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Some nonlinear elliptic equations in $\mathbb{R}^N$

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1. Introduction

Let us consider the following problem:

$$(\mathrm{P})\quad \begin{cases} -\Delta u + u + a(x)u = |u|^{p-2}u & \text{in } \mathbb{R}^N, \\ u>0 & \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases}$$

where $a(x)$ is a nonnegative function in $L^{N/2}(\mathbb{R}^N)$, $p>2$ and $p<2N/(N-2)$ if $N\geq 3$.

Problems of this kind arise in several contexts: for example, in the study of the standing waves solutions of nonlinear Schrödinger equations, or equations of Klein–Gordon type, or also in reaction–diffusion equations.

For example, if we look for standing waves solutions of the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi - |\psi|^{p-2}\psi, \tag{1.1}$$

i.e. solutions of the form

$$\psi(x,t) = \exp\left(-\frac{\mathrm{i}Et}{\hbar}\right) v(x),$$

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then one can easily verify that ψ satisfies Eq. (1.1) if and only if the function $v(x)$ solves the elliptic equation in $\mathbb{R}^N$

$$\frac{\hbar^2}{2m}\Delta v-(V(x)-E)v+|v|^{p-2}v=0. \tag{1.2}$$

In [16] Floer and Weinstein considered the case $N=1$ and $p=3$ with a potential V which is required to be a smooth function globally bounded in $\mathbb{R}^N$; for a given nondegenerate critical point of V they proved that, if $0<E<\inf_{\mathbb{R}^N}V$, then for $\hbar$ small enough there exists a solution of Eq. (1.2), which concentrates around the critical point as $\hbar\to 0$.

Their method (based on a Lyapunov–Schmidt reduction) was extended in several directions in order to obtain similar results for $N>1$ and $p\in\,]2,2N/(N-2)[$ or also to find solutions with multiple peaks, which concentrate around any prescribed finite set of nondegenerate critical points of V (see, for instance, [23–25]).

In [26] Rabinowitz used a global variational method to find solutions with minimal energy under the assumption $\liminf_{|x|\to\infty}V(x)>\inf_{\mathbb{R}^N}V$.

Other related results have been recently stated in [1, 2, 5, 14].

Notice that a simple change of variables shows that Eq. (1.2) is obviously equivalent to

$$-\Delta u+u+a(\lambda x)u=|u|^{p-2}u$$

with

$$\lambda=\frac{\hbar}{\sqrt{2m}\sqrt{\inf V-E}}\quad\text{and}\quad a(x)=\frac{V(x)-\inf V}{\inf V-E}\geq 0.$$

In this paper we are concerned with problem (P) in the case that $a(x)$ has the form

$$a(x)=\sum_{j=1}^{k}\lambda_j^2\alpha_j(\lambda_j(x-x_j)) \tag{1.3}$$

where $\lambda_j\in\mathbb{R}^+$, $x_j\in\mathbb{R}^N$ and $\alpha_j\in L^{N/2}(\mathbb{R}^N)$, with $\alpha_j\geq 0$ and $\alpha_j\not\equiv 0$.

Our aim is to give sufficient conditions on the numbers $\lambda_1,\dots,\lambda_k$ and the points $x_1,\dots,x_k$ in order to guarantee existence and multiplicity of solutions for (P).

We shall prove that, if $|x_i-x_j|$ is large enough for all $i\neq j$ $(i,j=1,\dots,k)$, then there exist at least $k-1$ solutions of (P); if, in addition, $\lambda_1,\dots,\lambda_k$ are small or large enough, then we have at least $2k-1$ solutions (see Theorems 2.1 and 4.1 and also Remark 4.2).

Let us denote by $H^{1,2}(\mathbb{R}^N)$ the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|=\left(\int_{\mathbb{R}^N}[|Du|^2+u^2]\,\mathrm{d}x\right)^{1/2}$$

and by $\|\cdot\|_{N/2}$ the usual norm in $L^{N/2}(\mathbb{R}^N)$.

The solutions of problem (P) correspond to the positive functions which are critical points for the functional $f_a : H^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$f_a(u) = \int_{\mathbb{R}^N} [|Du|^2 + (1 + a(x))u^2]\, dx, \tag{1.4}$$

constrained on the manifold

$$M = \left\{ u \in H^{1,2}(\mathbb{R}^N) \colon \int_{\mathbb{R}^N} |u|^p\, dx = 1 \right\}. \tag{1.5}$$

Since the embedding $H^{1,2}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is not compact, the Palais–Smale compactness condition for f_a constrained on M is not satisfied. Therefore, the classical variational methods cannot be applied in the usual way to find critical points. In particular, the infimum $\inf_M f_a$ is not achieved if $\|a\|_{N/2} \neq 0$ (see Proposition 2.2).

Similar difficulties also occur in the study of elliptic problems in other unbounded domains (exterior domains, for example) which have been investigated in several recent papers (see [3, 4, 6, 8–12, 15, 18, 19]).

Some methods elaborated in these papers apply in our problem too. In particular, using the concentration-compactness principle (see [21]), it is possible to analyse the obstructions to the compactness: in fact, it can be shown that every Palais–Smale sequence for f_a constrained on M either converges strongly to its weak limit or differs from it by one or more sequences, which, after suitable translations, converge to a solution of the limit problem

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \mathbb{R}^N, \\ u \in H^{1,2}(\mathbb{R}^N). \end{cases} \tag{1.6}$$

Taking also into account the uniqueness result of [20], this property allows us to find an energy range where the Palais–Smale condition is satisfied (see Proposition 2.3).

Notice that in this paper we only require $a(x) \in L^{N/2}(\mathbb{R}^N)$ and exploit the parameters $\lambda_1, \dots, \lambda_k$ in order to find critical values for f_a on M in the range where the Palais–Smale condition holds. However let us mention that, arguing as in [4], it is possible to find critical values, in the same energy range, under a suitable fast decay condition on $a(x)$ as $|x| \to \infty$ (see [22]); under this condition, in [22] it is proved the existence of $2k - 1$ solutions for problem (P) with $a(x)$ of the form

$$a(x) = \sum_{j=1}^{k} \alpha_j (x - x_j),$$

without introducing the parameters λ_j. Indeed there exist $k - 1$ "lower energy" solutions u_i ($i = 1, \dots, k - 1$), which (up to translations) converge to a positive solution of Eq. (1.6) as $|x_{i+1} - x_i| \to \infty$; moreover, there exist k "highest energy" solutions $\bar{u}_j$ ($j = 1, \dots, k$) which, as $\alpha_j \to +\infty$ a.e. in $\mathbb{R}^N$, tend to split as sum of two positive solutions of Eq. (1.6), sliding to infinity in opposite directions, while they converge (up to translations) to a positive solution of Eq. (1.6) as $\alpha_j \to 0$ in $L^{N/2}(\mathbb{R}^N)$.

Finally, let us remark that unlike [1, 2, 14, 16, 23–25], etc., our methods are not local in nature: we use global variational methods, inspired by [13, 4], which relate the number of critical points of f_a on M to the topological properties of its sublevels.

The paper is organized as follows: the main multiplicity results are stated in Theorems 2.1 and 4.1; in Section 2 we describe some preliminary properties of the functional f_a constrained on M; in Section 3 we obtain some asymptotic estimates which give informations on the topological properties of the sublevels of the functional f_a; in Section 4, we prove the main theorems and compare them with analogous multiplicity results one could obtain by Morse theory (see Remark 4.2).

2. Statement of the main theorem and preliminary results

We shall prove the following theorem.

Theorem 2.1. *Let* $p>2$ *and* $p<2N/(N-2)$ *if* $N\geq 3$. *Let* $\alpha_1,\dots,\alpha_k$ *be given nonnegative functions belonging to* $L^{N/2}(\mathbb{R}^N)$ *such that* $\|\alpha_j\|_{N/2}\neq 0$ *for all* $j=1,\dots,k$.

Then, there exist $\varepsilon_1>0$, $\varepsilon_2=\varepsilon_2(\lambda_1)>0$, $\varepsilon_3=\varepsilon_3(\lambda_1,\lambda_2)>0,\dots,\varepsilon_k=\varepsilon_k(\lambda_1,\dots,\lambda_{k-1})>0$ *and* $\varrho_1=\varrho_1(\lambda_1,\dots,\lambda_k,|x_1|)>0$, $\varrho_2=\varrho_2(\lambda_1,\dots,\lambda_k,|x_1|,|x_2|)>0,\dots,\varrho_{k-1}=\varrho_{k-1}(\lambda_1,\dots,\lambda_k,|x_1|,\dots,|x_{k-1}|)>0$ *such that*

$$\text{if}\quad \lambda_i<\varepsilon_i \quad\text{or}\quad \lambda_i>\frac{1}{\varepsilon_i}\quad \text{for any } i=1,\dots,k$$

and

$$|x_j|>\varrho_{j-1}\quad \text{for any } j=2,\dots,k,$$

then problem (P), *with* $a(x)$ *of form* (1.3), *has at least* $2k-1$ *distinct solutions.*

The proof is reported in Section 4.

