer the reader to [6, 42] for point concentration in the slightly subcritical case, when $q = \frac{n+2}{n-2} - \varepsilon$.

In the present paper we are concerning problem (1.1) by perturbation the exponent q slightly below from *second critical exponent* $\frac{n+1}{n-3}$, namely

$$-\Delta u + u = u^{\frac{n+1}{n-3}-\varepsilon} \quad \text{in } \Omega, \quad u > 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1.9}$$

where $\varepsilon > 0$ is a small parameter and Ω is a smooth bounded domain in $\mathbb{R}^n$. We will show that, if $n \geq 8$, then the problem admits a positive solution concentrating along a segment in the interior of Ω connecting two points of the boundary of Ω. Let us mention that to our knowledge no results for solutions to (1.9) concentrating along a high dimensional set on the boundary $\partial\Omega$ is known so far. For a related nonlinear boundary value problem, with Dirichlet boundary condition, at the second critical exponent, we refer the reader to [13].

1.4. *Main Results*

For notational convenience we define

$$n = N + 1,$$

and from now on write problem (1.9) in the form

$$\Delta u - u + u^{p-\varepsilon} = 0 \quad \text{in } \Omega, \quad u > 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1.10}$$

where Ω is a bounded smooth domain in $\mathbb{R}^{N+1}$ with $N \geq 7$. Here p is the critical Sobolev exponent in dimension N, namely $p = (N+2)/(N-2)$, which is often called the *second critical Sobolev exponent* in dimension $N+1$. Note that the function W in (1.15) has good decay under the technical assumption $N \geq 7$, i.e., $n \geq 8$ in (1.9), for our deriving of the linear resolution theory in Lemma 5.2.

Throughout the paper, we make the following assumptions and notation. The reader can refer the book [15] for some basic geometric results. Our candidate curve $\Gamma \in \Omega$ satisfies the following assumptions: The curvature of Γ is zero and we assume that in the $(\tilde{y}_1, \ldots, \tilde{y}_{N+1})$ coordinates, Γ is contained in the $\tilde{y}_{N+1}$ axis. After rescaling, we can always assume the arclength $|\Gamma| = 1$. Γ intersects $\partial\Omega$ at exactly two points, say,

$$\gamma_1 = (0, \ldots, 0, 1), \quad \gamma_0 = (0, \ldots, 0, 0),$$

and at these points $\Gamma \perp \partial\Omega$. Let us be more precise: we assume that in the small neighborhoods of γ_1 and γ_0, the boundary $\partial\Omega$ can be smoothly represented respectively as

$$\tilde{y}_{N+1} = \varphi_1(\tilde{y}_1, \ldots, \tilde{y}_N) \quad \text{and} \quad \tilde{y}_{N+1} = \varphi_0(\tilde{y}_1, \ldots, \tilde{y}_N).$$

Hence, there hold

$$\frac{\partial \varphi_0}{\partial \tilde{y}_i}(0, \ldots, 0) = 0, \quad \frac{\partial \varphi_1}{\partial \tilde{y}_i}(0, \ldots, 0) = 0, \quad i = 1, \ldots, N. \tag{1.11}$$

By defining a geometric eigenvalue problem, for $f = (f_1, \ldots, f_N)$

$$\begin{aligned} &f'' = \lambda f \quad \text{in } (0, 1), \\ &D^2\varphi_0(0)[f(0)] + f'(0) = 0, \quad D^2\varphi_1(0)[f(1)] + f'(1) = 0, \end{aligned} \tag{1.12}$$

we say that Γ is *non-degenerate* if (1.12) does not have a zero eigenvalue. This is equivalent to the following condition:

$$\det \begin{bmatrix} \mathrm{I} & D^2\varphi_0(0) \\ \mathrm{I} + |\Gamma| D^2\varphi_1(0) & D^2\varphi_1(0) \end{bmatrix} \neq 0, \tag{1.13}$$

where I denotes the Identity Matrix of dimension N.

For the following nonlinear elliptic problem

$$\Delta u + u^p = 0 \quad \text{in } \mathbb{R}^N, \tag{1.14}$$

it is well known that the problem has a solution $W(\mu, x)$ defined in the form

$$W(\mu, x) = \tau_N \left[\frac{\mu}{\mu^2 + |x|^2} \right]^{(N-2)/2}, \tag{1.15}$$

where $\tau_N = \left[N(N-2)\right]^{(N-2)/4}$ and μ is any positive parameter. For any point ξ in $\mathbb{R}^N$, the translated functions $W(\mu, y-\xi)$ are all and the only positive bounded solutions of problem (1.14) in the whole space $\mathbb{R}^N$.

Let μ_0 be the uniformly positive solution to the problem

$$\begin{aligned} &C_3\mu_0'' - C_1\mu_0 + C_2\,\mu_0^{-1} = 0 \quad \text{in } (0,1), \\ &\mu_0'(0) - b_{01}\,\kappa_0\,\mu_0(0) = 0, \quad \mu_0'(1) - b_{11}\,\kappa_1\,\mu_0(1) = 0, \end{aligned} \tag{1.16}$$

where C_1, C_2, C_3 and b_{11}, b_{01} are positive constants, given in (7.21) and (3.38)–(3.39). Here κ_0 and κ_1 are defined respectively as

$$\kappa_0 = \sum_{i=1}^{N} D_{ii}\varphi_0(0), \quad \kappa_1 = \sum_{i=1}^{N} D_{ii}\varphi_1(0) \tag{1.17}$$

the mean curvatures of $\partial\Omega$ at γ_0 and γ_1 with constraints

$$\kappa_0 > 0, \quad \kappa_1 < 0. \tag{1.18}$$

The solvability of problem (1.16) will be given in Lemma 3.1. Then we define a constant κ by

$$\kappa = \int_0^1 \frac{1}{\mu_0(\theta)}\, d\theta. \tag{1.19}$$

Our main theorem can be stated as follows:

Theorem 1.1. *Assume that* $\Omega \subset \mathbb{R}^{N+1}$ *with* $N \geq 7$ *and* (1.18) *holds, and the line segment* Γ *satisfy the non-degenerate condition* (1.13). *Given a small constant c, there exists* ε_0 *such that for all* $\varepsilon < \varepsilon_0$ *satisfying the following gap condition*

$$\left| \lambda_0\kappa^2 - \frac{j^2\pi^2\varepsilon}{|\Gamma|^2} \right| \geq c\sqrt{\varepsilon}, \quad \forall\, j \in \mathbb{N}, \tag{1.20}$$

problem (1.10) *has a positive solution* u_ε *concentrating along a curve* $\tilde{\Gamma}_\varepsilon$ *connecting the boundary* $\partial\Omega$. *Near* Γ, u_ε *takes the form*

$$u_\varepsilon(\tilde{y}) = W\Big(\mu\big(\sqrt{\varepsilon}\,\tilde{y}_{N+1}\big),\ dist(\tilde{y}', \tilde{\Gamma}_\varepsilon)/\sqrt{\varepsilon}\Big)(1 + o(1)), \tag{1.21}$$

where $\tilde{y} = (\tilde{y}', \tilde{y}_{N+1}) \in \Omega$ *and* $o(1)$ *denotes a smooth function which converges to* 0 *uniformly on compact sets of* $\Omega\backslash\Gamma$ *as* $\varepsilon \to 0$. *The parameter* μ *is a small perturbation of* μ_0. *Moreover, the curve* $\tilde{\Gamma}_\varepsilon$ *will collapse to* Γ *as* $\varepsilon \to 0$.

Let us mention that in the study of transition layers for the Allen-Cahn equation, a transition layer may occur at straight line segment contained in Ω which locally minimizes length among all curves nearby with endpoints lying on $\partial\Omega$. We refer to [11, 22, 24, 37] for related results in this direction.

Some words are in order on the proof of Theorem 1.1. The linear operator L_μ in (3.7) has a nontrivial kernel and also a positive first eigenvalue(cf. (3.8)–(3.9)). This leads to a complicated resonance for the construction of the solution. The proof of our result is based on a sort of *infinite Liapunov Schmidt reduction method*, used in other contexts like [10, 13], which is close in spirit to that of finite dimensional Liapunov Schmidt reduction. This method helps us deal with the complicated resonance, which also appears in the construction of concentration for the Schrödinger equation in [10]. Note that due to the homogeneous boundary condition, the concentration set of the solutions and boundary of the domain have a strong interaction. So we need to choose a suitable local coordinate system to decompose it in such a way that we can define an approximate solution to the problem, whose definition depends on a certain number of parameters that are smooth functions along the segment Γ. These parameters will give us freedom to deal with resonance in the reduction procedure. An actual solution to the problem is found as a small perturbation of such an approximate solution. To find the small perturbation, for the given parameters with some constraints we solve first a natural projected nonlinear problem where the linear operator is uniformly invertible. Then the resolution of the full problem is reduced to a nonlinear system of second order differential equations in the parameters introduced in the definition of the approximate solution. Such a system turns out to be solvable thanks to the assumptions made on the curve Γ.

The paper is organized as follows: to decompose the interaction of the boundary $\partial\Omega$ and the concentration set near Γ, in Section 2 we set up the problem by using local coordinates close to Γ (cf. (2.5)). Section 3 is devoted to construct a local approximation to the solutions, in a region close to the curve Γ. This is done in the following way. We first introduce some notation and prove the existence of the uniformly positive solution μ_0 to problem (1.16) under assumptions (1.18). Then we define an approximation to the solutions depending on several parameters μ, f, e(cf. (3.18)–(3.20)), which will be determined by the reduction procedure in Section 8. Later on, to improve the approximation, we estimate the errors inside the domain and on the boundary. In fact, we find that the boundary error can be partially improved by choosing suitable boundary conditions for the parameters(Subsection 3.3). We can add boundary correction terms to the previous approximate solution close to the boundary and define the basic approximate solution in (3.63). We end Section 3 with a final estimate of the error. Section 4 is devoted to what is called a gluing procedure (see [10]), that aims at connecting the problem in the whole domain to a problem locally close to the curve. In Section 5 we develop a solvability theory for a linear operator which will be used later on in Section 6 to solve a projected nonlinear problem. Section 7 is devoted to derive the system of ordinary differential equations of the parameters f, e, μ whose resolution will give the solvability of the whole problem. This is finally done in Section 8, which also contains the final proof of Theorem 1.1.

2. Setting Up the Problem in Local Coordinates

In this subsection, we focus on the procedure of setting up the problem near Γ. Globally in $\mathbb{R}^{N+1}$, we substitute

$$(y_1, \ldots, y_{N+1}) = \Big(\frac{\tilde{y}_1}{\sqrt{\varepsilon}}, \ldots, \frac{\tilde{y}_{N+1}}{\sqrt{\varepsilon}}\Big), \quad u(\sqrt{\varepsilon}y) = \varepsilon^{-(N-2)/4}v(y), \tag{2.1}$$

and then denote $\Gamma_\varepsilon = \frac{\Gamma}{\sqrt{\varepsilon}}$, $\gamma_{1\varepsilon} = (0, \ldots, 0, \frac{1}{\sqrt{\varepsilon}})$, $\gamma_{0\varepsilon} = (0, \ldots, 0, 0)$, $\Omega_\varepsilon = \frac{\Omega}{\sqrt{\varepsilon}}$ and ν_ε is the outward normal of $\partial\Omega_\varepsilon$. Problem (1.10) becomes

$$\triangle_y v - \varepsilon v + \varepsilon^{\frac{(N-2)\varepsilon}{4}} v^{p-\varepsilon} = 0 \quad \text{in } \Omega_\varepsilon, \quad \frac{\partial v}{\partial \nu_\varepsilon} = 0 \quad \text{on } \partial\Omega_\varepsilon, \tag{2.2}$$

where the differential operator is defined by $\triangle_y = \frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_{N+1}^2}$. In the sequel, we use the same notation v to denote the solutions in different coordinates.

Now, we define $\eta_\sigma(\zeta) = \eta(\sigma^{-1}\zeta)$ where η is a smooth cut-off function such that

$$\eta(\zeta) = 1 \quad \text{for } |\zeta| < 1 \quad \text{and} \quad \eta(\zeta) = 0 \quad \text{for } |\zeta| \geq 2. \tag{2.3}$$

The parameter σ is to be chosen later, and it will be of order $\varepsilon^{1/8}$. Let us use the notation

$$\begin{aligned} \alpha &= \eta_\sigma\Big(\varphi_0\big(\sqrt{\varepsilon}\, y_1, \ldots, \sqrt{\varepsilon}\, y_N\big) - \sqrt{\varepsilon}\, y_{N+1}\Big), \\ \beta &= \eta_\sigma\Big(\varphi_1\big(\sqrt{\varepsilon}\, y_1, \ldots, \sqrt{\varepsilon}\, y_N\big) - \sqrt{\varepsilon}\, y_{N+1}\Big). \end{aligned} \tag{2.4}$$

We introduce new coordinates near Γ_ε

$$\begin{aligned} &s_i = y_i, \quad i = 1, \ldots, N, \\ &z = y_{N+1} - \alpha \cdot \varphi_0(\sqrt{\varepsilon}\, y_1, \ldots, \sqrt{\varepsilon}\, y_N)/\sqrt{\varepsilon} - \beta \cdot \big[\, \varphi_1(\sqrt{\varepsilon}\, y_1, \ldots, \sqrt{\varepsilon}\, y_N) - 1 \,\big]/\sqrt{\varepsilon}, \end{aligned} \tag{2.5}$$

where $-\delta_0 < \sqrt{\varepsilon}s_1, \ldots, \sqrt{\varepsilon}s_N < \delta_0$ for small universal constant δ_0. Note that this transformation straighten the boundary, while keep the interior, far away from the boundary, unchanged. Now it is crucial to write the problem in the new coordinates. We call

$$Y(s, z) = \big(Y_1(s, z), \ldots, Y_{N+1}(s, z)\big),$$

the inverse of the transformation defined in (2.5). Here and in the sequel, we will denote $s = (s_1, \ldots, s_N)$. Note that the local coordinates $(s_1, \ldots, s_N, z)$ only hold in a region near the curve Γ_ε. The reader can also refer to [11].

We get that in a neighborhood of Γ_ε problem (2.2) takes the local form

$$\begin{aligned} &\triangle_s v + v_{zz} + B_1(v) - \varepsilon v + \varepsilon^{\frac{(N-2)\varepsilon}{4}} v^{p-\varepsilon} = 0, \\ &-\frac{\delta_0}{\sqrt{\varepsilon}} < s_1, \ldots, s_N < \frac{\delta_0}{\sqrt{\varepsilon}}, \; 0 < z < \frac{1}{\sqrt{\varepsilon}}, \end{aligned} \tag{2.6}$$

$$\check{D}_1(v) + v_z = 0, \quad -\frac{\delta_0}{\sqrt{\varepsilon}} < s_1, \ldots, s_N < \frac{\delta_0}{\sqrt{\varepsilon}}, \quad z = 0, \tag{2.7}$$

$$\widehat{D}_1(v) + v_z = 0, \quad -\frac{\delta_0}{\sqrt{\varepsilon}} < s_1, \ldots, s_N < \frac{\delta_0}{\sqrt{\varepsilon}}, \quad z = \frac{1}{\sqrt{\varepsilon}}, \tag{2.8}$$

where $\triangle_s v = \sum_{i=1}^N v_{s_i s_i}$ and the differential operators are defined by

$$B_1(v) = \mathbf{G}_1(v) + \mathbf{G}_2(v) + \mathbf{G}_3(v), \tag{2.9}$$

$$\check{D}_1(v) = -\sum_{i=1}^N \frac{\partial \varphi_0}{\partial y_i}\frac{\partial v}{\partial s_i} + \sum_{i=1}^N \left(\frac{\partial \varphi_0}{\partial y_i}\right)^2 \frac{\partial v}{\partial z}, \tag{2.10}$$

$$\widehat{D}_1(v) = -\sum_{i=1}^N \frac{\partial \varphi_1}{\partial y_i}\frac{\partial v}{\partial s_i} + \sum_{i=1}^N \left(\frac{\partial \varphi_1}{\partial y_i}\right)^2 \frac{\partial v}{\partial z}. \tag{2.11}$$

Here we have denoted the operators as

$$\mathbf{G}_1(\cdot) = -2\sum_{i=1}^N \left[\alpha\frac{\partial \varphi_0}{\partial y_i} + \beta\frac{\partial \varphi_1}{\partial y_i} + \varphi_0\frac{\partial \alpha}{\partial y_i} + (\varphi_1 - 1)\frac{\partial \beta}{\partial y_i}\right]\frac{\partial^2}{\partial s_i \partial z},$$

$$\mathbf{G}_2(\cdot) = \sum_{i=1}^N \left[\alpha\frac{\partial \varphi_0}{\partial y_i} + \beta\frac{\partial \varphi_1}{\partial y_i} + \varphi_0\frac{\partial \alpha}{\partial y_i} + (\varphi_1 - 1)\frac{\partial \beta}{\partial y_i}\right]^2 \frac{\partial^2}{\partial z^2}$$
$$+ \left[\varphi_0\frac{\partial \alpha}{\partial y_{N+1}} + (\varphi_1 - 1)\frac{\partial \beta}{\partial y_{N+1}}\right]^2 \frac{\partial^2}{\partial z^2},$$

$$\mathbf{G}_3(\cdot) = -\sum_{i=1}^N \left[\alpha\frac{\partial^2 \varphi_0}{\partial y_i^2} + \beta\frac{\partial^2 \varphi_1}{\partial y_i^2} + 2\frac{\partial \alpha}{\partial y_i}\frac{\partial \varphi_0}{\partial y_i} + 2\frac{\partial \beta}{\partial y_i}\frac{\partial \varphi_1}{\partial y_i} + \varphi_0\frac{\partial^2 \alpha}{\partial y_i^2} + (\varphi_1 - 1)\frac{\partial \beta}{\partial y_i^2}\right]\frac{\partial}{\partial z}$$
$$- \left[\varphi_0\frac{\partial^2 \alpha}{\partial y_{N+1}^2} + (\varphi_1 - 1)\frac{\partial^2 \beta}{\partial y_{N+1}^2}\right]\frac{\partial}{\partial z}.$$

We first introduce the following notation, for $i, j = 1, \ldots, N$,

$$D_{ij}\varphi = \frac{\partial^2 \varphi}{\partial \tilde{y}_i \partial \tilde{y}_j}(0),$$

as well as the $N \times N$ matrix

$$D^2\varphi = \big(D_{ij}\varphi\big).$$

Using condition (1.11), by Taylor's expansion, the problem near Γ_ε can be rewritten as

$$\triangle_s v + v_{zz} + B_2(v) - \varepsilon v + \varepsilon^{\frac{(N-2)\varepsilon}{4}} v^{p-\varepsilon} = 0,$$
$$-\frac{\delta_0}{\sqrt{\varepsilon}} < s_1, \ldots, s_N < \frac{\delta_0}{\sqrt{\varepsilon}},\ 0 < z < \frac{\delta_0}{\sqrt{\varepsilon}}, \tag{2.12}$$

with the following boundary condition

$$
\begin{aligned}
&\check{D}_2(v) + v_z = 0, \quad -\frac{\delta_0}{\sqrt{\varepsilon}} < s_1, \ldots, s_N < \frac{\delta_0}{\sqrt{\varepsilon}}, \quad z = 0, \\
&\widehat{D}_2(v) + v_z = 0, \qquad -\frac{\delta_0}{\sqrt{\varepsilon}} < s_1, \ldots, s_N < \frac{\delta_0}{\sqrt{\varepsilon}}, \quad z = \frac{1}{\sqrt{\varepsilon}}.
\end{aligned}
\tag{2.13}
$$

In the above, we have denoted

$$
\begin{aligned}
\check{D}_2(v) &= -\sqrt{\varepsilon}\sum_{i,j=1}^{N} D_{ij}\varphi_0\, s_j v_{s_i} - \frac{\varepsilon}{2}\sum_{i,j,k=1}^{N} D_{ijk}\varphi_0\, s_j s_k v_{s_i} \\
&\quad + \varepsilon\sum_{i=1}^{N}\left(\sum_{j=1}^{N} D_{ij}\varphi_0 s_j\right)^2 \frac{\partial v}{\partial z} + \check{D}_3(v), \\
\widehat{D}_2(v) &= -\sqrt{\varepsilon}\sum_{i,j=1}^{N} D_{ij}\varphi_1\, s_j v_{s_i} - \frac{\varepsilon}{2}\sum_{i,j,k=1}^{N} D_{ijk}\varphi_1\, s_j s_k v_{s_i} \\
&\quad + \varepsilon\sum_{i=1}^{N}\left(\sum_{j=1}^{N} D_{ij}\varphi_1 s_j\right)^2 \frac{\partial v}{\partial z} + \widehat{D}_3(v).
\end{aligned}
\tag{2.14}
$$

The other linear differential operator B_2 is defined by

$$
\begin{aligned}
B_2(v) = &-\sum_{i,j=1}^{N} 2\sqrt{\varepsilon}\big(\alpha D_{ij}\varphi_0 + \beta D_{ij}\varphi_1\big)s_j v_{zs_i} \\
&+ \sum_{i,j,k=1}^{N} \varepsilon\Big[\big(\alpha D_{ij}\varphi_0 + \beta D_{ij}\varphi_1\big)\big(\alpha D_{ik}\varphi_0 + \beta D_{ik}\varphi_1\big)s_j s_k\Big] u_{zz} \\
&- \sum_{i,j=1}^{N} \sqrt{\varepsilon}\big(\alpha D_{ii}\varphi_0 + \beta D_{ii}\varphi_1\big)s_j v_z + B_3(v).
\end{aligned}
\tag{2.15}
$$

Remark. For the brevity of notation, we have put all high order terms into $\widehat{D}_3(v)$, $\check{D}_3(v)$ and $B_3(v)$. We will show that these terms are high order terms measured by some norms.

For later use, for any positive solution v, we write the nonlinearity in the form

$$
\begin{aligned}
\mathbb{N}(v) &\equiv \varepsilon^{\frac{(N-2)\varepsilon}{4}} v^{p-\varepsilon} \\
&= v^p - \varepsilon v^p \log v + \frac{N-2}{4}\,\varepsilon(\log\varepsilon)\, v^p + O(\varepsilon^2\big|\log\varepsilon\big|^2) v^p \mathbb{N}_0(v),
\end{aligned}
\tag{2.16}
$$

and then denote

$$
\mathbb{S}(v) = \triangle_s v + v_{zz} + B_2(v) - \varepsilon v + \mathbb{N}(v). \tag{2.17}
$$

In the above, if v is uniformly bounded then $\mathbb{N}_0(v)$ is uniformly bounded.

3. Local Approximate Solution

3.1. *Preliminaries*

Recall that $p = (N+2)/(N-2)$. Consider the following nonlinear elliptic problem

$$\Delta u + u^p = 0 \quad \text{in } \mathbb{R}^N. \tag{3.1}$$

It is well known that the problem has a solution $W(\mu, x)$ defined in the form

$$W(\mu, x) = \tau_N \left[\frac{\mu}{\mu^2 + |x|^2} \right]^{(N-2)/2}, \tag{3.2}$$

where $\tau_N = \left[N(N-2)\right]^{(N-2)/4}$ and μ is any positive parameter. For any point ξ in $\mathbb{R}^N$, the translated functions $W(\mu, y-\xi)$ are all and the only positive bounded solutions of problem (3.1) in the whole space $\mathbb{R}^N$. Direct computation gives that

$$\begin{aligned} \int_{\mathbb{R}^N} \left|\nabla_y W(\mu, y-\xi)\right|^2 \mathrm{d}y &= \int_{\mathbb{R}^N} \left|W(\mu, y-\xi)\right|^{p+1} \mathrm{d}y \\ &= \left[N(N-2)\right]^{N/2} \int_{\mathbb{R}^N} \left(|t|^2+1\right)^{-N} \mathrm{d}t \equiv \check{C}, \\ \int_{\mathbb{R}^N} \left|W(\mu, y-\xi)\right|^2 \mathrm{d}y &= (\tau_N)^2 \mu^2 \int_{\mathbb{R}^N} \left(|t|^2+1\right)^{2-N} \mathrm{d}t = \widehat{C}\mu^2. \end{aligned} \tag{3.3}$$

Note that the constants $\check{C}$ and $\widehat{C}$ are independent of the parameter μ and the center ξ.

