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# DOUBLE BLOW-UP SOLUTIONS FOR A BREZIS–NIRENBERG TYPE PROBLEM

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In this paper we construct a domain $\Omega$ for which the problem

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} - \varepsilon u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a family of solutions which blow-up and concentrate in two different points of $\Omega$ as $\varepsilon$ goes to 0.

*Keywords*: Critical Sobolev exponent; blowing-up solution; Robin's function.

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## 0. Introduction

Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 3$ and let $p = \frac{N+2}{N-2}$ be the critical exponent for the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$.

In this paper we are concerned with the problem of existence and qualitative properties of solutions for the non linear elliptic problem

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} - \varepsilon u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{0.1}$$

where $\varepsilon$ is a positive parameter.

In the last years, several researches have been developed on the existence of solutions — not necessarily positive — of elliptic equations with a non linear term which is a perturbation of a critical non-linearity.

In the very celebrated paper [6], Brezis and Nirenberg study a critical elliptic problem with a general lower-order perturbation whose model is

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} + \lambda u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{0.2}$$

for an arbitrary parameter $\lambda$.

As the authors pointed out, solvability of (0.2) is strictly related to the sign of $\lambda$ and the dimension $N$.

A first general observation (see [6]) is that if $\lambda_1 \le \lambda$, $\lambda_1$ being the first eigenvalue of $(-\Delta)$ in $\Omega$ with Dirichlet boundary condition, then (0.2) does not have any solution.

On the other hand, if $\lambda < \lambda_1$ but still $\lambda > 0$, solvability of (0.2) depends on the dimension $N$. If $N \ge 4$ problem (0.2) has a solution, independently on $\Omega$. In $N = 3$, the problem turns out to be more delicate and in [6] a precise result is given in the case $\Omega$ is a ball: in this case, (0.2) has a solution if and only if $\lambda \in (\frac{1}{4}\lambda_1, \lambda_1)$.

Once established the solvability of (0.2), a natural direction of investigations was to study multiplicity and qualitative properties of solutions to (0.2); in particular to understand the concentration phenomena of the solutions for $\lambda > 0$ but close to 0.

In this context a crucial role is played by the Green's and Robin's functions of the domain play a crucial role. Let us recall their definitions.

Let $\Gamma_x(y) = \frac{\gamma_N}{|x-y|^{N-2}}$, for every $x$, $y \in \mathbb{R}^N$, be the fundamental solution for the Laplacian on entire $\mathbb{R}^N$. Here $\gamma_N$ is a positive constant which depends only on $N$. For every point $x \in \Omega \cup \partial\Omega$, let us define the regular part of the Green's function, $H_\Omega(x, \cdot)$, as the solution of the following Dirichlet problem

$$\begin{cases} \Delta_y H_\Omega(x,y) = 0 & \text{in } \Omega\,, \\ H_\Omega(x,y) = \Gamma_x(y) & \text{on } \partial\Omega\,. \end{cases} \tag{0.3}$$

The Green's function of the Dirichlet problem for the Laplacian is then defined by $G_x(y) = \Gamma_x(y) - H_\Omega(x,y)$ and it satisfies

$$\begin{cases} -\Delta_y G_x(y) = \delta_x(y) & \text{in } \Omega\,, \\ G_x(y) = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{0.4}$$

For every $x \in \Omega$ the leading term of the regular part of the Green's function, i.e. $x \to H_\Omega(x,x)$ is called *Robin function of* $\Omega$ *at the point* $x$.

In [21] it is proved that any nondegenerate critical point $x_0$ of the Robin's function generates a family of solutions of (0.2), for $\lambda = \varepsilon > 0$ and $N \ge 5$, concentrating around $x_0$ as $\varepsilon$ goes to 0 (see also [14]). Rey generalized this result in [22]. In [18] the authors constructed solutions which concentrate around $k \ge 1$ different points of $\Omega$ which are suitable critical points of the function $\Phi_k : \mathbb{R}^k_+ \times \Omega^k \longrightarrow \mathbb{R}$ defined by

$$\Phi_k(\Lambda, x) = \frac{1}{2}\,(M(x)\Lambda, \Lambda) - \frac{1}{2}\sum_{i=1}^{k} \Lambda_i^{\frac{4}{N-2}}\,, \tag{0.5}$$

where $\Lambda = (\Lambda_1, \dots, \Lambda_k)^T$ and $M(x) = (m_{ij}(x))_{1\le i,j\le k}$ is the matrix defined by

$$m_{ii}(x) = H(x_i, x_i)\,, \quad m_{ij}(x) = G(x_i, x_j) \quad \text{if } i \neq j\,. \tag{0.6}$$

Problem (0.2) becomes notably more delicate when $\lambda = 0$ or $\lambda < 0$, since in these cases its solvability depends also on the geometry and the topology of $\Omega$.

In fact, a Pohozaev's identity (see [20, 6]) yields that (0.2) has no solution when $\Omega$ is star-shaped (strictly star-shaped) and $\lambda < 0$ (respectively $\lambda = 0$). On the other hand, (0.2) has at least one solution if $\Omega$ is a symmetric anellus for any $\lambda \le 0$ (see [15]) or when $\Omega$ has a "small hole" for $\lambda = 0$ (see [8]). The most general result concerning existence of solution for (0.2) when $\lambda = 0$ is contained in [3]: Bahri and Coron showed that if some homology group of $\Omega$ with coefficients in $\mathbf{Z}_2$ is not trivial, then (0.2) has at least one non trivial solution.

In this paper we study solvability for problem (0.1) for $N \ge 5$. In particular, we are concerned with existence of solution which blow-up and concentrate in some points of $\Omega$ in the sense of the following definition.

**Definition 0.1.** Let $u_\varepsilon$ be a family of solutions for (0.1). We say that $u_\varepsilon$ blow-up and concentrate at $k$ points $x_1, \dots, x_k$ in $\Omega$ if there exist speeds of concentration $\mu_{1_\varepsilon}, \dots, \mu_{k_\varepsilon} > 0$, and points $x_{1_\varepsilon}, \dots, x_{k_\varepsilon} \in \Omega$ with $\lim_{\varepsilon\to 0} \mu_{i_\varepsilon} = 0$ and $\lim_{\varepsilon\to 0} x_{i_\varepsilon} = x_i$, $x_i \neq x_j$ for $i$, $j = 1, \dots, k$, $i \neq j$, such that

$$u_\varepsilon - \sum_{i=1}^{k} i^*_\Omega(U^p_{\mu_{i_\varepsilon}, x_{i_\varepsilon}}) \longrightarrow 0 \quad \text{in } \mathrm{H}^1_0(\Omega) \quad \text{as } \varepsilon \to 0$$

where $i^*_\Omega$ is the adjoint operator of the embedding $i_\Omega : H^1_0(\Omega) \to L^{p+1}(\Omega)$.

Such a definition is motivated by a blow-up analysis for solutions to problem (0.1), as it is performed in [23]. In [2], some links between the speeds of concentration and the points of concentration are established. Moreover it follows from [17] that the blow-up points remain far from each other and that the speeds of concentration are of the same order.

Here (see [1, 7] and [24])

$$U_{\lambda,y}(x) = c_N \frac{\lambda^{\frac{N-2}{2}}}{(\lambda^2 + |x-y|^2)^{\frac{N-2}{2}}}\,, \quad x \in \mathbb{R}^N\,,\ y \in \mathbb{R}^N\,,\ \lambda > 0\,,$$

with $c_N = [N(N-2)]^{(N-2)/4}$, are all the solutions of the equation

$$-\Delta u = u^{\frac{N+2}{N-2}} \quad \text{in } \mathbb{R}^N\,.$$

If $\mu_{1_\varepsilon}, \dots, \mu_{k_\varepsilon}$ are of order $\varepsilon^{\frac{1}{N-4}}$, namely $\lim_{\varepsilon\to 0} \varepsilon^{-\frac{1}{N-4}} \mu_{i_\varepsilon} = \lambda_i > 0$ for $i = 1, \dots, k$, then existence of solutions to (0.1) is related to existence of critical points for the function $\psi_k : \mathbb{R}^k_+ \times \Omega^k \longrightarrow \mathbb{R}$ defined by

$$\psi_k(\Lambda, x) = \frac{1}{2}\left(M(x)\Lambda, \Lambda\right) + \frac{1}{2}\sum_{i=1}^{k} \Lambda_i^{\frac{4}{N-2}}, \tag{0.7}$$

where the matrix $M(x)$ is defined in (0.6).

In the last part of Sec. 2 we will prove the following necessary condition.

**Theorem 0.1.** *Let $u_\varepsilon$ be a family of solution of* (0.1) (*as in Theorem* 2.1) *which blow-up and concentrate at $k$ different points $x_1, \dots, x_k$ of $\Omega$ with speed of concentration $\mu_{i_\varepsilon}$ such that* $\lim_{\varepsilon\to 0} \varepsilon^{-\frac{1}{N-4}} \mu_{i_\varepsilon} = \lambda_i > 0$ *for $i = 1, \dots, k$. Then $(\Lambda, x)$ is a critical point of $\psi_k$, where $\Lambda_i = c_n \lambda_i$ for $i = 1, \dots, k$* (*see* (2.23)).

A straightforward application of this theorem is a non-existence result.

**Theorem 0.2.** *There do not exist any family of solutions of* (0.1) (*as in Theorem* 2.1) *which blow-up and concentrate at a given point $x_0$ of $\Omega$.*

The crucial point is that the concentration point $x_0$ should be a critical point of the function $x \longrightarrow H(x, x)$ with $H(x_0, x_0) < 0$, which is not possible.

On the contrary, if $\Omega$ is a domain with a small "hole", we prove the existence of a solution which blow-up and concentrate in two points, showing that $\psi_2$ (see (0.7)) has a critical point of "min-max" type. Here we follow some ideas of [10] (see also [11]).

Our existence result is

**Theorem 0.3.** *Let $D$ be a bounded smooth domain in $\mathbb{R}^N$ which contains the origin* 0 *and let $N \geq 5$. There exists $\delta_0 > 0$ such that, if $0 < \delta < \delta_0$ is fixed and $\Omega$ is the domain given by $D \setminus \omega$ for any smooth domain $\omega \subset B(0, \delta)$, then there exists $\varepsilon_0 > 0$ such that problem* (0.1) *has a solution $u_\varepsilon$ for any $0 < \varepsilon < \varepsilon_0$. Moreover the family of solutions $u_\varepsilon$ blows-up and concentrates at two different points of $\Omega$ in the sense of Definition* 0.1, *with speeds of concentration of order $\varepsilon^{\frac{1}{N-4}}$.*

We would like to point out that it is known that functions similar to (0.5) and (0.7) play a crucial role in the concentration phenomena associated to the following supercritical and subcritical problems

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2} \pm \varepsilon} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{0.8}$$

More precisely in [4] the authors considered the subcritical case, i.e. $\frac{N+2}{N-2} - \varepsilon$, and they showed that existence of nondegenerate critical points of a suitable function, which involves the first eigenvalue of the matrix (0.6), allows to find solutions which concentrate in those points as $\varepsilon \to 0$.

In [10] the authors study the supercritical case, i.e. $\frac{N+2}{N-2}+\varepsilon$, and they exhibit a domain $\Omega$ such that problem (0.8) has a family of solutions which blow-up at exactly two different points of $\Omega$.

This paper is organized as follows. In Sec. 1 we reduce the problem to a finite dimensional one, using the usual Ljapunov–Schmidt procedure (see [2] and [12]). In Sec. 2 we work out the asymptotic expansion for a finite dimensional function which comes from the reduction and we prove Theorem 0.2. In Sec. 3 we set up a min-max scheme to find a critical point of the reduced function and we prove Theorem 0.3. Finally in Appendix A we make some technical computations.

## 1. The Finite-Dimensional Reduction

Let $\alpha$ be a fixed positive number which will be choosen later. Let us set

$$\Omega_\varepsilon := \Omega/\varepsilon^\alpha = \{x/\varepsilon^\alpha \mid x \in \Omega\}$$

and let us introduce the following problem

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} - \varepsilon^{2\alpha+1} u & \text{in } \Omega_\varepsilon \\ u > 0 & \text{in } \Omega_\varepsilon \\ u = 0 & \text{on } \partial\Omega_\varepsilon\,. \end{cases} \tag{1.1}$$

By a rescaling argument one sees that $u(x)$ is a solution of (0.1) if and only if $w(x) = \varepsilon^{\alpha\frac{N-2}{2}} u(\varepsilon^\alpha x)$ is a solution of (1.1).

