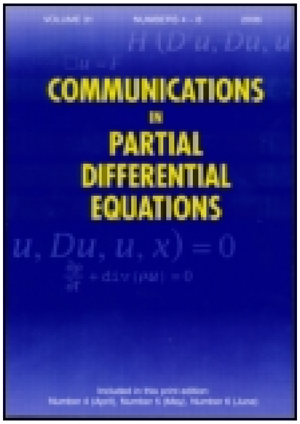

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NontriviaL solutions of some nonlinear elliptic problems

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NONTRIVIAL SOLUTIONS OF SOME NONLINEAR ELLIPTIC PROBLEMS

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Abstract. This paper is concerned with a class of semilinear elliptic Dirichlet problems approximating degenerate equations. The aim is to prove the existence of at least $4^k - 1$ nontrivial solutions when the degeneration set consists of k distinct connected components.

Key words. Semilinear elliptic equations. Degenerate equations. Variational methods. Nontrivial solutions.

1. Introduction

In this paper we deal with multiplicity of solutions for problem

$$\mathfrak{D}_\epsilon \qquad \begin{cases} div(a_\epsilon(x)Du) + g(x,u) = 0 & \text{in } \Omega \\ u \not\equiv 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$ ($N \geq 1$); for all $\epsilon > 0$ and $x \in \Omega$, $a_\epsilon(x)$ is a positive defined symmetric $N \times N$ matrix with coefficients $a_\epsilon^{i,j}(x) \in L^\infty(\Omega; \mathbb{R})$, for $i, j = 1, \ldots, N$, and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function, with superlinear and subcritical growth, such that $g(x, 0) = 0$ for all $x \in \Omega$.

We assume that the matrix $a_\epsilon(x)$ degenerates, as $\epsilon \to 0$, for all x in a suitable subset $\mathcal{D}$ (the degeneration set) of Ω.

As pointed out in [19], the behaviour of $a_\epsilon(x)$ as $\epsilon \to 0$ gives rise to a phenomenon of concentration of the solutions. Concentration phenomena of this kind also appear in other elliptic problems: for instance, in the case of elliptic equations with a critical or supercritical nonlinear term (see [1 - 3, 7 - 10, 20 - 25]); or in the case of elliptic equations with subcritical nonlinearity when certain parameters are sufficiently large (see, for example, [4 - 6, 11, 18]).

As shown by the previous examples, whenever a phenomenon of concentration of solutions occurs, the geometrical properties of the domain affect the solvability of the problem, the multiplicity and the qualitative properties of the solutions.

In this context we consider problem $\mathfrak{D}_\epsilon$. In particular, our aim is to study the solvability and the multiplicity of solutions for $\mathfrak{D}_\epsilon$ in dependence of the geometrical properties of the degeneration set $\mathcal{D}$, under suitable assumptions on the function $g(x, \tau)$.

The solvability of problem $\mathfrak{D}_\epsilon$ depends strongly on the sign of $g'(x, 0)$, where g' denotes the derivative of g with respect to the second variable.

In [17] we show that, if $g'(x, 0) > 0$ for all x in a subset of $\mathcal{D}$, then, for $\epsilon > 0$ small enough, $\mathfrak{D}_\epsilon$ cannot have any solution with constant sign in that subset; in particular, $\mathfrak{D}_\epsilon$ cannot have any positive solutions. On the other hand, under suitable symmetry assumptions on g, $\mathfrak{D}_\epsilon$ may have infinitely many solutions for all $\epsilon > 0$, but the sign of these solutions must change very rapidly in the subset of $\mathcal{D}$ where $g'(x,0) > 0$, when ϵ is close to 0.

On the contrary, when $g'(x,0) \leq 0$ in Ω, for all $\epsilon > 0$ $\mathfrak{D}_\epsilon$ has at least one positive solution u_ϵ such that

$$\lim_{\epsilon \to 0} \int_\Omega |Du_\epsilon|^2 \, dx = 0$$

and

$$\lim_{\epsilon \to 0} \Big(\int_\Omega |Du_\epsilon|^2 \, dx \Big)^{-1} \int_{\mathcal{D}} |Du_\epsilon|^2 \, dx = 1,$$

that is the solution u_ϵ tends to be localized near the degeneration set $\mathcal{D}$. Hence the following natural question arises: what happens when, for example, the degeneration set consists of k ($k > 1$) connected components?

The result we obtain shows that it is possible to relate the multiplicity of solutions to the number of connected components of $\mathcal{D}$.

Moreover, it is worth to remark that, if $\sup_{x \in \Omega} g'(x,0) < 0$, then the concentration phenomena are accentuated: not only the solutions tend to be localized near the degeneration set $\mathcal{D}$, but also they concentrate, as $\epsilon \to 0$, like Dirac mass near some points of $\mathcal{D}$. This property allows us to estimate the number of the positive solutions of $\mathfrak{D}_\epsilon$, for $\epsilon > 0$ small enough, by the Ljusternik-Schnirelman category of the degeneration set (see [16]); hence we may have more than one solution even if $\mathcal{D}$ is connected (i.e. $k = 1$) but has complex topology.

Let us specify the assumptions required on $a_\epsilon(x)$ and on g:

(a.1) for all $\epsilon > 0$ and for almost all $x \in \Omega$, there exist $\Lambda_1 = \Lambda_1(\epsilon, x) > 0$ and $\Lambda_2 = \Lambda_2(\epsilon, x) > 0$ such that

$$\Lambda_1|\xi|^2 \le a_\epsilon^{i,j}(x)\xi_i\xi_j \le \Lambda_2|\xi|^2 \quad \forall\, \xi \in \mathbb{R}^N$$

(here and later on we write, as it is usual, $a_\epsilon^{i,j}(x)\xi_i\xi_j$ instead of $\sum_{i,j=1}^{N} a_\epsilon^{i,j}(x)\xi_i\xi_j$);

(a.2)

$$\liminf_{\epsilon\to 0} \frac{1}{\epsilon} \inf_{x\in\Omega} \Lambda_1(\epsilon, x) > 0;$$

(a.3) there exist k nonempty subsets of Ω, we say $\Omega_1, \cdots, \Omega_k$ (the degeneration subsets for $a_\epsilon(x)$), such that

$$\limsup_{\epsilon\to 0} \frac{1}{\epsilon} \sup\{\Lambda_2(\epsilon, x) \,:\, x \in \bigcup_{t=1}^{k} \Omega_t\} < +\infty\,;$$

(a.4) for all $\eta > 0$

$$\liminf_{\epsilon\to 0} \inf\{\Lambda_1(\epsilon, x) :\; x \in \Omega\setminus \bigcup_{t=1}^{k} \Omega_t(\eta)\} > 0,$$

where $\Omega_t(\eta) = \{\, x \in \Omega :\; d(x, \Omega_t) < \eta\}$;

(D) $\Omega_1, \ldots, \Omega_k$ are smooth domains strictly contained in Ω (i.e. $\overline{\Omega}_t \subset \Omega\ \forall\ t = 1, \ldots, k$). For all $t = 1, \ldots, k$ let us denote by C_t the union of the connected components of $\overline{\Omega} \setminus \Omega_t$ which don't meet $\partial\Omega$ and set $\Omega_t' = \Omega_t \cup C_t$.

We require that the subsets $\overline{\Omega_1'}, \ldots, \overline{\Omega_k'}$ are pairwise disjoined and that every connected component of $\Omega\setminus \bigcup_{t=1}^{k} \Omega_t'$ meets $\partial\Omega$ (notice that $\Omega\setminus \bigcup_{t=1}^{k} \Omega_t'$ could have more than one connected component even if Ω is connected).

(g.1) For all $\tau \in \mathbb{R}$, $g(x,\tau)$ is measurable with respect to x; for almost all $x \in \Omega$, $g(x,\tau)$ is a C^1- function with respect to τ;

(g.2) there exist positive constants a and q, with $q < \frac{2N}{N-2}$ if $N \geq 3$, such that, for all $\tau \in \mathbb{R}$ and for almost all $x \in \Omega$,

$$|g(x,\tau)| \leq a + a|\tau|^{q-1}$$

and

$$|g'(x,\tau)| \leq a + a|\tau|^{q-2};$$

(g.3) $g(\cdot, 0) = 0$ a.e. in Ω and there exist $p > 2$, with $p < \frac{2N}{N-2}$ if $N \geq 3$, and a strictly positive function $\lambda : \Omega \to \mathbb{R}^+$, with $\lambda \in L^\infty(\Omega)$ and $\frac{1}{\lambda} \in L^\infty(\Omega)$, such that

$$\lim_{\tau \to 0} \frac{g'(x,\tau)}{(p-1)|\tau|^{p-2}} = \lambda(x) \text{ uniformly on } \Omega;$$

(g.4) there exists $\theta \in]0, \frac{1}{2}[$ such that

$$G(x,\tau) \leq \theta \tau g(x,\tau)$$

for all $\tau \in \mathbb{R}$ and for almost all $x \in \Omega$, where $G(x,\tau) = \int_0^\tau g(x,s)\, ds$.

Under the previous assumptions, there exists $\bar{\epsilon} > 0$ such that, for all $\epsilon \in]0, \bar{\epsilon}[$,

(1) $\mathfrak{D}_\epsilon$ has at least k positive solutions $u_{\epsilon,1}, \ldots, u_{\epsilon,k}$ (see [13]), satisfying the following properties:

$$\lim_{\epsilon \to 0} \Big(\int_\Omega |Du_{\epsilon,i}|^2\, dx \Big)^{-1} \int_{\Omega_i} |Du_{\epsilon,i}|^2\, dx = 1 \quad \forall\, i \in \{1, \ldots, k\};$$

(2) $\mathfrak{D}_\epsilon$ has at least $2^k - 1$ multibump positive solutions (see [15]). More precisely, if we choose arbitrarily r distinct subsets among $\Omega_1, \dots, \Omega_k$ (say $\Omega_{i_1}, \dots, \Omega_{i_r}$), one can construct a positive r-bumps solution $u_\epsilon^{i_1,\dots,i_r}$ with the following properties:

$$\lim_{\epsilon\to 0} \Big(\int_\Omega |Du_\epsilon^{i_1,\dots,i_r}|^2 \, dx \Big)^{-1} \int_{\Omega \setminus \bigcup_{s=1}^{r} \Omega_{i_s}} |Du_\epsilon^{i_1,\dots,i_r}|^2 \, dx = 0$$

and

$$\liminf_{\epsilon\to 0} \Big(\int_\Omega |Du_\epsilon^{i_1,\dots,i_r}|^2 \, dx \Big)^{-1} \int_{\Omega_{i_s}} |Du_\epsilon^{i_1,\dots,i_r}|^2 \, dx > 0 \quad \forall\, s \in \{1, \dots, r\};$$

(3) $\mathfrak{D}_\epsilon$ has at least k^2 sign changing solutions (see [14]), having exactly two nodal regions (i.e. both the supports of the positive and the negative parts of the solutions are connected subsets of Ω). Moreover the obtained solutions $u_{\epsilon,i,j}$, for all $i, j \in \{1, \dots, k\}$, have the following property:

$$\lim_{\epsilon\to 0} \Big(\int_\Omega |Du_{\epsilon,i,j}^+|^2 \, dx \Big)^{-1} \int_{\Omega_i} |Du_{\epsilon,i,j}^+|^2 \, dx = 1$$

and

$$\lim_{\epsilon\to 0} \Big(\int_\Omega |Du_{\epsilon,i,j}^-|^2 \, dx \Big)^{-1} \int_{\Omega_j} |Du_{\epsilon,i,j}^-|^2 \, dx = 1.$$

In this paper we obtain $4^k - 1$ nontrivial solutions by showing that, if we choose arbitrarily some subsets $\Omega_{i_1}, \dots, \Omega_{i_r}$ and $\Omega_{j_1}, \dots, \Omega_{j_s}$ among

$\Omega_1, \ldots, \Omega_k$, then we can construct a solution of $\mathfrak{D}_\epsilon$ whose positive (respectively negative) part can be decompose as sum of r (respectively s) non-negative functions which, as $\epsilon \to 0$, tend to be localized near the subsets $\Omega_{i_1}, \ldots, \Omega_{i_r}$ (respectively $\Omega_{j_1}, \ldots, \Omega_{j_s}$).

More precisely, we shall prove the following theorem.

Theorem 1.1. *Assume that conditions (a.1), ..., (a.4), (D), (g.1), ..., (g.4) are satisfied.*

Then there exists $\bar{\epsilon} > 0$ such that, for all $\epsilon \in]0, \bar{\epsilon}[$, $\mathfrak{D}_\epsilon$ has at least $4^k - 1$ distinct nontrivial solutions.

Indeed, for all T^+, T^- subsets of $\{1, \ldots, k\}$ and for every $\epsilon \in]0, \bar{\epsilon}[$, there exists a solution $v_\epsilon^{T^+,T^-}$ of $\mathfrak{D}_\epsilon$ such that

$$\begin{cases} \lim\limits_{\epsilon \to 0} \epsilon^{\frac{p}{2-p}} \displaystyle\int_{\Omega \setminus \bigcup_{t \in T^+} \Omega'_t} |(v_\epsilon^{T^+,T^-})^+|^p \, dx = 0, \\ \lim\limits_{\epsilon \to 0} \epsilon^{\frac{p}{2-p}} \displaystyle\int_{\Omega \setminus \bigcup_{t \in T^-} \Omega'_t} |(v_\epsilon^{T^+,T^-})^-|^p \, dx = 0, \end{cases} \tag{1.1}$$

and

$$\begin{cases} \liminf\limits_{\epsilon \to 0} \epsilon^{\frac{p}{2-p}} \displaystyle\int_{\Omega'_t} |(v_\epsilon^{T^+,T^-})^+|^p \, dx > 0 \quad \forall t \in T^+, \\ \liminf\limits_{\epsilon \to 0} \epsilon^{\frac{p}{2-p}} \displaystyle\int_{\Omega'_t} |(v_\epsilon^{T^+,T^-})^-|^p \, dx > 0 \quad \forall t \in T^-. \end{cases} \tag{1.2}$$

Moreover we have

$$0 < \liminf_{\epsilon \to 0} \epsilon^{\frac{2}{2-p}} \int_\Omega |Dv_\epsilon^{T^+,T^-}|^2 \, dx \le \limsup_{\epsilon \to 0} \epsilon^{\frac{2}{2-p}} \int_\Omega |Dv_\epsilon^{T^+,T^-}|^2 \, dx < +\infty$$

and

$$\lim_{\epsilon\to 0} \epsilon^{\frac{2}{2-p}} \int_{\Omega\setminus\bigcup_{t=1}^{k}\Omega_t} |Dv_\epsilon^{T^+,T^-}|^2\, dx = 0.$$

For the proof of this theorem we proceed as follows. We consider a functional f_ϵ (see (2.1)), whose critical points correspond to solutions of problem $\mathfrak{D}_\epsilon$ and, in order to obtain the solution $v_\epsilon^{T^+,T^-}$, we find a local minimum point $u_\epsilon^{T^+,T^-}$ for f_ϵ constrained on a suitable subset $M_\epsilon^{T^+,T^-}$ of $H_0^{1,2}(\Omega)$ (see Definition 2.2). Then we prove that, for ϵ small enough, $u_\epsilon^{T^+,T^-}$ is indeed a critical point for f_ϵ and so it gives rise to a solution $v_\epsilon^{T^+,T^-} = \epsilon^{\frac{1}{p-2}} u_\epsilon^{T^+,T^-}$ of $\mathfrak{D}_\epsilon$. However, it is worth to point out that $M_\epsilon^{T^+,T^-}$ is not a smooth manifold; hence the usual methods, which consist in proving that the Lagrange multipliers are zero, do not apply in our case. Therefore we need a specific device, based on topological arguments.

Finally we remark that the behaviour of $u_\epsilon^{T^+,T^-}$ as $\epsilon \to 0$ (see Proposition 3.10) shows that, for ϵ small enough, different solutions correspond to different choices of the pair T^+, T^-. Thus we have, on the whole, $4^k - 1$ distinct nontrivial solutions (notice that it is not required that $T^+ \cap T^- = \emptyset$).

2. Notations and some preliminary properties

Troughout the paper, $H_0^{1,2}(\Omega)$ will denote the usual Sobolev space endowed with the norm $||u|| = (\int_\Omega |Du|^2\, dx)^{\frac{1}{2}}$, while we will denote by $||u||_p = (\int_\Omega |u|^p\, dx)^{\frac{1}{p}}$ the usual norm in $L^p(\Omega)$.

In $L^p(\Omega)$ we also consider the following norm

$$||u||_{(\lambda,p)} = (\int_\Omega \lambda(x)|u(x)|^p\, dx)^{\frac{1}{p}}$$

where $\lambda(x)$ is the positive function which appears in $(g.3)$. Obviously $||\cdot||_{(\lambda,p)}$ and $||\cdot||_p$ are equivalent norms.

