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Sign changing solutions to a nonlinear elliptic problem involving the critical Sobolev exponent in pierced domains ☆

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Abstract

We consider the problem $\Delta u+|u|^{\frac{4}{N-2}}u=0$ in Ω_ε, $u=0$ on $\partial\Omega_\varepsilon$, where $\Omega_\varepsilon:=\Omega\setminus B(0,\varepsilon)$ and Ω is a bounded smooth domain in $\mathbb{R}^N$, which contains the origin and is symmetric with respect to the origin, $N\geqslant 3$ and ε is a positive parameter. As ε goes to zero, we construct sign changing solutions with multiple blow up at the origin.

Résumé

Nous considérons le problème $-\Delta u=|u|^{\frac{4}{N-2}}u$ dans Ω_ε, $u=0$ sur $\partial\Omega_\varepsilon$, où $\Omega_\varepsilon:=\Omega\setminus B(0,\varepsilon)$, Ω est un domaine borné de $\mathbb{R}^N$ contenant l'origine et symétrique par rapport à l'origine, $N\geqslant 3$; ε est un paramètre positif. Quand $\varepsilon\to 0$, nous construisons des solutions changeant de signe qui ressemblent à une superposition de solutions transitoires centrées dans l'origine.

MSC: 35J60; 35J25

Keywords: Critical Sobolev exponent; Sign changing solution; Multiple blow up

1. Introduction

Let D be a smooth bounded domain in $\mathbb{R}^N$, $N\geqslant 3$. Consider the following nonlinear elliptic problem:

$$\begin{cases}\Delta u+|u|^{\frac{4}{N-2}}u=0 & \text{in } D,\\ u=0 & \text{on } \partial Dt.\end{cases}\tag{1.1}$$

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It is well known that the Sobolev embedding $H_0^1(D) \hookrightarrow L^{\frac{2N}{N-2}}(D)$ is not compact and that this lack of compactness makes the question of solvability of (1.1) quite delicate.

Pohozaev's identity [31] shows that problem (1.1) has only the trivial solution if the domain D is assumed to be strictly starshaped. On the other hand, Kazdan and Warner showed in [25] that if D is an annulus then (1.1) has a (unique) positive solution in the class of functions with radial symmetry. In [7], the authors study the asymptotic behavior of this solution as the radius of the inner ball of the annulus tends to zero. In the nonsymmetric case, Coron [16] found via variational methods that (1.1) is solvable and that it admits a positive solution under the assumption that D is a domain with a small hole. Substantial improvement of this result was obtained by Bahri and Coron [5], showing that if some homology group of D with coefficients in $\mathbf{Z}_2$ is not trivial, then (1.1) has at least one positive solution. See also [4,6,11,20,22,32] for related results.

An interesting question is the study of the asymptotic behavior of Coron's solution as the size of the hole tends to zero. Under the assumption that the hole is symmetric (a ball of radius ρ), then the solution concentrates around the hole and it converges, as $\rho \to 0$, in the sense of measure to a Dirac delta centered at the center of the hole. In the literature this is what is known as a (simple) bubbling solution. We refer the reader to [26,28,33] where the study of existence of positive solutions to (1.1) in domains with several small symmetric holes and their asymptotic behavior as the size of the holes goes to zero is carried out.

When the hole in D is not symmetric as in Coron's setting, a recent result by Clapp and Weth [15] shows that, besides the positive solution discovered by Coron in [16], problem (1.1) has another solution. The argument in [15] is by contradiction and the authors can not describe the second solution, in particular they can not say whether it is positive or it changes sign. In fact, we believe that this second solution changes sign. Existence and qualitative behavior of sign changing solutions for elliptic problems with critical nonlinearity have been investigated by several authors in the last years. We refer the reader for instance to [1,2,8,9,13,14,23,24,29].

In this paper we treat the case of a bounded and smooth domain Ω in $\mathbb{R}^N$, $N \geqslant 3$, which contains the origin and is symmetric with respect to the origin, i.e. $x \in \Omega$ if and only if $-x \in \Omega$. We define $\Omega_\varepsilon = \Omega \setminus B(0,\varepsilon)$, for some $\varepsilon > 0$. We prove that problem

$$\begin{cases} \Delta u + |u|^{\frac{4}{N-2}} u = 0 & \text{in } \Omega_\varepsilon, \\ u = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{1.2}$$

admits any arbitrary number of sign changing solutions, provided ε is sufficiently small. Not only this, we construct sign changing solutions as a superposition of bubbles with alternating sign centered at the center of the hole, the origin.

The result is the following:

Theorem 1.1. *For any integer $k \geqslant 1$, there exists $\varepsilon_k > 0$ such that for any $\varepsilon \in (0, \varepsilon_k)$ there exists a pair of solutions u_ε and $-u_\varepsilon$ to problem* (1.2) *such that*

$$u_\varepsilon(x) = \alpha_N \sum_{i=1}^{k} (-1)^{i+1} \left(\frac{M_i \varepsilon^{\frac{2i-1}{2k}}}{M_i^2 \varepsilon^{2\frac{2i-1}{2k}} + |x|^2} \right)^{\frac{N-2}{2}} \big(1 + \mathrm{o}(1)\big),$$

where $\alpha_N := [N(N-2)]^{\frac{N-2}{4}}$, $M_1, \ldots, M_k$ are positive constants depending only on N and k and $\mathrm{o}(1) \to 0$ *uniformly on compact subsets of Ω, as $\varepsilon \to 0$. Moreover, u_ε is even with respect to the origin, i.e. $u_\varepsilon(x) = u_\varepsilon(-x)$.*

Observe that no positive solutions to (1.1) exhibiting this feature of superposition of bubbles can be found, as proved in [27]. The novelty of the result obtained in Theorem 1.1 is the existence of solutions for the problem at the critical Sobolev exponent which are superpositions of bubble of different heights and necessarily alternating sign: a first positive bubble, whose maximum is of size $\varepsilon^{-\frac{1}{k}}$, a second negative bubble, whose minimum is of order $\varepsilon^{-\frac{3}{k}}$, up to the last bubble, which in absolute value has maximum of order $\varepsilon^{-2+\frac{1}{k}}$.

We point out that in [30] the authors proved the existence of sign changing solutions with multiple blow up at the origin for a slightly sub critical problem in a symmetric domain. It seems that sign changing solutions that naturally appear in critical problems look like superpositions of bubbles. The phenomenon of superposition of bubbles for problems related with the critical Sobolev exponent is already known in the literature. Concerning superposition of positive bubbles, we refer the reader to [17–19], where the case of domains with symmetry is treated, and to [21], where the results obtained in [17] is generalized to a generic domain with no assumption on its symmetry.

In fact, in view of the results contained in [21,33], we conjecture that the following result holds true: Given a domain Ω not necessarily symmetric, if we drop a small ball with center in any arbitrary point inside Ω, then there exists a sign changing solution to problem (1.2) which looks like a superposition of an arbitrary number of bubbles with alternating sign and which concentrates around the center of the hole, provided the hole is sufficiently small.

We will prove Theorem 1.1 with the aim of a Liapunov–Schmidt reduction, which we describe, together with the scheme of the proof, in Section 2.

We would like to thank the referee for the helpful comments on the results contained in Theorem 1.1 and for the remarks on the organization of the paper, which helped us to improve its presentation.

2. Ansatz and sketch of the proof

Let us denote with p the critical Sobolev exponent $\frac{N+2}{N-2}$. The basic elements to construct sign changing solutions to (1.2) are the functions w_μ, defined by:

$$w_\mu(y) = \alpha_N \frac{\mu^{\frac{N-2}{2}}}{(\mu^2+|y|^2)^{\frac{N-2}{2}}}, \quad \mu > 0,$$

with $\alpha_N := [N(N-2)]^{\frac{N-2}{4}}$. It is well known (see [3,12,34]) that these functions are the only radial solutions of the equation $-\Delta u = u^p$ in $\mathbb{R}^N$. In order to fit the homogeneous Dirichlet boundary conditions in (1.2), we project w_μ onto $H_0^1(\Omega_\varepsilon)$. We define $\pi_{\mu,\varepsilon}$ to be the unique solution to the problem:

$$\begin{cases} \Delta \pi_{\mu,\varepsilon} = 0 & \text{in } \Omega_\varepsilon, \\ \pi_{\mu,\varepsilon} = -w_\mu & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Then the function $P_\varepsilon w_\mu := w_\mu + \pi_{\mu,\varepsilon}$ is the projection onto $\mathrm{H}_0^1(\Omega_\varepsilon)$ of the function w_μ, namely it satisfies:

$$\begin{cases} -\Delta P_\varepsilon w_\mu = w_\mu^p & \text{in } \Omega_\varepsilon, \\ P_\varepsilon w_\mu = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

As already observed in [17–19,30], a useful way to construct superpositions of bubbles, or tower of bubbles, is to rewrite problem (1.2) in different variables. We introduce spherical coordinates $y = y(\rho, \Theta)$ centered at the origin given by $\rho = |y|$ and $\Theta = \frac{y}{|y|}$. We define the transformation:

$$v(x,\Theta) = \mathcal{T}(u)(x,\Theta) := \left(\frac{p-1}{2}\right)^{\frac{2}{p-1}} \mathrm{e}^{-x} u\left(\mathrm{e}^{-\frac{p-1}{2}x}\Theta\right). \tag{2.1}$$

We denote by D_ε the subset of $S := \mathbb{R} \times S^{N-1}$ obtained from Ω_ε under the transformation (2.1), namely

$$D_\varepsilon := \left\{(x,\theta)\colon\ r_0(\theta) \leqslant x \leqslant r_\varepsilon,\ \theta \in S^{N-1}\right\},$$

where $r_0 : S^{N-1} \to \mathbb{R}$ is a continuous function. We remark that, after the change of variables, the ball $B(0,\varepsilon)$ becomes the set $[-\frac{N-2}{2}\log\varepsilon, +\infty) \times S^{N-1}$ and the domain Ω becomes a subset D of $\mathbb{R} \times S^{N-1}$, so that we can write:

$$D_\varepsilon = D \setminus [r_\varepsilon, +\infty) \times S^{N-1}, \quad \text{where } r_\varepsilon := -\frac{N-2}{2}\log\varepsilon.$$

After these changes of variables, problem (1.2) becomes:

$$\begin{cases} L_0(v) = |v|^{p-1}v & \text{in } D_\varepsilon, \\ v = 0 & \text{on } \partial D_\varepsilon, \end{cases} \tag{2.2}$$

where

$$L_0(v) = -\left(\frac{p-1}{2}\right)^2 \Delta_{S^{N-1}} v - v'' + v. \tag{2.3}$$

L_0 is the transformed operator associated to $-\Delta$. Here and in what follows, $' = \frac{\partial}{\partial x}$ and $\Delta_{S^{N-1}}$ denotes the Laplace–Beltrami operator on S^{N-1}.