Now we consider the linearization of the problem (3.1) at $W(\mu, x)$ for $\mu = 1$, say W_0. It is proved in [13] that there exists a unique positive eigenvalue λ_0 with corresponding eigenfunction Z_0 (even) in $L^2(\mathbb{R}^N)$ of the problem

$$L_0\phi \equiv \Delta_x \phi + pW_0^{p-1}\phi = \lambda\phi \quad \text{in } \mathbb{R}^N. \tag{3.4}$$

It is worth mentioning that $Z_0(x)$ has exponential decay of order $O(e^{-\sqrt{\lambda_0}|x|})$ at infinity. Moreover, the kernel of the operator L_0 in the space of bounded functions in $\mathbb{R}^N$ constitutes of

$$\widetilde{\mathbf{Z}}_1 = \frac{\partial W_0}{\partial x_1}, \ldots, \widetilde{\mathbf{Z}}_N = \frac{\partial W_0}{\partial x_N}, \quad \widetilde{\mathbf{Z}}_{N+1} = -x \cdot \nabla_x W_0 - \frac{N-2}{2} W_0. \tag{3.5}$$

For further references, we also denote $\widetilde{\mathbf{Z}}_0 = Z_0$. One can check that

$$\int_{\mathbb{R}^N} \widetilde{\mathbf{Z}}_i(x)\widetilde{\mathbf{Z}}_j(x)\,\mathrm{d}x = 0, \quad \forall\, i \neq j, \quad 0 \leq i, j \leq N+1. \tag{3.6}$$

It is easy to check that, for the linear operator at $W(\mu, x) = \mu^{-(N-2)/2} W_0\big(x/\mu\big)$, i.e.

$$L_\mu \phi \equiv \Delta_x \phi + p\big(W(\mu, x)\big)^{p-1}\phi \quad \text{in } \mathbb{R}^N, \tag{3.7}$$

the first eigenvalue and eigenfunction are

$$\lambda_\mu = \mu^{-2}\lambda_0, \quad \overline{\mathbf{Z}}_0(\mu, x) = \mu^{-\frac{N-2}{2}} Z_0\left(\frac{x}{\mu}\right). \tag{3.8}$$

The kernel of the operator L_μ in the space of bounded functions in $\mathbb{R}^N$ constitutes

$$\begin{aligned} \overline{\mathbf{Z}}_i(\mu, x) &= \frac{\partial W(\mu, x)}{\partial x_i}, \quad i = 1, \dots, N, \\ \overline{\mathbf{Z}}_{N+1}(\mu, x) &= \frac{\partial W(\mu, x)}{\partial \mu} = -\mu^{-1} x \cdot \nabla_x W(\mu, x) - \mu^{-1}\frac{N-2}{2} W(\mu, x). \end{aligned} \tag{3.9}$$

One can check that the following relations hold

$$\overline{\mathbf{Z}}_i(\mu, x) = \mu^{-N/2}\widetilde{\mathbf{Z}}_i\left(\frac{x}{\mu}\right), \quad \forall\, i = 1, \dots, N+1, \tag{3.10}$$

and

$$\int_{\mathbb{R}^N} \overline{\mathbf{Z}}_i(\mu, x)\overline{\mathbf{Z}}_j(\mu, x)\,\mathrm{d}x = 0, \quad \forall\, i \neq j,\ 0 \leq i, j \leq N+1. \tag{3.11}$$

In the final part of this subsection, by recalling the condition (1.18), we give the resolution theory of problem (1.16).

Lemma 3.1. *There is a positive solution μ_0 to problem* (1.16).

Proof. Consider the eigenvalue problem

$$\begin{aligned} &C_3\varphi'' - C_1\varphi = -\lambda\varphi \quad \text{in } (0, 1), \\ &\varphi'(0) - b_{01}\,\kappa_0\,\varphi(0) = 0, \qquad \varphi'(1) - b_{11}\,\kappa_1\,\varphi(1) = 0. \end{aligned}$$

Let φ_0 be the first eigenfunction corresponding to the first(positive) eigenvalue λ_0. It is well known that φ_0 is positive on $[0, 1]$. We set two positive functions

$$\mu_1 = \tau\varphi_0, \quad \mu_2 = \sqrt{C_2/C_1},$$

where τ is a positive parameter. By choosing τ small enough, we have that

$$\mu_1 < \mu_2 \quad \text{on } [0, 1],$$

and moreover

$$\begin{aligned} &C_3\mu_1'' - C_1\mu_1 + \frac{C_2}{\mu_1} = -\tau\lambda_0\varphi_0 + \frac{C_2}{\tau\varphi_0} > 0 \quad \text{in } (0, 1), \\ &\mu_1'(0) - b_{01}\,\kappa_0\,\mu_1(0) = 0, \quad \mu_1'(1) - b_{11}\,\kappa_1\,\mu_1(1) = 0. \end{aligned}$$

Due to the assumption $\kappa_1 < 0$ and $k_0 > 0$ in (1.18), μ_2 satisfies

$$\begin{aligned} &C_3\mu_2'' - C_1\mu_2 + \frac{C_2}{\mu_2} = 0 \quad \text{in } (0, 1), \\ &\mu_2'(0) - b_{01}\,\kappa_0\,\mu_2(0) < 0, \qquad \mu_2'(1) - b_{11}\,\kappa_1\,\mu_2(1) > 0. \end{aligned}$$

Hence, μ_1 and μ_2 are sub-solution and super-solution to problem (1.16). We conclude that there exists a solution $\mu_0(\theta) > 0$ for all $\theta \in [0, 1]$. Standard elliptic regularity gives that $\mu_0 \in C^2([0, 1])$. □

3.2. *The Approximations and Errors*

We assume that the location of the concentration layer of the solution is characterized by the curve $\widetilde{\Gamma}_\varepsilon$:

$$(s_1, \ldots, s_N) = \big(f_1(\sqrt{\varepsilon}\, z), \ldots, f_N(\sqrt{\varepsilon}\, z)\big),$$

in the $(s_1, \ldots, s_N, z)$ coordinates. For convenience of notation, we shall write $f = (f_1, \ldots, f_N)$ in the sequel. This set of functions f_j will be determined by a system of differential equations in the reduction method, see Sections 7-8. By using the block solution in (3.2), we then heuristically choose the *first approximate solution* by

$$\begin{aligned} v_1(s, z) &= (1+\varpi) W\Big(\mu\big(\sqrt{\varepsilon} z\big), s - f\big(\sqrt{\varepsilon} z\big)\Big) \\ &= (1+\varpi)\tau_N \left[\frac{\mu(\sqrt{\varepsilon} z)}{\mu^2(\sqrt{\varepsilon} z) + \ell^2}\right]^{(N-2)/2} \\ &\equiv U(s, z), \end{aligned} \tag{3.12}$$

where ℓ is a nonnegative scalar function defined by $\ell(s, z) = |s - f(\sqrt{\varepsilon} z)|$ and ϖ is a constant of order $O(\varepsilon|\log\varepsilon|)$ defined by the relation

$$\big(1+\varpi\big)^{p-1} = \left(1 + \frac{N-2}{4}\varepsilon\log\varepsilon\right)^{-1}. \tag{3.13}$$

In other words,

$$\varpi = \left(1 + \frac{N-2}{4}\varepsilon\log\varepsilon\right)^{\frac{N-2}{4}} - 1 = -\frac{(N-2)^2}{16}\,\varepsilon\log\varepsilon + O(\varepsilon^{\varrho}), \tag{3.14}$$

for a constant $\varrho > 1$. In the above, μ is a positive function to be determined in the reduction procedure, see Sections 7–8. In fact we will find that μ has the form

$$\mu = \mu_0 + \tilde{\mu}, \tag{3.15}$$

where μ_0 is a uniformly positive function defined by Lemma 3.1 and $\tilde{\mu}$ is a smooth perturbation with small norms in some sense to be made more precise in the sequel.

As we stated in the introduction, the problem has resonance caused by Z_0. So we introduce a new parameter e such that we can deal with it by the reduction method (see Sections 7-8), and then define the *second approximate solution* to the problem near Γ_ε as

$$v_2(s, z) = U(s, z) + \sqrt{\varepsilon}\, e(\sqrt{\varepsilon} z)\Psi(s, z), \tag{3.16}$$

where the correction layer Ψ is defined by

$$\Psi(s,z) = \overline{\mathbf{Z}}_0\Big(\mu(\sqrt{\varepsilon}z),\, s - f(\sqrt{\varepsilon}\, z)\Big). \tag{3.17}$$

In all what follows, we shall assume the validity of the following uniform constraints on the parameters f, e and $\tilde{\mu}$

$$\|f\|_a = \|f\|_{L^\infty(0,1)} + \|f'\|_{L^\infty(0,1)} + \|f''\|_{L^2(0,1)} \le \varepsilon^{\frac{1}{4}}, \tag{3.18}$$

$$\|e\|_b = \|e\|_{L^\infty(0,1)} + \sqrt{\varepsilon}\,\|e'\|_{L^2(0,1)} + \varepsilon\|e''\|_{L^2(0,1)} \le \varepsilon^{\frac{1}{4}}, \tag{3.19}$$

$$\|\tilde{\mu}\|_c = \|\tilde{\mu}\|_{L^\infty(0,1)} + \|\tilde{\mu}'\|_{L^2(0,1)} + \|\tilde{\mu}''\|_{L^2(0,1)} \le \varepsilon^{\frac{1}{4}}. \tag{3.20}$$

For further references, we introduce some notation below.

Notation:

(1) In rest of this paper, we shall use the translated variables

$$x = s - f, \quad \text{i.e.} \quad \ell = |x|. \tag{3.21}$$

(2) For simplicity, we also define

$$\digamma = \big\{ (f, e, \tilde{\mu}) : \text{the functions } f, e, \tilde{\mu} \text{ satisfy (3.18)–(3.20) respectively}\big\}. \tag{3.22}$$

(3) We use the notation

$$\begin{aligned}
\mathfrak{S} &= \left\{(s_1,\dots,s_N,z):\ s_i\in\mathbb{R},\ i=1,\dots,N,\ 0<z<\frac{1}{\sqrt{\varepsilon}}\right\},\\
\partial_1\mathfrak{S} &= \left\{(s_1,\dots,s_N,z):\ s_i\in\mathbb{R},\ i=1,\dots,N,\ z=\frac{1}{\sqrt{\varepsilon}}\right\},\\
\partial_0\mathfrak{S} &= \left\{(s_1,\dots,s_N,z):\ s_i\in\mathbb{R},\ i=1,\dots,N,\ z=0\right\}.
\end{aligned} \tag{3.23}$$

We fix a number $2 \le \sigma < N$. For any functions defined on $\mathfrak{S}$, we consider the following L^∞–weighted norms

$$\begin{aligned}
\|\phi\|_* &= \sup_{\mathfrak{S}}\Big(1+|s-f|^{\sigma-2}\Big)|\phi(s,z)| + \sup_{\mathfrak{S}}\Big(1+|s-f|^{\sigma-1}\Big)|D\phi(s,z)|,\\
\|h\|_{**} &= \sup_{\mathfrak{S}}\Big(1+|s-f|^{\sigma}\Big)|h(s,z)|.
\end{aligned} \tag{3.24}$$

(4) Introduce now the sets

$$\begin{aligned}
\mathfrak{S}_0 &= \left\{(x_1,\dots,x_N,z):\ (x_1,\dots,x_N)\in\mathbb{R}^N,\ 0<z<\frac{1}{\sqrt{\varepsilon}}\right\},\\
\partial_1\mathfrak{S}_0 &= \left\{(x_1,\dots,x_N,z):\ (x_1,\dots,x_N)\in\mathbb{R}^N,\ z=\frac{1}{\sqrt{\varepsilon}}\right\},\\
\partial_0\mathfrak{S}_0 &= \left\{(x_1,\dots,x_N,z):\ (x_1,\dots,x_N)\in\mathbb{R}^N,\ z=0\right\}.
\end{aligned} \tag{3.25}$$

For any function defined on $\mathfrak{S}_0$, we consider the following $L^\infty-$weighted norms

$$\|\tilde{\phi}\|_* = \sup_{\mathfrak{S}_0}\Big(1+|x|^{\sigma-2}\Big)|\tilde{\phi}(x,z)| + \sup_{\mathfrak{S}_0}\Big(1+|x|^{\sigma-1}\Big)|D\tilde{\phi}(x,z)|,$$
$$\|\tilde{h}\|_{**} = \sup_{\mathfrak{S}_0}\Big(1+|x|^{\sigma}\Big)|\tilde{h}(x,z)|. \tag{3.26}$$

We assume that all functions involved are smooth.

We start with the analysis of the error term. First we consider the *interior error*, namely $\mathscr{E}_1 := \mathbb{S}(v_2)$ where $\mathbb{S}$ is defined in (2.17) and v_2 in (3.16). We first dealt with the term $\partial_z\partial_z(e\Psi)$ in the error as follows

$$\begin{aligned}\sqrt{\varepsilon}\frac{\partial^2}{\partial z^2}\big(e\Psi\big) &= \varepsilon^{3/2}e''\Psi - 2\varepsilon^{3/2}e'\sum_{i=1}^{N}\Psi_{s_i}f_i' + 2\varepsilon^{3/2}e'\Psi_\mu\mu' + \varepsilon^{3/2}e\Psi_\mu\mu'' + \varepsilon^{3/2}e\Psi_{\mu\mu}(\mu')^2\\ &\quad - 2\varepsilon^{3/2}e\sum_{i}^{N}\Psi_{\mu s_i}f_i'\mu' + \varepsilon^{3/2}e\sum_{i,j=1}^{N}\Psi_{s_is_j}f_i'f_j' - \varepsilon^{3/2}e\sum_{i=1}^{N}\Psi_{s_i}f_i''\\ &\equiv \mathscr{E}_{00}.\end{aligned} \tag{3.27}$$

We further write the nonlinearity in the form

$$\begin{aligned}\mathbb{N}(U+\sqrt{\varepsilon}\,e\Psi) &= U^p + \sqrt{\varepsilon}pU^{p-1}e\Psi - \varepsilon U^p\log U + \mathscr{P}(U+\sqrt{\varepsilon}\,e\Psi) + \mathscr{Q}(U+\sqrt{\varepsilon}\,e\Psi)\\ &\quad + \frac{N-2}{4}\,\varepsilon(\log\varepsilon)\,U^p + \mathscr{R}(U+\sqrt{\varepsilon}\,e\Psi)\\ &\quad + O(\varepsilon^2\big|\log\varepsilon\big|^2)(U+\sqrt{\varepsilon}\,e\Psi)^p\,\mathbb{N}_0\big(U+\sqrt{\varepsilon}\,e\Psi\big),\end{aligned} \tag{3.28}$$

where we have defined the nonlinear operators

$$\begin{aligned}\mathscr{P}(U+\sqrt{\varepsilon}\,e\Psi) &= (U+\sqrt{\varepsilon}\,e\Psi)^p - U^p - \sqrt{\varepsilon}pU^{p-1}e\Psi,\\ \mathscr{Q}(U+\sqrt{\varepsilon}\,e\Psi) &= -\varepsilon(U+\sqrt{\varepsilon}e\Psi)^p\log(U+\sqrt{\varepsilon}e\Psi) + \varepsilon U^p\log U,\\ \mathscr{R}(U+\sqrt{\varepsilon}\,e\Psi) &= \frac{N-2}{4}\,\varepsilon(\log\varepsilon)\Big[(U+\sqrt{\varepsilon}\,e\Psi)^p - U^p\Big].\end{aligned} \tag{3.29}$$

Using equations (3.1) and (3.13), we have

$$\begin{aligned}&\triangle_s(U+\sqrt{\varepsilon}\,e\Psi) + \mathbb{N}(U+\sqrt{\varepsilon}\,e\Psi)\\ &\quad= \sqrt{\varepsilon}\mu^{-2}\lambda_0\,e\Psi + \sqrt{\varepsilon}ep\bigg[\Big(1+\frac{N-2}{4}\varepsilon\log\varepsilon\Big)^{-1} - 1\bigg]W^{p-1}\Psi\\ &\qquad - \varepsilon U^p\log U + \mathscr{P}(U+\sqrt{\varepsilon}\,e\Psi) + \mathscr{Q}(U+\sqrt{\varepsilon}\,e\Psi)\\ &\qquad + \mathscr{R}(U+\sqrt{\varepsilon}\,e\Psi) + O(\varepsilon^2\big|\log\varepsilon\big|^2)(U+\sqrt{\varepsilon}\,e\Psi)^p\,\mathbb{N}_0\big(U+\sqrt{\varepsilon}\,e\Psi\big)\\ &\quad\equiv\ \sqrt{\varepsilon}\mu^{-2}\lambda_0\,e\Psi + \mathscr{E}_{11}.\end{aligned} \tag{3.30}$$

On the other hand, direct computations give that

$$\begin{aligned} U_{zz} &= \varepsilon\frac{\partial U}{\partial\mu}\mu'' + \varepsilon\frac{\partial^2 U}{\partial\mu^2}(\mu')^2 - 2\varepsilon\frac{\partial^2 U}{\partial\mu\partial\ell}\mu'\sum_{i=1}^{N}\frac{(s_i - f_i)f_i'}{\ell} - \varepsilon\frac{\partial U}{\partial\ell}\sum_{i=1}^{N}\frac{(s_i - f_i)f_i''}{\ell} \\ &\quad + \varepsilon\frac{\partial U}{\partial\ell}\sum_{i=1}^{N}\frac{(f_i')^2}{\ell} + \varepsilon\frac{\partial U}{\partial\ell}\sum_{i=1}^{N}\frac{(s_i - f_i)^2(f_i')^2}{\ell^4} + \varepsilon\frac{\partial^2 U}{\partial\ell^2}\left[\sum_{i=1}^{N}\frac{(s_i - f_i)f_i'}{\ell}\right]^2 \\ &\equiv \mathscr{E}_{12}. \end{aligned} \tag{3.31}$$

For further references, we also denoted $\mathscr{E}_{01}$ by

$$\mathscr{E}_{01} \equiv \mathscr{E}_{00} + \sqrt{\varepsilon}\mu^{-2}\lambda_0 e\Psi. \tag{3.32}$$

Hence, the components of the interior error are decomposed as

$$\begin{aligned} \mathscr{E}_1 &= \mathbb{S}(U + \sqrt{\varepsilon}\, e\Psi) \\ &= \mathscr{E}_{01} + \mathscr{E}_{11} + \mathscr{E}_{12} + B_2(U + \sqrt{\varepsilon}\, e\Psi) - \varepsilon(U + \sqrt{\varepsilon}\, e\Psi), \end{aligned} \tag{3.33}$$

(see (3.32), (3.30), (3.31), (3.27) and (2.15)).

We next analyze the error terms on the boundary. By defining a component of the boundary error at $z = 0$ in the form

$$\begin{aligned} \check{\mathscr{B}}_0(s) &= \sqrt{\varepsilon}\,\frac{\partial U}{\partial\mu}\mu' - \sqrt{\varepsilon}\sum_{i,j=1}^{N} D_{ij}\varphi_0\,(s_j - f_j)U_{s_i} \\ &\quad - \sqrt{\varepsilon}\left[\sum_{j=1}^{N} U_{s_j}f_j' + \sum_{i,j=1}^{N} D_{ij}\varphi_0\, U_{s_i}f_j\right], \end{aligned} \tag{3.34}$$

the errors on the boundary take the form, for $z = 0$

$$\begin{aligned} \check{\mathscr{B}}_1 &= \check{\mathscr{B}}_0 + \varepsilon e'\Psi - \varepsilon e\sum_{i,j=1}^{N} D_{ij}\varphi_0\,(s_j - f_j)\Psi_{s_i} - \varepsilon e\sum_{i=1}^{N}\Psi_{s_i}f_i' - \varepsilon e\sum_{i,j=1}^{N} D_{ij}\varphi_0\, f_j\Psi_{s_i} \\ &\quad + \varepsilon e\Psi_\mu\mu' - \frac{\varepsilon}{2}\sum_{i,j,k=1}^{N} D_{ijk}\varphi_0\, s_js_k\frac{\partial}{\partial s_i}(U + \sqrt{\varepsilon}\, e\Psi) \\ &\quad + \varepsilon\sum_{i=1}^{N}\left(\sum_{j=1}^{N} D_{ij}\varphi_1 s_j\right)^2\frac{\partial}{\partial z}(U + \sqrt{\varepsilon}\, e\Psi) + \widehat{D}_3(U + \sqrt{\varepsilon}\, e\Psi). \end{aligned} \tag{3.35}$$

For $z = 1/\sqrt{\varepsilon}$, there also holds a similar boundary error $\hat{\mathscr{B}}_1$ with a component $\hat{\mathscr{B}}_0$ of order $O(\sqrt{\varepsilon})$.

3.3. *Derivation of Suitable Boundary Conditions for Parameters*

Note that $\check{\mathscr{B}}_0$ and $\hat{\mathscr{B}}_0$ in the boundary error are of order $O(\sqrt{\varepsilon}\,)$, which is not good enough for our further setting. Even worse, the approximate kernel of the linear operator of the problem linearized at v_2 is spanned by

$$\mathbf{Z}_i(s, z) = \overline{\mathbf{Z}}_i\big(\mu(\sqrt{\varepsilon}\, z), s - f(\sqrt{\varepsilon}\, z)\big) = \frac{1}{1 + \varpi}\frac{\partial U}{\partial s_i}, \quad i = 1, \ldots, N,$$

$$\mathbf{Z}_{N+1}(s,z)=\overline{\mathbf{Z}}_{N+1}\big(\mu(\sqrt{\varepsilon}\,z),\,s-f(\sqrt{\varepsilon}\,z)\big)=\frac{1}{1+\varpi}\frac{\partial U}{\partial\mu},\qquad(3.36)$$
$$\mathbf{Z}_0(s,z)=\overline{\mathbf{Z}}_0\big(\mu(\sqrt{\varepsilon}\,z),\,s-f(\sqrt{\varepsilon}\,z)\big)=\Psi,$$

and we then observe that $\check{\mathscr{B}}_0$ and $\hat{\mathscr{B}}_0$ are not orthogonal to the approximate kernel. So we impose the following further restrictions on the boundary, say at $z=0$

$$\begin{aligned}&\int_{\mathbb{R}^N}\Big[\frac{\partial U}{\partial\mu}\mu'-\sum_{i,j=1}^{N}D_{ij}\varphi_0\,(s_j-f_j)U_{s_i}\Big]\mathbf{Z}_{N+1}\,\mathrm{d}s=\;0,\\&\int_{\mathbb{R}^N}\Big[\sum_{j}^{N}U_{s_j}f_j'+\sum_{i,j=1}^{N}D_{ij}\varphi_0\,U_{s_i}f_j\Big]\mathbf{Z}_n\,\mathrm{d}s=\;0,\quad\forall\,n=1,\ldots,N,\end{aligned}\qquad(3.37)$$

with similar formulas on $z=1/\sqrt{\varepsilon}$.

Now, we compute the first formula in (3.37). It is obvious that

$$\begin{aligned}\mu'(0)\int_{\mathbb{R}^N}\frac{\partial U}{\partial\mu}\mathbf{Z}_{N+1}\,\mathrm{d}s&=\;\mu'(0)\frac{1}{(1+\varpi)}\int_{\mathbb{R}^N}\Big|\frac{\partial U}{\partial\mu}\Big|^2\,\mathrm{d}s\\&=\;\mu'(0)\frac{(1+\varpi)(N-2)^2(\tau_N)^2}{4}\int_{\mathbb{R}^N}\frac{\mu^{N-4}\big(\ell^2-\mu^2\big)^2}{\big(\mu^2+\ell^2\big)^N}\,\mathrm{d}s\\&=\;\frac{(1+\varpi)(N-2)^2\tau_N^2}{4}\mu'(0)\int_{\mathbb{R}^N}\frac{\big(|t|^2-1\big)^2}{\big(|t|^2+1\big)^N}\,\mathrm{d}t.\end{aligned}$$

Note that U is an even function for the variable s. We get

$$\begin{aligned}&-\int_{\mathbb{R}^N}\sum_{i,j=1}^{N}D_{ij}\varphi_0(s_j-f_j)U_{s_i}\mathbf{Z}_{N+1}\,\mathrm{d}s\\&\quad=-\frac{(1+\varpi)(N-2)^2\tau_N^2}{2}\int_{\mathbb{R}^N}\sum_{i=1}^{N}D_{ii}\varphi_0\frac{\mu^{N-3}(s_i-f_i)^2\big(\ell^2-\mu^2\big)}{\big(\mu^2+\ell^2\big)^N}\,\mathrm{d}s\\&\quad=-\frac{(1+\varpi)(N-2)^2\tau_N^2}{2}\,\mu(0)\sum_{i=1}^{N}D_{ii}\varphi_0\int_{\mathbb{R}^N}\frac{t_i^2\big(|t|^2-1\big)}{\big(|t|^2+1\big)^N}\,\mathrm{d}t.\end{aligned}$$

It is easy to check

$$\int_{\mathbb{R}^N}\frac{t_i^2\big(|t|^2-1\big)}{\big(|t|^2+1\big)^N}\,\mathrm{d}t=\frac{1}{N}\left|S^{N-1}\right|\int_1^\infty\frac{r^{N+1}(r^2-1)(1-r^{-6})}{\big(|r|^2+1\big)^N}\,\mathrm{d}r>0.$$

Hence the first equation in (3.37) is equivalent to the following boundary condition for μ at $z=0$

$$\mu'(0)-b_{01}\,\kappa_0\,\mu(0)=0,\qquad(3.38)$$

where

$$\kappa_0 = \sum_{i=1}^{N} D_{ii}\varphi_0(0).$$

Similarly, we can impose the boundary condition for μ at $z = 1$ of the form

$$\mu'(1) - b_{11}\,\kappa_1\,\mu(1) = 0. \tag{3.39}$$

In the above, b_{01} and b_{11} are two positive constants independent of ε, while κ_0 and κ_1 are the mean curvatures of the boundary $\partial\Omega$ at the intersection points with Γ.