Let $\mathrm{H}^1_0(\Omega_\varepsilon)$ be the Hilbert space equipped with the usual inner product

$$(u,v) = \int_{\Omega_\varepsilon} \nabla u \nabla v\,, \quad \text{which induces the norm } \|u\| = \left(\int_{\Omega_\varepsilon} |\nabla u|^2\right)^{1/2}.$$

It will be useful to rewrite problem (1.1) in a different setting. Let us then introduce the following operator.

**Definition 1.1.** Let $i^*_\varepsilon : \mathrm{L}^{\frac{2N}{N+2}}(\Omega_\varepsilon) \longrightarrow \mathrm{H}^1_0(\Omega_\varepsilon)$ be the adjoint operator of the immersion $i_\varepsilon : \mathrm{H}^1_0(\Omega_\varepsilon) \hookrightarrow \mathrm{L}^{\frac{2N}{N-2}}(\Omega_\varepsilon)$, i.e.

$$i^*_\varepsilon(u) = v \Longleftrightarrow (v,\varphi) = \int_{\Omega_\varepsilon} u(x)\varphi(x)\,dx \quad \forall\,\varphi \in \mathrm{H}^1_0(\Omega_\varepsilon)\,.$$

Observe that $i^*_\varepsilon : \mathrm{L}^{\frac{2N}{N+2}}(\Omega_\varepsilon) \longrightarrow \mathrm{H}^1_0(\Omega_\varepsilon)$ is continuous uniformly with respect to $\varepsilon$, i.e. there exists a constant $c > 0$ such that

$$\|i^*_\varepsilon(u)\| \le c\|u\|_{\frac{2N}{N+2}} \quad \forall\, u \in \mathrm{L}^{\frac{2N}{N+2}}(\Omega_\varepsilon)\,, \quad \forall\,\varepsilon > 0\,. \tag{1.2}$$

By means of the definition of the operator $i^*_\varepsilon$, problem (1.1) turns out to be equivalent to

$$\begin{cases} u = i^*_\varepsilon[f(u) - \varepsilon^{2\alpha+1}u] \\ u \in \mathrm{H}^1_0(\Omega_\varepsilon)\,, \end{cases} \tag{1.3}$$

where $f(s) = (s^+)^{\frac{N+2}{N-2}}$.

As $\varepsilon \to 0$, the limit problem associated to (1.1) is

$$-\Delta u = u^p \quad \text{in } \mathbb{R}^N \tag{1.4}$$

where $p = \frac{N+2}{N-2}$.

It is well known (see [1, 7, 24]) that all positive solutions to (1.4) are given by

$$U_{\lambda,y}(x) = c_N \left( \frac{\lambda}{\lambda^2 + |x-y|^2} \right)^{\frac{N-2}{2}}$$

where $c_N = [N(N-2)]^{\frac{(N-2)}{4}}$, $\lambda > 0$ and $y \in \mathbb{R}^N$.

It is then natural to look for solutions to (1.1) with $k$ blow-up points of the form

$$u = \sum_{j=1}^{k} P_\varepsilon U_{\lambda_j, y_j}(x) + \phi_\varepsilon(x) \tag{1.5}$$

where $P_\varepsilon$ denotes the orthogonal projection of $H_0^{1,2}(\mathbb{R}^N)$ onto $H_0^{1,2}(\Omega_\varepsilon)$, that is,

$$P_\varepsilon U_{\lambda_j, y_j}(x) = i_\varepsilon^*(U^p_{\lambda_j, y_j})(x) \quad x \in \Omega_\varepsilon\,, \tag{1.6}$$

for certain parameters $\lambda_j$ and points $y_j$. The function $\phi_\varepsilon$ in (1.5) is a lower order term given by a Ljapunov–Schmidt reduction.

For notation's convenience we call

$$U_j := U_{\lambda_j, y_j} \quad \text{and} \quad P_\varepsilon U_j := i_\varepsilon^*(U^p_{\lambda_j, y_j})\,.$$

In order to set the Liapunov–Schmidt reduction's scheme, we need to introduce the functions

$$\psi_i^0 := \frac{\partial U_{\lambda_i, y_i}}{\partial \lambda_i}\,, \quad \psi_i^j := \frac{\partial U_{\lambda_i, y_i}}{\partial y_i^j} \quad j = 1, \ldots, N\,,$$

and the corresponding projections onto $H_0^1(\Omega_\varepsilon)$, given by

$$P_\varepsilon \psi_i^j := i_\varepsilon^*(p U_i^{p-1} \psi_i^j)\,, \quad i = 1, \ldots, k,\ j = 0, \ldots, N\,.$$

We will first solve problem (1.1) over the set of functions orthogonal in $H_0^1(\Omega_\varepsilon)$ to $P_\varepsilon \psi_i^j$. For this purpose we need to introduce the following definitions.

**Definition 1.2.** For any $\varepsilon > 0$, $\lambda \in (\mathbb{R}^+)^k$ and $y \in \Omega_\varepsilon^k$ set

$$K^\varepsilon_{\lambda,y} = \{u \in \mathrm{H}_0^1(\Omega_\varepsilon) \mid (u, P_\varepsilon \psi_i^j) = 0,\ i = 1, \ldots, k,\ j = 0, 1, \ldots, N\}\,. \tag{1.7}$$

Let $\Pi^\varepsilon : (\mathbb{R}^+)^k \times \Omega_\varepsilon^k \times \mathrm{H}_0^1(\Omega_\varepsilon) \longrightarrow K^\varepsilon_{\lambda,y}$ be defined as

$$\Pi^\varepsilon(\lambda, y, u) := \Pi^\varepsilon_{\lambda,y}(u)\,,$$

where $\Pi^\varepsilon_{\lambda,y} : \mathrm{H}_0^1(\Omega_\varepsilon) \longrightarrow K^\varepsilon_{\lambda,y}$ denotes the orthogonal projection on $K^\varepsilon_{\lambda,y}$. Moreover let $L^\varepsilon_{\lambda,y} : \mathrm{H}_0^1(\Omega_\varepsilon) \longrightarrow K^\varepsilon_{\lambda,y}$ be the map defined by

$$L^\varepsilon_{\lambda,y}(\phi) = \Pi^\varepsilon_{\lambda,y} \left\{ \phi - i_\varepsilon^* \left[ f'\left( \sum_{i=1}^{k} P_\varepsilon U_i \right) \phi - \varepsilon^{2\alpha+1} \phi \right] \right\}. \tag{1.8}$$

The aim of the remaining part of this section is to show that there exists a unique solution $\phi \in K^{\varepsilon}_{\lambda,y}$ of the problem

$$\Pi^{\varepsilon}_{\lambda,y}\left\{\sum_{i=1}^{k} P_\varepsilon U_i + \phi - i^*_\varepsilon\left[\left(\sum_{i=1}^{k} P_\varepsilon U_i + \phi\right)^p - \varepsilon^{2\alpha+1}\left(\sum_{i=1}^{k} P_\varepsilon U_i + \phi\right)\right]\right\} = 0 \tag{1.9}$$

and to study how $\phi$ depends on $\varepsilon$, $\lambda$ and $y$.

Observe that (1.9) can be written in the form

$$L^{\varepsilon}_{\lambda,y}\left(\sum P_\varepsilon U_i + \phi\right)$$
$$= \Pi^{\varepsilon}_{\lambda,y} \circ i^*_\varepsilon\left[\left(\sum P_\varepsilon U_i + \phi\right)^p - p\left(\sum P_\varepsilon U_i\right)^{p-1}\left(\sum P_\varepsilon U_i + \phi\right)\right].$$

Hence we first need to study the invertibility and the regularity of the operator $L^{\varepsilon}_{\lambda,y}$, uniformly with respect to $\varepsilon$ and to the parameters $(\lambda, y)$ in a certain range.

From now on we will consider numbers $\lambda$ and points $y$ belonging to the set

$$\Theta^{\varepsilon}_{\delta} = \{(\lambda, y) \in (\mathbb{R}^+)^k \times \Omega^k_\varepsilon \mid y_i = x_i/\varepsilon^\alpha,\ i = 1, \ldots, k,\ (\lambda, x) \in \Theta_\delta\}, \tag{1.10}$$

where

$$\Theta_\delta = \{(\lambda, x) \in (\mathbb{R}^+)^k \times \Omega^k \mid \operatorname{dist}(x_i, \partial\Omega) \geq \delta,\ \delta < \lambda_i < 1/\delta, \tag{1.11}$$

$$|x_i - x_l| \geq \delta, \quad i = 1, \ldots, k,\ i \neq l\}. \tag{1.12}$$

**Lemma 1.1.** *The map $\Pi^\varepsilon$, given by Definition* 1.2, *is a $C^1$-map. Moreover for any $\delta > 0$ there exist $\varepsilon_0 > 0$ and $c > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, for any $(\lambda, y) \in \Theta^{\varepsilon}_{\delta}$ and for any $u \in \mathrm{H}^1_0(\Omega_\varepsilon)$ it holds*

$$\|\Pi^\varepsilon(\lambda, y, u)\| \leq c\|u\|,$$

$$\|D_\lambda \Pi^\varepsilon(\lambda, y, u)\|_{\mathcal{L}(\mathbb{R}^k, \mathrm{H}^1_0(\Omega_\varepsilon))} \leq c\|u\|,$$

$$\|D_y \Pi^\varepsilon(\lambda, y, u)\|_{\mathcal{L}(\mathbb{R}^{nk}, \mathrm{H}^1_0(\Omega_\varepsilon))} \leq c\|u\|,$$

$$\|D_u \Pi^\varepsilon(\lambda, y, u)\|_{\mathcal{L}(\mathbb{R}^{nk}, \mathrm{H}^1_0(\Omega_\varepsilon))} \leq c.$$

**Proof.** An application of the dominated convergence theorem and (1.2) yield that the maps

$$(\lambda, y) \longrightarrow P_\varepsilon U_i \quad \text{and} \quad (\lambda, y) \longrightarrow P_\varepsilon \psi^j_i$$

are $C^1$. Again by (1.2) and the linearity of differentiation, one gets

$$\|D_\lambda P_\varepsilon U_i\| \leq c \quad \text{and} \quad \|D_y P_\varepsilon U_i\| \leq c$$

and

$$\|D_\lambda P_\varepsilon \psi^j_i\| \leq c \quad \text{and} \quad \|D_y P_\varepsilon \psi^j_i\| \leq c,$$

uniformly for $\varepsilon$ small enough and $(\lambda, y) \in \Theta^{\varepsilon}_{\delta}$.

Now a direct computation yields the estimates we are looking for. □

**Lemma 1.2.** *For any $\delta > 0$ there exist $\varepsilon_0 > 0$ and $c > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and for any $(\lambda, y) \in \Theta_\delta^\varepsilon$ it holds*

$$\|L_{\lambda,y}^\varepsilon(\phi)\| \geq C\|\phi\| \quad \forall\, \phi \in K_{\lambda,y}^\varepsilon\,. \tag{1.13}$$

*Moreover the map $\mathcal{L}^\varepsilon : \Theta_\delta^\varepsilon \times K_{\lambda,y}^\varepsilon \longrightarrow K_{\lambda,y}^\varepsilon$ defined by*

$$\mathcal{L}^\varepsilon(\lambda, y, h) := \mathcal{L}_{\lambda,y}^\varepsilon(h) = (L_{\lambda,y}^\varepsilon)^{-1}(h) \tag{1.14}$$

*is of class $C^1$. Moreover for any $\varepsilon \in (0, \varepsilon_0)$, for any $(\lambda, y) \in \Theta_\delta^\varepsilon$ and for any $h \in K_{\lambda,y}^\varepsilon$ it holds*

$$\|D_y\mathcal{L}_{\lambda,y}^\varepsilon(\lambda, y, h)\| \leq C\|h\| \tag{1.15}$$

*and*

$$\|D_\lambda\mathcal{L}_{\lambda,y}^\varepsilon(\lambda, y, h)\| \leq C\|h\|\,. \tag{1.16}$$

**Proof.** Existence, uniqueness and estimate (1.13) can be proved arguing like in [18]. Let us show (1.15); estimate (1.16) can be obtained in a similar way.