We observe then that

$$\mathcal{T}(w_\mu)(x, \Theta) = W_\xi(x) := W(x - \xi),$$

where

$$W(x) := C_N \mathrm{e}^{-x}\left(1 + \mathrm{e}^{-\frac{4}{N-2}x}\right)^{-\frac{N-2}{2}}, \quad \text{with } \mu = \mathrm{e}^{-\frac{2}{N-2}\xi}, \tag{2.4}$$

and $C_N := (\frac{4N}{N-2})^{\frac{N-2}{4}}$. The function W is the unique solution of the problem:

$$\begin{cases} W'' - W + W^p = 0 & \text{in } \mathbb{R}, \\ W'(0) = 0, & W > 0, \\ W(x) \to 0 & \text{as } x \to \pm\infty. \end{cases} \tag{2.5}$$

We see also that setting

$$\Pi_{\xi,\varepsilon} = \mathcal{T}(\pi_{\mu,\varepsilon}), \quad \text{with } \mu = \mathrm{e}^{-\frac{2}{N-2}\xi}, \tag{2.6}$$

then $\Pi_{\xi,\varepsilon}$ solves the boundary problem:

$$\begin{cases} L_0(\Pi_{\xi,\varepsilon}) = 0 & \text{in } D_\varepsilon, \\ \Pi_{\xi,\varepsilon} = -W_\xi & \text{on } \partial D_\varepsilon. \end{cases} \tag{2.7}$$

Let ξ_i be positive points in $\mathbb{R}$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_k$ and $\mu_i = \mathrm{e}^{-\frac{2}{N-2}\xi_i}$. We look for a solution to (2.2) of the form:

$$v(x, \Theta) = \sum_{i=1}^{k} (-1)^i \left(W(x - \xi_i) + \Pi_{\xi_i,\varepsilon}(x, \Theta)\right) + \phi(x, \Theta), \tag{2.8}$$

where the rest term ϕ is a small function which is symmetric with respect to the variables $\Theta_1, \ldots, \Theta_N$. In original variables, the solution to (1.2) takes the form

$$u(y) = \sum_{i=1}^{k} (-1)^i \left(w_{\mu_i}(y) + \pi_{\mu_i,\varepsilon}(y)\right) + \psi(y), \tag{2.9}$$

where ψ is a small function which is symmetric with respect to the origin.

A crucial remark is that $v(x) \sim \sum_{i=1}^{k} (-1)^i W(x - \xi_i)$ solves (2.2) if and only if (going back in the change of variables)

$$u(y) \sim \alpha_N \sum_{i=1}^{k} (-1)^i \left(\frac{\mathrm{e}^{-\frac{2\xi_i}{N-2}}}{\mathrm{e}^{-\frac{4\xi_i}{N-2}} + |y|^2}\right)^{\frac{N-2}{2}}$$

solves (1.2). Therefore, the ansatz given for v provides (for large values of the ξ_i's) a sign changing bubble-tower solution for (1.2).

Let us write:

$$W_i(x) := W(x - \xi_i), \quad \Pi_i := \Pi_{\xi_i,\varepsilon}, \quad V_i = W_i + \Pi_i, \quad V := \sum_{i=1}^{k} (-1)^i V_i. \tag{2.10}$$

We consider the ansatz $v = V + \phi$. In terms of ϕ, problem (2.2) becomes:

$$\begin{cases} L(\phi) = N(\phi) + R & \text{in } D_\varepsilon, \\ \phi = 0 & \text{on } \partial D_\varepsilon, \end{cases} \tag{2.11}$$

where

$$L(\phi) := L_0(\phi) - f'(V)\phi, \tag{2.12}$$

$$N(\phi) := f(V+\phi) - f(V) - f'(V)\phi, \tag{2.13}$$

$$R := f(V) - \sum_{i=1}^{k} (-1)^i W_i^p. \tag{2.14}$$

Here, we set $f(s) := |s|^{p-1}s$.

For $i = 1, \dots, k$ we will choose points ξ_i such that

$$\mathrm{e}^{-\xi_i} = \varepsilon^{\alpha_i} \lambda_i, \quad \text{for some } \lambda_i > 0, \tag{2.15}$$

where

$$\alpha_i := \frac{2i-1}{k} \frac{N-2}{4}, \quad i = 1, \dots, k. \tag{2.16}$$

We choose the α_i's to be solutions to the linear system:

$$\begin{cases} 2\alpha_1 + 2\alpha_k = N-2, \\ 2\alpha_1 = \alpha_2 - \alpha_1, \\ \vdots \\ 2\alpha_1 = \alpha_k - \alpha_{k-1}. \end{cases} \tag{2.17}$$

Rather than solving (2.11) directly, we consider first the following intermediate problem: given points $\xi := (\xi_1, \dots, \xi_k) \in \mathbb{R}^k$, or equivalently parameters $\lambda := (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k_+$, find a function ϕ symmetric with respect to the variables $\Theta_1, \dots, \Theta_N$ such that, for certain constants c_i, it satisfies:

$$\begin{cases} L(\phi) = N(\phi) + R + \sum_{i=1}^{k} c_i Z_i & \text{in } D_\varepsilon, \\ \phi = 0 & \text{on } \partial D_\varepsilon, \\ \int_{D_\varepsilon} Z_i \phi \, \mathrm{d}x \, \mathrm{d}\Theta = 0 & \text{if } i = 1, \dots, k, \end{cases} \tag{2.18}$$

where the Z_i's are defined as follows. Let:

$$z_i(y) = \mu_i \frac{\partial}{\partial \mu_i} w_{\mu_i}(y) \quad \text{for } i = 1, \dots, k, \text{ with } \mu_i = \mathrm{e}^{-\frac{2}{N-2}\xi_i}.$$

Each z_i solves the linearized problem (see [10]):

$$-\Delta z = p w_{\mu_i}^{p-1} z \quad \text{in } \mathbb{R}^N.$$

Let $P_\varepsilon z_i$ be the projection onto $\mathrm{H}_0^1(\Omega_\varepsilon)$ of the function z_i, i.e. $\Delta P_\varepsilon z_i = \Delta z_i$ in Ω_ε, $P_\varepsilon z_i = 0$ on $\partial\Omega_\varepsilon$. Let $Z_i(x, \Theta) := \mathcal{T}(P_\varepsilon z_i)(x, \Theta)$. Then Z_i solves:

$$\begin{cases} L_0(Z_i) = p W_i^{p-1} Z_i & \text{in } D_\varepsilon, \\ Z_i = 0 & \text{on } \partial D_\varepsilon. \end{cases} \tag{2.19}$$

Unique solvability in certain class of functions for problem (2.18) is proved in Section 3.

According to (2.18), the problem has been reduced to that of finding points λ_i so that the constants c_i are all equal to zero. Thus, we need to solve the system of equations

$$c_i(\lambda) = 0 \quad \text{for any } i = 1, \dots, k. \tag{2.20}$$

If (2.20) holds, then $v = V + \phi$ will be a solution to (2.11) or equivalently to (2.2). In Section 4 we see that solving this system turns out to be equivalent to finding critical points of the function $(\lambda_1, \dots, \lambda_k) \to I_\varepsilon(V)$. Here I_ε is the energy functional given by:

$$I_\varepsilon(v) := \frac{1}{2}\left(\frac{p-1}{2}\right)^2 \int_{D_\varepsilon} |\nabla_\Theta v|^2 \, \mathrm{d}x \, \mathrm{d}\Theta + \frac{1}{2} \int_{D_\varepsilon} \left(|v'|^2 + |v|^2\right) \mathrm{d}x \, \mathrm{d}\Theta - \frac{1}{p+1} \int_{D_\varepsilon} |v|^{p+1} \, \mathrm{d}x \, \mathrm{d}\Theta \tag{2.21}$$

naturally associated to problem (2.2).

Using (2.15), (2.16), we compute the asymptotic expansion for $I_\varepsilon(V)$ and we conclude the proof of Theorem 1.1. This is done in Section 4. In the last section we collect some technical lemmas, whose results we use throughout the paper.

3. The reduction method

In order to solve problem (2.18), it is necessary to understand first its linear part. Given a function h, we consider the problem of finding ϕ such that for certain real numbers c_i the following is satisfied:

$$\begin{cases} L(\phi) = h + \sum_{i=1}^{k} c_i Z_i & \text{in } D_\varepsilon, \\ \phi = 0 & \text{on } \partial D_\varepsilon, \\ \int_{D_\varepsilon} Z_i \phi \, \mathrm{d}x \, \mathrm{d}\Theta = 0 & \text{if } i = 1, \dots, k, \end{cases} \tag{3.1}$$

where the linear operator L is defined in (2.12). We need uniformly bounded solvability in proper functional spaces for problem (3.1). To this end, it is convenient to introduce the following norm. Given an arbitrarily small but fixed number $\sigma > 0$, we define:

$$\|g\|_* := \sup_{(x,\Theta)\in D} \left(\sum_{i=1}^{k} \mathrm{e}^{-(1-\sigma)|x-\xi_i|} \right)^{-1} \left| g(x,\Theta) \right|. \tag{3.2}$$

Although this norm depends on σ and the numbers $0 < \xi_1 < \cdots < \xi_k$, we do not indicate this dependence in our notation. In fact, different choices of σ and $\lambda = (\lambda_1, \dots, \lambda_k)$ lead to equivalent norms. Let $\mathcal{C}_*$ be the Banach space of all continuous functions $g : D_\varepsilon \to \mathbb{R}$ which are symmetric with respect to the variables $\Theta_1, \dots, \Theta_N$ and for which $\|g\|_* < +\infty$.

First of all, we obtain the following result.

