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Multibump solutions for a class of nonlinear elliptic problems

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Abstract. The paper is concerned with a class of semilinear elliptic Dirichlet problems approximating degenerate equations. By using variational methods, it is proved that, if the degeneration set consists of k connected components, then there exist at least $2^k - 1$ multibump positive solutions.

1. Introduction and statement of the main theorem

Let Ω be a smooth bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ and $g(x,t)$ a given function which behaves like $t|t|^{p-2}$ as $t \to 0$, with $p > 2$ and $p < \frac{2N}{N-2}$ if $N \geq 3$.

We are concerned with the existence and multiplicity of nontrivial solutions for problems like

$$\begin{cases} div(a_\epsilon(x)Du) + g(x,u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where, for all $\epsilon > 0$ and $x \in \Omega$, $a_\epsilon(x) = (a_\epsilon^{i,j}(x))$ is a positive defined symmetric $N \times N$ matrix with coefficients $a_\epsilon^{i,j}$ belonging to $L^\infty(\Omega, \mathbb{R})$.

We will assume that the matrix $a_\epsilon(x)$ degenerates, as $\epsilon \to 0$, for all x in a suitable subset of Ω (the degeneration set). Our aim is to relate the number of nontrivial solutions, for $\epsilon > 0$ small enough, to the geometric properties of the degeneration set.

Some phenomena, pointed out in [31], describing the behaviour of the solutions as $\epsilon \to 0$, allow us to show that, if the degeneration set consists of k connected components, then there exist (for all $\epsilon > 0$ small enough) at least $k+1$ positive solutions (see [20]) and at least k^2 nodal solutions (see [21]) having exactly two nodal regions (i.e. both the positive and the negative part of the solutions have connected supports).

Similar phenomena also arise in some recent results concerning the spiked solutions to singularly perturbed semilinear equations, such as the nonlinear

Schrödinger equation (as in [17], [25], [26], [27], [32], [14], [23], [24], etc.) and the Ginzburg-Landau equations (see [8], [2], [35], [19], [15], etc.).

In several nonlinear problems, when it is possible to show that the solutions tend to be localized near some regions or points, one can relate the number of the solutions to the metric and topological properties of the domain (see, for example, [5], [6], [7], [11] and the references therein).

Concentration phenomena of this type play a fundamental role in existence, non existence and multiplicity results for elliptic problems involving critical or supercritical Sobolev exponents, that have been very much investigated in recent years (see, for example, [9], [10], [28], [29], [30], [16], [13], [33], [34], [3] and the references therein).

In our case, although the nonlinear term g has a subcritical growth, these phenomena occur because of the degeneration of the equation.

Let us specify the conditions on the matrix $a_\epsilon(x)$ we shall assume throughout the paper:

(a.1) for all $\epsilon > 0$ and for almost all $x \in \Omega$ there exist $\Lambda_1 = \Lambda_1(\epsilon, x) > 0$ and $\Lambda_2 = \Lambda_2(\epsilon, x) > 0$ such that

$$\Lambda_1 |\xi|^2 \le a_\epsilon^{i,j}(x)\xi_i\xi_j \le \Lambda_2|\xi|^2 \quad \forall\, \xi \in \mathbb{R}^N \tag{1.1}$$

(here and later on we write, as it is usual, $a_\epsilon^{i,j}(x)\xi_i\xi_j$ instead of $\sum_{i,j=1}^N a_\epsilon^{i,j}(x)\xi_i\xi_j$);

(a.2)

$$\liminf_{\epsilon\to 0} \frac{1}{\epsilon} \inf_{x\in\Omega} \Lambda_1(\epsilon, x) > 0; \tag{1.2}$$

(a.3) there exist k nonempty subsets of Ω, we say $\Omega_1, \cdots, \Omega_k$ (the degeneration subsets for $a_\epsilon(x)$), such that

$$\limsup_{\epsilon\to 0} \frac{1}{\epsilon} \sup \left\{ \Lambda_2(\epsilon, x) \,:\, x \in \bigcup_{s=1}^k \Omega_s \right\} < +\infty\,; \tag{1.3}$$

(a.4) for all $\eta > 0$

$$\liminf_{\epsilon\to 0} \inf \left\{ \Lambda_1(\epsilon, x) :\ x \in \Omega \setminus \bigcup_{t=1}^k \Omega_t(\eta) \right\} > 0, \tag{1.4}$$

where $\Omega_t(\eta) = \{x \in \Omega :\ d(x, \Omega_t) < \ \eta\}$;

(D) $\Omega_1, \ldots, \Omega_k$ are smooth domains strictly contained in Ω (i.e. $\overline{\Omega}_s \subset \Omega \;\forall\, s = 1, \ldots, k$). For all $s = 1, \ldots, k$ let us denote by C_s the union of the connected components of $\overline{\Omega} \setminus \Omega_s$ which don't meet $\partial\Omega$ and set $\Omega_s' = \Omega_s \cup C_s$.
We require that the subsets $\overline{\Omega_1'}, \ldots, \overline{\Omega_k'}$ are pairwise disjoined and that every connected component of $\Omega \setminus \bigcup_{t=1}^k \Omega_t'$ meets $\partial\Omega$.

Roughly speaking, condition (D) means that, although the disjoint components Ω_s ($s = 1, \dots, k$) of the degeneration set may have holes (i.e. $\Omega_s' \neq \Omega_s$), they are contained in pairwise disjoint subsets $\overline{\Omega_s'}$, without holes, whose union does not contain holes.

The positive solutions $u_{\epsilon,1}, \dots, u_{\epsilon,k+1}$ obtained in [20] have the following property:

$$\lim_{\epsilon\to 0} \left(\int_{\Omega} |u_{\epsilon,t}|^p \, dx \right)^{-1} \int_{\Omega_t'} |u_{\epsilon,t}|^p \, dx = 1 \quad \forall t \in \{1, \dots, k\}$$

and there exist at most two subsets among $\Omega_1', \dots, \Omega_k'$ (we say Ω_{t_1}' and Ω_{t_2}') such that

$$\lim_{\epsilon\to 0} \left(\int_{\Omega} |u_{\epsilon,k+1}|^p \, dx \right)^{-1} \int_{\Omega_{t_1}' \cup \Omega_{t_2}'} |u_{\epsilon,k+1}|^p \, dx = 1.$$

Analogous properties hold for the positive and the negative parts of the nodal solutions $u_{\epsilon,r,s}$ ($r, s \in \{1, \dots, k\}$) obtained in [21]. In fact we have for all $r, s \in \{1, \dots, k\}$:

$$\lim_{\epsilon\to 0} \left(\int_{\Omega} |u^+_{\epsilon,r,s}|^p \, dx \right)^{-1} \int_{\Omega_r'} |u^+_{\epsilon,r,s}|^p \, dx = 1$$

and

$$\lim_{\epsilon\to 0} \left(\int_{\Omega} |u^-_{\epsilon,r,s}|^p \, dx \right)^{-1} \int_{\Omega_s'} |u^-_{\epsilon,r,s}|^p \, dx = 1.$$

These properties show that, for $\epsilon > 0$ small enough, the solutions are localized near some of the subsets $\Omega_1', \dots, \Omega_k'$ of the degeneration set.

So a natural question arises: is it possible to find multibump solutions, i.e. solutions which can be decomposed as sum of functions localized near different subsets chosen among $\Omega_1', \dots, \Omega_k'$?

In this paper we answer this question obtaining positive solutions of this type.

In recent years several papers have been devoted to study multibump solutions for elliptic equations (see, for example, [33], [1], [18], [4]) as well as for hamiltonian systems (see [12] and the references therein).

Let us specify the assumptions required on the function $g : \Omega \times \mathbb{R}^+ \to \mathbb{R}$:

(g.1) for all $t \geq 0$, $g(x, t)$ is measurable with respect to x; for almost all $x \in \Omega$, $g(x, t)$ is a C^1- function with respect to t;

(g.2) there exist positive constants a and q, with $q < \frac{2N}{N-2}$ if $N \geq 3$, such that, for all $t \geq 0$ and for almost all $x \in \Omega$,

$$|g(x, t)| \leq a + at^{q-1} \tag{1.5}$$

and

$$|g'(x,t)| \le a + at^{q-2} \tag{1.6}$$

where $g'(x,t)$ denotes the derivative of g with respect to t;

(g.3) there exist $p > 2$, with $p < \frac{2N}{N-2}$ if $N \ge 3$, and a strictly positive function $\lambda : \Omega \to \mathbb{R}^+$, with $\lambda \in L^\infty(\Omega)$ and $\frac{1}{\lambda} \in L^\infty(\Omega)$, such that

$$\lim_{t\to 0^+} \frac{g'(x,t)}{(p-1)t^{p-2}} = \lambda(x) \text{ uniformly on } \Omega; \tag{1.7}$$

(g.4) there exists $\theta \in]0, \frac{1}{2}[$ such that

$$G(x,t) \le \theta t g(x,t) \tag{1.8}$$

for all $t \ge 0$ and for almost all $x \in \Omega$, where $G(x,t) = \int_0^t g(x,\tau)\,d\tau$.

We can now state the following multiplicity result:

Theorem 1.1. *Assume that conditions (a.1)–(a.4), (D), (g.1)–(g.4) are satisfied. Then there exists $\bar\epsilon > 0$ such that, for all $\epsilon \in]0, \bar\epsilon[$, the problem*

$$P_\epsilon \quad \begin{cases} div(a_\epsilon(x)Du) + g(x,u) = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least $2^k - 1$ distinct positive solutions.

Indeed, for every subset $\{t_1, \dots, t_r\}$ of $\{1, \dots, k\}$ and for all $\epsilon \in]0, \bar\epsilon[$, there exists a solution $v_\epsilon^{t_1,\dots,t_r}$ of P_ϵ such that

$$\lim_{\epsilon\to 0} \epsilon^{\frac{p}{2-p}} \int_{\Omega \setminus \bigcup_{s=1}^{r} \Omega'_{t_s}} (v_\epsilon^{t_1,\dots,t_r})^p \, dx = 0$$

$$\liminf_{\epsilon\to 0} \epsilon^{\frac{p}{2-p}} \int_{\Omega'_{t_s}} (v_\epsilon^{t_1,\dots,t_r})^p \, dx > 0 \quad \forall s \in \{1, \dots, r\}$$

(other properties of $v_\epsilon^{t_1,\dots,t_r}$ are described in Sect. 4).

Notice that it is possible to obtain more than one solution even if the degeneration set is connected (i.e. $k = 1$), but it is topologically complex: in [22] we estimate the number of solutions of P_ϵ by the Ljusternik-Schnirelman category of the degeneration set (under somehow different assumptions on g).

The paper is organized as follows. In Sect. 2 we state some preliminary results concerning the properties of the functional f_ϵ related to our problem. In Sect. 3, for every choice of the subsets $\{t_1, \dots, t_r\} \subseteq \{1, \dots, k\}$, we introduce suitable subsets of $H_0^{1,2}(\Omega)$ and prove the existence of the minimum for f_ϵ constrained on each of these subsets. Moreover we state some properties of the minimizing functions $u_\epsilon^{t_1,\dots,t_r}$, which are used in Sect. 4 in order to show, for $\epsilon > 0$ small enough, that $u_\epsilon^{t_1,\dots,t_r}$ is a local minimum point for f_ϵ constrained on a suitable subset $M_\epsilon^{t_1,\dots,t_r}$ of $H_0^{1,2}(\Omega)$, that $M_\epsilon^{t_1,\dots,t_r}$ is a smooth submanifold in a neighbourhood of $u_\epsilon^{t_1,\dots,t_r}$ and, finally, that $u_\epsilon^{t_1,\dots,t_r}$ is a critical point for f_ϵ, giving rise to a solution $v_\epsilon^{t_1,\dots,t_r} = \epsilon^{\frac{1}{p-2}} u_\epsilon^{t_1,\dots,t_r}$ of problem P_ϵ, satisfying the properties described in Theorem 1.1.

Thus we obtain, for all $r \in \{1,\dots,k\}$, at least $\binom{k}{r}$ solutions having r bumps; so, on the whole, we get $2^k - 1$ distinct positive solutions.

2. Preliminary results

Throughout the paper $H_0^{1,2}(\Omega)$ will denote the usual Sobolev space endowed with the norm $\|u\| = (\int_\Omega |Du|^2\,dx)^{\frac12}$, while we will denote by $\|u\|_p = (\int_\Omega |u|^p\,dx)^{\frac1p}$ the usual norm in $L^p(\Omega)$.

In $L^p(\Omega)$ we will also consider the following norm:

$$\|u\|_{(\lambda,p)} = (\int_\Omega \lambda(x)|u|^p\,dx)^{\frac1p}$$

where λ is the positive function which appears in (g.3). $\|\cdot\|_{(\lambda,p)}$ is equivalent to $\|\cdot\|_p$ in $L^p(\Omega)$.

A function $v \in H_0^{1,2}(\Omega)$ is a weak solution for P_ϵ if and only if $u = \epsilon^{-\frac{1}{p-2}} v$ is a nontrivial critical point for the functional $f_\epsilon : H_0^{1,2}(\Omega) \to \mathbb{R}$

$$f_\epsilon(u) = \frac12 \int_\Omega \frac{a_\epsilon^{ij}}{\epsilon} \partial_{x_i} u \partial_{x_j} u\,dx - \frac{1}{\epsilon^{\frac{p}{p-2}}} \int_\Omega G(x, \epsilon^{\frac{1}{p-2}} u)\,dx. \tag{2.1}$$

Here we intend that the function g is extended to $\Omega \times \mathbb{R}$ by setting $g(x,t) = 0$ for all $t \le 0$ and for $x \in \Omega$. Let us observe that (g.3) and (g.4) imply $g(x,t) > 0$ if $t > 0$. So every nontrivial critical point for f_ϵ is a positive function, which gives rise to a solution of problem P_ϵ by the maximum principle.

Definition 2.1. *Let $\Omega_1,\dots,\Omega_k, C_1\dots,C_k,\Omega_1',\dots\Omega_k'$ be as in condition (D) of Sect.* 1.

For every $u \in H_0^{1,2}(\Omega)$ and $\epsilon > 0$ let $P^\epsilon(u)$ be the function in $H_0^{1,2}(\Omega)$ such that

$$P^\epsilon(u) \equiv u \ \ in \ \ \Omega \setminus \bigcup_{t=1}^k \Omega_t'$$

$$\int_{\Omega_t'} a_\epsilon^{ij}(x)\partial_{x_i} P^\epsilon(u)\partial_{x_j} v\,dx = 0 \quad \forall v \in H_0^{1,2}(\Omega_t'), \quad \forall t = 1,\dots,k.$$

Set for all $t = 1,\dots,k$

$$P_t^\epsilon(u) = u - P^\epsilon(u) \quad in \quad \Omega_t', \quad P_t^\epsilon(u) = 0 \quad elsewhere.$$

Thus it results

$$u = P_1^\epsilon(u) + \ldots + P_k^\epsilon(u) + P^\epsilon(u) \quad \forall u \in H_0^{1,2}(\Omega)$$

with $P_t^\epsilon(u) \in H_0^{1,2}(\Omega_t')$ *for all* $t = 1, \ldots, k$ *(indeed* P_t^ϵ *is a projection of* $H_0^{1,2}(\Omega)$ *on* $H_0^{1,2}(\Omega_t')$*).*

Definition 2.2. *Let* r *be a fixed integer with* $1 \le r \le k$.

Let $t_1, \ldots, t_r$ *be* r *distinct integers such that* $1 \le t_i \le k$ *for all* $i = 1, \ldots, r$. *We set*

$$M_\epsilon^{t_1,\ldots,t_r} = \left\{ u \in H_0^{1,2}(\Omega) \,:\, f_\epsilon'(u)[P_t^\epsilon(u)] = 0 \quad \forall t \in \{t_1, \ldots, t_r\} \right\}.$$

We shall obtain $2^k - 1$ solutions for P_ϵ in the following way: first, for every choice of the subset $\{t_1, \ldots, t_r\} \subseteq \{1, \ldots, k\}$, we obtain a constrained minimum point for f_ϵ on $M_\epsilon^{t_1,\ldots,t_r}$, for all $\epsilon > 0$ small enough; then we show that, for $\epsilon > 0$ small enough, different minimum points correspond to different choices of the subsets $\{t_1, \ldots, t_r\}$; finally we prove that these minimum points are indeed critical points for f_ϵ for all $\epsilon > 0$ small enough; hence we obtain the desired number of solutions.

We need some further notations and definitions which will be useful in the next sections.

For simplicity of notation we write for all $u \in H_0^{1,2}(\Omega)$

$$G_\epsilon(x, u) = \frac{1}{\epsilon^{\frac{p}{p-2}}} G(x, \epsilon^{\frac{1}{p-2}} u), \tag{2.2}$$

$$g_\epsilon(x, u) = \frac{1}{\epsilon^{\frac{p-1}{p-2}}} g(x, \epsilon^{\frac{1}{p-2}} u) \tag{2.3}$$

and

$$g_\epsilon'(x, u) = \frac{1}{\epsilon} g'(x, \epsilon^{\frac{1}{p-2}} u). \tag{2.4}$$

Moreover, for every $u \in H_0^{1,2}(\Omega)$, u^+ and u^- denote respectively the positive and the negative part of u.