Let us recall some known facts which will be useful in the sequel.

Let us consider the functional $f: M\to\mathbb{R}$ (see Eq. (1.5)) defined by

$$f(u)=\int_{\mathbb{R}^N}[|Du|^2+u^2]\,\mathrm{d}x \tag{2.1}$$

and the following minimization problem

$$\mu=\inf\{f(u):\ u\in M\}. \tag{2.2}$$

It has been shown (see, for instance, [28, 8, 20, 17]) that μ is achieved by a positive function U which is unique modulo translation and radially symmetric with respect to 0, decreasing when the radial coordinate increases and such that

$$\lim_{|x|\to+\infty} U(x)|x|^{(N-1)/2}e^{|x|}=\eta_1>0, \tag{2.3}$$

$$\lim_{|x|\to+\infty} |DU(x)||x|^{(N-1)/2}e^{|x|}=\eta_2>0. \tag{2.4}$$

Moreover, it is well known (see [21]) that any minimizing sequence for μ in $H^{1,2}(\mathbb{R}^N)$ has the form

$$w_n(x) + U(x - y_n), \tag{2.5}$$

where $(w_n)_{n\geq 1}$ is a sequence of functions converging to 0 in $H^{1,2}(\mathbb{R}^N)$, $(y_n)_{n\geq 1}$ is a sequence of points in $\mathbb{R}^N$ and U is the positive function, spherically symmetric with respect to 0, that realizes μ.

Proposition 2.2. *Let $a \in L^{N/2}(\mathbb{R}^N)$ be a nonnegative function such that $\|a\|_{N/2} \neq 0$. Then*

$$\inf_M f_a = \mu \tag{2.6}$$

and the infimum is not achieved (*see* (1.4), (1.5), (2.2)).

Proof. Put $m_a = \inf_M f_a$. Clearly $\mu \leq m_a$.

Let us consider the sequence $(u_n)_{n\geq 1}$ in $H^{1,2}(\mathbb{R}^N)$ defined by $u_n(x) = U(x - y_n)$, where $U \in M$ is the positive function, spherically symmetric with respect to 0, that realizes μ (see Eq. (2.2)) and $(y_n)_{n\geq 1}$ is a sequence of points in $\mathbb{R}^N$ such that $\lim_{n\to\infty} |y_n| = +\infty$.

In order to obtain $\mu = m_a$, it suffices to show that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} a(x) u_n^2(x)\,\mathrm{d}x = 0. \tag{2.7}$$

Indeed, for any $\varrho > 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} a(x) u_n^2(x)\,\mathrm{d}x &= \int_{\mathbb{R}^N\setminus B(0,\varrho)} a(x) u_n^2(x)\,\mathrm{d}x + \int_{B(0,\varrho)} a(x) u_n^2(x)\,\mathrm{d}x \\ &\leq \left(\int_{\mathbb{R}^N\setminus B(0,\varrho)} a^{N/2}(x)\,\mathrm{d}x\right)^{2/N} \left(\int_{\mathbb{R}^N} |U(x)|^{2N/(N-2)}\,\mathrm{d}x\right)^{(N-2)/N} \\ &\quad + \int_{B(0,\varrho)} a(x) U^2(x - y_n)\,\mathrm{d}x.\end{aligned}$$

Now, since $|y_n| \to \infty$, by the Lebesgue convergence theorem we obtain

$$\lim_{n\to\infty} \int_{B(0,\varrho)} a(x) U^2(x - y_n)\,\mathrm{d}x = 0 \quad \forall \varrho > 0.$$

Moreover, we have

$$\lim_{\varrho\to+\infty} \int_{\mathbb{R}^N\setminus B(0,\varrho)} a^{N/2}(x)\,\mathrm{d}x = 0$$

because $a \in L^{N/2}(\mathbb{R}^N)$. Hence Eq. (2.7) is proved.

Now, let us argue by contradiction and assume that the infimum m_a is achieved by a function u. Without any loss of generality, we can also assume $u \geq 0$. Thus, we have

$$\mu \leq \int_{\mathbb{R}^N} [|Du|^2 + u^2]\,\mathrm{d}x \leq \int_{\mathbb{R}^N} [|Du|^2 + (1 + a(x))u^2]\,\mathrm{d}x = \mu,$$

which implies that $u(x) = U(x - y)$, for a suitable $y \in \mathbb{R}^N$, with $U(x - y) > 0$ for all $x \in \mathbb{R}^N$. Hence, we deduce

$$0 = \int_{\mathbb{R}^N} a(x)u^2(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} a(x)U^2(x - y)\,\mathrm{d}x > 0,$$

which is impossible. □

Proposition 2.3. *Let $a \in L^{N/2}(\mathbb{R}^N)$ be a nonnegative function. Let $(u_n)_{n\geq 1}$ be a sequence in M that satisfies*

$$\lim_{n\to\infty} f_a(u_n) = c \in]\mu, 2^{1-2/p}\mu[,$$
$$f'_{a|_M}(u_n) \to 0 \quad \textit{in } H^{-1,2}(\mathbb{R}^N)$$

(i.e. $(u_n)_{n\geq 1}$ is a Palais–Smale sequence for the functional f_a constrained on M at level c). Then $(u_n)_{n\geq 1}$ is relatively compact.

For the proof it suffices to argue as in [6].

Proposition 2.4. *Let $a \in L^{N/2}(\mathbb{R}^N)$ be a nonnegative function. If a function u is a critical point for f_a constrained on M, such that $f_a(u) < 2^{1-2/p}\mu$, then u has a constant sign.*

Proof. Let us suppose, by contradiction, that $u = u^+ - u^-$ $(u^+(x) = \max(u(x), 0)$, $u^-(x) = -\min(u(x), 0))$, with $u^+ \not\equiv 0$ and $u^- \not\equiv 0$.

Taking into account Eq. (2.6), we have

$$\|u^\pm\|_p^2\mu \leq \int_{\mathbb{R}^N} [|Du^\pm|^2 + (1 + a(x))|u^\pm|^2]\,\mathrm{d}x. \tag{2.8}$$

Moreover,

$$\int_{\mathbb{R}^N} [|Du^\pm|^2 + (1 + a(x))|u^\pm|^2]\,\mathrm{d}x = f_a(u)\|u^\pm\|_p^p \tag{2.9}$$

because u is a critical point for f_a on M.

Comparing Eqs. (2.8) and (2.9), we get $\|u^\pm\|_p^{p-2} \geq \mu/f_a(u)$ and so $f_a(u) \geq 2^{1-(2/p)}\mu$, contradicting our assumption. □

3. Some asymptotic estimates

In this section we will provide some estimates in order to get information on the topological properties of the sublevels of f_a constrained on M.

Lemma 3.1. *Let* α *be a nonnegative function in* $L^{N/2}(\mathbb{R}^N)$ *with* $\|\alpha\|_{N/2} \neq 0$. *Then we have*

$$\begin{aligned}
&\text{(a)} \quad \lim_{\lambda\to 0} \sup\left\{\lambda^2 \int_{\mathbb{R}^N} \alpha(\lambda x) U^2(x-y)\,\mathrm{d}x\colon\ y\in\mathbb{R}^N\right\} = 0,\\
&\text{(b)} \quad \lim_{\lambda\to +\infty} \sup\left\{\lambda^2 \int_{\mathbb{R}^N} \alpha(\lambda x) U^2(x-y)\,\mathrm{d}x\colon\ y\in\mathbb{R}^N\right\} = 0, \qquad (3.1)\\
&\text{(c)} \quad \lim_{R\to +\infty} \sup\left\{\lambda^2 \int_{\mathbb{R}^N} \alpha(\lambda x) U^2(x-y)\,\mathrm{d}x\colon\ \lambda>0,\ |y|=R\right\} = 0,
\end{aligned}$$

where U *is the positive function, spherically symmetric with respect to* 0, *that realizes* μ (*see Eq.* (2.2)).