We turn to the second formula in (3.37). For any $n = 1, \ldots, N$, we get

$$\int_{\mathbb{R}^N}\left[\sum_{i,j=1}^{N} D_{ij}\varphi_0 U_{s_i} f_j\right]\mathbf{Z}_n\,\mathrm{d}s = \frac{1}{1+\varpi}\sum_{j=1}^{N} D_{nj}\varphi_0\,f_j(0)\int_{\mathbb{R}^N}\left|\frac{\partial U}{\partial s_n}\right|^2 \mathrm{d}s,$$

as well as the estimate

$$\int_{\mathbb{R}^N}\left[\sum_{j=1}^{N} U_{s_j} f_j'\right]\mathbf{Z}_n\,\mathrm{d}s = \frac{1}{1+\varpi}\,f_n'(0)\int_{\mathbb{R}^N}\left|\frac{\partial U}{\partial s_n}\right|^2 \mathrm{d}s.$$

Hence, the second equation in (3.37) is equivalent to the boundary condition for $f = (f_1, \ldots, f_N)$ at $z = 0$ of the form

$$f'(0) + D^2\varphi_0\,f(0) = 0. \tag{3.40}$$

Similarly, we impose the boundary condition for $f = (f_1, \ldots, f_N)$ at $z = 1$ of the form

$$f'(1) + D^2\varphi_1\,f(1) = 0. \tag{3.41}$$

For further references, we also impose that at $z = 0$

$$\int_{\mathbb{R}^N}\left[e'\Psi - e\sum_{i,j=1}^{N} D_{ij}\varphi_0\,(s_j - f_j)\Psi_{s_i}\right]\mathbf{Z}_0\,\mathrm{d}s = 0,$$

which is equivalent to

$$e'(0) + b_{03}\,\kappa_0\,e(0) = 0. \tag{3.42}$$

Similarly, we impose the boundary condition for e at $z = 1$ of the form

$$e'(1) + b_{13}\,\kappa_1\,e(1) = 0. \tag{3.43}$$

Note that b_{03} and b_{13} are two nonzero constants independent of ε.

Thus, although the terms $\check{\mathscr{B}}_0$ and $\hat{\mathscr{B}}_0$ are of order $O(\sqrt{\varepsilon})$, they satisfy the following useful properties of orthogonality, for $i = 1, \ldots, N+1$

$$\int_{\mathbb{R}^N}\check{\mathscr{B}}_0\mathbf{Z}_i\,\mathrm{d}s = \int_{\mathbb{R}^N}\hat{\mathscr{B}}_0\mathbf{Z}_i\,\mathrm{d}s = 0. \tag{3.44}$$

3.4. *The Improvement of the Approximation*

To fulfill the object of canceling the terms of order $O(\sqrt{\varepsilon})$ in $\check{\mathscr{B}}_0$ and $\hat{\mathscr{B}}_0$ on the boundary, we will follow the methods in [11] and [45] to get improvements of the approximate solution.

Recall the definitions of W_0 and $\widetilde{\mathbf{Z}}_0$ in (3.4). As we have done in [45], define two constants d_0, d_1 as

$$d_0 = \sum_{i=1}^{N} D_{ii}\varphi_0 \int_{\mathbb{R}^N} x_i W_{0,x_i} \widetilde{\mathbf{Z}}_0 \,\mathrm{d}x, \qquad d_1 = \sum_{i=1}^{N} D_{ii}\varphi_1 \int_{\mathbb{R}^N} x_i W_{0,x_i} \widetilde{\mathbf{Z}}_0 \,\mathrm{d}x, \tag{3.45}$$

and a function $A(z)$ as

$$\begin{aligned} A(z) = {} & \frac{d_0 \cos\left[\kappa/\sqrt{\varepsilon}\right] - \left[\mu(1)\, d_1\right]/\mu(0)}{\left(\sqrt{\lambda_0}/\mu(0)\right)\, \sin\left[\kappa/\sqrt{\varepsilon}\right]} \cos\left[\frac{\sqrt{\lambda_0}\, t\left(\sqrt{\varepsilon}\, z\right)}{\sqrt{\varepsilon}}\right] \\ & + \frac{d_0}{\sqrt{\lambda_0}/\mu(0)} \sin\left[\frac{\sqrt{\lambda_0}\, t\left(\sqrt{\varepsilon}\, z\right)}{\sqrt{\varepsilon}}\right], \end{aligned} \tag{3.46}$$

where t is a function and κ is a constant defined respectively by

$$t(\theta) = \int_0^{\theta} \frac{1}{\mu(\zeta)}\,\mathrm{d}\zeta \quad \text{with } \kappa = \sqrt{\lambda_0}\int_0^1 \frac{1}{\mu(\zeta)}\,\mathrm{d}\zeta, \tag{3.47}$$

in such a way that

$$A'' = -\lambda_0 \mu^{-2} A + \sqrt{\varepsilon}\,\mathbf{A}_0(z), \quad A'(0) = d_0, \quad A'(1/\sqrt{\varepsilon}) = d_1, \tag{3.48}$$

In the above, we have defined the term $\mathbf{A}_0$ as

$$\begin{aligned} \mathbf{A}_0(z) = {} & \frac{d_0 \cos\left[\kappa/\sqrt{\varepsilon}\right] - \left[\mu(1)\, d_1\right]/\mu(0)}{\left(\sqrt{\lambda_0}/\mu(0)\right)\, \sin\left[\kappa/\sqrt{\varepsilon}\right]} \times \sqrt{\lambda_0}\, \mu^{-2}\mu' \sin\left[\frac{\sqrt{\lambda_0}\, t\left(\sqrt{\varepsilon}\, z\right)}{\sqrt{\varepsilon}}\right] \\ & + \frac{d_0}{\sqrt{\lambda_0}/\mu(0)} \times \sqrt{\lambda_0}\, \mu^{-2}\mu' \cos\left[\frac{\sqrt{\lambda_0}\, t\left(\sqrt{\varepsilon}\, z\right)}{\sqrt{\varepsilon}}\right]. \end{aligned} \tag{3.49}$$

The spectral gap condition in (1.20) and the uniform positivity of μ imply that A is bounded with respect to the parameter ε.

We define a smooth extension of the boundary error in the translated coordinates (x, z) and get a function g defined on the whole strip $\mathfrak{S}_0$. In other words, we define

$$\begin{aligned} g(x,z) = {} & \Bigg(\widetilde{\mathbf{Z}}_{N+1}(x)\mu'(0) - \mu(0) \sum_{i,j=1}^{N} D_{ij}\varphi_0\, x_j \widetilde{\mathbf{Z}}_i(x) - d_0\, \widetilde{\mathbf{Z}}_0(x) \\ & - \sum_{i,j=1}^{N} D_{ij}\varphi_0\, \widetilde{\mathbf{Z}}_i(x) f_j(0) - \sum_{j=1}^{N} \widetilde{\mathbf{Z}}_j(x) f_j'(0) \Bigg) \tilde{\eta}_0(2\sqrt{\varepsilon} z) \end{aligned} \tag{3.50}$$

$$+\left(\widetilde{\mathbf{Z}}_{N+1}(x)\mu'(1)-\mu(1)\sum_{i,j=1}^{N}D_{ij}\varphi_1\,x_j\widetilde{\mathbf{Z}}_i(x)-d_1\,\widetilde{\mathbf{Z}}_0(x)\right.$$
$$\left.-\sum_{j=1}^{N}\widetilde{\mathbf{Z}}_j(x)f_j'(1)-\sum_{i,j=1}^{N}D_{ij}\varphi_1\,\widetilde{\mathbf{Z}}_i(x)f_j(1)\right)\tilde{\eta}_1(2\sqrt{\varepsilon}z),$$

with suitable cutoff functions $\tilde{\eta}_0$ and $\tilde{\eta}_1$, in such a way that g satisfies the estimate

$$\|g\|_{**}\leq C,$$

with a generic constant C independent of ε, and also satisfies the boundary constraints

$$\big(\mu(\sqrt{\varepsilon}z)\big)^{-N/2}g\left(\frac{s-f}{\mu(\sqrt{\varepsilon}z)},\,z\right)+d_0\Psi=\frac{1}{1+\varpi}\check{\mathscr{B}}_0(s)\quad\text{for } z=0,$$
$$\big(\mu(\sqrt{\varepsilon}z)\big)^{-N/2}g\left(\frac{s-f}{\mu(\sqrt{\varepsilon}z)},\,z\right)+d_1\Psi=\frac{1}{1+\varpi}\hat{\mathscr{B}}_0(s)\quad\text{for } z=1/\sqrt{\varepsilon}.$$

In the last formula, we have used the relations in (3.10). We further make a decomposition of g of the form

$$g(x,z)=\sum_{i=0}^{N}g_i(x,z)+\sum_{i\neq j}\widetilde{g}_{ij}(x,z),\tag{3.51}$$

where g_0 is an even function in the variable x

$$g_0(x,z)=\left(\widetilde{\mathbf{Z}}_{N+1}(x)\mu'(0)-\mu(0)\sum_{i=1}^{N}D_{ii}\varphi_0\,x_i\widetilde{\mathbf{Z}}_i(x)-d_0\,\widetilde{\mathbf{Z}}_0(x)\right)\tilde{\eta}_0(2\sqrt{\varepsilon}z)$$
$$+\left(\widetilde{\mathbf{Z}}_{N+1}(x)\mu'(1)-\mu(1)\sum_{i=1}^{N}D_{ii}\varphi_1\,x_i\widetilde{\mathbf{Z}}_i(x)-d_1\,\widetilde{\mathbf{Z}}_0(x)\right)\tilde{\eta}_1(2\sqrt{\varepsilon}z),$$

and for fixed $i=1,\ldots,N,\ g_i$ is an odd function in the variable x_i

$$g_i(x,z)=\left(-\sum_{j=1}^{N}D_{ij}\varphi_0\,f_j(0)\,\widetilde{\mathbf{Z}}_i(x)-\widetilde{\mathbf{Z}}_i(x)f_i'(0)\right)\tilde{\eta}_0(2\sqrt{\varepsilon}z)$$
$$+\left(-\sum_{j=1}^{N}D_{ij}\varphi_1\,f_j(1)\,\widetilde{\mathbf{Z}}_i(x)-\widetilde{\mathbf{Z}}_i(x)f_i'(1)\right)\tilde{\eta}_1(2\sqrt{\varepsilon}z),$$

as well as for fixed $i\neq j,\ i,j=1,\ldots,N$ the function $\widetilde{g}_{ij}$ is odd both in the variables x_i and x_j

$$\widetilde{g}_{ij}(x,z)=-\mu(0)D_{ij}\varphi_0\,x_j\widetilde{\mathbf{Z}}_i(x)\tilde{\eta}_0(2\sqrt{\varepsilon}z)-\mu(1)D_{ij}\varphi_1\,x_j\widetilde{\mathbf{Z}}_i(x)\tilde{\eta}_1(2\sqrt{\varepsilon}z).$$

It is worth to mention that the requirements (3.38)–(3.39) and (3.40)–(3.41) imply that

$$\int_{\mathbb{R}^N} g_i(x,z)\widetilde{\mathbf{Z}}_k(x)\,\mathrm{d}x = 0, \quad \forall\, i = 0,\ldots,N,\ k = 0,\ldots,N+1. \tag{3.52}$$

$$\int_{\mathbb{R}^N} \widetilde{g}_{ij}(x,z)\widetilde{\mathbf{Z}}_k(x)\,\mathrm{d}x = 0, \quad \forall\, i,j = 1,\ldots,N \text{ and } \quad i \neq j, \quad k = 0,\ldots,N+1. \tag{3.53}$$

Here we also use the relations in (3.6) and the definition of d_0 and d_1 in (3.45).

For given g in (3.50), we consider the problem

$$\Delta_x \Phi + \mu^2 \Phi_{zz} + p W_0{}^{p-1}\Phi = 0 \quad \text{in } \widetilde{\mathfrak{S}}, \qquad \frac{\partial \Phi}{\partial z} = g \quad \text{on } \partial\widetilde{\mathfrak{S}}, \tag{3.54}$$

under the conditions

$$\int_{\mathbb{R}^N} \Phi(x,z)\widetilde{\mathbf{Z}}_i(x)\,\mathrm{d}x = 0, \quad i = 0,\ldots,N+1. \tag{3.55}$$

The resolution theory of the above problem reads:

Lemma 3.2. *There exists a solution to problem* (3.54)–(3.55) *such that*

$$\|\Phi\|_* \leq C\|g\|_{**}. \tag{3.56}$$

We will give the proof of this lemma after that of Lemma 5.2.

By the decomposition in (3.51) and the orthogonality conditions in (3.52), we can get functions Φ_i and Φ_{ij} by solving (3.54)–(3.55) with g replaced by g_i and $\widetilde{g}_{ij}$ respectively. Hence we obtain a decomposition of Φ

$$\Phi = \sum_{i=0}^{N} \Phi_i + \sum_{i\neq j,\, i,j=1}^{N} \Phi_{ij}, \tag{3.57}$$

where Φ_0 is an even function in the variable x, Φ_i is an odd function in the variable x_i, $i = 1,\ldots,N$ and for fixed $i \neq j$, Φ_{ij} are odd in the variables x_i and x_j.

Set the correction terms of the form

$$\begin{aligned}
\phi_1(s,z) &= \big(\mu(\sqrt{\varepsilon}z)\big)^{-N/2}\Phi\left(\frac{s - f(\sqrt{\varepsilon}\,z)}{\mu(\sqrt{\varepsilon}z)},\, z\right),\\
\phi_2(s,z) &= \big(\mu(\sqrt{\varepsilon}z)\big)^{-(N-2)/2} Z_0\left(\frac{s - f(\sqrt{\varepsilon}\,z)}{\mu(\sqrt{\varepsilon}z)}\right) A(z) = A(z)\Psi(s,z).
\end{aligned} \tag{3.58}$$

One checks that $\phi^* = \phi_1 + \phi_2$ solves the following problem

$$\begin{gathered}
\Delta_s \phi^* + \phi^*_{zz} + p W^{p-1}\phi^* = \mathscr{E}_2 \quad \text{in } \widetilde{\mathfrak{S}},\\
\frac{\partial \phi^*}{\partial z} = \frac{\hat{\mathscr{B}}_0}{\sqrt{\varepsilon}\,(1+\varpi)} + \hat{\mathscr{B}}_2 \quad \text{on } \partial_1\widetilde{\mathfrak{S}}, \qquad \frac{\partial \phi^*}{\partial z} = \frac{\check{\mathscr{B}}_0}{\sqrt{\varepsilon}\,(1+\varpi)} + \check{\mathscr{B}}_2 \quad \text{on } \partial_0\widetilde{\mathfrak{S}}.
\end{gathered} \tag{3.59}$$

Moreover, there hold the relations

$$\int_{\mathbb{R}^N} \phi_1 \mathbf{Z}_i \, \mathrm{d}s = 0, \quad i = 0, \ldots, N+1. \tag{3.60}$$

By using the expression of $\mathbf{A}_0$ in (3.49), the error $\mathscr{E}_2$ is

$$\begin{aligned}
\mathscr{E}_2 = & -2\sqrt{\varepsilon}\mu^{-(N+4)/2}\nabla_x \Phi_z \cdot \big(f'\mu + (s-f)\mu'\big) - N\sqrt{\varepsilon}\mu^{-(N+2)/2}\mu'\Phi_z \\
& - \varepsilon\nabla_s\phi_1 \cdot \big[f''\mu + (s-f)\mu'' - 2f'\mu' - 2(s-f)\mu^{-1}(\mu')^2 \big] \\
& + \varepsilon\triangle_s\phi_1\, \mu^{-4} \Big| f'\mu + (s-f)\mu' \Big|^2 + \varepsilon N\mu'\mu^{-3}\triangle_s\phi_1 \big[f'\mu + (s-f)\mu' \big] \\
& - \frac{N}{2}\,\varepsilon\mu''\mu^{-1}\phi_1 + \frac{N(N+2)}{4}\,\varepsilon|\mu'|^2\mu^{-2}\phi_1 \\
& - 2\sqrt{\varepsilon}\mu^{-(N+2)/2}\nabla_x Z_0 \cdot \big(f'\mu + (s-f)\mu'\big)A_z - (N-2)\sqrt{\varepsilon}\mu^{-N/2}\mu' Z_0 A_z \\
& + \sqrt{\varepsilon}\,\mathbf{A}_0 - \varepsilon\nabla_s\phi_2 \cdot \big[f''\mu + (s-f)\mu'' - 2f'\mu' - 2(s-f)\mu^{-1}(\mu')^2 \big] \\
& + \varepsilon\triangle_s\phi_2\, \mu^{-4} \Big| f'\mu + (s-f)\mu' \Big|^2 + \varepsilon(N-2)\mu'\mu^{-3}\triangle_s\phi_2 \big[f'\mu + (s-f)\mu' \big] \\
& - \frac{N-2}{2}\,\varepsilon\mu''\mu^{-1}\phi_2 + \frac{N(N-2)}{4}\,\varepsilon|\mu'|^2\mu^{-2}\phi_2,
\end{aligned} \tag{3.61}$$

and for $z = 1/\sqrt{\varepsilon}$ the boundary error is

$$\begin{aligned}
\hat{\mathscr{B}}_2 = & -\sqrt{\varepsilon}\,\mu^{-2}\nabla_x\phi_1 \cdot \big(f'\mu + (s-f)\mu'\big) - \frac{N}{2}\sqrt{\varepsilon}\mu^{-1}\mu'\phi_1 \\
& - \sqrt{\varepsilon}\,\mu^{-2}\nabla_x\phi_2 \cdot \big(f'\mu + (s-f)\mu'\big) - \frac{N}{2}\sqrt{\varepsilon}\mu^{-1}\mu'\phi_2,
\end{aligned} \tag{3.62}$$

with a similar expression of $\check{\mathscr{B}}_2$.

We also have the following estimates

$$\|\mathscr{E}_2\|_{**} \leq \varepsilon^{1/2}, \quad \|\hat{\mathscr{B}}_2\|_{**} \leq \varepsilon^{1/2}, \quad \|\check{\mathscr{B}}_2\|_{**} \leq \varepsilon^{1/2}.$$

Hence, we define the *basic approximate solution* as the function given by

$$v_3(s,z) = U(s,z) + \sqrt{\varepsilon}\, e\big(\sqrt{\varepsilon}\, z\big)\Psi(s,z) + \sqrt{\varepsilon}\,(1+\varpi)\phi^*(s,z). \tag{3.63}$$

3.5. *Local Setting Up of the Problem*

As we mentioned in Section 2, if we look for a solution of the form $v = v_3 + \phi$ locally close to Γ_ε, then the local problem (2.12)–(2.14) can be expanded as follows

$$\mathbb{S}(v_3 + \phi) = \mathbb{S}(v_3) + L(\phi) - \varepsilon\phi + B_2(\phi) + N(\phi) = 0, \tag{3.64}$$

with boundary condition

$$\check{D}_4(\phi) + \phi_z + \check{D}_3(v_3 + \phi) = \check{\mathscr{B}}_3 \quad \text{for } z = 0, \tag{3.65}$$

$$\widehat{D}_4(\phi) + \phi_z + \widehat{D}_3(v_3 + \phi) = \hat{\mathscr{B}}_3 \quad \text{for } z = 1/\sqrt{\varepsilon}. \tag{3.66}$$

Here, by recalling the notation in (2.14), (2.16) and (2.17), we have denoted

$$L(\phi) = \Delta_s \phi + \phi_{zz} + p v_3^{p-1} \phi, \tag{3.67}$$

$$N(\phi) = \mathbb{N}(v_3 + \phi) - \mathbb{N}(v_3) - p v_3^{p-1} \phi, \tag{3.68}$$

$$\begin{aligned} \check{D}_4(\phi) = &-\sqrt{\varepsilon} \sum_{i,j=1}^{N} D_{ij}\varphi_0 \, s_j \, \phi_{s_i} - \frac{1}{2}\varepsilon \sum_{i,j,k=1}^{N} D_{ijk}\varphi_0 \, s_j s_k \, \phi_{s_i} \\ &+ \varepsilon \sum_{i=1}^{N} \left(\sum_{j=1}^{N} D_{ij}\varphi_0 s_j \right)^2 \phi_z, \end{aligned} \tag{3.69}$$

with a similar expression for $\widehat{D}_4(\phi)$ at $z = 1/\sqrt{\varepsilon}$.

The new boundary error takes the form, at $z = 0$

$$\begin{aligned} -\check{\mathscr{B}}_3(x) = {} & \varepsilon e' \Psi - \varepsilon e \sum_{i,j=1}^{N} D_{ij}\varphi_0 \, (s_j - f_j) \Psi_{s_i} - \varepsilon e \sum_{i=1}^{N} \Psi_{s_i} f_i' + \varepsilon e \Psi_\mu \mu' \\ & - \varepsilon e \sum_{i,j=1}^{N} D_{ij}\varphi_0 \, f_j \Psi_{s_i} - \varepsilon \sum_{i,j=1}^{N} D_{ij}\varphi_0 \, s_j (1 + \varpi) \big(\phi_{1,s_i} + \phi_{2,s_i} \big) \\ & + \sqrt{\varepsilon}\,(1 + \varpi) \hat{\mathscr{B}}_2 - \frac{\varepsilon}{2} \sum_{i,j,k=1}^{N} D_{ijk}\varphi_0 \, s_j s_k \Big(U + \sqrt{\varepsilon}\, e\Psi + \sqrt{\varepsilon}\,(1 + \varpi)\phi^* \Big) \\ & + \varepsilon \sum_{i=1}^{N} \Big(D_{ij}\varphi_0 \Big)^2 \frac{\partial}{\partial z} \Big(U + \sqrt{\varepsilon}\, e\Psi + \sqrt{\varepsilon}\,(1 + \varpi)\phi^* \Big) \\ & + \check{D}_3 \Big(U + \sqrt{\varepsilon}\, e\Psi + \sqrt{\varepsilon}\,(1 + \varpi)\phi^* \Big), \end{aligned} \tag{3.70}$$

with a similar boundary error $\hat{\mathscr{B}}_3$ at $z = 1/\sqrt{\varepsilon}$. The interior error of the approximation is

$$\begin{aligned} \mathscr{E}_3 &= \mathbb{S}(v_3) \\ &= \mathscr{E}_1 + \sqrt{\varepsilon}\,(1 + \varpi)\mathscr{E}_2 + \sqrt{\varepsilon}\, B_2 \Big((1 + \varpi)\phi^* \Big) - \varepsilon^{3/2}\,(1 + \varpi)\phi^* + \mathbf{N}(\phi^*), \end{aligned} \tag{3.71}$$

where $\mathscr{E}_1$ is defined in (3.33) and the operator B_2 is given by (2.15). Moreover, the nonlinear term $\mathbf{N}$ in the error $\mathscr{E}_3$ can be written as of the form

$$\begin{aligned} \mathbf{N}(\phi^*) &= \mathbb{N}(v_3) - \mathbb{N}(v_2) - p U^{p-1} \sqrt{\varepsilon}\,(1 + \varpi)\phi^* \\ &= \big(U + \sqrt{\varepsilon} e\Psi + \sqrt{\varepsilon}\,(1 + \varpi)\phi^* \big)^p \\ &\quad - \big(U + \sqrt{\varepsilon} e\Psi \big)^p - p \big(U + \sqrt{\varepsilon} e\Psi \big)^{p-1} \sqrt{\varepsilon}\,(1 + \varpi)\phi^* \\ &\quad + p \big[\big(U + \sqrt{\varepsilon} e\Psi \big)^{p-1} - U^{p-1} \big] \sqrt{\varepsilon}\,(1 + \varpi)\phi^* \\ &\quad + \varepsilon \big(U + \sqrt{\varepsilon} e\Psi \big)^p \log(U + \sqrt{\varepsilon} e\Psi) \\ &\quad - \varepsilon \Big(U + \sqrt{\varepsilon} e\Psi + \sqrt{\varepsilon}\,(1 + \varpi)\phi^* \Big)^p \log \Big(U + \sqrt{\varepsilon} e\Psi + \sqrt{\varepsilon}\,(1 + \varpi)\phi^* \Big) \end{aligned}$$

$$
\begin{aligned}
&+\frac{N-2}{4}\varepsilon(\log\varepsilon)\Big(U+\sqrt{\varepsilon}e\Psi+\sqrt{\varepsilon}\,(1+\varpi)\phi^*\Big)^p\\
&-\frac{N-2}{4}\varepsilon(\log\varepsilon)\big(U+\sqrt{\varepsilon}e\Psi\big)^p\\
&+O(\varepsilon^2|\log\varepsilon|^2)\Big(U+\sqrt{\varepsilon}e\Psi+\sqrt{\varepsilon}\,(1+\varpi)\phi^*\Big)^p\\
&\times\mathbb{N}_0\Big(U+\sqrt{\varepsilon}e\Psi+\sqrt{\varepsilon}\,(1+\varpi)\phi^*\Big)\\
&-O(\varepsilon^2|\log\varepsilon|^2)\big(U+\sqrt{\varepsilon}e\Psi\big)^p\,\mathbb{N}_0\big(U+\sqrt{\varepsilon}e\Psi\big).
\end{aligned}
\tag{3.72}
$$

For further references, we also decompose

$$
\mathscr{E}_3=\mathscr{E}_{31}+\mathscr{E}_{32},
\tag{3.73}
$$

with the notation

$$
\mathscr{E}_{31}=\varepsilon^{3/2}e''\Psi+\sqrt{\varepsilon}\,\mu^{-2}\,\lambda_0 e\Psi \quad \text{and} \quad \mathscr{E}_{32}=\mathscr{E}_3-\mathscr{E}_{31}.
\tag{3.74}
$$

This decomposition will be useful in the future solvability theory of the full nonlinear problem of differential equations that we will deal with in Section 6.