Definition 2.3. *We call*

$$\underline{\Lambda} = \liminf_{\epsilon \to 0} \frac{1}{\epsilon} \inf \{\Lambda_1(\epsilon, x) \,:\, x \in \Omega\} \tag{2.5}$$

and

$$\overline{\Lambda} = \limsup_{\epsilon \to 0} \frac{1}{\epsilon} \sup \left\{ \Lambda_2(\epsilon, x) \,:\, x \in \bigcup_{t=1}^{k} \Omega_t \right\} \tag{2.6}$$

(see (a.1), (a.2), (a.3)).

Definition 2.4. *For all $t = 1, \dots, k$ let*

$$m_t = \inf \left\{ \int_{\Omega_t} |Du|^2 \, dx \; : \; u \in H_0^{1,2}(\Omega_t'), \right.$$

$$\left. \int_{\Omega_t' \setminus \Omega_t} |Du|^2 \, dx = 0 \, , \; \int_{\Omega_t'} \lambda(x) |u(x)|^p \, dx = 1 \right\}$$

and set $m = \min_{t \in \{1,\dots,k\}} m_t$.

Notice that $\left\{ u \in H_0^{1,2}(\Omega_t') : \int_{\Omega_t' \setminus \Omega_t} |Du|^2 \, dx = 0, \int_{\Omega_t'} \lambda(x)|u(x)|^p \, dx = 1 \right\} \neq \emptyset$ *and the infimum m_t is achieved since $p < \frac{2N}{N-2}$. For all $t \in \{1, \dots, k\}$, let $v_t \in H_0^{1,2}(\Omega_t')$, with $\int_{\Omega_t' \setminus \Omega_t} |Dv_t|^2 \, dx = 0$ and $\int_{\Omega_t'} \lambda(x)|v_t(x)|^p \, dx = 1$, be a positive function that realizes m_t.*

The imposition of vanishing of Du on the set $\Omega_t' \setminus \Omega_t$ (the union of the holes of Ω_t) will be clear below: it is related to the fact that, because of condition (a.4), the critical points for f_ϵ we shall find are functions which tend to be flat in $\Omega \setminus \bigcup_{t=1}^{k} \Omega_t$ as $\epsilon \to 0$; moreover, since they belong to $H_0^{1,2}(\Omega)$, they converge to zero in $\Omega \setminus \bigcup_{t=1}^{k} \Omega_t'$, because of condition (D), and the limit function can be decomposed as sum of functions in $H_0^{1,2}(\Omega_t')$ ($t = 1, \dots, k$).

Definition 2.5. *Let $\Gamma_\epsilon : \Omega \times \mathbb{R} \to \mathbb{R}$, $\gamma_\epsilon : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\tilde{\gamma}_\epsilon : \Omega \times \mathbb{R} \to \mathbb{R}$ be the functions defined by*

$$\Gamma_\epsilon(x,t) = \begin{cases} \frac{G_\epsilon(x,t)}{t^p} & \text{for } t > 0 \\ \frac{\lambda(x)}{p} & \text{for } t = 0 \\ 0 & \text{for } t < 0, \end{cases} \tag{2.7}$$

$$\gamma_\epsilon(x,t) = \begin{cases} \frac{g_\epsilon(x,t)}{t^{p-1}} & \text{for } t > 0 \\ \lambda(x) & \text{for } t = 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{2.8}$$

and

$$\tilde{\gamma}_\epsilon(x,t) = \begin{cases} \frac{g_\epsilon'(x,t)}{t^{p-2}} & \text{for } t > 0 \\ (p-1)\lambda(x) & \text{for } t = 0 \\ 0 & \text{for } t < 0. \end{cases} \tag{2.9}$$

Because of (1.7), Γ_ϵ, γ_ϵ and $\tilde{\gamma}_\epsilon$ are continuous for $t \to 0^+$ uniformly in Ω.

We now prove some properties of f_ϵ' which are used to obtain the existence of local minimum points for f_ϵ constrained on $M_\epsilon^{t_1,\dots,t_r}$ (see Definition 2.2).

Lemma 2.6. *For all* $\mu > 0$ *and* $\varrho \in]0, (\underline{\Lambda} m)^{\frac{1}{p-2}}[$, *there exist* $B > 0$ *and* $\overline{\epsilon} > 0$ *such that, if* $w \in H_0^{1,2}(\Omega)$ *with* $\|w\| \leq B$ *and* $\epsilon \in]0, \overline{\epsilon}[$, *then*

$$\inf \left\{ f_\epsilon'(u + w)[u] \, : \, u \in H_0^{1,2}(\Omega_t'), \left(\int_{\Omega_t'} |Du|^2 \, dx \right)^{\frac{1}{2}} \leq \mu, \; \|u^+\|_{(\lambda,p)} = \varrho \right\} > 0,$$

for all $t \in \{1, \dots k\}$.

Proof. By contradiction, suppose that for all $B > 0$ and $\overline{\epsilon} > 0$ there exist $\epsilon \in]0, \overline{\epsilon}[$ and $w \in H_0^{1,2}(\Omega)$, with $\|w\| \leq B$, such that the assertion does not hold.

Hence there exist an infinitesimal sequence $(\epsilon_n)_{n \geq 1}$ of positive numbers and a sequence $(w_n)_{n \geq 1}$ of functions in $H_0^{1,2}(\Omega)$, with $\|w_n\| \to 0$ as $n \to \infty$, such that

$$\inf \left\{ f_{\epsilon_n}'(u + w_n)[u] \, : \, u \in H_0^{1,2}(\Omega_t') \, , \left(\int_{\Omega_t'} |Du|^2 \, dx \right)^{\frac{1}{2}} \leq \mu, \; \|u^+\|_{(\lambda,p)} = \varrho \right\} \leq 0 \quad \forall n \geq 1 \, . \tag{2.10}$$

It follows that one can find a sequence $(u_n)_{n \geq 1}$ of functions in $H_0^{1,2}(\Omega_t')$, with $\int_{\Omega_t'} |Du_n|^2 \, dx \leq \mu^2$ and $\|u_n^+\|_{(\lambda,p)} = \varrho$ for all $n \geq 1$, such that

$$\lim_{n \to \infty} f_{\epsilon_n}'(u_n + w_n)[u_n] \leq 0. \tag{2.11}$$

Since $(u_n)_{n \geq 1}$ is bounded in $H_0^{1,2}(\Omega_t')$, up to a subsequence $u_n \to u \in H_0^{1,2}(\Omega_t')$ weakly in $H_0^{1,2}(\Omega_t')$, in $L^p(\Omega_t')$, in $L^q(\Omega_t')$ and a.e. in Ω_t', with $\int_{\Omega_t'} |Du|^2 \, dx \leq \mu^2$ and $\|u^+\|_{(\lambda,p)} = \varrho$.

Moreover, since $w_n \to 0$ in $H_0^{1,2}(\Omega)$ and

$$\liminf_{\epsilon \to 0} \frac{1}{\epsilon} \inf \left\{ \Lambda_1(\epsilon, x) \, : \, x \in \bigcup_{j=1}^k \Omega_j(\eta) \right\} = +\infty \;\; \forall \; \eta > 0,$$

(2.11) implies

$$\int_{\Omega_t' \setminus \Omega_t} |Du|^2 \, dx = 0. \tag{2.12}$$

Since $\lim_{n \to \infty} \|w_n\| = 0$ and $u_n \to u$ a.e. in Ω_t' and since (g.3) holds, from (2.8) we have

$$\begin{aligned} &\lim_{n \to \infty} g_{\epsilon_n}(x, (u_n + w_n))u_n \\ &= \lim_{n \to \infty} \gamma_{\epsilon_n}(x, (u_n + w_n))|u_n + w_n|^{p-1} u_n = \lambda(x)(u^+)^p \quad \text{a.e. in } \Omega_t' \, ; \end{aligned}$$

moreover, assuming $q \geq p$, (g.2) and (g.3) imply the existence of $C > 0$ and $\eta > 0$ such that

$$|\gamma_{\epsilon_n}(x, u_n + w_n)|u_n + w_n|^{p-1}u_n| \leq$$

$$\begin{cases} \frac{(\lambda(x)+C)}{p}((p-1)|u_n+w_n|^p + |u_n|^p) & \text{if } |\epsilon_n^{\frac{1}{p-2}}(u_n+w_n)| \leq \eta; \\ \frac{a}{p\eta^{p-1}}((p-1)|u_n+w_n|^p + |u_n|^p) & \\ +\frac{a}{q}\epsilon_n^{\frac{q-p}{p-2}}((q-1)|u_n+w_n|^q + |u_n|^q) & \text{if } |\epsilon_n^{\frac{1}{p-2}}(u_n+w_n)| > \eta. \end{cases}$$

Since $u_n \to u$ and $w_n \to 0$ in L^p and L^q, from the generalized Lebesgue theorem we infer that

$$\lim_{n\to\infty} \int_{\Omega'_t} g_{\epsilon_n}(x, u_n + w_n)u_n \, dx = \int_{\Omega'_t} \lambda(x)(u^+)^p \, dx. \tag{2.13}$$

From (2.11), by using (2.5), (2.12), (2.13) and Definition 2.4, we obtain

$$\begin{aligned} 0 \quad &\geq \lim_{n\to\infty} f'_{\epsilon_n}(u_n + w_n)[u_n] \geq \underline{\Lambda} \liminf_{n\to\infty} \int_{\Omega'_t} |Du_n|^2 \, dx - \int_{\Omega'_t} \lambda(x)(u^+)^p \, dx \\ &\geq \underline{\Lambda} \frac{\int_{\Omega_t} |Du|^2 \, dx}{(\int_{\Omega'_t} \lambda(x)|u|^p \, dx)^{\frac{2}{p}}} (\int_{\Omega'_t} \lambda(x)|u|^p \, dx)^{\frac{2}{p}} - \int_{\Omega'_t} \lambda(x)(u^+)^p \, dx \\ &\geq \underline{\Lambda} m (\int_{\Omega'_t} \lambda(x)(u^+)^p \, dx)^{\frac{2}{p}} - \int_{\Omega'_t} \lambda(x)(u^+)^p \, dx \\ &= \underline{\Lambda} m \|u^+\|^2_{(\lambda,p)} - \|u^+\|^p_{(\lambda,p)} \\ &= \underline{\Lambda} m \varrho^2 - \varrho^p > 0 \end{aligned}$$

where the last inequality is due to the choice of ϱ.

Hence we get a contradiction. □

Definition 2.7. *For all $\epsilon > 0$ let $\Phi_\epsilon : H_0^{1,2}(\Omega) \to \mathbb{R}$ be the functional defined by*

$$\Phi_\epsilon(u) = \frac{1}{2} \int_\Omega g_\epsilon(x, u)u \, dx - \int_\Omega G_\epsilon(x, u) \, dx$$

(see (2.2) and (2.3)).

For all $\epsilon > 0$, Φ_ϵ belongs to $C^1(H_0^{1,2}(\Omega); \mathbb{R})$ and

$$\Phi'_\epsilon(u)[u] = \frac{1}{2} \left\{ \int_\Omega g'_\epsilon(x, u)u^2 \, dx - \int_\Omega g_\epsilon(x, u)u \, dx \right\}$$

(see (2.4)).

Lemma 2.8. *For all $\overline{\mu} > 0$ and $\overline{\varrho} > 0$, there exist $B > 0$ and $\overline{\epsilon} > 0$ such that, if $w \in H_0^{1,2}(\Omega)$ with $\|w\| \leq B$ and $\epsilon \in]0, \overline{\epsilon}[$, then*

$$\inf \left\{ \Phi_\epsilon'(u + w)[u] \,:\, u \in H_0^{1,2}(\Omega),\ \|u\| \leq \overline{\mu},\ \|u^+\|_{(\lambda,p)} \geq \overline{\varrho} \right\} > 0.$$

Proof. By contradiction, suppose that for all $B > 0$ and for all $\overline{\epsilon} > 0$ there exist $\epsilon \in]0, \overline{\epsilon}[$ and $w \in H_0^{1,2}(\Omega)$ with $\|w\| \leq B$, such that the claim does not hold.

Hence there exist an infinitesimal sequence $(\epsilon_n)_{n \geq 1}$ of positive numbers and a sequence $(w_n)_{n \geq 1}$ in $H_0^{1,2}(\Omega)$, with $\|w_n\| \to 0$ as $n \to \infty$, such that

$$\inf \left\{ \Phi_{\epsilon_n}'(u + w_n)[u] \,:\, u \in H_0^{1,2}(\Omega),\ \|u\| \leq \overline{\mu},\ \|u^+\|_{(\lambda,p)} \geq \overline{\varrho} \right\} \leq 0. \qquad (2.14)$$

It follows that there exists a sequence $(u_n)_{n \geq 1}$ of functions in $H_0^{1,2}(\Omega)$, with $\|u_n\| \leq \overline{\mu}$ and $\|u_n^+\|_{(\lambda,p)} \geq \overline{\varrho}$, such that

$$\lim_{n \to \infty} \Phi_{\epsilon_n}'(u_n + w_n)[u_n] \leq 0. \qquad (2.15)$$

Since $(u_n)_{n \geq 1}$ is bounded in $H_0^{1,2}(\Omega)$, up to a subsequence, $u_n \to u \in H_0^{1,2}(\Omega)$ weakly in $H_0^{1,2}(\Omega)$, in $L^p(\Omega)$, in $L^q(\Omega)$ and a.e. in Ω. It is clear that $\|u\| \leq \overline{\mu}$ and $\|u^+\|_{(\lambda,p)} \geq \overline{\varrho}$.

Since $\|w_n\| \to 0$ as $n \to \infty$, (g.3) yields

$$\begin{aligned} \lim_{n \to \infty} g_{\epsilon_n}(x, u_n + w_n) u_n &= \lim_{n \to \infty} \gamma_{\epsilon_n}(x, u_n + w_n) |u_n + w_n|^{p-1} u_n \\ &= \lambda(x)(u^+)^p \quad \text{a.e. in } \Omega \end{aligned} \qquad (2.16)$$

(see (2.8)) and

$$\begin{aligned} \lim_{n \to \infty} g_{\epsilon_n}'(x, u_n + w_n) u_n^2 &= \lim_{n \to \infty} \tilde{\gamma}_{\epsilon_n}(x, u_n + w_n) |u_n + w_n|^{p-2} u_n^2 \\ &= (p-1)\lambda(x)(u^+)^p \quad \text{a.e. in } \Omega \end{aligned} \qquad (2.17)$$

(see (2.9)).

Moreover, assuming $q \geq p$, (g.2) and (g.3) imply that there exist some constants $C > 0$ and $\eta > 0$ such that

$$|\gamma_{\epsilon_n}(x, u_n + w_n)|u_n + w_n|^{p-1} u_n| \leq$$

$$\begin{cases} \frac{(\lambda(x)+C)}{p}((p-1)|u_n + w_n|^p + |u_n|^p) & \text{if } |\epsilon_n^{\frac{1}{p-2}}(u_n + w_n)| \leq \eta \\ \frac{a}{p\eta^{p-1}}((p-1)|u_n + w_n|^p + |u_n|^p) & \\ +\frac{a}{q}\epsilon_n^{\frac{q-p}{p-2}}((q-1)|u_n + w_n|^q + |u_n|^q) & \text{if } |\epsilon_n^{\frac{1}{p-2}}(u_n + w_n)| > \eta; \end{cases}$$

and

$$|\tilde{\gamma}_{\epsilon_n}(x, u_n + w_n)|u_n + w_n|^{p-2} u_n^2| \leq$$

$$\begin{cases} (p-1)\frac{(\lambda(x)+C)}{p}((p-2)|u_n + w_n|^p + 2|u_n|^p) & \text{if } |\epsilon_n^{\frac{1}{p-2}}(u_n + w_n)| \leq \eta \\ \frac{a}{p\eta^{p-2}}((p-2)|u_n + w_n|^p + 2|u_n|^p) & \\ +\frac{a}{q}\epsilon_n^{\frac{q-p}{p-2}}((q-2)|u_n + w_n|^q + 2|u_n|^q) & \text{if } |\epsilon_n^{\frac{1}{p-2}}(u_n + w_n)| > \eta. \end{cases}$$

From (2.15), since $u_n \to u$ and $w_n \to 0$ in $L^p(\Omega)$ and in $L^q(\Omega)$ and since (2.17), (2.18) hold, the generalized Lebesgue theorem implies

$$0 \geq \lim_{n\to\infty} \Phi'_{\epsilon_n}(u_n + w_n)[u_n]$$
$$= \frac{1}{2}\left\{(p-1)\int_{\Omega'_t} \lambda(x)(u^+)^p\,dx - \int_{\Omega'_t} \lambda(x)(u^+)^p\,dx\right\} \geq \frac{p-2}{2}\overline{\varrho}^p > 0,$$

which is a contradiction. □

3. Constrained minimum points

Let us introduce two useful notations.