Proof. In order to prove Eq. (3.1)(a) we argue by contradiction: suppose there exist a sequence $(y_n)_{n\geq 1}$ of points in $\mathbb{R}^N$ and a sequence $(\lambda_n)_{n\geq 1}$ of positive numbers such that $\lim_{n\to\infty} \lambda_n = 0$ and

$$\lim_{n\to\infty} \lambda_n^2 \int_{\mathbb{R}^N} \alpha(\lambda_n x) U^2(x-y_n)\,\mathrm{d}x > 0. \qquad (3.2)$$

For all $n \geq 1$, Hölder inequality implies that

$$\begin{aligned}
&\lambda_n^2 \int_{\mathbb{R}^N} \alpha(\lambda_n x) U^2(x-y_n)\,\mathrm{d}x\\
&\quad = \lambda_n^2 \int_{B(y_n,1/\sqrt{\lambda_n})} \alpha(\lambda_n x) U^2(x-y_n)\,\mathrm{d}x\\
&\qquad + \lambda_n^2 \int_{\mathbb{R}^N\setminus B(y_n,1/\sqrt{\lambda_n})} \alpha(\lambda_n x) U^2(x-y_n)\,\mathrm{d}x\\
&\quad \leq \lambda_n^2 \left(\int_{B(y_n,1/\sqrt{\lambda_n})} \alpha^{N/2}(\lambda_n x)\,\mathrm{d}x\right)^{2/N}\\
&\qquad \times \left(\int_{B(y_n,1/\sqrt{\lambda_n})} |U(x-y_n)|^{2N/(N-2)}\,\mathrm{d}x\right)^{(N-2)/N}\\
&\qquad + \lambda_n^2 \left(\int_{\mathbb{R}^N\setminus B(y_n,1/\sqrt{\lambda_n})} \alpha^{N/2}(\lambda_n x)\,\mathrm{d}x\right)^{2/N}\\
&\qquad \times \left(\int_{\mathbb{R}^N\setminus B(y_n,1/\sqrt{\lambda_n})} |U(x-y_n)|^{2N/(N-2)}\,\mathrm{d}x\right)^{(N-2)/N}
\end{aligned}$$

$$\leq \left(\int_{B(\lambda_n y_n, \sqrt{\lambda_n})} \alpha^{N/2}(x)\,\mathrm{d}x\right)^{2/N} \left(\int_{\mathbb{R}^N} |U(x)|^{2N/(N-2)}\,\mathrm{d}x\right)^{(N-2)/N}$$
$$+ \left(\int_{\mathbb{R}^N} \alpha^{N/2}(x)\,\mathrm{d}x\right)^{2/N} \left(\int_{\mathbb{R}^N\setminus B(0,1/\sqrt{\lambda_n})} |U(x)|^{2N/(N-2)}\,\mathrm{d}x\right)^{(N-2)/N}.$$

Since $\alpha \in L^{N/2}(\mathbb{R}^N)$ and $\lambda_n \to 0$ as $n\to\infty$, we have

$$\lim_{n\to\infty} \int_{B(\lambda_n y_n, \sqrt{\lambda_n})} \alpha^{N/2}(x)\,\mathrm{d}x = 0;$$

moreover, since $U \in H^{1,2}(\mathbb{R}^N)$, we have that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N\setminus B(0,1/\sqrt{\lambda_n})} |U(x)|^{2N/(N-2)}\,\mathrm{d}x = 0.$$

The previous computations imply that

$$\lim_{n\to\infty} \lambda_n^2 \int_{\mathbb{R}^N} \alpha(\lambda_n x)U^2(x-y_n)\,\mathrm{d}x = 0,$$

which is a contradiction with Eq. (3.2).

Let us now prove 3.1(b). By contradiction, let us suppose that there exist a sequence $(y_n)_{n\geq 1}$ of points in $\mathbb{R}^N$ and a sequence $(\lambda_n)_{n\geq 1}$ of positive numbers such that $\lim_{n\to\infty} \lambda_n = +\infty$ and Eq. (3.2) holds.

For all $n \geq 1$, by using Hölder inequality, we get

$$\lambda_n^2 \int_{\mathbb{R}^N} \alpha(\lambda_n x)U^2(x-y_n)\,\mathrm{d}x$$
$$= \lambda_n^2 \int_{B(0,1/\sqrt{\lambda_n})} \alpha(\lambda_n x)U^2(x-y_n)\,\mathrm{d}x$$
$$+ \lambda_n^2 \int_{\mathbb{R}^N\setminus B(0,1/\sqrt{\lambda_n})} \alpha(\lambda_n x)U^2(x-y_n)\,\mathrm{d}x$$
$$\leq \lambda_n^2 \left(\int_{B(0,1/\sqrt{\lambda_n})} \alpha^{N/2}(\lambda_n x)\,\mathrm{d}x\right)^{2/N}$$
$$\times \left(\int_{B(0,1/\sqrt{\lambda_n})} |U(x-y_n)|^{2N/(N-2)}\,\mathrm{d}x\right)^{(N-2)/N}$$
$$+\lambda_n^2 \left(\int_{\mathbb{R}^N\setminus B(0,1/\sqrt{\lambda_n})} \alpha^{N/2}(\lambda_n x)\,\mathrm{d}x\right)^{2/N}$$
$$\times \left(\int_{\mathbb{R}^N\setminus B(0,1/\sqrt{\lambda_n})} |U(x-y_n)|^{2N/(N-2)}\,\mathrm{d}x\right)^{(N-2)/N}$$

$$\leq \left(\int_{B(0,\sqrt{\lambda_n})} \alpha^{N/2}(x)\,\mathrm{d}x\right)^{2/N} \left(\int_{B(0,1/\sqrt{\lambda_n})} |U(x-y_n)|^{2N/(N-2)}\,\mathrm{d}x\right)^{(N-2)/N}$$
$$+\left(\int_{\mathbb{R}^N\setminus B(0,\sqrt{\lambda_n})} \alpha^{N/2}(x)\,\mathrm{d}x\right)^{2/N} \left(\int_{\mathbb{R}^N} |U(x)|^{2N/(N-2)}\,\mathrm{d}x\right)^{(N-2)/N}.$$

Since $\alpha\in L^{N/2}(\mathbb{R}^N)$, we have

$$\lim_{n\to\infty}\int_{B(0,\sqrt{\lambda_n})} \alpha^{N/2}(x)\,\mathrm{d}x=\int_{\mathbb{R}^N}\alpha^{N/2}(x)\,\mathrm{d}x<+\infty$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N\setminus B(0,\sqrt{\lambda_n})} \alpha^{N/2}(x)\,\mathrm{d}x=0.$$

Moreover, since $U\in H^{1,2}(\mathbb{R}^N)$,

$$\lim_{n\to\infty}\int_{B(0,1/\sqrt{\lambda_n})} |U(x-y_n)|^{2N/(N-2)}\,\mathrm{d}x=\lim_{n\to\infty}\int_{B(y_n,1/\sqrt{\lambda_n})} |U(x)|^{2N/(N-2)}\,\mathrm{d}x=0.$$

Hence, we get

$$\lim_{n\to\infty}\lambda_n^2\int_{\mathbb{R}^N}\alpha(\lambda_n x)U^2(x-y_n)\,\mathrm{d}x=0,$$

that is a contradiction with Eq. (3.2).

Finally, let us argue again by contradiction to prove 3.1(c): suppose that there exist a sequence $(\lambda_n)_{n\geq 1}$ of positive numbers and a sequence $(y_n)_{n\geq 1}$ of points in $\mathbb{R}^N$ such that $\lim_{n\to\infty}|y_n|=+\infty$ and Eq. (3.2) holds.

From 3.1 (a) and (b) we get

$$0<\liminf_{n\to\infty}\lambda_n\leq\limsup_{n\to\infty}\lambda_n<+\infty;$$

hence, it is not restrictive to assume that $\lim_{n\to\infty}\lambda_n=\bar{\lambda}\in\,]0,+\infty[$.

By Hölder inequality we can write, for any n and $\varrho>0$,

$$\lambda_n^2\int_{\mathbb{R}^N}\alpha(\lambda_n x)U^2(x-y_n)\,\mathrm{d}x$$
$$=\lambda_n^2\int_{B(0,\varrho)}\alpha(\lambda_n x)U^2(x-y_n)\,\mathrm{d}x$$
$$+\lambda_n^2\int_{\mathbb{R}^N\setminus B(0,\varrho)}\alpha(\lambda_n x)U^2(x-y_n)\,\mathrm{d}x$$
$$\leq\lambda_n^2\int_{B(0,\varrho)}\alpha(\lambda_n x)U^2(x-y_n)\,\mathrm{d}x$$
$$+\left(\int_{\mathbb{R}^N\setminus B(0,\varrho\lambda_n)}\alpha^{N/2}(x)\,\mathrm{d}x\right)^{2/N}\left(\int_{\mathbb{R}^N}|U(x)|^{2N/(N-2)}\,\mathrm{d}x\right)^{(N-2)/N}.$$

Since $\lim_{n\to\infty}\lambda_n=\bar\lambda\in\,]0,+\infty[$ and $\lim_{n\to\infty}|y_n|=+\infty$, we have

$$\lim_{n\to\infty}\int_{B(0,\varrho)} a(\lambda_n x)U^2(x-y_n)\,\mathrm{d}x=0\quad\forall\varrho>0.$$

It follows that

$$\limsup_{n\to\infty}\lambda_n^2\int_{\mathbb{R}^N}\alpha(\lambda_n x)U^2(x-y_n)\,\mathrm{d}x\le C\left(\int_{\mathbb{R}^N\setminus B(0,\varrho\bar\lambda)}\alpha^{N/2}(x)\,\mathrm{d}x\right)^{2/N}\qquad\forall\varrho>0,$$

with $C>0$.