Let us call $\phi = \mathcal{L}_{\lambda,y}^\varepsilon(\lambda, y, h)$. By differentiating with respect to $y$ the following expression

$$\Pi_{\lambda,y}^\varepsilon \left\{ \phi - i_\varepsilon^* \left[ p \left( \sum_{i=1}^k P_\varepsilon U_i \right)^{p-1} \phi - \varepsilon^{2\alpha+1}\phi \right] \right\} = h$$

we easily get

$$L_{\lambda,y}^\varepsilon(D_y\phi) = \Pi_{\lambda,y}^\varepsilon \left[ i_\varepsilon^* \left( p(p-1) \left( \sum_{i=1}^k P_\varepsilon U_i \right)^{p-2} D_y \left( \sum_{i=1}^k P_\varepsilon U_i \right) \phi \right) \right]$$
$$- (D_y\Pi_{\lambda,y}^\varepsilon) \left\{ \phi - i_\varepsilon^* \left[ p \left( \sum_{i=1}^k P_\varepsilon U_i \right)^{p-1} \phi - \varepsilon^{2\alpha+1}\phi \right] \right\}. \tag{1.17}$$

Now set

$$D_y\phi = (D_y\phi)^\perp + \sum b_{ij} P_\varepsilon \psi_i^j\,, \quad \text{with } (D_y\phi)^\perp \in K_{\lambda,y}^\varepsilon\,.$$

First of all we claim that

$$|b_{ij}| = O(\|\phi\|)\,. \tag{1.18}$$

In fact, since $\phi \in K_{\lambda,y}^\varepsilon$, we have $(\phi, P_\varepsilon\psi_i^j) = 0\ \forall\, i,\ j$, which becomes by means of differentiation $(\phi, D_y P_\varepsilon \psi_i^j) = -(D_y\phi, P_\varepsilon\psi_i^j)$. Then the numbers $b_{ij}$ are solutions of the following algebraic system

$$\sum b_{ij}(P_\varepsilon\psi_i^j, P_\varepsilon\psi_h^k) = -(\phi, D_y P_\varepsilon \psi_h^k)$$

and (1.18) follows. Summing up all the above information, we see that $(D_y\phi)^\perp \in K^\varepsilon_{\lambda,y}$ satisfies the following relation

$$\begin{aligned} L^\varepsilon_{\lambda,y}((D_y\phi)^\perp) &= -L^\varepsilon_{\lambda,y}\left(\sum b_{ij}P_\varepsilon\psi_i^j\right) \\ &\quad + \Pi^\varepsilon_{\lambda,y}\left(i^*_\varepsilon\left(p(p-1)\left(\sum_{i=1}^k P_\varepsilon U_i\right)^{p-2} D_y\left(\sum_{i=1}^k P_\varepsilon U_i\right)\phi\right)\right) \\ &\quad - (D_y\Pi^\varepsilon_{\lambda,y})\left(\phi - i^*_\varepsilon\left(p\left(\sum_{i=1}^k P_\varepsilon U_i\right)^{p-1}\phi - \varepsilon^{2\alpha+1}\phi\right)\right). \end{aligned} \tag{1.19}$$

From (1.19) and (1.13), we can argue that

$$\begin{aligned} \|(D_y\phi)^\perp\| &\le C\|L^\varepsilon_{\lambda,y}((D_y\phi)^\perp)\| \\ &\le C\left\|\sum b_{ij}P_\varepsilon\psi_i^j - i^*_\varepsilon\left(p\left(\sum_{i=1}^k P_\varepsilon U_i\right)^{p-1}\left(\sum b_{ij}P_\varepsilon\psi_i^j\right)\right.\right. \\ &\quad \left.\left. - \varepsilon^{2\alpha+1}\sum b_{ij}P_\varepsilon\psi_i^j\right)\right\| + C\left\|i^*_\varepsilon\left(\left(\sum_{i=1}^k P_\varepsilon U_i\right)^{p-1}\phi\right)\right\| \\ &\quad + C\left\|\phi - i^*_\varepsilon\left(\left(\sum_{i=1}^k P_\varepsilon U_i\right)^{p-1}\phi - \varepsilon^{2\alpha+1}\phi\right)\right\| \\ &\le C\left\|\sum b_{ij}P_\varepsilon\psi_i^j\right\| + \left\|\left(\sum_{i=1}^k P_\varepsilon U_i\right)^{p-1}\phi\right\|_{\frac{2N}{N+2}} + \|\phi\| \\ &\le C\left\{\sum |b_{ij}| + \|\phi\|\right\} \le C\|\phi\| \end{aligned} \tag{1.20}$$

where we have used (1.18) and the property that for any $u \in H^{1,2}_0(\Omega_\varepsilon)$ it holds

$$\left\|u - i^*_\varepsilon\left(p\left(\sum_{i=1}^k P_\varepsilon U_i\right)^{p-1}u - \varepsilon^{2\alpha+1}u\right)\right\| \le C\|u\|$$

as follows from simple computations. Hence from (1.18), (1.19), (1.20) we get

$$\|D_y\phi\| \le \|(D_y\phi)^\perp\| + \left\|\sum b_{ij}P_\varepsilon\psi_i^j\right\| \le C\|\phi\|$$

and (1.15) follows. □

We have now all elements to solve (1.3) over the set $K^\varepsilon_{\lambda,y}$.

**Proposition 1.1.** *Let* $\alpha = \frac{1}{N-4}$. *For any* $\delta > 0$ *there exists* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *and for any* $(\lambda, y) \in \Theta_\delta^\varepsilon$, *there exists a unique* $\phi_{\lambda,y}^\varepsilon \in K_{\lambda,y}^\varepsilon$ *such that*

$$\Pi_{\lambda,y}^\varepsilon \left\{ \sum_{i=1}^k P_\varepsilon U_i + \phi - i_\varepsilon^* \left[ \left( \sum_{i=1}^k P_\varepsilon U_i + \phi \right)^p - \varepsilon^{2\alpha+1} \left( \sum_{i=1}^k P_\varepsilon U_i + \phi \right) \right] \right\} = 0 \tag{1.21}$$

*and*

$$\|\phi\| \le C\varepsilon^\mu \,, \tag{1.22}$$

*where*

$$\mu = \begin{cases} 2\alpha + \dfrac{1}{2} & \text{if } N \ge 6 \\[2ex] 2\alpha + \dfrac{1}{4} & \text{if } N = 5 \end{cases} \tag{1.23}$$

*Moreover the map* $\phi^\varepsilon : \Theta_\delta^\varepsilon \longrightarrow K_{\lambda,y}^\varepsilon$ *defined by*

$$\phi^\varepsilon(\lambda, y) := \phi_{\lambda,y}^\varepsilon \tag{1.24}$$

*is of class* $C^1$. *Moreover for any* $\varepsilon \in (0, \varepsilon_0)$ *and for any* $(\lambda, y) \in \Theta_\delta^\varepsilon$

$$\|D_\lambda \phi^\varepsilon(\lambda, y)\| \le C\varepsilon^\mu \tag{1.25}$$

*and*

$$\|D_y \phi^\varepsilon(\lambda, y)\| \le C\varepsilon^\mu \,. \tag{1.26}$$

**Proof.** Existence, uniqueness of $\phi_{\lambda,y}^\varepsilon$ and estimate (1.22) follow arguing like in [18].

For notation's convenience we will write $\phi = \phi_{\lambda,y}^\varepsilon$.

By definition, the function $\phi$, is a zero of the map $B : \Theta_\delta^\varepsilon \times K_{\lambda,y}^\varepsilon \longrightarrow K_{\lambda,y}^\varepsilon$ defined by

$$B(\lambda, y, \phi) = \phi - \mathcal{L}_{\lambda,y}^\varepsilon \circ \Pi_{\lambda,y}^\varepsilon \circ i_\varepsilon^*[N_\varepsilon(\lambda, y, \phi)] \tag{1.27}$$

where $N_\varepsilon : (\mathbb{R}^+)^k \times \Omega_\varepsilon^k \times \mathrm{H}_0^1(\Omega_\varepsilon) \longrightarrow K_{\lambda,y}^\varepsilon$ is given by

$$N_\varepsilon(\lambda, y, u) = \left[ f\left( \sum_{i=1}^k P_\varepsilon U_i + u \right) - \sum_{i=1}^k f(U_i) - f'\left( \sum_{i=1}^k P_\varepsilon U_i \right) u \right.$$
$$\left. - \varepsilon^{2\alpha+1} \left( \sum_{i=1}^k P_\varepsilon U_i \right) \right].$$

Observe that $N_\varepsilon$ depends continuously on its parameters.

Differentiating (1.27) with respect to $\phi$ we see that for any $\theta \in \mathrm{H}_0^1(\Omega_\varepsilon)$

$$D_\phi B(\lambda, y, \phi)[\theta] = \theta - \mathcal{L}_{\lambda,y}^\varepsilon \circ \Pi_{\lambda,y}^\varepsilon \circ i_\varepsilon^*[D_\phi N_\varepsilon(\lambda, y, \phi)\theta] \,. \tag{1.28}$$

By Lemma 1.1 we deduce

$$\begin{aligned}
&\|\mathcal{L}^{\varepsilon}_{\lambda,y} \circ \Pi^{\varepsilon}_{\lambda,y} \circ i^*_{\varepsilon}[D_\phi N_\varepsilon(\lambda, y, \phi)\theta]\| \\
&\quad \le C\|D_\phi N_\varepsilon(\lambda, y, \phi)\theta\|_{\frac{2N}{N+2}} \le C\|D_\phi N_\varepsilon(\lambda, y, \phi)\|_{\frac{N}{2}}\|\theta\|_{\frac{2N}{N-2}} \\
&\quad \le C\varepsilon^{2\alpha}\|\theta\|\,, \qquad (1.29)
\end{aligned}$$

where we used that

$$\|D_\phi N_\varepsilon(\lambda, y, \phi)\|_{\frac{N}{2}} = \|f'\left(\sum_{i=1}^{k} P_\varepsilon U_i + \phi\right) - f'\left(\sum_{i=1}^{k} P_\varepsilon U_i\right)\|_{\frac{N}{2}} \le C\varepsilon^{2\alpha}$$

with a constant $C$ independent of $\varepsilon$ and $(\lambda, y) \in \Theta^{\varepsilon}_{\delta}$ (see [18, Lemma 5.3]). From (1.28) and (1.29) it follows that $D_\phi B(\lambda, y, \phi)$ is invertible with uniformly bounded inverse; moreover by Lemmas 1.1, 1.2 and (1.28) it follows that $D_\phi B(\lambda, y, \phi)$ is a $C^1$-map.

Let us now differentiate with respect to $y$

$$\begin{aligned}
&D_y B(\lambda, y, \phi) = -D_y \mathcal{L}^{\varepsilon}_{\lambda,y}[\Pi^{\varepsilon}_{\lambda,y} \circ i^*_\varepsilon(N_\varepsilon(\lambda, y, \phi))] \\
&\circ D_y \Pi^{\varepsilon}_{\lambda,y}[i^*_\varepsilon(N_\varepsilon(\lambda, y, \phi))] \circ i^*_\varepsilon \\
&(D_y N_\varepsilon(\lambda, y, \phi))\,, \qquad (1.30)
\end{aligned}$$

while

$$\begin{aligned}
D_{y_a^b} N_\varepsilon(\lambda, y, \phi) = {} & f'\left(\sum_{i=1}^{k} P_\varepsilon U_i + \phi\right) D_{y_a^b} P_\varepsilon U_a - f'(U_a) D_{y_a^b} U_a \\
& - f''\left(\sum_{i=1}^{k} P_\varepsilon U_i\right) D_{y_a^b} P_\varepsilon U_a \phi - \varepsilon^{2\alpha+1} D_{y_a^b} P_\varepsilon U_a\,. \qquad (1.31)
\end{aligned}$$

Since $D_y B(\lambda, y, \phi)$ depends continuously on $(\lambda, y, \phi)$, the implicit function Theorem let us conclude that $\phi^\varepsilon$ is a $C^1$-map and also that

$$D_{\lambda,y}\phi^\varepsilon(\lambda, y) = -(D_\phi B(\lambda, y, \phi))^{-1} \circ [D_{\lambda,y} B(\lambda, y, \phi)]\,. \qquad (1.32)$$

Now let us prove (1.26). (1.25) can be proved in a similar way.