Proposition 3.1. *For any $\delta > 0$, there exist $\varepsilon_0 > 0$ and $C > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$ and if $\delta < \lambda_i < \delta^{-1}$, $i = 1, \dots, k$, then for any $h \in \mathcal{C}_*$ problem* (3.1) *admits a unique solution $T_\varepsilon(\lambda, h) \in \mathcal{C}_*$, with*

$$\left\| T_\varepsilon(\lambda, h) \right\|_* \leqslant C \|h\|_* \quad \textit{and} \quad |c_i| \leqslant C \|h\|_*.$$

Proof. The proof of this proposition consists of 2 steps.

Step 1. An a-priori estimate. Assume the existence of sequences of numbers $\varepsilon_n \to 0$, c_i^n, functions ϕ_n and h_n, with $\|h_n\|_* \to 0$ so that

$$\begin{cases} L(\phi_n) = h_n + \sum_{i=1}^{k} c_i^n Z_i & \text{in } D_{\varepsilon_n}, \\ \phi_n = 0 & \text{on } \partial D_{\varepsilon_n}, \\ \int_{D_{\varepsilon_n}} Z_i \phi_n \, \mathrm{d}x \, \mathrm{d}\Theta = 0 & \text{if } i = 1, \dots, k. \end{cases} \tag{3.3}$$

Then necessarily $\|\phi_n\|_* \to 0$. We will first show that $\|\phi_n\|_\infty \to 0$. Assume by contradiction that $\|\phi_n\|_\infty = 1$. A first fact to observe is that $c_i^n \to 0$ as $n \to \infty$ for all i. This follows by testing equation in (3.3) against Z_i and integrating by parts. These two operations give that the constants c_i^n solve an almost diagonal system. This fact, together with (2.19), implies that $\lim_{n\to\infty} c_i^n = 0$. Now let $(x_n, \theta_n) \in D_{\varepsilon_n}$ be such that $\phi_n(x_n, \theta_n) = 1$. We claim that, for n large enough, there exist $R > 0$ and $i \in \{1, \dots, k\}$ such that $|x_n - \xi_i^n| < R$. We argue by contradiction and suppose that $|x_n - \xi_i^n| \to +\infty$ as $n \to +\infty$ for any $i = 1, \dots, k$. Then either $|x_n| \to +\infty$ or $|x_n|$ remains bounded. Assume first that $|x_n| \to +\infty$.

Let us define:

$$\tilde{\phi}_n(x, \Theta) = \phi_n(x + x_n, \Theta).$$

We may assume that, up to subsequences, $\tilde{\phi}_n$ converges uniformly over compact sets to a function $\tilde{\phi}$. Define $\tilde{\psi} = \mathcal{T}^{-1}(\tilde{\phi})$. Then, from elliptic estimates, $\tilde{\psi}$ satisfies:

$$\Delta\tilde{\psi} = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Moreover, $\|\tilde{\phi}\|_\infty = 1$, translates into $|\tilde{\psi}(y)| \leqslant |y|^{-(N-2)/2}$. It follows that $\tilde{\psi}$ extends smoothly to 0, to a harmonic function in $\mathbb{R}^N$ with this decay condition, hence $\tilde{\psi} \equiv 0$, yielding a contradiction.

Arguing in a similar way one shows that $|x_n|$ cannot be bounded. Hence, there exists an integer $l \in \{1, \dots, k\}$ and a positive number $R > 0$ such that, for n sufficiently large, $|x_n - \xi_l^n| \leqslant R$. Let again $\tilde{\phi}_n(x, \Theta) \equiv \phi_n(x + \xi_l^n, \Theta)$. This relation implies that $\tilde{\phi}_n$ converges uniformly over compacts to a function $\tilde{\phi}$. Define again $\tilde{\psi} = \mathcal{T}^{-1}(\tilde{\phi})$. Then $\tilde{\psi}$ is a nontrivial solution of

$$\Delta\tilde{\psi} + p\, w_1^{p-1}\tilde{\psi} = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}$$

with $|\tilde{\psi}(y)| \leqslant C\,|y|^{-\frac{N-2}{2}}$ for all y. Thus we get a classical solution in $\mathbb{R}^N \setminus \{0\}$ which decays at infinity and hence equals a linear combination of the z_m's. It follows that ϕ is a linear combination of the Z_m's. But then the orthogonality relations imply $\tilde{\phi} = 0$, again a contradiction. We have thus proven $\|\phi_n\|_\infty \to 0$ as $n \to \infty$. Next we shall establish that $\|\phi_n\|_* \to 0$. We write the equation in (3.3) as $L_0(\phi_n) = g_n$ with $g_n = f'(V)\phi_n + h_n + \sum c_i^n Z_i$ and we observe that the operator L_0 satisfies the maximum principle. Furthermore we have that

$$|g_n| \leqslant G_n \equiv m_n \sum \mathrm{e}^{-(1-\sigma)|x-\xi_j|}$$

for some constant $m_n \to 0$. Let us now define $\psi_n = K G_n$. Direct substitution shows that $L_0(\psi_n) \leqslant -G_n$ in weak sense, provided that K is chosen large enough but independent of n. From Maximum Principle, we obtain then that $\phi_n \leqslant \psi_n$. Similarly we obtain $\phi_n \geqslant -\psi_n$. Since, as well $\psi_n \leqslant C\, G_n$, this shows that $\|\phi_n\|_* \to 0$, and the proof of Step 1 is thus concluded.

Step 2. Let us consider the space:

$$H = \left\{\phi \in H_0^1(D_\varepsilon) \colon \int_{D_\varepsilon} Z_i \phi \,\mathrm{d}x = 0 \ \forall i\right\},$$

endowed with the usual inner product:

$$[\phi, \psi] = \frac{1}{2}\left(\frac{p-1}{2}\right)^2 \int_{D_\varepsilon} \nabla_\Theta \phi \cdot \nabla_\Theta \psi + \frac{1}{2}\int_{D_\varepsilon} (\phi'\psi' + \phi\psi)\,.$$

Problem (3.1) expressed in weak form is equivalent to that of finding a $\phi \in H$ such that

$$[\phi, \psi] = \int \big[f'(V)\phi + h\big]\psi \,\mathrm{d}x \quad \forall \psi \in H.$$

With the aid of Riesz's representation theorem, this equation gets rewritten in H in the operational form $\phi = K(\phi) + \tilde{h}$, for certain $\tilde{h} \in H$, where K is a compact operator in H. Fredholm's alternative guarantees unique solvability of this problem for any h provided that the homogeneous equation $\phi = K(\phi)$ has only the zero solution in H. Let us observe that this last equation is precisely equivalent to (3.1) with $h \equiv 0$. Thus existence of a unique solution follows. The bounded solvability in the sense of the $\|\ \|_*$-norm follows after an indirect argument from Step 1. The bound on the constants c_i is obtained arguing as in the first part of Step 1. □

Remark 3.2. Under the same assumptions of Proposition 3.1 we have that the map $\lambda \to T_\varepsilon(\lambda, h)$, with values in $\mathcal{L}(\mathcal{C}_*)$, is of class C^1 and

$$\big\|D_\lambda T_\varepsilon(\lambda, h)\big\|_{\mathcal{L}(\mathcal{C}_*)} \leqslant C$$

uniformly in λ.

Now, we are ready to solve problem (2.18). Just observe that, directly from the definitions (2.13) and (2.14), we get:

$$\big\|N(\phi)\big\|_* \leqslant C\|\phi\|_*^2, \qquad \|R\|_* \leqslant C\varepsilon^{\frac{N-2}{2k}(1-\sigma)}. \tag{3.4}$$

Proposition 3.2. *For any $\delta > 0$, there exist $\varepsilon_0 > 0$ and $C > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$ and if $\delta < \lambda_i < \delta^{-1}$, $i = 1, \dots, k$, there exists a unique solution $\phi = \phi(\lambda)$, $c = (c_1(\lambda), \dots, c_k(\lambda))$ to problem* (2.18) *which satisfies $\|\phi\|_* \leqslant C\varepsilon^{\frac{N-2}{2k}(1-\sigma)}$. Moreover, the map $\lambda \to \phi(\lambda)$ is of class C^1 for the $\|\cdot\|_*$norm and $\|D_\lambda \phi\|_* \leqslant C\varepsilon^{\frac{N-2}{2k}(1-\sigma)}$.*

Proof. In terms of the operator T defined in Proposition 3.1, problem (2.18) becomes:

$$\phi = T\big(N(\phi) + R\big) \equiv A(\phi), \tag{3.5}$$

where $N(\phi)$ and R where defined in (2.13) and (2.14). For a given M, let us consider the region:

$$\mathcal{F} \equiv \big\{\phi \in C\big(\bar{D}\big):\ \|\phi\|_* \leqslant M\varepsilon^{\frac{N-2}{2k}(1-\sigma)}\big\}.$$

From Proposition 3.1, we get:

$$\big\|A(\phi)\big\|_* \leqslant C\big[\big\|N(\phi)\big\|_* + \|R\|_*\big].$$

By (3.4), we thus obtain:

$$\big\|A(\phi)\big\|_* \leqslant M\varepsilon^{\frac{N-2}{2k}(1-\sigma)},$$

provided M is chosen large, but fixed.

On the other hand we can easily check that N satisfies, for $\phi_1, \phi_2 \in \mathcal{F}$,

$$\big\|N(\phi_1) - N(\phi_2)\big\|_* \leqslant C_1 M\varepsilon^{\frac{N-2}{2k}(1-\sigma)}\|\phi_1 - \phi_2\|_*.$$

We conclude that A is a contraction mapping of $\mathcal{F}$, and therefore a unique fixed point of A exists in this region.