Now let us remark that

$$\lim_{\varrho\to+\infty}\int_{\mathbb{R}^N\setminus B(0,\varrho\bar\lambda)}\alpha^{N/2}(x)\,\mathrm{d}x=0$$

because $\bar\lambda>0$ and $\alpha\in L^{N/2}(\mathbb{R}^N)$. Hence we infer that $[\lim_{n\to\infty}\lambda_n^2\int_{\mathbb{R}^N}\alpha(\lambda_n x)U^2(x-y_n)\,\mathrm{d}x=0]$, contradicting Eq. (3.2). □

Let us define $\gamma:\mathbb{R}^N\to\mathbb{R}^N$, $\beta:M\to\mathbb{R}^N$ and, for any $j=1,\dots,k$, $\beta_j:M\to\mathbb{R}^N$ in the following way:

$$\gamma(x)=\frac{x}{1+|x|},\tag{3.3}$$

$$\beta(u)=\int_{\mathbb{R}^N}\frac{x}{1+|x|}|u(x)|^p\,\mathrm{d}x\tag{3.4}$$

and

$$\beta_j(u)=\int_{\mathbb{R}^N}\frac{x-x_j}{1+|x-x_j|}|u(x)|^p\,\mathrm{d}x\tag{3.5}$$

where x_j is the point of $\mathbb{R}^N$ which appears in Eq. (1.3).

Moreover, for any $j=1,\dots,k$, we define $f_j:H^{1,2}(\mathbb{R}^N)\to\mathbb{R}$ and $\tilde f_j:H^{1,2}(\mathbb{R}^N)\to\mathbb{R}$ to be, respectively,

$$f_j(u)=\int_{\mathbb{R}^N}[|Du|^2+u^2+\lambda_j^2\alpha_j(\lambda_j(x-x_j))u^2]\,\mathrm{d}x\tag{3.6}$$

and

$$\tilde f_j(u)=\int_{\mathbb{R}^N}[|Du|^2+u^2+\lambda_j^2\alpha_j(\lambda_j x)u^2]\,\mathrm{d}x\tag{3.7}$$

where α_j, λ_j and x_j are defined as in Eq. (1.3).

Proposition 3.2. *Assume that* $\|\alpha_j\|_{N/2} \neq 0$ *for any* $j = 1, \ldots, k$.
Then

$$\inf\{\tilde{f}_j(u)\colon\ u \in M,\ \ \beta(u) = 0\} > \mu \tag{3.8}$$

(*see Eqs.* (3.7), (1.3), (3.4)).

Proof. Let us notice that Proposition 2.2 implies that

$$\inf\{\tilde{f}_j(u)\colon\ u \in M,\ \ \beta(u) = 0\} \geq \mu. \tag{3.9}$$

By contradiction, let us suppose that equality holds in Eq. (3.9).

Hence there exists a sequence $(u_n)_{n\geq 1}$ of functions belonging to M such that $\beta(u_n) = 0\ \forall n \geq 1$ and

$$\lim_{n\to\infty} \tilde{f}_j(u_n) = \mu.$$

Then (see [19] and Eq. (2.5)) there exist a sequence $(y_n)_{n\geq 1}$ of points in $\mathbb{R}^N$ and a sequence $(w_n)_{n\geq 1}$ in $H^{1,2}(\mathbb{R}^N)$, with $w_n \to 0$ in $H^{1,2}(\mathbb{R}^N)$, such that

$$u_n(x) = U(x - y_n) + w_n(x).$$

First, let us remark that $(y_n)_{n\geq 1}$ must be a bounded sequence in $\mathbb{R}^N$: otherwise, we should have (up to a subsequence) $|y_n| \to +\infty$, which implies

$$\lim_{n\to\infty} \left| \beta(u_n) - \frac{y_n}{1 + |y_n|} \right| = 0$$

and, as a consequence, $\lim_{n\to\infty} |\beta(u_n)| = 1$. But this is impossible since $\beta(u_n) = 0$ $\forall n \geq 1$.

Now let us prove that $y_n \to 0$: in fact assume, by contradiction, that (up to a subsequence) $y_n \to \overline{y} \neq 0$; then, $\beta(u_n) \to \beta(U(\cdot - \overline{y}))$, with $\beta(U(\cdot - \overline{y})) \neq 0$ if $\overline{y} \neq 0$ (as follows easily taking into account Eq. (3.4) and the radial symmetry of U with respect to zero); but $\beta(u_n) = 0 \quad \forall n \geq 1$, so we must have $y_n \to 0$. It follows that

$$\lim_{n\to\infty} \lambda_j^2 \int_{\mathbb{R}^N} \alpha_j(\lambda_j x)[U(x - y_n) + w_n(x)]^2\,\mathrm{d}x = \lambda_j^2 \int_{\mathbb{R}^N} \alpha_j(\lambda_j x) U^2(x)\,\mathrm{d}x > 0,$$

where the inequality holds because $U(x) > 0$ for all $x \in \mathbb{R}^N$, $\alpha_j \geq 0$ in $\mathbb{R}^N$ and $\alpha_j \not\equiv 0$.

Therefore, we have

$$\begin{aligned}\mu = \lim_{n\to\infty} \tilde{f}_j(u_n) &= \lim_{n\to\infty} \int_{\mathbb{R}^N} [|D[U(x - y_n) + w_n(x)]|^2 + |U(x - y_n) + w_n(x)|^2]\,\mathrm{d}x \\ &\quad + \lambda_j^2 \int_{\mathbb{R}_N} \alpha_j(\lambda_j x)|U(x - y_n) + w_n(x)|^2\,\mathrm{d}x \\ &= \mu + \lambda_j^2 \int_{\mathbb{R}^N} \alpha_j(\lambda_j x) U^2(x)\,\mathrm{d}x > \mu,\end{aligned}$$

which is impossible. □

Remark 3.3. *Taking into account the radial symmetry of U with respect to zero and the definition of* β *(see Eq.* (3.4)), *it is easy to verify that* $\beta(U)=0$ *and*

$$\beta(U(\cdot - y)) = \tau(|y|)y \quad \forall y \in \mathbb{R}^N \tag{3.10}$$

where $\tau : \mathbb{R}^+ \to \mathbb{R}$ *is a continuous function satisfying* $\tau(\varrho)>0$ *for all* $\varrho>0$.

Hence, for any $R>0$, *the map* $y \to \beta(U(\cdot - y))$ *is homotopically equivalent in* $\mathbb{R}^N\backslash\{0\}$ *to the identity map of* $\partial B(0,R)$.

For any $h=2,\dots,k$, let $f_{1,\dots,h} : H^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ be the functional defined by

$$f_{1,\dots,h}(u) = \int_{\mathbb{R}^N} \left[|Du|^2 + |u|^2 + \sum_{j=1}^{h} \lambda_j^2 \alpha_j(\lambda_j(x-x_j))u^2 \right] \mathrm{d}x. \tag{3.11}$$

Now, for all $z \in \mathbb{R}^N$, let us define $\Sigma_z : \mathbb{R}^N \to \mathbb{R}^N$ by

$$\Sigma_z(x) = x - \varphi_z(x \cdot z)z, \tag{3.12}$$

where $\varphi_z : \mathbb{R} \to \mathbb{R}$ is the following function:

$$\varphi_z(t) = \begin{cases} 1 & \text{if } t \geq |z|^2, \\ \frac{t}{|z|^2} & \text{if } |t| \leq |z|^2, \\ -1 & \text{if } t \leq -|z|^2; \end{cases} \tag{3.13}$$

moreover, for any $j=2,\dots,k$, let $\beta_{j-1,j} : M \to \mathbb{R}^N$ be the map defined by

$$\beta_{j-1,j}(u) = \int_{\mathbb{R}^N} (\gamma \circ \Sigma_{(x_j - x_{j-1})/2}) \left(x - \frac{x_{j-1}+x_j}{2} \right) |u(x)|^p \, \mathrm{d}x \tag{3.14}$$

(see Eq. (3.3)).

Finally, for $j=2,\dots,k$, if $x_j \neq x_{j-1}$, let us put

$$S_{j-1,j} = \left\{ \frac{x_j - x_{j-1}}{2|x_j - x_{j-1}|}(2t-1) \colon \; t \in [0,1] \right\} \tag{3.15}$$

and

$$T_{j-1,j} = \left\{ x \in \mathbb{R}^N \colon \; \left(x - \frac{x_{j-1}+x_j}{2} \right) \cdot (x_j - x_{j-1}) = 0 \right\}. \tag{3.16}$$

The following proposition can be proved arguing as in [19].