We have

$$\begin{aligned}
\|D_y\phi\| &\le C\|D_y B(\lambda, y, \phi)\| \\
&\le C\{\|i^*_\varepsilon(N_\varepsilon(\lambda, y, \phi))\| + \|i^*_\varepsilon((D_y N_\varepsilon)(\lambda, y, \phi))\|\} \\
&\quad C\{\|N_\varepsilon(\lambda, y, \phi)\|_{\frac{2N}{N+2}} + \|(D_y N_\varepsilon)(\lambda, y, \phi)\|_{\frac{2N}{N+2}}\} \\
&\le C\varepsilon^\mu\,, \qquad (1.33)
\end{aligned}$$

where the last inequality follows from the estimates (see [18, Appendix A] and [21])

$$\begin{aligned}
&\|N_\varepsilon(\lambda, y, \phi)\|_{\frac{2N}{N+2}} \\
&\quad \le C\left\{\left\| f'\left(\sum_{i=1}^{k} P_\varepsilon U_i + \phi\right) - f'\left(\sum_{i=1}^{k} U_i\right)\right\|_{\frac{N}{2}} \|\phi\| + \varepsilon^{2\alpha+1}\|P_\varepsilon U_i\|_{\frac{2N}{N+2}}\right\} \\
&\quad \le C(\varepsilon^{2\alpha+\mu} + \varepsilon^{2\alpha+1})
\end{aligned} \tag{1.34}$$

and

$$\begin{aligned}
&\|(D_y N_\varepsilon)(\lambda, y, \phi)\|_{\frac{2N}{N+2}} \\
&\quad \le \left\|\left[f'\left(\sum_{i=1}^{k} P_\varepsilon U_i + \phi\right) - f'\left(\sum_{i=1}^{k} P_\varepsilon U_i\right) - f''\left(\sum_{i=1}^{k} P_\varepsilon U_i\right)\phi\right] D_{y_a^b} P_\varepsilon U_a\right\|_{\frac{2N}{N+2}} \\
&\qquad + \left\|\left[f'\left(\sum_{i=1}^{k} P_\varepsilon U_i\right) - f'(U_a)\right] D_{y_a^b} P_\varepsilon U_a\right\|_{\frac{2N}{N+2}} \\
&\qquad + \|f'(U_a)[D_{y_a^b} P_\varepsilon U_a - U_a]\|_{\frac{2N}{N+2}} + \varepsilon^{2\alpha+1}\|D_{y_a^b} P_\varepsilon U_a\|_{\frac{2N}{N+2}} \\
&\quad \le C(\|\phi\|^{\min\{2,p\}} + \varepsilon^{\alpha\frac{N+2}{2}} + \varepsilon^{2\alpha+1}\varepsilon^{-\frac{\alpha}{2}}) \\
&\quad \le C\varepsilon^{\mu}\,.
\end{aligned} \tag{1.35}$$

□

## 2. The Reduced Functional

From Proposition 1.1 we can deduce that the function $w_\varepsilon = \sum_{i=1}^{k} P_\varepsilon U_{\lambda_i, y_i} + \phi^\varepsilon_{\lambda,y}$ is a solution of (1.3) if and only if $(\lambda, y) \in \Theta^\varepsilon_\delta$ are such that for any $i = 1, \ldots, k$ and $j = 0, \ldots, N$

$$\begin{aligned}
0 = &\left(\sum_{i=1}^{k} P_\varepsilon U_{\lambda_i, y_i} + \phi^\varepsilon_{\lambda,y}, P_\varepsilon \psi^j_{\lambda_i^\varepsilon, y_i^\varepsilon}\right) - \left(i^*_\varepsilon\left[\left(\sum_{i=1}^{k} P_\varepsilon U_{\lambda_i, y_i} + \phi^\varepsilon_{\lambda,y}\right)^p\right.\right. \\
&\left.\left. - \varepsilon^{2\alpha+1}\left(\sum_{i=1}^{k} P_\varepsilon U_{\lambda_i, y_i} + \phi^\varepsilon_{\lambda,y}\right)\right], P_\varepsilon \psi^j_{\lambda_i^\varepsilon, y_i^\varepsilon}\right).
\end{aligned} \tag{2.1}$$

We prove the following

**Lemma 2.1.** *The function* $w_\varepsilon = \sum_{i=1}^{k} P_\varepsilon U_{\lambda_i, y_i} + \phi^\varepsilon_{\lambda,y}$ *is a solution for* (1.1) *if and only if* $(\lambda, x) \in \Theta_\delta$, $x = \varepsilon^\alpha y$ *(see* (1.12), (1.10)), *is a critical point for the function* $F_\varepsilon(\lambda, x)$ *defined by*

$$F_\varepsilon(\lambda, x) = J_\varepsilon\left(\sum_{i=1}^{k} P_\varepsilon U_{\lambda_i, y_i} + \phi^\varepsilon_{\lambda,y}\right), \tag{2.2}$$

*where* $J_\varepsilon : \mathrm{H}_0^1(\Omega_\varepsilon) \longrightarrow \mathbb{R}$ *is defined by*

$$J_\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon} |Du|^2\,dy - \frac{1}{p+1}\int_{\Omega_\varepsilon} u^{p+1}\,dy + \frac{\varepsilon^{2\alpha+1}}{2}\int_{\Omega_\varepsilon} u^2\,dy\,.$$

**Proof.** Observe that

$$\frac{\partial F_\varepsilon}{\partial x_i^j}(\lambda, x) = 0\,, \quad \frac{\partial F_\varepsilon}{\partial \lambda}(\lambda, x) = 0$$

is equivalent to

$$DJ_\varepsilon\left(\sum P_\varepsilon U_i + \phi^\varepsilon_{\lambda,y}\right)\left[\frac{\partial}{\partial y_i^j}\left(\sum P_\varepsilon U_i\right) + \frac{\partial}{\partial y_i^j}\,\phi^\varepsilon_{\lambda,y}\right] = 0 \tag{2.3}$$

and

$$DJ_\varepsilon\left(\sum P_\varepsilon U_i + \phi^\varepsilon_{\lambda,y}\right)\left[\frac{\partial}{\partial \lambda_i}\left(\sum P_\varepsilon U_i\right) + \frac{\partial}{\partial \lambda_i}\,\phi^\varepsilon_{\lambda,y}\right] = 0\,. \tag{2.4}$$

Since

$$\frac{\partial}{\partial y_i^j}\left(\sum P_\varepsilon U_i\right) = P_\varepsilon \psi_i^j + o(1)\,, \quad \frac{\partial}{\partial \lambda_i}\left(\sum P_\varepsilon U_i\right) = P_\varepsilon \psi_i^0 + o(1)$$

and

$$\left\|\frac{\partial \phi^\varepsilon_{\lambda,y}}{\partial y_i^j}\right\| \le C\varepsilon^{\alpha+1}\,, \quad \left\|\frac{\partial \phi^\varepsilon_{\lambda,y}}{\partial \lambda_i}\right\| \le C\varepsilon^{\alpha+1}$$

(see Proposition 1.1), Eqs. (2.3) and (2.4) read

$$DJ_\varepsilon\left(\sum P_\varepsilon U_i + \phi^\varepsilon_{\lambda,y}\right)[P_\varepsilon \psi_i^j + o(1)] = 0\,.$$

Observe now that for a given function $\psi \in H_0^1(\Omega_\varepsilon)$, we can uniquely decompose $\psi$ in the following way

$$\psi = \Pi^\varepsilon_{\lambda,y}\psi + \sum_{ij} b_{ij} P_\varepsilon \psi$$

for certain unique constants $b_{ij}$; obviously $\Pi^\varepsilon_{\lambda,y}\psi \in K^\varepsilon_{\lambda,y}$.

On the other hand, from the definition of $\phi^\varepsilon_{\lambda,y}$ we have that

$$DJ_\varepsilon\left(\sum P_\varepsilon U_i + \phi^\varepsilon_{\lambda,y}\right)[\theta] = 0 \quad \forall\,\theta \in K^\varepsilon_{\lambda,y}\,.$$

Hence

$$\nabla F_\varepsilon(\lambda, x) = 0$$

is equivalent to

$$DJ_\varepsilon\left(\sum P_\varepsilon U_i + \phi^\varepsilon_{\lambda,y}\right)[P_\varepsilon \psi_i^j + o(1)\psi] = 0$$

$$DJ_\varepsilon\left(\sum P_\varepsilon U_i + \phi^\varepsilon_{\lambda,y}\right)\left[P_\varepsilon \psi_i^j + o(1)\left(\sum_{i,j} P_\varepsilon \psi_i^j\right)\right] = 0\,,$$

that turns out to be

$$DJ_\varepsilon\left(\sum P_\varepsilon U_i + \phi^\varepsilon_{\lambda,y}\right)[P_\varepsilon \psi_i^j] = 0\,; \tag{2.5}$$

finally, Eq. (2.5) is precisely (2.1). □

We want now to work out a precise expansion for

$$F_\varepsilon(\lambda, x) = J_\varepsilon\left(\sum_{i=1}^k P_\varepsilon U_{\lambda_i, y_i} + \phi^\varepsilon_{\lambda,y}\right).$$

**Lemma 2.2.** *Let* $\alpha = \frac{1}{N-4}$. *We have*

$$F_\varepsilon(\lambda, x) = kC_N + \left[\frac{1}{2}A^2(M(x)\lambda^{\frac{N-2}{2}}, \lambda^{\frac{N-2}{2}}) + \frac{1}{2}B\left(\sum_{i=1}^k \lambda_i^2\right)\right]\varepsilon^{\frac{N-2}{N-4}} + o(\varepsilon^{\frac{N-2}{N-4}}) \tag{2.6}$$

*uniformly in* $C^1$*-norm with respect to* $(\lambda, x) \in \Theta_\delta$. *Here*

$$C_N = \frac{1}{2}\int_{\mathbb{R}^N} |DU|^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} U^{p+1} \tag{2.7}$$

*and*

$$A = \int_{\mathbb{R}^N} U^p \quad \text{and} \quad B = \int_{\mathbb{R}^N} U^2\,. \tag{2.8}$$

**Proof.** The proof of this lemma is based on the following estimates

$$P_\varepsilon U_{\lambda_j, y_j}(z) = \varepsilon^{\alpha(N-2)} G(\varepsilon^\alpha z, x)\lambda_j^{\frac{N-2}{2}}\left(\int_{\mathbb{R}^N} U^p\right) + o(\varepsilon^{\alpha(N-2)}) \tag{2.9}$$

away from $z = y$, and

$$\phi_{\lambda_j, y_j}(z) = \varepsilon^{\alpha(N-2)} H(\varepsilon^\alpha z, x)\lambda_j^{\frac{N-2}{2}}\left(\int_{\mathbb{R}^N} U^p\right) + o(\varepsilon^{\alpha(N-2)}) \tag{2.10}$$

uniformly for $z$ on each compact subset of $\Omega_\varepsilon$, where $\phi_{\lambda_j, y_j}(z) = U_{\lambda_j, y_j} - P_\varepsilon U_{\lambda_j, y_j}$, i.e. $\phi_{\lambda_j, y_j}(z)$ solves the equation

$$\begin{cases} -\Delta\phi_{\lambda_j, y_j}(z) = 0 & \text{in } \Omega_\varepsilon \\ \phi_{\lambda_j, y_j} = U_{\lambda_j, y_j} & \text{on } \partial\Omega_\varepsilon\,. \end{cases}$$

The functions $G$ and $H$ are respectively the Green function and the Robin function of the Laplacian with Dirichlet boundary condition on $\Omega$. In fact, we want to work out an expansion of $F_\varepsilon(\lambda, x)$ in term of $G$ and $H$.