Definition 3.1. *Let $r \in \{1,\dots,k\}$ and $\{t_1,\dots,t_r\} \subseteq \{1,\dots,k\}$ with $t_i \neq t_j$ if $i \neq j$. For all $\varrho \in]0, (\underline{\Lambda} m)^{\frac{1}{p-2}}[$ (see Definition 2.1) let us set*

$$K^{t_1,\dots,t_r}_{\epsilon,\varrho} = \left\{u \in H^{1,2}_0(\Omega) : \|(P^\epsilon_{t_s} u)^+\|_{(\lambda,p)} \geq \varrho \quad \forall \quad s = 1,\dots,r\right\}.$$

Definition 3.2. *Let $B > 0$, $r \in \{1,\dots,k\}$ and $\{t_1,\dots,t_r\} \subseteq \{1,\dots,k\}$ with $t_i \neq t_j$ if $i \neq j$. Let us define*

$$N^{t_1,\dots,t_r}_{\epsilon,B} = \left\{u \in H^{1,2}_0(\Omega) : \int_\Omega |DP^\epsilon u|^2\,dx + \sum_{t\notin\{t_1,\dots,t_r\}} \int_{\Omega'_t} |DP^\epsilon_t u|^2\,dx \leq B^2\right\}$$

and

$$\partial N^{t_1,\dots,t_r}_{\epsilon,B} = \left\{u \in H^{1,2}_0(\Omega) : \int_\Omega |DP^\epsilon u|^2\,dx + \sum_{t\notin\{t_1,\dots,t_r\}} \int_{\Omega'_t} |DP^\epsilon_t u|^2\,dx = B^2\right\}.$$

Proposition 3.3. *Let $\{t_1,\dots,t_r\} \subseteq \{1,\dots,k\}$. For all $s = 1,\dots,r$, let v_{t_s} be the positive function in $H^{1,2}_0(\Omega'_{t_s})$, with $\int_{\Omega'_{t_s}\setminus\Omega_{t_s}} |Dv_{t_s}|^2\,dx = 0$ and $\int_{\Omega'_{t_s}} \lambda(x)v^p_{t_s}\,dx = 1$, which achieves m_{t_s}.*

Then, for all $\epsilon > 0$, there exist some positive numbers $\alpha_{\epsilon,1},\dots,\alpha_{\epsilon,r}$ such that the function

$$v_\epsilon = \sum_{s=1}^r \alpha_{\epsilon,s} v_{t_s} \in M^{t_1,\dots,t_r}_\epsilon.$$

Moreover, for all $s = 1,\dots,r$,

$$(\underline{\Lambda} m_{t_s})^{\frac{1}{p-2}} \leq \liminf_{\epsilon\to 0} \alpha_{\epsilon,s} \leq \limsup_{\epsilon\to 0} \alpha_{\epsilon,s} \leq (\overline{\Lambda} m_{t_s})^{\frac{1}{p-2}}$$

(see Definitions 2.2, 2.3, 2.4).

Proof. For all $s = 1, \ldots, r$ let us consider the mapping $z \in \mathbb{R}^+ \to f_\epsilon(z v_{t_s})$. From $(g.3)$ and (g.4) it follows that it has a local minimum in $z = 0$ and $\lim_{z\to+\infty} f_\epsilon(z v_{t_s}) = -\infty$.

Then there exists a maximum point $\alpha_{\epsilon,s}$ such that $f_\epsilon(\alpha_{\epsilon,s} v_{t_s}) > 0$ and $f_\epsilon'(\alpha_{\epsilon,s} v_{t_s}) \cdot[\alpha_{\epsilon,s} v_{t_s}] = 0$ for all $s = 1, \ldots r$, i.e. $v_\epsilon \in M_\epsilon^{t_1,\ldots,t_r}$.

Let us show that $\forall s = 1, \ldots, r \quad \lim_{\epsilon\to0} \epsilon^{\frac{1}{p-2}} \alpha_{\epsilon,s} = 0$; in fact, by contradiction, suppose that there exists an infinitesimal sequence $(\epsilon_n)_{n\geq1}$ of positive numbers such that $\lim_{n\to\infty} \epsilon_n^{\frac{1}{p-2}} \alpha_{\epsilon_n,s} = \alpha_s > 0$ for some $s \in \{1, \ldots r\}$. Then, since $G(x,t) > 0$ for $t > 0$ (by (g.3) and (g.4)), we get

$$\limsup_{n\to\infty} f_{\epsilon_n}(\alpha_{\epsilon_n,s} v_{t_s}) \leq \limsup_{n\to\infty} \frac{1}{\epsilon_n^{\frac{2}{p-2}}} \left\{ {\alpha_s}^2 \frac{\overline{\Lambda} m_{t_s}}{2} - \frac{1}{\epsilon_n} \int_{\Omega_{t_s}} G(x, \alpha_s v_{t_s})\, dx \right\} = -\infty$$

which is a contradiction with the fact that $f_{\epsilon_n}(\alpha_{\epsilon_n,s} v_{t_s}) > 0 \quad \forall n \geq 1$.

Because of (g.2) and (g.3), there exist $C > 0$ and $\eta > 0$ such that, for all $s = 1, \ldots, r$,

$$|\gamma_\epsilon(x, \alpha_{\epsilon,s} v_{t_s}) v_{t_s}^p| \leq \begin{cases} (\lambda(x) + C) v_{t_s}^p & \text{if } |\epsilon^{\frac{1}{p-2}} \alpha_{\epsilon,s} v_{t_s}| \leq \eta \\ \frac{a}{\eta^{p-1}} v_{t_s}^p + a(\epsilon^{\frac{1}{p-2}} \alpha_{\epsilon,s})^{q-p} v_{t_s}^q & \text{if } |\epsilon^{\frac{1}{p-2}} \alpha_{\epsilon,s} v_{t_s}| > \eta. \end{cases}$$

Since $\lim_{\epsilon\to0} \epsilon^{\frac{1}{p-2}} \alpha_{\epsilon,s} = 0 \quad \forall s = 1, \ldots, r$, it follows that

$$\begin{aligned} &\lim_{\epsilon\to0} \frac{1}{{\alpha_{\epsilon,s}}^p} \int_{\Omega'_{t_s}} g_\epsilon(x, \alpha_{\epsilon,s} v_{t_s}) \alpha_{\epsilon,s} v_{t_s}\, dx \\ &= \lim_{\epsilon\to0} \int_{\Omega'_{t_s}} \gamma_\epsilon(x, \alpha_{\epsilon,s} v_{t_s}) v_{t_s}^p\, dx = \|v_{t_s}\|_{(\lambda,p)}^p = 1 \quad \forall s = 1, \ldots, r\,. \end{aligned}$$

Moreover $f_\epsilon'(\alpha_{\epsilon,s} v_{t_s})[\alpha_{\epsilon,s} v_{t_s}] = 0$ implies

$$\frac{1}{\alpha_{\epsilon,s}^p} \int_{\Omega'_{t_s}} g_\epsilon(x, \alpha_{\epsilon,s} v_{t_s}) \alpha_{\epsilon,s} v_{t_s}\, dx = \frac{1}{\alpha_{\epsilon,s}^{p-2}} \int_{\Omega'_{t_s}} \frac{a_\epsilon^{i,j}(x)}{\epsilon} \partial_{x_i} v_{t_s} \partial_{x_j} v_{t_s}\, dx \quad \forall s = 1, \ldots r.$$

Thus, taking into account Definitions 2.3 and 2.4, we obtain, for all $s = 1, \ldots, r$,

$$(\underline{\Lambda} m_{t_s})^{\frac{1}{p-2}} \leq \liminf_{\epsilon\to0} \alpha_{\epsilon,s} \leq \limsup_{\epsilon\to0} \alpha_{\epsilon,s} \leq (\overline{\Lambda} m_{t_s})^{\frac{1}{p-2}}.$$

□

Definition 3.4. *For all $\epsilon > 0$ let $Q_\epsilon : H_0^{1,2}(\Omega) \to \mathbb{R}$ be defined by*

$$Q_\epsilon(u) = \int_\Omega \frac{a_\epsilon^{i,j}}{\epsilon} \partial_{x_i} u \partial_{x_j} u\, dx\,.$$

Lemma 3.5. *Let* $\{t_1,\dots,t_r\}\subseteq\{1,\dots,k\}$ *and* $\varrho\in]0,(\underline{\Lambda}m)^{\frac{1}{p-2}}[$.
Then there exists $\bar\epsilon>0$ *such that*

$$\left\{u\in M_\epsilon^{t_1,\dots,t_r}\cap K_{\epsilon,\varrho}^{t_1,\dots,t_r}\ :\ P_t^\epsilon(u)\equiv 0\quad\forall t\notin\{t_1,\dots,t_r\}\ ,\ P^\epsilon(u)\equiv 0\right\}\neq\emptyset$$
$$\forall\epsilon\in]0,\bar\epsilon[$$

and we have

$$\begin{aligned}\limsup_{\epsilon\to0}\inf\big\{f_\epsilon(u)\ :\ u\in M_\epsilon^{t_1,\dots,t_r}\cap K_{\epsilon,\varrho}^{t_1,\dots,t_r}\ ,\ P_t^\epsilon(u)\equiv 0\\ \forall t\notin\{t_1,\dots,t_r\}\ ,\ P^\epsilon(u)\equiv 0\big\}\\ \leq(\frac12-\frac1p)\sum_{s=1}^r\left(\overline{\Lambda}m_{t_s}\right)^{\frac{p}{p-2}}<+\infty\,.\end{aligned}$$

Proof. First let us observe that, since $\varrho<(\underline{\Lambda}m)^{\frac{1}{p-2}}$, Proposition 3.3 implies that $v_\epsilon\in K_{\epsilon,\varrho}^{t_1,\dots,t_r}$ for all $\epsilon>0$ small enough.

Let us now consider an infinitesimal sequence $(\epsilon_n)_{n\geq1}$ of positive numbers such that

$$\begin{aligned}&\limsup_{\epsilon\to0}\inf\big\{f_\epsilon(u):\ u\in M_\epsilon^{t_1,\dots,t_r}\cap K_{\epsilon,\varrho}^{t_1,\dots,t_r},\ P_t^\epsilon(u)\equiv 0\\ &\forall t\notin\{t_1,\dots,t_r\}\ ,\ P^\epsilon(u)\equiv 0\big\}\\ &\ =\ \lim_{n\to\infty}\inf\big\{f_{\epsilon_n}(u):\ u\in M_{\epsilon_n}^{t_1,\dots,t_r}\cap K_{\epsilon_n,\varrho}^{t_1,\dots,t_r}\ ,\\ &\qquad P_t^{\epsilon_n}(u)\equiv 0\ \forall t\notin\{t_1,\dots,t_r\}\ ,\ P^{\epsilon_n}(u)\equiv 0\big\}\,.\end{aligned}\tag{3.1}$$

From Proposition 3.3, for all n large enough there exist $\alpha_{n,1},\dots,\alpha_{n,r}>0$ such that $v_{\epsilon_n}=\sum_{s=1}^r\alpha_{n,s}v_{t_s}\in M_{\epsilon_n}^{t_1,\dots,t_r}\cap K_{\epsilon_n,\varrho}^{t_1,\dots,t_r}$.

Moreover $P_t^{\epsilon_n}(v_{\epsilon_n})\equiv 0\quad\forall\quad t\notin\{t_1,\dots,t_r\}$ and $P^{\epsilon_n}(v_{\epsilon_n})\equiv 0$.

As in the proof of Proposition 3.3 one can show that $\lim_{n\to\infty}\epsilon_n{}^{\frac{1}{p-2}}\alpha_{n,s}=0$ for all $s=1,\dots,r$ and, up to a subsequence,

$$\lim_{n\to\infty}\alpha_{n,s}^{p-2}=\lim_{n\to\infty}\int_{\Omega_{t_s}}\frac{a_{\epsilon_n}^{i,j}}{\epsilon_n}\partial_{x_i}v_{t_s}\partial_{x_j}v_{t_s}\,dx\quad\forall\quad s=1,\dots,r\tag{3.2}$$

by using $v_{\epsilon_n}\in M_{\epsilon_n}^{t_1,\dots,t_r}$ $\forall$ n large enough, (g.2), (g.3) and the Lebesgue theorem. Moreover, $\forall$ $s=1,\dots,r$, it is $\lim_{n\to\infty}\frac{1}{\alpha_{n,s}{}^p}G_{\epsilon_n}(x,\alpha_{n,s}v_{t_s})=\lim_{n\to\infty}\Gamma_{\epsilon_n}(x,\alpha_{n,s}v_{t_s})v_{t_s}^p=\frac{\lambda(x)}{p}v_{t_s}^p$ a.e. in Ω_{t_s}' and from (g.2) and (g.3), assuming $q\geq p$, we can find $C>0,\eta>0$ and $b>0$ such that

$$|\Gamma_{\epsilon_n}(x,\alpha_{n,s}v_{t_s})v_{t_s}^p|\leq\begin{cases}(\frac{\lambda(x)}{p}+C)v_{t_s}^p & \text{if } |\epsilon_n{}^{\frac{1}{p-2}}\alpha_{n,s}v_{t_s}|\leq\eta\\ \frac{b}{\eta^p}v_{t_s}^p+b(\epsilon_n{}^{\frac{1}{p-2}}\alpha_{n,s})^{q-p}v_{t_s}^q & \text{if } |\epsilon_n{}^{\frac{1}{p-2}}\alpha_{n,s}v_{t_s}|>\eta;\end{cases}$$

hence, from the generalized Lebesgue theorem, (3.1) and (3.2) imply

$$
\begin{aligned}
&\limsup_{\epsilon\to 0} \inf\left\{f_\epsilon(u) : u \in M_\epsilon^{t_1,\dots,t_r} \cap K_{\epsilon,\varrho}^{t_1,\dots,t_r}, P_t^\epsilon(u) \equiv 0 \quad \forall t \notin \{t_1,\dots,t_r\},\ P^\epsilon(u)\equiv 0\right\} \\
&\leq \lim_{n\to\infty} f_{\epsilon_n}(v_{\epsilon_n}) \\
&= \lim_{n\to\infty} \sum_{s=1}^r \left\{ \frac{\alpha_{n,s}^2}{2} \int_{\Omega_{t_s}} \frac{a_{\epsilon_n}^{i,j}}{\epsilon_n} \partial_{x_i} v_{t_s} \partial_{x_j} v_{t_s}\, dx - \int_{\Omega'_{t_s}} G_{\epsilon_n}(x, \alpha_{n,s} v_{t_s})\, dx \right\} \\
&= \lim_{n\to\infty} \sum_{s=1}^r \left\{ \frac{\alpha_{n,s}^2}{2} \int_{\Omega_{t_s}} \frac{a_{\epsilon_n}^{i,j}}{\epsilon_n} \partial_{x_i} v_{t_s} \partial_{x_j} v_{t_s}\, dx - \alpha_{n,s}^p \int_{\Omega'_{t_s}} \frac{G_{\epsilon_n}(x, \alpha_{n,s} v_{t_s})}{(\alpha_{n,s} v_{t_s})^p} v_{t_s}^p\, dx \right\} \\
&= \left(\frac{1}{2} - \frac{1}{p}\right) \lim_{n\to\infty} \sum_{s=1}^r \left(Q_{\epsilon_n}(v_{t_s})\right)^{\frac{p}{p-2}} \leq \left(\frac{1}{2} - \frac{1}{p}\right) \limsup_{\epsilon\to 0} \sum_{s=1}^r \left(Q_\epsilon(v_{t_s})\right)^{\frac{p}{p-2}} \\
&\leq \left(\frac{1}{2} - \frac{1}{p}\right) \sum_{s=1}^r \left(\overline{\Lambda} m_{t_s}\right)^{\frac{p}{p-2}} < +\infty\,.
\end{aligned}
$$

□

Lemma 3.6. *Let* $B > 0$ *and* $\varrho \in]0, (\underline{\Lambda} m)^{\frac{1}{p-2}}[$*. Then for all* $\{t_1,\dots,t_r\} \subseteq \{1,\dots,k\}$ *we have*

$$
\limsup_{\epsilon\to 0} \inf\left\{f_\epsilon(u) : u \in M_\epsilon^{t_1,\dots,t_r} \cap K_{\epsilon,\varrho}^{t_1,\dots,t_r} \cap N_{\epsilon,B}^{t_1,\dots,t_r}\right\} \leq \left(\frac{1}{2} - \frac{1}{p}\right) \sum_{s=1}^r \left(\overline{\Lambda} m_{t_s}\right)^{\frac{p}{p-2}}
$$

(see Definitions 2.2*,* 3.1*,* 3.2 *and* 3.4*).*

For the proof it suffices to remark that $v_\epsilon \in N_{\epsilon,B}^{t_1,\dots,t_r}$ and argue as in the proof of Lemma 3.5.