3.6. *The Accuracy of the Error*

For the estimate $\|\mathbf{N}(\phi^*)\|_{**}$, we first consider a component

$$
\begin{aligned}
\mathbf{N}_1(\phi^*)=&\Big(U+\sqrt{\varepsilon}\,e\Psi+\sqrt{\varepsilon}\,(1+\varpi)\phi^*\Big)^p\\
&-\big(U+\sqrt{\varepsilon}\,e\Psi\big)^p-p\sqrt{\varepsilon}\big(U+\sqrt{\varepsilon}\,e\Psi\big)^{p-1}(1+\varpi)\phi^*.
\end{aligned}
$$

If $|x|\equiv|s-f|\le\delta\varepsilon^{-1/2}$, we have that

$$
\big|\mathbf{N}_1(\phi^*)\big|\le CU^{p-2}\big|\sqrt{\varepsilon}\,(1+\varpi)\phi^*\big|^2.
$$

Then in this region, we have that

$$
\sup_{|x|\le\delta\varepsilon^{-1/2}}\Big|(1+|x|)^{N-2}\mathbf{N}_1(\phi^*)\Big|\le C\varepsilon.
$$

If $|x|\equiv|s-f|>\delta\varepsilon^{-1/2}$, then

$$
\big|\mathbf{N}_1(\phi^*)\big|\le C\big|\sqrt{\varepsilon}\,\phi^*\big|^p.
$$

Then in this region, we have that

$$
\sup_{|x|>\delta\varepsilon^{-1/2}}\Big|(1+|x|)^{N-2}\mathbf{N}_1(\phi^*)\Big|\le C\varepsilon^{p/2}\sup_{|x|>\delta\varepsilon^{-1/2}}\Big|(1+|x|)^{-2+\frac{8}{N-2}}\Big|\le C\varepsilon.
$$

Other terms can be estimated in a similar way and we get

$$
\|\mathbf{N}(\phi^*)\|_{**}\le C\varepsilon.
$$

From the uniform bound of e in (3.19), it is easy to see that

$$\|\mathscr{E}_{31}\|_{**} \le C\varepsilon^{\frac{1}{2}}. \tag{3.75}$$

Since $\sqrt{\varepsilon}\,\phi^*$ and $\sqrt{\varepsilon}\,e\Psi$ are of size $O(\sqrt{\varepsilon}\,)$, all terms in $\mathscr{E}_{32}$ carry ε in front. One checks

$$\|\mathscr{E}_{32}\|_{**} \le C\varepsilon. \tag{3.76}$$

Similarly, we have the following estimates

$$\|\hat{\mathscr{B}}_3\|_{**} + \|\check{\mathscr{B}}_3\|_{**} \le C\varepsilon. \tag{3.77}$$

Note that these errors take the unknown functions μ, e, f as parameters. Direct computations will give that

$$\begin{aligned} &\|\check{\mathscr{B}}_3(\mu_1, e_1, f_1) - \check{\mathscr{B}}_3(\mu_2, e_2, f_2)\|_{**} + \|\hat{\mathscr{B}}_3(\mu_1, e_1, f_1) - \hat{\mathscr{B}}_3(\mu_2, e_2, f_2)\|_{**} \\ &\quad \le C\varepsilon\big[\, \|f_1 - f_2\|_a + \|e_1 - e_2\|_b + \|\mu_1 - \mu_2\|_c \,\big]. \end{aligned} \tag{3.78}$$

and

$$\begin{aligned} &\|\mathscr{E}_{32}(\mu_1, e_1, f_1) - \mathscr{E}_{32}(\mu_2, e_2, f_2)\|_{**} \\ &\quad \le C\varepsilon\big[\, \|f_1 - f_2\|_a + \|e_1 - e_2\|_b + \|\mu_1 - \mu_2\|_c \,\big]. \end{aligned} \tag{3.79}$$

4. The Gluing Procedure

In this section, we use a gluing technique (as in [12, 13], to reduce the problem (2.2) in Ω_ε to a projected nonlinear problem on the infinite strip $\mathfrak{S}$ defined in (3.23) with the coordinates (s, z) defined in (2.5).

Let $\delta < \delta_0/100$ be a fixed number, where δ_0 is a constant defined in (2.5). We consider a smooth cut-off function $\eta_\delta(t)$ where $t \in \mathbb{R}_+$ such that $\eta_\delta(t) = 1$ for $0 \le t \le \delta$ and $\eta(t) = 0$ for $t > 2\delta$. Set $\eta_\delta^\varepsilon(s) = \eta_\delta(\sqrt{\varepsilon}\,|s|)$, where s is the normal coordinate to Γ_ε. Let $v_3(s, z)$ denote the approximate solution defined in (3.63) and constructed near the curve Γ_ε in the coordinates (s, z), which were introduced in (2.5). We define our first global approximation to be simply

$$\mathbb{W} = \eta_{3\delta}^\varepsilon(s) v_3. \tag{4.1}$$

In the coordinates $(y_1, \dots, y_{N+1})$ introduced in (2.2), $\mathbb{W}$ is a function defined on Ω_ε which is extended globally as 0 beyond the $6\delta/\varepsilon$-neighborhood of Γ_ε.

For $v = \mathbb{W} + \hat{\phi}$ where $\hat{\phi}$ is globally defined in Ω_ε, we call

$$S(v) = \triangle_y v - \varepsilon v + \varepsilon^{\frac{(N-2)\varepsilon}{4}} v^{p-\varepsilon} \quad \text{in } \Omega_\varepsilon.$$

Then v satisfies (2.2) if and only if

$$\widetilde{\mathscr{L}}(\hat{\phi}) = -\widetilde{E} - \widetilde{N}(\hat{\phi}) \quad \text{in } \Omega_\varepsilon, \tag{4.2}$$

with boundary condition

$$\frac{\partial \hat{\phi}}{\partial \nu_\varepsilon} + \frac{\partial \mathbb{W}}{\partial \nu_\varepsilon} = 0 \quad \text{on } \partial\Omega_\varepsilon, \tag{4.3}$$

where we have denoted

$$\widetilde{E} = S(\mathbb{W}), \qquad \widetilde{\mathscr{L}}(\hat{\phi}) = \triangle_y \hat{\phi} - \varepsilon\hat{\phi} + p\mathbb{W}^{p-1}\hat{\phi},$$
$$\widetilde{N}(\hat{\phi}) = \mathbb{N}\big(\mathbb{W} + \hat{\phi}\big) - \mathbb{N}\big(\mathbb{W}\big) - p\mathbb{W}^{p-1}\hat{\phi}.$$

In the above formula $\mathbb{N}$ is the nonlinear operator defined in (2.16). We further separate $\hat{\phi}$ in the following form

$$\hat{\phi} = \eta_{3\delta}^\varepsilon \phi + \psi,$$

where, in the coordinates (s, z) of the form (2.5), we assume that ϕ is defined in the whole strip $\mathfrak{S}$ (see (3.23)). Obviously, (4.2)–(4.3) is equivalent to the following system of differential equations in ψ and ϕ

$$\eta_{3\delta}^\varepsilon\Big(\triangle_y\phi - \varepsilon\phi + p\mathbb{W}^{p-1}\phi\Big) = \eta_\delta^\varepsilon\Big[-\widetilde{N}(\eta_{3\delta}^\varepsilon\phi + \psi) - \widetilde{E} - p\mathbb{W}^{p-1}\psi\Big], \tag{4.4}$$

$$\begin{aligned}\triangle_y\psi - \varepsilon\psi + (1-\eta_\delta^\varepsilon)p\mathbb{W}^{p-1}\psi = {} & -(\triangle_y\eta_{3\delta}^\varepsilon)\phi - 2(\nabla_y\eta_{3\delta}^\varepsilon)(\nabla_y\phi)\\ & -(1-\eta_\delta^\varepsilon)\widetilde{N}(\eta_\delta^\varepsilon\phi + \psi) - -(1-\eta_\delta^\varepsilon)\widetilde{E}.\end{aligned} \tag{4.5}$$

On the boundary, we get the boundary conditions

$$\eta_{3\delta}^\varepsilon\frac{\partial\phi}{\partial\nu_\varepsilon} + \eta_\delta^\varepsilon\frac{\partial\mathbb{W}}{\partial\nu_\varepsilon} = 0\,, \tag{4.6}$$

$$\frac{\partial\psi}{\partial\nu_\varepsilon} + (1-\eta_\delta^\varepsilon)\frac{\partial\mathbb{W}}{\partial\nu_\varepsilon} + \sqrt{\varepsilon}\,\frac{\partial\eta_{3\delta}^\varepsilon}{\partial\nu_\varepsilon}\phi = 0\,. \tag{4.7}$$

The key observation is that, for given ϕ, if we solve (4.5) and (4.7) in ψ, we substitute back in (4.4)–(4.6), we get that the problem can be transformed to the following nonlinear problem in the unknown ϕ involving the parameter ψ on $\mathfrak{S}$

$$\widetilde{\mathscr{L}}(\phi) = \eta_\delta^\varepsilon\Big[-\widetilde{N}(\eta_{3\delta}^\varepsilon\phi + \psi) - \widetilde{E} - p\mathbb{W}^{p-1}\psi\Big] \quad \text{in } \mathfrak{S}, \tag{4.8}$$

$$\frac{\partial\phi}{\partial\nu} + \eta_\delta^\varepsilon\frac{\partial\mathbb{W}}{\partial\nu} = 0 \quad \text{on } \partial\mathfrak{S}. \tag{4.9}$$

Notice that the operator $\widetilde{\mathscr{L}}$ in Ω_ε may be taken as any compatible extension outside the $6\delta/\varepsilon$-neighborhood of Γ_ε in the strip $\mathfrak{S}$ and the operator $\frac{\partial}{\partial\nu}$ may be taken as any compatible extension outside the $6\delta/\varepsilon$-neighborhood of Γ_ε on the boundary $\partial\mathfrak{S}$.

First, we solve, given a small ϕ, problem (4.5) and (4.7) for ψ. The solvability can be done in the following way: let us observe that $\mathbb{W}$ is small for $|s| > \delta/\varepsilon$, where s is the normal coordinate to Γ_ε. Then the problem

$$\triangle\psi - \big[\varepsilon - (1-\eta_\delta^\varepsilon)p\mathbb{W}^{p-1}\big]\psi = h \quad \text{in } \Omega_\varepsilon,$$
$$\frac{\partial\psi}{\partial\nu_\varepsilon} = -(1-\eta_\delta^\varepsilon)\frac{\partial\mathbb{W}}{\partial\nu_\varepsilon} - \sqrt{\varepsilon}\,\frac{\partial\eta_{3\delta}^\varepsilon}{\partial\nu_\varepsilon}\phi \quad \text{on } \partial\Omega_\varepsilon,$$

has a unique bounded solution ψ whenever $\|h\|_\infty \leq +\infty$. Moreover, there holds

$$\|\psi\|_\infty \leq C\|h\|_\infty.$$

Let us observe that, for instance

$$\|(\triangle_y \eta^\varepsilon_{3\delta})\phi\|_\infty \ \leq\ C\varepsilon\|\phi\|_{L^\infty(|y|>\delta/\sqrt{\varepsilon})},$$

and

$$\|(\nabla_y \eta^\varepsilon_{3\delta})\nabla_y\phi\|_\infty \ \leq\ C\sqrt{\varepsilon}\,\|\nabla\phi\|_{L^\infty(|y|>\delta/\sqrt{\varepsilon})}.$$

Since $\widetilde{N}$ is power-like with power greater than one, a direct application of contraction mapping principle yields that (4.5) and (4.7) has a unique (small) solution $\psi = \psi(\phi)$ with

$$\|\psi(\phi)\|_{L^\infty} \leq C\sqrt{\varepsilon}\big[\,\|\phi\|_{L^\infty(|s|>\delta/\varepsilon)} + \|\nabla\phi\|_{L^\infty(|s|>\delta/\varepsilon)}\,\big] + \|\widetilde{E}\|_{L^\infty(|s|>\delta/\varepsilon)}, \tag{4.10}$$

where $|s| > \delta/\varepsilon$ denotes the complement in Ω_ε of δ/ε-neighborhood of Γ_ε. Moreover, the nonlinear operator ψ satisfies a Lipschitz condition of the form

$$\|\psi(\phi_1) - \psi(\phi_2)\|_{L^\infty} \leq C\sqrt{\varepsilon}\big[\,\|\phi_1 - \phi_2\|_{L^\infty(|s|>\delta/\varepsilon)} + \|\nabla\phi_1 - \nabla\phi_2\|_{L^\infty(|s|>\delta/\varepsilon)}\,\big]. \tag{4.11}$$

Therefore, from the above discussion, the full problem has been reduced to solving the following (nonlocal) problem in the infinite strip $\mathfrak{S}$

$$\mathscr{L}(\phi) = \eta^\varepsilon_\delta\Big[-\widetilde{N}(\eta^\varepsilon_{3\delta}\phi + \psi(\phi)) - \widetilde{E} - p\mathbb{W}^{p-1}\psi(\phi)\Big] \quad \text{in } \mathfrak{S}, \tag{4.12}$$

$$\mathscr{B}(\phi) + \eta^\varepsilon_\delta\frac{\partial\mathbb{W}}{\partial\nu_\varepsilon} = 0 \quad \text{on } \partial\mathfrak{S}. \tag{4.13}$$

Here $\mathscr{L}$ denotes a linear operator that coincides with $\widetilde{\mathscr{L}}$ on the region $|s| < 8\delta/\varepsilon$, $\mathscr{B}$ denotes the outward normal derivatives of $\mathfrak{S}$ that coincides with outward normal $\frac{\partial}{\partial\nu_\varepsilon}$ of Ω_ε on the region $|s| < 8\delta/\varepsilon$.

The definitions of these operators can be showed as follows. The local form of the problem (4.12)–(4.13) for $|s| < 8\delta/\varepsilon$ is given in coordinates (s, z) by formula (3.64)–(3.66). We extend it for functions ϕ defined in the whole strip $\mathfrak{S}$ in terms of (s, z) as the following

$$\begin{aligned}\mathscr{L}(\phi) &= \Delta_s\phi + \phi_{zz} + pv_3{}^{p-1}\phi - \varepsilon\phi + \eta^\varepsilon_{6\delta}B_2(\phi)\\ &= L(\phi) - \varepsilon\phi + \eta^\varepsilon_{6\delta}B_2(\phi) \quad \text{in } \mathfrak{S},\end{aligned} \tag{4.14}$$

where L and B_2 are the operators defined in (3.67) and (2.15). Similarly, the boundary conditions can be written as

$$\begin{aligned}\phi_z + \eta^\varepsilon_{6\delta}\check{D}_4(\phi) + \eta^\varepsilon_{6\delta}\check{D}_3(\phi + v_3) &= \eta^\varepsilon_\delta\,\check{\mathscr{B}}_3 \quad \text{on } \partial_0\mathfrak{S},\\ \phi_z + \eta^\varepsilon_{6\delta}\widehat{D}_4(\phi) + \eta^\varepsilon_{6\delta}\widehat{D}_3(\phi + v_3) &= \eta^\varepsilon_\delta\,\hat{\mathscr{B}}_3 \quad \text{on } \partial_1\mathfrak{S},\end{aligned} \tag{4.15}$$

where the operators $\check{D}_4$ and $\widehat{D}_4$ are defined in (3.69), as well as $\check{D}_3$ and $\widehat{D}_3$ in (2.14). The boundary errors of local form $\check{\mathscr{B}}_3$ and $\hat{\mathscr{B}}_3$ are also given in (3.70).

Recall the approximate kernel of the linear operator defined in (3.36) and $\mathcal{F}$ in (3.22). Rather than solving problem (4.12)–(4.13), we deal with the following projected problem by mode out the approximate kernel: for each set of parameters f, μ and e in $\mathcal{F}$, finding functions $\phi \in H^2(\mathfrak{S})$ with multiplies $c_0, \ldots, c_{N+1}, \Lambda_0, \ldots, \Lambda_{N+1}$ such that

$$\mathscr{L}(\phi) = -\mathscr{E} - \mathscr{N}(\phi) + \sum_{i=0}^{N+1} c_i \mathbf{Z}_i \quad \text{in } \mathfrak{S}, \tag{4.16}$$

$$\phi_z + \eta^\varepsilon_{6\delta} \widehat{D}_4(\phi) + \eta^\varepsilon_{6\delta} \widehat{D}_3(\phi + v_3) = \eta^\varepsilon_\delta \hat{\mathscr{B}}_3 \quad \text{on } \partial_1 \mathfrak{S}, \tag{4.17}$$

$$\phi_z + \eta^\varepsilon_{6\delta} \check{D}_4(\phi) + \eta^\varepsilon_{6\delta} \check{D}_3(\phi + v_3) = \eta^\varepsilon_\delta \check{\mathscr{B}}_3 \quad \text{on } \partial_0 \mathfrak{S}, \tag{4.18}$$

$$\int_{\mathbb{R}^N} \phi(s, z) \mathbf{Z}_i \, \mathrm{d}s = \Lambda_i(z), \quad i = 0, \ldots, N+1, \ 0 < z < 1/\sqrt{\varepsilon}, \tag{4.19}$$

where we have denoted

$$\mathscr{N}(\phi) = \eta^\varepsilon_{8\delta} \widetilde{N}\big(\eta^\varepsilon_{3\delta} \phi + \psi(\phi)\big) + \eta^\varepsilon_{8\delta} p \mathbb{W}^{p-1} \psi(\phi), \quad \mathscr{E} = \eta^\varepsilon_{8\delta} \widetilde{E}. \tag{4.20}$$

After the development of the linear resolution theory in Proposition 5.3, we will prove, in Section 6, that this problem has a unique solution ϕ whose norm is controlled by the L^2-norm of $\mathscr{E}_{32}$, $\hat{B}_3$, $\check{B}_3$. The final complete statement for the solvability of this full nonlinear projected problem is concluded in Proposition 6.1. After this has been done, our task is to adjust the parameters e, f and μ such that the functions $c_0, \ldots, c_{N+1}$ in (4.16) are identically zero. It turns out that this procedure is equivalent to solving a nonlocal, nonlinear coupled second order system of differential equations for the unknowns (f, e, μ) with suitable boundary conditions in (3.40)–(3.41), (3.42)–(3.43), (3.38)–(3.39). In fact, in Sections 7 and 8 we will derive and then solve this system for e, f, μ in $\mathcal{F}$.

5. Linear Theory with Weighted Norms

In this section we will use the weighted space in [13] to develop the linear resolution theory and also the method in [11] to deal with the boundary error.

Recall the linear operator L_0 defined in (3.4)

$$L_0(\tilde{\phi}) \equiv \Delta_x \tilde{\phi} + p {W_0}^{p-1} \tilde{\phi}.$$

We consider the resolution theory of L_0, which was stated in the following lemma in [13].

Lemma 5.1. *Assume that $\xi \neq 0, \pm\sqrt{\lambda_0}$. Then given $h \in L^\infty(\mathbb{R}^N)$, there exists a unique bounded solution of*

$$(L_0 - |\xi|^2)\psi = h \quad \text{in } \mathbb{R}^N.$$

Moreover, there holds

$$\|\psi\|_{L^\infty} \leq C_\xi \|h\|_{L^\infty}$$

for some constant C_ξ only depending on ξ.

Recall the translated variables x in (3.21) and the notation in (3.23)–(3.24). We define an operator

$$\mathscr{L}_1(\phi) = \Delta_x \phi + b\,\phi_{zz} + pW_0{}^{p-1}\phi - \varepsilon b\,\phi, \tag{5.1}$$

where $b = \mu^2$ with the asymptotic formula $b \sim \mu_0^2$. From the composition of μ in (3.15), assume that for a number $m > 0$ we have that

$$m \leq b \leq m^{-1}, \quad |\partial_\theta b(\sqrt{\varepsilon}\,z)| \leq \varepsilon^{\varrho}, \quad \forall\, z \in [0, 1/\sqrt{\varepsilon}], \tag{5.2}$$

for some universal positive constant ϱ.

We deal with the following projected problem: for given functions $h \in C(\mathfrak{S}_0)$ and $g \in C(\bar{\mathfrak{S}}_0)$, finding function ϕ with multiplies $c_0, \ldots, c_{N+1}$ such that

$$\mathscr{L}_1(\phi) = h + \sum_{i=0}^{N+1} c_i \widetilde{\mathbf{Z}}_i \quad \text{in } \mathfrak{S}_0, \tag{5.3}$$

$$\phi_z = g \quad \text{on } \partial\mathfrak{S}_0, \tag{5.4}$$

$$\int_{\mathbb{R}^N} \phi(x,z)\widetilde{\mathbf{Z}}_i \,\mathrm{d}x = \tilde{\Lambda}_i(z), \quad i = 0, \ldots, N+1,\ 0 < z < 1/\sqrt{\varepsilon}. \tag{5.5}$$

In the above, we have chosen suitable $\tilde{\Lambda}_i$'s such that

$$\tilde{\Lambda}_i'(z) = \int_{\mathbb{R}^N} g(x,z)\widetilde{\mathbf{Z}}_i(x)\,\mathrm{d}x, \quad \forall\, z \in \big[\,0, 1/\sqrt{\varepsilon}\,\big]. \tag{5.6}$$

The existence and uniform *a priori* estimates for problem (5.3)–(5.5) reads as follows.

Lemma 5.2. *Assume that $N \geq 7$, $N - 2 \leq \sigma < N$. There exists a number δ such that if*

$$|\partial_\theta b(\sqrt{\varepsilon}\,z)| \leq \delta, \quad \forall\, z \in [0, 1/\sqrt{\varepsilon}\,], \tag{5.7}$$

then for any h, g with $\|h\|_{**} < +\infty$ and $\|g\|_{**} < +\infty$, there exists a unique solution $\phi = T_0(h, g)$ with property

$$\|\phi\|_* \leq C\big(\|h\|_{**} + \|g\|_{**}\big). \tag{5.8}$$

Proof. The proof will be carried out in three steps.