Concerning now the differentiability of the function $\phi(\lambda)$, let us write:

$$B(\lambda, \phi) := \phi - T\big(N(\phi) + R\big).$$

Of course we have $B(\lambda, \phi) = 0$. Now we write:

$$D_\phi B(\lambda, \phi)[\theta] = \theta - T\big(\theta D_\phi\big(N(\phi)\big)\big) =: \theta + M(\theta).$$

It is not hard to check that the following estimate holds:

$$\big\|M(\theta)\big\|_* \leqslant C\varepsilon^{\frac{N-2}{2k}(1-\sigma)}\|\theta\|_*.$$

It follows that for small ε, the linear operator $D_\phi B(\lambda, \phi)$ is invertible in $\mathcal{C}_*$, with uniformly bounded inverse. It also depends continuously on its parameters. Let us differentiate with respect to λ. We have:

$$D_\lambda B(\lambda, \phi) = -(D_\lambda T)\big(N(\phi) + R\big) - T\big((D_\lambda N)(\lambda, \phi) + D_\lambda R\big),$$

where all the previous expressions depend continuously on their parameters. Hence the implicit function theorem yields that $\phi(\lambda)$ is a C^1 function into $\mathcal{C}_*$. Moreover, we have:

$$D_\lambda \phi = -\big(D_\phi B(\lambda, \phi)\big)^{-1}\big[D_\lambda B(\lambda, \phi)\big],$$

so that

$$\|D_\lambda \phi\|_* \leqslant C\big(\big\|N(\phi) + R\big\|_* + \big\|D_\lambda N(\lambda, \phi)\big\|_*\big) \leqslant C\,\varepsilon^{\frac{N-2}{2k}(1-\sigma)}.$$

This concludes the proof of the proposition. □

4. Estimates for the reduced functional

Let $\phi = \phi(\lambda)$ and $c_i = c_i(\lambda)$ be solutions to (2.18) given by Proposition 3.2. Our problem reduces to

$$c_i(\lambda) = 0 \quad \text{for any } i = 1, \dots, k. \tag{4.1}$$

This system turns out to be equivalent to a variational problem, related to the functional (2.21) associated to problem (2.2). Indeed, by the same (standard) arguments as given on p. 301 in [17], the following result is proved.

Lemma 4.1. *The function $V+\phi$ is a solution to* (2.2) *if λ is a critical point of the function,*

$$\tilde{I}_\varepsilon(\lambda) := I_\varepsilon(V+\phi),$$

where $\phi=\phi(\lambda)$ is given by Proposition 3.2 *and I_ε is defined in* (2.21).

The following estimate is crucial to find critical points of $\tilde{I}_\varepsilon$.

Lemma 4.2. *The following expansion holds*:

$$\tilde{I}_\varepsilon(\lambda) = I_\varepsilon(V) + \mathrm{O}\big(\varepsilon^{\frac{N-2}{k}(1-\sigma)}\big),$$

where the term $\mathrm{O}(\varepsilon^{\frac{N-2}{k}(1-\sigma)})$ is uniform over all λ_i's satisfying $\delta<\lambda_i<\delta^{-1}$, $i=1,\dots,k$, for some given $\delta>0$.

Proof. Taking into account that $0 = DI_\varepsilon(V+\phi)[\phi]$, a Taylor expansion gives:

$$\begin{aligned} I_\varepsilon(V+\phi) - I_\varepsilon(V) &= \int_0^1 D^2 I_\varepsilon(V+t\phi)\big[\phi^2\big] t\,\mathrm{d}t \\ &= \int_0^1 \left(\int_{D_\varepsilon} \big[N(\phi)+R\big]\phi + \int_{D_\varepsilon} p\big[V^{p-1} - (V+t\phi)^{p-1}\big]\phi^2 \right) t\,\mathrm{d}t. \end{aligned}$$

Since $\|\phi\|_* = \mathrm{O}(\varepsilon^{\frac{N-2}{2k}(1-\sigma)})$, we get:

$$\tilde{I}_\varepsilon(\lambda) - I_\varepsilon(V) = \mathrm{O}\big(\varepsilon^{\frac{N-2}{k}(1-\sigma)}\big),$$

uniformly on points satisfying $0<\lambda_i<\delta^{-1}$. Differentiating now with respect to the λ variables, we get that

$$D_\lambda\big[\tilde{I}_\varepsilon(\lambda) - I_\varepsilon(V)\big] = \int_0^1 \left(\int_{D_\varepsilon} D_\lambda\big[\big(N(\phi)+R\big)\phi\big] t\,\mathrm{d}t + p\int_0^\infty D_\lambda\big[\big((V+t\phi)^{p-1} - V^{p-1}\big)\phi^2\big] \right).$$

Using the computations in the proof of Proposition 3.2, we get that the first integral can be estimated by $\mathrm{O}(\varepsilon^{\frac{N-2}{k}(1-\sigma)})$, so does the second. Hence the proof of Lemma 4.2 is complete. □

The advantage of the choice of ξ_i's made in (2.15) is the validity of the expansion of the functional (2.21) given in the following proposition.

Proposition 4.3. *For any $\delta>0$, there exists $\varepsilon_0>0$ such that for any $\varepsilon\in(0,\varepsilon_0)$ the following expansion holds*:

$$I_\varepsilon(V) = ka_1 + \varepsilon^{\frac{N-2}{2k}}\Psi_k(\lambda_1,\dots,\lambda_k) + \mathrm{o}\big(\varepsilon^{\frac{N-2}{2k}}\big), \tag{4.2}$$

where

$$\Psi_k(\lambda_1,\dots,\lambda_k) := a_2\lambda_1^2 + a_3\frac{1}{\lambda_k^2} + a_4\sum_{i=1}^{k-1}\frac{\lambda_{i+1}}{\lambda_i}, \tag{4.3}$$

and as $\varepsilon\to 0$ the term $\mathrm{o}(\varepsilon^{\frac{N-2}{2k}})$ converges to 0 uniformly on the set of λ_i's with $\delta<\lambda_i<\delta^{-1}$, $i=1,\dots,k$. Here, we have:

$$a_1 := \frac{1}{N}\omega_{N-1}\int\limits_{\mathbb{R}} W^{p+1}(y)\,\mathrm{d}y, \tag{4.4}$$

$$a_2 := H(0,0)\frac{C_N}{2}\omega_{N-1}\int\limits_{\mathbb{R}} \mathrm{e}^{-y}W^p(y)\,\mathrm{d}y, \tag{4.5}$$

$$a_3 := \frac{C_N}{2}\omega_{N-1}\int\limits_{\mathbb{R}} \mathrm{e}^{y}W^p(y)\,\mathrm{d}y, \tag{4.6}$$

$$a_4 := C_N\omega_{N-1}\int\limits_{\mathbb{R}} \mathrm{e}^{-y}W^p(y)\,\mathrm{d}y, \tag{4.7}$$

and ω_{N-1} is the surface area of S^{N-1}.

The proof of this expansion relies on arguments inspired by [17–19]. For the convenience of the reader, we partly reproduce the proofs here. As a first step, we collect some asymptotic estimates in the following two lemmata.

Lemma 4.4. *It holds*:

$$\sum_{i=1}^{k} I_\varepsilon(V_i) = ka_1 + a_2\mathrm{e}^{-2\xi_1} + a_3\mathrm{e}^{-2(r_\varepsilon-\xi_k)} + \mathrm{o}\big(\mathrm{e}^{-2\xi_1}\big), \tag{4.8}$$

where a_1, a_2 and a_3 are given in (4.4)*,* (4.5) *and* (4.6)*.*

Proof. By definition of the function V_i and using the mean value theorem, we deduce that

$$I_\varepsilon(V_i) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int\limits_{D_\varepsilon} |W_i|^{p+1}\,\mathrm{d}x\,\mathrm{d}\theta - \frac{1}{2}\int\limits_{D_\varepsilon} \Pi_i W_i^p\,\mathrm{d}x\,\mathrm{d}\theta - \frac{p}{2}\int\limits_{D_\varepsilon} |W_i + t_i\Pi|^{p-1}\Pi_i^2\,\mathrm{d}x\,\mathrm{d}\theta, \tag{4.9}$$

for some $t_i = t_i(x) \in [0,1]$.

We see first that

$$\begin{aligned}\int\limits_{D_\varepsilon} |W_i|^{p+1}\,\mathrm{d}x\,\mathrm{d}\theta &= \int\limits_{D_\varepsilon-\xi_i} W^{p+1}(y)\,\mathrm{d}y\,\mathrm{d}\theta\\ &= \omega_{N-1}\int\limits_{\mathbb{R}} W^{p+1} + \mathrm{O}\big(\mathrm{e}^{-(p+1)\xi_i} + \mathrm{e}^{-(p+1)(r_\varepsilon-\xi_i)}\big)\\ &= \omega_{N-1}\int\limits_{\mathbb{R}} W^{p+1} + \mathrm{o}\big(\varepsilon^{2\alpha_1}\big).\end{aligned} \tag{4.10}$$

Here and in the following $D_\varepsilon - \xi_i = \{(y,\theta)\colon\ r_0(\theta)-\xi_i \leqslant y \leqslant r_\varepsilon - \xi_i,\ \theta\in S^{N-1}\}$. By (A.5) we deduce:

$$\begin{aligned}-\int\limits_{D_\varepsilon} \Pi_i W_i^p\,\mathrm{d}x\,\mathrm{d}\theta &= C_N\int\limits_{D_\varepsilon} \mathrm{e}^{-x-\xi_i}H\big(\mathrm{e}^{-\frac{p-1}{2}x}\theta,0\big)W_i^p(x)\,\mathrm{d}x\,\mathrm{d}\theta\\ &\quad + C_N\int\limits_{D_\varepsilon} \mathrm{e}^{-2r_\varepsilon+\xi_i+x}W_i^p(x)\,\mathrm{d}x\,\mathrm{d}\theta + \int\limits_{D_\varepsilon} \rho_i(x,\theta)W_i^p\,\mathrm{d}x\,\mathrm{d}\theta.\end{aligned} \tag{4.11}$$

We have (setting $x = y + \xi_i$):

$$\int_{D_\varepsilon} \mathrm{e}^{-x-\xi_i} H\big(\mathrm{e}^{-\frac{p-1}{2}x}\theta, 0\big) W_i^p(x)\,\mathrm{d}x\,\mathrm{d}\theta$$
$$= \int_{D_\varepsilon-\xi_i} \mathrm{e}^{-y-2\xi_i} H\big(\mathrm{e}^{-\frac{p-1}{2}(y+\xi_i)}\theta, 0\big) W^p(y)\,\mathrm{d}y\,\mathrm{d}\theta$$
$$= \mathrm{e}^{-2\xi_i}\bigg(H(0,0)\omega_{N-1}\int_{\mathbb{R}} \mathrm{e}^{-y} W^p(y)\,\mathrm{d}y + \mathrm{o}(1)\bigg), \tag{4.12}$$

and

$$\int_{D_\varepsilon} \mathrm{e}^{-2r_\varepsilon+\xi_i+x} W_i^p(x)\,\mathrm{d}x\,\mathrm{d}\theta = \int_{D_\varepsilon-\xi_i} \mathrm{e}^{y-2r_\varepsilon+2\xi_i} W^p(y)\,\mathrm{d}y\,\mathrm{d}\theta$$
$$= \mathrm{e}^{-2(r_\varepsilon-\xi_i)}\bigg(\omega_{N-1}\int_{\mathbb{R}} \mathrm{e}^{y} W^p(y)\,\mathrm{d}y + \mathrm{o}(1)\bigg). \tag{4.13}$$