Let

$$F^*_\varepsilon(\lambda, x) = J_\varepsilon\left(\sum_{i=1}^k P_\varepsilon U_{\lambda_i, y_i}\right).$$

First of all we prove that

$$F_\varepsilon^*(\lambda, x) = kC_N + \left[\frac{1}{2} A(M(x)\lambda^{\frac{N-2}{2}}, \lambda^{\frac{N-2}{2}})\varepsilon^{\alpha(N-2)} + \frac{1}{2}B\left(\sum_{i=1}^k \lambda_i^2\right)\right]\varepsilon^{2\alpha+1} + o(\varepsilon^{2\alpha+1}) \tag{2.11}$$

uniformly in $C^1$-norm with respect to $(\lambda, x) \in \Theta_\delta$. Arguing like in [10], we have

$$\frac{1}{2}\int_{\Omega_\varepsilon}\left|D\left(\sum_{i=1}^k P_\varepsilon U_{\lambda_i,y_i}\right)\right|^2 - \frac{1}{p+1}\int_{\Omega_\varepsilon}\left(\sum_{i=1}^k P_\varepsilon U_{\lambda_i,y_i}\right)^{p+1} = kC_N + \frac{1}{2}A(M(x)\lambda^{\frac{N-2}{2}}, \lambda^{\frac{N-2}{2}})\varepsilon^{\alpha(N-2)} \tag{2.12}$$

uniformly in $C^1$-norm with respect to $(\lambda, x) \in \Theta_\delta$.

We need now to evaluate

$$\frac{\varepsilon^{2\alpha+1}}{2}\int_{\Omega_\varepsilon}\left[\sum_{i=1}^k P_\varepsilon U_{\lambda_i,y_i}\right]^2 dx = \frac{\varepsilon^{2\alpha+1}}{2}\left\{\sum_{i=1}^k\int_{\Omega_\varepsilon}(P_\varepsilon U_{\lambda_i,y_i})^2\,dx + 2\sum_{i<j}\int_{\Omega_\varepsilon}P_\varepsilon U_{\lambda_i,y_i}P_\varepsilon U_{\lambda_j,y_j}\right\}. \tag{2.13}$$

For $i = 1, \dots, k$ we get

$$\begin{aligned}\int_{\Omega_\varepsilon}(P_\varepsilon U_{\lambda_i,y_i})^2\,dx &= \int_{\Omega_\varepsilon}[(P_\varepsilon U_{\lambda_i,y_i})^2 - (U_{\lambda_i,y_i})^2]dx + \int_{\Omega_\varepsilon}(U_{\lambda_i,y_i})^2\,dx\\ &= \int_{\Omega_\varepsilon}[\phi^2_{\lambda_i,y_i} - 2U_{\lambda_i,y_i}\phi_{\lambda_i,y_i}]\,dx + \int_{\Omega_\varepsilon}(U_{\lambda_i,y_i})^2\,dx\,.\end{aligned} \tag{2.14}$$

Since $N > 4$, we have

$$\int_{\Omega_\varepsilon}U^2_{\lambda_i,y_i}\,dx = \int_{\Omega_\varepsilon}\left[\frac{\lambda_i}{(\lambda_i)^2 + |y-y_i|^2}\right]^{N-2}dy = \lambda_i^2\int_{\mathbb{R}^N}U^2\,dx + O(\varepsilon^{\alpha(N-2)})\,. \tag{2.15}$$

From (2.10) we get

$$\int_{\Omega_\varepsilon}\phi^2_{\lambda_i,y_i}\,dx = \lambda_i^{N-2}\varepsilon^{\alpha(N-2)}\left(\int_{\mathbb{R}^N}U^p\,dx\right)^2\int_\Omega H(x,x_i)^2\,dx + o(\varepsilon^{\alpha(N-2)}) = O(\varepsilon^{\alpha(N-2)})\,, \tag{2.16}$$

$$\int_{\Omega_\varepsilon} \phi_{\lambda_i,y_i} U_{\lambda_i,y_i}\, dx = -2\lambda_i^{N-2}\left(\int_{\mathbb{R}^N} U^p\, dx\right)^2 \int_\Omega \frac{H(x,x_i)}{|x-x_i|}\, dx + o(\varepsilon^{\alpha(N-2)})$$
$$= O(\varepsilon^{\alpha(N-2)}) \tag{2.17}$$

and if $j \neq i$

$$\int_{\Omega_\varepsilon} P_\varepsilon U_{\lambda_i,y_i} P_\varepsilon U_{\lambda_j,y_j}\, dx = O(\varepsilon^{\alpha(N-2)}) \tag{2.18}$$

uniformly with respect to $(\lambda, x) \in \Theta_\delta$. By means of (2.14)–(2.18) we conclude that

$$\frac{\varepsilon^{2\alpha+1}}{2}\int_{\Omega_\varepsilon}\left(\sum_{i=1}^k P_\varepsilon U_{\lambda_i,y_i}\right)^2 dx = \frac{1}{2}\left(\int_{\mathbb{R}^N} U^2\right)\left(\sum_{i=1}^k \lambda_i^2\right)\varepsilon^{2\alpha+1} + o(\varepsilon^{2\alpha+1})\,. \tag{2.19}$$

Therefore the claim follows by (2.13) and (2.19).

After having found the expansion of $F_\varepsilon^*$, we need to show that the functions $F_\varepsilon$ and $F_\varepsilon^*$ are $C^1$-close, that is

$$F_\varepsilon(\lambda,x) - F_\varepsilon^*(\lambda,x) = o(\varepsilon^{\alpha(N-2)}) \tag{2.20}$$

and

$$D(F_\varepsilon(\lambda,x) - F_\varepsilon^*(\lambda,x)) = o(\varepsilon^{\alpha(N-2)}) \tag{2.21}$$

uniformly for $(\lambda, x) \in \Theta_\delta$.

By Taylor expansion, we have

$$F_\varepsilon(\lambda,x) - F_\varepsilon^*(\lambda,x) = J_\varepsilon\left(\sum_{i=1}^k P_\varepsilon U_{\lambda_i,y_i} + \tilde\phi\right) - J_\varepsilon\left(\sum_{i=1}^k P_\varepsilon U_{\lambda_i,y_i}\right)$$
$$= \int_0^1 tdtD^2 J_\varepsilon\left(\sum_{i=1}^k P_\varepsilon U_{\lambda_i,y_i} + t\tilde\phi\right)[\tilde\phi]^2\,, \tag{2.22}$$

where we used that $DJ_\varepsilon(\sum_{i=1}^k P_\varepsilon U_{\lambda_i,y_i} + \phi)[\phi] = 0$ from definition of $\phi$. We have, in particular,

$$\int_0^1 tdtD^2 J_\varepsilon\left(\sum_{i=1}^k P_\varepsilon U_{\lambda_i,y_i} + t\phi\right)[\phi]^2$$
$$= \int_0^1 tdt\left[\int_{\Omega_\varepsilon}\left(|D\phi|^2 - p\left(\sum_{i=1}^k P_\varepsilon U_{\lambda_i,y_i} + t\phi\right)^{p-1}\phi^2 + \varepsilon^{2\alpha+1}\phi^2\right)dx\right]$$
$$= \int_0^1 tdt\left[\int_{\Omega_\varepsilon}\left(\left(\sum_{i=1}^k P_\varepsilon U_{\lambda_i,y_i}\right)^p\phi - \left(\sum_{i=1}^k P_\varepsilon U_{\lambda_i,y_i} + \phi\right)^p\phi\right.\right.$$
$$\left.\left. - p\left(\sum_{i=1}^k P_\varepsilon U_{\lambda_i,y_i} + t\phi\right)^{p-1}\phi^2 + \varepsilon^{2\alpha+1}\sum_{i=1}^k P_\varepsilon U_{\lambda_i,y_i}\phi\right)dx\right].$$

So we can conclude that

$$\begin{aligned}
&|F_\varepsilon(\lambda, x) - F_\varepsilon^*(\lambda, x)| \\
&\quad \le C\left(\int_{\Omega_\varepsilon}\left(\sum_{i=1}^k P_\varepsilon U_{\lambda_i,y_i}\right)^{p-1}\phi^2 + \varepsilon^{2\alpha+1}\int_{\Omega_\varepsilon}\left(\sum_{i=1}^k P_\varepsilon U_{\lambda_i,y_i}\right)\phi\right) \\
&\quad \le C\left[\left(\int_{\Omega_\varepsilon}\left(\sum_{i=1}^k P_\varepsilon U_{\lambda_i,y_i}\right)^{\frac{2N}{N-2}} dx\right)^{\frac{2}{N}}\|\phi\|^2 + \varepsilon^{2\alpha+1}\left\|\sum_{i=1}^k P_\varepsilon U_{\lambda_i,y_i}\right\|_{\frac{2N}{N+2}}\|\phi\|\right] \\
&\quad = o(\varepsilon^{\alpha(N-2)}),
\end{aligned}$$

so we get (2.20).

In order to obtain (2.21), we observe that

$$\begin{aligned}
&D_x[F_\varepsilon(\lambda, x) - F_\varepsilon^*(\lambda, x)] \\
&\quad = \varepsilon^{-\alpha}\Bigg\{\int_0^1 t dt\Bigg[\int_{\Omega_\varepsilon} D_y\Bigg[\left(\sum_{i=1}^k P_\varepsilon U_{\lambda_i,y_i}\right)^p\phi - \left(\sum_{i=1}^k P_\varepsilon U_{\lambda_i,y_i} + \phi\right)^p\phi \\
&\qquad \not{p}\left(\sum_{i=1}^k P_\varepsilon U_{\lambda_i,y_i} + t\phi\right)^{p-1}\phi^2\Bigg] + \Bigg[\varepsilon^{2\alpha+1}\int_{\Omega_\varepsilon} D_{y_a^b}\left(\sum_{i=1}^k P_\varepsilon U_{\lambda_i,y_i}\phi\right)\Bigg] dx\Bigg]\Bigg\}.
\end{aligned}$$

Arguing like in Lemma 1.2 and taking into account (1.15), we get

$$|D_{x_a^b}(F_\varepsilon(\lambda, x) - F_\varepsilon^*(\lambda, x))| = o(\varepsilon^{\alpha(N-2)})$$

uniformly on $(\lambda, x) \in \Theta_\delta$. The corresponding estimate for the derivative with respect to $\lambda$ can be obtain in a similar way. □

Let us now introduce new parameters $\Lambda$ defined by

$$A^2\lambda_i^{N-2} = B\Lambda_i^2 \quad \text{for } i = 1, \ldots, k \tag{2.23}$$

and the function $\psi_k : \mathbb{R}_+^k \times \Omega^k \longrightarrow \mathbb{R}$ defined by

$$\psi_k(\Lambda, x) = \frac{1}{2}(M(x)\Lambda, \Lambda) + \frac{1}{2}\sum_{i=1}^k \Lambda_i^{\frac{4}{N-2}}, \tag{2.24}$$

where $M(x) = (m_{ij}(x))_{1\le i,j\le k}$ is the matrix defined by

$$m_{ii}(x) = H(x_i, x_i), \quad m_{ij}(x) = G(x_i, x_j) \quad \text{if } i \ne j. \tag{2.25}$$

**Theorem 2.1.** *Let* $u_\varepsilon = \sum_{i=1}^k P_\varepsilon U_{\lambda_{i_\varepsilon},y_{i_\varepsilon}} + \phi^\varepsilon_{\lambda_\varepsilon,y_\varepsilon}$ *be a family of solution of* (1.1) *such that* $\lim_{\varepsilon\to 0}\lambda_\varepsilon = \lambda_0 > 0$ *and* $\lim_{\varepsilon\to 0}\varepsilon^{\frac{1}{N-4}}y_\varepsilon = x_0$ *with* $(\lambda_0, x_0) \in \mathcal{O}_\delta$ *for some* $\delta > 0$. *Then* $(\Lambda_0, x_0)$ (*see* (2.23)) *is a critical point of* $\psi_k$.