Step 1: Let us assume that in problem (5.3)–(5.5) the terms $g, \Lambda_0, \ldots, \Lambda_{N+1}$, $c_0, \ldots, c_{N+1}$ are identically zero. Arguing as in [13], we have that δ as in the above statement can be chosen so that for any h with $\|h\|_{**} < +\infty$ and any solution ϕ of problem (5.3)–(5.5) with $\|\phi\|_* < +\infty$ we have

$$\|\phi\|_* \leq \|h\|_{**}.$$

Step 2: We claim that the a priori estimate obtained in Step 1 is in reality valid for the full problem (5.3)–(5.5). Let ϕ_0 be the solution of

$$\Delta_x \phi_0 + b\,\phi_{0,zz} - \varepsilon\phi_0 = 0 \quad \text{in } \mathfrak{S}_0, \qquad \frac{\partial \phi_0}{\partial v} = g \quad \text{on } \partial\mathfrak{S}_0.$$

Note that we have

$$\|\phi_0\|_* \leq \|g\|_{**}.$$

Since $N \geq 7$, for any z, the following integral

$$\overline{\Lambda}_i(z) = \int_{\mathbb{R}^N} \phi_0(x,z)\widetilde{\mathbf{Z}}_i(x)\,\mathrm{d}x$$

is well defined. Thus, to prove the general case it suffices to apply the argument with

$$\bar{\phi} = \phi - \phi_0 + \sum_{i=0}^{N+1} \frac{\big(\tilde{\Lambda}_i(z) - \overline{\Lambda}_i(z)\big)\widetilde{\mathbf{Z}}_i}{\int_{\mathbb{R}^N} |\widetilde{\mathbf{Z}}_i|^2\,\mathrm{d}x}.$$

Then $\bar{\phi}$ satisfies a problem of a similar form with homogeneous Neumann boundary condition and orthogonality condition, as well as h replaced by a function $\bar{h}$ with norm bounded by

$$\|\bar{h}\|_{**} \;\leq\; C\big[\,\|h\|_{**} + \|g\|_{**}\,\big].$$

In other words, we shall consider the problem

$$\mathscr{L}_1(\bar{\phi}) = \bar{h} + \sum_{i=0}^{N+1} c_i\widetilde{\mathbf{Z}}_i \quad \text{in } \mathfrak{S}_0, \tag{5.9}$$

$$\bar{\phi}_z = 0 \quad \text{on } \partial\mathfrak{S}_0, \tag{5.10}$$

$$\int_{\mathbb{R}^N} \bar{\phi}(x,z)\widetilde{\mathbf{Z}}_i(x)\,\mathrm{d}x = 0, \quad i = 0,\ldots,N+1,\ 0 < z < 1/\sqrt{\varepsilon}. \tag{5.11}$$

We will prove that for any solution $\bar{\phi}$ to (5.9)–(5.11) with $\|\bar{\phi}\|_* < \infty$, the following estimate holds

$$|c_i|_\infty + \|\bar{\phi}\|_* \leq C\|\bar{h}\|_{**}.$$

Besides there holds

$$c_i(z)\int_{\mathbb{R}^N} |\widetilde{\mathbf{Z}}_i|^2\,\mathrm{d}x = -\int_{\mathbb{R}^N} \bar{h}\widetilde{\mathbf{Z}}_i\,\mathrm{d}x + o(1)\|\bar{h}\|_{**},$$

where $o(1) \to 0$ as $\varepsilon \to 0$.

Testing the equation against $\widetilde{\mathbf{Z}}_i$ and integrating only in x, we find

$$c_i(z)\int_{\mathbb{R}^N} |\widetilde{\mathbf{Z}}_i|^2\,\mathrm{d}x = -\int_{\mathbb{R}^N} \bar{h}\widetilde{\mathbf{Z}}_i\,\mathrm{d}x, \tag{5.12}$$

where we have used the equation for $\widetilde{\mathbf{Z}}_i$ and the orthogonal condition (5.11). It is obvious that

$$\left| \int_{\mathbb{R}^N} \bar{h} \widetilde{\mathbf{Z}}_i \, \mathrm{d}x \right| \leq C || \bar{h} ||_{**}.$$

Plugging all the above estimates into (5.12), we obtain

$$|c_i| \leq C\big(\|\bar{h}\|_{**} + \varepsilon^{\varrho_3} \|\bar{\phi}\|_* \big),$$

for some small positive constant ϱ_3. On the other hand, Lemma 5.1 implies that

$$\|\bar{\phi}\|_* \leq C\big(\|\bar{h}\|_{**} + \|c_i \mathbf{Z_i}\|_{**} + \varepsilon^{\varrho_4} \|\bar{\phi}\|_* \big) \leq C\big(\|\bar{h}\|_{**} + \|c_i\|_\infty + \varepsilon^{\varrho_4} \|\bar{\phi}\|_* \big),$$

for some small positive constant ϱ_4. We complete the proof by combining the last two estimates.

Step 3: We now prove the existence part of our statement. As we have stated in Step 2, we only need to consider the case that $g, \Lambda_0, \ldots, \Lambda_{N+1}$ are identically zero. We look for a weak solution ϕ in the space H defined as the subspace of functions ψ which are in H^1 such that the homogeneous Neumann boundary condition holds and that

$$\int_{\mathbb{R}^N} \psi(x, z) \widetilde{\mathbf{Z}}_i \, \mathrm{d}x = 0, \quad i = 0, \ldots, N+1, \quad 0 < z < 1/\sqrt{\varepsilon}.$$

The bilinear form defined on H is

$$B(\phi, \psi) = \int_{\mathfrak{S}_0} \psi \mathscr{L}_1(\phi).$$

Problem (5.3)–(5.5) gets weakly formulated as that of finding $\phi \in H$ such that

$$B(\phi, \psi) = \int_{\mathfrak{S}_0} h\psi \quad \text{for all } \psi \in H.$$

If h is smooth, elliptic regularity yields that a weak solution is a classical one. The weak formulation can readily be put into the form

$$\phi + \mathbb{K}\phi = \hat{h},$$

in H, where $\hat{h}$ is a linear operator of h and $\mathbb{K}$ is compact. The a priori estimate of Step 2 yields that for $h = 0$ only the trivial solution is present. Fredholm alternative thus applies yielding that problem is solvable. □

As an application the previous Lemma, we will give the proof for Lemma 3.2 below.

Proof of Lemma 3.2:. To get the validity of (3.56), by the explicit formula g and the orthogonality conditions (3.52), we can choose $\tilde{\Lambda}_i$'s identically zero. The existence and *a priori* estimate in Lemma 3.2 are direct application of Lemma 5.2. For $i = 0, \ldots, N+1$, multiplying the equation against $\widetilde{\mathbf{Z}}_i$, and then integrating by parts, one can show $c_i \equiv 0$ with the help of the orthogonality conditions in (3.52). □

We now shall develop the resolution theory for the linear operator $\mathscr{L}$ defined in (4.14), i.e.

$$\mathscr{L}(\phi) = \Delta_s \phi + \phi_{zz} + p{v_3}^{p-1}\phi - \varepsilon\phi + \eta^{\varepsilon}_{6\delta}B_2(\phi),$$

where B_2 is a linear differential operator defined in (2.15), v_3 is defined in (3.63) and $\eta^{\varepsilon}_{6\delta}$ defined as at the beginning of Section 4. We deal with the following projected problem: for given functions $h \in C(\mathfrak{S})$ and $g \in C(\bar{\mathfrak{S}})$, find a function ϕ with multiplies $c_0, \ldots, c_{N+1}$, $\Lambda_0, \ldots, \Lambda_{N+1}$ such that

$$\mathscr{L}(\phi) = h + \sum_{i=0}^{N+1} c_i \mathbf{Z}_i \quad \text{in } \mathfrak{S} \tag{5.13}$$

$$\phi_z = g \quad \text{on } \partial\mathfrak{S}, \tag{5.14}$$

$$\int_{\mathbb{R}^N} \phi(s, z)\mathbf{Z}_i \, \mathrm{d}s = \Lambda_i(z), \quad i = 0, \ldots, N+1, \; 0 < z < 1/\sqrt{\varepsilon}. \tag{5.15}$$

The existence and uniform *a priori* estimates for problem (5.13)–(5.15) read.

Proposition 5.3. *Assume that $N \geq 7$, $\sigma = N - 2 + \tilde{\sigma}$, for some small but fixed $\tilde{\sigma}$. Given parameters $\mu = \mu_0 + \tilde{\mu}$, f, e in* (3.22)*, for any $h \in C(\mathfrak{S})$ and $g \in C(\bar{\mathfrak{S}})$ with $\|h\|_{**} < +\infty$ and $\|g\|_{**} < +\infty$, we can find $c_0, \ldots, c_{N+1}$, $\Lambda_0, \ldots, \Lambda_{N+1}$ such that there exists a unique solution $\phi = T(h, g)$ to* (5.13)–(5.14) *with property*

$$\|\phi\|_* \leq C\big(\|h\|_{**} + \|g\|_{**}\big), \quad \|\Lambda_i\|_\infty \leq C\big(\|h\|_{**} + \|g\|_{**}\big), \; \forall\, i = 0, \ldots, N+1. \tag{5.16}$$

Moreover there hold the decomposition

$$\Lambda_i'(z) = \Theta(z) + \int_{\mathbb{R}^N} g(x, z)\widetilde{\mathbf{Z}}_i(x)\, \mathrm{d}x, \quad \forall\, z \in \big[0, 1/\sqrt{\varepsilon}\,\big], \tag{5.17}$$

with

$$\|\Theta(z)\|_\infty + \|\Theta'(z)\|_\infty \leq C\sqrt{\varepsilon}\,\big(\|h\|_{**} + \|g\|_{**}\big).$$

Proof. We observe that the assumption $N \geq 7$ is needed to ensure that the integral in (5.15) is finite for functions ϕ with $\|\phi\|_*$ bounded. We recall the following relations

$$U(s, z) = (1 + \varpi)\big(\mu(\sqrt{\varepsilon}\, z)\big)^{(2-N)/2} W_0\left(\frac{s - f(\sqrt{\varepsilon}z)}{\mu(\sqrt{\varepsilon}\, z)}\right),$$

$$\mathbf{Z}_i(s, z) = \big(\mu(\sqrt{\varepsilon}\, z)\big)^{-N/2}\widetilde{\mathbf{Z}}_i\left(\frac{s - f(\sqrt{\varepsilon}z)}{\mu(\sqrt{\varepsilon}\, z)}\right), \quad \forall\, i = 1, \ldots, N+1,$$

$$\Psi = \mathbf{Z}_0(s, z) = \big(\mu(\sqrt{\varepsilon}\, z)\big)^{(2-N)/2}\widetilde{\mathbf{Z}}_0\left(\frac{s - f(\sqrt{\varepsilon}z)}{\mu(\sqrt{\varepsilon}\, z)}\right).$$

For a function $\xi(s, z)$ defined in $\mathfrak{S}$, by the translation $s = \mu(\sqrt{\varepsilon}\, z)x + f(\sqrt{\varepsilon} z)$, we define a type of new function on $\mathfrak{S}_0$ below

$$\tilde{\xi}(x, z) = \xi\Big(\mu(\sqrt{\varepsilon}\, z)x + f(\sqrt{\varepsilon} z),\ z\Big)\big(\mu(\sqrt{\varepsilon}\, z)\big)^{(N-2)/2}. \tag{5.18}$$

Note that

$$\mathscr{L}(\phi) = \Delta_s \phi + \phi_{zz} + p v_3{}^{p-1}\phi - \varepsilon\phi + \eta^{\varepsilon}_{6\delta} B_2(\phi),$$

where B_2 is a linear differential operator defined in (2.15) and v_3 is defined in (3.63).

Direct computation gives that problem (5.13)–(5.15) is equivalent to

$$\Delta_x \tilde{\phi} + \mu^2 \tilde{\phi}_{zz} + B_3(\tilde{\phi}) - \varepsilon\mu^2\tilde{\phi} + pW_0^{p-1}\tilde{\phi} + B_4(\tilde{\phi}) + \eta^{\varepsilon}_{6\delta}\tilde{B}_2(\tilde{\phi}) = \mu^2 \tilde{h} + \sum_{j=0}^{N+1} \tilde{c}_j \widetilde{\mathbf{Z}}_j \quad \text{in } \mathfrak{S}_0,$$

$$\frac{\partial \tilde{\phi}}{\partial z} = \tilde{g} + B_5(\tilde{\phi}) \quad \text{on } \partial\mathfrak{S}_0,$$

$$\int_{\mathbb{R}^N} \tilde{\phi}(x, z)\, \widetilde{\mathbf{Z}}_j \,\mathrm{d}x = \mu^{-1}\Lambda_j(z), \quad j = 1, \ldots, N+1, \quad 0 < z < \frac{1}{\sqrt{\varepsilon}},$$

$$\int_{\mathbb{R}^N} \tilde{\phi}(x, z)\, \widetilde{\mathbf{Z}}_0 \,\mathrm{d}x = \mu^{-2}\Lambda_0(z), \quad 0 < z < \frac{1}{\sqrt{\varepsilon}},$$

where we have denoted

$$B_4(\tilde{\phi}) = p\Big[\big(1+\varpi\big)^{4/(N-2)} - 1\Big] W_0^{p-1}\tilde{\phi} - p(1+\varpi)^{p-1} W_0^{p-1}\tilde{\phi}$$
$$+ \Big[(1+\varpi)W_0 + \sqrt{\varepsilon}\, e\widetilde{\mathbf{Z}}_0 + \sqrt{\varepsilon}\,\mu^{-1}\Psi + \sqrt{\varepsilon}\, A\widetilde{\mathbf{Z}}_0\Big]^{p-1}\tilde{\phi},$$

$$B_5(\tilde{\phi}) = \sqrt{\varepsilon}\,\mu^{-1} f' \cdot \nabla_x \tilde{\phi} + \sqrt{\varepsilon}\,\mu^{-1}\mu' x \cdot \nabla_x \tilde{\phi} - \sqrt{\varepsilon}\,\frac{2-N}{2}\,\mu^{-2}\mu'\tilde{\phi}.$$

The linear operator B_3 is a small perturbation of $\partial_z\partial_z$ and the linear operator $\tilde{B}_2$ is the counterpart of B_2 after the changing of variables. In the above, we also have the relations

$$c_0 = \tilde{c}_0\,\mu^{-2}, \quad c_i = \tilde{c}_i\,\mu^{-1}, \quad \forall\, i = 1, \ldots, N+1.$$

The problem is then equivalent to the fixed point linear problem

$$\tilde{\phi} = T_0\Big(\tilde{h} - B_3(\tilde{\phi}) - B_4(\tilde{\phi}) - \eta^{\varepsilon}_{6\delta}\tilde{B}_2(\tilde{\phi}),\ \tilde{g} + B_5(\tilde{\phi})\Big),$$

where T_0 is the linear operator defined by Lemma 3.2. Note that the linear operator B_2 is a linear combination of the differential operators such as $\partial^2_{zs_i}$, ∂^2_{zz} and ∂_z. Moreover, all terms in $B_2(\bar{\phi})$ carry the coefficients $\sqrt{\varepsilon}$. Then we use the property of the cut-off function $\eta^{\varepsilon}_{6\delta}$ to get

$$\Big|\int_{\mathbb{R}^N} \eta^{\varepsilon}_{6\delta}\tilde{B}_2(\tilde{\phi})\widetilde{\mathbf{Z}}_i \,\mathrm{d}x\Big| \le C\varepsilon^{\varrho_2}||\phi||_*,$$

for some small positive constant ϱ_2. The linear operators B_3 and B_4 are small in the sense that

$$\|B_3(\tilde{\phi})\|_{**} + \|B_4(\tilde{\phi})\|_{**} + \|\eta^{\varepsilon}_{6\delta}\tilde{B}_2(\tilde{\phi})\|_{**} + \|B_5(\tilde{\phi})\|_{**} \le o(1)\,\|\tilde{\phi}\|_*,$$

with $o(1) \to 0$ as $\sqrt{\varepsilon} \to 0$. From this, unique solvability of the problem and the desired estimate immediately follow. In fact, we can work in the space of the form

$$\mathscr{F} = \big\{\tilde{\phi} : \|\tilde{\phi}\|_* \le C\big(\|h\|_{**} + \|g\|_{**}\big)\big\}.$$

For any given $\tilde{\phi} \in \mathscr{F}$, as we have done in (5.6) we can choose suitable $\tilde{\Lambda}_i = \mu^{-1}\Lambda_j$, $j = 1, \ldots, N+1$ and $\tilde{\Lambda}_0 = \mu^{-2}\Lambda_0$ in such a way that

$$\|\Lambda_i\|_\infty \le C\big(\|h\|_{**} + \|g\|_{**}\big), \quad i = 0, \ldots, N+1.$$

Moreover, by the using the decomposition of μ, we get the expression of Λ_i's in (6.12) and the estimates of its components. Then we can find

$$\tilde{v} = T_0\Big(\tilde{h} - B_3(\tilde{\phi}) - B_4(\tilde{\phi}) - \eta^{\varepsilon}_{6\delta}\tilde{B}_2(\tilde{\phi}),\ \tilde{g} + B_5(\tilde{\phi})\Big)$$

by Lemma 5.2. Moreover, $\|\tilde{v}\|_* \le C\big(\|h\|_{**} + \|g\|_{**}\big)$. Then the fixed point theory will fulfill the proof the result. □

6. Solving the Nonlinear Projected Problem (4.16)–(4.19)

In this section, we will solve (4.16)–(4.19) in $\mathfrak{S}$, see (3.23). A first elementary, but crucial observation is the following: the term

$$\mathscr{E}_{31} = \varepsilon^{3/2} e''\Psi + \sqrt{\varepsilon}\,\mu^{-2}\lambda_0 e\Psi,$$

in the decomposition of $\mathscr{E}_3$, has precisely the form $c_0\Psi$ and can be absorbed in that term $c_0\Psi$. Then, the equivalent equation of (4.16) is

$$\mathscr{L}(\phi) \;=\; -\tilde{\mathscr{E}} - \widetilde{\mathscr{N}}(\phi) + \sum_{i=0}^{N+1} c_i\mathbf{Z}_i,$$

where $\tilde{\mathscr{E}}$ is the extension of $\eta^{\varepsilon}_{8\delta}\widetilde{E}$ without the component $\mathscr{E}_{31}$ and $\widetilde{\mathscr{N}}(\phi)$ is defined by

$$\widetilde{\mathscr{N}}(\phi) = \eta^{\varepsilon}_{8\delta}\widetilde{N}\big(\eta^{\varepsilon}_{3\delta}\phi + \psi(\phi)\big) + \eta^{\varepsilon}_{8\delta}p\,\mathbb{W}^{p-1}\psi(\phi) + p\big(v_3^{p-1} - W^{p-1}\big)\phi.$$

As we have done in (3.50), we also extend $\check{\mathscr{B}}_3$ and $\hat{\mathscr{B}}_3$ to a function $\mathscr{G}$ defined on $\mathfrak{S}$

$$\mathscr{G} = \tilde{\eta}_1(z)\,\hat{\mathscr{B}}_3 + \tilde{\eta}_0(z)\,\check{\mathscr{B}}_3. \tag{6.1}$$

Let T be the bounded operator defined by Proposition 5.3. Then problem (4.16)–(4.19) is equivalent to the following fixed point problem

$$\phi = T\big(h, g\big) \equiv \mathscr{A}(\phi), \tag{6.2}$$

where we have denoted

$$h = -\tilde{\mathscr{E}} - \widetilde{\mathscr{N}}(\phi), \tag{6.3}$$

and the boundary term g is defined by

$$g = \mathscr{G} - \tilde{\eta}_1(z)\left(\eta^{\varepsilon}_{6\delta}\widehat{D}_4(\phi) - \eta^{\varepsilon}_{6\delta}\widehat{D}_3(\phi + v_3)\right) - \tilde{\eta}_0(z)\left(\eta^{\varepsilon}_{6\delta}\check{D}_4(\phi) - \eta^{\varepsilon}_{6\delta}\check{D}_3(\phi + v_3)\right). \tag{6.4}$$

We collect some useful facts to find the domain of the operator $\mathscr{A}$ such that the nonlinear operator $\mathscr{A}$ becomes a contraction mapping. The big difference between $\mathscr{E}_{31}$ and $\mathscr{E}_{32}$ is their sizes. From (3.75) and (3.76)

$$\|\mathscr{E}_{32}\|_{**} \le c_*\,\varepsilon, \tag{6.5}$$

while $\mathscr{E}_{31}$ is only of size $O(\varepsilon^{1/2})$. Similarly, by the extension (6.1) and (3.77), we have

$$\|\mathscr{G}\|_{**} \le c_*\,\varepsilon. \tag{6.6}$$

Recall that the operator $\psi(\phi)$ satisfies, as seen directly from its definition

$$\|\psi(\phi)\|_{L^\infty} \;\le\; C\sqrt{\varepsilon}\,\big|\big|\,\big(|\phi| + |\nabla\phi|\big)\,\big|\big|_{L^\infty(|x|>20\delta/\varepsilon)} + \|\widetilde{E}\|_{L^\infty(|x|>\delta/\varepsilon)}, \tag{6.7}$$

and a Lipschitz condition of the form

$$\|\psi(\phi_1) - \psi(\phi_2)\|_{L^\infty} \;\le\; C\sqrt{\varepsilon}\Big[\big|\big|\,|\phi_1 - \phi_2| + |\nabla(\phi_1 - \phi_2)|\,\big|\big|_{L^\infty(|x|>20\delta/\varepsilon)}\Big]. \tag{6.8}$$

Now, the facts above will allow us to construct a region where contraction mapping principle applies and then solve the problem (4.16)–(4.19). Consider the following closed, bounded subset

$$\mathfrak{D} = \{\phi \mid \; \|\phi\|_* \le \tau\varepsilon\}. \tag{6.9}$$

We claim that we can choose a suitable constant τ such that the map $\mathscr{A}$ defined in (6.2) is a contraction from $\mathfrak{D}$ into itself. Let us analyze the characters of the nonlinear operator involved in $\mathscr{A}$ for functions $\phi \in \mathfrak{D}$, namely

$$\begin{aligned}\widetilde{\mathscr{N}}(\phi) &= \eta^{\varepsilon}_{8\delta}\widetilde{N}\big(\eta^{\varepsilon}_{3\delta}\phi + \psi(\phi)\big) + \eta^{\varepsilon}_{8\delta}p\mathbb{W}^{p-1}\psi(\phi) + p\big(v_3^{p-1} - U^{p-1}\big)\phi\\ &\equiv \widetilde{\mathscr{N}}_1(\phi) + \widetilde{\mathscr{N}}_2(\phi) + \widetilde{\mathscr{N}}_3(\phi).\end{aligned}$$

For the estimate of the term $\widetilde{\mathscr{N}}_1(\varphi)$, we first consider one of its component of the form

$$\widetilde{\mathscr{N}}_{11}(\phi) = \eta^{\varepsilon}_{8\delta}\Big[\big(\mathbb{W} + \eta^{\varepsilon}_{3\delta}\phi + \psi(\phi)\big)^p - \mathbb{W}^p - p\mathbb{W}^{p-1}\big(\eta^{\varepsilon}_{3\delta}\phi + \psi(\phi)\big)\Big]$$

we get

$$\|\widetilde{\mathscr{N}}_{11}(\phi)\|_{**} \le C \sup_{|x|\le\delta\varepsilon^{-1/2}} \Big|(1 + |x|)^{N-2}\mathbb{W}^{p-2}(\phi + \psi)^2\Big|$$

$$
\begin{aligned}
&+ C \sup_{|x|\geq \delta\varepsilon^{-1/2}} \Big|(1+|x|)^{N-2}(|\phi|^p + |\psi|^p)\Big| \\
&\leq C\varepsilon^4.
\end{aligned}
$$

The other components in $\widetilde{\mathcal{N}}_1(\varphi)$ can be estimated in the same way. There also holds

$$
\begin{aligned}
\|\widetilde{\mathcal{N}}_1(\phi)\|_{**} &\leq \|(U + \sqrt{\varepsilon}e\Psi + \sqrt{\varepsilon}\phi^*)^{p-1} - U^{p-1}\|_{**} \\
&\leq C\|U^{p-2}(\sqrt{\varepsilon}e\Psi + \sqrt{\varepsilon}\phi^*)\phi\|_{**} \\
&\leq C\sqrt{\varepsilon}\,\|\phi\|_*.
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
\|\eta^{\varepsilon}_{8\delta}\mathbb{W}^{p-1}\psi(\phi)\|_{**} &\leq \varepsilon^{N-3-\frac{2}{N-2}} \sup_{|x|\leq \delta\varepsilon^{-N-1N-2}} (1+|x|)^{N-6}\|\phi\|_* \\
&\leq C\varepsilon^{2+\frac{2}{N-2}}\|\phi\|_*.
\end{aligned}
$$

Hence, from the properties of $\mathbb{W}$ and $\psi(\phi)$, we obtain for $\phi \in \mathfrak{D}$

$$
\|\widetilde{\mathcal{N}}(\phi)\|_{**} \leq C\varepsilon^{3/2}. \tag{6.10}
$$

Let $\phi \in \mathfrak{D}$ and $v = \mathcal{A}(\phi)$, then from (6.5)–(6.6) and the property of T, we have $v \in \mathfrak{D}$ provided τ is chosen large enough.