Lemma 3.7. *Let* $\varrho \in]0, (\underline{\Lambda} m)^{\frac{1}{p-2}}[$*. Then there exist* $\overline{\epsilon} > 0$ *and* $B > 0$ *such that for all* $\epsilon \in]0, \overline{\epsilon}[$

$$
\inf\left\{f_\epsilon(u) : u \in M_\epsilon^{t_1,\dots,t_r} \cap K_{\epsilon,\varrho}^{t_1,\dots,t_r} \cap N_{\epsilon,B}^{t_1,\dots,t_r}\right\}
$$

is achieved (see Definitions 2.2*,* 3.1*,* 3.2*).*

Proof. Let $0 < B \leq 1$ and choose $\overline{\epsilon}_1 > 0$ small enough that

$$
M_\epsilon^{t_1,\dots,t_r} \cap K_{\epsilon,\varrho}^{t_1,\dots,t_r} \cap N_{\epsilon,B}^{t_1,\dots,t_r} \neq \emptyset \quad \forall\, \epsilon \in]0, \overline{\epsilon}_1[\tag{3.3}
$$

(see Lemma 3.5),

$$
\inf\left\{\frac{\Lambda_1(\epsilon, x)}{\epsilon} : x \in \Omega\right\} > \frac{\underline{\Lambda}}{2} \quad \forall\, \epsilon \in]0, \overline{\epsilon}_1[\tag{3.4}
$$

(see (2.5)) and

$$\sup_{\epsilon\in]0,\overline{\epsilon}_1[}\inf\left\{f_\epsilon(u)\ :\ u\in M_\epsilon^{t_1,\dots,t_r}\cap K_{\epsilon,\varrho}^{t_1,\dots,t_r}\cap N_{\epsilon,B}^{t_1,\dots,t_r}\right\}<+\infty \tag{3.5}$$

(see Lemma 3.6).

For all $\epsilon\in]0,\overline{\epsilon}_1[$, let $(u_n^\epsilon)_{n\geq 1}$ be a sequence of functions in $M_\epsilon^{t_1,\dots,t_r}\cap K_{\epsilon,\varrho}^{t_1,\dots,t_r}\cap N_{\epsilon,B}^{t_1,\dots,t_r}$ such that

$$\lim_{n\to\infty}f_\epsilon(u_n^\epsilon)=\inf\left\{f_\epsilon(u)\ :\ u\in M_\epsilon^{t_1,\dots,t_r}\cap K_{\epsilon,\varrho}^{t_1,\dots,t_r}\cap N_{\epsilon,B}^{t_1,\dots,t_r}\right\}.$$

First we prove that $(u_n^\epsilon)_{n\geq 1}$ is bounded in $H_0^{1,2}(\Omega)$.

If we write $u_n^\epsilon=P_1^\epsilon(u_n^\epsilon)+\dots+P_k^\epsilon(u_n^\epsilon)+P^\epsilon(u_n^\epsilon)$ (see Definition 2.1), then we obtain

$$\begin{aligned}
&f_\epsilon(u_n^\epsilon)\\
&\quad=\ \frac{1}{2}\int_\Omega\frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i}u_n^\epsilon\partial_{x_j}u_n^\epsilon\,dx-\int_\Omega G_\epsilon(x,u_n^\epsilon)\,dx\\
&\quad=\ \sum_{s=1}^r\left\{\frac{1}{2}\int_{\Omega'_{t_s}}\frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i}P_{t_s}^\epsilon u_n^\epsilon\partial_{x_j}P_{t_s}^\epsilon u_n^\epsilon\,dx-\int_{\Omega'_{t_s}}G_\epsilon(x,P_{t_s}^\epsilon u_n^\epsilon+P^\epsilon u_n^\epsilon)\,dx\right\}\\
&\qquad+\sum_{t\notin\{t_1,\dots,t_r\}}\left\{\frac{1}{2}\int_{\Omega'_t}\frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i}P_t^\epsilon u_n^\epsilon\partial_{x_j}P_t^\epsilon u_n^\epsilon\,dx-\int_{\Omega'_t}G_\epsilon(x,P_t^\epsilon u_n^\epsilon+P^\epsilon u_n^\epsilon)\,dx\right\}\\
&\qquad+\frac{1}{2}\int_\Omega\frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i}P^\epsilon u_n^\epsilon\partial_{x_j}P^\epsilon u_n^\epsilon\,dx-\int_{\Omega\setminus\bigcup_{t=1}^k\Omega'_t}G_\epsilon(x,P^\epsilon u_n^\epsilon)\,dx\,.
\end{aligned}\tag{3.6}$$

Notice that the sets $\{P^\epsilon u_n^\epsilon\ :\ n\geq 1,\ \epsilon\in]0,\overline{\epsilon}_1[\}$ and $\{P_t^\epsilon u_n^\epsilon\ :\ n\geq 1,\ \epsilon\in]0,\overline{\epsilon}_1[\}$, for all $t\notin\{t_1,\dots,t_r\}$, are bounded in $H_0^{1,2}(\Omega)$ since $u_n^\epsilon\in N_{\epsilon,B}^{t_1,\dots,t_r}$ with $B\leq 1$.

For all $t\in\{t_1,\dots,t_r\}$, since $u_n^\epsilon\in M_\epsilon^{t_1,\dots,t_r}$, we get from (g.4)

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega'_t}\frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i}P_t^\epsilon u_n^\epsilon\partial_{x_j}P_t^\epsilon u_n^\epsilon\,dx-\int_{\Omega'_t}G_\epsilon(x,P_t^\epsilon u_n^\epsilon+P^\epsilon u_n^\epsilon)\,dx\\
&\quad\geq\ \frac{1}{2}\int_{\Omega'_t}\frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i}P_t^\epsilon u_n^\epsilon\partial_{x_j}P_t^\epsilon u_n^\epsilon\,dx-\theta\int_{\Omega'_t}g_\epsilon(x,P_t^\epsilon u_n^\epsilon+P^\epsilon u_n^\epsilon)(P_t^\epsilon u_n^\epsilon+P^\epsilon u_n^\epsilon)\,dx\\
&\quad=\ \frac{1}{2}(\frac{1}{2}-\theta)\int_{\Omega'_t}\frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i}P_t^\epsilon u_n^\epsilon\partial_{x_j}P_t^\epsilon u_n^\epsilon\,dx\\
&\qquad+\int_{\Omega'_t}g_\epsilon(x,P_t^\epsilon u_n^\epsilon+P^\epsilon u_n^\epsilon)[\frac{1}{2}(\frac{1}{2}-\theta)P_t^\epsilon u_n^\epsilon-\theta P^\epsilon u_n^\epsilon]\,dx\,.
\end{aligned}$$

If we set $\Omega_n = \left\{ x \in \Omega \,:\, -P^\epsilon u_n^\epsilon(x) \le P_t^\epsilon u_n^\epsilon(x) \le \frac{4\theta}{1-2\theta} P^\epsilon u_n^\epsilon(x) \right\}$, since $g_\epsilon(x,t) \ge 0$ and $g_\epsilon(x,t) = 0$ if $t \le 0$, we obtain

$$\int_{\Omega_t'} g_\epsilon(x, P_t^\epsilon u_n^\epsilon + P^\epsilon u_n^\epsilon)[\frac{1}{2}(\frac{1}{2} - \theta) P_t^\epsilon u_n^\epsilon - \theta P^\epsilon u_n^\epsilon]\, dx$$
$$\ge \int_{\Omega_n} g_\epsilon(x, P_t^\epsilon u_n^\epsilon + P^\epsilon u_n^\epsilon)[\frac{1}{2}(\frac{1}{2} - \theta) P_t^\epsilon u_n^\epsilon - \theta P^\epsilon u_n^\epsilon]\, dx \ge -C \tag{3.7}$$

for a suitable constant $C > 0$, because $(P^\epsilon u_n^\epsilon)_{n\ge1}$ is bounded in $H_0^{1,2}(\Omega)$ and (g.2) holds.

Hence we get for all $t \in \{t_1, \dots, t_r\}$

$$\frac{1}{2}\int_{\Omega_t'} \frac{a_\epsilon^{i,j}}{\epsilon} \partial_{x_i} P_t^\epsilon u_n^\epsilon \partial_{x_j} P_t^\epsilon u_n^\epsilon\, dx - \int_{\Omega_t'} G_\epsilon(x, P_t^\epsilon u_n^\epsilon + P^\epsilon u_n^\epsilon)\, dx \ge \frac{1}{C_1}\|P_t^\epsilon u_n^\epsilon\|^2 - C_1. \tag{3.8}$$

for a suitable constant $C_1 > 0$.

From (3.6) and (3.8), taking into account that $u_n^\epsilon \in N_{\epsilon,B}^{t_1,\dots,t_r}$ with $B \le 1$, we infer that there exists a constant $C_2 > 0$ such that

$$f_\epsilon(u_n^\epsilon) \ge \frac{1}{C_2}\|u_n^\epsilon\|^2 - C_2 \quad \forall\, n \ge 1,\ \forall\, \epsilon \in]0, \overline{\epsilon}_1[. \tag{3.9}$$

It follows that $(u_n^\epsilon)_{n\ge1}$ is bounded in $H_0^{1,2}(\Omega)$ for all $\epsilon \in]0, \overline{\epsilon}_1[$; hence, up to a subsequence, $(u_n^\epsilon)_{n\ge1}$ converges to $u_\epsilon \in H_0^{1,2}(\Omega)$ weakly in $H_0^{1,2}(\Omega)$, in $L^p(\Omega)$, in $L^q(\Omega)$ and a.e. in Ω. So we have $u_\epsilon \in K_{\epsilon,\varrho}^{t_1,\dots,t_r} \cap N_{\epsilon,B}^{t_1,\dots,t_r}$, $f_\epsilon'(u_\epsilon)[P_{t_s}^\epsilon u_\epsilon] \le 0$ for $s = 1, \dots, r$ and

$$f_\epsilon(u_\epsilon) \le \lim_{n\to\infty} f_\epsilon(u_n^\epsilon) = \inf\left\{ f_\epsilon(u) \,:\, u \in M_\epsilon^{t_1,\dots,t_r} \cap K_{\epsilon,\varrho}^{t_1,\dots,t_r} \cap N_{\epsilon,B}^{t_1,\dots,t_r} \right\}. \tag{3.10}$$

Moreover (3.5), (3.8) and (3.9) imply

$$\sup_{0<\epsilon<\overline{\epsilon}_1} \|P_t^\epsilon u_\epsilon\| < +\infty \quad \forall\, t \in \{t_1, \dots, t_r\}. \tag{3.11}$$

Now, choose $\overline{\varrho} \in]0, \varrho[$ and $\mu, \overline{\mu}$ greater than $\sup_{0<\epsilon<\overline{\epsilon}_1} \|P_t^\epsilon u_\epsilon\|$ for all $t \in \{t_1, \dots t_r\}$, fix $\overline{\epsilon} \in]0, \overline{\epsilon}_1[$ and $B \in]0, 1]$ such that the assertions of Lemma 2.6 and 2.8 hold.

In order to prove that, for all $\epsilon \in]0, \overline{\epsilon}[$, u_ϵ is a function that realizes

$$\inf\left\{ f_\epsilon(u) \,:\, u \in M_\epsilon^{t_1,\dots,t_r} \cap K_{\epsilon,\varrho}^{t_1,\dots,t_r} \cap N_{\epsilon,B}^{t_1,\dots,t_r} \right\},$$

it remains to show that $u_\epsilon \in M_\epsilon^{t_1,\dots,t_r}$.

By contradiction, suppose there exists $I \subseteq \{t_1, \dots, t_r\}$, $I \ne \emptyset$, such that $f_\epsilon'(u_\epsilon)[P_t^\epsilon u_\epsilon] < 0$ for all $t \in I$.

Let us fix $t \in I$. Since $\|(P_t^\epsilon u_\epsilon)^+\|_{(\lambda,p)} \ge \varrho$, there exists $\overline{\xi}_t \in]0, 1]$ such that $\varrho = \|\overline{\xi}_t (P_t^\epsilon u_\epsilon)^+\|_{(\lambda,p)}$; since (3.11) holds, $\|P^\epsilon u_\epsilon\| \le B$ and $\|\overline{\xi}_t P_t^\epsilon u_\epsilon\| \le \mu$, then Lemma 2.6 implies that $f_\epsilon'(P^\epsilon u_\epsilon + \overline{\xi}_t P_t^\epsilon u_\epsilon)[\overline{\xi}_t P_t^\epsilon u_\epsilon] > 0$.

On the other hand, $f_\epsilon'(P^\epsilon u_\epsilon + P_t^\epsilon u_\epsilon)[P_t^\epsilon u_\epsilon] < 0$.
Hence there exists $\xi_t \in]\overline{\xi}_t, 1[$ such that $f_\epsilon'(P^\epsilon u_\epsilon + \xi_t P_t^\epsilon u_\epsilon)[\xi_t P_t^\epsilon u_\epsilon] = 0$, i.e. $v_\epsilon = \sum_{t\in\{1,\dots k\}\setminus I} P_t^\epsilon(u_\epsilon) + \sum_{t\in I} \xi_t P_t^\epsilon u_\epsilon + P^\epsilon u_\epsilon \in M_\epsilon^{t_1,\dots,t_r}$.
For every $F \subseteq \Omega$, let us set

$$f_{\epsilon_{|F}}(u) = \frac{1}{2}\int_F \frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i} u \partial_{x_j} u\, dx - \int_F G_\epsilon(x,u)\, dx\,.$$

Since $u_n^\epsilon \in M_\epsilon^{t_1,\dots,t_r}$, we obtain, for all $t \in I$,

$$\begin{aligned}
&\lim_{n\to\infty} f_{\epsilon_{|\Omega_t'}}(u_n^\epsilon)\\
&= \lim_{n\to\infty}\Bigg\{\frac{1}{2}\int_{\Omega_t'} \frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i} P^\epsilon u_n^\epsilon \partial_{x_j} P^\epsilon u_n^\epsilon\, dx\\
&\quad + \frac{1}{2}\int_{\Omega_t'} \frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i} P_t^\epsilon u_n^\epsilon \partial_{x_j} P_t^\epsilon u_n^\epsilon\, dx - \int_{\Omega_t'} G_\epsilon(x, P_t^\epsilon u_n^\epsilon + P^\epsilon u_n^\epsilon)\, dx\Bigg\}\\
&= \lim_{n\to\infty}\Bigg\{\frac{1}{2}\int_{\Omega_t'} \frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i} P^\epsilon u_n^\epsilon \partial_{x_j} P^\epsilon u_n^\epsilon\, dx + \frac{1}{2}\int_{\Omega_t'} g_\epsilon(x, P_t^\epsilon u_n^\epsilon + P^\epsilon u_n^\epsilon) P_t^\epsilon u_n^\epsilon\, dx\\
&\quad - \int_{\Omega_t'} G_\epsilon(x, P_t^\epsilon u_n^\epsilon + P^\epsilon u_n^\epsilon)\, dx\Bigg\}\\
&\geq \frac{1}{2}\int_{\Omega_t'} \frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i} P^\epsilon u_\epsilon \partial_{x_j} P^\epsilon u_\epsilon\, dx + \frac{1}{2}\int_{\Omega_t'} g_\epsilon(x, P_t^\epsilon u_\epsilon + P^\epsilon u_\epsilon)(P_t^\epsilon u_\epsilon)\, dx\\
&\quad - \int_{\Omega_t'} G_\epsilon(x, P_t^\epsilon u_\epsilon + P^\epsilon u_\epsilon)\, dx\\
&\quad = \frac{1}{2}\int_{\Omega_t'} \frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i} P^\epsilon u_\epsilon \partial_{x_j} P^\epsilon u_\epsilon\, dx + \frac{1}{2}\int_{\Omega_t'} g_\epsilon(x, \xi_t P_t^\epsilon u_\epsilon + P^\epsilon u_\epsilon)(\xi_t P_t^\epsilon u_\epsilon)\, dx\\
&\quad - \int_{\Omega_t'} G_\epsilon(x, \xi_t P_t^\epsilon u_\epsilon + P^\epsilon u_\epsilon)\, dx + (1-\xi_t)\Phi_\epsilon'(x, \xi P_t^\epsilon u_\epsilon + P^\epsilon u_\epsilon)[P_t^\epsilon u_\epsilon]
\end{aligned}$$

for a suitable $\xi \in]\xi_t, 1[$.

Since $\|\xi(P_t^\epsilon u_\epsilon)^+\|_{(\lambda,p)} \geq \varrho \geq \overline{\varrho}$, $\|\xi P_t^\epsilon u_\epsilon\| \leq \overline{\mu}$ and $\|P^\epsilon u_\epsilon\| \leq B$, then Lemma 2.8 can be applied obtaining

$$\lim_{n\to\infty} f_{\epsilon_{|\Omega_t'}}(u_n^\epsilon) \quad > \quad \frac{1}{2}\int_{\Omega_t'} \frac{a_\epsilon^{i,j}}{\epsilon}\partial_{x_i} P^\epsilon u_\epsilon \partial_{x_j} P^\epsilon u_\epsilon\, dx$$

$$
\begin{aligned}
&+\frac{1}{2}\int_{\Omega_t'} g_\epsilon(x,\xi_t P_t^\epsilon u_\epsilon + P^\epsilon u_\epsilon)(\xi_t P_t^\epsilon u_\epsilon)\,dx \\
&-\int_{\Omega_t'} G_\epsilon(x,\xi_t P_t^\epsilon u_\epsilon + P^\epsilon u_\epsilon)\,dx \\
=\;& f_{\epsilon_{|\Omega_t'}}(\xi_t P_t^\epsilon u_\epsilon + P^\epsilon u_\epsilon)\,,
\end{aligned}
$$

where the last equality holds because $f_\epsilon'(\xi_t P_t^\epsilon u_\epsilon + P^\epsilon u_\epsilon)[\xi_t P_t^\epsilon u_\epsilon] = 0$.