Notice that a function $v \in H_0^{1,2}(\Omega)$, $v \not\equiv 0$, is a weak solution for $\mathfrak{D}_\epsilon$ if and only if $u = \epsilon^{-\frac{1}{p-2}} v$ is a critical point for the functional $f_\epsilon : H_0^{1,2}(\Omega) \to \mathbb{R}$ defined by

$$f_\epsilon(u) = \frac{1}{2}\int_\Omega \frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i} u \partial_{x_j} u\, dx - \frac{1}{\epsilon^{\frac{p}{p-2}}}\int_\Omega G(x, \epsilon^{\frac{1}{p-2}} u)\, dx. \tag{2.1}$$

Let us introduce some useful tools.

Definition 2.1. *Let $\Omega_1, \ldots, \Omega_k, C_1 \ldots, C_k, \Omega_1', \ldots \Omega_k'$ be as in condition (D).*

For every $u \in H_0^{1,2}(\Omega)$ and $\epsilon > 0$ let $P^\epsilon(u)$ be the function in $H_0^{1,2}(\Omega)$ such that

$$P^\epsilon(u) \equiv u \ \ in \ \ \Omega \setminus \bigcup_{t=1}^k \Omega_t'$$

and

$$\int_{\Omega_t'} a_\epsilon^{i,j}(x)\partial_{x_i} P^\epsilon(u) \partial_{x_j} v\, dx = 0 \quad \forall v \in H_0^{1,2}(\Omega_t'), \quad \forall t = 1, \ldots, k.$$

Set for all $t = 1, \ldots, k$

$$P_t^\epsilon(u) = u - P^\epsilon(u) \quad in \quad \Omega_t', \quad P_t^\epsilon(u) = 0 \quad in \quad \Omega \setminus \Omega_t'.$$

Thus it results

$$u = P_1^\epsilon(u) + \cdots + P_k^\epsilon(u) + P^\epsilon(u) \quad \forall u \in H_0^{1,2}(\Omega)$$

with $P_t^\epsilon(u) \in H_0^{1,2}(\Omega_t')$ *for all* $t = 1, \dots, k$ *(indeed* P_t^ϵ *is a projection of* $H_0^{1,2}(\Omega)$ *on* $H_0^{1,2}(\Omega_t')$*).*

Definition 2.2. *For all* T^+, T^-, *subsets of* $\{1, \dots, k\}$, *let us set*

$$M_\epsilon^{T^+,T^-} = \Big\{u \in H_0^{1,2}(\Omega) \ : f_\epsilon'(u)[(P_t^\epsilon u)^+] = 0 \quad \forall t \in T^+, \\ f_\epsilon'(u)[(P_t^\epsilon u)^-] = 0 \quad \forall t \in T^-\Big\}.$$

For simplicity of notation, we write for all $u \in H_0^{1,2}(\Omega)$

$$G_\epsilon(x,u) = \frac{1}{\epsilon^{\frac{p}{p-2}}} G(x, \epsilon^{\frac{1}{p-2}} u), \tag{2.2}$$

$$g_\epsilon(x,u) = \frac{1}{\epsilon^{\frac{p-1}{p-2}}} g(x, \epsilon^{\frac{1}{p-2}} u) \tag{2.3}$$

and

$$g_\epsilon'(x,u) = \frac{1}{\epsilon} g'(x, \epsilon^{\frac{1}{p-2}} u). \tag{2.4}$$

Definition 2.3. *Let us set*

$$\underline{\Lambda} = \liminf_{\epsilon \to 0} \frac{1}{\epsilon} \inf\{\Lambda_1(\epsilon, x) \ : \ x \in \Omega\} \tag{2.5}$$

and

$$\overline{\Lambda} = \limsup_{\epsilon\to 0} \frac{1}{\epsilon} \inf\{\Lambda_2(\epsilon, x) \; : \; x \in \bigcup_{t=1}^{k} \Omega_t\} \tag{2.6}$$

(see (a.1), (a.2) and (a.3)).

Moreover, for all $t = 1, \dots, k$, we set

$$\mu_t = \inf\{\int_{\Omega_t} |Du|^2\, dx \; : \; u \in H_0^{1,2}(\Omega_t'), \; \int_{\Omega_t'\setminus\Omega_t} |Du|^2\, dx = 0\,, \; \int_{\Omega_t'} \lambda(x)|u(x)|^p\, dx = 1 \;\}$$

and $\tilde{\mu} = \min_{t\in\{1,\dots,k\}} \mu_t$.

Definition 2.4. *Let $\Gamma_\epsilon : \Omega \times \mathbb{R} \to \mathbb{R}$, $\gamma_\epsilon : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\tilde{\gamma}_\epsilon : \Omega \times \mathbb{R} \to \mathbb{R}$ be the functions defined by*

$$\Gamma_\epsilon(x,\tau) = \begin{cases} \frac{G_\epsilon(x,\tau)}{|\tau|^p} & \text{for } \tau \neq 0 \\ \frac{\lambda(x)}{p} & \text{for } \tau = 0 \end{cases} \tag{2.8}$$

$$\gamma_\epsilon(x,\tau) = \begin{cases} \frac{g_\epsilon(x,\tau)}{|\tau|^{p-2}\tau} & \text{for } \tau \neq 0 \\ \lambda(x) & \text{for } \tau = 0 \end{cases} \tag{2.9}$$

and

$$\tilde{\gamma}_\epsilon(x,\tau) = \begin{cases} \frac{g_\epsilon'(x,\tau)}{|\tau|^{p-2}} & \text{for } \tau \neq 0 \\ (p-1)\lambda(x) & \text{for } \tau = 0. \end{cases} \tag{2.10}$$

Because of $(g.3)$, for almost all $x \in \Omega$, Γ_ϵ, γ_ϵ and $\tilde{\gamma}_\epsilon$ are continuous function with respect to τ.

Lemma 2.5. *Let $\varrho \in]0, (\underline{\Lambda}\tilde{\mu})^{\frac{1}{p-2}}[$ and choose $\mu > 0$ large enough in such a way that*

$$\{u \in H_0^{1,2}(\Omega_t') : \int_{\Omega_t'} |Du|^2\, dx \le \mu^2,\ ||u||_{(\lambda,p)} = \varrho\} \ne \emptyset$$

for all $t \in \{1,\ldots,k\}$. Then there exist $B > 0$ and $\bar{\epsilon} > 0$ such that, if $w \in H_0^{1,2}(\Omega)$ with $||w|| \le B$ and $\epsilon \in]0,\bar{\epsilon}[$, we have

$$\inf\{f_\epsilon'(u+w)[u^+] : u \in H_0^{1,2}(\Omega_t'), \int_{\Omega_t'} |Du|^2\, dx \le \mu^2, ||u^+||_{(\lambda,p)} = \varrho\} > 0$$

and

$$\inf\{f_\epsilon'(u+w)[-u^-] : u \in H_0^{1,2}(\Omega_t'), \int_{\Omega_t'} |Du|^2\, dx \le \mu^2, ||u^-||_{(\lambda,p)} = \varrho\} > 0,$$

for all $t \in \{1,\ldots,k\}$.

Proof. Arguing by contradiction, assume that for all $B > 0$ and $\bar{\epsilon} > 0$ there exist $\epsilon \in]0,\bar{\epsilon}[$ and $w \in H_0^{1,2}(\Omega)$, with $||w|| \le B$, such that the assertion does not hold. Hence there exist a sequence $(\epsilon_n)_{n\ge1}$ of positive numbers, converging to zero, a sequence of functions $(w_n)_{n\ge1}$, converging to zero in $H_0^{1,2}(\Omega)$, and, for some $t \in \{1,\ldots,k\}$, a sequence of functions $(u_n)_{n\ge1}$ in $H_0^{1,2}(\Omega_t')$, with $\int_{\Omega_t'} |Du_n|^2\, dx \le \mu^2$ for all $n \ge 1$, such that

$$||u_n^+||_{(\lambda,p)} = \varrho \quad \forall n \ge 1 \quad \text{and} \quad \lim_{n\to\infty} f_{\epsilon_n}'(u_n+w_n)[u_n^+] \le 0 \tag{2.11}$$

or

$$||u_n^-||_{(\lambda,p)} = \varrho \quad \forall n \geq 1 \quad \text{and} \quad \lim_{n\to\infty} f'_{\epsilon_n}(u_n + w_n)[-u_n^-] \leq 0. \tag{2.12}$$

Let us consider, for example, the case that (2.11) holds (analogous arguments can be used in the case (2.12)).

Up to a subsequence, $u_n \to u \in H_0^{1,2}(\Omega_t')$, weakly in $H_0^{1,2}(\Omega_t')$, in $L^p(\Omega_t')$, in $L^q(\Omega_t')$ and a.e. in Ω_t', with $\int_{\Omega_t'} |Du|^2\, dx \leq \mu^2$ and $||u^+||_{(\lambda,p)} = \varrho$.

Moreover (2.11) and condition $(a.4)$ imply that

$$\int_{\Omega_t'\setminus\Omega_t} |Du^+|^2\, dx = 0. \tag{2.13}$$

Notice that condition $(g.3)$ implies

$$\begin{aligned} \lim_{n\to\infty} g_{\epsilon_n}(x, u_n + w_n)u_n^+ &= \lim_{n\to\infty} \gamma_{\epsilon_n}(x, u_n + w_n)|u_n + w_n|^{p-2}(u_n + w_n)u_n^+ \\ &= \lambda(x)(u^+)^p \quad \text{a.e. in} \quad \Omega_t'. \end{aligned} \tag{2.14}$$

Hence $(g.2)$ and $(g.3)$ (where we can assume $q \geq p$) allow us to apply the Lebesgue convergence theorem to obtain

$$\lim_{n\to\infty} \int_{\Omega_t'} g_{\epsilon_n}(x, u_n + w_n)u_n^+\, dx = \int_{\Omega_t'} \lambda(x)|u^+|^p\, dx. \tag{2.15}$$

Then, from (2.11), (2.13) and (2.15), we infer that

$$\begin{aligned} 0 &\geq \lim_{n\to\infty} f'_{\epsilon_n}(u_n + w_n)[u_n^+] \geq \underline{\Lambda} \int_{\Omega_t'} |Du^+|^2\, dx - \int_{\Omega_t'} \lambda(x)(u^+)^p\, dx \\ &\geq \underline{\Lambda}\tilde{\mu}||u^+||^2_{(\lambda,p)} - ||u^+||^p_{(\lambda,p)} = \underline{\Lambda}\tilde{\mu}\varrho^2 - \varrho^p > 0, \end{aligned}$$

where the last inequality is due to the choice of ϱ. Hence we get a contradiction. □

Definition 2.6. *For all $\epsilon > 0$, let Ψ_ϵ be the functional defined by*

$$\Psi_\epsilon(w,u) = \frac{1}{2}\int_\Omega g_\epsilon(x, w+u)u\,dx - \int_\Omega G_\epsilon(x,w+u)\,dx \quad \forall w,u \in H_0^{1,2}(\Omega).$$

Notice that Ψ_ϵ is a C^1 functional for all $\epsilon > 0$ and for all $w, u, v \in H_0^{1,2}(\Omega)$

$$\frac{\partial}{\partial u}\Psi_\epsilon(w,u)[v] = \frac{1}{2}\{\int_\Omega g_\epsilon'(x,w+u)uv\,dx - \int_\Omega g_\epsilon(x,w+u)v\,dx\}.$$

Lemma 2.7. *Let $\overline{\varrho} > 0$ and choose $\overline{\mu} > 0$ large enough in such a way that*

$$\{u \in H_0^{1,2}(\Omega) \, : \, ||u|| \leq \overline{\mu}, \, ||u||_{(\lambda,p)} \geq \overline{\varrho}\} \neq \emptyset.$$

Then there exist $B > 0$ and $\overline{\epsilon} > 0$ such that, if $w \in H_0^{1,2}(\Omega)$ with $||w|| \leq B$ and $\epsilon \in]0, \overline{\epsilon}[$, we have

$$\inf\{\frac{\partial}{\partial u}\Psi_\epsilon(w,u)[u^+] \, : \, u \in H_0^{1,2}(\Omega), \, ||u|| \leq \overline{\mu}, \, ||u^+||_{(\lambda,p)} \geq \overline{\varrho}\} > 0$$

and

$$\inf\{\frac{\partial}{\partial u}\Psi_\epsilon(w,u)[-u^-] \, : \, u \in H_0^{1,2}(\Omega), \, ||u|| \leq \overline{\mu}, \, ||u^-||_{(\lambda,p)} \geq \overline{\varrho}\} > 0.$$

Proof. Suppose, by contradiction, that for all $B > 0$ and $\bar{\epsilon} > 0$ there exist $\epsilon \in]0, \bar{\epsilon}[$ and $w \in H_0^{1,2}(\Omega)$, with $||w|| \leq B$, such that the first or the second inequality do not hold. Hence there exist a sequence $(\epsilon_n)_{n\geq 1}$ of positive numbers, converging to zero, a sequence $(w_n)_{n\geq 1}$ of functions in $H_0^{1,2}(\Omega)$, converging to zero in $H_0^{1,2}(\Omega)$, and a sequence of functions $(u_n)_{n\geq 1}$ in $H_0^{1,2}(\Omega)$, with $||u_n|| \leq \overline{\mu}$ for all $n \geq 1$, such that

$$||u_n^+||_{(\lambda,p)} \geq \overline{\varrho} \quad \forall n \geq 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\partial}{\partial u} \Psi_{\epsilon_n}(w_n, u_n)[u_n^+] \leq 0 \tag{2.16}$$

or

$$||u_n^-||_{(\lambda,p)} \geq \overline{\varrho} \quad \forall n \geq 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\partial}{\partial u} \Psi_{\epsilon_n}(w_n, u_n)[-u_n^-] \leq 0. \tag{2.17}$$

Let us consider the case (2.16) (analogous arguments hold in the case (2.17)). Up to a subsequence, $u_n \to u \in H_0^{1,2}(\Omega)$ weakly in $H_0^{1,2}(\Omega)$, in $L^p(\Omega)$, in $L^q(\Omega)$ and a.e. in Ω, with $||u|| \leq \overline{\mu}$ and $||u^+||_{(\lambda,p)} \geq \overline{\varrho}$.
By $(g.3)$, we have, for almost all $x \in \Omega$,

$$\begin{aligned} \lim_{n\to\infty} g_{\epsilon_n}(x, w_n + u_n)u_n^+ &= \lim_{n\to\infty} \gamma_{\epsilon_n}(x, w_n + u_n)|w_n + u_n|^{p-2}(w_n + u_n)u_n^+ \\ &= \lambda(x)(u^+)^p, \end{aligned} \tag{2.18}$$

$$\begin{aligned} \lim_{n\to\infty} g'_{\epsilon_n}(x, w_n + u_n)u_n u_n^+ &= \lim_{n\to\infty} \tilde{\gamma}_{\epsilon_n}(x, w_n + u_n)|w_n + u_n|^{p-2}u_n u_n^+ \\ &= (p-1)\lambda(x)(u^+)^p. \end{aligned} \tag{2.19}$$

Taking into account $(g.2)$ and $(g.3)$, we can apply the Lebesgue convergence theorem and obtain

$$\lim_{n\to\infty}\frac{\partial}{\partial u}\Psi_{\epsilon_n}(w_n,u_n)[u_n^+]=\frac{1}{2}(p-2)||u^+||^p_{(\lambda,p)}\geq\frac{p-2}{2}\overline{\varrho}^p>0,$$

which contradicts (2.16). □

3. Local minima and proof of the main result

In this section T^+ and T^- are fixed subsets of $\{1,\dots,k\}$.

Definition 3.1. *For all $\varrho>0$, let us set*

$$K_{\epsilon,\varrho}^{T^+,T^-}=\{u\in H_0^{1,2}(\Omega):||(P_t^\epsilon u)^+||_{(\lambda,p)}\geq\varrho\quad\forall t\in T^+,\\ ||(P_t^\epsilon u)^-||_{(\lambda,p)}\geq\varrho\quad\forall t\in T^-\}.$$

Definition 3.2. *For all $B>0$, let us define*

$$N_{\epsilon,B}^{T^+,T^-}=\{u\in H_0^{1,2}(\Omega):\int_\Omega|DP^\epsilon u|^2\,dx+\sum_{t\notin T^+}\int_{\Omega_t'}|D(P_t^\epsilon u)^+|^2\,dx\\ +\sum_{t\notin T^-}\int_{\Omega_t'}|D(P_t^\epsilon u)^-|^2\,dx\leq B^2\}$$

and

$$\partial N_{\epsilon,B}^{T^+,T^-}=\{u\in H_0^{1,2}(\Omega):\int_\Omega|DP^\epsilon u|^2\,dx+\sum_{t\notin T^+}\int_{\Omega_t'}|D(P_t^\epsilon u)^+|^2\,dx\\ +\sum_{t\notin T^-}\int_{\Omega_t'}|D(P_t^\epsilon u)^-|^2\,dx=B^2\}.$$

Definition 3.3. *For all $\epsilon > 0$, let $Q_\epsilon : H_0^{1,2}(\Omega) \to \mathbb{R}$ be defined by*

$$Q_\epsilon(u) = \int_\Omega \frac{a_\epsilon^{i,j}}{\epsilon} \partial_{x_i} u \partial_{x_j} u \, dx.$$

Definition 3.4. *For all $t \in \{1, \ldots, k\}$, let us choose a function $v_t \in H_0^{1,2}(\Omega_t)$ such that $||v_t^+||_{(\lambda,p)} = ||v_t^-||_{(\lambda,p)} = 1$ (we consider v_t extended in Ω by setting $v_t = 0$ in $\Omega \setminus \Omega_t$).*
Then we put

$$M = \sum_{t=1}^{k} \left(||v_t^+||^{\frac{2p}{p-2}} + ||v_t^-||^{\frac{2p}{p-2}} \right).$$

Let us remark that, for all $t \in \{1, \ldots, k\}$, $||v_t^+|| \geq \mu_t$ and $||v_t^-|| \geq \mu_t$ (see Definition 2.3).