We next prove that $\mathcal{A}$ is a contraction mapping, so that the fixed point problem can be uniquely solved in $\mathcal{D}$. This fact is a direct consequence of (6.8). Indeed, arguing as in the estimates above

$$
\|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)\|_* \;\leq\; C\|\widetilde{\mathcal{N}}_1(\phi_1) - \widetilde{\mathcal{N}}_1(\phi_2)\|_{**} \;\leq\; o(1)\|\phi_1 - \phi_2\|_*,
$$

where $o(1) \to 0$ as $\varepsilon \to 0$.

It is worth to mention that the error $\mathcal{E}_{32}$ and the operator T itself carry the functions f, e and μ as parameters. For future reference, we should consider their Lipschitz dependence on these parameters. (3.79) is just the formula about the Lipschitz dependence of error $\mathcal{E}_{32}$ on these parameters. The other task can be realized by careful and direct computations of all terms involved in the differential operator which will show that this dependence is indeed Lipschitz.

For the linear operator T, we have the following Lipschitz dependence

$$
\|T(\mu_1, f_1) - T(\mu_2, f_2)\| \;\leq\; C\varepsilon\big(\|f_1 - f_2\|_a + \|\mu_1 - \mu_2\|_b\big).
$$

Moreover, the operator $\widetilde{\mathcal{N}}$ also has Lipschitz dependence on (f, e, μ). It is easily checked that for $\phi \in \mathfrak{D}$, see (6.9), we have, with obvious notation

$$
\|\widetilde{\mathcal{N}}_{(f_1,e_1,\mu_1)}(\phi) - \widetilde{\mathcal{N}}_{(f_2,e_2,\mu_2)}(\phi)\|_{L^2(\mathfrak{S})} \leq C\varepsilon\Big[\|f_1 - f_2\|_a + \|e_1 - e_2\|_b + \|\mu_1 - \mu_2\|_c\Big].
$$

Hence, from the fixed point characterization we get that

$$
\begin{aligned}
&\|\phi(f_1, e_1, \mu_1) - \phi(f_2, e_2, \mu_2)\|_{H^2(\mathfrak{S})} \\
&\quad \leq C\varepsilon\big[\,\|f_1 - f_2\|_a + \|e_1 - e_2\|_b + \|\mu_1 - \mu_2\|_c\,\big].
\end{aligned} \tag{6.11}
$$

We make a conclusion of this section that

Proposition 6.1. *There is a number $C > 0$ such that for all small ε and all parameters (f, e, μ) in F, problem* (4.16)–(4.19) *has a unique solution $\phi = \phi(f, e, \mu)$ which satisfies*

$$\sum_{i=1}^{3} \|\Lambda_i\|_\infty + \|\phi\|_* \le C\varepsilon.$$

Moreover, ϕ depends Lipschitz-continuously on the parameters f, e and μ in the sense of the estimate (6.11). *Moreover there holds the decomposition, for $i = 1, 2, 3$*

$$\Lambda_i'(z) = \Theta_i(z) + \int_{\mathbb{R}^N} \mathscr{G}(x, z)\widetilde{\mathbf{Z}}_i(x)\,\mathrm{d}x, \quad \forall\, z \in \left[0, 1/\sqrt{\varepsilon}\right], \tag{6.12}$$

with the estimate

$$\|\Theta_i(z)\|_\infty + \|\Theta_i'(z)\|_\infty \le C\sqrt{\varepsilon}\left(\|\bar{\mathscr{E}}\|_{**} + \|\mathscr{G}\|_{**}\right).$$

7. The Reduction Procedure

In this section, we will set up the equations for the parameters $e, f = (f_1, \ldots, f_N)$ and μ which are equivalent to making the multipliers c_i, $i = 0, \ldots, N+1$ identically zero in the system (4.16)–(4.19). These equations are obtained by simply integrating the equation (4.16) (only in s) against $\mathbf{Z}_i$'s respectively. Using the definitions of $\mathbf{Z}_i$'s and the formula (4.19), it is easy to derive the following equations, for $i = 1, \ldots, N+1$,

$$\int_{\mathbb{R}^N} \left[\mathscr{E} + \mathscr{N}(\phi) + \phi_{zz} - \varepsilon\phi + \eta_{6\delta}^{\varepsilon} B_2(\phi) + p\left(v_3^{p-1} - W^{p-1}\right)\phi\right]\mathbf{Z}_i\,\mathrm{d}s = 0, \tag{7.1}$$

as well as

$$\int_{\mathbb{R}^N} \left[\mathscr{E} + \mathscr{N}(\phi) + \mu^{-2}\lambda_0\phi + \phi_{zz} - \varepsilon\phi + \eta_{6\delta}^{\varepsilon} B_2(\phi) \right.$$
$$\left. + p\left(v_3^{p-1} - W^{p-1}\right)\phi\right]\mathbf{Z}_0\,\mathrm{d}s = 0. \tag{7.2}$$

By the properties of the cut-off function carried in $\mathscr{E}$, it is therefore of crucial importance to carry out computations of the estimates of the terms

$$\int_{\mathbb{R}^N} \mathscr{E}_3 \mathbf{Z}_i\,\mathrm{d}s, \quad \forall\, i = 0, \ldots, N+1,$$

and similarly, some other terms involving ϕ. The symmetry of the function U will play an important role in the computation in the sense that it makes some terms identically zero, which we may not state explicitly. We will also use the fact that the eigenfunctions associated to different eigenvalues of L_μ in (3.7) are orthogonal. By the formula (3.71), we make another decomposition below

$$\mathscr{E}_3 = \mathscr{E}_1 + \overline{\mathscr{E}}_1.$$

For the set (μ, f, e) in (3.22), denote by $b_{1\varepsilon}$ and $b_{2\varepsilon}$, generic, uniformly bounded continuous functions of the form

$$b_{l\varepsilon} = b_{l\varepsilon}\Big(z,\ \mu,\ f,\ e,\ \mu',\ f',\ e'\Big), \quad l = 1, 2,$$

where $b_{1\varepsilon}$ is uniformly Lipschitz in its four last arguments. We also introduce the generic functions of the type

$$h_\varepsilon = h_{1\varepsilon}\Big(z,\ \mu,\ f,\ e,\ \mu',\ f',\ \sqrt{\varepsilon}e'\Big) + o(1)h_{2\varepsilon}\Big(z,\ \mu,\ f,\ e,\ \mu',\ f',\ \sqrt{\varepsilon}e',\ \mu'',\ f'',\ \varepsilon e''\Big),$$

where $h_{1\varepsilon}$ and $h_{2\varepsilon}$ are smooth functions of their arguments, uniformly bounded in ε and $o(1) \to 0$ as $\varepsilon \to 0$. Moreover, $h_{2\varepsilon}$ depends linearly on the arguments f'', e'', μ''. In the sequel, we will also denote ϱ a generic positive constant greater than 1.

7.1. *Derivation of the Equation for f*

For any fixed $m = 1, \ldots, N$, we will derive the equation for f_m and then divide the computation into two parts below.

Part 1: As we stated in the above, to derive the main components of the equation for the unknown parameter f_m, we multiply $\mathscr{E}_3$ by $\mathbf{Z}_m = U_{s_m}$ and integrate against the variable s on $\mathbb{R}^N$

$$\int_{\mathbb{R}^N} \mathscr{E}_3 \frac{\partial U}{\partial s_m}\,\mathrm{d}s = \int_{\mathbb{R}^N} \Big(\mathscr{E}_1 + \overline{\mathscr{E}}_1\Big)\frac{\partial U}{\partial s_m}\,\mathrm{d}s.$$

For the component of $\mathscr{E}_1$ in (3.33), the estimates can be done as follows. The first term is

$$\begin{aligned}
\mathscr{J}_1 &\equiv \int_{\mathbb{R}^N} \mathscr{E}_{01}\frac{\partial U}{\partial s_m}\,\mathrm{d}s \\
&= \int_{\mathbb{R}^N} \mathscr{E}_{00}\frac{\partial U}{\partial s_m}\,\mathrm{d}s \\
&= -2\varepsilon^{3/2}e'\sum_{i=1}^N f_i'\int_{\mathbb{R}^N}\frac{\partial \Psi}{\partial s_i}\frac{\partial U}{\partial s_m}\,\mathrm{d}s - 2\varepsilon^{3/2}e\mu'\sum_{i=1}^N f_i'\int_{\mathbb{R}^N}\frac{\partial^2\Psi}{\partial\mu\partial s_i}\frac{\partial U}{\partial s_m}\,\mathrm{d}s \\
&\quad + \varepsilon^{3/2}e\sum_{i,j=1}^N f_i'f_j'\int_{\mathbb{R}^N}\frac{\partial^2\Psi}{\partial s_i\partial s_j}\frac{\partial U}{\partial s_m}\,\mathrm{d}s - \varepsilon^{3/2}e\int_{\mathbb{R}^N}\sum_{i,j=1}^N f_i''\frac{\partial\Psi}{\partial s_i}\frac{\partial U}{\partial s_m}\,\mathrm{d}s \\
&= \varepsilon^{3/2}\big[\tilde{b}_1 e'|f'| + \tilde{b}_1 e\mu'|f'| + \tilde{b}_3 e|f'|^2 + \tilde{b}_3 e|f''|\big].
\end{aligned}$$

Again using the definition of $\mathscr{E}_{11}$ in (3.30), we get the estimate of the following

$$\mathscr{J}_2 \equiv \int_{\mathbb{R}^N} \mathscr{E}_{11}\frac{\partial U}{\partial s_m}\,\mathrm{d}s = \varepsilon^2\log\varepsilon\, b_{2\varepsilon},$$

where we have used the definitions of $\mathscr{P}$, $\mathscr{Q}$, $\mathscr{R}$ in (3.29).

For the next component, we also obtain its estimate

$$
\begin{aligned}
\mathcal{J}_3 &\equiv \int_{\mathbb{R}^N} \mathcal{E}_{12} \frac{\partial U}{\partial s_m}\, \mathrm{d}s \\
&= \varepsilon \int_{\mathbb{R}^N} \Bigg[-2\mu' \frac{\partial^2 U}{\partial \mu \partial \ell} \sum_{i=1}^{N} \frac{(s_i - f_i) f_i'}{\ell} - \frac{\partial U}{\partial \ell} \sum_{i=1}^{N} \frac{\big(s_i - f_i\big) f_i''}{\ell} \\
&\quad + \frac{\partial^2 U}{\partial \ell^2} \bigg(\sum_{i=1}^{N} \frac{\big(s_i - f_i\big) f_i'}{\ell} \bigg)^2 \Bigg] \frac{\partial U}{\partial s_m}\, \mathrm{d}s \\
&= \varepsilon \big((N-2)\tau_N\big)^2 f_m'' \int_{\mathbb{R}^N} \frac{\mu^{N-2}(s_m - f_m)^2 \ell}{\big(\mu^2 + \ell^2\big)^N}\, \mathrm{d}s + \varepsilon b_{1\varepsilon} \mu' |f'| + \varepsilon b_{1\varepsilon} |f'|^2 \\
&= \varepsilon \big((N-2)\tau_N\big)^2 f_m''\, \mu \int_{\mathbb{R}^N} \frac{|t_m|^2 |t|}{\big(1 + |t|^2\big)^N}\, \mathrm{d}t + \varepsilon b_{1\varepsilon} \mu' |f'| + \varepsilon b_{1\varepsilon} |f'|^2 \\
&= \varepsilon\, d_1\, \mu\, f_m'' + \varepsilon b_{1\varepsilon} \mu' |f'| + \varepsilon b_{1\varepsilon} |f'|^2,
\end{aligned}
$$

where d_1 is a positive constant.

By using of (3.11), it is easy to show that

$$
\mathcal{J}_4 \equiv -\varepsilon \int_{\mathbb{R}^N} \Big(U + \sqrt{\varepsilon} e \Psi \Big) \frac{\partial U}{\partial s_m}\, \mathrm{d}s = 0.
$$

We now deal with the term $\mathcal{J}_5$ of the form

$$
\mathcal{J}_5 = \int_{\mathbb{R}^N} B_2 \big(U + \sqrt{\varepsilon} e \Psi \big) \frac{\partial U}{\partial s_m}\, \mathrm{d}s.
$$

Note that B_2 is a small perturbation of the operator $\triangle$, and each component contains the multiplying of the cut-off functions α, β in (2.4) and the derivatives (first or second order) with respect to z. We claim that

$$
\mathcal{J}_5 = \varepsilon^{\varrho} h_\varepsilon, \tag{7.3}
$$

with the constant $\varrho > 1$. In fact, for example, we consider the following term in $\mathcal{J}_5$

$$
\begin{aligned}
\mathcal{J}_{5,1} &= \sqrt{\varepsilon} \sum_{i,j=1}^{N} D_{ij} \varphi_0 \int_{\mathbb{R}^N} \alpha\, s_j \frac{\partial^2 U}{\partial z \partial s_i} \frac{\partial U}{\partial s_m}\, \mathrm{d}s \\
&= \varepsilon \sum_{i,j=1}^{N} D_{ij} \varphi_0 \int_{\mathbb{R}^N} s_j\, \eta_\sigma\big(\sqrt{\varepsilon}\, z + (\alpha - 1)\varphi_0\big) \Bigg[\frac{\partial^2 U}{\partial \mu \partial s_i} \mu' - \sum_{k=1}^{N} \frac{\partial U}{\partial s_k \partial s_i} f_k' \Bigg] \frac{\partial U}{\partial s_m}\, \mathrm{d}s \\
&= -\varepsilon \sum_{i,j=1}^{N} D_{ij} \varphi_0 \int_{\mathbb{R}^N} \big(s_j - f_j\big) \Bigg(\eta_\sigma(\sqrt{\varepsilon}\, z) + \sigma^{-1} \sqrt{\varepsilon}\, \eta_\sigma'(\sqrt{\varepsilon}\, z)(\alpha - 1) \sum_{l,\theta=1}^{N} D_{l\theta} \varphi_0 \Bigg) \\
&\qquad \times \Bigg[\frac{\partial^2 U}{\partial \mu \partial s_i} \mu' - \sum_{k=1}^{N} \frac{\partial U}{\partial s_k \partial s_i} f_k' \Bigg] \frac{\partial U}{\partial s_m}\, \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&- \varepsilon \sum_{i,j=1}^{N} D_{ij}\varphi_0 f_j \int_{\mathbb{R}^N} \left(\eta_\sigma(\sqrt{\varepsilon}\, z) + \sigma^{-1}\sqrt{\varepsilon}\, \eta'_\sigma(\sqrt{\varepsilon}\, z)(\alpha - 1) \sum_{l,\theta=1}^{N} D_{l\theta}\varphi_0 \right) \\
&\qquad \times \left[\frac{\partial^2 U}{\partial \mu \partial s_i}\mu' - \sum_{k=1}^{N} \frac{\partial U}{\partial s_k \partial s_i} f'_k \right] \frac{\partial U}{\partial s_m}\, \mathrm{d}s \\
&= \varepsilon\, b_{1\varepsilon,1}|f'|\mu' + \varepsilon\, b_{1\varepsilon,2}|f'|^2 + \varepsilon^{\varrho}\, b_{1\varepsilon,3}\, \mu' + \varepsilon^{\varrho}\, b_{1\varepsilon,4}\, |f'|,
\end{aligned}
$$

where $\varrho > 1$. In the above, we have used the definition of α in (2.4) and taken the Taylor expansion to α. We finish the computation for the components with $\mathscr{E}_1$ involved.

As we have stated in the above, Ψ and U_{s_m} are orthogonal. By using the orthogonal condition (3.55) and the decomposition in (3.71), it is easy to check that

$$
\begin{aligned}
\mathscr{J}_6 &\equiv \int_{\mathbb{R}^N} \bar{\mathscr{E}}_1 \frac{\partial U}{\partial s_m}\, \mathrm{d}s \\
&= \sqrt{\varepsilon}\,(1+\varpi) \int_{\mathbb{R}^N} \mathscr{E}_2 \frac{\partial U}{\partial s_m}\, \mathrm{d}s + \sqrt{\varepsilon}\,(1+\varpi)\int_{\mathbb{R}^N} B_2(\phi^*) \frac{\partial U}{\partial s_m}\, \mathrm{d}s + \int_{\mathbb{R}^N} \mathbf{N}(\phi^*) \frac{\partial U}{\partial s_m}\, \mathrm{d}s \\
&\equiv \mathscr{J}_{6,1} + \mathscr{J}_{6,2} + \mathscr{J}_{6,3}.
\end{aligned}
$$

By the formula (3.61) here we compute the first term

$$
\begin{aligned}
\mathscr{J}_{6,1}(z) &\equiv \sqrt{\varepsilon}\,(1+\varpi) \int_{\mathbb{R}^N} \mathscr{E}_2 \frac{\partial U}{\partial s_m}\, \mathrm{d}s \\
&= -2\varepsilon\,(1+\varpi) \int_{\mathbb{R}^N} \mu^{-(N+4)/2} \nabla_x \Phi_z \cdot \Big(f'\mu + (s-f)\mu' \Big) \frac{\partial U}{\partial s_m}\, \mathrm{d}s \\
&\quad - 2\varepsilon\,(1+\varpi) A_z \int_{\mathbb{R}^N} \mu^{-(N+2)/2} \nabla_x Z_0 \cdot \Big(f'\mu + (s-f)\mu' \Big) \frac{\partial U}{\partial s_m}\, \mathrm{d}s + \varepsilon^{3/2} h_\varepsilon \\
&= -2\varepsilon\,(1+\varpi) \int_{\mathbb{R}^N} \nabla_t \Phi_z(t,z) \cdot \big[f'\big(\sqrt{\varepsilon}\, z\big) + t\mu'\big(\sqrt{\varepsilon}\, z\big) \big] \frac{\partial}{\partial t_m} W_0(t)\, \mathrm{d}t \\
&\quad - 2\varepsilon\,(1+\varpi) A_z\, \mu\big(\sqrt{\varepsilon}\, z\big) \int_{\mathbb{R}^N} \nabla_t Z_0(t) \cdot \big[f'\big(\sqrt{\varepsilon}\, z\big) + t\mu'\big(\sqrt{\varepsilon}\, z\big) \big] \frac{\partial}{\partial t_m} W_0(t)\, \mathrm{d}t \\
&\quad + \varepsilon^{3/2} h_\varepsilon \\
&\equiv \varepsilon \sum_{i=1}^{N} \zeta_{1,i}(z) f'_i + \varepsilon \zeta_2(z) f'_m + \varepsilon \zeta_3(z)\mu' + \varepsilon^{\varrho} h_\varepsilon,
\end{aligned}
\tag{7.4}
$$

where we have denoted

$$
\begin{aligned}
\zeta_{1,i}(z) &= -2 \int_{\mathbb{R}^N} \frac{\partial}{\partial t_i} \Phi_z(t,z) \cdot \frac{\partial}{\partial t_m} W_0(t)\, \mathrm{d}t, \\
\zeta_2(z) &= -2 A_z\, \mu\big(\sqrt{\varepsilon}\, z\big) \int_{\mathbb{R}^N} \frac{\partial}{\partial t_m} Z_0(t) \cdot \frac{\partial}{\partial t_m} W_0(t)\, \mathrm{d}t,
\end{aligned}
\tag{7.5}
$$

$$\zeta_3(z) = -2\int_{\mathbb{R}^N}\Big(t\cdot\nabla_t\Phi_z(t,z)\Big)\frac{\partial}{\partial t_m}W_0(t)\,\mathrm{d}t.$$

The other term is

$$\begin{aligned}
\mathcal{J}_{6,3}(z) &\equiv \int_{\mathbb{R}^N}\mathbf{N}(\phi^*)\frac{\partial U}{\partial s_m}\,\mathrm{d}s\\
&= \int_{\mathbb{R}^N}\Big[\big(U+\sqrt{\varepsilon}e\Psi+\sqrt{\varepsilon}\,(1+\varpi)\phi^*\big)^p-\big(U+\sqrt{\varepsilon}e\Psi\big)^p\\
&\quad - p\big(U+\sqrt{\varepsilon}e\Psi\big)^{p-1}\sqrt{\varepsilon}\,(1+\varpi)\phi^*\Big]\frac{\partial U}{\partial s_m}\,\mathrm{d}s\\
&\quad + p\int_{\mathbb{R}^N}\Big[\big(U+\sqrt{\varepsilon}e\Psi\big)^{p-1}-U^{p-1}\Big]\sqrt{\varepsilon}\,(1+\varpi)\phi^*\frac{\partial U}{\partial s_m}\,\mathrm{d}s+\varepsilon^{3/2}h_\varepsilon\\
&= \varepsilon p(p-1)\int_{\mathbb{R}^N}U^{p-2}\big(\phi^*\big)^2\frac{\partial U}{\partial s_m}\,\mathrm{d}s+\varepsilon p(p-1)e\int_{\mathbb{R}^N}U^{p-2}\,\phi^*\Psi\,\frac{\partial U}{\partial s_m}\,\mathrm{d}s+\varepsilon^{3/2}h_\varepsilon\\
&\equiv \varepsilon\rho_1+\varepsilon\rho_2 e+\varepsilon^{\varrho}h_\varepsilon,
\end{aligned}\tag{7.6}$$

where we have used the definition of ϕ^* in (3.59). In fact, ρ_1 and ρ_2 have the following form

$$\begin{aligned}
\rho_1 &= \mu^{-3}\int_{\mathbb{R}^N}\frac{1}{\big(1+t^2\big)^3}\Big[\Phi_0(t,z)\Phi_m(t,z)+\sum_{i=1}^N\Phi_i(t,z)\Phi_{in}(t,z)+\mu\Phi_m(t,z)Z_0(t)A\Big]\,\mathrm{d}t\\
&\quad + \mu^{-2}\int_{\mathbb{R}^N}\frac{1}{\big(1+t^2\big)^3}\Phi_m(t,z)Z_0(t)A\,\mathrm{d}t, \qquad (7.7)\\
\rho_2 &= \mu^{-3}\int_{\mathbb{R}^N}\frac{t_m}{\big(1+t^2\big)^3}\Phi_m(t,z)Z_0(t)\,\mathrm{d}t,
\end{aligned}$$

We shall show the terms $\sum_{i=1}^N\zeta_{1,i}(z)f_i'+\zeta_2(z)f_m'+\zeta_3(z)\mu'$ and $\rho_1+\rho_2 e$ are small enough for our further setting in some sense in Section 8. The term $\mathcal{J}_{6,2}$ can be estimated as $\mathcal{J}_5$, and we omit it here.

As a conclusion, we sum up all estimates together and then get

$$\begin{aligned}
\int_{\mathbb{R}^N}\mathscr{E}_3\frac{\partial U}{\partial s_m}\,\mathrm{d}s &= \varepsilon\, d_1\mu f_m''+\varepsilon\sum_{i=1}^N\zeta_{1,i}(z)f_i'+\varepsilon\zeta_2(z)f_m'+\varepsilon\zeta_3(z)\mu'+\varepsilon\rho_1+\varepsilon\rho_2 e\\
&\quad+\varepsilon b_{1\varepsilon}\mu'|f'|+\varepsilon b_{1\varepsilon}|f'|^2\\
&\quad+\varepsilon^{3/2}\big[\tilde{b}_1e'f'+\tilde{b}_1e\mu'f'+\tilde{b}_3e|f'|^2+\tilde{b}_3ef''\big].
\end{aligned}\tag{7.8}$$

In the above, we have use the decomposition of μ in (3.15) and the constraint for $\tilde{\mu}$ in (3.22).