**Proof.** Set $x_{i_\varepsilon} = \varepsilon^{\frac{1}{N-4}} y_{i_\varepsilon}$ and $\Lambda_{i_\varepsilon} = AB^{-1/2}\lambda_{i_\varepsilon}^{\frac{N-2}{2}}$ for $i = 1, \dots, k$. From Lemmas 2.1 and 2.2 we deduce the estimates

$$0 = \nabla F_\varepsilon(\lambda_\varepsilon, x_\varepsilon) = [\nabla\psi_k(\Lambda_\varepsilon, x_\varepsilon) + o(1)]\varepsilon^{\frac{N-2}{N-4}}\,, \tag{2.26}$$

which hold uniformly with respect to $(\lambda, x)$ in $\mathcal{O}_\delta$. By passing to the limit as $\varepsilon$ goes to zero in (2.26) we get the claim. □

**Proof of Theorem 2.1.** It follows from Theorem 2.1. □

In particular, as far as it concerns the existence of solutions which blow-up and concentrate at one point, i.e. $k = 1$, we can prove the result of non-existence contained in Theorem 0.2.

**Proof of Theorem 2.2.** Let $u_\varepsilon$ be a family of solutions which blow-up and concentrate at $x_0 \in \Omega$. Arguing as in [4], one can prove that the speeds of concentration are of order $\varepsilon^{\frac{1}{N-4}}$. Then we apply Theorem 0.2, taking into account that if $k = 1$ the function $\psi_1 : \mathbb{R}^+ \times \Omega \longrightarrow \mathbb{R}$ reduces to

$$\psi_1(\Lambda, x) = \frac{1}{2}\,H(x,x)\Lambda^2 + \frac{1}{2}\,\Lambda^{\frac{4}{N-2}}$$

and it does not have any critical point, since $H(x,x) > 0$ for any $x \in \Omega$. □



## 3. Existence of a Two-Spike Solution

In this section we construct a domain $\Omega$ for which problem (0.1) has a family of solutions which blow-up and concentrate at two different points of $\Omega$ in the sense of Definition 0.1. Here we follow the ideas of [10].

Let $D$ be a bounded domain with smooth boundary in $\mathbb{R}^N$ which contains the origin 0. The following result holds (see [10, Corollary 2.1])

**Corollary 3.1.** *For any (fixed) sufficiently small $\sigma > 0$ there exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$ and for any smooth domain $\omega \subset B(0,\delta)$ it holds*

$$\lambda_1(M(x)) < 0 \quad \forall\, x \in \mathcal{S}\,,$$

*where the manifold $\mathcal{S}$ is defined by*

$$\mathcal{S} = \{(x_1, x_2) \in \Omega^2 \mid |x_1| = |x_2| = \sigma\}$$

*and the domain $\Omega$ is given by*

$$\Omega = D \setminus \omega\,.$$

Here $\lambda_1(M(x))$ denotes the first eigenvalue of the matrix $M(x)$ associated with the domain $\Omega$.

In order to find a solution with two blow-up points in $\Omega$ of (1.1), in virtue of Lemmas 2.1 and 2.2, it is enough to find a "sufficiently stable" critical point of the function $\psi$ defined by

$$\psi(\Lambda, x) = \frac{1}{2}\left[H(x_1, x_1)\Lambda_1^2 + H(x_2, x_2)\Lambda_2^2 - 2G(x_1, x_2)\Lambda_1\Lambda_2\right] + \frac{1}{2}\left[\Lambda_1^\gamma + \Lambda_2^\gamma\right],$$

where $\gamma = \frac{4}{N-2}$.

In the following we will construct a critical point of "min-max" type of the function $\psi$.

Let us now introduce for $l > 0$ and $\rho > 0$ the following manifold

$$W_\rho^l = \{x \in \Omega^2 \mid \lambda_1(M(x)) < -l\} \cap V_\rho\,,$$

where

$$V_\rho = \{(x_1, x_2) \in \Omega^2 \mid \mathrm{dist}(x_1, \partial\Omega) > \rho,\ \mathrm{dist}(x_2, \partial\Omega) > \rho,\ |x_1 - x_2| > \rho\}\,.$$

**Lemma 3.1.** *There exist $\rho_0 > 0$ and $l_0 > 0$ such that for any $\rho \in (0, \rho_0)$ and $l \in (0, l_0)$ it holds $\mathcal{S} \subset W_\rho^l$.*

**Proof.** It is enough to take $\lambda_0 = -\max_{x\in\mathcal{S}^2} \lambda_1(M(x))$ and $\rho_0 = \mathrm{dist}(\mathcal{S}, \partial\Omega)$. □

**Lemma 3.2.** *There exists $R > 0$ such that it holds*

$$\max_{\substack{x\in\mathcal{S}^2 \\ 0\le r\le R}} \psi(re(x), x) > \max_{\substack{x\in\mathcal{S}^2 \\ r=0,R}} \psi(re(x), x) = 0\,, \tag{3.1}$$

*where $e(x) = (e_1(x), e_2(x)) \in \mathbb{R}_+^2$ is an eigenvector associated with $\lambda_1(M(x))$ with $|e(x)| = 1$.*

**Proof.** It follows from Corollary 3.1, since $\gamma < 2$. □

Now let $a$ and $b$ be fixed so that

$$b = \max_{\substack{x\in\mathcal{S}^2 \\ 0\le r\le R}} \psi(re(x), x) > a > \max_{\substack{x\in\mathcal{S}^2 \\ r=0,R}} \psi(re(x), x) = 0\,. \tag{3.2}$$

**Lemma 3.3.** *There exists $R > 0$ and for any $\rho \in (0, \rho_0)$ there exists $\tau = \tau(\rho) > 0$ such that for any $l \in (0, l_0)$ it holds*

$$\begin{aligned} b = \max_{\substack{x\in\mathcal{S}^2 \\ 0\le r\le R}} \psi(x, re(x)) &\ge \min_{\substack{x\in\mathcal{S}^2 \\ \Lambda\in I_\tau}} \psi(x, \Lambda) \\ &\ge \min_{\substack{x\in W_\rho^l \\ \Lambda\in I_\tau}} \psi(x, \Lambda) > a > \max_{\substack{x\in\mathcal{S}^2 \\ r=0,R}} \psi(x, re(x)) = 0\,, \end{aligned} \tag{3.3}$$

*where $I_\tau$ is the hyperbola in $\mathbb{R}_+^2$ defined by $I_\tau = \{\Lambda \in \mathbb{R}_+^2 \mid \Lambda_1\Lambda_2 = \tau\}$.*

**Proof.** For any $\Lambda \in I_\tau$, we have

$$\psi(x,\Lambda) \geq -G(x_1,x_2)\tau + \frac{1}{2}\left[\Lambda_1^\gamma + \left(\frac{\tau}{\Lambda_1}\right)^\gamma\right]$$
$$\geq -\frac{1}{\rho^{N-2}}\,\tau + \frac{1}{2}\left[\Lambda_1^\gamma + \left(\frac{\tau}{\Lambda_1}\right)^\gamma\right] > a\,, \tag{3.4}$$

provided that $\tau$ is choosen small enough, since $\gamma < 2$.

Finally (3.3) follows from (3.1), (3.4) and Lemma 3.1. □

**Lemma 3.4.** *For any $0 < a < b$ and $l \in (0, l_0)$ there exists $\hat\rho_0 > 0$ such that for any $\rho \in (0,\hat\rho_0)$ and for any $(\Lambda, x) \in \mathbb{R}^2_+ \times W^l_\rho$ with $\psi(x,\Lambda) \in [a,b]$, $\nabla_\Lambda \psi(\Lambda,x) = 0$ and $x \in \partial V_\rho$ there exists a vector $T$ tangent to $\mathbb{R}^2_+ \times \partial V_\rho$ at the point $(\Lambda, x)$ such that*

$$\nabla\psi(\Lambda,x)\cdot T \neq 0\,.$$

**Proof.** *Step 1.* We argue by contradiction. Let $(\Lambda_\rho, x_\rho) \in \mathbb{R}^2_+ \times \Omega^2$ be such that $\psi(\Lambda_\rho,x_\rho) \in [a,b]$, $\nabla_\Lambda\psi(\Lambda_\rho,x_\rho) = 0$, $\lambda_1(M(x_\rho)) < -l < 0$, $\mathrm{dist}(x_{1_\rho},\partial\Omega) = \rho$, $\mathrm{dist}(x_{2_\rho},\partial\Omega) \geq \rho$, $|x_{1_\rho} - x_{2_\rho}| \geq \rho$ and for any vector $T$ tangent to $\mathbb{R}^2_+ \times \partial V_\rho$ at the point $(\Lambda_\rho, x_\rho)$ it holds

$$\nabla\psi(\Lambda_\rho,x_\rho)\cdot T = 0\,. \tag{3.5}$$

Set $\Omega_\rho = \frac{\Omega}{\rho}$, $y = \frac{x}{\rho}$ and $\mu_\rho = \rho^{-\frac{N-2}{2-\gamma}}\Lambda_\rho$. We will use the notation of the Appendix A.

Then

$$\mathrm{dist}(y_{1_\rho},\partial\Omega_\rho) = 1\,, \quad \mathrm{dist}(y_{2_\rho},\partial\Omega_\rho) \geq 1\,, \quad |y_{1_\rho} - y_{2_\rho}| \geq 1\,.$$

After a rotation and a translation we may assume that $y_{1_\rho} \to (0,1)$ as $\rho \to 0$, where $0 = 0_{\mathbb{R}^{N-1}}$ and that the domain $\Omega_\rho$ becomes the half-space $P = \{(y^1,\ldots,y^N) \in \mathbb{R}^N : y^N > 0\}$.

First of all we claim that

$$0 < c_1 \leq \Lambda_{1_\rho}, \Lambda_{2_\rho} \leq c_2 \quad \text{as } \rho \to 0\,. \tag{3.6}$$

It is easy to check that $0 < c_1 \leq |\Lambda_\rho| \leq c_2$. In fact since $\nabla_\Lambda\psi(\Lambda_\rho,x_\rho) = 0$ we have that

$$\psi(\Lambda_\rho,x_\rho) = \frac{2-\gamma}{4}\,(\Lambda_{1_\rho}^\gamma + \Lambda_{2_\rho}^\gamma) \in [a,b]$$

and so if $|\Lambda_\rho| \longrightarrow +\infty$ or $|\Lambda_\rho| \longrightarrow 0$ and a contradiction arises.

Assume that $\lim_\rho \Lambda_{1_\rho} = 0$. Since $\nabla_\Lambda\psi(\Lambda_\rho,x_\rho) = 0$, we have that

$$0 = \rho^{-(N-2)}\partial_{\Lambda_1}\psi(\Lambda_\rho,x_\rho)$$
$$= H_\rho(y_{1_\rho},y_{1_\rho})\Lambda_{1_\rho} - G_\rho(y_{1_\rho},y_{2_\rho})\Lambda_{2_\rho} + \frac{\gamma}{2}\,[\Lambda_{1_\rho}\rho^{-(N-2)}]\Lambda_{1_\rho}^{\gamma-2}\,,$$

with $H_\rho(y_{1_\rho}, y_{1_\rho}) \le 1$ and $G_\rho(y_{1_\rho}, y_{2_\rho}) \le 1$. If $\gamma - 1 \le 0$ or $\liminf_{\rho\to 0} \Lambda_{1_\rho}\rho^{-(N-2)} \ge c > 0$ by passing to the limit we deduce immediately that $\Lambda_{2_\rho} \to +\infty$ and a contradiction arises. Assume $\gamma - 1 > 0$ and $\liminf_{\rho\to 0} \Lambda_{1_\rho}\rho^{-(N-2)} = 0$. Then also $\liminf_{\rho\to 0} H(x_{1_\rho}, x_{1_\rho})\Lambda_{1_\rho} = 0$ and by

$$0 = \partial_{\Lambda_1}\psi(\Lambda_\rho, x_\rho) = H(x_{1_\rho}, x_{1_\rho})\Lambda_{1_\rho} - G(x_{1_\rho}, x_{2_\rho})\Lambda_{2_\rho} + \frac{\gamma}{2}\,\Lambda_1^{\gamma-1}\,,$$

we deduce $\liminf_{\rho\to 0} G(x_{1_\rho}, x_{2_\rho})\Lambda_{2_\rho} = 0$. On the other hand since $\lambda_1(M(x_\rho)) \le -l$ and $H(x_{1_\rho}, x_{1_\rho}) \to +\infty$ as $\rho \to 0$, we obtain that also $G(x_{1_\rho}, x_{2_\rho}) \to +\infty$ as $\rho \to 0$. In conclusion it must be $\Lambda_{2_\rho} \to 0$ and a contradiction again arises.