Proposition 3.5. *For all $t \in \{1, \ldots, k\}$, let v_t be the function introduced in Definition 3.4. Then, for all $\epsilon > 0$, there exist some positive numbers $\alpha_{\epsilon,t}$, for $t \in T^+$, and $\beta_{\epsilon,t}$, for $t \in T^-$, such that*

$$v_\epsilon = \sum_{t \in T^+} \alpha_{\epsilon,t} v_t^+ - \sum_{t \in T^-} \beta_{\epsilon,t} v_t^- \in M_\epsilon^{T^+,T^-}$$

(see Definition 3.4). Moreover, we have

$$(\underline{\Lambda} ||v_t^+||^2)^{\frac{1}{p-2}} \leq \liminf_{\epsilon \to 0} \alpha_{\epsilon,t} \leq \limsup_{\epsilon \to 0} \alpha_{\epsilon,t} \leq (\overline{\Lambda} ||v_t^+||^2)^{\frac{1}{p-2}} \quad \forall t \in T^+$$

and

$$(\underline{\Lambda} ||v_t^-||^2)^{\frac{1}{p-2}} \leq \liminf_{\epsilon \to 0} \beta_{\epsilon,t} \leq \limsup_{\epsilon \to 0} \beta_{\epsilon,t} \leq (\overline{\Lambda} ||v_t^-||^2)^{\frac{1}{p-2}} \quad \forall t \in T^-.$$

Proof. For all $t \in T^+$, let us consider the mapping $z \in \mathbb{R}^+ \to f_\epsilon(zv_t^+)$. Because of $(g.3)$ and $(g.4)$, this mapping has a local minimum in $z = 0$ and $\lim_{z\to+\infty} f_\epsilon(zv_t^+) = -\infty$. Then there exists a maximum point $\alpha_{\epsilon,t} > 0$ such that $f_\epsilon(\alpha_{\epsilon,t}v_t^+) > 0$ and $f_\epsilon'(\alpha_{\epsilon,t}v_t^+)[\alpha_{\epsilon,t}v_t^+] = 0$.

In an analogous way, if we consider, for all $t \in T^-$, the mapping $z \in \mathbb{R}^+ \to f_\epsilon(-zv_t^-)$, we find a maximum point $\beta_{\epsilon,t} > 0$ such that $f_\epsilon(-\beta_{\epsilon,t}v_t^-) > 0$ and $f_\epsilon'(-\beta_{\epsilon,t}v_t^-)[\beta_{\epsilon,t}v_t^-] = 0$.

Now let us consider $v_\epsilon = \sum_{t\in T^+} \alpha_{\epsilon,t}v_t^+ - \sum_{t\in T^-} \beta_{\epsilon,t}v_t^-$.

Taking into account the properties of $\alpha_{\epsilon,t}$ and $\beta_{\epsilon,t}$, it is easy to verify that

$$v_\epsilon \in M_\epsilon^{T^+,T^-}.$$

Let us show that $\lim_{\epsilon\to 0} \epsilon^{\frac{1}{p-2}}\alpha_{\epsilon,t} = 0$, for all $t \in T^+$; in fact, by contradiction, suppose that there exists a sequence $(\epsilon_n)_{n\geq 1}$ of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$ and $\lim_{n\to\infty} \epsilon_n^{\frac{1}{p-2}}\alpha_{\epsilon_n,t} = \alpha_t > 0$ for some $t \in T^+$. Then, since under our assumptions $G(x,t) > 0$ for $t > 0$, we get

$$\limsup_{n\to\infty} f_{\epsilon_n}(\alpha_{\epsilon_n,t}v_t^+) \leq \limsup_{n\to\infty} \frac{1}{\epsilon_n^{\frac{2}{p-2}}}\Big\{\alpha_t^2\frac{\overline{\Lambda}||v_t^+||^2}{2} - \frac{1}{\epsilon_n}\int_{\Omega_t} G(x,\alpha_t v_t^+)\,dx\Big\} = -\infty,$$

which is in contradiction with the fact that $f_{\epsilon_n}(\alpha_{\epsilon_n,t}v_t^+) > 0$ for all $n \geq 1$.

Since $\lim_{\epsilon\to 0} \epsilon^{\frac{1}{p-2}}\alpha_{\epsilon,t} = 0$ for all $t \in T^+$ and since $(g.2)$, $(g.3)$ allow us to apply the Lebesgue convergence theorem, it follows that

$$\lim_{\epsilon\to 0} \frac{1}{\alpha_{\epsilon,t}{}^p}\int_{\Omega_t'} g_\epsilon(x,\alpha_{\epsilon,t}v_t^+)\alpha_{\epsilon,t}v_t^+\,dx = \lim_{\epsilon\to 0}\int_{\Omega_t'} \gamma_\epsilon(x,\alpha_{\epsilon,t}v_t^+)(v_t^+)^p\,dx = ||v_t^+||^p_{(\lambda,p)} = 1$$

Moreover, $f_\epsilon'(\alpha_{\epsilon,t}v_t^+)[\alpha_{\epsilon,t}v_t^+] = 0$ means that

$$\frac{1}{\alpha_{\epsilon,t}{}^p}\int_{\Omega_t} g_\epsilon(x,\alpha_{\epsilon,t}v_t^+)\alpha_{\epsilon,t}v_t^+\,dx = \frac{1}{\alpha_{\epsilon,t}{}^{p-2}}\int_{\Omega_t}\frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i}v_t^+\partial_{x_j}v_t^+\,dx \quad \forall t\in T^+.$$

Thus, taking into account Definition 2.3, we obtain for all $t \in T^+$

$$(\underline{\Lambda}||v_t^+||^2)^{\frac{1}{p-2}} \le \liminf_{\epsilon\to 0}\alpha_{\epsilon,t} \le \limsup_{\epsilon\to 0}\alpha_{\epsilon,t} \le (\overline{\Lambda}||v_t^+||^2)^{\frac{1}{p-2}}.$$

In analogous way one can prove that, for all $t\in T^-$, $\lim_{\epsilon\to 0}\epsilon^{\frac{1}{p-2}}\beta_{\epsilon,t}=0$ and

$$(\underline{\Lambda}||v_t^-||^2)^{\frac{1}{p-2}} \le \liminf_{\epsilon\to 0}\beta_{\epsilon,t} \le \limsup_{\epsilon\to 0}\beta_{\epsilon,t} \le (\overline{\Lambda}||v_t^-||^2)^{\frac{1}{p-2}}.$$

□

Lemma 3.6. *Let $\varrho\in]0,(\underline{\Lambda}\tilde{\mu})^{\frac{1}{p-2}}[$. Then there exists $\overline{\epsilon}>0$ such that*

$$\{u\in M_\epsilon^{T^+,T^-}\cap K_{\epsilon,\varrho}^{T^+,T^-} : (P_t^\epsilon u)^+\equiv 0\ \forall t\notin T^+,\ (P_t^\epsilon u)^-\equiv 0\ \forall t\notin T^-,\ \text{and}\quad P^\epsilon u\equiv 0\}\neq\emptyset \quad \forall\epsilon\in]0,\overline{\epsilon}[;$$

moreover

$$\limsup_{\epsilon\to 0}\inf\{f_\epsilon(u) : u\in M_\epsilon^{T^+,T^-}\cap K_{\epsilon,\varrho}^{T^+,T^-}\ (P_t^\epsilon u)^+\equiv 0\ \forall t\notin T^+,\ (P_t^\epsilon u)^-\equiv 0\ \forall t\notin T^-,\ \text{and}\quad P^\epsilon u\equiv 0\} \le (\frac{1}{2}-\frac{1}{p})\overline{\Lambda}^{\frac{p}{p-2}}M$$

(see Definitions 2.3, 3.1 and 3.4).

Proof. Because of the definition of $\tilde{\mu}$ and since $\varrho<(\underline{\Lambda}\tilde{\mu})^{\frac{1}{p-2}}$, Proposition 3.5 implies that $v_\epsilon\in K_{\epsilon,\varrho}^{T^+,T^-}$ for all $\epsilon>0$ small enough.

Let us now consider a sequence $(\epsilon_n)_{n\ge 1}$ of positive numbers such that $\lim_{n\to\infty}\epsilon_n=0$ and

$$\limsup_{\epsilon\to 0}\inf\{f_\epsilon(u)\ : u\in M_\epsilon^{T^+,T^-}\cap K_{\epsilon,\varrho}^{T^+,T^-},\ (P_t^\epsilon u)^+\equiv 0\ \forall t\notin T^+,$$
$$(P_t^\epsilon u)^-\equiv 0\ \forall t\notin T^-,\ \text{and}\quad P^\epsilon u\equiv 0\}$$
$$=\lim_{n\to\infty}\inf\{f_{\epsilon_n}(u)\ : u\in M_{\epsilon_n}^{T^+,T^-}\cap K_{\epsilon_n,\varrho}^{T^+,T^-},\ (P_t^{\epsilon_n} u)^+\equiv 0\ \forall t\notin T^+,$$
$$(P_t^{\epsilon_n} u)^-\equiv 0\ \forall t\notin T^-,\ \text{and}\quad P^{\epsilon_n} u\equiv 0\}. \tag{3.1}$$

From Proposition 3.5 we infer that, for all n large enough, there exist $\alpha_{\epsilon_n,t}$, for $t\in T^+$,and $\beta_{\epsilon_n,t}$, for $t\in T^-$, such that

$$v_{\epsilon_n}=\sum_{t\in T^+}\alpha_{\epsilon_n,t}v_t^+-\sum_{t\in T^-}\beta_{\epsilon_n,t}v_t^-\in M_{\epsilon_n}^{T^+,T^-}\cap K_{\epsilon_n,\varrho}^{T^+,T^-}$$

(see Definition 3.4).

Moreover, by construction, we have that $(P_t^{\epsilon_n}v_{\epsilon_n})^+\equiv 0$ for all $t\notin T^+$, $(P_t^{\epsilon_n}v_{\epsilon_n})^-\equiv 0$ for all $t\notin T^-$ and $P^{\epsilon_n}v_{\epsilon_n}\equiv 0$.

Arguing as in the proof of Proposition 3.5, one can show that $\lim_{n\to\infty}\epsilon_n^{\frac{1}{p-2}}\alpha_{\epsilon_n,t}=0$ for all $t\in T^+$, $\lim_{n\to\infty}\epsilon_n^{\frac{1}{p-2}}\beta_{\epsilon_n,t}=0$ for all $t\in T^-$ and, up to a subsequence,

$$\lim_{n\to\infty}\alpha_{\epsilon_n,t}^{p-2}=\lim_{n\to\infty}\int_{\Omega_t}\frac{a_{\epsilon_n}^{i,j}}{\epsilon_n}\partial_{x_i}v_t^+\partial_{x_j}v_t^+\,dx\quad\forall t\in T^+, \tag{3.2}$$

$$\lim_{n\to\infty}\beta_{\epsilon_n,t}^{p-2}=\lim_{n\to\infty}\int_{\Omega_t}\frac{a_{\epsilon_n}^{i,j}}{\epsilon_n}\partial_{x_i}v_t^-\partial_{x_j}v_t^-\,dx\quad\forall t\in T^-. \tag{3.3}$$

Moreover we have, for all $t\in T^+$,

$$\lim_{n\to\infty}\frac{1}{\alpha^p_{\epsilon_n,t}}G_{\epsilon_n}(x,\alpha_{\epsilon_n,t}v_t^+)=\lim_{n\to\infty}\Gamma_{\epsilon_n}(x,\alpha_{\epsilon_n,t}v_t^+)(v_t^+)^p$$
$$=\frac{\lambda(x)}{p}(v_t^+(x))^p \quad \text{a.e. in} \quad \Omega_t'$$

and, for all $t\in T^-$,

$$\lim_{n\to\infty}\frac{1}{\beta^p_{\epsilon_n,t}}G_{\epsilon_n}(x,\beta_{\epsilon_n,t}v_t^-)=\lim_{n\to\infty}\Gamma_{\epsilon_n}(x,\beta_{\epsilon_n,t}v_t^-)(v_t^-)^p$$
$$=\frac{\lambda(x)}{p}(v_t^-(x))^p \quad \text{a.e. in} \quad \Omega_t'.$$

Taking into account that $(g.2)$ and $(g.3)$ allow us to apply the Lebesgue convergence theorem, (3.1), (3.2) and (3.3) imply (see Definition 3.3)

$$\begin{aligned}
&\limsup_{\epsilon\to 0}\inf\{f_\epsilon(u)\ :\ u\in M_\epsilon^{T^+,T^-}\cap K_{\epsilon,\varrho}^{T^+,T^-},\ (P_t^\epsilon u)^+\equiv 0\ \forall t\notin T^+,\\
&(P_t^\epsilon u)^-\equiv 0\quad \forall t\notin T^-,\ P^\epsilon u\equiv 0\}\\
&\le \lim_{n\to\infty} f_{\epsilon_n}(v_{\epsilon_n})\\
&=\lim_{n\to\infty}\{\sum_{t\in T^+}\{\frac{\alpha^2_{\epsilon_n,t}}{2}\int_{\Omega_t}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}v_t^+\partial_{x_j}v_t^+\,dx-\int_{\Omega_t}G_{\epsilon_n}(x,\alpha_{\epsilon_n,t}v_t^+)\,dx\}\\
&+\sum_{t\in T^-}\{\frac{\beta^2_{\epsilon_n,t}}{2}\int_{\Omega_t}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}v_t^-\partial_{x_j}v_t^-\,dx-\int_{\Omega_t}G_{\epsilon_n}(x,-\beta_{\epsilon_n,t}v_t^-)\,dx\}\}\\
&=(\frac{1}{2}-\frac{1}{p})\lim_{n\to\infty}\{\sum_{t\in T^+}[Q_{\epsilon_n}(v_t^+)]^{\frac{p}{p-2}}+\sum_{t\in T^-}[Q_{\epsilon_n}(v_t^-)]^{\frac{p}{p-2}}\}\\
&\le(\frac{1}{2}-\frac{1}{p})\limsup_{\epsilon\to 0}\{\sum_{t\in T^+}[Q_\epsilon(v_t^+)]^{\frac{p}{p-2}}+\sum_{t\in T^-}[Q_\epsilon(v_t^-)]^{\frac{p}{p-2}}\}\\
&\le(\frac{1}{2}-\frac{1}{p})\{\sum_{t\in T^+}(\overline{\Lambda}\|v_t^+\|^2)^{\frac{p}{p-2}}+\sum_{t\in T^-}(\overline{\Lambda}\|v_t^-\|^2)^{\frac{p}{p-2}}\}\\
&=(\frac{1}{2}-\frac{1}{p})\overline{\Lambda}^{\frac{p}{p-2}}\{\sum_{t\in T^+}\|v_t^+\|^{\frac{2p}{p-2}}+\sum_{t\in T^-}\|v_t^-\|^{\frac{2p}{p-2}}\}\\
&\le(\frac{1}{2}-\frac{1}{p})\overline{\Lambda}^{\frac{p}{p-2}}M.
\end{aligned}$$

□

Corollary 3.7. *Let $B > 0$ and $\varrho \in]0, (\underline{\Lambda}\tilde{\mu})^{\frac{1}{p-2}}[$.*
Then we have

$$\limsup_{\epsilon \to 0} \inf\{f_\epsilon(u) \,:\, u \in M_\epsilon^{T^+,T^-} \cap K_{\epsilon,\varrho}^{T^+,T^-} \cap N_{\epsilon,B}^{T^+,T^-}\}$$
$$\leq (\frac{1}{2} - \frac{1}{p})\overline{\Lambda}^{\frac{p}{p-2}} M$$

(see Definitions 2.3, 3.2 and 3.4).

Lemma 3.8. *Let $\varrho \in]0, (\underline{\Lambda}\tilde{\mu})^{\frac{1}{p-2}}[$. Then there exist $\overline{\epsilon} > 0$ and $B > 0$ such that, for all $\epsilon \in]0, \overline{\epsilon}[$,*

$$\inf\{f_\epsilon(u) \,:\, u \in M_\epsilon^{T^+,T^-} \cap K_{\epsilon,\varrho}^{T^+,T^-} \cap N_{\epsilon,B}^{T^+,T^-}\}$$

is achieved by a function u_ϵ, which satisfies $\sup_{0<\epsilon<\overline{\epsilon}} ||u_\epsilon|| < +\infty$.