Since $f_\epsilon = \sum_{t\in I} f_{\epsilon_{|\Omega_t'}} + f_{\epsilon_{|\Omega\setminus\bigcup_{t\in I}\Omega_t'}}$, we have found a function $v_\epsilon \in M_\epsilon^{t_1,\dots,t_r}$ such that

$$
f_\epsilon(v_\epsilon) = \sum_{t\in I} f_{\epsilon_{|\Omega_t'}}(v_\epsilon) + f_{\epsilon_{|\Omega\setminus\bigcup_{t\in I}\Omega_t'}}(v_\epsilon) < \lim_{n\to\infty} f_\epsilon(u_n^\epsilon),
$$

which is a contradiction with (3.10), since $v_\epsilon \in K_{\epsilon,\varrho}^{t_1,\dots,t_r} \cap N_{\epsilon,B}^{t_1,\dots,t_r}$.

Hence $I = \emptyset$. □

Lemma 3.8. *Let* $0 < \varrho < (\underline{\Lambda} m)^{\frac{1}{p-2}}$ *and* $0 < B < \left(\underline{\Lambda}\frac{p}{2} m^{\frac{p}{2}}\right)^{\frac{1}{p-2}}$. *Then there exists* $\bar\epsilon > 0$ *such that, for all* $\{t_1,\dots,t_r\} \subseteq \{1,\dots,k\}$ *and for all* $\epsilon \in]0,\bar\epsilon[$,

$$
\begin{aligned}
&\inf\left\{f_\epsilon(u) : u \in M_\epsilon^{t_1,\dots,t_r} \cap K_{\epsilon,\varrho}^{t_1,\dots,t_r} \cap N_{\epsilon,B}^{t_1,\dots,t_r}\right\} \\
&< \quad \inf\left\{f_\epsilon(u) : u \in M_\epsilon^{t_1,\dots,t_r} \cap K_{\epsilon,\varrho}^{t_1,\dots,t_r} \cap \partial N_{\epsilon,B}^{t_1,\dots,t_r}\right\}
\end{aligned}
$$

(see Definitions 2.3, 3.1, 3.2*).*

Proof. Suppose, by contradiction, there exists an infinitesimal sequence $(\epsilon_n)_{n\geq 1}$ of positive numbers such that

$$
\begin{aligned}
&\inf\left\{f_{\epsilon_n}(u) : u \in M_{\epsilon_n}^{t_1,\dots,t_r} \cap K_{\epsilon_n,\varrho}^{t_1,\dots,t_r} \cap \partial N_{\epsilon_n,B}^{t_1,\dots,t_r}\right\} \\
&\leq \quad \inf\left\{f_{\epsilon_n}(u) : u \in M_{\epsilon_n}^{t_1,\dots,t_r} \cap K_{\epsilon_n,\varrho}^{t_1,\dots,t_r} \cap N_{\epsilon_n,B}^{t_1,\dots,t_r}\right\}, \qquad (3.12)
\end{aligned}
$$

for some $\{t_1,\dots,t_r\}$ fixed as in our assumption.

Hence there exists a sequence of functions $(u_n)_{n\geq 1}$ in $M_{\epsilon_n}^{t_1,\dots,t_r} \cap K_{\epsilon_n,\varrho}^{t_1,\dots,t_r} \cap \partial N_{\epsilon_n,B}^{t_1,\dots t_r}$ such that

$$
\limsup_{n\to\infty} f_{\epsilon_n}(u_n) \leq \limsup_{n\to\infty} \inf\left\{f_{\epsilon_n}(u) : u \in M_{\epsilon_n}^{t_1,\dots,t_r} \cap K_{\epsilon_n,\varrho}^{t_1,\dots,t_r} \cap N_{\epsilon_n,B}^{t_1,\dots t_r}\right\}. \qquad (3.13)
$$

The proof consists of 3 steps.

STEP 1. $(u_n)_{n\geq 1}$ is bounded in $H_0^{1,2}(\Omega)$.

From Proposition 3.5, from (3.12) and (3.13), we have

$$
\limsup_{n\to\infty} f_{\epsilon_n}(u_n) \leq \left(\frac{1}{2} - \frac{1}{p}\right) \sum_{s=1}^{r} \left(\overline{\Lambda} m_{t_s}\right)^{\frac{p}{p-2}} < +\infty. \qquad (3.14)
$$

Since $u_n \in M_{\epsilon_n}^{t_1,\dots t_r}$, for all $s = 1, \dots, r$, we have

$$\begin{aligned}
&\liminf_{n\to\infty} \left\{ \frac{1}{2} \int_{\Omega'_{t_s}} \frac{a^{i,j}_{\epsilon_n}}{\epsilon_n} \partial_{x_i} P^{\epsilon_n} u_n \partial_{x_j} P^{\epsilon_n} u_n \, dx + \frac{1}{2} \int_{\Omega'_{t_s}} \frac{a^{i,j}_{\epsilon_n}}{\epsilon_n} \partial_{x_i} P^{\epsilon_n}_{t_s} u_n \partial_{x_j} P^{\epsilon_n}_{t_s} u_n \, dx \right. \\
&\qquad \left. - \int_{\Omega'_{t_s}} G_{\epsilon_n}(x, P^{\epsilon_n}_{t_s} u_n + P^{\epsilon_n} u_n) \, dx \right\} \\
&\geq \liminf_{n\to\infty} \left\{ \frac{1}{2} \int_{\Omega'_{t_s}} \frac{a^{i,j}_{\epsilon_n}}{\epsilon_n} \partial_{x_i} P^{\epsilon_n} u_n \partial_{x_j} P^{\epsilon_n} u_n \, dx \right. \\
&\qquad + \frac{1}{2}(\frac{1}{2} - \theta) \int_{\Omega'_{t_s}} \frac{a^{i,j}_{\epsilon_n}}{\epsilon_n} \partial_{x_i} P^{\epsilon_n}_{t_s} u_n \partial_{x_j} P^{\epsilon_n}_{t_s} u_n \, dx \\
&\qquad \left. + \int_{\Omega'_{t_s}} g_{\epsilon_n}(x, P^{\epsilon_n}_{t_s} u_n + P^{\epsilon_n} u_n) [\frac{1}{2}(\frac{1}{2} - \theta) P^{\epsilon_n}_{t_s} u_n - \theta P^{\epsilon_n} u_n] \, dx \right\} \\
&\geq (\frac{1}{2} - \theta) \frac{\underline{\Lambda}}{2} \liminf_{n\to\infty} \int_{\Omega'_{t_s}} [|DP^{\epsilon_n} u_n|^2 + |DP^{\epsilon_n}_{t_s} u_n|^2] \, dx - C \qquad (3.15)
\end{aligned}$$

where the last inequality holds with a suitable constant $C > 0$, since $u_n \in N_{\epsilon,B}^{t_1,\dots,t_r}$ and so $(P^{\epsilon_n} u_n)_{n\geq 1}$ is bounded in $H_0^{1,2}(\Omega)$.

Taking into account that $(P_t^{\epsilon_n} u_n)_{n\geq 1}$ is bounded in $H_0^{1,2}(\Omega)$ for all $t \in \{1, \dots, k\} \setminus \{t_1, \dots t_r\}$, (3.15) implies

$$\limsup_{n\to\infty} f_{\epsilon_n}(u_n) \geq \frac{1}{2}(\frac{1}{2} - \theta) \underline{\Lambda} \liminf_{n\to\infty} \int_{\Omega} |Du_n|^2 \, dx - C_1$$

for a suitable constant C_1.

Thus, because of (3.14), $(u_n)_{n\geq 1}$ is bounded in $H_0^{1,2}(\Omega)$; as a consequence, $u_n \to u \in H_0^{1,2}(\Omega)$ weakly in $H_0^{1,2}(\Omega)$, in $L^p(\Omega)$, in $L^q(\Omega)$ and a.e. in Ω .
Now, since $(f_{\epsilon_n}(u_n))_{n\geq 1}$ is bounded and for all $\eta > 0$

$$\liminf_{\epsilon\to 0} \frac{1}{\epsilon} \inf \left\{ \Lambda_1(\epsilon, x) \, : \, x \in \bigcup_{t=1}^{k} \Omega_t(\eta) \right\} = +\infty,$$

it must be $\int\limits_{\Omega\setminus\bigcup_{t=1}^k \Omega_t} |Du|^2\,dx = 0$; hence $u \equiv 0$ in $\Omega \setminus \bigcup_{t=1}^k \Omega_t'$ and so we can write $u = u_1 + \cdots + u_k$ with $u_t \in H_0^{1,2}(\Omega_t')$, $\int\limits_{\Omega_t'\setminus\Omega_t} |Du_t|^2\,dx = 0$ for all $t \in \{1,\ldots,k\}$. Moreover we have $\|(u_{t_s})^+\|_{(\lambda,p)} \geq \varrho$ for all $s = 1,\ldots,r$ and $\sum\limits_{t\notin\{t_1,\ldots t_r\}} \int\limits_{\Omega_t'} |Du_t|^2\,dx \leq B^2$.

STEP 2. We prove that (up to a subsequence)

$$\begin{aligned} \lim_{n\to\infty} f_{\epsilon_n}(u_n) = \lim_{n\to\infty} \Bigg\{ &(\frac{1}{2}-\frac{1}{p})\sum_{s=1}^r Q_{\epsilon_n}(P_{t_s}^{\epsilon_n}(u_n)) \\ &+ \sum_{t\notin\{t_1,\ldots,t_r\}} \Bigg[\frac{1}{2}\int\limits_{\Omega_t'} \frac{a_{\epsilon_n}^{i,j}}{\epsilon_n}\partial_{x_i}P_t^{\epsilon_n}u_n\partial_{x_j}P_t^{\epsilon_n}u_n\,dx \\ &- \int\limits_{\Omega_t'} G_{\epsilon_n}(x, P_t^{\epsilon_n}u_n + P^{\epsilon_n}u_n)\,dx \Bigg] \\ &+ \frac{1}{2}\int\limits_{\Omega} \frac{a_{\epsilon_n}^{i,j}}{\epsilon_n}\partial_{x_i}P^{\epsilon_n}u_n\partial_{x_j}P^{\epsilon_n}u_n\,dx \Bigg\}. \end{aligned} \tag{3.16}$$

Since $u_n \in M_{\epsilon_n}^{t_1,\ldots,t_r}$, for all $s = 1,\ldots,r$ we have

$$\begin{aligned} &\int\limits_{\Omega_{t_s}'} \frac{a_{\epsilon_n}^{i,j}}{\epsilon_n}\partial_{x_i}P_{t_s}^{\epsilon_n}u_n\partial_{x_j}P_{t_s}^{\epsilon_n}u_n\,dx \\ &\quad= \int\limits_{\Omega_{t_s}'} g_{\epsilon_n}(x, P_{t_s}^{\epsilon_n}u_n + P^{\epsilon_n}u_n)P_{t_s}^{\epsilon_n}u_n\,dx \\ &\quad= \int\limits_{\Omega_{t_s}'} \gamma_{\epsilon_n}(x, P_{t_s}^{\epsilon_n}u_n + P^{\epsilon_n}u_n)|P_{t_s}^{\epsilon_n}u_n + P^{\epsilon_n}u_n|^{p-1}P_{t_s}^{\epsilon_n}u_n\,dx\,; \end{aligned} \tag{3.17}$$

as $\limsup\limits_{n\to\infty} \int\limits_{\Omega_{t_s}'} \frac{a_{\epsilon_n}^{i,j}}{\epsilon_n}\partial_{x_i}P_{t_s}^{\epsilon_n}u_n\partial_{x_j}P_{t_s}^{\epsilon_n}u_n\,dx < +\infty$, we can assume that (up to a subsequence) there exists $\lim\limits_{n\to\infty} \int\limits_{\Omega_{t_s}'} \frac{a_{\epsilon_n}^{i,j}}{\epsilon_n}\partial_{x_i}P_{t_s}^{\epsilon_n}u_n\partial_{x_j}P_{t_s}^{\epsilon_n}u_n\,dx$.

As in the proof of Proposition 3.3, since $\lim\limits_{n\to\infty} \epsilon_n^{\frac{1}{p-2}}(P_{t_s}^{\epsilon_n}u_n + P^{\epsilon_n}u_n)(x) = 0$ a.e. in Ω_{t_s}' $\quad\forall\quad s = 1,\ldots,r$, from (g.3) we get

$$\lim_{n\to\infty} \gamma_{\epsilon_n}(x, P_{t_s}^{\epsilon_n}u_n + P^{\epsilon_n}u_n)|P_{t_s}^{\epsilon_n}u_n + P^{\epsilon_n}u_n|^{p-1}P_{t_s}^{\epsilon_n}u_n = \lambda(x)(u_{t_s}^+)^p \quad \text{a.e. in} \quad \Omega_{t_s}'.$$

Moreover, (g.2) and (g.3) (assuming $q \geq p$) imply that there exist $C > 0$ and $\eta > 0$ such that

$$|\gamma_{\epsilon_n}(x, P_{t_s}^{\epsilon_n} u_n + P^{\epsilon_n} u_n)|P_{t_s}^{\epsilon_n} u_n + P^{\epsilon_n} u_n|^{p-1} P_{t_s}^{\epsilon_n} u_n| \leq$$

$$\begin{cases} \frac{(\lambda(x)+C)}{p}((p-1)|P_{t_s}^{\epsilon_n} u_n + P^{\epsilon_n} u_n|^p + |P_{t_s}^{\epsilon_n} u_n|^p) & \text{if } |\epsilon_n^{\frac{1}{p-2}}(P_{t_s}^{\epsilon_n} u_n + P^{\epsilon_n} u_n)| \leq \eta \\ \frac{a}{p\eta^{p-1}}((p-1)|P_{t_s}^{\epsilon_n} u_n + P^{\epsilon_n} u_n|^p + |P_{t_s}^{\epsilon_n} u_n|^p) + & \\ \frac{a}{q}\epsilon_n^{\frac{q-p}{p-2}}((q-1)|P_{t_s}^{\epsilon_n} u_n + P^{\epsilon_n} u_n|^q + |P_{t_s}^{\epsilon_n} u_n|^q) & \text{if } |\epsilon_n^{\frac{1}{p-2}}(P_{t_s}^{\epsilon_n} u_n + P^{\epsilon_n} u_n)| > \eta. \end{cases}$$

Since $P_{t_s}^{\epsilon_n} u_n + P^{\epsilon_n} u_n \to u_{t_s}$ in $L^p(\Omega_{t_s}')$ and in $L^q(\Omega_{t_s}')$, it follows

$$\lim_{n\to\infty} \int_{\Omega_{t_s}'} \frac{a_{\epsilon_n}^{i,j}}{\epsilon_n} \partial_{x_i} P_{t_s} u_n \partial_{x_j} P_{t_s} u_n \, dx = \|(P_{t_s} u)^+\|_{(\lambda,p)}^p \quad \forall \quad s = 1, \ldots, r. \tag{3.18}$$

Moreover, from (g.3), we get

$$\lim_{n\to\infty} \Gamma_{\epsilon_n}(x, P_{t_s}^{\epsilon_n} u_n + P^{\epsilon_n} u_n)|P_{t_s}^{\epsilon_n} u_n + P^{\epsilon_n} u_n|^p = \frac{\lambda(x)}{p} u_{t_s}^+ \quad \text{a.e. in } \Omega_{t_s}' \, \forall s = 1, \ldots, r\,;$$

besides, (g.2) and (g.3) imply the existence of $C > 0$, $\eta > 0$, and $b > 0$ such that, for all $s = 1, \ldots, r$

$$|\Gamma_{\epsilon_n}(x, P_{t_s}^{\epsilon_n} u_n + P^{\epsilon_n} u_n)|P_{t_s}^{\epsilon_n} u_n + P^{\epsilon_n} u_n|^p| \leq$$

$$\begin{cases} (\frac{\lambda(x)}{p} + C)|P_{t_s}^{\epsilon_n} u_n + P^{\epsilon_n} u_n|^p & \text{if } |\epsilon_n^{\frac{1}{p-2}}(P_{t_s}^{\epsilon_n} u_n + P^{\epsilon_n} u_n)| \leq \eta \\ \frac{b}{\eta^p}|P_{t_s}^{\epsilon_n} u_n + P^{\epsilon_n} u_n|^p & \\ +b\epsilon_n^{\frac{q-p}{p-2}}|P_{t_s}^{\epsilon_n} u_n + P^{\epsilon_n} u_n|^q & \text{if } |\epsilon_n^{\frac{1}{p-2}}(P_{t_s}^{\epsilon_n} u_n + P^{\epsilon_n} u_n)| > \eta. \end{cases}$$