Part 2: In this part we will deal with the components with ϕ involved. Using the quadratic nature of $\mathcal{N}(\phi)$ and Proposition 6.1, we get the estimates for the terms below

$$\int_{\mathbb{R}^N}\left[\mathcal{N}(\phi)+\eta_{6\delta}^{\varepsilon}B_2(\phi)+p\big(v_3^{p-1}-W^{p-1}\big)\right]\frac{\partial U}{\partial s_m}\,\mathrm{d}s=\varepsilon^{3/2}r,$$

where r is the sum of the form

$$\begin{aligned}&h_0(\sqrt{\varepsilon}\,z)\big[\,h_1\big(z,\ \mu,\ f,\ e,\ \mu',\ f',\ \sqrt{\varepsilon}e'\big)\\&\quad+o(1)h_2\big(z,\ \mu,\ f,\ e,\ \mu',\ f',\ \sqrt{\varepsilon}e',\ \mu'',\ f'',\ \varepsilon e''\big)\big],\end{aligned}$$

where h_0 is a smooth function uniformly bounded in ε, h_1 depends smoothly on μ, f, e and their first derivative, it is bounded in the sense that

$$\|h_1\|_\infty \le c\|(\mu, f, e)\|,$$

and it is compact, as a direct application of Ascoli-Arzela Theorem. The function h_2 depends on (μ, f, e), together with their first and second derivatives. An important remark is that h_2 depends linearly on μ'', f'', e''. Furthermore, it is Lipschitz with

$$\|h_2(\mu_1, f_1, e_1)-h_2(\mu_2, f_2, e_2)\| \le o(1)\|(\mu_1-\mu_2, f_1-f_2, e_1-e_2)\|. \tag{7.9}$$

The estimates for other terms can be done as follows

$$\begin{aligned}\mathcal{I}_1&=\int_{\mathbb{R}^N}\big(\phi_{zz}-\varepsilon\phi\big)\frac{\partial U}{\partial s_m}\,\mathrm{d}s\\&=\frac{\partial^2}{\partial z^2}\int_{\mathbb{R}^N}\phi\frac{\partial U}{\partial s_m}\,\mathrm{d}s-2\int_{\mathbb{R}^N}\frac{\partial}{\partial z}\phi\cdot\frac{\partial^2 U}{\partial s_m\partial z}\,\mathrm{d}s\\&\quad-\int_{\mathbb{R}^N}\phi\cdot\frac{\partial^3 U}{\partial s_m\partial z^2}\,\mathrm{d}s-\varepsilon\int_{\mathbb{R}^N}\phi\frac{\partial U}{\partial s_m}\,\mathrm{d}s\\&\equiv\Lambda_m''+\mathcal{I}_{11}+\mathcal{I}_{12}-\varepsilon\Lambda_m.\end{aligned}\tag{7.10}$$

From (6.12), there holds

$$\Lambda_m''=\Theta'(z)+\frac{\mathrm{d}}{\mathrm{d}z}\int_{\mathbb{R}^N}\mathcal{G}(x,z)\widetilde{\mathbf{Z}}_i(x)\,\mathrm{d}x,$$

with the function $\mathcal{G}$ defined in (6.1) and $|\Theta'(z)|_\infty \le \varepsilon^{3/2}$. Note that the components of $\mathcal{G}$ without involving $\sqrt{\varepsilon}\,z$ are the following two terms

$$-\varepsilon\sum_{i,j=1}^{N}D_{ij}\varphi_0\,s_j(1+\varpi)\big(\phi_{1,s_i}+\phi_{2,s_i}\big),\quad \sqrt{\varepsilon}\,(1+\varpi)\hat{\mathscr{B}}_2.$$

By using the expression (3.62), the estimates related to these two terms can be handled as that in (7.4). We omit the details here. With the help of the Proposition 6.1, we have the similar estimate as above

$$\mathcal{I}_{11}+\mathcal{I}_{12}-\varepsilon\Lambda_m=\varepsilon^{3/2}r.$$

7.2. Derivation of the Equation for e

In this subsection, we derive the equation for e. We only compute the main components, while the other parts can dealt similarly as those in previous subsection.

To derive the main components of the equation for the unknown parameter e, we multiply $\mathscr{E}_3$ by $\mathbf{Z}_0 = \Psi$ and integrate against the variable s on $\mathbb{R}^N$

$$\int_{\mathbb{R}^N} \mathscr{E}_3 \Psi \, \mathrm{d}s = \int_{\mathbb{R}^N} \Big(\mathscr{E}_1 + \overline{\mathscr{E}}_1 \Big) \Psi \, \mathrm{d}s.$$

For the components of $\mathscr{E}_1$ in (3.33), the estimates can be done as follows.

$$\begin{aligned}
\mathscr{J}_7 &\equiv \int_{\mathbb{R}^N} \mathscr{E}_{01} \Psi \, \mathrm{d}s \\
&= \int_{\mathbb{R}^N} \mathscr{E}_{00} \Psi \, \mathrm{d}s + \sqrt{\varepsilon}\, \mu^{-2} \lambda_0 \, e \int_{\mathbb{R}^N} \Psi^2 \, \mathrm{d}s \\
&= \Big(\sqrt{\varepsilon}\, \mu^{-2} \lambda_0 \, e + \varepsilon^{3/2} e'' \Big) \int_{\mathbb{R}^N} \Psi^2 \, \mathrm{d}s + 2\varepsilon^{3/2} e' \mu' \int_{\mathbb{R}^N} \frac{\partial \Psi}{\partial \mu} \Psi \, \mathrm{d}s + \varepsilon^{3/2} e \mu'' \int_{\mathbb{R}^N} \frac{\partial \Psi}{\partial \mu} \Psi \, \mathrm{d}s \\
&\quad + \varepsilon^{3/2} e (\mu')^2 \int_{\mathbb{R}^N} \frac{\partial^2 \Psi}{\partial \mu^2} \Psi \, \mathrm{d}s + \varepsilon^{3/2} e \sum_{i,j=1}^{N} f_i' f_j' \int_{\mathbb{R}^N} \frac{\partial^2 \Psi}{\partial s_i \partial s_j} \Psi \, \mathrm{d}s \\
&= \sqrt{\varepsilon}\, d_2 \mu^2 \Big(\varepsilon e'' + \mu^{-2} \lambda_0 \, e \Big) + \varepsilon^{3/2} \big[\tilde{b}_1 (e' \mu' + e \mu'') + \tilde{b}_2 e (\mu')^2 + \tilde{b}_3 e |f'|^2 \big],
\end{aligned}$$

where we have denoted the constant d_2 by the relation

$$d_2 \mu^2 \equiv \int_{\mathbb{R}^N} \Psi^2 \, \mathrm{d}s = \mu^2 \int_{\mathbb{R}^N} |Z_0(t)|^2 \, \mathrm{d}t > 0. \tag{7.11}$$

Again using (3.30), we get the estimate of the firs component $\mathscr{J}_{8,1}$ in

$$\mathscr{J}_8 \equiv \int_{\mathbb{R}^N} \mathscr{E}_{11} \Psi \, \mathrm{d}s,$$

which was

$$\begin{aligned}
\mathscr{J}_{8,1} &= -\varepsilon \int_{\mathbb{R}^N} U^p \log U \Psi \, \mathrm{d}s \\
&= -\varepsilon \frac{(N-2)}{2} \tau_N^{-\frac{N+2}{N-2}} \int_{\mathbb{R}^N} \frac{\mu^{\frac{N+2}{2}} \mu^{\frac{-N+2}{2}}}{\big(\mu^2 + \ell^2\big)^{(N+2)/2}} \log \frac{\mu}{\big(\mu^2 + \ell^2\big)} Z_0\Big(\frac{\ell}{\mu}\Big) \, \mathrm{d}s \\
&= \varepsilon \frac{(N-2)}{2} \tau_N^{-\frac{N+2}{N-2}} \int_{\mathbb{R}^N} \frac{\mu^{-\frac{N-2}{2}} \log \mu}{\big(|t|^2 + 1\big)^{\frac{N+2}{2}}} Z_0(t) \, \mathrm{d}t \\
&\quad + \varepsilon \frac{(N-2)}{2} \tau_N^{-\frac{N+2}{N-2}} \int_{\mathbb{R}^N} \frac{\mu^{-\frac{N-2}{2}}}{\big(|t|^2 + 1\big)^{\frac{N+2}{2}}} \log \big(|t|^2 + 1\big) Z_0(t) \, \mathrm{d}t \\
&= \varepsilon \, d_3 \mu^{-\frac{N-2}{2}} \log \mu + \varepsilon \, d_4 \mu^{-\frac{N-2}{2}}.
\end{aligned}$$

where d_3 and d_4 are two positive constants. On the other hand, the second part is

$$\begin{aligned}\mathcal{J}_{8,2} &= \int_{\mathbb{R}^N} \left[\mathcal{P}(U+\sqrt{\varepsilon}\, e\Psi) + \mathcal{Q}(U+\sqrt{\varepsilon}\, e\Psi) + \mathcal{R}(U+\sqrt{\varepsilon}\, e\Psi)\right]\Psi\, \mathrm{d}s \\ &= \varepsilon b_{1\varepsilon} e^2 + \varepsilon^{3/2} b_{2\varepsilon} + \varepsilon^{3/2}\log\varepsilon\, b_{1\varepsilon}\, e + \varepsilon^2 \log\varepsilon\, b_{2\varepsilon}.\end{aligned}$$

The third part is

$$\begin{aligned}\mathcal{J}_{8,3} = \int_{\mathbb{R}^N} \Bigg[& \sqrt{\varepsilon} ep\left(\left(1+\frac{N-2}{4}\varepsilon\log\varepsilon\right)^{-1} - 1\right) W^{p-1}\Psi \\ & + O(\varepsilon^2 \big|\log\varepsilon\big|^2)(U+\sqrt{\varepsilon}\, e\Psi)^p\, \mathbb{N}_0\big(U+\sqrt{\varepsilon}\, e\Psi\big)\Bigg]\Psi\, \mathrm{d}s \\ = {} & \varepsilon^{3/2}\log\varepsilon\, b_{2\varepsilon}.\end{aligned}$$

Hence, there holds

$$\begin{aligned}\mathcal{J}_8 = \int_{\mathbb{R}^N} \mathcal{E}_{11}\Psi\, \mathrm{d}s = {} & \varepsilon\, d_3 \mu^{-\frac{N-2}{2}}\log\mu + \varepsilon\, d_4\mu^{-\frac{N-2}{2}} + \varepsilon b_{1\varepsilon} e^2 \\ & + \varepsilon^{3/2} b_{2\varepsilon} + \varepsilon^{3/2}\log\varepsilon\, b_{1\varepsilon}\, e + \varepsilon^2\log\varepsilon\, b_{2\varepsilon}.\end{aligned}$$

For the next component, we also obtain its estimate

$$\begin{aligned}\mathcal{J}_9 &\equiv \int_{\mathbb{R}^N} \mathcal{E}_{12}\Psi\, \mathrm{d}s \\ &= \varepsilon \int_{\mathbb{R}^N} \Bigg[\frac{\partial^2 U}{\partial\mu^2}(\mu')^2 + \frac{\partial U}{\partial\ell}\sum_{i=1}^N \frac{(f_i')^2}{\ell} + \frac{\partial U}{\partial \ell}\sum_{i=1}^N \frac{(s_i-f_i)^2(f_i')^2}{\ell^4} \\ &\quad + \frac{\partial^2 U}{\partial\ell^2}\left(\sum_{i=1}^N \frac{(s_i-f_i)f_i'}{\ell}\right)^2\Bigg]\Psi\, \mathrm{d}s \\ &\equiv \varepsilon\, b_{1\varepsilon}|\mu'|^2 + \varepsilon\, b_{2\varepsilon}|f'|^2.\end{aligned}$$

We now deal with the term $\mathcal{J}_{10}$ of the form

$$\mathcal{J}_{10} = \int_{\mathbb{R}^N} B_2\big(U+\sqrt{\varepsilon} e\Psi\big)\Psi\, \mathrm{d}s.$$

This term can be dealt with as that for $\mathcal{J}_5$.

By using of (3.17), it is easy to show that

$$\begin{aligned}\mathcal{J}_{11} &\equiv -\varepsilon\int_{\mathbb{R}^N}\Big(U+\sqrt{\varepsilon} e\Psi\Big)\Psi\, \mathrm{d}s \\ &= -\varepsilon\tau_N\mu^2\int_{\mathbb{R}^N}\frac{|Z_0(t)|^2}{\big(|t|^2+1\big)^{\frac{N-2}{2}}}\, \mathrm{d}t - \varepsilon^{3/2}\mu^2\,\tilde{b}_0\, e \\ &\equiv -\varepsilon\, C_1\,\mu^2 - \varepsilon^{3/2}\mu^2\,\tilde{b}_0\, e \quad \text{with } C_1 > 0.\end{aligned}$$

One can check that

$$
\begin{aligned}
\mathcal{J}_{12} &\equiv \int_{\mathbb{R}^N} \bar{\mathscr{E}}_1 \Psi \,\mathrm{d}s \\
&= \sqrt{\varepsilon}(1+\varpi)\int_{\mathbb{R}^N} \mathscr{E}_2 \Psi \,\mathrm{d}s + \sqrt{\varepsilon}\int_{\mathbb{R}^N} B_2(\phi^*)\Psi \,\mathrm{d}s + \int_{\mathbb{R}^N} \mathbf{N}(\phi^*)\Psi \,\mathrm{d}s \\
&= \mathcal{J}_{12,1} + \mathcal{J}_{12,2} + \mathcal{J}_{12,3}.
\end{aligned}
$$

By the formula (3.61) here we only compute the first term

$$
\begin{aligned}
\mathcal{J}_{12,1}(z) &\equiv \sqrt{\varepsilon}\,(1+\varpi)\int_{\mathbb{R}^N} \mathscr{E}_2 \Psi \,\mathrm{d}s \\
&= -2\varepsilon\,(1+\varpi)\int_{\mathbb{R}^N} \mu^{-(N+4)/2}\nabla_x \Phi_z \cdot \Big(f'\mu + (s-f)\mu'\Big)\Psi \,\mathrm{d}s \\
&\quad - 2\varepsilon\,(1+\varpi)A_z \int_{\mathbb{R}^N} \mu^{-(N+2)/2}\nabla_x Z_0 \cdot \Big(f'\mu + (s-f)\mu'\Big)\Psi \,\mathrm{d}s + \varepsilon^{3/2} h_\varepsilon \\
&= -2\varepsilon\,(1+\varpi)\int_{\mathbb{R}^N} \nabla_t \Phi_z(t,z) \cdot \Big[f'\big(\sqrt{\varepsilon}\,z\big) + t\mu'\big(\sqrt{\varepsilon}\,z\big)\Big] Z_0(t)\,\mathrm{d}t \\
&\quad - 2\varepsilon\,(1+\varpi)A_z\,\mu\big(\sqrt{\varepsilon}\,z\big)\int_{\mathbb{R}^N} \nabla_t Z_0(t) \cdot \Big[f'\big(\sqrt{\varepsilon}\,z\big) + t\mu'\big(\sqrt{\varepsilon}\,z\big)\Big] Z_0(t)\,\mathrm{d}t \\
&\quad + \varepsilon^{3/2} h_\varepsilon \\
&\equiv \varepsilon \sum_{i=1}^{N} \zeta_{4,i}(z) f_i' + \varepsilon\zeta_5(z)\mu' + \varepsilon^{\varrho} h_\varepsilon,
\end{aligned}
\tag{7.12}
$$

where we have denoted

$$
\begin{aligned}
\zeta_{4,i}(z) &= -2\int_{\mathbb{R}^N} \frac{\partial}{\partial t_i}\Phi_z(t,z)\cdot Z_0(t)\,\mathrm{d}t \\
\zeta_5(z) &= -2\int_{\mathbb{R}^N} \Big(t\cdot\nabla_t\Phi_z(t,z)\Big)Z_0(t)\,\mathrm{d}t - 2A_z\,\mu\big(\sqrt{\varepsilon}\,z\big)\int_{\mathbb{R}^N}\Big(t\cdot\nabla_t Z_0(t)\Big)Z_0(t)\,\mathrm{d}t.
\end{aligned}
\tag{7.13}
$$

The other term is

$$
\begin{aligned}
\mathcal{J}_{12,3}(z) &\equiv \int_{\mathbb{R}^N} \mathbf{N}(\phi^*)\Psi \,\mathrm{d}s \\
&= \int_{\mathbb{R}^N} \Big[\big(U + \sqrt{\varepsilon}e\Psi + \sqrt{\varepsilon}\,(1+\varpi)\phi^*\big)^p - \big(U+\sqrt{\varepsilon}e\Psi\big)^p \\
&\qquad - p\big(U+\sqrt{\varepsilon}e\Psi\big)^{p-1}\sqrt{\varepsilon}\,(1+\varpi)\phi^*\Big]\Psi\,\mathrm{d}s \\
&\qquad + p\int_{\mathbb{R}^N}\Big[\big(U+\sqrt{\varepsilon}e\Psi\big)^{p-1} - U^{p-1}\Big]\sqrt{\varepsilon}\,(1+\varpi)\phi^*\Psi\,\mathrm{d}s + \varepsilon^{3/2}h_\varepsilon \\
&= \varepsilon p(p-1)\int_{\mathbb{R}^N} U^{p-2}\Big(\phi^*\Big)^2\Psi\,\mathrm{d}s + \varepsilon p(p-1)e\int_{\mathbb{R}^N} U^{p-2}\,\phi^*\Psi^2\,\mathrm{d}s + \varepsilon^{3/2}h_\varepsilon \\
&\equiv \varepsilon\rho_3 + \varepsilon\rho_4 e + \varepsilon^{\varrho}h_\varepsilon,
\end{aligned}
\tag{7.14}
$$

where we have used the definition of ϕ^* in (3.59). In fact, ρ_3 and ρ_4 have the following forms

$$\rho_3 = \mu^{-2}\int_{\mathbb{R}^N} \frac{(1+t^2)^{N/2}}{(1+t^2)^3}\Big[\sum_{i=0}^{N}\Phi_i^2(t,z)+\sum_{i\neq j}^{N}\Phi_i^2(t,z)\Big]Z_0(t)\,\mathrm{d}t$$
$$+\mu^{-1}\int_{\mathbb{R}^N}\frac{(1+t^2)^{N/2}}{(1+t^2)^3}\Phi_0(t,z)\big(Z_0(t)\big)^2A\,\mathrm{d}t+\int_{\mathbb{R}^N}\frac{(1+t^2)^{N/2}}{(1+t^2)^3}\big(Z_0(t)\big)^3A^2\,\mathrm{d}t, \quad (7.15)$$
$$\rho_4 = \mu^{-1}\int_{\mathbb{R}^N}\frac{(1+t^2)^{N/2}}{(1+t^2)^3}\Phi_0(t,z)\big(Z_0(t)\big)^2\,\mathrm{d}t+\int_{\mathbb{R}^N}\frac{(1+t^2)^{N/2}}{(1+t^2)^3}\big(Z_0(t)\big)^3A\,\mathrm{d}t.$$

We shall show the terms $\varepsilon\sum_{i=1}^N\zeta_{4,i}(z)f_i'+\varepsilon\zeta_5(z)\mu'$ and $\rho_3+\rho_4 e$ are small enough for our further setting in some sense in section 8. The term $\mathcal{J}_{12,2}$ can be estimated as $\mathcal{J}_5$, and we omit it here.

We make a conclusion that

$$\int_{\mathbb{R}^N}\mathscr{E}_3\Psi\,\mathrm{d}s = \sqrt{\varepsilon}\,d_2\,\mu^2\big[\varepsilon e''+\mu^{-2}\lambda_0 e\big]+\varepsilon\,d_3\mu^{-\frac{N-2}{2}}\log\mu+\varepsilon\,d_4\mu^{-\frac{N-2}{2}}-\varepsilon\,C_1\,\mu^2$$
$$+\varepsilon b_{1\varepsilon}e^2+\varepsilon\,b_{1\varepsilon}|\mu'|^2+\varepsilon\,b_{2\varepsilon}|f'|^2+\varepsilon\sum_{i=1}^N\zeta_{4,i}(z)f_i'+\varepsilon\zeta_5(z)\mu'$$
$$+\varepsilon\rho_3+\varepsilon\rho_4 e+\varepsilon^{\varrho}\,r. \quad (7.16)$$

There are also some terms with involving ϕ, whose estimates can done as the same for the equation f. We omit the details here.

7.3. *Derivation of the Equation for* μ

As we sated in the above, to derive the main components of the equation for the unknown parameter μ, we multiply $\mathscr{E}_3$ by $\mathbf{Z}_{N+1}=U_\mu$ and integrate against the variable s on $\mathbb{R}^N$

$$\int_{\mathbb{R}^N}\mathscr{E}_3\frac{\partial U}{\partial\mu}\,\mathrm{d}s=\int_{\mathbb{R}^N}\mathscr{E}_3=\big[\mathscr{E}_1+\bar{\mathscr{E}}_1\big]\frac{\partial U}{\partial\mu}\,\mathrm{d}s.$$

For the component of $\mathscr{E}_1=\mathscr{E}_{01}+\mathscr{E}_{11}+\mathscr{E}_{12}$ in (3.33), the estimates can be done as follows. We start from the estimate

$$\begin{aligned}\mathcal{J}_{13} &\equiv \int_{\mathbb{R}^N}\mathscr{E}_{01}\frac{\partial U}{\partial\mu}\,\mathrm{d}s\\ &= 2\varepsilon^{3/2}e'\mu'\int_{\mathbb{R}^N}\frac{\partial\Psi}{\partial\mu}\frac{\partial U}{\partial\mu}\,\mathrm{d}s+\varepsilon^{3/2}e\mu''\int_{\mathbb{R}^N}\frac{\partial\Psi}{\partial\mu}\frac{\partial U}{\partial\mu}\,\mathrm{d}s\\ &\quad+\varepsilon^{3/2}e(\mu')^2\int_{\mathbb{R}^N}\frac{\partial^2\Psi}{\partial\mu^2}\frac{\partial U}{\partial\mu}\,\mathrm{d}s+\varepsilon^{3/2}e\sum_{i,j=1}^N f_i'f_j'\int_{\mathbb{R}^N}\frac{\partial^2\Psi}{\partial s_i\partial s_j}\frac{\partial U}{\partial\mu}\,\mathrm{d}s\\ &= \varepsilon^{3/2}\big[d_3(e'\mu'+e\mu'')+d_4e(\mu')^2+\tilde{b}_3e|f'|^2\big].\end{aligned}$$

Again using (3.30), we get the estimate of the firs component $\mathcal{J}_{14,1}$ in

$$\mathcal{J}_{14} \equiv \int_{\mathbb{R}^N} \mathcal{E}_{11} \frac{\partial U}{\partial \mu} \, \mathrm{d}s,$$

which was

$$\begin{aligned}
\mathcal{J}_{14,1} &= -\varepsilon \int_{\mathbb{R}^N} U^p \log U \frac{\partial U}{\partial \mu} \, \mathrm{d}s \\
&= -\varepsilon \frac{N-2}{2(p+1)} \log \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \int_{\mathbb{R}^N} U^{p+1} \, \mathrm{d}s \\
&\quad + \varepsilon \frac{N-2}{2} \int_{\mathbb{R}^N} U^p \frac{\partial U}{\partial \mu} \log(\mu^2 + \ell^2) \, \mathrm{d}s \\
&= \varepsilon \frac{(N-2)^2 \tau_N}{4} \int_{\mathbb{R}^N} \frac{\mu^{N-1}(\ell^2 - \mu^2)}{(\mu^2 + \ell^2)^{N+1}} \log(\mu^2 + \ell^2) \, \mathrm{d}s \\
&= \varepsilon \frac{(N-2)^2 \tau_N}{4\mu} \int_{\mathbb{R}^N} \frac{(|t|^2 - 1)}{(|t|^2 + 1)^{N+1}} \log(|t|^2 + 1) \, \mathrm{d}t.
\end{aligned}$$

It is easy to prove that

$$\int_{\mathbb{R}^N} \frac{(|t|^2 - 1)}{(|t|^2 + 1)^{N+1}} \log(|t|^2 + 1) \, \mathrm{d}t = 2|S^{N-1}| \int_1^{+\infty} \frac{(r^2 - 1) r^{N-1}}{(r^2 + 1)^{N+1}} \log r \, \mathrm{d}r > 0,$$

where $|S^{N-1}|$ is the area of unit sphere in $\mathbb{R}^N$. On the other hand, the second part is

$$\begin{aligned}
\mathcal{J}_{14,2} &= \int_{\mathbb{R}^N} \left[\mathcal{P}(U + \sqrt{\varepsilon}\, e\Psi) + \mathcal{Q}(U + \sqrt{\varepsilon}\, e\Psi) + \mathcal{R}(U + \sqrt{\varepsilon}\, e\Psi) \right] \frac{\partial U}{\partial \mu} \, \mathrm{d}s \\
&= \varepsilon b_{1\varepsilon} e^2 + \varepsilon^{3/2} b_{2\varepsilon} + \varepsilon^{3/2} \log \varepsilon \, b_{1\varepsilon} \, e + \varepsilon^2 \log \varepsilon \, b_{2\varepsilon}.
\end{aligned}$$

The third part is

$$\begin{aligned}
\mathcal{J}_{14,3} = \int_{\mathbb{R}^N} \Bigg[& \sqrt{\varepsilon} e p \left(\left(1 + \frac{N-2}{4} \varepsilon \log \varepsilon \right)^{-1} - 1 \right) W^{p-1} \Psi \\
& + O(\varepsilon^2 |\log \varepsilon|^2)(U + \sqrt{\varepsilon}\, e\Psi)^p \, \mathbb{N}_0(U + \sqrt{\varepsilon}\, e\Psi) \Bigg] \frac{\partial U}{\partial \mu} \, \mathrm{d}s \\
= {} & \varepsilon^{3/2} \log \varepsilon \, b_{2\varepsilon}.
\end{aligned}$$

Hence, there holds

$$\mathcal{J}_{14} = \int_{\mathbb{R}^N} \mathcal{E}_{11} \frac{\partial U}{\partial \mu} \, \mathrm{d}s = \varepsilon \, C_2 \mu^{-1} + \varepsilon b_{1\varepsilon} e^2 + \varepsilon^{3/2} b_{2\varepsilon} + \varepsilon^{3/2} \log \varepsilon \, b_{1\varepsilon} \, e + \varepsilon^2 \log \varepsilon \, b_{2\varepsilon}.$$

for some constant $C_2 > 0$.