Proof. Let $B > 0$ and choose $\overline{\epsilon}_1 > 0$ so small that

$$M_\epsilon^{T^+,T^-} \cap K_{\epsilon,\varrho}^{T^+,T^-} \cap N_{\epsilon,B}^{T^+,T^-} \neq \emptyset \quad \forall \epsilon \in]0, \overline{\epsilon}_1[\tag{3.4}$$

(see Lemma 3.6),

$$\inf\{\frac{\Lambda_1(\epsilon, x)}{\epsilon} \,:\, x \in \Omega\} \geq \frac{\underline{\Lambda}}{2} \quad \forall \epsilon \in]0, \overline{\epsilon}_1[\tag{3.5}$$

and

$$\sup_{\epsilon \in]0,\overline{\epsilon}_1[} \inf\{f_\epsilon(u) \,:\, u \in M_\epsilon^{T^+,T^-} \cap K_{\epsilon,\varrho}^{T^+,T^-} \cap N_{\epsilon,B}^{T^+,T^-}\} < +\infty \tag{3.6}$$

(see Corollary 3.7).

For all $\epsilon \in]0, \bar{\epsilon}_1[$, let $(u_n^\epsilon)_{n\geq 1}$ be a sequence of functions in $M_\epsilon^{T^+,T^-} \cap K_{\epsilon,\varrho}^{T^+,T^-} \cap N_{\epsilon,B}^{T^+,T^-}$ such that

$$\lim_{n\to\infty} f_\epsilon(u_n^\epsilon) = \inf\{f_\epsilon(u) \,:\, u \in M_\epsilon^{T^+,T^-} \cap K_{\epsilon,\varrho}^{T^+,T^-} \cap N_{\epsilon,B}^{T^+,T^-}\}.$$

First we prove that $(u_n^\epsilon)_{n\geq 1}$ is bounded in $H_0^{1,2}(\Omega)$.

Taking into account that $u_n^\epsilon \in N_{\epsilon,B}^{T^+,T^-}$ for all $n \geq 1$, we have that the sets $\{P^\epsilon u_n^\epsilon \,:\, n \geq 1, \epsilon \in]0, \bar{\epsilon}_1[\}$, $\{(P_t^\epsilon u_n^\epsilon)^+ \,:\, n \geq 1, \epsilon \in]0, \bar{\epsilon}_1[\}$, for $t \notin T^+$, and $\{(P_t^\epsilon u_n^\epsilon)^- \,:\, n \geq 1, \epsilon \in]0, \bar{\epsilon}_1[\}$, for $t \notin T^-$, are bounded in $H_0^{1,2}(\Omega)$. Hence $(g.2)$ implies

$$\begin{aligned} f_\epsilon(u_n^\epsilon) &= \frac{1}{2}\int_\Omega \frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i} u_n^\epsilon \partial_{x_j} u_n^\epsilon \, dx - \int_\Omega G_\epsilon(x, u_n^\epsilon)\, dx \\ &\geq \sum_{t\in T^+} [\frac{1}{2}\int_{\Omega_t'} \frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i}(P_t^\epsilon u_n^\epsilon)^+ \partial_{x_j}(P_t^\epsilon u_n^\epsilon)^+ \, dx \\ &\quad - \int_{\Omega_t'} G_\epsilon(x, (P_t^\epsilon u_n^\epsilon)^+ + P^\epsilon u_n^\epsilon)\, dx] \\ &\quad + \sum_{t\in T^-} [\frac{1}{2}\int_{\Omega_t'} \frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i}(P_t^\epsilon u_n^\epsilon)^- \partial_{x_j}(P_t^\epsilon u_n^\epsilon)^- \, dx \\ &\quad - \int_{\Omega_t'} G_\epsilon(x, -(P_t^\epsilon u_n^\epsilon)^- + P^\epsilon u_n^\epsilon)\, dx] - C_0 \end{aligned} \tag{3.7}$$

for a suitable constant $C_0 \geq 0$.

For all $t \in T^+$, since $u_n^\epsilon \in M_\epsilon^{T^+,T^-}$ and since $(g.4)$ holds, we have

$$\frac{1}{2}\int_{\Omega_t'} \frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i}(P_t^\epsilon u_n^\epsilon)^+ \partial_{x_j}(P_t^\epsilon u_n^\epsilon)^+ \, dx - \int_{\Omega_t'} G_\epsilon(x, (P_t^\epsilon u_n^\epsilon)^+ + P^\epsilon u_n^\epsilon)\, dx$$

$$\geq \frac{1}{2}\int_{\Omega'_t} \frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i}(P_t^\epsilon u_n^\epsilon)^+\partial_{x_j}(P_t^\epsilon u_n^\epsilon)^+\,dx$$

$$-\theta\int_{\Omega'_t} g_\epsilon(x,(P_t^\epsilon u_n^\epsilon)^+ + P^\epsilon u_n^\epsilon)[(P_t^\epsilon u_n^\epsilon)^+ + P^\epsilon u_n^\epsilon]\,dx$$

$$=\frac{1}{2}(\frac{1}{2}-\theta)\int_{\Omega'_t} \frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i}(P_t^\epsilon u_n^\epsilon)^+\partial_{x_j}(P_t^\epsilon u_n^\epsilon)^+\,dx$$

$$+\int_{\Omega'_t} g_\epsilon(x,(P_t^\epsilon u_n^\epsilon)^+ + P^\epsilon u_n^\epsilon)[\frac{1}{2}(\frac{1}{2}-\theta)(P_t^\epsilon u_n^\epsilon)^+ - \theta P^\epsilon u_n^\epsilon]\,dx. \tag{3.8}$$

If we set $\Omega_n^1 = \{x\in\Omega : -P^\epsilon u_n^\epsilon(x) \leq (P_t^\epsilon u_n^\epsilon)^+(x) \leq \frac{4\theta}{1-2\theta}P^\epsilon u_n^\epsilon(x)\}$ and $\Omega_n^2 = \{x\in\Omega : \frac{4\theta}{1-2\theta}P^\epsilon u_n^\epsilon(x) \leq (P_t^\epsilon u_n^\epsilon)^+(x) \leq -P^\epsilon u_n^\epsilon(x)\}$, since under our assumptions $g_\epsilon(x,s)\geq 0$ for all $s\geq 0$ and $g_\epsilon(x,s)\leq 0$ for all $s\leq 0$, we obtain

$$\int_{\Omega'_t} g_\epsilon(x,(P_t^\epsilon u_\epsilon^n)^+ + P^\epsilon u_n^\epsilon)[\frac{1}{2}(\frac{1}{2}-\theta)(P_t^\epsilon u_n^\epsilon)^+ - \theta P^\epsilon u_n^\epsilon]\,dx$$

$$\geq \sum_{i=1}^{2}\int_{\Omega'_t\cap\Omega_n^i} g_\epsilon(x,(P_t^\epsilon u_\epsilon^n)^+ + P^\epsilon u_n^\epsilon)[\frac{1}{2}(\frac{1}{2}-\theta)(P_t^\epsilon u_n^\epsilon)^+ - \theta P^\epsilon u_n^\epsilon]\,dx$$

$$\geq -C_1 \quad \forall t\in T^+ \tag{3.9}$$

for a suitable constant $C_1>0$, as $(P^\epsilon u_n^\epsilon)_{n\geq 1}$ is bounded in $H_0^{1,2}(\Omega)$ and $(g.2)$ holds.

Hence from (3.8) and (3.9) it follows that there exists a suitable constant $\overline{C}>0$ such that, for all $t\in T^+$,

$$\frac{1}{2}\int_{\Omega'_t} \frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i}(P_t^\epsilon u_n^\epsilon)^+\partial_{x_j}(P_t^\epsilon u_n^\epsilon)^+\,dx - \int_{\Omega'_t} G_\epsilon(x,(P_t^\epsilon u_n^\epsilon)^+ + P^\epsilon u_n^\epsilon)\,dx$$

$$\geq \frac{1}{\overline{C}}||(P_t^\epsilon u_n^\epsilon)^+||^2 - \overline{C}. \tag{3.10}$$

In analogous way, for all $t \in T^-$, we have

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega_t'} \frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i}(P_t^\epsilon u_n^\epsilon)^-\partial_{x_j}(P_t^\epsilon u_n^\epsilon)^-\,dx - \int_{\Omega_t'} G_\epsilon(x, -(P_t^\epsilon u_n^\epsilon)^- + P^\epsilon u_n^\epsilon)\,dx \\
&\geq \frac{1}{2}\int_{\Omega_t'} \frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i}(P_t^\epsilon u_n^\epsilon)^-\partial_{x_j}(P_t^\epsilon u_n^\epsilon)^-\,dx \\
&-\theta\int_{\Omega_t'} g_\epsilon(x, -(P_t^\epsilon u_n^\epsilon)^- + P^\epsilon u_n^\epsilon)[-(P_t^\epsilon u_n^\epsilon)^- + P^\epsilon u_n^\epsilon]\,dx \\
&= \frac{1}{2}(\frac{1}{2}-\theta)\int_{\Omega_t'} \frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i}(P_t^\epsilon u_n^\epsilon)^-\partial_{x_j}(P_t^\epsilon u_n^\epsilon)^-\,dx \qquad (3.11)\\
&-\int_{\Omega_t'} g_\epsilon(x, -(P_t^\epsilon u_\epsilon^n)^- + P^\epsilon u_n^\epsilon)[\frac{1}{2}(\frac{1}{2}-\theta)(P_t^\epsilon u_n^\epsilon)^- + \theta P^\epsilon u_n^\epsilon]\,dx.
\end{aligned}
$$

If we set $\Omega_n^3 = \{x \in \Omega : -P^\epsilon u_n^\epsilon(x) \leq -(P_t^\epsilon u_n^\epsilon)^-(x) \leq \frac{4\theta}{1-2\theta}P^\epsilon u_n^\epsilon(x)\}$ and $\Omega_n^4 = \{x \in \Omega : \frac{4\theta}{1-2\theta}P^\epsilon u_n^\epsilon(x) \leq -(P_t^\epsilon u_n^\epsilon)^-(x) \leq -P^\epsilon u_n^\epsilon(x)\}$, arguing as before, we find a positive constant C_2 such that

$$
\begin{aligned}
&-\int_{\Omega_t'} g_\epsilon(x, -(P_t^\epsilon u_\epsilon^n)^- + P^\epsilon u_n^\epsilon)[\frac{1}{2}(\frac{1}{2}-\theta)(P_t^\epsilon u_n^\epsilon)^- + \theta P^\epsilon u_n^\epsilon]\,dx \\
&\geq -\sum_{i=3}^{4}\int_{\Omega_t'\cap\Omega_n^i} g_\epsilon(x, -(P_t^\epsilon u_\epsilon^n)^- + P^\epsilon u_n^\epsilon)[\frac{1}{2}(\frac{1}{2}-\theta)(P_t^\epsilon u_n^\epsilon)^- + \theta P^\epsilon u_n^\epsilon]\,dx \\
&\geq -C_2. \qquad (3.12)
\end{aligned}
$$

From (3.11) and (3.12) it follows that there exists a constant $\tilde{C} > 0$ such that, for all $t \in T^-$,

$$
\frac{1}{2}\int_{\Omega_t'} \frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i}(P_t^\epsilon u_n^\epsilon)^-\partial_{x_j}(P_t^\epsilon u_n^\epsilon)^-\,dx - \int_{\Omega_t'} G_\epsilon(x, -(P_t^\epsilon u_n^\epsilon)^- + P^\epsilon u_n^\epsilon)\,dx
$$

$$\geq \frac{1}{\tilde{C}}||(P_t^\epsilon u_n^\epsilon)^-||^2 - \tilde{C}. \qquad (3.13)$$

Taking into account (3.7), (3.10) and (3.13), we can find a positive constant $C > 0$ such that

$$f_\epsilon(u_n^\epsilon) \geq \frac{1}{C}||u_n^\epsilon||^2 - C \quad \forall n \geq 1, \forall \epsilon \in]0, \overline{\epsilon}_1[. \qquad (3.14)$$

Because of (3.6), $(u_n^\epsilon)_{n\geq 1}$ is bounded in $H_0^{1,2}(\Omega)$ for all $\epsilon \in]0, \overline{\epsilon}_1[$; hence, up to a subsequence, $(u_n^\epsilon)_{n\geq 1}$ converges to $u_\epsilon \in H_0^{1,2}(\Omega)$ weakly in $H_0^{1,2}(\Omega)$, in $L^p(\Omega)$, in $L^q(\Omega)$ and a.e. in Ω. So we have $u_\epsilon \in K_{\epsilon,\varrho}^{T^+,T^-} \cap N_{\epsilon,B}^{T^+,T^-}$, $f_\epsilon'(u_\epsilon)[(P_t^\epsilon u_\epsilon)^+] \leq 0$ for all $t \in T^+$, $f_\epsilon'(u_\epsilon)[-(P_t^\epsilon u_\epsilon)^-] \leq 0$ for all $t \in T^-$ and

$$f_\epsilon(u_\epsilon) \leq \lim_{n\to\infty} f_\epsilon(u_n^\epsilon) = \inf\{f_\epsilon(u) \, : \, u \in M_\epsilon^{T^+,T^-} \cap K_{\epsilon,\varrho}^{T^+,T^-} \cap N_{\epsilon,B}^{T^+,T^-}\}. \qquad (3.15)$$

Furthermore, from (3.6) and (3.14) we get

$$\sup_{0<\epsilon<\overline{\epsilon}_1} ||(P_t^\epsilon u_\epsilon)^+|| < +\infty \quad \forall t \in T^+ \qquad (3.16)$$

and

$$\sup_{0<\epsilon<\overline{\epsilon}_1} ||(P_t^\epsilon u_\epsilon)^-|| < +\infty \quad \forall t \in T^-. \qquad (3.17)$$

Now choose $\overline{\varrho} \in]0, \varrho[$ and $\mu, \overline{\mu}$ greater than $\max_{t\in T^+} \sup_{0<\epsilon<\overline{\epsilon}_1} ||(P_t^\epsilon u_\epsilon)^+||$ and $\max_{t\in T^-} \sup_{0<\epsilon<\overline{\epsilon}_1} ||(P_t^\epsilon u_\epsilon)^-||$; then we can fix $\overline{\epsilon} \in]0, \overline{\epsilon}_1[$ and $B > 0$ such that the assertions of Lemma 2.5 and Lemma 2.7 hold.

In order to prove that, for all $\epsilon \in]0, \overline{\epsilon}[$, u_ϵ realizes

$$\inf\{f_\epsilon(u) \,:\, u \in M_\epsilon^{T^+,T^-} \cap K_{\epsilon,\varrho}^{T^+,T^-} \cap N_{\epsilon,B}^{T^+,T^-}\},$$

it remains to show that $u_\epsilon \in M_\epsilon^{T^+,T^-}$.

Denote by I^+ the subset of all $t \in T^+$ such that $f_\epsilon'(u_\epsilon)[(P_t^\epsilon u_\epsilon)^+] < 0$ and by I^- the subset of all $t \in T^-$ such that $f_\epsilon'(u_\epsilon)[-(P_t^\epsilon u_\epsilon)^-] < 0$. Arguing by contradiction, suppose that $I^+ \neq \emptyset$ or $I^- \neq \emptyset$.

For all $t \in I^+$, since $||(P_t^\epsilon u_\epsilon)^+||_{(\lambda,p)} \geq \varrho$, there exists $\overline{\xi}_t \in]0,1]$ such that $||\overline{\xi}_t(P_t^\epsilon u_\epsilon)^+||_{(\lambda,p)} = \varrho$; taking into account that (3.16) holds, that $||P^\epsilon u_\epsilon|| \leq B$ and that $||\overline{\xi}_t(P_t^\epsilon u_\epsilon)^+|| \leq \mu$, Lemma 2.5 implies $f_\epsilon'(P^\epsilon u_\epsilon + \overline{\xi}_t(P_t^\epsilon u_\epsilon)^+)$ $[\overline{\xi}_t(P_t^\epsilon u_\epsilon)^+] > 0$. On the other hand, $f_\epsilon'(u_\epsilon)[(P_t^\epsilon u_\epsilon)^+] < 0$. Hence there exists $\xi_t \in]\overline{\xi}_t, 1[$ such that $f_\epsilon'(P^\epsilon u_\epsilon + \xi_t(P_t^\epsilon u_\epsilon)^+)[\xi_t(P_t^\epsilon u_\epsilon)^+] = 0$.

In analogous way, for all $t \in I^-$ one can find $\overline{\eta}_t \in]0,1]$ such that $\varrho = ||\overline{\eta}_t(P_t^\epsilon u_\epsilon)^-||_{(\lambda,p)}$; since (3.17) holds, since $||P^\epsilon u_\epsilon|| \leq B$ and $||\overline{\eta}_t(P_t^\epsilon u_\epsilon)^-|| \leq \mu$, from Lemma 2.5 we get $f_\epsilon'(P^\epsilon u_\epsilon - \overline{\eta}_t(P_t^\epsilon u_\epsilon)^-)[-\overline{\eta}_t(P_t^\epsilon u_\epsilon)^-] > 0$. Again, as $f_\epsilon'(u_\epsilon)[-(P_t^\epsilon u_\epsilon)^-] < 0$, there exists $\eta_t \in]\overline{\eta}_t, 1[$ such that $f_\epsilon'(P^\epsilon u_\epsilon - \eta_t(P_t^\epsilon u_\epsilon)^-)$ $[-\eta_t(P_t^\epsilon u_\epsilon)^-] = 0$.