So we obtain

$$\begin{aligned} &\lim_{n\to\infty} \int_{\Omega_{t_s}'} G_{\epsilon_n}(x, P_{t_s}^{\epsilon_n} u_n + P^{\epsilon_n} u_n)\, dx \\ &= \lim_{n\to\infty} \int_{\Omega_{t_s}'} \Gamma_{\epsilon_n}(x, P_{t_s}^{\epsilon_n} u_n + P^{\epsilon_n} u_n)|P_{t_s}^{\epsilon_n} u_n + P^{\epsilon_n} u_n|^p \, dx \\ &= \frac{1}{p}\|u_{t_s}^+\|_{(\lambda,p)}^p\,. \end{aligned} \tag{3.19}$$

Since $P^{\epsilon_n} u_n \to 0$ in $L^q(\Omega)$, from (3.18) and (3.19) we get (up to a subsequence)

$$\lim_{n\to\infty} f_{\epsilon_n}(u_n) = \lim_{n\to\infty} \Bigg\{ \sum_{s=1}^{r} \Big[\frac{1}{2} \int_{\Omega'_{t_s}} \frac{a^{i,j}_{\epsilon_n}}{\epsilon_n} \partial_{x_i} P^{\epsilon_n}_{t_s} u_n \partial_{x_j} P^{\epsilon_n}_{t_s} u_n \, dx$$

$$- \int_{\Omega'_{t_s}} G_{\epsilon_n}(x, P^{\epsilon_n}_{t_s} u_n + P^{\epsilon_n} u_n)\, dx \Big]$$

$$+ \sum_{t \notin \{t_1,\dots,t_r\}} \left[\frac{1}{2} \int_{\Omega'_t} \frac{a^{i,j}_{\epsilon_n}}{\epsilon_n} \partial_{x_i} P^{\epsilon_n}_t u_n \partial_{x_j} P^{\epsilon_n}_t u_n \, dx - \int_{\Omega'_t} G_{\epsilon_n}(x, P^{\epsilon_n}_t u_n + P^{\epsilon_n} u_n)\, dx \right]$$

$$+ \frac{1}{2} \int_{\Omega} \frac{a^{i,j}_{\epsilon_n}}{\epsilon_n} \partial_{x_i} P^{\epsilon_n} u_n \partial_{x_j} P^{\epsilon_n} u_n \, dx - \int_{\Omega \setminus \bigcup_{t=1}^{k} \Omega'_t} G_{\epsilon_n}(x, P^{\epsilon_n} u_n)\, dx \Bigg\}$$

$$= \lim_{n\to\infty} \Bigg\{ \sum_{s=1}^{r} \Big[\Big(\frac{1}{2} - \frac{1}{p}\Big) \int_{\Omega'_{t_s}} \frac{a^{i,j}_{\epsilon_n}}{\epsilon_n} \partial_{x_i} P^{\epsilon_n}_{t_s} u_n \partial_{x_j} P^{\epsilon_n}_{t_s} u_n \, dx \Big]$$

$$+ \sum_{t \notin \{t_1,\dots,t_r\}} \Big[\frac{1}{2} \int_{\Omega'_t} \frac{a^{i,j}_{\epsilon_n}}{\epsilon_n} \partial_{x_i} P^{\epsilon_n}_t u_n \partial_{x_j} P^{\epsilon_n}_t u_n \, dx$$

$$- \int_{\Omega'_t} G_{\epsilon_n}(x, P^{\epsilon_n}_t u_n + P^{\epsilon_n} u_n)\, dx \Big] + \frac{1}{2} \int_{\Omega} \frac{a^{i,j}_{\epsilon_n}}{\epsilon_n} \partial_{x_i} P^{\epsilon_n} u_n \partial_{x_j} P^{\epsilon_n} u_n \, dx \Bigg\}.$$

STEP 3. We arrive at a contradiction.

We have

$$\|P^{\epsilon_n} u_n\|^2 + \sum_{t \notin \{t_1,\dots,t_r\}} \|P^{\epsilon_n}_t u_n\|^2 = B^2 \tag{3.20}$$

as $u_n \in \partial N^{t_1,\dots,t_r}_{\epsilon_n,B}$ for all $n \geq 1$. Since for all $s = 1, \dots, r$ $\quad P^{\epsilon_n}_{t_s} u_n + P^{\epsilon_n} u_n \to u_{t_s}$ in $L^p(\Omega'_{t_s})$ and a.e. in Ω'_{t_s}, arguing as in the proof of Proposition 3.3 one can prove that for all n large enough there exist $\xi_{1,n}, \dots, \xi_{s,n} > 0$ such that

$$z_n = \sum_{s=1}^{r} \xi_{s,n} P^{\epsilon_n}_{t_s} u_n \in M^{t_1,\dots,t_r}_{\epsilon_n} \cap K^{t_1,\dots,t_r}_{\epsilon_n,\varrho} \tag{3.21}$$

and

$$f_{\epsilon_n}(\xi_{s,n} P^{\epsilon_n}_{t_s} u_n) > 0 \quad \forall\ s = 1, \dots, r.$$

By definition, $z_n \in N^{t_1,\dots t_r}_{\epsilon_n,B}$ as $P^{\epsilon_n} z_n \equiv 0$ and $P^{\epsilon_n}_t z_n \equiv 0 \quad \forall \quad t \notin \{t_1, \dots, t_r\}$.

From (3.13) (up to a subsequence) we obtain

$$\lim_{n\to\infty} f_{\epsilon_n}(u_n) \leq \lim_{n\to\infty} f_{\epsilon_n}(z_n). \tag{3.22}$$

As in the proof of Proposition 3.3, $f_{\epsilon_n}(\xi_{s,n}P^{\epsilon_n}_{t_s}u_n) > 0$ implies $\lim_{n\to\infty} \epsilon_n^{\frac{1}{p-2}}\xi_{s,n} = 0$ for all $s = 1, \dots, r$.

Since $z_n \in M^{t_1,\dots,t_r}_{\epsilon_n,\varrho}$, for all $s = 1, \dots, r$ we have

$$\xi_{s,n}{}^2 \int_{\Omega'_{t_s}} \frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}P^{\epsilon_n}_{t_s}u_n\partial_{x_j}P^{\epsilon_n}_{t_s}u_n\,dx = \int_{\Omega'_{t_s}} g_{\epsilon_n}(x,\xi_{s,n}P^{\epsilon_n}_{t_s}u_n)\xi_{s,n}P^{\epsilon_n}_{t_s}u_n\,dx$$
$$= \xi_{s,n}{}^p \int_{\Omega'_{t_s}} \gamma_{\epsilon_n}(x,\xi_{s,n}P^{\epsilon_n}_{t_s}u_n)|P^{\epsilon_n}_{t_s}u_n|^p\,dx\,. \tag{3.23}$$

From (g.3) we have $\lim_{n\to\infty}\gamma_{\epsilon_n}(x,\xi_{s,n}P^{\epsilon_n}_{t_s}u_n(x))|P^{\epsilon_n}_{t_s}u_n(x)|^p = \lambda(x)(u^+_{t_s})^p$ a.e. in Ω'_{t_s} for all $s = 1, \dots, r$.

Assuming $q \geq p$, (g.2) and (g.3) imply

$$|\gamma_{\epsilon_n}(x,\xi_{s,n}P^{\epsilon_n}_{t_s}u_n)|P^{\epsilon_n}_{t_s}u_n|^p| \leq$$
$$\begin{cases} (\lambda(x)+C)|P^{\epsilon_n}_{t_s}u_n|^p & \text{if } |\epsilon_n^{\frac{1}{p-2}}\xi_{s,n}P^{\epsilon_n}_{t_s}u_n| \leq \eta \\ \frac{a}{\eta^{p-1}}|P^{\epsilon_n}_{t_s}u_n|^p + a(\epsilon_n^{\frac{1}{p-2}}\xi_{s,n})^{q-p}|P^{\epsilon_n}_{t_s}u_n|^q & \text{if } |\epsilon_n^{\frac{1}{p-2}}\xi_{s,n}P^{\epsilon_n}_{t_s}u_n| > \eta, \end{cases}$$

for suitable positive constants C and η.

Hence, since $P^{\epsilon_n}_{t_s}u_n \to u_{t_s}$ in $L^q(\Omega'_{t_s})$ for all $s = 1, \dots, r$, from (3.23) we obtain

$$\lim_{n\to\infty}\int_{\Omega'_{t_s}} \frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}P^{\epsilon_n}_{t_s}u_n\partial_{x_j}P^{\epsilon_n}_{t_s}u_n\,dx = \|(u_{t_s})^+\|^p_{(\lambda,p)}\lim_{n\to\infty}\xi_{s,n}{}^{p-2}. \tag{3.24}$$

Comparing (3.18) and (3.24), we get

$$\lim_{n\to\infty}\xi_{s,n} = 1 \quad \text{for all} \quad s = 1, \dots, r. \tag{3.25}$$

Moreover, (g.3) implies $\lim_{n\to\infty}\Gamma_{\epsilon_n}(x,\xi_{s,n}P^{\epsilon_n}_{t_s}u_n)|P^{\epsilon_n}_{t_s}u_n|^p = \lambda(x)(u^+_{t_s})^p$ a.e. in Ω'_{t_s} and, using (g.2) and (g.3) as before, one can show that

$$\lim_{n\to\infty}\int_{\Omega'_{t_s}} \Gamma_{\epsilon_n}(x,\xi_{s,n}P^{\epsilon_n}_{t_s}u_n)|P^{\epsilon_n}_{t_s}u_n|^p\,dx = \frac{1}{p}\|u^+_{t_s}\|^p_{(\lambda,p)}.$$

Thus, from (3.24) and (3.25), we get

$$\lim_{n\to\infty} f_{\epsilon_n}(z_n)$$
$$= \lim_{n\to\infty}\sum_{s=1}^r\left\{\frac{\xi^2_{s,n}}{2}\int_{\Omega'_{t_s}}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}P^{\epsilon_n}_{t_s}u_n\partial_{x_j}P^{\epsilon_n}_{t_s}u_n\,dx - \int_{\Omega'_{t_s}} G_{\epsilon_n}(x,\xi_{s,n}P^{\epsilon_n}_{t_s}u_n)\,dx\right\}$$

$$= \lim_{n\to\infty}\sum_{s=1}^{r}\left\{\frac{\xi_{s,n}^2}{2}\int_{\Omega'_{t_s}}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}P^{\epsilon_n}_{t_s}u_n\partial_{x_j}P^{\epsilon_n}_{t_s}u_n\,dx\right.$$

$$\left.-\xi^p_{s,n}\int_{\Omega'_{t_s}}\Gamma_{\epsilon_n}(x,\xi_{s,n}P^{\epsilon_n}_{t_s}u_n)|P^{\epsilon_n}_{t_s}u_n|^p\,dx\right\}$$

$$= \lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{p}\right)\sum_{s=1}^{r}\int_{\Omega'_{t_s}}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}P^{\epsilon_n}_{t_s}u_n\partial_{x_j}P^{\epsilon_n}_{t_s}u_n\,dx\,. \tag{3.26}$$

By using (3.16), (3.26), (3.20) and the fact that $u_n\in\partial N^{t_1,\dots t_r}_{\epsilon_n,B}$, we obtain

$$\lim_{n\to\infty}\left[f_{\epsilon_n}(u_n)-f_{\epsilon_n}(z_n)\right]$$

$$= \lim_{n\to\infty}\left\{\sum_{s=1}^{r}\left[\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\Omega'_{t_s}}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}P^{\epsilon_n}_{t_s}u_n\partial_{x_j}P^{\epsilon_n}_{t_s}u_n\,dx\right]\right.$$

$$+\sum_{t\notin\{t_1,\dots,t_r\}}\left[\frac{1}{2}\int_{\Omega'_t}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}P^{\epsilon_n}_{t}u_n\partial_{x_j}P^{\epsilon_n}_{t}u_n\,dx-\int_{\Omega'_t}G_{\epsilon_n}(x,P^{\epsilon_n}_t u_n+P^{\epsilon_n}u_n)\,dx\right]$$

$$+\frac{1}{2}\int_{\Omega}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}P^{\epsilon_n}u_n\partial_{x_j}P^{\epsilon_n}u_n\,dx$$

$$\left.-\sum_{s=1}^{r}\left[\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\Omega'_{t_s}}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}P^{\epsilon_n}_{t_s}u_n\partial_{x_j}P^{\epsilon_n}_{t_s}u_n\,dx\right]\right\}$$

$$= \lim\left\{\sum_{t\notin\{t_1,\dots,t_r\}}\left[\frac{1}{2}\int_{\Omega'_t}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}P^{\epsilon_n}_{t}u_n\partial_{x_j}P^{\epsilon_n}_{t}u_n\,dx\right.\right.$$

$$\left.\left.-\int_{\Omega'_t}G_{\epsilon_n}(x,P^{\epsilon_n}_t u_n+P^{\epsilon_n}u_n)\,dx\right]+\frac{1}{2}\int_{\Omega}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}P^{\epsilon_n}u_n\partial_{x_j}P^{\epsilon_n}u_n\,dx\right\}$$

$$\geq \liminf_{n\to\infty}\left\{\sum_{t\notin\{t_1,\dots,t_r\}}\left[\frac{\Lambda}{2}\int_{\Omega'_t}|DP^{\epsilon_n}_t u_n|^2\,dx\right.\right.$$

$$\left.\left.-\int_{\Omega'_t}G_{\epsilon_n}(x,P^{\epsilon_n}_t u_n+P^{\epsilon_n}u_n)\,dx\right]+\frac{\Lambda}{2}\int_{\Omega}|DP^{\epsilon_n}u_n|^2\,dx\right\}$$

$$
\begin{aligned}
&= \frac{\underline{\Lambda}}{2}B^2 - \lim_{n\to\infty} \sum_{t\notin\{t_1,\dots t_r\}} \int_{\Omega_t'} G_{\epsilon_n}(x, P_t^{\epsilon_n} u_n + P^{\epsilon_n} u_n)\, dx \\
&= \frac{\underline{\Lambda}}{2}B^2 - \sum_{t\notin\{t_1,\dots t_r\}} \frac{\|u_t\|^p_{(\lambda,p)}}{p} \\
&\geq \frac{\underline{\Lambda}}{2}B^2 - \sum_{t\notin\{t_1,\dots t_r\}} \frac{(\|u_t\|^2)^{\frac{p}{2}}}{p m_t^{\frac{p}{2}}} \geq \frac{\underline{\Lambda}}{2}B^2 - \frac{1}{p m^{\frac{p}{2}}} \Big[\sum_{t\notin\{t_1,\dots t_r\}} \|u_t\|^2 \Big]^{\frac{p}{2}} \\
&\geq \frac{\underline{\Lambda}}{2}B^2 - \frac{1}{p m^{\frac{p}{2}}} B^p > 0
\end{aligned}
$$

where the last inequality is due to the fact that $0 < B < \left(\underline{\Lambda}\frac{p}{2} m^{\frac{p}{2}}\right)^{\frac{1}{p-2}}$.

We have just found a contradiction with (3.22). □

Proposition 3.9. *Let* $0 < \varrho < (\underline{\Lambda} m)^{\frac{1}{p-2}}$ *and* $0 < B < \left(\underline{\Lambda}\frac{p}{2} m^{\frac{p}{2}}\right)^{\frac{1}{p-2}}$*. Assume* B *small enough, such that the inequality in Lemma 2.6 holds. Then there exists* $\overline{\epsilon} > 0$ *such that, for all* $\epsilon \in]0, \overline{\epsilon}[$ *and for all* $\{t_1, \dots, t_r\} \subseteq \{1, \dots, k\}$*, there exists a function* $u_\epsilon^{t_1,\dots,t_r}$ *which minimizes the functional* f_ϵ *in the set* $M_\epsilon^{t_1,\dots,t_r} \cap K_{\epsilon,\varrho}^{t_1,\dots,t_r} \cap N_{\epsilon,B}^{t_1,\dots,t_r}$ *with* $\|(P_{t_s} u_\epsilon^{t_1,\dots,t_r})^+\|_{(\lambda,p)} > \varrho$ *for all* $s = 1, \dots, r$ *and* $\|P u_\epsilon^{t_1,\dots,t_r}\|^2 + \sum_{t\notin\{t_1,\dots t_r\}} \|P_t u_\epsilon^{t_1,\dots,t_r}\|^2 < B^2$.

Moreover $u_\epsilon^{t_1,\dots,t_r}$ *verifies the following properties:*

(I)

$$
\lim_{\epsilon\to 0} \int_{\Omega} |D P^\epsilon u_\epsilon^{t_1,\dots,t_r}|^2\, dx = 0 \tag{3.27}
$$

and

$$
\lim_{\epsilon\to 0} \int_{\Omega_t'} |D P_t^\epsilon u_\epsilon^{t_1,\dots,t_r}|^2\, dx = 0 \quad \forall \quad t \in \{1, \dots, k\} \setminus \{t_1, \dots t_r\}\,; \tag{3.28}
$$

(II)

$$
\lim_{\epsilon\to 0} \int_{\Omega_t' \setminus \Omega_t} |D u_\epsilon^{t_1,\dots,t_r}|^2\, dx = 0; \tag{3.29}
$$

(III)

$$
\liminf_{\epsilon\to 0} \Big(\int_{\Omega_{t_s}'} \lambda(x) |(u_\epsilon^{t_1,\dots,t_r})^+|^p\, dx \Big)^{\frac{1}{p}} \geq (\underline{\Lambda} m)^{\frac{1}{p-2}} \quad \forall \quad s = 1, \dots, r. \tag{3.30}
$$

Proof. Let $\{t_1, \dots, t_r\} \subseteq \{1, \dots, k\}$. The existence of $\overline{\epsilon} > 0$ and $u_\epsilon^{t_1,\dots,t_r} \in M_\epsilon^{t_1,\dots,t_r} \cap K_{\epsilon,\varrho}^{t_1,\dots,t_r} \cap N_{\epsilon,B}^{t_1,\dots,t_r}$ for all $\epsilon \in]0, \overline{\epsilon}[$ follows from Lemma 3.7.