Proposition 3.4. *Assume* $\|\alpha_j\|_{N/2} \neq 0$ *and* $\lambda_j > 0$ *for* $j=1,\dots,k$. *Then we have* (*see Eqs.* (3.11)–(3.16)):

(a) *for any* $h=2,\dots,k$ *and* $j=2,\dots,h$,

$$\inf \left\{ f_{1,\dots,h}(u) \colon \; u \in M, \; \beta_{j-1,j}(u) = \pm \frac{x_j - x_{j-1}}{2|x_j - x_{j-1}|}, x_j, x_{j-1} \in \mathbb{R}^N, \; x_j \neq x_{j-1} \right\} > \mu; \tag{3.17}$$

(b) *for any* $h=2,\dots,k$ *and* $j=2,\dots,h$, *if* $x_j \neq x_{j-1}$,

$$\inf\{f_{1,\dots,h}(u):\ u\in M,\ \beta_{j-1,j}(u)\in S_{j-1,j}\}>\mu; \tag{3.18}$$

(c) *for any* $j=2,\dots,k$,

$$\lim_{R\to+\infty}\sup_{z\in\partial B(0,1)}\left|\beta_{j-1,j}\left(U\left(\cdot-\frac{x_{j-1}+x_j}{2}-Rz\right)\right)-z\right|=0. \tag{3.19}$$

Remark 3.5. *Taking into account the radial symmetry of U with respect to zero and Eqs.* (3.12)–(3.16), *it is easy to verify that, for* $j=2,\dots,k$,

$$\beta_{j-1,j}\left(U\left(\cdot-\frac{x_{j-1}+x_j}{2}\right)\right)=0$$

and

$$\beta_{j-1,j}(U(\cdot-z))=\overline{\tau}\left(\left|z-\frac{x_{j-1}+x_j}{2}\right|\right)\left(z-\frac{x_{j-1}+x_j}{2}\right)\quad\forall z\in T_{j-1,j},$$

where $\overline{\tau}(\varrho)>0\ \forall\varrho>0$.

4. Proof of the main result

Proof of Theorem 2.1. The idea is to choose the parameters $\lambda_1,\dots,\lambda_k$, $x_1,\dots,x_k$ in order to obtain suitable inequalities which describe the topological properties of the sublevels of f_a constrained on M and give rise to $2k-1$ distinct critical values.

More precisely, we choose consecutively $\lambda_1,\dots,\lambda_k$, $|x_2-x_1|,|x_3-x_2|,\dots,|x_k-x_{k-1}|$ in such a way that every choice does not modify the inequalities previously stated and produces new inequalities.

The proof consists of four steps.

Step 1: Choice of the parameters $\lambda_1,\dots,\lambda_k$.

Lemma 3.1 implies that there exists $\varepsilon_1>0$ such that, if $\lambda_1<\varepsilon_1$ or $\lambda_1>1/\varepsilon_1$, then

$$\sup\{\tilde f_1(U(\cdot-y)):\ y\in B(0,R)\}<2^{1-2/p}\mu\quad\forall R>0. \tag{4.1}$$

Taking into account Proposition 3.2, it follows that we can choose a positive number R_1 sufficiently large such that

$$\begin{aligned}\sup\{\tilde f_1(U(\cdot-y)):\ y\in\partial B(0,R_1)\}&<\inf\{\tilde f_1(u):\ u\in M,\beta(u)=0\}\\ &\le\sup\{\tilde f_1(U(\cdot-y)):\ y\in B(0,R_1)\}<2^{1-2/p}\mu.\end{aligned} \tag{4.2}$$

Now, using again Lemma 3.1, we can fix $\varepsilon_2=\varepsilon_2(\lambda_1)>0$ such that, if $\lambda_2<\varepsilon_2$ or $\lambda_2>1/\varepsilon_2$, then

$$\sup\{\tilde{f}_2(U(\cdot-y)):\ y\in B(0,R)\}<\inf\{\tilde{f}_1(u):\ u\in M,\beta(u)=0\}\quad \forall R>0. \tag{4.3}$$

From Proposition 3.2, we infer that there exists $R_2>0$ sufficiently large such that

$$\begin{aligned}\sup\{\tilde{f}_2(U(\cdot-y)):\ y\in\partial B(0,R_2)\} &< \inf\{\tilde{f}_2(u):\ u\in M,\beta(u)=0\}\\ &\le \sup\{\tilde{f}_2(U(\cdot-y)):\ y\in B(0,R_2)\}\\ &< \inf\{\tilde{f}_1(u):\ u\in M,\beta(u)=0\}.\end{aligned} \tag{4.4}$$

Iterating this procedure, we can choose $\lambda_3,\dots,\lambda_k$; in particular, we have that there exist positive numbers ε_1, $\varepsilon_2=\varepsilon_2(\lambda_1)>0,\dots,\varepsilon_k=\varepsilon_k(\lambda_1,\dots,\lambda_{k-1})>0$ and $R_1,\dots,R_k$ such that, if $\lambda_j<\varepsilon_j$ or $\lambda_j>1/\varepsilon_j$, then

$$\begin{aligned}\mu &< \sup\{\tilde{f}_j(U(\cdot-y)):\ y\in\partial B(0,R_j)\}<\inf\{\tilde{f}_j(u):\ u\in M,\beta(u)=0\}\\ &\le \sup\{\tilde{f}_j(U(\cdot-y)):\ y\in B(0,R_j)\}<2^{1-2/p}\mu\quad \text{for any } j=1,\dots,k\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}&\sup\{\tilde{f}_j(U(\cdot-y)):\ y\in B(0,R_j)\}\\ &\quad<\inf\{\tilde{f}_{j-1}(u):\ u\in M,\beta(u)=0\}\quad \text{for any } j=2,\dots,k.\end{aligned} \tag{4.6}$$

We shall consider $\lambda_1,\dots,\lambda_k$ fixed as before.

Step 2. Behaviour of the functional with respect to the parameters $x_1,\dots,x_k$.

Let us first observe that

$$\lim_{|x_2-x_1|\to\infty}\sup\left\{\lambda_i^2\int_{\mathbb{R}^N}\alpha_i(\lambda_i(x-x_i))U^2(x-y)\,\mathrm{d}x:\ y\in T_{1,2}\right\}=0\quad \text{for } i=1,2, \tag{4.7}$$

since

$$\lim_{|x_2-x_1|\to\infty} dist(x_1,T_{1,2})=\lim_{|x_2-x_1|\to\infty} dist(x_2,T_{1,2})=+\infty.$$

Therefore, Eqs. (4.7) and (3.8) imply that, if $|x_2-x_1|$ is sufficiently large, then

$$\mu<\sup\{f_{1,2}(U(\cdot-y)):\ y\in T_{1,2}\}<\inf\{\tilde{f}_k(u):\ u\in M,\beta(u)=0\}. \tag{4.8}$$

Moreover, from Eqs. (4.7), (3.17), (3.18) and Remark 3.5 it follows that, if $|x_2-x_1|$ is large enough, then

$$\begin{aligned}\mu &< \inf\{f_{1,2}(u):\ u\in M,\beta_{1,2}(u)\in S_{1,2}\}\le\sup\{f_{1,2}(U(\cdot-y)):\ y\in T_{1,2}\}\\ &< \inf\left\{f_{1,2}(u):\ u\in M,\beta_{1,2}(u)=\pm\frac{x_2-x_1}{2|x_2-x_1|}\right\}.\end{aligned} \tag{4.9}$$

Arguing as in the proof of Lemma 3.1(c) one can show that, if r_1 is sufficiently large, then

$$\mu < \sup\left\{ f_{1,2}(U(\cdot - y)):\ y\in\partial B\left(\frac{x_1+x_2}{2}, r_1\right)\right\}$$
$$< \inf\{ f_{1,2}(u):\ u\in M, \beta_{1,2}(u)\in S_{1,2}\}. \tag{4.10}$$

Finally, Eq. (3.19) and Remark 3.5 imply that, if r_1 is large enough,

$$\text{the map } y\to\beta_{1,2}\left(U\left(\cdot - \frac{x_1+x_2}{2} - r_1 y\right)\right) \text{ is homotopically equivalent in}$$
$$\mathbb{R}^N \Big\backslash \left\{\pm\frac{x_2-x_1}{2|x_2-x_1|}\right\} \text{ to the identity map on } \partial B(0,1)\cup\left(T_{1,2} - \frac{x_1+x_2}{2}\right). \tag{4.11}$$

Analogous properties hold when $|x_j - x_{j-1}|$ is large enough for $j=2,\dots,k$.

Step 3. Choice of the points $x_1,\dots,x_k$ in $\mathbb{R}^N$.