Let us define

$$v_\epsilon = \sum_{t \notin I^+} (P_t^\epsilon u_\epsilon)^+ - \sum_{t \notin I^-} (P_t^\epsilon u_\epsilon)^- + \sum_{t \in I^+} \xi_t (P_t^\epsilon u_\epsilon)^+ - \sum_{t \in I^-} \eta_t (P_t^\epsilon u_\epsilon)^- + P^\epsilon u_\epsilon.$$

It is clear that

$$v_\epsilon \in M_\epsilon^{T^+,T^-} \cap K_{\epsilon,\varrho}^{T^+,T^-} \cap N_{\epsilon,B}^{T^+,T^-}.$$

Thus, for all $t \in I^+$, we have

$$\lim_{n\to\infty}\Big\{\frac{1}{2}\int_{\Omega_t'}\frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i}(P_t^\epsilon u_n^\epsilon)^+\partial_{x_j}(P_t^\epsilon u_n^\epsilon)^+\,dx$$
$$-\int_{\Omega_t'}G_\epsilon(x,P^\epsilon u_n^\epsilon+(P_t^\epsilon u_n^\epsilon)^+)\,dx\Big\}$$
$$=\frac{1}{2}\int_{\Omega_t'}g_\epsilon(x,(P_t^\epsilon u_\epsilon)^++P^\epsilon u_\epsilon)(P_t^\epsilon u_\epsilon)^+\,dx$$
$$-\int_{\Omega_t'}G_\epsilon(x,(P_t^\epsilon u_\epsilon)^++P^\epsilon u_\epsilon)\,dx$$
$$=\frac{1}{2}\int_{\Omega_t'}g_\epsilon(x,\xi_t(P_t^\epsilon u_\epsilon)^++P^\epsilon u_\epsilon)\xi_t(P_t^\epsilon u_\epsilon)^+\,dx$$
$$-\int_{\Omega_t'}G_\epsilon(x,\xi_t(P_t^\epsilon u_\epsilon)^++P^\epsilon u_\epsilon)\,dx$$
$$+(1-\xi_t)\frac{\partial}{\partial u}\Psi_\epsilon(P^\epsilon u_\epsilon,\xi(P_t^\epsilon u_\epsilon)^+)[(P_t^\epsilon u_\epsilon)^+]$$

(see Definition 2.6) for a suitable $\xi\in]\xi_t,1[$.

Taking into account that $||\xi(P_t^\epsilon u_\epsilon)^+||_{(\lambda,p)}\geq\varrho\geq\overline{\varrho}$, $||\xi(P_t^\epsilon u_\epsilon)^+||\leq\overline{\mu}$ and $||P^\epsilon u_\epsilon||\leq B$, Lemma 2.7 implies

$$\lim_{n\to\infty}\Big\{\frac{1}{2}\int_{\Omega_t'}\frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i}(P_t^\epsilon u_n^\epsilon)^+\partial_{x_j}(P_t^\epsilon u_n^\epsilon)^+\,dx-\int_{\Omega_t'}G_\epsilon(x,P^\epsilon u_n^\epsilon+(P_t^\epsilon u_n^\epsilon)^+)\,dx\Big\}$$
$$>\frac{1}{2}\int_{\Omega_t'}g_\epsilon(x,\xi_t(P_t^\epsilon u_\epsilon)^++P^\epsilon u_\epsilon)\xi_t(P_t^\epsilon u_\epsilon)^+\,dx-\int_{\Omega_t'}G_\epsilon(x,\xi_t(P_t^\epsilon u_\epsilon)^++P^\epsilon u_\epsilon)\,dx$$
$$=\frac{1}{2}\xi_t^2\int_{\Omega_t'}\frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i}(P_t^\epsilon u_\epsilon)^+\partial_{x_j}(P_t^\epsilon u_\epsilon)^+\,dx$$
$$-\int_{\Omega_t'}G_\epsilon(x,\xi_t(P_t^\epsilon u_\epsilon)^++P^\epsilon u_\epsilon)\,dx,\tag{3.18}$$

where the last equality follows from the fact that $f_\epsilon'(P^\epsilon u_\epsilon + \xi_t(P_t^\epsilon u_\epsilon)^+)$ $[\xi_t(P_t^\epsilon u_\epsilon)^+] = 0$.

Now for all $t \in I^-$ we have

$$
\begin{aligned}
\lim_{n\to\infty} \Big\{ \frac{1}{2} \int_{\Omega_t'} & \frac{a_\epsilon^{i,j}}{\epsilon} \partial_{x_i} (P_t^\epsilon u_n^\epsilon)^- \partial_{x_j} (P_t^\epsilon u_n^\epsilon)^- \, dx \\
& - \int_{\Omega_t'} G_\epsilon(x, -(P_t^\epsilon u_n^\epsilon)^- + P^\epsilon u_n^\epsilon)\, dx \Big\} \\
& = \frac{1}{2} \int_{\Omega_t'} g_\epsilon(x, -(P_t^\epsilon u_\epsilon)^- + P^\epsilon u_\epsilon)[-(P_t^\epsilon u_\epsilon)^-]\, dx \\
& - \int_{\Omega_t'} G_\epsilon(x, -(P_t^\epsilon u_\epsilon)^- + P^\epsilon u_\epsilon)\, dx \\
& = \frac{1}{2} \int_{\Omega_t'} g_\epsilon(x, -\eta_t(P_t^\epsilon u_\epsilon)^- + P^\epsilon u_\epsilon)[-\eta_t(P_t^\epsilon u_\epsilon)^-]\, dx \\
& - \int_{\Omega_t'} G_\epsilon(x, -\eta_t(P_t^\epsilon u_\epsilon)^- + P^\epsilon u_\epsilon)\, dx \\
& + (1-\eta_t)\frac{\partial}{\partial u} \Psi_\epsilon(P^\epsilon u_\epsilon, -\eta(P_t^\epsilon u_\epsilon)^-)[-(P_t^\epsilon u_\epsilon)^-]
\end{aligned}
$$

(see Definition 2.6) for a suitable $\eta \in]\eta_t, 1[$.

Since $||\eta(P_t^\epsilon u_\epsilon)^-||_{(\lambda,p)} \geq \varrho \geq \overline{\varrho}$, $||\eta(P_t^\epsilon u_\epsilon)^-|| \leq \overline{\mu}$ and $||P^\epsilon u_\epsilon|| \leq B$, Lemma 2.7 implies

$$
\begin{aligned}
\lim_{n\to\infty} \Big\{ \frac{1}{2} \int_{\Omega_t'} & \frac{a_\epsilon^{i,j}}{\epsilon} \partial_{x_i} (P_t^\epsilon u_n^\epsilon)^- \partial_{x_j} (P_t^\epsilon u_n^\epsilon)^- \, dx \\
& - \int_{\Omega_t'} G_\epsilon(x, P^\epsilon u_n^\epsilon - (P_t^\epsilon u_n^\epsilon)^-)\, dx \Big\} \\
& > \frac{1}{2} \int_{\Omega_t'} g_\epsilon(x, -\eta_t(P_t^\epsilon u_\epsilon)^- + P^\epsilon u_\epsilon)[-\eta_t(P_t^\epsilon u_\epsilon)^-]\, dx
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{\Omega_t'} G_\epsilon(x, -\eta_t (P_t^\epsilon u_\epsilon)^- + P^\epsilon u_\epsilon)\, dx \\
&= \frac{1}{2}\eta_t^2 \int_{\Omega_t'} \frac{a_\epsilon^{i,j}}{\epsilon} \partial_{x_i} (P_t^\epsilon u_\epsilon)^- \partial_{x_j} (P_t^\epsilon u_\epsilon)^-\, dx \\
&- \int_{\Omega_t'} G_\epsilon(x, -\eta_t (P_t^\epsilon u_\epsilon)^- + P^\epsilon u_\epsilon)\, dx, \qquad (3.19)
\end{aligned}
$$

where the last equality follows from the fact that $f_\epsilon'(P^\epsilon u_\epsilon - \eta_t(P_t^\epsilon u_\epsilon)^-)$ $[\eta_t(P_t^\epsilon u_\epsilon)^-] = 0$.

Hence, by using (3.18) and (3.19), we conclude that, if $I^+ \neq \emptyset$ or $I^- \neq \emptyset$, then

$$
\begin{aligned}
&\inf\{f_\epsilon(u) \,:\, u \in M_\epsilon^{T^+,T^-} \cap K_{\epsilon,\varrho}^{T^+,T^-} \cap N_{\epsilon,B}^{T^+,T^-}\} = \lim_{n\to\infty} f_\epsilon(u_n^\epsilon) \\
&> \frac{1}{2}\int_\Omega \frac{a_\epsilon^{i,j}}{\epsilon} \partial_{x_i} P^\epsilon u_\epsilon \partial_{x_j} P^\epsilon u_\epsilon\, dx - \int_\Omega G_\epsilon(x, P^\epsilon u_\epsilon)\, dx \\
&+ 2\sum_{t=1}^k \int_{\Omega_t'} G_\epsilon(x, P^\epsilon u_\epsilon)\, dx \\
&+ \sum_{t\in I^+} \{ \frac{\xi_t{}^2}{2} \int_{\Omega_t'} \frac{a_\epsilon^{i,j}}{\epsilon} \partial_{x_i} (P_t^\epsilon u_\epsilon)^+ \partial_{x_j} (P_t^\epsilon u_\epsilon)^+\, dx \\
&- \int_{\Omega_t'} G_\epsilon(x, \xi_t (P_t^\epsilon u_\epsilon)^+ + P^\epsilon u_\epsilon)\, dx \} \\
&+ \sum_{t\in I^-} \{ \frac{\eta_t{}^2}{2} \int_{\Omega_t'} \frac{a_\epsilon^{i,j}}{\epsilon} \partial_{x_i} (P_t^\epsilon u_\epsilon)^- \partial_{x_j} (P_t^\epsilon u_\epsilon)^-\, dx \\
&- \int_{\Omega_t'} G_\epsilon(x, -\eta_t (P_t^\epsilon u_\epsilon)^- + P^\epsilon u_\epsilon)\, dx \} \\
&+ \sum_{t\notin I^+} \{ \frac{1}{2} \int_{\Omega_t'} \frac{a_\epsilon^{i,j}}{\epsilon} \partial_{x_i} (P_t^\epsilon u_\epsilon)^+ \partial_{x_j} (P_t^\epsilon u_\epsilon)^+\, dx
\end{aligned}
$$

$$- \int_{\Omega_t'} G_\epsilon(x, (P_t^\epsilon u_\epsilon)^+ + P^\epsilon u_\epsilon)\, dx\}$$

$$+ \sum_{t \notin I^-} \{ \frac{1}{2} \int_{\Omega_t'} \frac{a_\epsilon^{i,j}}{\epsilon} \partial_{x_i} (P_t^\epsilon u_\epsilon)^- \partial_{x_j} (P_t^\epsilon u_\epsilon)^- \, dx$$

$$- \int_{\Omega_t'} G_\epsilon(x, -(P_t^\epsilon u_\epsilon)^- + P^\epsilon u_\epsilon)\, dx\}$$

$$= f_\epsilon(v_\epsilon),$$

which is impossible since $v_\epsilon \in M_\epsilon^{T^+,T^-} \cap K_{\epsilon,\varrho}^{T^+,T^-} \cap N_{\epsilon,B}^{T^+,T^-}$. Hence $I^+ = \emptyset$ and $I^- = \emptyset$.

Finally it remains to remark that $\sup_{0<\epsilon<\bar{\epsilon}} ||u_\epsilon|| < +\infty$, which follows easily from (3.16) and (3.17), taking into account that $u_\epsilon \in N_{\epsilon,B}^{T^+,T^-}$. □

Lemma 3.9. *Let* $0 < \varrho < (\underline{\Lambda}\tilde{\mu})^{\frac{1}{p-2}}$ *and* $0 < B < (\underline{\Lambda}\frac{\varrho}{2}\tilde{\mu}^{\frac{p}{2}})^{\frac{1}{p-2}}$. *Then there exists* $\bar{\epsilon} > 0$ *such that, for all* $\epsilon \in]0, \bar{\epsilon}[$,

$$\inf\{f_\epsilon(u) : u \in M_\epsilon^{T^+,T^-} \cap K_{\epsilon,\varrho}^{T^+,T^-} \cap N_{\epsilon,B}^{T^+,T^-}\}$$
$$< \inf\{f_\epsilon(u) : u \in M_\epsilon^{T^+,T^-} \cap K_{\epsilon,\varrho}^{T^+,T^-} \cap \partial N_{\epsilon,B}^{T^+,T^-}\}.$$

Proof. Let us argue by contradiction: suppose there exists a sequence $(\epsilon_n)_{n\geq 1}$ of positive numbers converging to zero such that

$$\inf\{f_{\epsilon_n}(u) : u \in M_{\epsilon_n}^{T^+,T^-} \cap K_{\epsilon_n,\varrho}^{T^+,T^-} \cap \partial N_{\epsilon_n,B}^{T^+,T^-}\}$$
$$\leq \inf\{f_{\epsilon_n}(u) : u \in M_{\epsilon_n}^{T^+,T^-} \cap K_{\epsilon_n,\varrho}^{T^+,T^-} \cap N_{\epsilon_n,B}^{T^+,T^-}\}. \quad (3.20)$$

Hence there exists a sequence of functions $(u_n)_{n\geq 1}$ in $M_{\epsilon_n}^{T^+,T^-} \cap K_{\epsilon_n,\varrho}^{T^+,T^-} \cap \partial N_{\epsilon_n,B}^{T^+,T^-}$ such that

$$\limsup_{n\to\infty} f_{\epsilon_n}(u_n) \le \limsup_{n\to\infty} \inf\{f_{\epsilon_n}(u) \,:\, u \in M_{\epsilon_n}^{T^+,T^-} \cap K_{\epsilon_n,\varrho}^{T^+,T^-} \cap N_{\epsilon_n,B}^{T^+,T^-}\}. \tag{3.21}$$

The proof consists of 3 steps.

STEP 1. The sequence $(u_n)_{n\ge 1}$ is bounded in $H_0^{1,2}(\Omega)$.

From Corollary 3.7, from (3.20) and (3.21) we have

$$\limsup_{n\to\infty} f_{\epsilon_n}(u_n) \le (\frac{1}{2} - \frac{1}{p})\overline{\Lambda}^{\frac{p}{p-2}} M < +\infty. \tag{3.22}$$

Since $u_n \in M_{\epsilon_n}^{T^+,T^-} \cap N_{\epsilon_n,B}^{T^+,T^-}$ for all $n \ge 1$, arguing as in Lemma 3.8 one can prove that the sequences $\big((P_t^{\epsilon_n} u_n)^+\big)_{n\ge 1}$, for $t \in T^+$, and $\big((P_t^{\epsilon_n} u_n)^-\big)$, for $t \in T^-$, are bounded in $H_0^{1,2}(\Omega)$. Hence the sequence $(u_n)_{n\ge 1}$ is bounded in $H_0^{1,2}(\Omega)$, as $u_n \in \partial N_{\epsilon_n,B}^{T^+,T^-}$; it follows that, up to a subsequence, $u_n \to u \in H_0^{1,2}(\Omega)$ weakly in $H_0^{1,2}(\Omega)$, in $L^p(\Omega)$, in $L^q(\Omega)$ and a.e. in Ω.

Now, since $(f_{\epsilon_n}(u_n))_{n\ge 1}$ is bounded, from $(a.4)$ we infer that $\int_{\Omega\setminus\bigcup_{t=1}^k \Omega_t} |Du|^2\,dx = 0$; hence, $u \equiv 0$ in $\Omega \setminus \bigcup_{t=1}^k \Omega_t'$ and so we can write $u = u_1 + \cdots + u_k$ with $u_t \in H_0^{1,2}(\Omega_t')$ and $\int_{\Omega_t'\setminus\Omega_t} |Du_t|^2\,dx = 0$ for all $t \in \{1,\ldots,k\}$.

Moreover $\|u_t^+\|_{(\lambda,p)} \ge \varrho$ for all $t \in T^+$, $\|u_t^-\|_{(\lambda,p)} \ge \varrho$ for all $t \in T^-$ and $\sum_{t\notin T^+} \int_{\Omega_t'} |Du_t^+|^2\,dx + \sum_{t\notin T^-} \int_{\Omega_t'} |Du_t^-|^2\,dx \le B^2$.