By (A.6) we get (setting $x = y + \xi_i$):

$$\bigg|\int_{D_\varepsilon} \rho_i(x,\theta) W_i^p\,\mathrm{d}x\,\mathrm{d}\theta\bigg| \leqslant c \int_{D_\varepsilon-\xi_i} \big(\mathrm{e}^{-y-(p+1)\xi_i} + \mathrm{e}^{-2r_\varepsilon+y}\big) W^p(y)\,\mathrm{d}y\,\mathrm{d}\theta$$
$$= \mathrm{O}\big(\mathrm{e}^{-(p+1)\xi_i} + \mathrm{e}^{-2r_\varepsilon}\big) = \mathrm{O}\big(\varepsilon^{(p+1)\alpha_i} + \varepsilon^{N-2}\big) = \mathrm{o}\big(\varepsilon^{2\alpha_1}\big). \tag{4.14}$$

Finally, by (4.1) and (4.2) we deduce:

$$\int_{D_\varepsilon} |W_i + t_i\Pi_i|^{p-1}\Pi_i^2\,\mathrm{d}x\,\mathrm{d}\theta = \mathrm{o}\big(\varepsilon^{2\alpha_1}\big). \tag{4.15}$$

The claim follows by all the estimates (4.9)–(4.15). □

Lemma 4.5. *Considering the numbers $\chi_1 = 0$, $\chi_l = \frac{\xi_{l-1}+\xi_l}{2}$, $l = 2,\dots,k$, $\chi_{k+1} = r_\varepsilon$ and setting $D_\varepsilon^l = \{(x,\Theta) \in D_\varepsilon$: $\chi_l \leqslant x < \chi_{l+1}\}$, we have for $l = 1,\dots,k$,*

$$\int_{D_\varepsilon^l} W_i^p W_j = \mathrm{o}\big(\varepsilon^{2\alpha_1(1-\sigma)}\big) \quad \text{if } i \neq l, \tag{4.16}$$

$$\int_{D_\varepsilon^l} W_l^p W_j = a_4 \mathrm{e}^{-|\xi_j-\xi_l|} + \mathrm{o}\big(\varepsilon^{2\alpha_1(1-\sigma)}\big) \quad \text{if } j \neq l, \tag{4.17}$$

$$\int_{D_\varepsilon^l} \bigg[V_l^{p+1} - |V|^{p+1} + (p+1)V_l^p \sum_{j\neq l}(-1)^{l+j} V_j\bigg] = \mathrm{o}\big(\varepsilon^{2\alpha_1(1-\sigma)}\big), \tag{4.18}$$

$$\int_{D_\varepsilon^l} \big(W_i^p - V_i^p\big) V_j = \mathrm{o}\big(\varepsilon^{2\alpha_1(1-\sigma)}\big) \quad \text{if } i \neq j, \tag{4.19}$$

where a_4 is given in (4.7).

Proof. Throughout this proof, C stands for a generic constant depending only on N and k whose value may change in every step of the calculation. From (2.4) we directly deduce the estimate:

$$|W(x)| \leqslant C\mathrm{e}^{-|x|} \quad \text{in } D_\varepsilon \tag{4.20}$$

which will be frequently used in the following:

We start by verifying (4.16), first for $i \neq l$ and $j \neq l$. In this case (4.20) implies:

$$\begin{aligned}
\int\limits_{D_\varepsilon^l} W_i^p W_j &\leqslant C \int\limits_{\chi_l}^{\chi_{l+1}} \mathrm{e}^{-p|x-\xi_i|}\mathrm{e}^{-|x-\xi_j|} \\
&\leqslant C(\chi_{l+1}-\chi_l)\max\bigl\{\mathrm{e}^{-p|\chi_l-\xi_i|}, \mathrm{e}^{-p|\chi_{l+1}-\xi_i|}\bigr\}\max\bigl\{\mathrm{e}^{-|\chi_l-\xi_j|}, \mathrm{e}^{-|\chi_{l+1}-\xi_j|}\bigr\} \\
&\leqslant C\frac{\xi_{l+1}-\xi_{l-1}}{2}\mathrm{e}^{-\frac{p|\xi_l-\xi_i|}{2}}\mathrm{e}^{-\frac{|\xi_l-\xi_j|}{2}} \leqslant C(-\log\varepsilon)\varepsilon^{|\alpha_l-\alpha_j|+\frac{p}{2}|\alpha_l-\alpha_i|} \\
&= \mathrm{o}\bigl(\varepsilon^{2\alpha_1(1-\sigma)}\bigr).
\end{aligned}$$

Next we consider $j = l$. By (4.20) for $i < l$ (so that $\xi_i \leqslant \xi_{l-1}$) setting $x - \xi_l = y$, we get:

$$\begin{aligned}
\int\limits_{D_\varepsilon^l} W_i^p W_l &\leqslant C \int\limits_{\chi_l}^{\chi_{l+1}} \mathrm{e}^{-p(x-\xi_i)}\mathrm{e}^{x-\xi_l}\,\mathrm{d}x = C\mathrm{e}^{-p(\xi_l-\xi_i)} \int\limits_{\frac{\xi_{l-1}-\xi_l}{2}}^{\frac{\xi_{l+1}-\xi_l}{2}} \mathrm{e}^{-(p-1)y}\,\mathrm{d}y \\
&\leqslant C\mathrm{e}^{-p(\xi_l-\xi_i)}\mathrm{e}^{(p-1)\frac{\xi_l-\xi_{l-1}}{2}} \leqslant C\mathrm{e}^{-\frac{p+1}{2}(\xi_l-\xi_{l-1})} \\
&\leqslant C\varepsilon^{\frac{p+1}{2}(\alpha_l-\alpha_{l-1})} = \mathrm{o}\bigl(\varepsilon^{2\alpha_1(1-\sigma)}\bigr).
\end{aligned}$$

For $i > l$ (so that $\xi_i \leqslant \xi_{l-1}$) setting $x - \xi_l = y$, we find similarly,

$$\begin{aligned}
\int\limits_{D_\varepsilon^l} W_i^p W_l &\leqslant C \int\limits_{\chi_l}^{\chi_{l+1}} \mathrm{e}^{p(x-\xi_i)}\mathrm{e}^{-(x-\xi_l)}\,\mathrm{d}x = C\mathrm{e}^{p(\xi_l-\xi_i)} \int\limits_{\frac{\xi_{l-1}-\xi_l}{2}}^{\frac{\xi_{l+1}-\xi_l}{2}} \mathrm{e}^{(p-1)y}\,\mathrm{d}y \\
&\leqslant C\mathrm{e}^{p(\xi_l-\xi_i)}\mathrm{e}^{(p-1)\frac{\xi_{l+1}-\xi_l}{2}} \leqslant C\mathrm{e}^{-\frac{p+1}{2}(\xi_{l+1}-\xi_l)} \\
&\leqslant C\varepsilon^{\frac{p+1}{2}(\alpha_{l+1}-\alpha)} = \mathrm{o}\bigl(\varepsilon^{2\alpha_1(1-\sigma)}\bigr),
\end{aligned}$$

and thus (4.16) is proved in all cases.

Next we derive (4.17) for $j < l$, using the definition of W given in (2.4) and setting $x - \xi_l = y$ and $D_\varepsilon^l - \xi_i = \{(y,\theta)\colon (y+\xi_l,\theta) \in D_\varepsilon^l\}$;

$$\begin{aligned}
\int\limits_{D_\varepsilon^l} W_l^p(x)W_j(x)\,\mathrm{d}x\,\mathrm{d}\theta &= \int\limits_{D_\varepsilon^l} W^p(y)W(y+\xi_l-\xi_j)\,\mathrm{d}y\,\mathrm{d}\theta \\
&= C_N\mathrm{e}^{-(\xi_l-\xi_j)} \int\limits_{D_\varepsilon^l} W^p(y)\mathrm{e}^{-y}\bigl(1+\mathrm{e}^{-\frac{4}{N-2}(y+\xi_l-\xi_j)}\bigr)^{-\frac{N-2}{2}}\,\mathrm{d}y\,\mathrm{d}\theta \\
&= C_N\mathrm{e}^{-(\xi_l-\xi_j)}\left(\omega_{N-1}\int\limits_{\mathbb{R}} W^p(y)\mathrm{e}^{-y}\,\mathrm{d}y + \mathrm{o}(1)\right) \\
&= a_3\mathrm{e}^{-|\xi_j-\xi_l|} + \mathrm{o}\bigl(\varepsilon^{\alpha_l-\alpha_j}\bigr) = a_3\mathrm{e}^{-|\xi_j-\xi_l|} + \mathrm{o}\bigl(\varepsilon^{2\alpha_1(1-\sigma)}\bigr).
\end{aligned}$$

The proof for $j > l$ is similar, since $W(-x) = W(x)$ for all $x \in \mathbb{R}$. In particular, (4.16) and (4.17) yield:

$$\int\limits_{D_\varepsilon^l} W_i^p W_j = \mathrm{O}\bigl(\varepsilon^{2\alpha_1(1-\sigma)}\bigr) \quad \text{if } i \neq l \text{ or } j \neq l. \tag{4.21}$$

Let us show (4.18). Via a Taylor expansion, we get:

$$\begin{aligned}
&\int_{D_\varepsilon^l}\Big(V_l^{p+1}-|V|^{p+1}+(p+1)V_l^p\sum_{j\neq l}(-1)^{l+j}V_j\Big)\\
&\quad=\int_{D_\varepsilon^l}\Big(V_l^{p+1}-\Big|V_l+\sum_{j\neq l}(-1)^{l+j}V_j\Big|^{p+1}+(p+1)V_l^p\sum_{j\neq l}(-1)^{l+j}V_j\Big)\\
&\quad\leqslant\frac{p(p+1)}{2}\int_{D_\varepsilon^l}\Big(\sum_{j=1}^k V_j\Big)^{p-1}\Big(\sum_{j\neq l}^k V_j\Big)^2\leqslant C\max_{\substack{i,j\\ j\neq l}}\int_{D_\varepsilon^l}V_i^{p-1}V_j^2\\
&\quad\leqslant C\max_{\substack{i,j\\ j\neq l}}\Big(\int_{D_\varepsilon^l}V_i^pV_j\Big)^{\frac{p-1}{p}}\Big(\int_{D_\varepsilon^l}V_j^{p+1}\Big)^{\frac1p}\\
&\quad\leqslant C\max_{\substack{i,j\\ j\neq l}}\Big(\int_{D_\varepsilon^l}W_i^pW_j\Big)^{\frac{p-1}{p}}\Big(\int_{D_\varepsilon^l}W_j^{p+1}\Big)^{\frac1p}=\mathrm{o}\big(\varepsilon^{2\alpha_1(1-\sigma)}\big),
\end{aligned}$$

where (4.16) and (4.21) was used in the last line. Hence (4.18) is proved.