Let x_1 be a fixed point in $\mathbb{R}^N$. Arguing as in the proof of Lemma 3.1(c), one can show that for $i=1,2$,

$$\lim_{|x_2|\to\infty} \sup\{ f_{1,2}(U(\cdot - y)):\ y\in\partial B(x_i,R_i)\}$$
$$= \sup\{ \tilde{f}_i(U(\cdot - y)):\ y\in\partial B(0,R_i)\} \tag{4.12}$$

and

$$\lim_{|x_2|\to\infty} \sup\{ f_{1,2}(U(\cdot - y)):\ y\in B(x_i,R_i)\}$$
$$= \sup\{ \tilde{f}_i(U(\cdot - y)):\ y\in B(0,R_i)\}, \tag{4.13}$$

where R_1 and R_2 are the positive numbers fixed in Step 1.

Moreover, it is clear that for $i=1,2$,

$$\inf\{ \tilde{f}_i(u):\ u\in M, \beta(u)=0\} = \inf\{ f_i(u):\ u\in M, \beta_i(u)=0\}$$
$$\le \inf\{ f_{1,2}(u):\ u\in M, \beta_i(u)=0\}. \tag{4.14}$$

Consequently, from Eqs. (4.5), (4.6), (4.12)–(4.14) we infer that, if $|x_2|$ is sufficiently large,

$$\mu < \sup\{ f_{1,2}(U(\cdot - y)):\ y\in\partial B(x_2,R_2)\} < \inf\{ f_{1,2}(u):\ u\in M, \beta_2(u)=0\}$$
$$\le \sup\{ f_{1,2}(U(\cdot - y)):\ y\in B(x_2,R_2)\} < \inf\{ f_{1,2}(u):\ u\in M, \beta_1(u)=0\} \tag{4.15}$$

and

$$\sup\{ f_{1,2}(U(\cdot - y)):\ y\in\partial B(x_1,R_1)\} < \inf\{ f_{1,2}(u):\ u\in M, \beta_1(u)=0\}$$
$$\le \sup\{ f_{1,2}(U(\cdot - y)):\ y\in B(x_1,R_1)\}. \tag{4.16}$$

We can now conclude that there exists $\varrho_1 = \varrho_1(\lambda_1, \dots, \lambda_k, |x_1|)$ such that, if $|x_2| > \varrho_1$, Eqs. (4.15), (4.16) and all the properties stated in Step 2 hold.

Now fix $x_2 \in \mathbb{R}^N$ as before and let $|x_3| \to \infty$.

One can easily verify that

$$\lim_{|x_3|\to\infty} \sup\{f_{1,2,3}(U(\cdot - y)):\ y \in \partial B(x_3, R_3)\}$$
$$= \sup\{\tilde{f}_3(U(\cdot - y)):\ y \in \partial B(0, R_3)\}, \tag{4.17}$$

$$\lim_{|x_3|\to\infty} \sup\{f_{1,2,3}(U(\cdot - y)):\ y \in B(x_3, R_3)\}$$
$$= \sup\{\tilde{f}_3(U(\cdot - y)):\ y \in B(0, R_3)\} \tag{4.18}$$

and, for $i = 1, 2$,

$$\lim_{|x_3|\to\infty} \sup\{f_{1,2,3}(U(\cdot - y)):\ y \in \partial B(x_i, R_i)\}$$
$$= \sup\{f_{1,2}(U(\cdot - y)):\ y \in \partial B(x_i, R_i)\} \tag{4.19}$$

and

$$\lim_{|x_3|\to\infty} \sup\{f_{1,2,3}(U(\cdot - y)):\ y \in B(x_i, R_i)\}$$
$$= \sup\{f_{1,2}(U(\cdot - y)):\ y \in B(x_i, R_i)\}, \tag{4.20}$$

where R_1, R_2 and R_3 are the positive numbers fixed in Step 1.

It is obvious that

$$\inf\{\tilde{f}_3(u):\ u \in M, \beta(u) = 0\} = \inf\{f_3(u):\ u \in M, \beta_3(u) = 0\}$$
$$\leq \inf\{f_{1,2,3}(u):\ u \in M, \beta_3(u) = 0\} \tag{4.21}$$

and, for $i = 1, 2$,

$$\inf\{f_{1,2}(u):\ u \in M, \beta_i(u) = 0\} \leq \inf\{f_{1,2,3}(u):\ u \in M, \beta_i(u) = 0\}. \tag{4.22}$$

Taking into account Eqs. (4.5), (4.6), (4.17)–(4.22), we obtain

$$\mu < \sup\{f_{1,2,3}(U(\cdot - y)):\ y \in \partial B(x_i, R_i)\} < \inf\{f_{1,2,3}(u):\ u \in M, \beta_i(u) = 0\}$$
$$\leq \sup\{f_{1,2,3}(U(\cdot - y)):\ y \in B(x_i, R_i)\} \quad \text{for } i = 1, 2, 3 \tag{4.23}$$

and

$$\sup\{f_{1,2,3}(U(\cdot - y)):\ y \in B(x_i, R_i)\}$$
$$< \inf\{f_{1,2,3}(u):\ u \in M, \beta_{i-1}(u) = 0\} \quad \text{for } i = 2, 3. \tag{4.24}$$

Let $r_1 > 0$ be fixed in such a way that Eqs. (4.10) and (4.11) hold; one can easily verify that

$$\lim_{|x_3|\to\infty} \sup\left\{ f_{1,2,3}(U(\cdot - y)):\ y \in \partial B\left(\frac{x_1+x_2}{2}, r_1\right)\right\}$$
$$= \sup\left\{ f_{1,2}(U(\cdot - y)):\ y \in \partial B\left(\frac{x_1+x_2}{2}, r_1\right)\right\} \tag{4.25}$$

and

$$\lim_{|x_3|\to\infty} \sup\left\{ f_{1,2,3}(U(\cdot - y)):\ y \in T_{1,2} \cap B\left(\frac{x_1+x_2}{2}, r_1\right)\right\}$$
$$= \sup\left\{ f_{1,2}(U(\cdot - y)):\ y \in T_{1,2} \cap B\left(\frac{x_1+x_2}{2}, r_1\right)\right\}. \tag{4.26}$$

Taking into account Eqs. (4.8)–(4.10), it follows that, for $|x_3|$ sufficiently large,

$$\sup\left\{ f_{1,2,3}(U(\cdot - y)):\ y \in T_{1,2} \cap B\left(\frac{x_1+x_2}{2}, r_1\right)\right\}$$
$$< \inf\{ \tilde{f}_k(u):\ u \in M, \beta(u) = 0\} \tag{4.27}$$

and

$$\mu < \sup\left\{ f_{1,2,3}(U(\cdot - y)):\ y \in \partial B\left(\frac{x_1+x_2}{2}, r_1\right)\right\}$$
$$< \inf\{ f_{1,2,3}(u):\ u \in M, \beta_{1,2}(u) \in S_{1,2}\}$$
$$\leq \sup\left\{ f_{1,2,3}(U(\cdot - y)):\ y \in T_{1,2} \cap B\left(\frac{x_1+x_2}{2}, r_1\right)\right\}$$
$$< \inf\left\{ f_{1,2,3}(u):\ u \in M, \beta_{1,2}(u) = \pm \frac{x_2 - x_1}{2|x_2 - x_1|}\right\}. \tag{4.28}$$

Now, arguing as in Step 2, one can verify that there exists $\varrho_2 = \varrho_2(\lambda_1, \ldots, \lambda_k, |x_1|, |x_2|) > 0$ such that, if $|x_3| > \varrho_2$, then Eqs. (4.23), (4.24), (4.27) and (4.28) hold,

$$\sup\{ f_{1,2,3}(U(\cdot - y)):\ y \in T_{2,3}\} < \inf\{ f_{1,2,3}(u):\ u \in M, \beta_{1,2}(u) \in S_{1,2}\} \tag{4.29}$$

and, for r_2 sufficiently large,

$$\mu < \sup\left\{ f_{1,2,3}(U(\cdot - y)):\ y \in \partial B\left(\frac{x_2+x_3}{2}, r_2\right)\right\}$$
$$< \inf\{ f_{1,2,3}(u):\ u \in M, \beta_{2,3}(u) \in S_{2,3}\}$$
$$\leq \sup\{ f_{1,2,3}(U(\cdot - y)):\ y \in T_{2,3}\}$$
$$< \inf\left\{ f_{1,2,3}(u):\ u \in M, \beta_{2,3}(u) = \pm \frac{x_3 - x_2}{2|x_3 - x_2|}\right\}. \tag{4.30}$$

Furthermore, r_2 can be chosen large enough such that the map

$$y \to \beta_{2,3}\left(U\left(\cdot - \frac{x_2+x_3}{2} - yr_2\right)\right)$$

is homotopically equivalent in $\mathbb{R}^N\setminus\{\pm(x_3-x_2)/(2|x_3-x_2|)\}$ to the identity map on $\partial B(0,1)\cup(T_{2,3}-(x_2+x_3)/2)$.