Moreover, for $\overline{\epsilon}$ small enough, Lemma 3.8 implies

$$\|P^{\epsilon} u_{\epsilon}^{t_1,\dots,t_r}\|^2 + \sum_{t\notin\{t_1,\dots t_r\}} \|P_t^{\epsilon} u_{\epsilon}^{t_1,\dots,t_r}\|^2 < B^2 \ ,$$

while Lemma 2.6 implies $\|(P_{t_s}^{\epsilon} u_{\epsilon}^{t_1,\dots,t_r})^+\|_{(\lambda,p)} > \varrho$ for all $s = 1,\dots,r$.

(I) Proving (3.27) and (3.28) is equivalent to show that

$$\lim_{\epsilon\to 0} \left\{ \|P^{\epsilon} u_{\epsilon}^{t_1,\dots,t_r}\|^2 + \sum_{t\notin\{t_1,\dots,t_r\}} \|P_t^{\epsilon} u_{\epsilon}^{t_1,\dots,t_r}\|^2 \right\} = 0.$$

By contradiction, suppose there exist $\beta \in]0, B]$ and a sequence $(\epsilon_n)_{n\geq 1}$ of positive numbers, with $\lim\limits_{n\to\infty} \epsilon_n = 0$, such that

$$\lim_{n\to\infty} \left\{ \|P^{\epsilon_n} u_{\epsilon_n}^{t_1,\dots,t_r}\|^2 + \sum_{t\notin\{t_1,\dots,t_r\}} \|P_t^{\epsilon_n} u_{\epsilon_n}^{t_1,\dots,t_r}\|^2 \right\} = \beta^2. \tag{3.31}$$

From Lemma 3.6 it follows that

$$\limsup_{n\to\infty} f_{\epsilon_n}(u_{\epsilon_n}^{t_1,\dots,t_r}) \leq (\frac{1}{2} - \frac{1}{p}) \sum_{s=1}^{r} \left(\overline{\Lambda} m_{t_s}\right)^{\frac{p}{p-2}} < +\infty. \tag{3.32}$$

Since $u_{\epsilon_n}^{t_1,\dots,t_r} \in N_{\epsilon_n,B}^{t_1,\dots t_r}$ for all $n \geq 1$, $(P^{\epsilon_n} u_{\epsilon_n}^{t_1,\dots,t_r})_{n\geq 1}$ and $(P_t^{\epsilon_n} u_{\epsilon_n}^{t_1,\dots,t_r})_{n\geq 1}$ (for all $t \in \{1,\dots,k\} \setminus \{t_1,\dots,t_r\}$) are bounded in $H_0^{1,2}(\Omega)$; since $u_{\epsilon_n}^{t_1,\dots,t_r} \in M_{\epsilon_n}^{t_1,\dots,t_r}$ for all $n \geq 1$ and (3.32) holds, arguing as in the proof of Lemma 3.8 one can show that $(u_{\epsilon_n}^{t_1,\dots t_r})_{n\geq 1}$ is bounded in $H_0^{1,2}(\Omega)$. Hence, up to a subsequence, $u_{\epsilon_n}^{t_1,\dots,t_r} \to u^{t_1,\dots,t_r} \in H_0^{1,2}(\Omega)$, weakly in $H_0^{1,2}(\Omega)$, in $L^p(\Omega)$, in $L^q(\Omega)$ and a.e. in Ω.

Moreover, since $\forall \quad \eta > 0$

$$\liminf_{\epsilon\to 0} \frac{1}{\epsilon} \inf \left\{ \Lambda_1(\epsilon, x) \ : \ x \in \bigcup_{j=1}^{k} \Omega_j(\eta) \right\} = +\infty,$$

from (3.32) we get $\int_{\Omega\setminus\bigcup_{t=1}^{k}\Omega_t} |Du|^2 \, dx = 0$; it follows that one can write $u^{t_1,\dots,t_r} = u_1 + \dots + u_k$ with $u_t \in H_0^{1,2}(\Omega_t')$ for all $t \in \{1,\dots,k\}$.

Arguing as in STEP 2 of Lemma 3.8, one can see that (up to a subsequence)

$$\begin{aligned} \lim_{n\to\infty} f_{\epsilon_n}(u_{\epsilon_n}^{t_1,\dots,t_r}) \quad = \quad & \lim_{n\to\infty} \left\{ (\frac{1}{2} - \frac{1}{p}) \sum_{s=1}^{r} Q_{\epsilon_n}(P_{t_s}^{\epsilon_n} u_{\epsilon_n}^{t_1,\dots,t_r}) \right. \\ & + \sum_{t\notin\{t_1,\dots,t_r\}} \Big[\frac{1}{2} \int_{\Omega_t'} \frac{a_{\epsilon_n}^{i,j}}{\epsilon_n} \partial_{x_i} P_t^{\epsilon_n} u_{\epsilon_n}^{t_1,\dots,t_r} \partial_{x_j} P_t^{\epsilon_n} u_{\epsilon_n}^{t_1,\dots,t_r} \, dx \\ & - \int_{\Omega_t'} G_{\epsilon_n}(x, P_t^{\epsilon_n} u_{\epsilon_n}^{t_1,\dots,t_r} + P^{\epsilon_n} u_{\epsilon_n}^{t_1,\dots,t_r}) \, dx \Big] \end{aligned}$$

$$+\frac{1}{2}\int_{\Omega}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}P^{\epsilon_n}u^{t_1,\dots,t_r}_{\epsilon_n}\partial_{x_j}P^{\epsilon_n}u^{t_1,\dots,t_r}_{\epsilon_n}\,dx\Bigg\}\,. \qquad (3.33)$$

Let us now define (z_n) (for n large enough) in the following way: $z_n = \sum_{s=1}^{r}\xi_{n,s}P^{\epsilon_n}_{t_s}u^{t_1,\dots,t_r}_{\epsilon_n}$, with $\xi_{n,s} > 0 \quad \forall \quad s = 1,\dots,r$ such that $z_n \in M^{t_1,\dots,t_r}_{\epsilon_n} \cap K^{t_1,\dots,t_r}_{\epsilon_n,\varrho}$ and $f_{\epsilon_n}(z_n) > 0$; the existence of such numbers, for n large enough, follows arguing as in the proof of Proposition 3.3. It is clear that $z_n \in N^{t_1,\dots,t_r}_{\epsilon_n,B}$, and so (for n large enough)

$$f_{\epsilon_n}(u^{t_1,\dots,t_r}_{\epsilon_n}) \le f_{\epsilon_n}(z_n). \qquad (3.34)$$

Since $z_n \in M^{t_1,\dots,t_r}_{\epsilon_n} \cap K^{t_1,\dots t_r}_{\epsilon_n,\varrho}$ and $P^{\epsilon_n}_{t_s}z_n \to u_{t_s}$ a.e. in Ω'_{t_s}, in $L^p(\Omega'_{t_s})$, in $L^q(\Omega'_{t_s})$, arguing as in STEP 3 of Lemma 3.8, one can prove that for all $s = 1,\dots,r$

$$\lim_{n\to\infty}\xi_{n,s} = 1. \qquad (3.35)$$

Hence, from (3.31), (3.33), (3.35), we get (up to a subsequence)

$$\begin{aligned}
&\lim_{n\to\infty}\left[f_{\epsilon_n}(u^{t_1,\dots,t_r}_{\epsilon_n}) - f_{\epsilon_n}(z_n)\right]\\
&= \lim_{n\to\infty}\Bigg\{\sum_{s=1}^{k}\Big(\frac{1}{2}-\frac{1}{p}\Big)Q_{\epsilon_n}(P^{\epsilon_n}_{t_s}u^{t_1,\dots,t_r}_{\epsilon_n})\\
&\quad + \sum_{t\notin\{t_1,\dots,t_r\}}\Big[\frac{1}{2}\int_{\Omega'_t}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}P^{\epsilon_n}_t u^{t_1,\dots,t_r}_{\epsilon_n}\partial_{x_j}P^{\epsilon_n}_t u^{t_1,\dots,t_r}_{\epsilon_n}\,dx\\
&\quad - \int_{\Omega'_t}G_{\epsilon_n}(x, P^{\epsilon_n}_t u^{t_1,\dots,t_r}_{\epsilon_n} + P^{\epsilon_n}u^{t_1,\dots,t_r}_{\epsilon_n})\,dx\Big]\\
&\quad + \frac{1}{2}\int_{\Omega}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}P^{\epsilon_n}u^{t_1,\dots,t_r}_{\epsilon_n}\partial_{x_j}P^{\epsilon_n}u^{t_1,\dots,t_r}_{\epsilon_n}\,dx\\
&\quad - \sum_{s=1}^{k}\Big(\frac{1}{2}-\frac{1}{p}\Big)\xi^2_{n,s}Q_{\epsilon_n}(P^{\epsilon_n}_{t_s}u^{t_1,\dots,t_r}_{\epsilon_n})\Bigg\}\\
&\ge \lim_{n\to\infty}\Bigg\{\sum_{t\notin\{t_1,\dots,t_r\}}\Big[\frac{1}{2}\int_{\Omega'_t}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}P^{\epsilon_n}_t u^{t_1,\dots,t_r}_{\epsilon_n}\partial_{x_j}P^{\epsilon_n}_t u^{t_1,\dots,t_r}_{\epsilon_n}\,dx\\
&\quad - \int_{\Omega'_t}G_{\epsilon_n}(x, P^{\epsilon_n}_t u^{t_1,\dots,t_r}_{\epsilon_n} + P^{\epsilon_n}u^{t_1,\dots,t_r}_{\epsilon_n})\,dx\Big]\\
&\quad + \frac{1}{2}\int_{\Omega}\frac{a^{i,j}_{\epsilon_n}}{\epsilon_n}\partial_{x_i}P^{\epsilon_n}u^{t_1,\dots,t_r}_{\epsilon_n}\partial_{x_j}P^{\epsilon_n}u^{t_1,\dots,t_r}_{\epsilon_n}\,dx\Bigg\}
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{2}\underline{\Lambda} \lim_{n\to\infty} \left\{ \|P^{\epsilon_n} u_{\epsilon_n}^{t_1,\dots,t_r}\|^2 + \sum_{t\notin\{t_1,\dots,t_r\}} \|P_t^{\epsilon_n} u_{\epsilon_n}^{t_1,\dots,t_r}\|^2 \right\} \\
&\quad - \sum_{t\notin\{t_1,\dots,t_r\}} \frac{1}{p}\|u_t\|_{(\lambda,p)}^p \\
&\geq \frac{\underline{\Lambda}}{2}\beta^2 - \frac{1}{pm^{\frac{p}{2}}} \left(\sum_{t\notin\{t_1,\dots,t_r\}} \|u_t\|^2 \right)^{\frac{p}{2}} \\
&\geq \frac{\underline{\Lambda}}{2}\beta^2 - \frac{1}{pm^{\frac{p}{2}}}\beta^p > 0
\end{aligned}$$

where the last inequality is due to the fact that $0 < \beta \leq B < (\underline{\Lambda}\frac{p}{2} m^{\frac{p}{2}})^{\frac{1}{p-2}}$.

This is a contradiction with (3.34).

(II) (3.29) follows from assumption $(a.4)$ taking into account that the minimizing function $u_\epsilon^{t_1,\dots,t_r} \in M_\epsilon^{t_1,\dots,t_r}$ and

$$\limsup_{\epsilon\to 0} f_\epsilon(u_\epsilon^{t_1,\dots,t_r}) \leq (\frac{1}{2} - \frac{1}{p}) \sum_{s=1}^{r} \left(\overline{\Lambda} m_{t_s}\right)^{\frac{p}{p-2}} < +\infty$$

(see Definitions 2.4, 3.4 and Lemma 3.6).

(III) By contradiction, suppose there exist $I \subseteq \{1,\dots,r\}$, with $I \neq \emptyset$, and an infinitesimal sequence $(\epsilon_n)_{n\geq 1}$ of positive numbers such that, for $s \in I$,

$$\lim_{n\to\infty} \left(\int_{\Omega'_{t_s}} \lambda(x) |(u_{\epsilon_n}^{t_1,\dots,t_r})^+|^p \, dx \right)^{\frac{1}{p}} < (\underline{\Lambda} m)^{\frac{1}{p-2}}. \tag{3.36}$$

Arguing as in (I), one can prove that $(u_{\epsilon_n}^{t_1,\dots,t_r})_{n\geq 1}$ is bounded in $H_0^{1,2}(\Omega)$; hence, up to a subsequence, $u_{\epsilon_n}^{t_1,\dots,t_r}$ converges weakly in $H_0^{1,2}(\Omega)$, in $L^p(\Omega)$, in $L^q(\Omega)$ and a.e. in Ω to a function $u^{t_1,\dots t_r}$ which can be written as $u^{t_1,\dots t_r} = u_1 + \dots + u_k$ with $u_t \in H_0^{1,2}(\Omega'_t)$ and $\int_{\Omega'_t\setminus\Omega_t} |Du_t|^2 \, dx = 0$ for all $t \in \{1,\dots,k\}$.

As showed in (I), we have $P^{\epsilon_n} u_{\epsilon_n}^{t_1,\dots,t_r} \to 0$ and $P_t^{\epsilon_n} u_{\epsilon_n}^{t_1,\dots,t_r} \to 0$ for all $t \notin \{t_1,\dots,t_r\}$ in $H_0^{1,2}(\Omega)$; so $u_t \equiv 0 \quad \forall\, t \in \{1,\dots,k\} \setminus \{t_1,\dots,t_r\}$.

Moreover we have $\|u_{t_s}^+\|_{(\lambda,p)} \geq \varrho$ for all $s = 1,\dots,r$ and, by (3.36), $\|u_{t_s}^+\|_{(\lambda,p)} < (\underline{\Lambda} m)^{\frac{1}{p-2}}$ for all $s \in I$.

Notice that

$$f'_{\epsilon_n}(u_{\epsilon_n}^{t_1,\dots,t_r})[P_{t_s}^{\epsilon_n} u_{\epsilon_n}^{t_1,\dots,t_r}] = 0 \quad \forall \quad s = 1,\dots,r, \ \forall\, n \geq 1$$

since $u_{\epsilon_n}^{t_1,\dots,t_r} \in M_{\epsilon_n}^{t_1,\dots,t_r}$.

Since again $P_{t_s}^{\epsilon_n} u_{\epsilon_n}^{t_1,\dots,t_r} + P^{\epsilon_n} u_{\epsilon_n}^{t_1,\dots,t_r} \to u_{t_s}$ in $L^p(\Omega)$, in $L^q(\Omega)$, a.e. in Ω, using (g.2), (g.3) as before, we obtain, for all $s \in I$,

$$
\begin{aligned}
0 &= \lim_{n\to\infty} f'_{\epsilon_n}(u_{\epsilon_n}^{t_1,\dots,t_r})[P_{t_s}^{\epsilon_n} u_{\epsilon_n}^{t_1,\dots,t_r}] \geq \underline{\Lambda} \int_{\Omega_{t_s}} |Du_{t_s}|^2\,dx - \int_{\Omega'_{t_s}} \lambda(x)|u_{t_s}^+|^p\,dx \\
&\geq \underline{\Lambda} \frac{\int_{\Omega_{t_s}} |Du_{t_s}|^2\,dx}{(\int_{\Omega'_{t_s}} \lambda(x)|u_{t_s}|^p\,dx)^{\frac{2}{p}}} (\int_{\Omega'_{t_s}} \lambda(x)|u_{t_s}^+|^p\,dx)^{\frac{2}{p}} - \int_{\Omega'_{t_s}} \lambda(x)|u_{t_s}^+|^p\,dx \\
&\geq \underline{\Lambda} m \|u_{t_s}^+\|^2_{(\lambda,p)} - \|u_{t_s}^+\|^p_{(\lambda,p)} > 0,
\end{aligned}
$$

where the last inequality is due to the fact that $0 < \varrho \leq \|u_{t_s}^+\|_{(\lambda,p)} < (\underline{\Lambda} m)^{\frac{1}{p-2}}$.

Hence we have obtained a contradiction. □

4. Existence of multibump solutions

In order to get our main result, it remains to show that the constrained minimum points, obtained in the previous section, are critical points for the functional f_ϵ and give rise to multibump solutions of our problem.