Second we prove that

$$|y_{2_\rho}| \le C \quad \text{as } \rho \to 0\,. \tag{3.7}$$

Assume by contradiction that $|y_{1_\rho} - y_{2_\rho}| \longrightarrow +\infty$ as $\rho \to 0$. We have

$$G_\rho(y_{1_\rho}, y_{2_\rho}) \le |y_{1_\rho} - y_{2_\rho}|^{-(N-2)} \longrightarrow 0$$

and by (A.6)

$$H_\rho(y_{1_\rho}, y_{1_\rho}) \longrightarrow H_P(0,1;0,1) > 0\,.$$

Then, since $\nabla_\mu \psi_\rho(\mu_\rho, y_\rho) = 0$ (see (A.1) and (A.3)), we have

$$\begin{aligned}\psi_\rho(\mu_\rho, y_\rho) &= \frac{\gamma-2}{2\gamma}\,(M_\rho(y_\rho)\mu_\rho, \mu_\rho)\\ &= \frac{\gamma-2}{2\gamma}\,\rho^{-2\frac{N-2}{2-\gamma}}[H_\rho(y_{1_\rho}, y_{1_\rho})\Lambda_{1_\rho}^2 + H_\rho(y_{2_\rho}, y_{2_\rho})\Lambda_{2_\rho}^2\\ &\quad - 2G_\rho(y_{1_\rho}, y_{2_\rho})\Lambda_{1_\rho}\Lambda_{2_\rho}]\end{aligned}$$

and therefore using (3.6)

$$\limsup_{\rho\to 0} \psi_\rho(\mu_\rho, y_\rho) \le 0\,.$$

On the other hand by (A.2) we get

$$\rho^{\gamma\frac{N-2}{2-\gamma}}\psi_\rho(\mu_\rho, y_\rho) = \psi(\Lambda_\rho, x_\rho) \in [a,b]$$

and so a contradiction arises.

Third we prove that

$$\begin{cases} \text{There exist } \hat{y} = (0,1;y',\beta) \text{ with } (0,1) \ne (y',\beta),\ 0,\ y' \in \mathbb{R}^{N-1} \text{ and } 1,\ \beta \in \mathbb{R}\,,\\ \text{and } \hat{\mu} = (\hat{\mu}_1, \hat{\mu}_2) \in \mathbb{R}^2_+ \text{ such that } M_P(\hat{y})\hat{\mu} = 0\\ T \cdot \nabla_y \psi_P(\hat{\mu}, \hat{y}) = 0 \quad \forall\, T \in \mathbb{R}^{N-1} \times \{0\} \times \mathbb{R}^N\,. \end{cases} \tag{3.8}$$

By (3.7) we deduce that, up to a subsequence, $\hat{y}_2 = \lim_\rho y_{2_\rho}$, with $\mathrm{dist}(\hat{y}_2, \partial P) \ge 1$ and $|\hat{y}_1 - \hat{y}_2| \ge 1$, where $\hat{y}_1 = (0,1)$. Moreover from (3.6) it follows that

$\lim_{\rho\to 0}|\mu_\rho| = +\infty$, then up to a subsequence we can assume that $\hat\mu = \lim_{\rho\to 0}\frac{\mu_\rho}{|\mu_\rho|}$. It holds $|\hat\mu| = 1$. Now, since $\nabla_\mu\psi_\rho(\mu_\rho, y_\rho) = 0$, we have

$$M_\rho(y_\rho)\frac{\mu_\rho}{|\mu_\rho|} + \frac{\gamma}{2}\left(\frac{\mu_{1_\rho}^{\gamma-1}}{|\mu_\rho|} + \frac{\mu_{2_\rho}^{\gamma-1}}{|\mu_\rho|}\right) = 0$$

and passing to the limit we get $M_P(\hat y)\hat\mu = 0$. (If $\gamma < 1$ we used the fact that both $\mu_{1_\rho}$ and $\mu_{2_\rho}$ tend to $+\infty$.) Therefore $\hat\mu$ is an eigenvector associated with the first eigenvalue of the matrix $M_P(\hat y)$ and by [4] it follows that $\hat\mu_1 > 0$ and $\hat\mu_2 > 0$. Finally from (A.5) we get $\nabla_y\psi_P(\hat\mu, \hat y) = \lim_{\rho\to 0}\frac{1}{|\mu_\rho|^2}\nabla_\mu\psi_\rho(\mu_\rho, y_\rho)$ and the last statement follows from the assumption.

Finally we prove that by (3.8) we get a contradiction with (3.5). We write now the function $\psi_P$ explicitly:

$$\psi_P(\mu, y) = \frac{1}{2}\left(\frac{1}{(2y_1^N)^{N-2}}\,\mu_1^2 + \frac{1}{(2y_2^N)^{N-2}}\,\mu_2^2 - 2G(y_1, y_2)\mu_1\mu_2\right) + \frac{1}{2}\left(\mu_1^\gamma + \mu_2^\gamma\right),$$

where

$$G(y_1, y_2) = \frac{1}{|y_1 - y_2|^{N-2}} - \frac{1}{|y_1 - \bar y_2|^{N-2}}, \quad \bar y_2 = (y_2', -y_2^N).$$

We have $\hat y_1 = (0, 1)$ and $\hat y_2 = (\hat y_2', \beta)$. If $\hat y_2' \neq 0$ then

$$\begin{aligned}\hat y_2' \cdot \nabla_{y_{2'}}\psi_P(\hat\mu, \hat y) &= -\hat y_2' \cdot \nabla_{y_{2'}} G(\hat y_1, \hat y_2)\hat\mu_1\hat\mu_2 \\ &= -|y_2'|^2\left[\frac{1}{|(y_2', \beta - 1)|^N} - \frac{1}{|(y_2', \beta + 1)|^N}\right]\hat\mu_1\hat\mu_2 \neq 0\end{aligned}$$

and a contradiction arises.

If $\hat y_2' = 0$ then $\beta > 1$ and

$$0 = \nabla_{y_2^N}\psi_P(\hat\mu, \hat y) = (N-2)\hat\mu_2\left[\Gamma_{N-1}(\beta)\hat\mu_1 - \frac{1}{(2\beta)^{N-1}}\,\hat\mu_2\right],$$

where

$$\Gamma_{N-1}(\beta) = \frac{1}{(\beta - 1)^{N-1}} - \frac{1}{(\beta + 1)^{N-1}} > 0.$$

We deduce that

$$\hat\mu_2 = (2\beta)^{N-1}\Gamma_{N-1}(\beta)\hat\mu_1. \tag{3.9}$$

On the other hand by the condition $M_P(\hat y)\hat\mu = 0$, we get

$$\begin{cases}\dfrac{1}{2^{N-2}}\,\hat\mu_1 - \Gamma_{N-2}(\beta)\hat\mu_2 = 0, \\[2ex] -\Gamma_{N-2}(\beta)\hat\mu_1 + \dfrac{1}{(2\beta)^{N-2}}\hat\mu_2 = 0,\end{cases} \tag{3.10}$$

where

$$\Gamma_{N-2}(\beta) = \frac{1}{(\beta - 1)^{N-2}} - \frac{1}{(\beta + 1)^{N-2}} > 0.$$

By (3.9) and (3.10) we get

$$[2\beta\Gamma_{N-1}(\beta) - \Gamma_{N-2}(\beta)]\hat{\mu}_1 = 0$$

and a contradiction arises since $2\Gamma_{N-1}(\beta) - \Gamma_{N-2}(\beta) > 0$.

*Step 2.* We argue by contradiction. Let $(\Lambda_\rho, x_\rho) \in \mathbb{R}^2_+ \times \Omega^2$ be such that $\psi(\Lambda_\rho, x_\rho) \in [a,b]$, $\nabla_\Lambda \psi(\Lambda_\rho, x_\rho) = 0$, $\lambda_1(M(x_\rho)) < -l < 0$, $\operatorname{dist}(x_{1_\rho}, \partial\Omega) \geq \rho$, $\operatorname{dist}(x_{2_\rho}, \partial\Omega) \geq \rho$, $|x_{1_\rho} - x_{2_\rho}| = \rho$ and for any vector $T$ tangent to $\mathbb{R}^2_+ \times \partial V_\rho$ at the point $(\Lambda_\rho, x_\rho)$ it holds

$$\nabla\psi(\Lambda_\rho, x_\rho) \cdot T = 0\,. \tag{3.11}$$

We use the same notation of *Step 1*. First of all arguing as in *Step 1* we prove that $0 < c_1 \leq |\Lambda_\rho| \leq c_2$. Secondly we prove that

$$1 \leq \frac{\operatorname{dist}(x_{i_\rho}, \partial\Omega)}{\rho} \leq c \quad \text{for } i = 1 \text{ or } i = 2\,. \tag{3.12}$$

Assume by contradiction that for $i = 1, 2$ $\operatorname{dist}(x_{i_\rho}, \partial\Omega)/\rho \longrightarrow +\infty$. Then as $\rho \to 0$ we get

$$H_\rho(y_{i_\rho}, y_{i_\rho}) = \rho^{N-2} H(x_{i_\rho}, x_{i_\rho}) \leq \left(\frac{\rho}{\operatorname{dist}(x_{i_\rho}, \partial\Omega)}\right)^{N-2} \longrightarrow 0 \quad \text{for } i = 1, 2 \tag{3.13}$$

and

$$G_\rho(y_{1_\rho}, y_{2_\rho}) = \rho^{N-2} G(x_{1_\rho}, x_{2_\rho}) \longrightarrow 1 \tag{3.14}$$

(since $2H(x_{1_\rho}, x_{2_\rho}) \leq (x_{1_\rho}, x_{1_\rho}) + H(x_{2_\rho}, x_{2_\rho})$). Using (3.13) and (3.14) and arguing as in the proof of (3.6) we can show that $\Lambda_{i_\rho} \longrightarrow \Lambda_i > 0$ for $i = 1, 2$. Therefore

$$\psi(\Lambda_\rho, x_\rho) = \frac{\gamma - 2}{2\gamma}\,(M(x_\rho)\Lambda_\rho, \Lambda_\rho) \longrightarrow +\infty \quad \text{as } \rho \to 0\,,$$

and a contradiction arises, since $\psi(\Lambda_\rho, x_\rho) \in [a,b]$.

Next arguing as in *Step 1*, without loss of generality, we can assume that (up to a subsequence) $\Omega_\rho$ becomes the half-space $P$ and $\hat{y}_1 = \lim_\rho y_{1_\rho}$, $\hat{y}_1 = (0, \alpha)$ with $0 \in \mathbb{R}^{N-1}$ and $\alpha \geq 1$, $\hat{y}_2 = \lim_\rho y_{2_\rho}$ with $\operatorname{dist}(\hat{y}_2, \partial P) \geq 1$ and $|\hat{y}_1 - \hat{y}_2| = 1$.