Iterating this procedure we obtain that there exist $r_i>0$ and $\varrho_i=\varrho_i(\lambda_1,\ldots,\lambda_k,|x_1|,\ldots,|x_i|)>0$, for $i=1,\ldots,k-1$, such that, if $|x_{i+1}|>\varrho_i$, then the functional $f_a=f_{1,\ldots,k}$ satisfies

$$\begin{aligned}\mu &< \inf\{f_a(u)\colon\ u\in M, \beta_{i,i+1}(u)\in S_{i,i+1}\}\\ &\le \sup\left\{f_a(U(\cdot-y))\colon\ y\in T_{i,i+1}\cap B\left(\frac{x_i+x_{i+1}}{2},r_i\right)\right\}\\ &< \inf\{f_a(u)\colon\ u\in M,\beta_{i-1,i}(u)\in S_{i-1,i}\}\\ &\le \sup\left\{f_a(U(\cdot-y))\colon\ y\in T_{i-1,i}\cap B\left(\frac{x_{i-1}+x_i}{2},r_{i-1}\right)\right\}\\ &< \inf\{f_a(u)\colon\ u\in M,\beta_k(u)=0\}\quad \text{for } i=2,\ldots,k-1 \end{aligned}\tag{4.31}$$

and, for $i=1,\ldots,k-1$,

$$\begin{aligned}\mu &< \sup\left\{f_a(U(\cdot-y))\colon\ y\in \partial B\left(\frac{x_i+x_{i+1}}{2},r_i\right)\right\}\\ &< \inf\{f_a(u)\colon\ u\in M,\beta_{i,i+1}(u)\in S_{i,i+1}\}\\ &\le \sup\left\{f_a(U(\cdot-y))\colon\ y\in T_{i,i+1}\cap B\left(\frac{x_i+x_{i+1}}{2},r_i\right)\right\}\\ &< \inf\left\{f_a(u)\colon\ u\in M,\beta_{i,i+1}(u)=\pm\frac{x_{i+1}-x_i}{2|x_{i+1}-x_i|}\right\}<2^{1-(2/p)}\mu,\end{aligned}\tag{4.32}$$

where r_i is large enough, such that

$$\text{the map } y\to\beta_{i,i+1}\left(U\left(\cdot-\frac{x_i+x_{i+1}}{2}-r_iy\right)\right) \text{ is homotopically equivalent in } \mathbb{R}^N\Bigg\backslash\left\{\pm\frac{x_{i+1}-x_i}{2|x_{i+1}-x_i|}\right\} \text{ to the identity map on } \partial B(0,1)\cup\left(T_{i,i+1}-\frac{(x_i+x_{i+1})}{2}\right).\tag{4.33}$$

Furthermore, we have

$$\begin{aligned}\mu &< \sup\left\{f_a(U(\cdot-y))\colon\ y\in T_{1,2}\cap B\left(\frac{x_1+x_2}{2},r_1\right)\right\}\\ &< \inf\{f_a(u)\colon\ u\in M,\beta_{i+1}(u)=0\}\end{aligned}$$

$$\le \sup\{f_a(U(\cdot - y)):\ y\in B(x_{i+1},R_{i+1})\}$$
$$< \inf\{f_a(u):\ u\in M, \beta_i(u)=0\}$$
$$\le \sup\{f_a(U(\cdot - y)):\ y\in B(x_i,R_i)\}<2^{1-(2/p)}\mu \quad \text{for } i=1,\dots,k-1, \tag{4.34}$$

$$\mu < \sup\{f_a(U(\cdot - y)):\ y\in \partial B(x_j,R_j)\}<\inf\{f_a(u):\ u\in M, \beta_j(u)=0\}$$
$$\le \sup\{f_a(U(\cdot - y)):\ y\in B(x_j,R_j)\}<2^{1-(2/p)}\mu \quad \text{for } j=1,\dots,k \tag{4.35}$$

and (see Remark 3.3)

$$\text{the map } y\to \beta_j(U(\cdot - x_j - R_j y)) \text{ is homotopically equivalent in } \mathbb{R}^N\setminus\{0\}$$
$$\text{to the identity map on } \partial B(0,1). \tag{4.36}$$

Step 4. Our aim is now to prove the existence of $2k-1$ critical points for f_a constrained on M.

Let us define, for $i=1,\dots,k-1$,

$$b_i=\inf\{f_a(u):\ u\in M, \beta_{i,i+1}(u)\in S_{i,i+1}\},$$
$$d_i=\sup\left\{f_a(U(\cdot - y)):\ y\in T_{i,i+1}\cap B\left(\frac{x_i+x_{i+1}}{2},r_i\right)\right\}$$

and, for $j=1,\dots,k$,

$$e_j=\inf\{f_a(u):\ u\in M, \beta_j(u)=0\},$$
$$g_j=\sup\{f_a(U(\cdot - y)):\ y\in B(x_j,R_j)\}.$$

Using the inequalities stated in the previous steps, we can now show the existence of a critical value in $[b_i,d_i]$, for any $i=1,\dots,k-1$, and the existence of a critical value in $[e_j,g_j]$, for any $j=1,\dots,k$.

Let us observe that, since

$$\mu < b_{k-1}\le d_{k-1}<b_{k-2}\le d_{k-2}<\cdots<b_1\le d_1$$
$$< e_k\le g_k<e_{k-1}\le g_{k-1}<\cdots<e_1\le g_1<2^{1-(2/p)}\mu,$$

the critical values we shall find are pairwise distinct; so they correspond to $2k-1$ distinct critical points for f_a constrained on M; moreover, these critical points are nonnegative functions, because of Proposition 2.4.

Let us now fix $i\in\{1,\dots,k-1\}$ and let us show the existence of a critical value in $[b_i,d_i]$.

Arguing by contradiction, assume that $[b_i,d_i]$ does not contain any critical value. Since $\mu<b_i\le d_i<2^{1-(2/p)}\mu$ and the Palais–Smale condition holds in $]\mu,2^{1-(2/p)}\mu[$, it follows that there exists $\eta>0$ such that (see, for instance, [27]) the sublevel $f_a^{b_i-\eta}$ is a deformation retract of the sublevel $f_a^{d_i}$ (as usual, we set $f_a^c=\{u\in M:\ f_a(u)\le c\}$

$\forall c \in \mathbb{R}$). This means, in particular, that there exists a continuous map $\chi_i : [0,1] \times f_a^{d_i} \to f_a^{d_i}$ such that

$$\chi_i(0,u) = u \quad \forall u \in f_a^{d_i},$$

$$\chi_i(1,u) \in f_a^{b_i - \eta} \quad \forall u \in f_a^{d_i}.$$

Let

$$\Psi_i : [0,1] \times \left(\partial B(0,1) \cup \left(T_{i,i+1} - \frac{x_i + x_{i+1}}{2} \right) \right) \to \mathbb{R}^N \setminus \left\{ \pm \frac{x_{i+1} - x_i}{2|x_{i+1} - x_i|} \right\}$$

be a homotopy such that (see (3.19), Remark 3.5 and Eq. (4.33))

$$\Psi_i(0,y) = y \quad \text{and} \quad \Psi_i(1,y) = \beta_{i,i+1}\left(U\left(\cdot - \frac{x_i + x_{i+1}}{2} - r_i y \right) \right)$$

for all $y \in \partial B(0,1) \cup (T_{i,i+1} - (x_i + x_{i+1})/2)$.

Now define

$$\Gamma_i : [0,1] \times \left(\partial B(0,1) \cup \left(T_{i,i+1} - \frac{x_i + x_{i+1}}{2} \right) \cap B(0,1) \right) \to \mathbb{R}^N$$

by

$$\Gamma_i(t,y) = \begin{cases} \Psi_i(2t,y) & \text{if } t \in [0, \frac{1}{2}], \\ \beta_{i,i+1}\left(\chi_i\left(2t-1, U\left(\cdot - \frac{x_i + x_{i+1}}{2} - r_i y \right) \right) \right) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

The function Γ_i is well defined (because of Eq. (4.32)), it is continuous and, for all $y \in \partial B(0,1) \cup (T_{i,i+1} - (x_i + x_{i+1})/2) \cap B(0,1)$, it satisfies

$$\Gamma_i(0,y) = y, \quad \Gamma_i(t,y) \neq \pm \frac{x_{i+1} - x_i}{2|x_{i+1} - x_i|} \quad \forall t \in [0,1]$$

(as we infer from Eq. (4.32) and the properties of Ψ_i and χ_i) and, moreover,

$$\Gamma_i(1,y) \notin S_{i,i+1}$$

since

$$\chi_i\left(1, U\left(\cdot - \frac{x_i + x_{i+1}}{2} - r_i y \right) \right) \in f_a^{b_i - \eta}.$$

Thus Γ_i is a continuous deformation in

$$\mathbb{R}^N \setminus \left\{ \pm \frac{x_{i+1} - x_i}{2|x_{i+1} - x_i|} \right\} \text{ from } \left(\partial B(0,1) \cup \left(T_{i,i+1} - \frac{x_i + x_{i+1}}{2} \right) \cap B(0,1) \right)$$

to a set which does not intersect $S_{i,i+1}$, which is impossible.