STEP 2. We prove that (up to a subsequence)

$$\begin{aligned}\lim_{n\to\infty} f_{\epsilon_n}(u_n) = \lim_{n\to\infty} \{ (\frac{1}{2} - \frac{1}{p})[\sum_{t\in T^+} Q_{\epsilon_n}((P_t^{\epsilon_n} u_n)^+) \\ + \sum_{t\in T^-} Q_{\epsilon_n}((P_t^{\epsilon_n} u_n)^-)] \\ + \sum_{t\notin T^+} [\frac{1}{2}\int_{\Omega_t'} \frac{a_{\epsilon_n}^{i,j}}{\epsilon_n} \partial_{x_i}(P_t^{\epsilon_n} u_n)^+ \partial_{x_j}(P_t^{\epsilon_n} u_n)^+\,dx\end{aligned}$$

$$- \int_{\Omega_t'} G_{\epsilon_n}(x, (P_t^{\epsilon_n} u_n)^+ + P^{\epsilon_n} u_n)\, dx]$$

$$+ \sum_{t \notin T^-} [\frac{1}{2} \int_{\Omega_t'} \frac{a_{\epsilon_n}^{i,j}}{\epsilon_n} \partial_{x_i} (P_t^{\epsilon_n} u_n)^- \partial_{x_j} (P_t^{\epsilon_n} u_n)^-\, dx$$

$$- \int_{\Omega_t'} G_{\epsilon_n}(x, -(P_t^{\epsilon_n} u_n)^- + P^{\epsilon_n} u_n)\, dx]$$

$$+ \frac{1}{2} \int_{\Omega} \frac{a_{\epsilon_n}^{i,j}}{\epsilon_n} \partial_{x_i} P^{\epsilon_n} u_n \partial_{x_j} P^{\epsilon_n} u_n\, dx\} \tag{3.23}$$

(see Definition 3.3). Since $u_n \in M_{\epsilon_n}^{T^+,T^-}$, we have, for $t \in T^+$,

$$\begin{aligned} &\int_{\Omega_t'} \frac{a_{\epsilon_n}^{i,j}}{\epsilon_n} \partial_{x_i} (P_t^{\epsilon_n} u_n)^+ \partial_{x_j} (P_t^{\epsilon_n} u_n)^+\, dx \\ &= \int_{\Omega_t'} g_{\epsilon_n}(x, (P_t^{\epsilon_n} u_n)^+ + P^{\epsilon_n} u_n)(P_t^{\epsilon_n} u_n)^+\, dx \qquad (3.24) \\ &= \int_{\Omega_t'} \gamma_{\epsilon_n}(x, (P_t^{\epsilon_n} u_n)^+ + P^{\epsilon_n} u_n) |(P_t^{\epsilon_n} u_n)^+ + P^{\epsilon_n} u_n|^{p-1} (P_t^{\epsilon_n} u_n)^+\, dx \end{aligned}$$

and, for $t \in T^-$,

$$\begin{aligned} &\int_{\Omega_t'} \frac{a_{\epsilon_n}^{i,j}}{\epsilon_n} \partial_{x_i} (P_t^{\epsilon_n} u_n)^- \partial_{x_j} (P_t^{\epsilon_n} u_n)^-\, dx \\ &= \int_{\Omega_t'} g_{\epsilon_n}(x, -(P_t^{\epsilon_n} u_n)^- + P^{\epsilon_n} u_n)[-(P_t^{\epsilon_n} u_n)^-]\, dx \qquad (3.25) \\ &= \int_{\Omega_t'} \gamma_{\epsilon_n}(x, -(P_t^{\epsilon_n} u_n)^- + P^{\epsilon_n} u_n) |-(P_t^{\epsilon_n} u_n)^- + P^{\epsilon_n} u_n|^{p-1} [-(P_t^{\epsilon_n} u_n)^-]\, dx. \end{aligned}$$

Since

$$\limsup_{n\to\infty} \int_{\Omega_t'} \frac{a_{\epsilon_n}^{i,j}}{\epsilon_n} \partial_{x_i}(P_t^{\epsilon_n} u_n)^+ \partial_{x_j}(P_t^{\epsilon_n} u_n)^+ \, dx < +\infty \quad \forall \quad t \in T^+$$

and

$$\limsup_{n\to\infty} \int_{\Omega_t'} \frac{a_{\epsilon_n}^{i,j}}{\epsilon_n} \partial_{x_i}(P_t^{\epsilon_n} u_n)^- \partial_{x_j}(P_t^{\epsilon_n} u_n)^- \, dx < +\infty \quad \forall \quad t \in T^-,$$

we can assume that (up to a subsequence) the limits
$\lim_{n\to\infty} \int_{\Omega_t'} \frac{a_{\epsilon_n}^{i,j}}{\epsilon_n} \partial_{x_i}(P_t^{\epsilon_n} u_n)^+ \partial_{x_j}(P_t^{\epsilon_n} u_n)^+ \, dx$,
$\lim_{n\to\infty} \int_{\Omega_t'} \frac{a_{\epsilon_n}^{i,j}}{\epsilon_n} \partial_{x_i}(P_t^{\epsilon_n} u_n)^- \partial_{x_j}(P_t^{\epsilon_n} u_n)^- \, dx$ do exist.
Taking into account that

$$\lim_{n\to\infty} \epsilon_n^{\frac{1}{p-2}} ((P_t^{\epsilon_n} u_n)^+ + P^{\epsilon_n} u_n)(x) = 0 \quad \text{a.e. in} \quad \Omega_t', \quad \forall t \in T^+$$

and

$$\lim_{n\to\infty} \epsilon_n^{\frac{1}{p-2}} (-(P_t^{\epsilon_n} u_n)^- + P^{\epsilon_n} u_n)(x) = 0 \quad \text{a.e. in} \quad \Omega_t', \quad \forall t \in T^-,$$

from $(g.2)$, $(g.3)$, (3.24) and (3.25) it follows that

$$\lim_{n\to\infty} \int_{\Omega_t'} \frac{a_{\epsilon_n}^{i,j}}{\epsilon_n} \partial_{x_i}(P_t^{\epsilon_n} u_n)^+ \partial_{x_j}(P_t^{\epsilon_n} u_n)^+ \, dx = ||u_t^+||_{(\lambda,p)}^p \quad \forall t \in T^+ \tag{3.26}$$

and

$$\lim_{n\to\infty}\int\limits_{\Omega_t'} \frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}(P_t^{\epsilon_n}u_n)^-\partial_{x_j}(P_t^{\epsilon_n}u_n)^-\,dx = ||u_t^-||_{(\lambda,p)} \quad \forall t\in T^-. \tag{3.27}$$

Again, by using $(g.2)$ and $(g.3)$, one can prove that

$$\lim_{n\to\infty}\int\limits_{\Omega_t'} G_{\epsilon_n}(x,(P_t^{\epsilon_n}u_n)^+ + P^{\epsilon_n}u_n)\,dx = \frac{1}{p}||u_t^+||^p_{(\lambda,p)} \quad \forall t\in T^+ \tag{3.28}$$

and

$$\lim_{n\to\infty}\int\limits_{\Omega_t'} G_{\epsilon_n}(x,-(P_t^{\epsilon_n}u_n)^- + P^{\epsilon_n}u_n)\,dx = \frac{1}{p}||u_t^-||^p_{(\lambda,p)} \quad \forall t\in T^-. \tag{3.29}$$

Since $(a.4)$ implies $P^{\epsilon_n}u_n \to 0$ in $L^q(\Omega)$, (3.23) follows from (3.26), (3.27), (3.28) and (3.29).

STEP 3. We arrive at a contradiction.

Since $u_n \in \partial N^{T^+,T^-}_{\epsilon_n,B}$ for all $n\geq 1$, we have

$$||P^{\epsilon_n}u_n||^2 + \sum_{t\notin T^+}||(P_t^{\epsilon_n}u_n)^+||^2 + \sum_{t\notin T^-}||(P_t^{\epsilon_n}u_n)^-||^2 = B^2. \tag{3.30}$$

Taking into account that $P^{\epsilon_n}u_n + (P_t^{\epsilon_n}u_n)^+ \to u_t^+$ for all $t\in T^+$ and, for all $t\in T^-$, $P^{\epsilon_n}u_n + (P_t^{\epsilon_n}u_n)^- \to u_t^-$ in $L^p(\Omega_t')$ and a.e., arguing as in Proposition 3.5, one can prove that, for n large enough, there exist some positive numbers $\alpha_{t,n}$ for $t\in T^+$ and $\beta_{t,n}$ for $t\in T^-$, such that

$$z_n = \sum_{t\in T^+}\alpha_{t,n}(P_t^{\epsilon_n}u_n)^+ - \sum_{t\in T^-}\beta_{t,n}(P_t^{\epsilon_n}u_n)^- \in M^{T^+,T^-}_{\epsilon_n}\cap K^{T^+,T^-}_{\epsilon_n,\varrho} \tag{3.31}$$

and

$$f_{\epsilon_n}(\alpha_{t,n}(P_t^{\epsilon_n}u_n)^+) > 0 \quad \forall t \in T^+,\ f_{\epsilon_n}(-\beta_{t,n}(P_t^{\epsilon_n}u_n)^-) > 0 \quad \forall t \in T^-. \tag{3.32}$$

Clearly $z_n \in N_{\epsilon_n,B}^{T^+,T^-}$; hence, (3.21) implies that, up to a subsequence,

$$\lim_{n\to\infty} f_{\epsilon_n}(u_n) \le \lim_{n\to\infty} f_{\epsilon_n}(z_n). \tag{3.33}$$

Arguing as in the proof of Proposition 3.5, from (3.32) one can easily prove that $\lim_{n\to\infty} \epsilon_n^{\frac{1}{p-2}}\ \alpha_{t,n} = 0\ \forall t \in T^+$ and $\lim_{n\to\infty} \epsilon_n^{\frac{1}{p-2}} \beta_{t,n} = 0\ \forall t \in T^-$.
Since $(g.2)$ and $(g.3)$ hold, by using the facts that $z_n \in M_{\epsilon_n}^{T^+,T^-}$ for all n large enough and that $u_n \to u_1 + \cdots + u_k$ (with $u_t \in H_0^{1,2}(\Omega_t')$ for $t = 1, \ldots, k$) in $L^p(\Omega)$ and in $L^q(\Omega)$, one can prove that

$$\lim_{n\to\infty} \int_{\Omega_t'} \frac{a_{\epsilon_n}^{i,j}}{\epsilon_n} \partial_{x_i}(P_t^{\epsilon_n}u_n)^+ \partial_{x_j}(P_t^{\epsilon_n}u_n)^+ \, dx = ||u_t^+||_{(\lambda,p)}^p \lim_{n\to\infty} \alpha_{t,n}^{p-2} \quad \forall t \in T^+ \tag{3.34}$$

and

$$\lim_{n\to\infty} \int_{\Omega_t'} \frac{a_{\epsilon_n}^{i,j}}{\epsilon_n} \partial_{x_i}(P_t^{\epsilon_n}u_n)^- \partial_{x_j}(P_t^{\epsilon_n}u_n)^- \, dx = ||u_t^-||_{(\lambda,p)}^p \lim_{n\to\infty} \alpha_{t,n}^{p-2} \quad \forall t \in T^-. \tag{3.35}$$

Comparing respectively (3.26) with (3.34) and (3.27) with (3.35), it follows that

$$\lim_{n\to\infty} \alpha_{t,n} = 1 \quad \forall t \in T^+ \quad \text{and} \quad \lim_{n\to\infty} \beta_{t,n} = 1 \quad \forall t \in T^-. \tag{3.36}$$

Thus, from $(g.2)$, $(g.3)$ and (3.36) we get

$$\lim_{n\to\infty} f_{\epsilon_n}(z_n) = \lim_{n\to\infty}(\frac{1}{2}-\frac{1}{p})\{\sum_{t\in T^+}\int_{\Omega'_t}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}(P^{\epsilon_n}_t u_n)^+\partial_{x_j}(P^{\epsilon_n}_t u_n)^+\,dx$$
$$+\sum_{t\in T^-}\int_{\Omega'_t}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}(P^{\epsilon_n}_t u_n)^-\partial_{x_j}(P^{\epsilon_n}_t u_n)^-\,dx\}. \tag{3.37}$$

By using (3.23), (3.37) and (3.30), we obtain (up to a subsequence)

$$\lim_{n\to\infty}[f_{\epsilon_n}(u_n) - f_{\epsilon_n}(z_n)] = \lim_{n\to\infty}\{(\frac{1}{2}-\frac{1}{p})$$
$$[\sum_{t\in T^+}\int_{\Omega'_t}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}(P^{\epsilon_n}_t u_n)^+\partial_{x_j}(P^{\epsilon_n}_t u_n)^+\,dx$$
$$+\sum_{t\in T^-}\int_{\Omega'_t}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}(P^{\epsilon_n}_t u_n)^-\partial_{x_j}(P^{\epsilon_n}_t u_n)^-\,dx]$$
$$+\sum_{t\notin T^+}[\frac{1}{2}\int_{\Omega'_t}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}(P^{\epsilon_n}_t u_n)^+\partial_{x_j}(P^{\epsilon_n}_t u_n)^+\,dx$$
$$-\int_{\Omega'_t}G_{\epsilon_n}(x,(P^{\epsilon_n}_t u_n)^+ + P^{\epsilon_n}u_n)\,dx]$$
$$+\sum_{t\notin T^-}[\frac{1}{2}\int_{\Omega'_t}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}(P^{\epsilon_n}_t u_n)^-\partial_{x_j}(P^{\epsilon_n}_t u_n)^-\,dx$$
$$-\int_{\Omega'_t}G_{\epsilon_n}(x,-(P^{\epsilon_n}_t u_n)^- + P^{\epsilon_n}u_n)\,dx]$$
$$+\frac{1}{2}\int_{\Omega}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}P^{\epsilon_n}u_n\partial_{x_j}P^{\epsilon_n}u_n\,dx$$

$$-\left(\frac{1}{2}-\frac{1}{p}\right)\Big[\sum_{t\in T^+}\int_{\Omega'_t}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}(P^{\epsilon_n}_t u_n)^+\partial_{x_j}(P^{\epsilon_n}_t u_n)^+\,dx$$

$$+\sum_{t\in T^-}\int_{\Omega'_t}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}(P^{\epsilon_n}_t u_n)^-\partial_{x_j}(P^{\epsilon_n}_t u_n)^-\,dx\Big]\Big\}$$

$$\geq \liminf_{n\to\infty}\Big\{\frac{\Lambda}{2}\Big[\sum_{t\notin T^+}||(P^{\epsilon_n}_t u_n)^+||^2+\sum_{t\notin T^-}||(P^{\epsilon_n}_t u_n)^-||^2+||P^{\epsilon_n}u_n||^2\Big]$$

$$-\sum_{t\notin T^+}\int_{\Omega'_t}G_{\epsilon_n}(x,(P^{\epsilon_n}_t u_n)^+ + P^{\epsilon_n}u_n)\,dx$$

$$-\sum_{t\notin T^-}\int_{\Omega'_t}G_{\epsilon_n}(x,-(P^{\epsilon_n}_t u_n)^- + P^{\epsilon_n}u_n)\,dx\Big\}$$

$$=\frac{\Lambda}{2}B^2-\sum_{t\notin T^+}\frac{||u_t^+||^p_{(\lambda,p)}}{p}-\sum_{t\notin T^-}\frac{||u_t^-||^p_{(\lambda,p)}}{p}$$

$$\geq\frac{\Lambda}{2}B^2-\sum_{t\notin T^+}\frac{(||u_t^+||^2)^{\frac{p}{2}}}{p\mu_t^{\frac{p}{2}}}-\sum_{t\notin T^-}\frac{(||u_t^-||^2)^{\frac{p}{2}}}{p\mu_t^{\frac{p}{2}}}$$

$$\geq\frac{\Lambda}{2}B^2-\frac{1}{p\tilde\mu^{\frac{p}{2}}}\Big[\sum_{t\notin T^+}||u_t^+||^2+\sum_{t\notin T^-}||u_t^-||^2\Big]^{\frac{p}{2}}$$

$$\geq\frac{\Lambda}{2}B^2-\frac{B^p}{p\tilde\mu^{\frac{p}{2}}}>0,$$

where the last inequality is due to the fact that $0<B<\left(\Lambda\frac{p}{2}\tilde\mu^{\frac{p}{2}}\right)^{\frac{1}{p-2}}$.