For the next component, we also obtain its estimate

$$
\begin{aligned}
\mathcal{J}_{15} &\equiv \int_{\mathbb{R}^N} \mathscr{E}_{12} \frac{\partial U}{\partial \mu}\, \mathrm{d}s \\
&= \varepsilon \int_{\mathbb{R}^N} \left[\frac{\partial U}{\partial \mu}\mu'' + \frac{\partial^2 U}{\partial \mu^2}(\mu')^2 + \frac{\partial U}{\partial \ell} \sum_{i=1}^N \frac{(f_i')^2}{\ell} + \frac{\partial U}{\partial \ell} \sum_{i=1}^N \frac{(s_i - f_i)^2 (f_i')^2}{\ell^4} \right] \frac{\partial U}{\partial \mu}\, \mathrm{d}s \\
&\quad + \varepsilon \int_{\mathbb{R}^N} \frac{\partial^2 U}{\partial \ell^2} \left[\sum_{i=1}^N \frac{(s_i - f_i) f_i'}{\ell} \right]^2 \frac{\partial U}{\partial \mu}\, \mathrm{d}s \\
&= \varepsilon \frac{(N-2)^2 \tau_N^2}{4} \frac{(\mu')^2}{\mu} \int_{\mathbb{R}^N} \frac{(|t|^2 - 1)(2 - 6|t|^2)}{(|t|^2+1)^{N+1}}\, \mathrm{d}t \\
&\quad + \varepsilon \frac{(N-2)^2 \tau_N^2}{4} \mu'' \int_{\mathbb{R}^N} \frac{(|t|^2 - 1)}{(|t|^2+1)^{N}}\, \mathrm{d}t \\
&\quad + \varepsilon \frac{(N-4)(N-2)^2 \tau_N^2}{16} \frac{(\mu')^2}{\mu} \int_{\mathbb{R}^N} \frac{(|t|^2 - 1)^3}{(|t|^2+1)^{N+1}}\, \mathrm{d}t + \varepsilon b_{1\varepsilon} |f'|^2 \\
&\equiv \varepsilon\, C_3 \mu'' + \varepsilon\, C_4 (\mu')^2 \mu^{-1} + \varepsilon b_{1\varepsilon} |f'|^2.
\end{aligned}
$$

It can also be verified that

$$
\int_{\mathbb{R}^N} \frac{(|t|^2 - 1)}{(|t|^2+1)^{N}}\, \mathrm{d}t = \left|S^{N-1}\right| \int_1^{+\infty} \frac{(r^2-1)^2 r^{N-3}}{(r^2+1)^N}\, \mathrm{d}r > 0,
$$

which implies that $C_3 > 0$.

We now deal with the term $\mathcal{J}_{16}$ of the form

$$
\mathcal{J}_{16} = \int_{\mathbb{R}^N} B_2\big(U + \sqrt{\varepsilon} e \Psi\big) \frac{\partial U}{\partial \mu}\, \mathrm{d}s.
$$

This term can be estimated as $\mathcal{J}_5$.

By using of (3.3), it is easy to show that

$$
\begin{aligned}
\mathcal{J}_{17} &\equiv -\varepsilon \int_{\mathbb{R}^N} \Big(U + \sqrt{\varepsilon} e \Psi\Big) \frac{\partial U}{\partial \mu}\, \mathrm{d}s \\
&= -\frac{\varepsilon}{2} \frac{\mathrm{d}}{\mathrm{d}\mu} \int_{\mathbb{R}^N} U^2\, \mathrm{d}s \\
&= -\varepsilon (\tau_N)^2 \mu \int_{\mathbb{R}^N} \frac{1}{(|t|^2+1)^{N-2}}\, \mathrm{d}t \\
&\equiv -\varepsilon\, C_1\, \mu \quad \text{with } C_1 > 0.
\end{aligned}
$$

As we have stated in the above, Ψ and U_μ are orthogonal. By using the orthogonal condition (3.55), it is easy to check that

$$
\mathcal{J}_{18} \equiv \int_{\mathbb{R}^N} \bar{\mathscr{E}}_1 \frac{\partial U}{\partial \mu}\, \mathrm{d}s
$$

$$
\begin{aligned}
&= \sqrt{\varepsilon}(1+\varpi)\int_{\mathbb{R}^N} \mathscr{E}_2 \frac{\partial U}{\partial \mu}\,\mathrm{d}s + \sqrt{\varepsilon}\int_{\mathbb{R}^N} B_2(\phi^*)\frac{\partial U}{\partial \mu}\,\mathrm{d}s + \int_{\mathbb{R}^N} \mathbf{N}(\phi^*)\frac{\partial U}{\partial \mu}\,\mathrm{d}s \\
&= \mathscr{J}_{18,1} + \mathscr{J}_{18,2} + \mathscr{J}_{18,3}.
\end{aligned}
$$

Hence, by the formula (3.61) here we only compute the first term

$$
\begin{aligned}
\mathscr{J}_{18,1}(z) &\equiv \sqrt{\varepsilon}\,(1+\varpi)\int_{\mathbb{R}^N} \mathscr{E}_2 \frac{\partial U}{\partial \mu}\,\mathrm{d}s \\
&= -2\varepsilon\,(1+\varpi)\int_{\mathbb{R}^N} \mu^{-(N+4)/2}\nabla_x \Phi_z \cdot \Big(f'\mu + (s-f)\mu'\Big)\frac{\partial U}{\partial \mu}\,\mathrm{d}s \\
&\quad - 2\varepsilon\,(1+\varpi)A_z \int_{\mathbb{R}^N} \mu^{-(N+2)/2}\nabla_x Z_0 \cdot \Big(f'\mu + (s-f)\mu'\Big)\frac{\partial U}{\partial \mu}\,\mathrm{d}s + \varepsilon^{3/2}h_\varepsilon \\
&= -2\varepsilon\,(1+\varpi)\int_{\mathbb{R}^N} \nabla_t \Phi_z(t,z)\cdot \Big[f'\big(\sqrt{\varepsilon}\,z\big) + t\mu'\big(\sqrt{\varepsilon}\,z\big)\Big]\tilde{Z}_{N+1}(t)\,\mathrm{d}t \\
&\quad - 2\varepsilon\,(1+\varpi)A_z\,\mu\big(\sqrt{\varepsilon}\,z\big)\int_{\mathbb{R}^N} \nabla_t Z_0(t)\cdot \Big[f'\big(\sqrt{\varepsilon}\,z\big) + t\mu'\big(\sqrt{\varepsilon}\,z\big)\Big]\tilde{Z}_{N+1}(t)\,\mathrm{d}t \\
&\quad + \varepsilon^{3/2}h_\varepsilon \\
&\equiv \varepsilon\sum_{i=1}^{N}\zeta_{6,i}(z)f_i' + \varepsilon\zeta_7(z)\mu' + \varepsilon^{\varrho}h_\varepsilon,
\end{aligned}
\tag{7.17}
$$

where we have denoted

$$
\begin{aligned}
\zeta_{6,i}(z) &= -2\int_{\mathbb{R}^N} \frac{\partial}{\partial t_i}\Phi_z(t,z)\cdot \tilde{Z}_{N+1}(t)\,\mathrm{d}t \\
\zeta_7(z) &= -2\int_{\mathbb{R}^N} \Big(t\cdot \nabla_t \Phi_z(t,z)\Big)\tilde{Z}_{N+1}(t)\,\mathrm{d}t \\
&\quad - 2A_z\,\mu\big(\sqrt{\varepsilon}\,z\big)\int_{\mathbb{R}^N} \Big(t\cdot \nabla_t Z_0(t)\Big)\tilde{Z}_{N+1}(t)\,\mathrm{d}t.
\end{aligned}
\tag{7.18}
$$

The other term is

$$
\begin{aligned}
\mathscr{J}_{18,3}(z) &\equiv \int_{\mathbb{R}^N} \mathbf{N}(\phi^*)\frac{\partial U}{\partial \mu}\,\mathrm{d}s \\
&= \int_{\mathbb{R}^N} \Big[\big(U + \sqrt{\varepsilon}e\Psi + \sqrt{\varepsilon}\,(1+\varpi)\phi^*\big)^p - \big(U + \sqrt{\varepsilon}e\Psi\big)^p \\
&\quad - p\big(U + \sqrt{\varepsilon}e\Psi\big)^{p-1}\sqrt{\varepsilon}\,(1+\varpi)\phi^*\Big]\frac{\partial U}{\partial \mu}\,\mathrm{d}s \\
&\quad + p\int_{\mathbb{R}^N}\Big[\big(U + \sqrt{\varepsilon}e\Psi\big)^{p-1} - U^{p-1}\Big]\sqrt{\varepsilon}\,(1+\varpi)\phi^*\frac{\partial U}{\partial \mu}\,\mathrm{d}s + \varepsilon^{3/2}h_\varepsilon \\
&= \varepsilon p(p-1)\int_{\mathbb{R}^N} U^{p-2}\Big(\phi^*\Big)^2\frac{\partial U}{\partial \mu}\,\mathrm{d}s + \varepsilon p(p-1)e\int_{\mathbb{R}^N} U^{p-2}\,\phi^*\Psi\,\frac{\partial U}{\partial \mu}\,\mathrm{d}s + \varepsilon^{3/2}h_\varepsilon \\
&\equiv \varepsilon\rho_5 + \varepsilon\rho_6 e + \varepsilon^{\varrho}h_\varepsilon,
\end{aligned}
\tag{7.19}
$$

where we have used the definition of ϕ^* in (3.59). In fact, ρ_5 and ρ_6 have the following forms

$$\rho_5 = \mu^{-3}\int_{\mathbb{R}^N} \frac{t^2-1}{(1+t^2)^3}\left[\sum_{i=0}^{N}\Phi_i^2(t,z) + \sum_{i\neq j}^{N}\Phi_i^2(t,z)\right]\mathrm{d}t + \mu^{-1}\int_{\mathbb{R}^N}\frac{t^2-1}{(1+t^2)^3}\big(Z_0(t)\big)^2 A^2\,\mathrm{d}t$$
$$+\mu^{-2}\int_{\mathbb{R}^N}\frac{t^2-1}{(1+t^2)^3}\Phi_0(t,z)Z_0(t)A\,\mathrm{d}t,$$
$$\rho_6 = \mu^{-2}\int_{\mathbb{R}^N}\frac{t^2-1}{(1+t^2)^3}\Phi_0(t)Z_0(t,z)A\,\mathrm{d}t + \mu^{-1}\int_{\mathbb{R}^N}\frac{t^2-1}{(1+t^2)^3}\big(Z_0(t)\big)^2 A\,\mathrm{d}t. \tag{7.20}$$

We shall show the terms $\varepsilon\sum_{i=1}^N \zeta_{6,i}(z)f_i' + \varepsilon\zeta_7(z)\mu'$ and $\rho_5+\rho_6 e$ are small enough for our further setting in some sense in section 8. The term $\mathcal{J}_{18,2}$ can be estimated as $\mathcal{J}_5$, and we omit it here.

As a conclusion, we get

$$\int_{\mathbb{R}^N}\mathcal{E}_3\frac{\partial U}{\partial\mu}\,\mathrm{d}s = \varepsilon\, C_3\mu'' - \varepsilon\, C_1\mu + \varepsilon\, C_2\,\mu^{-1} + \varepsilon\, C_4(\mu')^2\mu^{-1} + \varepsilon b_{1\varepsilon}e^2 + \varepsilon b_{1\varepsilon}|f'|^2$$
$$+\varepsilon\rho_5 + \varepsilon\rho_6 e + \varepsilon\sum_{i=1}^N \zeta_{6,i}(z)f_i' + \varepsilon\zeta_7(z)\mu' + \varepsilon^{\varrho} r. \tag{7.21}$$

There are also some terms with involving ϕ, whose estimates can done as the same for the equation f. We omit the details here.

8. Solving the Reduced System

Recall that the boundary conditions for the unknown parameters f, e, μ are given in (3.40)–(3.41), (3.42)–(3.43), (3.38)–(3.39). Using the estimates in previous section, we find the following nonlinear, nonlocal system of differential equations for the parameters (f, e, μ) in the variable $\theta = \varepsilon z$ with $\theta \in (0, 1)$

$$\mathcal{L}_1^*(f) \equiv f''(\theta) + \big[\zeta(\theta) + \zeta_2(\theta)\mathrm{I}\big]f' = \sqrt{\varepsilon}\,M_{1\varepsilon}, \tag{8.1}$$
$$\mathcal{L}_2^*(e) \equiv \varepsilon\mu^2 e''(\theta) + \lambda_0 e(\theta) = \sqrt{\varepsilon}\rho(\theta) + \sqrt{\varepsilon}\,M_{2\varepsilon}, \tag{8.2}$$
$$\mathcal{L}_3^*(\mu) \equiv C_3\mu''(\theta) - C_1\mu(\theta) + C_2\,\mu^{-1} + C_4(\mu')^2\mu^{-1} = \sqrt{\varepsilon}\,M_{3\varepsilon}, \tag{8.3}$$

with the boundary conditions

$$f'(1) + D^2\varphi_1 f(1) = 0, \quad f'(0) + D^2\varphi_0 f(0) = 0; \tag{8.4}$$
$$e'(1) + b_{13}\kappa_1 e(1) = 0, \quad e'(0) + b_{03}\kappa_0 e(0) = 0; \tag{8.5}$$
$$\mu'(1) - b_{11}\,\kappa_1\,\mu(1) = 0, \quad \mu'(0) - b_{01}\,\kappa_0\,\mu(0) = 0, \tag{8.6}$$

where b_{01}, b_{11} are positive constants, b_{13}, b_{03} are two constants, and κ_0, κ_1 are the mean curvatures of the boundary $\partial\Omega$ at the intersection points with Γ. In the above, I is an $N\times N$ identity matrix and the $n-$th row of the $N\times N$ matrix ζ is a vector of the form

$$\Big(\zeta_{n,1}(z), \ldots, \zeta_{n,N}(z)\Big),$$

with the function $\zeta_{1,i}$'s and ζ_2 defined in (7.5). The function $\rho(\theta)$ is defined by

$$\rho = d_3(\mu_0)^{-\frac{N-2}{2}} \log \mu_0 + d_4(\mu_0)^{-\frac{N-2}{2}} - C_1 (\mu_0)^2,$$

where we have used the formula (7.16) and the decomposition of μ in (3.15). Note that μ_0 is a given smooth positive function defined in Lemma 3.1.

The nonlinear operators $M_{1\varepsilon}$, $M_{2\varepsilon}$ and $M_{3\varepsilon}$ can be decomposed in the following form

$$M_{l\varepsilon}(f, e, \mu) = A_{l\varepsilon}(f, e, \mu) + K_{l\varepsilon}(f, e, \mu), \quad l = 1, 2, 3,$$

where $K_{l\varepsilon}$ is uniformly bounded in $L^2(0, 1)$ for (f, e, μ) in $\mathcal{F}$ and is also compact. The operator $A_{l\varepsilon}$ is Lipschitz in this region,

$$\begin{aligned} &\|A_{l\varepsilon}(f_1, e_1, \mu_1) - A_{l\varepsilon}(f_2, e_2, \mu_2)\|_{L^2(0,1)} \\ &\quad \le C\big[\, \|f_1 - f_2\|_a + \|e_1 - e_2\|_b + \|\mu_1 - \mu_2\|_c \,\big]. \end{aligned} \tag{8.7}$$

8.1. *The Resolution Theory*

Before solving (8.1)–(8.6), some basic facts about the invertibility of corresponding linear operators are derived. Firstly, we consider the following problem

$$\begin{aligned} &f''(\theta) + \Big[\zeta(\theta) + \zeta_2(\theta)\mathrm{I}\Big] f' = h(\theta), \quad 0 < \theta < 1, \\ &f'(1) + D^2\varphi_0 f(1) = 0, \quad f'(0) + D^2\varphi_1 f(0) = 0. \end{aligned} \tag{8.8}$$

This problem can be uniquely solved under the nondegeneracy condition by the following lemma.

Lemma 8.1. *Under the non-degenerate condition* (1.13), *if* $h \in L^2(0, 1)$ *then problem* (8.8) *has a unique solution* $f \in H^2(0, 1)$ *which satisfies*

$$\|f\|_a \le C\|h\|_{L^2(0,1)}.$$

Proof. The proof of this Lemma is the same as that in Lemma 7.1 in [46]. The key step in [46] is to show the term $\zeta(\theta) + \zeta_2(\theta)$ is small. We omit it here. □

Secondly, we consider the following problem

$$\begin{aligned} &\varepsilon\mu^2 e''(\theta) + \lambda_0 e(\theta) = g(\theta), \quad 0 < \theta < 1, \\ &e'(1) + b_{13}\,\kappa_1\, e(1) = 0, \quad e'(0) + b_{03}\,\kappa_0\, e(0) = 0. \end{aligned} \tag{8.9}$$

We follow the proof of Lemma 8.1 in [10] and then get.

Lemma 8.2. *If* $g \in L^2(0, 1)$ *then for all small* ε *satisfying the gap condition* (1.20) *there is a unique solution* $e \in H^2(0, 1)$ *to problem* (8.9) *which satisfies*

$$\sqrt{\varepsilon}\, \|e\|_b \le C\, \|g\|_{L^2(0,1)}.$$

Moreover, if $g \in H^2(0,1)$ *then there holds*

$$\varepsilon \|e''\|_{L^2(0,1)} + \|e'\|_{L^2(0,1)} + \|e\|_{L^\infty(0,1)} \le C\|g\|_{H^2(0,1)}.$$

Proof. Consider the following transformation

$$\ell = \int_0^1 \frac{1}{\mu(\theta)}\,\mathrm{d}\theta, \quad t(\theta) = \frac{1}{\ell}\int_0^\theta \frac{1}{\mu(\varsigma)}\,\mathrm{d}\varsigma, \quad \tilde{\lambda}_0 = \frac{\ell^2\lambda_0}{\pi^2},$$

and

$$y(t) = e(\theta).$$

Then the problem reduces to

$$\begin{aligned} &\varepsilon y''(t) + \tilde{\lambda}_0 y(t) = \tilde{g}(t), \quad 0 < t < 1,\\ &y'(1) + b_{13}\,\kappa_1\, y(1) = 0, \quad y'(0) + b_{03}\,\kappa_0\, y(0) = 0. \end{aligned}$$

Then we can follow the arguments in the proof for Lemma 8.1 in [10]. □

Finally, we consider the following nonlinear problem

$$\begin{aligned} &\mathscr{L}_3^*(\mu) \equiv C_3\mu''(\theta) - C_1\mu(\theta) + \frac{C_2}{\mu(\theta)} = g,\\ &\mu'(1) - b_{11}\,\kappa_1\,\mu(1) = 0, \quad \mu'(0) - b_{01}\,\kappa_0\,\mu(0) = 0, \end{aligned} \tag{8.10}$$

where $C_1, C_2, C_3, b_{11}, b_{01}$ are positive constants and κ_0, κ_1 are the mean curvatures.

Lemma 8.3. *If* $g \in L^2(0,1)$ *is small then for all small* ε *there is a unique solution* $\mu = \mu_0 + \hat{\mu} \in H^2(0,1)$, *where* μ_0 *is the positive function defined in* (1.16), *to problem* (8.10). *Moreover there holds the estimate*

$$\|\hat{\mu}\|_b \le C\,\|g\|_{L^2(0,1)}.$$

Proof. By setting $\mu = \mu_0 + \hat{\mu}$, the nonlinear problem reduces to

$$\begin{aligned} &C_3\hat{\mu}'' - \left(\frac{C_2}{\mu_0^2} + C_1\right)\hat{\mu} + \hat{N}(\hat{\mu}) = g,\\ &\hat{\mu}'(1) - b_{11}\,\kappa_1\,\hat{\mu}(1) = 0, \quad \hat{\mu}'(0) - b_{01}\,\kappa_0\,\hat{\mu}(0) = 0, \end{aligned}$$

where the nonlinear operator is defined by

$$\hat{N}(\hat{\mu}) = C_2\left[(\mu_0 + \hat{\mu})^{-1} - \mu_0^{-1} + \frac{\hat{\mu}}{\mu_0^2}\right].$$

Since C_1, C_2, C_3 are positive constants and μ_0 is a uniform positive function, by the quadratic property of the nonlinear operator $\hat{N}$ we can find a solution $\hat{\mu}$ with the property

$$\|\hat{\mu}\|_b \leq C\, \|g\|_{L^2(0,1)}.$$

□

8.2. ***Solving the Nonlinear Nonlocal System and Proof of Theorem 1.1:***

Let $\hat{e}$ solves

$$\mathscr{L}_2^*(\hat{e}) = \sqrt{\varepsilon}\,\rho, \quad 0 < \theta < 1$$
$$\hat{e}'(1) + b_{13}\hat{e}(1) = 0, \quad \hat{e}'(0) + b_{03}\hat{e}(0) = 0.$$

By Lemma 8.2, we get

$$\|\hat{e}\|_b \leq C\sqrt{\varepsilon}.$$

Setting $e = \hat{e} + \tilde{e}$, the system (8.1)–(8.6) keeps the same form except that the term $\sqrt{\varepsilon}\,\rho$ disappear. Let $(\tilde{f}, \tilde{e}, \tilde{\mu}) \in \mathcal{F}$, where $\mathcal{F}$ is defined in (3.22), and define

$$\begin{aligned}
&\big(h_1(f, e, \mu),\ h_2(f, e, \mu),\ h_3(f, e, \mu)\big)\\
&\quad = \Big(\varepsilon A_{1\varepsilon}(f, e, \mu) + \varepsilon K_{1\varepsilon}(\tilde{f}, \tilde{e}),\ \varepsilon^2 A_{2\varepsilon}(f, e, \mu) + \varepsilon^2 K_{2\varepsilon}(\tilde{f}, \tilde{e}),\\
&\qquad \varepsilon^2 A_{2\varepsilon}(f, e, \mu) + \varepsilon^2 K_{2\varepsilon}(\tilde{f}, \tilde{e})\Big).
\end{aligned}$$

From (8.7), $A_{1\varepsilon}$ and $A_{2\varepsilon}$ are contraction mappings of its arguments in $\mathcal{F}$. By Banach Contraction Mapping theorem and Lemmas 8.1-8.3, we can solve the nonlinear problem defined on the region $\mathcal{F}$

$$\mathscr{L}(f, e, \mu) \equiv \Big(\mathscr{L}_1^*(f),\ \mathscr{L}_2^*(e), \mathscr{L}_2^*(e)\Big) = (h_1, h_2, h_3),$$

with the boundary conditions defined. Hence, we can define a new operator $\mathfrak{T}$ from $\mathcal{F}$ into $\mathcal{F}$ by $\mathfrak{T}(\tilde{f}, \tilde{e}, \tilde{\mu}) = (f, e, \mu)$. Finding a solution to the problem (8.1)–(8.6) is equivalent to locating a fixed point of $\mathfrak{T}$. Schauder's fixed point theorem applies to finish the proof of its existence. Hence, by Proposition 6.1 and the arguments in the last graph of Section 4, we complete the existence part of Theorem 1.1. Other properties of u_ε in Theorem 1.1 can be showed easily. □

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