Therefore $[b_i, d_i]$ must contain a critical value c_i.

On the whole we get $k-1$ distinct critical points for f_a constrained on M, say $v_1, \ldots, v_{k-1}$, such that $\mu < b_i \leq f_a(v_i) \leq d_i < e_k < 2^{1-(2/p)}\mu$ for $i = 1, \ldots, k-1$.

Let us now fix $j \in \{1,\dots,k\}$ and prove that there exists a critical value $\bar{c}_j \in [e_j, g_j]$.

Assume, by contradiction, that $[e_j, g_j]$ does not contain any critical value for f_a constrained on M. Since the Palais–Smale condition holds in $]\mu, 2^{1-(2/p)}\mu[$, taking into account Eq. (4.35), it follows that there exists $\xi > 0$ such that the sublevel $f_a^{e_j-\xi}$ is a deformation retract of $f_a^{g_j}$ and

$$\sup\{f_a(U(\cdot - y)):\ y \in \partial B(x_j, R_j)\} < e_j - \xi. \tag{4.37}$$

Thus, in particular, there exists a continuous function $\gamma_j : f_a^{g_j} \to f_a^{e_j-\xi}$ such that

$$\gamma_j(u) = u \quad \forall u \in f_a^{e_j-\xi}. \tag{4.38}$$

Moreover, we have

$$\{U(\cdot - y):\ y \in \partial B(x_j, R_j)\} \subseteq f_a^{e_j-\xi} \tag{4.39}$$

(because of Eq. (4.37)) and

$$\{U(\cdot - y):\ y \in B(x_j, R_j)\} \subseteq f_a^{g_j}, \tag{4.40}$$

as follows from the definition of g_j.

Hence, we can consider the map $\Theta_j : [0,1] \times \partial B(0,1) \to \mathbb{R}^N$ defined, for all $z \in \partial B(0,1)$, by

$$\Theta_j(t,z) = \begin{cases} (1-2t)z + 2t\beta_j(U(\cdot - x_j - R_j z)) & \text{for } t \in [0, \frac{1}{2}], \\ \beta_j \circ \gamma_j(U(\cdot - x_j - 2(1-t)R_j z)) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

The function Θ_j is well defined (because of Eq. (4.40)), it is continuous (because of Eqs. (4.38) and (4.39)) and it satisfies

$$\Theta_j(0,z) = z \quad \text{and} \quad \Theta_j(1,z) = \beta_j \circ \gamma_j(U(\cdot - x_j)) \quad \forall z \in \partial B(0,1). \tag{4.41}$$

Moreover, taking into account Remark 3.3, Eq. (4.35), the definition of e_j and the properties of γ_j, we infer that

$$\Theta_j(t,z) \neq 0 \quad \forall t \in [0,1],\ \forall z \in \partial B(0,1). \tag{4.42}$$

It is clear that Eqs. (4.41) and (4.42) give a contradiction, since $\partial B(0,1)$ is not contractible in $\mathbb{R}^N \setminus \{0\}$. Hence $[e_j, g_j]$ must contain a critical value $\bar{c}_j$.

Since $j \in \{1,\dots,k\}$, we have k distinct critical points for f_a constrained on M, say $\bar{v}_1,\dots,\bar{v}_k$, such that $\mu < d_{k-1} < e_j \le f_a(\bar{v}_j) \le g_j < 2^{1-(2/p)}\mu$.

Summarizing, if we set

$$u_i(x) = [f_a(v_i)]^{1/(p-2)} v_i(x) \quad \text{for } i = 1,\dots,k-1$$

and

$$\bar{u}_i(x) = [f_a(\bar{v}_j)]^{1/(p-2)} \bar{v}_j(x) \quad \text{for } j = 1,\dots,k,$$

we have on the whole $2k-1$ distinct solutions of problem (P). □

The proof of Theorem 2.1 suggests that a weaker multiplicity result can be stated even when no assumption is required on the positive numbers $\lambda_1,\dots,\lambda_k$, which appears in Eq. (1.3).

In fact, missing Step 1 and arguing as in the other steps of the proof of Theorem 2.1, one can prove the following:

Theorem 4.1. *Let $p>2$ and $p<2N/(N-2)$ if $N\geq 3$. Let $\alpha_1,\dots\alpha_k$ be given nonnegative functions belonging to $L^{N/2}(\mathbb{R}^N)$ such that $\|\alpha_j\|_{N/2}\neq 0$ for all $j=1,\dots,k$.*

Then there exist $\varrho_1=\varrho_1(|x_1|)>0,\dots,\varrho_{k-1}=\varrho_{k-1}(|x_1|,\dots,|x_{k-1}|)>0$ such that, if $|x_j|>\varrho_{j-1}$ for $j=2,\dots,k$, problem (P) *with $a(x)$ of the form*

$$a(x)=\sum_{j=1}^{k}\alpha_j(x-x_j) \tag{4.43}$$

has at least $k-1$ distinct solutions.

Remark 4.2. *Let us notice that the multiplicity results stated in this paper show some possible way to choose the positive numbers $\lambda_1,\dots,\lambda_k$ and the points $x_1,\dots,x_k$ in order to obtain distinct solutions of problem* (P) (*indeed distinct critical values of the corresponding functional*).

On the other hand, using Morse theory (*see, for example,* [7]), *it is possible to obtain the same number of solutions, choosing $\lambda_1,\dots,\lambda_k$ large or small enough and $|x_i-x_j|$ sufficiently large for $i\neq j$ $(i,j=1,\dots,k)$, without any other relation between them. But let us point out that the solutions one could obtain by means of Morse theory are not really distinct*: *they are counted with their own multiplicity* (*defined in a suitable way*).

For example, results like the following ones could be proved by means of Morse theory.

Let $p>2$ and $p<2N/(N-2)$ if $N\geq 3$. Let $\alpha_1,\dots,\alpha_k$ be given nonnegative functions belonging to $L^{N/2}(\mathbb{R}^N)$, such that $\|\alpha_j\|_{N/2}\neq 0$ for all $j=1,\dots,k$.

Then:

(a) there exists $\varrho>0$ such that, if

$$|x_i-x_j|>\varrho \quad \text{for } i\neq j \ (i,j=1,\dots,k), \tag{4.44}$$

then problem (P), with $a(x)$ of the form Eq. (4.43), has at least $k-1$ solutions, which are counted with their multiplicity;

(b) there exist $\varepsilon>0$ and $\varrho=\varrho(\lambda_1,\dots,\lambda_k)$ such that, if

$$\lambda_i<\varepsilon \text{ or } \lambda_i>\frac{1}{\varepsilon} \quad \text{for all } i=1,\dots,k,$$

and Eq. (4.44) holds, then problem (P) with $a(x)$ of the form Eq. (1.3) has at least $2k-1$ solutions, which are counted with their own multiplicity.

Finally, let us remark that the proof of Theorem 2.1 gives some information about the behaviour of the solutions $u_1,\dots,u_{k-1}$, $\overline{u}_1,\dots,\overline{u}_k$. In fact, one can infer that

$$f_a\left(\frac{u_i}{\|u_i\|_p}\right)\to\mu \text{ as } |x_{i+1}-x_i|\to\infty \quad \text{for } i=1,\dots,k-1$$

and

$$f_a\left(\frac{\overline{u}_j}{\|\overline{u}_j\|_p}\right)\to\mu \quad \text{as } \lambda_j\to 0 \text{ or } \lambda_j\to\infty \quad \text{for } j=1,\dots,k.$$

Therefore, taking into account [21], we obtain:

(a) if $|x_{i+1}-x_i|\to\infty$, there exist $z_i\in\mathbb{R}^N$ and $w_i\to 0$ in $H^{1,2}(\mathbb{R}^N)$ such that

$$u_i(x)=\mu^{1/(p-2)}[U(x-z_i)+w_i(x)];$$

(b) if $\lambda_j\to 0$ or $\lambda_j\to\infty$, there exist $\overline{z}_j\in\mathbb{R}^N$ and $\overline{w}_j\to 0$ in $H^{1,2}(\mathbb{R}^N)$ such that

$$\overline{u}_j(x)=\mu^{1/(p-2)}[U(x-\overline{z}_i)+\overline{w}_i(x)].$$

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