Lemma 4.1. *Let $0 < \varrho < (\underline{\Lambda} m)^{\frac{1}{p-2}}$ and $0 < B < \left(\underline{\Lambda}\frac{p}{2} m^{\frac{p}{2}}\right)^{\frac{1}{p-2}}$. Assume B small enough, such that the inequality in Lemma 2.6 holds. Then there exists $\bar{\epsilon} > 0$ such that, for all $\epsilon \in]0, \bar{\epsilon}[$, for all subset $\{t_1, \dots, t_r\}$ of $\{1, \dots k\}$, the function $u_\epsilon^{t_1,\dots,t_r}$, which achieves the minimum for f_ϵ constrained on $M_\epsilon^{t_1,\dots,t_r} \cap K_{\epsilon,\varrho}^{t_1,\dots,t_r} \cap N_{\epsilon,B}^{t_1,\dots,t_r}$ (see Lemma 3.7 and Proposition 3.9), is a critical point for f_ϵ.*

Proof. Let $\{t_1, \dots, t_r\} \subseteq \{1, \dots, k\}$. Lemma 3.7 and Proposition 3.9 imply that, for all $\epsilon > 0$ small enough, there exists $u_\epsilon^{t_1,\dots,t_r}$ which achieves $\inf\left\{f_\epsilon(u) : u \in M_\epsilon^{t_1,\dots,t_r} \cap K_{\epsilon,\varrho}^{t_1,\dots,t_r} \cap N_{\epsilon,B}^{t_1,\dots,t_r}\right\}$ and satisfies $\|(P_{t_s}^\epsilon u_\epsilon^{t_1,\dots,t_r})^+\|_{(\lambda,p)} > \varrho$ for all $s = 1, \dots, r$ and $\|P^\epsilon u_\epsilon^{t_1,\dots,t_r}\|^2 + \sum_{t \notin \{t_1,\dots,t_r\}} \|P_t^\epsilon u_\epsilon^{t_1,\dots,t_r}\|^2 < B^2$. Hence there exists a neighbourhood of $u_\epsilon^{t_1,\dots,t_r}$ in $H_0^{1,2}(\Omega)$ which is contained in $K_{\epsilon,\varrho}^{t_1,\dots,t_r} \cap N_{\epsilon,B}^{t_1,\dots,t_r}$, i.e. $u_\epsilon^{t_1,\dots,t_r}$ is a local minimum point for f_ϵ constrained on $M_\epsilon^{t_1,\dots,t_r}$.

We shall prove first that $M_\epsilon^{t_1,\dots,t_r}$ is a smooth C^1- manifold in a neighbouhood of $u_\epsilon^{t_1,\dots,t_r}$; then we will show that all the Lagrange multipliers are zero, i.e. that $u_\epsilon^{t_1,\dots,t_r}$ is a critical point for f_ϵ.

For simplicity of notation let us write u_ϵ instead of $u_\epsilon^{t_1,\dots,t_r}$.

For all $t \in \{1, \dots, k\}$, let $h_{t,\epsilon} : H_0^{1,2}(\Omega) \to \mathbb{R}$ be the functional defined by

$$h_{t,\epsilon}(u) = f'_\epsilon(u)[P_t^\epsilon u].$$

Under our assumptions, $h_{t,\epsilon} \in C^1(H_0^{1,2}(\Omega); \mathbb{R})$. Notice that

$$M_\epsilon^{t_1,\dots,t_r} = \left\{u \in H_0^{1,2}(\Omega) : h_{t,\epsilon}(u) = 0 \quad \forall \quad t \in \{t_1, \dots, t_r\}\right\}.$$

In order to prove that $M_\epsilon^{t_1,\dots,t_r}$ is a smooth manifold in a neighbourhood of u_ϵ, by using the implicit function theorem, it suffices to verify that

$$\sum_{s=1}^{r} \mu_s h'_{s,\epsilon}(u_\epsilon) = 0 \quad \Rightarrow \quad \mu_s = 0 \ \forall s = 1, \dots, r.$$

This property holds if $\epsilon > 0$ is small enough. In fact

$$\sum_{s=1}^{r} \mu_s h'_{s,\epsilon}(u_\epsilon) = 0 \quad \Rightarrow \quad \sum_{s=1}^{r} \mu_s h'_{s,\epsilon}(u_\epsilon)[P^\epsilon_{t_\sigma} u_\epsilon] = 0.$$

Therefore, since $u_\epsilon \in M_\epsilon^{t_1,\dots,t_r}$, we obtain for $\sigma \in \{1, \dots, r\}$

$$\begin{aligned}
0 &= \sum_{s=1}^{r} \mu_s \left\{ f''_\epsilon(u_\epsilon)[P_{t_s} u_\epsilon][P_{t_\sigma} u_\epsilon] + f'_\epsilon(u_\epsilon)[P^\epsilon_{t_s} P^\epsilon_{t_\sigma} u_\epsilon] \right\} \\
&= \mu_\sigma \left\{ f''_\epsilon(u_\epsilon)[P^\epsilon_{t_\sigma} u_\epsilon]^2 + f'_\epsilon(u_\epsilon)[P^\epsilon_{t_\sigma} u_\epsilon] \right\} \\
&= \mu_\sigma f''_\epsilon(u_\epsilon)[P^\epsilon_{t_\sigma} u_\epsilon]^2.
\end{aligned}$$

The claim will be obtained if we shall prove that

$$\limsup_{\epsilon\to 0} f''_\epsilon(u_\epsilon)[P^\epsilon_{t_\sigma} u_\epsilon]^2 \le (2-p)\varrho^p < 0 \ \forall \ \sigma \in \{1, \dots, r\}. \tag{4.1}$$

Taking into account that $u_\epsilon \in M_\epsilon^{t_1,\dots,t_r}$, we have

$$\begin{aligned}
f''_\epsilon(u_\epsilon)[P^\epsilon_{t_\sigma} u_\epsilon]^2 &= \int_{\Omega'_{t_\sigma}} g_\epsilon(x, u_\epsilon) P^\epsilon_{t_\sigma} u_\epsilon \, dx - \int_{\Omega'_{t_\sigma}} g'_\epsilon(x, u_\epsilon)[P^\epsilon_{t_\sigma} u_\epsilon]^2 \, dx \\
&= \int_{\Omega'_{t_\sigma}} \gamma_\epsilon(x, u_\epsilon)(u^+_\epsilon)^{p-1} P^\epsilon_{t_\sigma} u_\epsilon \, dx \qquad (4.2)\\
&\quad - \int_{\Omega'_{t_\sigma}} \tilde{\gamma}_\epsilon(x, u_\epsilon)(u^+_\epsilon)^{p-2} [P^\epsilon_{t_\sigma} u_\epsilon]^2 \, dx
\end{aligned}$$

(see notations (2.3), (2.4), (2.8), (2.9)).

Assume, by contradiction, that for some $\sigma \in \{1, \dots, r\}$ there exists a sequence of positive numbers $(\epsilon_n)_{n\ge 1} \to 0$ such that

$$\lim_{n\to\infty} f''_{\epsilon_n}(u_{\epsilon_n})[P^{\epsilon_n}_{t_\sigma} u_{\epsilon_n}]^2 > (2-p)\varrho^p. \tag{4.3}$$

From Proposition 3.9 we infer that, up to a subsequence, $P^{\epsilon_n} u_{\epsilon_n} \to 0$ and $P^{\epsilon_n}_{t_\sigma} u_{\epsilon_n} \to u_{t_\sigma} \in H^{1,2}_0(\Omega'_{t_\sigma})$ in $L^p(\Omega'_{t_\sigma})$, in $L^q(\Omega'_{t_\sigma})$ and a.e. in Ω'_{t_σ}.

Notice that $\|u^+_{t_\sigma}\|_{(\lambda,p)} \ge \varrho$ since $u_{\epsilon_n} \in K^{t_1,\dots,t_r}_{\epsilon_n,\varrho}$ for all $n \ge 1$. Moreover

$$\gamma_{\epsilon_n}(x, u_{\epsilon_n})(u^+_{\epsilon_n})^{p-1} P^{\epsilon_n}_{t_\sigma} u_{\epsilon_n} \to \lambda(x)(u^+_{t_\sigma})^p$$

and

$$\tilde{\gamma}_{\epsilon_n}(x, u_{\epsilon_n})(u^+_{\epsilon_n})^{p-2}[P^{\epsilon_n}_{t_\sigma} u_{\epsilon_n}]^2 \to (p-1)\lambda(x)(u^+_{t_\sigma})^p$$

a.e. in Ω'_{t_σ}.

Hence assumptions (g.2) and (g.3) allow us to apply the Lebesgue theorem as above and to obtain

$$\lim_{n\to\infty} f''_{\epsilon_n}(u_{\epsilon_n})[P^{\epsilon_n}_{t_\sigma} u_{\epsilon_n}]^2 = (2-p) \int_{\Omega'_{t_\sigma}} \lambda(x)(u^+_{t_\sigma})^p \, dx \leq (2-p)\varrho^p,$$

which is in contradiction with (4.3).

Thus (4.1) holds and so $M^{t_1,\dots,t_r}_\epsilon$ is a smooth manifold in a neighbourhood of u_ϵ for all $\epsilon > 0$ small enough.

Since u_ϵ is a local minimum point for f_ϵ on $M^{t_1,\dots,t_r}_\epsilon$, there exist some constants $\lambda_{1,\epsilon}, \dots \lambda_{r,\epsilon}$ (the Lagrange multipliers) such that

$$f'_\epsilon(u_\epsilon) = \lambda_{1,\epsilon} h'_{1,\epsilon}(u_\epsilon) + \cdots + \lambda_{r,\epsilon} h'_{r,\epsilon}(u_\epsilon)$$

i.e.

$$f'_\epsilon(u_\epsilon)[\varphi] = \lambda_{1,\epsilon} h'_{1,\epsilon}(u_\epsilon)[\varphi] + \cdots + \lambda_{r,\epsilon} h'_{r,\epsilon}(u_\epsilon)[\varphi] \quad \forall \quad \varphi \in H^{1,2}_0(\Omega).$$

Let us choose $\varphi = P^\epsilon_{t_\sigma} u_\epsilon$; since $u_\epsilon \in M^{t_1,\dots,t_r}_\epsilon$, we obtain

$$\begin{aligned} 0 &= f'_\epsilon(u_\epsilon)[P^\epsilon_{t_\sigma}(u_\epsilon)] \\ &= \sum_{s=1}^{r} \lambda_{s,\epsilon} \left\{ f''_\epsilon(u_\epsilon)[P^\epsilon_{t_s} u_\epsilon][P^\epsilon_{t_\sigma} u_\epsilon] + f'_\epsilon(u_\epsilon)[P^\epsilon_{t_s}(P^\epsilon_{t_\sigma} u_\epsilon)] \right\} \\ &= \lambda_{\sigma,\epsilon} f''_\epsilon(u_\epsilon)[P^\epsilon_{t_\sigma} u_\epsilon]^2 , \end{aligned}$$

which implies $\lambda_{\sigma,\epsilon} = 0$ for $\epsilon > 0$ small enough, because of (4.1).

Hence there exists $\bar{\epsilon} > 0$ such that, for all $\epsilon \in]0, \bar{\epsilon}[$, $\lambda_{1,\epsilon} = \cdots = \lambda_{r,\epsilon} = 0$, i.e. u_ϵ is a critical point for f_ϵ. □

Proof of Theorem 1.1. If u_ϵ is a nontrivial critical point for f_ϵ, then $v_\epsilon = \epsilon^{\frac{1}{p-2}} u_\epsilon$ is a solution of problem P_ϵ by the maximum principle, since $g(x,t) = 0$ for $t \leq 0$ and $g(x,t) > 0$ for $t > 0$.

The behaviour of the functions $u^{t_1,\dots,t_r}_\epsilon$ as $\epsilon \to 0$ follows from Proposition 3.9. In particular we have that, for $\epsilon > 0$ small enough, different choice of the subset $\{t_1, \dots, t_r\}$ produce (by Lemma 4.1) different solutions of our problem.

Hence, for every $r \in \{1, \dots, k\}$ there exist $\binom{k}{r}$ $r-$bump solutions; so, on the whole, we get $2^k - 1$ distinct positive solutions. □

Finally in the following proposition we summarize the main properties of the obtained solutions.

Proposition 4.2. *For all* $\epsilon \in]0, \bar{\epsilon}[$ *and for all* $\{t_1, \dots, t_r\} \subseteq \{1, \dots, k\}$ *there exists a solution* $v^{t_1,\dots,t_r}_\epsilon$ *of problem* P_ϵ *such that*

(I)

$$\limsup_{\epsilon\to 0} \epsilon^{\frac{1}{2-p}} \|v_\epsilon^{t_1,\dots,t_r}\| < +\infty;$$

(II)

$$\liminf_{\epsilon\to 0} \epsilon^{\frac{1}{2-p}} \left(\int_{\Omega'_{t_s}} \lambda(x)|v_\epsilon^{t_1,\dots,t_r}|^p \, dx \right)^{\frac{1}{p}} \geq (\underline{\Lambda} m)^{\frac{1}{p-2}} \quad \forall s \in \{1,\dots,r\};$$

(III)

$$\lim_{\epsilon\to 0} \epsilon^{\frac{2}{2-p}} \int_{\Omega\setminus\bigcup_{s=1}^r \Omega_{t_s}} |Dv_\epsilon^{t_1,\dots,t_r}|^2 \, dx = \lim_{\epsilon\to 0} \epsilon^{\frac{p}{2-p}} \int_{\Omega\setminus\bigcup_{s=1}^r \Omega'_{t_s}} |v_\epsilon^{t_1,\dots,t_r}|^p \, dx = 0.$$

The proof follows easily from Proposition 3.9 and Lemma 3.7 and 4.1, taking into account the proof of Theorem 1.1.

Remark 4.3. Let $v_\epsilon^{t_1,\dots,t_r}$ be the solution of problem P_ϵ given by Theorem 1.1. The method used in the proof shows that, up to a subsequence, the function $u_\epsilon^{t_1,\dots,t_r} = \epsilon^{\frac{1}{2-p}} v_\epsilon^{t_1,\dots,t_r}$ converges, as $\epsilon \to 0$, to a function $u^{t_1,\dots,t_r}$, which can be written as follows:

$$u^{t_1,\dots,t_r} = \sum_{s=1}^r u_{t_s},$$

where, for all $s = 1,\dots,r$, u_{t_s} is a positive function in $H_0^{1,2}(\Omega'_{t_s})$ such that $Du_{t_s} \equiv 0$ in $\Omega'_{t_s} \setminus \Omega_{t_s}$.

Moreover, if in addition we assume that $\frac{a_\epsilon}{\epsilon}(x) \to A(x)$ in $L^\infty(\Omega_{t_s})$ as $\epsilon \to 0$, then u_{t_s} satisfies the equation

$$div(A(x)Du_{t_s}) + \lambda(x)u_{t_s}^{p-1} = 0 \quad \forall x \in \Omega_{t_s}$$

(with other conditions on the boundary of $\Omega'_{t_s}\setminus\Omega_{t_s}$) and the function $\overline{u}_{t_s} = \frac{u_{t_s}}{\|u_{t_s}\|_{(\lambda,p)}}$ realizes the minimum

$$\min\left\{ \int_{\Omega_{t_s}} A^{ij}(x)\partial_{x_i}u\partial_{x_j}u \, dx \; : \; u \in H_0^{1,2}(\Omega'_{t_s}), \|u\|_{(\lambda,p)} = 1, \int_{\Omega'_{t_s}\setminus\Omega_{t_s}} |Du|^2 \, dx = 0 \right\}.$$

Remark 4.4. The method used in the proof of Theorem 1.1 can be adapted to state analogous multiplicity results when the coefficient matrix $a_\epsilon(x)$ degenerates only at a finite collection of points $x_1,\dots,x_k$ belonging to Ω.

In this case the degree of vanishing near the degeneration points plays an important role (as in [31]).

For example, let us consider the problem

$$\tilde{P}_\epsilon \quad \begin{cases} div(\Lambda_\epsilon(x)Du) + g(x,u) = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where, for all $\epsilon > 0$, $\Lambda_\epsilon : \Omega \to \mathbb{R}$ is defined by

$$\Lambda_\epsilon(x) = \max\{\epsilon, \Lambda(x)\}$$

and $\Lambda(x) \in L^\infty(\Omega)$ is a positive function which behaves as $|x - x_t|^\alpha$, with $\alpha > \frac{2N+2p-Np}{p}$, near the degeneration points x_t $(t = 1, \ldots, k)$.

Then it is possible to find $\bar{\epsilon} > 0$ such that, for all $\epsilon \in]0, \bar{\epsilon}[$, $\tilde{P}_\epsilon$ has at least $2^k - 1$ multispike solutions $u_\epsilon^{t_1,\ldots,t_r}$ $(\{t_1, \ldots, t_r\} \subseteq \{1, \ldots, k\})$ such that $u_\epsilon^{t_1,\ldots,t_r}$ can be decomposed in the following way:

$$u_\epsilon^{t_1,\ldots,t_r} = \sum_{s=1}^{r} u_\epsilon^{t_s}$$

where, for all $s = 1, \ldots, r$, $u_\epsilon^{t_s} \to 0$ in $H_0^{1,2}(\Omega)$ as $\epsilon \to 0$ and $\frac{u_\epsilon^{t_s}}{\|u_\epsilon^{t_1,\ldots,t_r}\|_{(\lambda,p)}}$ concentrates like Dirac mass near the degeneration point x_{t_s}.

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