Let us prove (4.18). By the mean value theorem and (A.7), we get for $i\neq j$,

$$0\leqslant\int_{D_\varepsilon^l}\big(W_i^p-V_i^p\big)V_j\leqslant-p\int_{D_\varepsilon^l}W_i^{p-1}\Pi_iV_j\leqslant c\int_{D_\varepsilon^l}\big(\mathrm{e}^{-\xi_i-x}+\mathrm{e}^{\xi_i-2r_\varepsilon+x}\big)W_i^{p-1}W_j\,\mathrm{d}x. \tag{4.22}$$

By Hölder's inequality and by estimates (4.16) and (4.17), we get:

$$\begin{aligned}
\int_{D_\varepsilon^l}\mathrm{e}^{-\xi_i-x}W_i^{p-1}W_j\,\mathrm{d}x&\leqslant\Big(\int_{D_\varepsilon^l}W_i^pW_j\,\mathrm{d}x\Big)^{\frac{p-1}{p}}\Big(\int_{D_\varepsilon^l}\mathrm{e}^{-p(\xi_i+x)}W_j\,\mathrm{d}x\Big)^{\frac1p}\\
&=\mathrm{O}\big(\varepsilon^{2\alpha_1\frac{p-1}{p}}\big)\Big(\int_{D_\varepsilon^l}\mathrm{e}^{-p(\xi_i+x)}W_j\,\mathrm{d}x\Big)^{\frac1p}.
\end{aligned}\tag{4.23}$$

If $j<l$ (so that $\xi_j\leqslant\xi_{l-1}$) and setting $x-\xi_l=y$, we get:

$$\begin{aligned}
\int_{D_\varepsilon^l}\mathrm{e}^{-p(\xi_i+x)}W_j\,\mathrm{d}x&\leqslant c\int_{\chi_l}^{\chi_{l+1}}\mathrm{e}^{-p\xi_i}\mathrm{e}^{-(x-\xi_j)}\,\mathrm{d}x=\int_{\frac{\xi_{l-1}-\xi_l}{2}}^{\frac{\xi_{l+1}-\xi_l}{2}}\mathrm{e}^{-p\xi_i}\mathrm{e}^{-(y+\xi_l-\xi_j)}\,\mathrm{d}y\\
&\leqslant c\mathrm{e}^{-p\xi_i-\xi_l+\xi_j-\frac{\xi_{l-1}-\xi_l}{2}}\leqslant c\mathrm{e}^{-p\xi_1+\frac{\xi_{l-1}-\xi_l}{2}}\\
&=\mathrm{O}\big(\varepsilon^{p\alpha_1+\frac{\alpha_l-\alpha_{l-1}}{2}}\big)=\mathrm{o}\big(\varepsilon^{2\alpha_1(1-\sigma)}\big).
\end{aligned}\tag{4.24}$$

If $j>l$ (so that $\xi_j\geqslant\xi_{l+1}$) and setting $x-\xi_l=y$, we get:

$$\begin{aligned}
&\int_{D_\varepsilon^l}\mathrm{e}^{-p(\xi_i+x)}W_j\,\mathrm{d}x\leqslant c\int_{\chi_l}^{\chi_{l+1}}\mathrm{e}^{-p\xi_i}\mathrm{e}^{x-\xi_j}\,\mathrm{d}x=\int_{\frac{\xi_{l-1}-\xi_l}{2}}^{\frac{\xi_{l+1}-\xi_l}{2}}\mathrm{e}^{-p\xi_i}\mathrm{e}^{y+\xi_l-\xi_j}\,\mathrm{d}y\\
&\quad\leqslant c\mathrm{e}^{-p\xi_i+\xi_l-\xi_j+\frac{\xi_{l+1}-\xi_l}{2}}\leqslant c\mathrm{e}^{-p\xi_1-\frac{\xi_{l+1}-\xi_l}{2}}=\mathrm{O}\big(\varepsilon^{p\alpha_1+\frac{\alpha_{l+1}-\alpha_l}{2}}\big)=\mathrm{o}\big(\varepsilon^{2\alpha_1(1-\sigma)}\big).
\end{aligned}\tag{4.25}$$

If $j=l$, we get:

$$\int_{D_\varepsilon^l} \mathrm{e}^{-p(\xi_i+x)} W_j\,\mathrm{d}x \leqslant c \int_{\chi_l}^{\chi_{l+1}} \mathrm{e}^{-p(\xi_i+x)} \mathrm{e}^{x-\xi_l}\,\mathrm{d}x = c\mathrm{e}^{-p\xi_i-\xi_l-} \int_{\chi_l}^{\chi_{l+1}} \mathrm{e}^{-(p-1)x}\,\mathrm{d}x$$
$$\leqslant c\mathrm{e}^{-(p+1)\xi_1} \mathrm{e}^{-(p-1)\frac{\xi_l-\xi_{l-1}}{2}} = \mathrm{O}\big(\varepsilon^{(p+1)\alpha_1+(p-1)\frac{\alpha_l-\alpha_{l-1}}{2}}\big) = \mathrm{o}\big(\varepsilon^{2\alpha_1(1-\sigma)}\big). \tag{4.26}$$

By (4.23)–(4.26) we deduce that

$$\int_{D_\varepsilon^l} \mathrm{e}^{-\xi_i-x} W_i^{p-1} W_j\,\mathrm{d}x = \mathrm{o}\big(\varepsilon^{2\alpha_1(1-\sigma)}\big). \tag{4.27}$$

Now, let us estimate:

$$\int_{D_\varepsilon^l} \mathrm{e}^{\xi_i-2r_\varepsilon+x} W_i^{p-1} W_j\,\mathrm{d}x \leqslant c \int_{\chi_l}^{\chi_{l+1}} \mathrm{e}^{\xi_i-2r_\varepsilon+x} \mathrm{e}^{-(x-\xi_j)}\,\mathrm{d}x$$
$$\leqslant c\mathrm{e}^{\xi_i-2r_\varepsilon+x}(\chi_{l+1}-\chi_l) \leqslant c\varepsilon^{\alpha_i-\alpha_j+N-2}|\log\varepsilon| = \mathrm{o}\big(\varepsilon^{2\alpha_1(1-\sigma)}\big). \tag{4.28}$$

Finally, by (4.22), (4.27) and (4.28) we deduce (4.18). □

That concludes the proof.

Proof of Proposition 4.3 (completed). Direct computations yield:

$$I_\varepsilon(V) - \sum_{i=1}^k I_\varepsilon(V_i) - \frac{1}{p+1}\int_{D_\varepsilon}\left[\sum_{i=1}^k V_i^{p+1} - |V|^{p+1}\right] = \sum_{\substack{i,j=1\\ i>j}}^k (-1)^{i+j} \int_{D_\varepsilon} W_i^p V_j.$$

Let χ_l and D_ε^l be defined as in Lemma 4.5. Since $0\leqslant V_i\leqslant W_i$ for all i, we can replace the letter W by V once or twice in the estimate (4.16), and thus we obtain:

$$\begin{aligned}
I_\varepsilon(V) - \sum_{i=1}^k I_\varepsilon(V_i) &= \frac{1}{p+1}\sum_{l=1}^k \int_{D_\varepsilon^l}\left[V_l^{p+1} - |V|^{p+1} + (p+1)\sum_{l>j}(-1)^{l+j} W_l^p V_j\right] + \mathrm{o}\big(\varepsilon^{2\alpha_1}\big)\\
&= \frac{1}{p+1}\sum_{l=1}^k \int_{D_\varepsilon^l}\left[V_l^{p+1} - |V|^{p+1} + (p+1)\sum_{j\neq l}(-1)^{l+j} V_l^p V_j\right]\\
&\quad + \sum_{l=1}^k\sum_{j\neq l}(-1)^{l+j}\int_{D_\varepsilon^l}(W_l^p - V_l^p)V_j - \sum_{l=1}^k\sum_{j>l}(-1)^{l+j}\int_{D_\varepsilon^l} W_l^p(V_j-W_j)\\
&\quad - \sum_{l=1}^k\sum_{j>l}(-1)^{l+j}\int_{D_\varepsilon^l} W_l^p W_j + \mathrm{o}\big(\varepsilon^{2\alpha_1}\big) \quad \text{(using (4.18), (4.19) and (A.14))}\\
&= -\sum_{l=1}^k\sum_{j>l}(-1)^{l+j}\int_{D_\varepsilon^l} W_l^p W_j + \mathrm{o}\big(\varepsilon^{2\alpha_1}\big) \quad \text{(using (4.16) and (4.17))}\\
&= \sum_{l=1}^{k-1}\int_{D_\varepsilon^l} W_l^p W_{l+1} + \mathrm{o}\big(\varepsilon^{2\alpha_1}\big) = a_3\sum_{l=1}^{k-1}\mathrm{e}^{-|\xi_{l+1}-\xi_l|} + \mathrm{o}\big(\varepsilon^{2\alpha_1}\big).
\end{aligned} \tag{4.29}$$

Combining estimates (4.29) and (4.8) with the choice in (2.15), we deduce that

$$\begin{aligned} I_\varepsilon(V) &= ka_1 + a_2 e^{-2\xi_1} + a_3 \varepsilon^{N-2} e^{2\xi_k} + a_4 \sum_{i=1}^{k-1} e^{-|\xi_{i+1}-\xi_i|} + o(\varepsilon^{2\alpha_1}) \\ &= ka_1 + a_2 \varepsilon^{2\alpha_1} \lambda_1^2 + a_3 \varepsilon^{N-2-2\alpha_k} \frac{1}{\lambda_k^2} + a_4 \sum_{i=1}^{k-1} \varepsilon^{\alpha_{i+1}-\alpha_i} \frac{\lambda_{i+1}}{\lambda_i} + o(\varepsilon^{2\alpha_1}), \end{aligned}$$

and the claim follows, because of the choice (2.16) for the α_i's. □

Let us complete the proof of the existence of sign changing-bubble tower solutions to problem (1.2).