Thus we have a contradiction with (3.33). □

Proposition 3.10. *Let $0<\varrho<(\Lambda\tilde\mu)^{\frac{1}{p-2}}$. Then there exist $B>0$ and $\bar\epsilon>0$ such that, for all $\epsilon\in]0,\bar\epsilon[$ and for all T^+ and T^- subsets of $\{1,\dots,k\}$, there exists a function $u_\epsilon^{T^+,T^-}$ which minimizes the functional f_ϵ in the set $M_\epsilon^{T^+,T^-}\cap K_{\epsilon,\varrho}^{T^+,T^-}\cap N_{\epsilon,B}^{T^+,T^-}$ and such that*

$$||(P_t^\epsilon u_\epsilon^{T^+,T^-})^+||_{(\lambda,p)}>\varrho\quad\forall t\in T^+,$$

$$||(P_t^\epsilon u_\epsilon^{T^+,T^-})^-||_{(\lambda,p)}>\varrho\quad\forall t\in T^-,$$

$$||P^\epsilon u_\epsilon^{T^+,T^-}||^2 + \sum_{t \notin T^+} ||(P_t^\epsilon u_\epsilon^{T^+,T^-})^+||^2 + \sum_{t \notin T^-} ||(P_t^\epsilon u_\epsilon^{T^+,T^-})^-||^2 < B^2.$$

Moreover $u_\epsilon^{T^+,T^-}$ satisfies the following properties:

(1)

$$\lim_{\epsilon \to 0} \int_\Omega |DP^\epsilon u_\epsilon^{T^+,T^-}|^2 \, dx = 0,$$

$$\lim_{\epsilon \to 0} \int_{\Omega_t'} |D(P_t^\epsilon u_\epsilon^{T^+,T^-})^+|^2 \, dx = 0 \quad \forall t \notin T^+,$$

$$\lim_{\epsilon \to 0} \int_{\Omega_t'} |D(P_t^\epsilon u_\epsilon^{T^+,T^-})^-|^2 \, dx = 0 \quad \forall t \notin T^-.$$

(2)

$$\lim_{\epsilon \to 0} \int_{\Omega_t' \setminus \Omega_t} |Du_\epsilon^{T^+,T^-}|^2 \, dx = 0 \quad \forall \, t = 1, \ldots, k,$$

(3)

$$\liminf_{\epsilon \to 0} \Big(\int_{\Omega_t'} \lambda(x) |(u_\epsilon^{T^+,T^-})^+|^p \, dx \Big)^{\frac{1}{p}} \geq (\underline{\Lambda} \tilde{\mu})^{\frac{1}{p-2}} \quad \forall t \in T^+$$

$$\liminf_{\epsilon \to 0} \Big(\int_{\Omega_t'} \lambda(x) |(u_\epsilon^{T^+,T^-})^-|^p \, dx \Big)^{\frac{1}{p}} \geq (\underline{\Lambda} \tilde{\mu})^{\frac{1}{p-2}} \quad \forall t \in T^-.$$

Proof. Let ϱ, T^+ and T^- be as in our assumptions. From Lemma 3.8 we infer that, if $B > 0$ and $\bar{\epsilon} > 0$ are small enough, for all $\epsilon \in]0, \bar{\epsilon}[$, there exists $u_\epsilon^{T^+,T^-}$ minimizing f_ϵ in $M_\epsilon^{T^+,T^-} \cap K_{\epsilon,\varrho}^{T^+,T^-} \cap N_{\epsilon,B}^{T^+,T^-}$ and satisfying $\sup_{0<\epsilon<\bar\epsilon} \|u_\epsilon^{T^+,T^-}\| < +\infty$.

Moreover Lemma 2.5 implies

$$\|(P_t^\epsilon u_\epsilon^{T^+,T^-})^+\|_{(\lambda,p)} > \varrho \quad \forall t \in T^+, \ \|(P_t^\epsilon u_\epsilon^{T^+,T^-})^-\|_{(\lambda,p)} > \varrho \quad \forall t \in T^-.$$

If, in addition, we choose $B < (\underline{\Lambda}_2^{\frac{p}{2}} \tilde\mu^{\frac{p}{2}})^{\frac{1}{p-2}}$, Lemma 3.9 implies

$$\|P^\epsilon u_\epsilon^{T^+,T^-}\|^2 + \sum_{t \notin T^+} \|(P_t^\epsilon u_\epsilon^{T^+,T^-})^+\|^2 + \sum_{t \notin T^-} \|(P_t^\epsilon u_\epsilon^{T^+,T^-})^-\|^2 < B^2.$$

Proving part (1) is equivalent to show that

$$\lim_{\epsilon\to 0} \{\|P^\epsilon u_\epsilon^{T^+,T^-}\|^2 + \sum_{t \notin T^+} \|(P_t^\epsilon u_\epsilon^{T^+,T^-})^+\|^2 + \sum_{t \notin T^-} \|(P_t^\epsilon u_\epsilon^{T^+,T^-})^-\|^2\} = 0.$$

By contradition, assume that there exist $\beta \in]0, B]$ and a sequence $(\epsilon_n)_{n\geq 1}$ of positive numbers converging to 0, such that

$$\lim_{n\to\infty} \{\|P^{\epsilon_n} u_{\epsilon_n}^{T^+,T^-}\|^2 + \sum_{t \notin T^+} \|(P_t^{\epsilon_n} u_{\epsilon_n}^{T^+,T^-})^+\|^2 + \sum_{t \notin T^-} \|(P_t^{\epsilon_n} u_{\epsilon_n}^{T^+,T^-})^-\|^2\} = \beta^2. \tag{3.38}$$

Notice that the sequence $(u_{\epsilon_n}^{T^+,T^-})_{n\geq 1}$ is bounded in $H_0^{1,2}(\Omega)$. Hence, up to a subsequence, $u_{\epsilon_n}^{T^+,T^-}$ converges to $u^{T^+,T^-} \in H_0^{1,2}(\Omega)$ weakly in $H_0^{1,2}(\Omega)$, in $L^p(\Omega)$, in $L^q(\Omega)$ and a.e. in Ω. Moreover, from assumption $(a.4)$ and

Corollary 3.7, it follows that $\int\limits_{\Omega\setminus\bigcup_{t=1}^{k}\Omega_t} |Du|^2\,dx = 0$; therefore we can write $u = \sum_{t=1}^{k} u_t$, with $u_t \in H_0^{1,2}(\Omega_t')$ for all $t \in \{1,\dots,k\}$.

Arguing as in STEP 2 of Lemma 3.9, one can obtain that (up to a subsequence)

$$\begin{aligned}
\lim_{n\to\infty} f_{\epsilon_n}(u_{\epsilon_n}^{T^+,T^-}) = \lim_{n\to\infty}\Big\{&\Big(\frac{1}{2}-\frac{1}{p}\Big)\Big[\sum_{t\in T^+} Q_{\epsilon_n}((P_t^{\epsilon_n}u_{\epsilon_n}^{T^+,T^-})^+)\\
&+\sum_{t\in T^-} Q_{\epsilon_n}((P_t^{\epsilon_n}u_{\epsilon_n}^{T^+,T^-})^-)\Big]\\
&+\sum_{t\notin T^+}\Big[\frac{1}{2}\int\limits_{\Omega_t'}\frac{a_{\epsilon_n}^{i,j}}{\epsilon_n}\partial_{x_i}(P_t^{\epsilon_n}u_{\epsilon_n}^{T^+,T^-})^+\partial_{x_j}(P_t^{\epsilon_n}u_{\epsilon_n}^{T^+,T^-})^+\,dx\\
&-\int\limits_{\Omega_t'} G_{\epsilon_n}(x,(P_t^{\epsilon_n}u_{\epsilon_n}^{T^+,T^-})^+ + P^{\epsilon_n}u_{\epsilon_n}^{T^+,T^-})\,dx\Big]\\
&+\sum_{t\notin T^-}\Big[\frac{1}{2}\int\limits_{\Omega_t'}\frac{a_{\epsilon_n}^{i,j}}{\epsilon_n}\partial_{x_i}(P_t^{\epsilon_n}u_{\epsilon_n}^{T^+,T^-})^-\partial_{x_j}(P_t^{\epsilon_n}u_{\epsilon_n}^{T^+,T^-})^-\,dx\\
&-\int\limits_{\Omega_t'} G_{\epsilon_n}(x,-(P_t^{\epsilon_n}u_{\epsilon_n}^{T^+,T^-})^- + P^{\epsilon_n}u_{\epsilon_n}^{T^+,T^-})\,dx\Big]\\
&+\frac{1}{2}\int\limits_{\Omega}\frac{a_{\epsilon_n}^{i,j}}{\epsilon_n}\partial_{x_i}P^{\epsilon_n}u_{\epsilon_n}^{T^+,T^-}\,\partial_{x_j}P^{\epsilon_n}u_{\epsilon_n}^{T^+,T^-}\,dx\Big\}.
\end{aligned}\tag{3.39}$$

Now set, for n large enough,

$$z_n = \sum_{t\in T^+}\alpha_{t,n}(P_t^{\epsilon_n}u_{\epsilon_n}^{T^+,T^-})^+ - \sum_{t\in T^-}\beta_{t,n}(P_t^{\epsilon_n}u_{\epsilon_n}^{T^+,T^-})^- \tag{3.40}$$

with $\alpha_{t,n} > 0$ for all $t \in T^+$, $\beta_{t,n} > 0$ for all $t \in T^-$, such that

$$z_n \in M_{\epsilon_n}^{T^+,T^-} \cap K_{\epsilon_n,\varrho}^{T^+,T^-}$$

and $f_{\epsilon_n}(z_n) > 0$ (the existence of these numbers follows arguing as in Proposition 3.5).

By definition, $z_n \in N_{\epsilon_n,B}^{T^+,T^-}$ and so

$$f_{\epsilon_n}(u_{\epsilon_n}^{T^+,T^-}) \leq f_{\epsilon_n}(z_n). \tag{3.41}$$

Moreover, arguing as in STEP 3 of Lemma 3.9, one can prove that

$$\lim_{n\to\infty} \alpha_{t,n} = 1 \quad \forall t \in T^+ \quad \text{and} \quad \lim_{n\to\infty} \beta_{t,n} = 1 \quad \forall t \in T^-. \tag{3.42}$$

Hence, from (3.39), (3.38) and (3.42), we infer that

$$\begin{aligned}
&\lim_{n\to\infty} [f_{\epsilon_n}(u_{\epsilon_n}^{T^+,T^-}) - f_{\epsilon_n}(z_n)] \\
&\geq \lim_{n\to\infty} \{ \sum_{t\notin T^+} [\frac{1}{2} \int_{\Omega'_t} \frac{a_{\epsilon_n}^{i,j}}{\epsilon_n} \partial_{x_i}(P_t^{\epsilon_n} u_{\epsilon_n}^{T^+,T^-})^+ \partial_{x_j}(P_t^{\epsilon_n} u_{\epsilon_n}^{T^+,T^-})^+ \, dx \\
&- \int_{\Omega'_t} G_{\epsilon_n}(x, (P_t^{\epsilon_n} u_{\epsilon_n}^{T^+,T^-})^+ + P^{\epsilon_n} u_{\epsilon_n}^{T^+,T^-}) \, dx] \\
&+ \sum_{t\notin T^-} [\frac{1}{2} \int_{\Omega'_t} \frac{a_{\epsilon_n}^{i,j}}{\epsilon_n} \partial_{x_i}(P_t^{\epsilon_n} u_{\epsilon_n}^{T^+,T^-})^- \partial_{x_j}(P_t^{\epsilon_n} u_{\epsilon_n}^{T^+,T^-})^- \, dx \\
&- \int_{\Omega'_t} G_{\epsilon_n}(x, -(P_t^{\epsilon_n} u_{\epsilon_n}^{T^+,T^-})^- + P^{\epsilon_n} u_{\epsilon_n}^{T^+,T^-}) \, dx] \\
&+ \frac{1}{2} \int_{\Omega} \frac{a_{\epsilon_n}^{i,j}}{\epsilon_n} \partial_{x_i} P^{\epsilon_n} u_{\epsilon_n}^{T^+,T^-} \partial_{x_j} P^{\epsilon_n} u_{\epsilon_n}^{T^+,T^-} \, dx \} \\
&\geq \frac{\Lambda}{2} \lim_{n\to\infty} \{ \sum_{t\notin T^+} ||(P_t^{\epsilon_n} u_{\epsilon_n}^{T^+,T^-})^+||^2 + \sum_{t\notin T^-} ||(P_t^{\epsilon_n} u_{\epsilon_n}^{T^+,T^-})^-||^2 \\
&+ ||P^{\epsilon_n} u_{\epsilon_n}^{T^+,T^-}||^2 \} - \frac{1}{p} \{ \sum_{t\notin T^+} ||u_t^+||_{(\lambda,p)}^p + \sum_{t\notin T^-} ||u_t^-||_{(\lambda,p)}^p \}
\end{aligned}$$

$$\geq \frac{\Lambda}{2}\beta^2 - \frac{1}{p\tilde{\mu}^{\frac{p}{2}}}\Big(\sum_{t\notin T^+}||u_t^+||^2 + \sum_{t\notin T^-}||u_t^-||^2\Big)^{\frac{p}{2}}$$

$$\geq \frac{\Lambda}{2}\beta^2 - \frac{1}{p\tilde{\mu}^{\frac{p}{2}}}\beta^p > 0,$$

where the last inequality is due to the fact that $0 < \beta \leq B < (\underline{\Lambda}\frac{p}{2}\tilde{\mu}^{\frac{p}{2}})^{\frac{1}{p-2}}$. Thus we have obtained a contradiction with (3.41).

Part (2) follows easily from assumption $(a.4)$ and Corollary 3.7, taking into account that $u_\epsilon^{T^+,T^-} \in M_\epsilon^{T^+,T^-}$.

In order to prove part (3) we argue by contradiction and assume that there exist $I^+ \subseteq T^+$, $I^+ \neq \emptyset$, or $I^- \subseteq T^-$, $I^- \neq \emptyset$, and a sequence $(\epsilon_n)_{n\geq 1}$ of positive numbers converging to zero, such that

$$\lim_{n\to\infty} \int_{\Omega_t'} \lambda(x)|(u_{\epsilon_n}^{T^+,T^-})^+|^p\,dx < (\underline{\Lambda}\tilde{\mu})^{\frac{p}{p-2}} \quad \forall t \in I^+ \tag{3.43}$$

and

$$\lim_{n\to\infty} \int_{\Omega_t'} \lambda(x)|(u_{\epsilon_n}^{T^+,T^-})^-|^p\,dx < (\underline{\Lambda}\tilde{\mu})^{\frac{p}{p-2}} \quad \forall t \in I^-. \tag{3.44}$$

Arguing as before, we have (up to a subsequence) $u_{\epsilon_n}^{T^+,T^-} \to u^{T^+,T^-} \in H_0^{1,2}(\Omega)$ weakly in $H_0^{1,2}(\Omega)$, in $L^p(\Omega)$, in $L^q(\Omega)$ and a.e. in Ω; moreover, $u^{T^+,T^-} = \sum_{t=1}^k u_t$, with $u_t \in H_0^{1,2}(\Omega_t')$ and $\int_{\Omega_t'\setminus\Omega_t} |Du|^2\,dx = 0$ for all $t = 1,\dots,k$.

Notice that $||u_t^+||_{(\lambda,p)} \geq \varrho \quad \forall t \in T^+$ and $||u_t^-||_{(\lambda,p)} \geq \varrho \quad \forall t \in T^-$ because $u_{\epsilon_n}^{T^+,T^-} \in K_{\epsilon_n,\varrho}^{T^+,T^-}$.

Since $u_{\epsilon_n}^{T^+,T^-} \in M_{\epsilon_n}^{T^+,T^-}$ for all $n \geq 1$, $(g.2)$ and $(g.3)$ imply

$$0 = \lim_{n\to\infty} f_{\epsilon_n}'(u_{\epsilon_n}^{T^+,T^-})[(P_t^{\epsilon_n} u_{\epsilon_n}^{T^+,T^-})^+] \geq \underline{\Lambda}\int_{\Omega_t} |Du_t^+|^2\,dx - \int_{\Omega_t'} \lambda(x)|u_t^+|^p\,dx$$

$$\geq \underline{\Lambda}\tilde{\mu}||u_t^+||_{(\lambda,p)}^2 - ||u_t^+||_{(\lambda,p)}^p > 0 \quad \forall t \in I^+ \tag{3.45}$$

and

$$0 = \lim_{n\to\infty} f'_{\epsilon_n}(u_{\epsilon_n}^{T^+,T^-})[(P_t^{\epsilon_n} u_{\epsilon_n}^{T^+,T^-})^-] \geq \underline{\Lambda} \int_{\Omega_t} |Du_t^-|^2\, dx - \int_{\Omega'_t} \lambda(x)|u_t^-|^p\, dx$$

$$\geq \underline{\Lambda}\tilde{\mu}||u_t^-||^2_{(\lambda,p)} - ||u_t^-||^p_{(\lambda,p)} > 0 \quad \forall t \in I^-, \tag{3.46}$$

where the last inequalities in (3.45) and (3.46) follow respectively from (3.43) and (3.44). Hence we get a contradiction.

□

For the proof of Theorem 1.1 we need the following result.