Moreover we can show that there exists $\hat{\mu} = (\hat{\mu}_1, \hat{\mu}_2) \in \mathbb{R}^2_+$ such that $T \cdot \nabla_y \psi_P(\hat{y}, \hat{\mu}) = 0$ for any $T \in \mathbb{R}^{N-1} \times \{0\} \times \mathbb{R}^N$ and $M_P(\hat{y})\hat{\mu} = 0$ where $\hat{y} = (\hat{y}_1, \hat{y}_2)$. Finally, again arguing as in *Step 1*, we get a contradiction with (3.11). □

**Lemma 3.5.** *There exist* $l_0 > 0$ *and* $\rho_0 > 0$ *such that for any* $l \in (0, l_0)$ *and* $\rho \in (0, \rho_0)$ *the function* $\psi$ *satisfies the following property*:

*for any sequence* $(\Lambda_n, x_n)$ *in* $\mathbb{R}^2_+ \times W^l_\rho$ *such that* $\lim_n(\Lambda_n, x_n) = (\Lambda, x) \in \partial(\mathbb{R}^2_+ \times W^l_\rho)$ *and* $\psi(\Lambda_n, x_n) \in [a,b]$ *there exists a vector* $T$ *tangent to* $\mathbb{R}^2_+ \times \partial(W^l_\rho)$ *at* $(\Lambda, x)$, *such that*

$$\nabla\psi(\Lambda, x) \cdot T \neq 0\,.$$

**Proof.** First of all we prove that $\Lambda_n$ is component-wise bounded from below and from above by a positive constant. We have that $|\Lambda_n| \longrightarrow +\infty$ and $|\Lambda_n| \longrightarrow 0$ yield respectively to $|\psi(\Lambda_n, x_n)| \longrightarrow +\infty$ and $|\psi(\Lambda_n, x_n)| \longrightarrow 0$, which is impossible.

Let $\Lambda = \lim_n \Lambda_n$ and $x = \lim_n x_n$.

If $\nabla_\Lambda \psi(\Lambda, x) \neq 0$, then $T$ can be chosen parallel to $\nabla_\Lambda \psi(\Lambda, x)$. If $\nabla_\Lambda \psi(\Lambda, x) = 0$, then $\Lambda \in \mathbb{R}^2_+$. In fact if $\Lambda_2 = 0$ by

$$0 = \partial_{\Lambda_1} \psi(\Lambda, x) = H(x_1, x_1)\Lambda_1 + \frac{\gamma}{2}\Lambda_1^{\gamma-1},$$

we get a contradiction. Analogously $\Lambda_1 \neq 0$.

Thus $(\Lambda, x) \in \mathbb{R}^2_+ \times \partial W^l_\rho$.

Now we claim that there exists $l_0 > 0$ such that

$$\lambda_1(M(x)) < -l_0 . \tag{3.15}$$

In fact, since $\nabla_\Lambda \psi(\Lambda, x) = 0$, we have

$$\psi(\Lambda, x) = \frac{2-\gamma}{4}(\Lambda_1^\gamma + \Lambda_2^\gamma) = \frac{\gamma - 2}{2\gamma}(M(x)\Lambda, \Lambda),$$

and since $\psi(\Lambda, x) \in [a, b]$ we deduce that

$$|\Lambda|^2 \leq 2\left(\frac{4}{2-\gamma}\right)^{2/\gamma} b^{2/\gamma} \quad \text{and} \quad (M(x)\Lambda, \Lambda) \leq -\frac{2\gamma}{2-\gamma}\,a,$$

which implies (3.15) because $(M(x)\Lambda, \Lambda) \geq \lambda_1(M(x))|\Lambda|^2$.

Therefore we have that $x \in \partial V_\rho$ and we can apply Lemma 3.4 to conclude the proof. □

**Lemma 3.6.** *The function $\psi$ constrained to $\mathbb{R}^2_+ \times W^l_\rho$ satisfies the Palais–Smale condition in $[a, b]$.*

**Proof.** Let $(\Lambda_n, x_n)$ in $\mathbb{R}^2_+ \times W^l_\rho$ be such that $\lim_n \psi(\Lambda_n, x_n) = c > 0$ and $\lim_n \nabla\psi(\Lambda_n, x_n) = 0$. Arguing as in the proof of Lemma 3.4 it can be shown that $\Lambda_n$ remains bounded component-wise from above and below by a positive constant. □

**Proposition 3.1.** *There exists a critical level for $\psi$ between $a$ and $b$.*

**Proof.** Assume by contradiction that there are no critical levels in the interval $[a, b]$. We can define an appropriate negative gradient flow that will remain in $\mathbb{R}^2_+ \times W^l_\rho$ at any level $c \in [a, b]$. Moreover the Palais–Smale condition holds in $[a, b]$. Hence there exists a continuous deformation

$$\eta : [0, 1] \times \psi^b \longrightarrow \psi^b$$

such that for some $a' \in (0, a)$

$$\eta(0,u) = u \quad \forall\, u \in \psi^b$$
$$\eta(t,u) = u \quad \forall\, u \in \psi^{a'}$$
$$\eta(1,u) \in \psi^{a'}\,.$$

Let us call

$$\mathcal{A} = \{(\Lambda, x) \in \mathbb{R}^2_+ \times W^l_\rho \mid x \in \mathcal{S},\ \Lambda = re(x),\ 0 \le r \le R\}\,,$$
$$\partial\mathcal{A} = \{(\Lambda, x) \in \mathbb{R}^2_+ \times W^l_\rho \mid x \in \mathcal{S},\ \Lambda = 0 \text{ or } \Lambda = \mathrm{Re}(x)\}\,,$$
$$\mathcal{C} = I_\tau \times W^l_\rho\,.$$

From (3.3) we deduce that $\mathcal{A} \subset \psi^b$, $\partial\mathcal{A} \subset \psi^{a'}$ and $\psi^{a'} \cap \mathcal{C} = \emptyset$. Therefore

$$\begin{aligned} &\eta(0,u) = u \quad \forall\, u \in \mathcal{A}\,,\\ &\eta(t,u) = u \quad \forall\, u \in \partial\mathcal{A}\,,\\ &\eta(1,\mathcal{A}) \cap \mathcal{C} = \emptyset\,. \end{aligned} \tag{3.16}$$

For any $(\Lambda, x) \in \mathcal{A}$ and for any $t \in [0,1]$ we denote

$$\eta(t,(\Lambda,x)) = (\tilde{\Lambda}(\Lambda,x,t), \tilde{x}(\Lambda,x,t)) \in \mathbb{R}^2_+ \times W^l_\rho\,.$$

We define the set

$$\mathcal{B} = \{(\Lambda,x) \in \mathcal{A} \mid \tilde{\Lambda}(x,\Lambda,1) \in I_\tau\}\,.$$

Since $\eta(1,\mathcal{A}) \cap \mathcal{C} = \emptyset$ it holds $\mathcal{B} = \emptyset$. Now let $\mathcal{U}$ be a neighborhood of $\mathcal{B}$ in $W^l_\rho \times \mathbb{R}^2_+$ such that $H^*(\mathcal{U}) = H^*(\mathcal{B})$. If $\pi : \mathcal{U} \longrightarrow \mathcal{S}$ denotes the projection, arguing like in Lemma 7.1 of [10] we can show that

$$\pi^* : H^*(\mathcal{S}) \longrightarrow H^*(\mathcal{U}) \quad \text{is a monomorphism}\,.$$

This condition provides a contradiction, since $H^*(\mathcal{U}) = \{0\}$ and $H^*(\mathcal{S}) \neq \{0\}$. □

**Proof of Theorem 0.3.** Arguing as in [10] and using Lemma 2.2 and Proposition 3.1, it is possible to construct a critical point of the function $F_\varepsilon$ (see (2.2)) for $\varepsilon$ small enough. Therefore by Lemma 2.1 the claim follows. □

## Appendix A

Consider, for small $\rho$, the modified domain $\Omega_\rho = \Omega/\rho$. We can assume, without loss of generality, that as $\rho$ tends to 0 the domain $\Omega_\rho$ becomes the half-space $P = \{(y^1, \ldots, y^N) \in \mathbb{R}^N \mid y^N > 0\}$. We observe that if $G_\rho$ and $H_\rho$ are the Green's function and the regular part associated to the domain $\Omega_\rho$ then

$$G_\rho(y_1,y_2) = \rho^{N-2} G(\rho y_1, \rho y_2)\,, \quad H_\rho(y_1,y_2) = \rho^{N-2} H(\rho y_1, \rho y_2)\,.$$

Moreover, if $M_\rho$ denotes the matrix associated to the domain $\Omega_\rho$,

$$M_\rho(y) = \rho^{N-2} M(\rho y) \quad \text{and} \quad \lambda_1(M_\rho(y)) = \rho^{N-2}\lambda_1(M(\rho y))\,.$$

Let

$$\psi_\rho(\mu, y) = \frac{1}{2}\left[H_\rho(y_1,y_1)\mu_1^2 + H_\rho(y_2,y_2)\mu_2^2 - 2G_\rho(y_1,y_2)\mu_1\mu_2\right] + \frac{1}{2}[\mu_1^\gamma + \mu_2^\gamma]\,, \tag{A.1}$$

where $\gamma = \frac{4}{N-2}$. We remark that if $\mu = \rho^{-\frac{N-2}{2-\gamma}}\Lambda$ and $y = x/\rho$ then

$$\psi_\rho(\mu, y) = \rho^{-\gamma\frac{N-2}{2-\gamma}}\psi(\Lambda, x) \tag{A.2}$$

and

$$\nabla_\Lambda \psi(\Lambda, x) = 0 \quad \text{if and only if } \nabla_\mu \psi_\rho(\mu, y) = 0\,. \tag{A.3}$$

**Lemma A.1.** *It holds*

$$M_{\Omega_\rho} \longrightarrow M_P$$

$$C^1\text{-}\textit{uniformly on compact sets of } \{(y_1,y_2) \in P^2 \mid y_1 \neq y_2\}\,. \tag{A.4}$$

*Moreover*

$$\frac{1}{|\mu|^2}\nabla_y \psi_\rho(\mu, y) \longrightarrow \frac{1}{|\mu|^2}\nabla_y \psi_P(\mu, y)$$

$$C^1\text{-}\textit{uniformly on compact sets of } \{(y_1,y_2) \in P^2 \mid y_1 \neq y_2\} \times \mathbb{R}_+^2\,. \tag{A.5}$$

**Proof.** First of all we point out the following results

$$\lim_{\rho\to 0} H_\rho(y,y) = H_P(y,y)$$

$$C^1\text{-uniformly on compact sets of } P \tag{A.6}$$

and

$$\lim_{\rho\to 0} G_\rho(y_1,y_2) = G_P(y_1,y_2)$$

$$C^1\text{-uniformly on compact sets of } \{(y_1,y_2) \in P^2 \mid y_1 \neq y_2\}\,. \tag{A.7}$$

Let us prove (A.6). The proof of (A.7) is similar.

For any $y_1 \in P$ and $y_2 \in P$ we have, by a comparison argument, that $H_\rho(y_1,y_2)$ is increasing with respect to $\rho$ and $H_P(y_1,y_2) \le H_\rho(y_1,y_2) \le H_\Omega(y_1,y_2)$. Then $H_\rho(y_1,y_2)$ converges decreasingly as $\rho$ decreases to 0. By harmonicity the pointwise limit of $H_\rho(\cdot\,,\cdot)$ in $P^2$ is therefore uniform on compact sets of $P^2$ as $\rho$ goes to zero. Moreover for any $y \in P$ the resulting limit is an harmonic function with respect to $y$ in $P$ which coincides with $\frac{1}{|y_1-y_2|^{N-2}}$ on $\partial P$, namely the resulting limit is $H_P(y,\cdot)$.

Moreover if $K$ is a compact set of $P^2$ we have the following interior derivative estimate (see [13, Theorem (2.10)])

$$\max_{(y_1,y_2)\in K} |\nabla H_P(y_1,y_2) - \nabla H_P(y_1,y_2)|$$
$$\le \frac{N}{\operatorname{dist}(K,\partial(P^2))} \max_{(y_1,y_2)\in K} |H_P(y_1,y_2) - H_P(y_1,y_2)|\,,$$

which proves our claim.

Therefore (A.4) follows by (A.6) and (A.7).

Let us prove (A.5). Let $K$ be a compact set of $\{(y_1,y_2)\in P^2 \mid y_1\neq y_2\}$. It holds

$$\sup_{\substack{y\in K\\ \mu\neq 0}} \frac{1}{|\mu|^2} |\nabla_y\psi_\rho(\mu,y) - \nabla_y\psi_P(\mu,y)| = \sup_{\substack{y\in K\\ \mu\neq 0}} \frac{1}{2|\mu|^2} |([M'_\rho(y) - M'_P(y)]\mu,\mu)|$$
$$\le C\sup_{y\in K} \|M'_\rho(y) - M'_P(y)\|$$

and the claim follows by (A.4). □

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