Proof of Theorem 1.1. In virtue of Lemma 4.1, we need to find a critical point of the function $\tilde{I}_\varepsilon$. From Lemma 4.2 and Proposition 4.3, we get:

$$\tilde{I}_\varepsilon(\lambda) = ka_1 + \varepsilon^{\frac{N-2}{2k}} \big(\Psi_k(\Lambda) + o(1)\big),$$

where the term o(1) is uniform.

We claim that the function Ψ_k has a minimum point $\lambda^* = (\lambda_1^*, \ldots, \lambda_k^*)$, where $\lambda_i^* > 0$, $i = 1, \ldots, k$. In fact, it holds

$$\lim_{|\lambda| \to +\infty} \Psi_k(\lambda) = +\infty, \tag{4.30}$$

and

$$\lim_{\lambda_i \to 0} \Psi_k(\lambda) = +\infty \quad \text{for any } i = 1, \ldots, k. \tag{4.31}$$

Let us prove (4.30). Assume $|\lambda| \to +\infty$. If $\lambda_1 \to +\infty$, then the claim follows, since $\Psi_k(\lambda) \geqslant a_2 \lambda_1^2$. If λ_1 is bounded then $\frac{\lambda_{i+1}}{\lambda_i} \to +\infty$ for some i (otherwise $|\lambda|$ will be bounded!) and the claim again follows, since $\Psi_k(\lambda) \geqslant a_3 \frac{\lambda_{i+1}}{\lambda_i}$. Let us prove (4.31). If $\lambda_k \to 0$, then the claim follows, since $\Psi_k(\lambda) \geqslant \frac{1}{2} a_2 \frac{1}{\lambda_k^2}$. If $\lambda_j \to 0$ for some $j = 1, \ldots, k-1$, then either $\frac{\lambda_{i+1}}{\lambda_i} \to +\infty$ for some $i \geqslant j$ or $\frac{\lambda_{i+1}}{\lambda_i}$ is bounded for any $i = j, \ldots, k-1$. In the first case, it is clear that the claim immediately follows. In the second case, we deduce that $\lambda_k \to 0$ and the claim again follows.

Since λ^* is stable with respect to uniform perturbation, for ε small enough there exists $\lambda^\varepsilon = (\lambda_1^\varepsilon, \ldots, \lambda_k^\varepsilon)$ critical point of $\tilde{I}_\varepsilon(\lambda)$, such that $\lambda_i^\varepsilon \to \lambda_i^*$ as $\varepsilon \to 0$, for $i = 1, \ldots, k$.

Therefore, the function $V + \phi(\xi^\varepsilon)$ is a solution to (2.2).

The claim follows, since $\mu_i = e^{-\frac{2}{N-2}\xi_i} = \lambda_i^{\frac{2}{N-2}} \varepsilon^{\frac{2i-1}{2k}}$ and $M_i := \lambda_i^{\frac{1}{N-2}}$. □

Appendix A

Lemma A.1. *It holds*:

$$P_\varepsilon w_\mu(x) = w_\mu(x) - \alpha_N \mu^{\frac{N-2}{2}} H(x,0) - \alpha_N \left(\frac{\varepsilon^2 \mu}{\varepsilon^2 + \mu^2}\right)^{\frac{N-2}{2}} \frac{1}{|x|^{N-2}} + R_{\varepsilon,\mu}(x),$$

where

$$0 \leqslant R_{\varepsilon,\mu}(x) \leqslant c\mu^{N-2}\left[\frac{\varepsilon^{N-2}}{|x|^{N-2}} + \mu^2 + \varepsilon^{N-2}\right], \quad x \in \Omega_\varepsilon,$$

for some positive and fixed constant c.

Proof. The function $R_{\varepsilon,\mu}$ solves:

$$\begin{cases} -\Delta R_{\varepsilon,\mu} = 0, & x \in \Omega_\varepsilon, \\ R_{\varepsilon,\mu}(x) = \alpha_N \left[-\dfrac{\mu^{\frac{N-2}{2}}}{(\mu^2 + |x|^2)^{\frac{N-2}{2}}} + \dfrac{\mu^{\frac{N-2}{2}}}{|x|^{N-2}} + \left(\dfrac{\varepsilon^2 \mu}{\varepsilon^2 + \mu^2}\right)^{\frac{N-2}{2}} \dfrac{1}{|x|^{N-2}}\right], & x \in \partial\Omega, \\ R_{\varepsilon,\mu}(x) = \alpha_N \mu^{\frac{N-2}{2}} H(x,0), & x \in \partial B_\varepsilon. \end{cases}$$

Let $\widehat{R}_{\varepsilon,\mu}(y) := \mu^{-\frac{N-2}{2}} R_{\varepsilon,\mu}(\varepsilon y),\ y \in \Omega/\varepsilon \setminus B_1$. Then $\widehat{R}_{\varepsilon,\mu}$ solves:

$$\begin{cases} -\Delta \widehat{R}_{\varepsilon,\mu} = 0, & y \in \Omega/\varepsilon \setminus B_1, \\ \widehat{R}_{\varepsilon,\mu}(y) = \alpha_N \Big[-\frac{1}{(\mu^2+\varepsilon^2|y|^2)^{\frac{N-2}{2}}} + \frac{1}{\varepsilon^{N-2}|y|^{N-2}} + \frac{1}{(\varepsilon^2+\mu^2)^{\frac{N-2}{2}}} \frac{1}{|y|^{N-2}} \Big], & y \in \partial\Omega/\varepsilon, \\ R_{\varepsilon,\mu}(x) = \alpha_N H(\varepsilon y, 0), & y \in \partial B_1. \end{cases} \tag{A.1}$$

In particular, there exists a positive constant c such that

$$0 \leqslant \widehat{R}_{\varepsilon,\mu}(y) \leqslant c\mu^2, \quad \forall y \in \partial\Omega/\varepsilon, \qquad 0 \leqslant \widehat{R}_{\varepsilon,\mu}(y) \leqslant c, \quad \forall y \in \partial B_1. \tag{A.2}$$

Let Ψ be the solution to

$$\begin{cases} -\Delta \Psi = 0, & y \in B_{d/\varepsilon} \setminus B_1, \\ \Psi(y) = c\mu^2, & y \in \partial B_{d/\varepsilon}, \\ \Psi(y) = c, & y \in \partial B_1, \end{cases} \tag{A.3}$$

where $d := \operatorname{diam} \Omega$, so that $\Omega/\varepsilon \subset B_{d/\varepsilon}$. An easy computation gives that

$$\Psi = \frac{c}{d^{N-2} - \varepsilon^{N-2}} \left[\frac{d^{N-2}(1-\mu^2)}{|y|^{N-2}} + d^{N-2}\mu^2 - \varepsilon^{N-2} \right]. \tag{A.4}$$

By (A.1)–(A.3), using maximum principle, we deduce that $0 \leqslant \widehat{R}_{\varepsilon,\mu}(y) \leqslant \Psi(y)$ for any $y \in \Omega/\varepsilon \setminus B_1$ and by (A.4) the claim follows. □

It is useful to write Lemma A.1 in the following equivalent way.

Lemma A.2. *It holds*:

$$\Pi_i(x,\theta) = -C_N \mathrm{e}^{-x-\xi_i} H\big(\mathrm{e}^{-\frac{p-1}{2}x}\theta, 0\big) - C_N \mathrm{e}^{-2r_\varepsilon+\xi_i+x} + \rho_i(x,\theta), \tag{A.5}$$

where

$$\big|\rho_i(x,\theta)\big| \leqslant c\big(\mathrm{e}^{-x-p\xi_i} + \mathrm{e}^{-2r_\varepsilon-\xi_i+x}\big), \quad (x,\theta) \in D_\varepsilon. \tag{A.6}$$

In particular

$$\big|\Pi_i(x,\theta)\big| \leqslant c\big(\mathrm{e}^{-x-\xi_i} + \mathrm{e}^{-2r_\varepsilon+\xi_i+x}\big), \quad (x,\theta) \in D_\varepsilon. \tag{A.7}$$

Lemma A.3. *It holds*:

$$\int\limits_{D_\varepsilon} |W_i|^{p-1} \mathrm{e}^{-2(x+\xi_i)}\,\mathrm{d}x\,\mathrm{d}\theta = \begin{cases} \mathrm{O}(\mathrm{e}^{-(p+1)\xi_i}) & \textit{if } N \geqslant 5, \\ \mathrm{O}(r_\varepsilon \mathrm{e}^{-4\xi_i}) & \textit{if } N = 4, \\ \mathrm{O}(\mathrm{e}^{-4\xi_i}) & \textit{if } N = 3, \end{cases} \tag{A.8}$$

and

$$\int\limits_{D_\varepsilon} |W_i|^{p-1} \mathrm{e}^{2(-2r_\varepsilon+x+\xi_i)}\,\mathrm{d}x\,\mathrm{d}\theta = \begin{cases} \mathrm{O}(\mathrm{e}^{-(p+1)(r_\varepsilon-\xi_i)}) & \textit{if } N \geqslant 5, \\ \mathrm{O}(r_\varepsilon \mathrm{e}^{-4(r_\varepsilon-\xi_i)}) & \textit{if } N = 4, \\ \mathrm{O}(\mathrm{e}^{-4(r_\varepsilon-\xi_i)}) & \textit{if } N = 3. \end{cases} \tag{A.9}$$

Proof. Let us prove (A.8). We have:

$$\int\limits_{D_\varepsilon} |W_i|^{p-1} \mathrm{e}^{-2(x+\xi_i)}\,\mathrm{d}x\,\mathrm{d}\theta \leqslant c \int\limits_{D_\varepsilon} \mathrm{e}^{-(p-1)(x-\xi_i)} \big(1+\mathrm{e}^{-(p-1)(x-\xi_i)}\big)^{-2} \mathrm{e}^{-2(x+\xi_i)}\,\mathrm{d}x\,\mathrm{d}\theta. \tag{A.10}$$