Lemma 3.11. *Let $\varrho, B, \bar{\epsilon}, T^+, T^-$ be as in Proposition 3.10. For all $\epsilon \in]0, \bar{\epsilon}[$, let $u_\epsilon^{T^+,T^-}$ be a function which minimizes the functional f_ϵ in the set $M_\epsilon^{T^+,T^-} \cap K_{\epsilon,\varrho}^{T^+,T^-} \cap N_{\epsilon,B}^{T^+,T^-}$.*
Then we have

$$\limsup_{\epsilon\to 0} f''_\epsilon(u_\epsilon^{T^+,T^-})[(P_t^\epsilon u_\epsilon^{T^+,T^-})^+]^2 \leq (2-p)\varrho^p < 0 \quad \forall t \in T^+ \tag{3.47}$$

and

$$\limsup_{\epsilon\to 0} f''_\epsilon(u_\epsilon^{T^+,T^-})[(P_t^\epsilon u_\epsilon^{T^+,T^-})^-]^2 \leq (2-p)\varrho^p < 0 \quad \forall t \in T^-. \tag{3.48}$$

Proof. Taking into account that $u_\epsilon^{T^+,T^-} \in M_\epsilon^{T^+,T^-}$, we have

$$
\begin{aligned}
f_\epsilon''(u_\epsilon^{T^+,T^-})[(P_t^\epsilon u_\epsilon^{T^+,T^-})^+]^2 &= \int\limits_{\Omega_t'} g_\epsilon(x, u_\epsilon^{T^+,T^-})(P_t^\epsilon u_\epsilon^{T^+,T^-})^+ \, dx \\
&- \int\limits_{\Omega_t'} g_\epsilon'(x, u_\epsilon^{T^+,T^-})[(P_t^\epsilon u_\epsilon^{T^+,T^-})^+]^2 \, dx \\
&= \int\limits_{\Omega_t'} \gamma_\epsilon(x, u_\epsilon^{T^+,T^-})|u_\epsilon^{T^+,T^-}|^{p-1}(P_t^\epsilon u_\epsilon^{T^+,T^-})^+ \, dx \\
&- \int\limits_{\Omega_t'} \tilde{\gamma}_\epsilon(x, u_\epsilon^{T^+,T^-})|u_\epsilon^{T^+,T^-}|^{p-2}[(P_t^\epsilon u_\epsilon^{T^+,T^-})^+]^2 \, dx \quad \forall \quad t \in T^+ \qquad (3.49)
\end{aligned}
$$

and

$$
\begin{aligned}
f_\epsilon''(u_\epsilon^{T^+,T^-})[(P_t^\epsilon u_\epsilon^{T^+,T^-})^-]^2 &= - \int\limits_{\Omega_t'} g_\epsilon(x, u_\epsilon^{T^+,T^-})(P_t^\epsilon u_\epsilon^{T^+,T^-})^- \, dx \\
&- \int\limits_{\Omega_t'} g_\epsilon'(x, u_\epsilon^{T^+,T^-})[(P_t^\epsilon u_\epsilon^{T^+,T^-})^-]^2 \, dx \\
&= - \int\limits_{\Omega_t'} \gamma_\epsilon(x, u_\epsilon^{T^+,T^-})|u_\epsilon^{T^+,T^-}|^{p-1}(P_t^\epsilon u_\epsilon^{T^+,T^-})^- \, dx \\
&- \int\limits_{\Omega_t'} \tilde{\gamma}_\epsilon(x, u_\epsilon^{T^+,T^-})|u_\epsilon^{T^+,T^-}|^{p-2}[(P_t^\epsilon u_\epsilon^{T^+,T^-})^-]^2 \, dx \quad \forall \quad t \in T^- \qquad (3.50)
\end{aligned}
$$

(see Definition 2.4).

Arguing by contradiction, assume that there exists a sequence of positive numbers $(\epsilon_n)_{n\geq 1}$, converging to zero, such that

$$
\lim_{n\to\infty} f_{\epsilon_n}''(u_{\epsilon_n}^{T^+,T^-})[(P_t^{\epsilon_n} u_{\epsilon_n}^{T^+,T^-})^+]^2 > (2-p)\varrho^p \quad \text{for some} \quad t \in T^+ \qquad (3.51)
$$

or

$$\lim_{n\to\infty} f''_{\epsilon_n}(u_{\epsilon_n}^{T^+,T^-})[(P_t^{\epsilon_n} u_{\epsilon_n}^{T^+,T^-})^-]^2 > (2-p)\varrho^p \quad \text{for some} \quad t \in T^-. \quad (3.52)$$

From Lemma 3.9 and Proposition 3.10 we have (up to a subsequence) P^{ϵ_n} $u_{\epsilon_n}^{T^+,T^-} \to 0$ and $P_t^{\epsilon_n} u_{\epsilon_n}^{T^+,T^-} \to u_t \in H_0^{1,2}(\Omega_t')$ in $L^p(\Omega_t')$, in $L^q(\Omega_t')$ and a.e. in Ω_t' for all $t \in \{1,\dots,k\}$.

Let us consider, for example, the case that (3.51) holds for some $t \in T^+$. Notice that $||u_t^+||_{(\lambda,p)} \geq \varrho$ since $u_{\epsilon_n}^{T^+,T^-} \in K_{\epsilon_n,\varrho}^{T^+,T^-}$ for all $n \geq 1$. Moreover assumptions $(g.2)$ and $(g.3)$ allow us to apply the Lebesgue convergence theorem obtaining

$$\lim_{n\to\infty} \int_{\Omega_t'} \gamma_{\epsilon_n}(x, u_{\epsilon_n}^{T^+,T^-})|u_{\epsilon_n}^{T^+,T^-}|^{p-1}(P_t^{\epsilon_n} u_{\epsilon_n}^{T^+,T^-})^+ \, dx = \int_{\Omega_t'} \lambda(x)|u_t^+|^p \, dx$$

and

$$\lim_{n\to\infty} \int_{\Omega_t'} \tilde{\gamma}_{\epsilon_n}(x, u_{\epsilon_n}^{T^+,T^-})|u_{\epsilon_n}^{T^+,T^-}|^{p-2}[(P_t^{\epsilon_n} u_{\epsilon_n}^{T^+,T^-})^+]^2 \, dx$$
$$= (p-1) \int_{\Omega_t'} \lambda(x)|u_t^+|^p \, dx.$$

Hence we have

$$\lim_{n\to\infty} f''_{\epsilon_n}(u_{\epsilon_n}^{T^+,T^-})[(P_t^{\epsilon_n} u_{\epsilon_n}^{T^+,T^-})^+]^2 = (2-p)||u_t^+||_{(\lambda,p)}^p \leq (2-p)\varrho^p,$$

which contradicts (3.51).

An analogous contradiction could be found in the case that (3.52) holds for some $t \in T^-$.

□

Proof of Theorem 1.1. In order to prove this theorem, it suffices to show that for all T^+, T^-, subsets of $\{1, \dots, k\}$, and for all $\epsilon > 0$ small enough, every function $u_\epsilon^{T^+,T^-}$, which minimizes f_ϵ in the set $M_\epsilon^{T^+,T^-} \cap K_{\epsilon,\varrho}^{T^+,T^-} \cap N_{\epsilon,B}^{T^+,T^-}$, is a critical point for f_ϵ. The behaviour of $u_\epsilon^{T^+,T^-}$ as $\epsilon \to 0$ (see Proposition 3.10) guarantees that, for ϵ small enough, different solutions correspond to different choices of the pair T^+, T^-.

Notice that, for ϵ sufficiently small, we have

$$f_\epsilon''(u_\epsilon^{T^+,T^-})[(P_t^\epsilon u_\epsilon^{T^+,T^-})^+]^2 < 0 \quad \forall t \in T^+ \tag{3.53}$$

and

$$f_\epsilon''(u_\epsilon^{T^+,T^-})[(P_t^\epsilon u_\epsilon^{T^+,T^-})^-]^2 < 0 \quad \forall t \in T^-. \tag{3.54}$$

Set $T^+ = \{l_1, \dots, l_r\}$, $T^- = \{m_1, \dots, m_s\}$ (with $r, s \le k$ and $l_i \ne l_j$, $m_i \ne m_j$ for $i \ne j$) and consider the functions σ_ϵ and ψ_ϵ defined as follows. The function $\sigma_\epsilon : \mathbb{R}^{r+s} \to H_0^{1,2}(\Omega)$ is defined, for all $\xi = (\xi_1^+, \dots, \xi_r^+, \xi_1^-, \dots, \xi_s^-)$, by

$$\sigma_\epsilon(\xi) = u_\epsilon^{T^+,T^-} + \sum_{t=1}^{r} \xi_t^+ (P_{l_t}^\epsilon u_\epsilon^{T^+,T^-})^+ - \sum_{t=1}^{s} \xi_t^- (P_{m_t}^\epsilon u_\epsilon^{T^+,T^-})^-, \tag{3.55}$$

while $\psi_\epsilon : H_0^{1,2}(\Omega) \to \mathbb{R}^{r+s}$ is the function defined by

$$\begin{aligned} \psi_\epsilon(u) = -\big(&f_\epsilon'(u)[(P_{l_1}^\epsilon u)^+], \dots, f_\epsilon'(u)[(P_{l_r}^\epsilon u)^+], \\ &f_\epsilon'(u)[(P_{m_1}^\epsilon u)^-], \dots, f_\epsilon'(u)[(P_{m_s}^\epsilon u)^-]\big). \end{aligned} \tag{3.56}$$

Notice that $\psi_\epsilon(u) = 0$ if and only if $u \in M_\epsilon^{T^+,T^-}$.

For all $\delta > 0$, let us set $B(\delta) = \{\xi \in \mathbb{R}^{r+s} : |\xi| \leq \delta\}$ and $\partial B(\delta) = \{\xi \in \mathbb{R}^{r+s} : |\xi| = \delta\}$. Because of (3.53) and (3.54), since $u_\epsilon^{T^+,T^-} \in M_\epsilon^{T^+,T^-}$, there exists $\overline{\delta} > 0$, sufficiently small, such that

$$f_\epsilon \circ \sigma_\epsilon(0) > f_\epsilon \circ \sigma_\epsilon(\xi) \quad \forall \xi \in B(\overline{\delta}) \setminus \{0\} \tag{3.57}$$

and

$$(\xi \cdot (\psi_\epsilon \circ \sigma_\epsilon(\xi))) > 0 \quad \forall \xi \in \partial B(\overline{\delta}) \tag{3.58}$$

(here $(\cdot)$ denotes the usual scalar product in $\mathbb{R}^{r+s}$).
Moreover, taking also into account Proposition 3.10, $\overline{\delta}$ can be chosen small enough, in such a way to have, in addition,

$$\sigma_\epsilon(\xi) \in K_{\epsilon,\varrho}^{T^+,T^-} \cap N_{\epsilon,B}^{T^+,T^-} \quad \forall \xi \in B(\overline{\delta}).$$

In order to show that $u_\epsilon^{T^+,T^-}$ is a critical point for f_ϵ, that is

$$f_\epsilon'(u_\epsilon^{T^+,T^-})[w] = 0 \quad \forall w \in H_0^{1,2}(\Omega), \tag{3.59}$$

we assume, by contradiction, that there exists $\overline{w} \in H_0^{1,2}(\Omega)$ such that $f_\epsilon'(u_\epsilon^{T^+,T^-})[\overline{w}] < 0$. Since $f_\epsilon'(u)[\overline{w}]$ depends continuously on u, there exists a neighbourhood $I_\epsilon(u_\epsilon^{T^+,T^-})$ of $u_\epsilon^{T^+,T^-}$ in $H_0^{1,2}(\Omega)$ such that

$$f_\epsilon'(u)[\overline{w}] < 0 \quad \forall u \in I_\epsilon(u_\epsilon^{T^+,T^-}).$$

For all $\delta > 0$, let $z_\delta \in C^0(\mathbb{R}^{r+s}; \mathbb{R})$ be a function satisfying the following conditions:

$$0 \leq z_\delta(\xi) \leq \delta \quad \forall \xi \in \mathbb{R}^{r+s}, \tag{3.60}$$

$$z_\delta(\xi) = \delta \quad \text{if} \quad |\xi| \leq \delta, \tag{3.61}$$

$$z_\delta(\xi) = 0 \quad \text{if} \quad |\xi| \geq 2\delta. \tag{3.62}$$

Hence we can choose $\delta > 0$ small enough, such that

$$z_\delta(\xi) = 0 \quad \forall \xi \in \partial B(\bar{\delta}), \tag{3.63}$$

$$\sigma_\epsilon(\xi) + \tau z_\delta(\xi)\overline{w} \in I_\epsilon(u_\epsilon^{T^+,T^-}) \quad \forall \xi \in B(2\delta), \quad \forall \tau \in [0,1], \tag{3.64}$$

$$\sigma_\epsilon(\xi) + \tau z_\delta(\xi)\overline{w} \in K_{\epsilon,\varrho}^{T^+,T^-} \cap N_{\epsilon,B}^{T^+,T^-} \quad \forall \xi \in B(\bar{\delta}), \quad \forall \tau \in [0,1]. \tag{3.65}$$

Now, since $f_\epsilon'(u)[\overline{w}] < 0$ for all $u \in I_\epsilon(u_\epsilon^{T^+,T^-})$, it follows that

$$\frac{d}{d\tau} f_\epsilon(\sigma_\epsilon(\xi) + \tau z_\delta(\xi)\overline{w}) \leq 0 \quad \forall \xi \in B(\bar{\delta}), \quad \forall \tau \in [0,1]$$

and

$$\frac{d}{d\tau} f_\epsilon(\sigma_\epsilon(\xi) + \tau z_\delta(\xi)\overline{w}) < 0 \quad \forall \xi \in B(\delta), \quad \forall \tau \in [0,1].$$

Therefore

$$f_\epsilon(\sigma_\epsilon(\xi) + z_\delta(\xi)\overline{w}) \leq f_\epsilon(\sigma_\epsilon(\xi)) \quad \forall \xi \in B(\overline{\delta})$$

and

$$f_\epsilon(\sigma_\epsilon(\xi) + z_\delta(\xi)\overline{w}) < f_\epsilon(\sigma_\epsilon(\xi)) \quad \forall \xi \in B(\delta).$$

Since 0 is the unique maximum point for $f_\epsilon \circ \sigma_\epsilon$ in $B(\overline{\delta})$, it follows that

$$f_\epsilon(\sigma_\epsilon(\xi) + z_\delta(\xi)\overline{w}) < f_\epsilon(\sigma_\epsilon(0)) \quad \forall \xi \in B(\overline{\delta}). \tag{3.66}$$

Taking into account that $\sigma_\epsilon(0) = u_\epsilon^{T^+,T^-}$ minimizes f_ϵ in $M_\epsilon^{T^+,T^-} \cap K_{\epsilon,\varrho}^{T^+,T^-} \cap N_{\epsilon,B}^{T^+,T^-}$ and that $\sigma_\epsilon(\xi) + z_\delta(\xi)\overline{w} \in K_{\epsilon,\varrho}^{T^+,T^-} \cap N_{\epsilon,B}^{T^+,T^-} \quad \forall \xi \in B(\overline{\delta})$, from (3.66) we infer that

$$\{\sigma_\epsilon(\xi) + z_\delta(\xi)\overline{w} \, : \, \xi \in B(\overline{\delta})\} \cap M_\epsilon^{T^+,T^-} = \emptyset. \tag{3.67}$$

Now, consider the continuous function $\varphi_\epsilon : B(\overline{\delta}) \to \mathbb{R}^{r+s}$, defined by

$$\varphi_\epsilon(\xi) = \psi_\epsilon(\sigma_\epsilon(\xi) + z_\delta(\xi)\overline{w})$$

(see (3.56)).

Because of (3.63) and (3.58) we have

$$(\xi \cdot \varphi_\epsilon(\xi)) > 0 \quad \forall \xi \in \partial B(\overline{\delta}).$$

Hence, by well known topological arguments (see, for example, [12]), we infer that there exists $\overline{\xi} \in B(\overline{\delta})$ such that $\varphi_\epsilon(\overline{\xi}) = 0$, i.e.

$$\sigma_\epsilon(\bar{\xi}) + z_\delta(\bar{\xi})\overline{w} \in M_\epsilon^{T^+,T^-},$$

in contradiction with (3.67).

All the qualitative properties of the solutions $v_\epsilon^{T^+,T^-} = \epsilon^{\frac{1}{p-2}} u_\epsilon^{T^+,T^-}$ follow easily from Proposition 3.10.

□

Remark 3.12. Under the assumption that $g(x,\cdot)$ is an odd function, well known results guarantee the existence of infinitely many solutions for $\mathfrak{D}_\epsilon$. On the contrary, no symmetry assumption is required in Theorem 1.1. The multiplicity result and the qualitative properties of the solutions of $\mathfrak{D}_\epsilon$, for ϵ small enough, depend only on the geometrical properties of the degeneration set $\mathcal{D}$ and on the behaviour of $g(x,\tau)$ as $\tau \to 0$. Moreover, let us point out that condition $(g.3)$ could also be weakened by requiring only that there exist $p^+, p^- > 2$, with $p^+, p^- < \frac{2N}{N-2}$ if $N \geq 3$, and two strictly positive functions $\lambda^+, \lambda^- : \Omega \to \mathbb{R}$, with $\lambda^+, \lambda^-, \frac{1}{\lambda^+}, \frac{1}{\lambda^-} \in L^\infty(\Omega)$, such that

$$\lim_{\tau \to 0^+} \frac{g'(x,\tau)}{(p^+ - 1)|\tau|^{p^+-2}} = \lambda^+(x) \quad \text{and} \quad \lim_{\tau \to 0^-} \frac{g'(x,\tau)}{(p^- - 1)|\tau|^{p^--2}} = \lambda^-(x).$$

In fact it is easy to verify that all the claims and the proofs continue to hold (with obvious modifications).

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