If $N=3$ we set $x - \xi_i = y$ and, we get:

$$\begin{aligned} &\int\limits_{D_\varepsilon} \mathrm{e}^{-(p-1)(x-\xi_i)} \big(1+\mathrm{e}^{-(p-1)(x-\xi_i)}\big)^{-2} \mathrm{e}^{-2(x+\xi_i)}\,\mathrm{d}x\,\mathrm{d}\theta \\ &\quad = \mathrm{e}^{-4\xi_i} \int\limits_{D_\varepsilon-\xi_i} \mathrm{e}^{-(p+1)y} \big(1+\mathrm{e}^{-(p-1)y}\big)^{-2} \mathrm{e}^{-2(x+\xi_i)}\,\mathrm{d}y\,\mathrm{d}\theta = \mathrm{O}\big(\mathrm{e}^{-4\xi_i}\big). \end{aligned}$$

Here $D_\varepsilon - \xi_i = \{(y,\theta)\colon\ r_0(\theta) - \xi_i \leqslant y \leqslant \xi_i,\ \theta \in S^{N-1}\}$. If $N \geqslant 5$

$$\begin{aligned}
&\int\limits_{D_\varepsilon} \mathrm{e}^{-(p-1)(x-\xi_i)}\bigl(1+\mathrm{e}^{-(p-1)(x-\xi_i)}\bigr)^{-2}\mathrm{e}^{-2(x+\xi_i)}\,\mathrm{d}x\,\mathrm{d}\theta\\
&\quad\leqslant \int\limits_{D_\varepsilon} \mathrm{e}^{-(p-1)(x-\xi_i)}\mathrm{e}^{2(p-1)(x-\xi_i)}\mathrm{e}^{-2(x+\xi_i)}\,\mathrm{d}x\,\mathrm{d}\theta\\
&\quad= \mathrm{e}^{-(p+1)\xi_i}\int\limits_{D_\varepsilon} \mathrm{e}^{(p-3)x}\,\mathrm{d}x\,\mathrm{d}\theta = \mathrm{O}\bigl(\mathrm{e}^{-(p+1)\xi_i}\bigr).
\end{aligned}$$

The case $N = 4$ can be deduced by previous estimate.

Let us prove (A.9). We have:

$$c\int\limits_{D_\varepsilon} |W_i|^{p-1}\mathrm{e}^{2(-2r_\varepsilon+x+\xi_i)}\,\mathrm{d}x\,\mathrm{d}\theta \leqslant c\int\limits_{D_\varepsilon} \mathrm{e}^{-(p-1)(x-\xi_i)}\bigl(1+\mathrm{e}^{-(p-1)(x-\xi_i)}\bigr)^{-2}\mathrm{e}^{2(-2r_\varepsilon+x+\xi_i)}\,\mathrm{d}x\,\mathrm{d}\theta. \tag{A.11}$$

If $N = 3$ we set $x - \xi_i = y$ and, we get:

$$\begin{aligned}
&\int\limits_{D_\varepsilon} \mathrm{e}^{-(p-1)(x-\xi_i)}\bigl(1+\mathrm{e}^{-(p-1)(x-\xi_i)}\bigr)^{-2}\mathrm{e}^{2(-2r_\varepsilon+x+\xi_i)}\,\mathrm{d}x\,\mathrm{d}\theta\\
&\quad= \mathrm{e}^{-4(r_\varepsilon-\xi_i)}\int\limits_{D_\varepsilon-\xi_i} \mathrm{e}^{-(p-3)y}\bigl(1+\mathrm{e}^{-(p-1)y}\bigr)^{-2}\,\mathrm{d}y\,\mathrm{d}\theta = \mathrm{O}\bigl(\mathrm{e}^{-4(r_\varepsilon-\xi_i)}\bigr).
\end{aligned}$$

Here $D_\varepsilon - \xi_i = \{(y,\theta)\colon\ r_0(\theta) - \xi_i \leqslant y \leqslant \xi_i,\ \theta \in S^{N-1}\}$.

If $N \geqslant 5$ we set $x - r_\varepsilon = y$ and, we get:

$$\begin{aligned}
&\int\limits_{D_\varepsilon} \mathrm{e}^{-(p-1)(x-\xi_i)}\bigl(1+\mathrm{e}^{-(p-1)(x-\xi_i)}\bigr)^{-2}\mathrm{e}^{2(-2r_\varepsilon+x+\xi_i)}\,\mathrm{d}x\,\mathrm{d}\theta\\
&\quad= \mathrm{e}^{-(p+1)(r_\varepsilon-\xi_i)}\int\limits_{D_\varepsilon-r_\varepsilon} \mathrm{e}^{-(p-3)y}\bigl(1+\mathrm{e}^{-(p-1)(y+r_\varepsilon-\xi-i)}\bigr)^{-2}\,\mathrm{d}y\,\mathrm{d}\theta\\
&\quad\leqslant \mathrm{e}^{-(p+1)(r_\varepsilon-\xi_i)}\int\limits_{D_\varepsilon-r_\varepsilon} \mathrm{e}^{-(p-3)y}\,\mathrm{d}y\,\mathrm{d}\theta = \mathrm{O}\bigl(\mathrm{e}^{-(p+1)(r_\varepsilon-\xi_i)}\bigr).
\end{aligned}$$

Here $D_\varepsilon - r_\varepsilon = \{(y,\theta)\colon\ r_0(\theta) - r_\varepsilon \leqslant y \leqslant r_\varepsilon,\ \theta \in S^{N-1}\}$.

The case $N = 4$ can be deduced by previous estimate. □

Lemma A.4. *It holds*:

$$\int\limits_{D_\varepsilon} W_i^{p-1}\Pi_i^2\,\mathrm{d}x\,\mathrm{d}\theta = \mathrm{o}\bigl(\varepsilon^{2\alpha_1}\bigr), \tag{A.12}$$

$$\int\limits_{D_\varepsilon} |\Pi_i|^{p+1}\,\mathrm{d}x\,\mathrm{d}\theta = \mathrm{o}\bigl(\varepsilon^{2\alpha_1}\bigr), \tag{A.13}$$

$$\int\limits_{D_\varepsilon} W_i^{p}\Pi_j\,\mathrm{d}x\,\mathrm{d}\theta = \mathrm{o}\bigl(\varepsilon^{2\alpha_1}\bigr), \quad i \neq j. \tag{A.14}$$

Proof. Let us prove (A.12). By (A.7), using estimates (A.8) and (A.9), we get:

$$\int_{D_\varepsilon} |W_i|^{p-1}\Pi_i^2\,\mathrm{d}x\,\mathrm{d}\theta \leqslant c\int_{D_\varepsilon} |W_i|^{p-1}\mathrm{e}^{-2(x+\xi_i)}\,\mathrm{d}x\,\mathrm{d}\theta + c\int_{D_\varepsilon}|W_i|^{p-1}\mathrm{e}^{2(-2r_\varepsilon+x+\xi_i)}\,\mathrm{d}x\,\mathrm{d}\theta$$
$$= \mathrm{o}\big(\mathrm{e}^{-2\xi_i}\big) + \mathrm{o}\big(\mathrm{e}^{-2(r_\varepsilon-\xi_i)}\big).$$

Let us prove (A.13). By (A.7) we get (setting in the second integral $x - r_\varepsilon = y$):

$$\int_{D_\varepsilon} |\Pi_i|^{p+1}\,\mathrm{d}x\,\mathrm{d}\theta \leqslant c\int_{D_\varepsilon} \mathrm{e}^{-(p+1)(\xi_i+x)}\,\mathrm{d}x\,\mathrm{d}\theta + c\int_{D_\varepsilon}\mathrm{e}^{(p+1)(-2r_\varepsilon+\xi_i+x)}\,\mathrm{d}x\,\mathrm{d}\theta$$
$$\leqslant c\mathrm{e}^{-(p+1)\xi_i}\int_{D_\varepsilon}\mathrm{e}^{-(p+1)x}\,\mathrm{d}x\,\mathrm{d}\theta + c\mathrm{e}^{-(p+1)(r_\varepsilon-\xi_i)}\int_{D_\varepsilon-r_\varepsilon}\mathrm{e}^{(p+1)y}\,\mathrm{d}y\,\mathrm{d}\theta$$
$$= \mathrm{O}\big(\mathrm{e}^{-(p+1)\xi_i} + \mathrm{e}^{-(p+1)(r_\varepsilon-\xi_i)}\big) = \mathrm{O}\big(\varepsilon^{(p+1)\alpha_i} + \varepsilon^{(p+1)(\frac{N-2}{2}-\alpha_i)}\big) = \mathrm{o}\big(\varepsilon^{2\alpha_1}\big). \tag{A.15}$$

Here $D_\varepsilon - r_\varepsilon = \{(y,\theta):\ r_0(\theta) - r_\varepsilon \leqslant y \leqslant 0,\ \theta\in S^{N-1}\}$. Let us prove (A.14). By (A.7), setting $x - \xi_i = y$ we deduce that

$$\int_{D_\varepsilon} W_i^p(W_j - V_j)\,\mathrm{d}x \leqslant \int_{D_\varepsilon}\big(\mathrm{e}^{\xi_j-2r_\varepsilon+x} + \mathrm{e}^{-\xi_j-x}\big)W_i^p(x)\,\mathrm{d}x$$
$$\leqslant c\int_{D_\varepsilon-\xi_i}\big(\mathrm{e}^{\xi_j-2r_\varepsilon+\xi_i+y} + \mathrm{e}^{-\xi_j-\xi_i-y}\big)W^p(y)\,\mathrm{d}y$$
$$\leqslant c\mathrm{e}^{\xi_j-2r_\varepsilon+\xi_i}\int_{\mathbb{R}}\mathrm{e}^{y}W^p(y)\,\mathrm{d}y + c\mathrm{e}^{-\xi_j-\xi_i}\int_{\mathbb{R}}\mathrm{e}^{-y}W^p(y)\,\mathrm{d}y$$
$$= \mathrm{O}\big(\mathrm{e}^{\xi_j-2r_\varepsilon+\xi_i}\big) + \mathrm{O}\big(\mathrm{e}^{-\xi_j-\xi_i}\big) = \mathrm{o}\big(\varepsilon^{2\alpha_1}\big). \tag{A.16}$$

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