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NONLINEAR ELLIPTIC PROBLEMS APPROXIMATING DEGENERATE EQUATIONS

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1. INTRODUCTION

This paper is concerned with a class of semilinear elliptic problems of the form

$$P_\epsilon \qquad \begin{cases} div(a_\epsilon(x)Du) + g(x,u) = 0 & \text{in } \Omega \\ u \not\equiv 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega , \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$, for all $\epsilon > 0$ and $x \in \Omega$ $a_\epsilon(x)$ is a positive defined symmetric $N \times N$ matrix, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a given function such that $g(x,0) = 0$ $\forall x \in \Omega$. We shall assume that the matrix $a_\epsilon(x)$ degenerates as $\epsilon \to 0$ for all x in a subset $\mathcal{D}$ of Ω. Our aim is to study existence, nonexistence and multiplicity of solutions of P_ϵ under suitable assumptions on the degeneration set $\mathcal{D}$ and the function $g(x,t)$.

Let us specify the assumptions required on $a_\epsilon(x)$ and on g:

(a.1) for all $\epsilon > 0$ and almost all $x \in \Omega$ there exist $\Lambda_1 = \Lambda_1(\epsilon, x) > 0$ and $\Lambda_2 = \Lambda_2(\epsilon, x) > 0$ such that

$$\Lambda_1|\xi|^2 \leq \sum_{i,j=1}^{N} a_\epsilon^{i,j}(x)\xi_i\xi_j \leq \Lambda_2|\xi|^2 \quad \forall\, \xi \in \mathbb{R}^N;$$

(a.2)

$$\liminf_{\epsilon\to 0} \frac{1}{\epsilon} \inf_{x\in\Omega} \Lambda_1(\epsilon, x) > 0;$$

(a.3) there exists a smooth domain $\mathcal{D} \subseteq \Omega$, $\mathcal{D} \neq \emptyset$, such that

$$\limsup_{\epsilon\to 0} \frac{1}{\epsilon} \sup\{\Lambda_2(\epsilon, x) \,:\, x \in \mathcal{D}\} < +\infty$$

and, if $\mathcal{D} \neq \Omega$, for all $\eta > 0$

$$\liminf_{\epsilon\to 0} \inf\{\Lambda_1(\epsilon, x) \,:\, x \in \Omega \setminus \mathcal{D}(\eta)\,\} > 0$$

where

$$\mathcal{D}(\eta) := \{x \in \Omega \, : \, dist(x, \mathcal{D}) < \eta\};$$

(g.1) g and g_t (the derivative of g with respect to the second variable) are Caratheodory functions and there exist positive constants a and q, with $q < \frac{2N}{N-2}$ if $N \geq 3$, such that, for all $t \in \mathbb{R}$ and almost all $x \in \Omega$,

$$|g(x,t)| \leq a + a|t|^{q-1}$$

and

$$|g_t(x,t)| \leq a + a|t|^{q-2};$$

(g.2) for $t \neq 0$ and for almost all $x \in \Omega$

$$t\frac{d}{dt}\left[\frac{g(x,t)}{t}\right] > 0;$$

(g.3) there exists $\theta \in]0, \frac{1}{2}[$ such that

$$H(x,t) \leq \theta t h(x,t)$$

for all $t \in \mathbb{R}$ and almost all $x \in \Omega$, where $h(x,t) = g(x,t) - g_t(x,0)t$ and $H(x,t) = \int_0^t h(x,\tau)\, d\tau$.

The sign of $g_t(x,0)$ seems to play a key role for the solvability of P_ϵ: if $g_t(x,0) > 0$ in a subset of $\mathcal{D}$, then we shall prove a general nonexistence result (Theorem 2.1), which, in particular, implies that no positive solution of P_ϵ can exist for $\epsilon > 0$ small enough.
On the contrary, if $g_t(x,0) \leq 0$ for all $x \in \Omega$, under our assumptions it is easy to prove that P_ϵ has at least one positive solution for all $\epsilon > 0$.
In [5] it is proved that the solutions tend to concentrate near the degeneration set as $\epsilon \to 0$. Concentration phenomena of this type play a fundamental role in existence, non existence and multiplicity of solutions for elliptic problems involving critical and supercritical exponents and allow to relate the number of solutions to the shape of the domain. In our case, although the nonlinear term g has subcritical growth, these phenomena occur because of the degeneration of the equations and allow us to relate the number of solutions to the geometric properties of the degeneration set.
In fact, in the case $g_t(x,0) \leq 0 \; \forall x \in \Omega$, if there exist smooth domains $\mathcal{D}_1, \ldots, \mathcal{D}_k, \Omega_1, \ldots, \Omega_k$ satisfying:

(1) $\mathcal{D} = \bigcup_{i=1}^{k} \mathcal{D}_i$,

(2) $\emptyset \neq \mathcal{D}_i \subset \Omega_i \subset\subset \Omega \quad \forall i = 1, \ldots, k$,

(3) $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$ if $i \neq j$,

(4) every connected component of $\overline{\Omega} \setminus \bigcup_{i=1}^{k} \Omega_i$ meets $\partial\Omega$,

then there exists $\bar{\epsilon} > 0$, such that, for all $\epsilon \in]0, \bar{\epsilon}[$, the following multiplicity results hold:

(1) P_ϵ has at least k positive solutions $u_{\epsilon,1}, \dots, u_{\epsilon,k}$ (see [1]), satisfying the following properties:

$$\lim_{\epsilon \to 0} \int_\Omega |Du_{\epsilon,i}|^2 \, dx = 0$$

and

$$\lim_{\epsilon \to 0} \Big(\int_\Omega |Du_{\epsilon,i}|^2 \, dx \Big)^{-1} \int_{\Omega_i} |Du_{\epsilon,i}|^2 \, dx = 1 \ \forall \ i \in \{1, \dots, k\}.$$

(2) P_ϵ has multibump positive solutions (see [3]). More precisely, if we choose arbitrarily r distinct subsets among $\Omega_1, \dots, \Omega_k$ (say $\Omega_{i_1}, \dots, \Omega_{i_r}$), we can construct a positive r-bumps solution $u_\epsilon^{i_1,\dots,i_r}$ with the following properties:

$$\lim_{\epsilon \to 0} \int_\Omega |Du_\epsilon^{i_1,\dots,i_r}|^2 \, dx = 0,$$

$$\lim_{\epsilon \to 0} \Big(\int_\Omega |Du_\epsilon^{i_1,\dots,i_r}|^2 \, dx \Big)^{-1} \int_{\Omega \setminus \bigcup_{s=1}^{r} \Omega_{i_s}} |Du_\epsilon^{i_1,\dots,i_r}|^2 \, dx = 0$$

and

$$\liminf_{\epsilon \to 0} \Big(\int_\Omega |Du_\epsilon^{i_1,\dots,i_r}|^2 \, dx \Big)^{-1} \int_{\Omega_{i_s}} |Du_\epsilon^{i_1,\dots,i_r}|^2 \, dx > 0 \ \forall \ s \in \{1, \dots, r\}.$$

Thus, for all $r \in \{1, \dots, k\}$, we obtain at least $\binom{k}{r}$ positive solutions having r bumps; so, on the whole, we get at least $2^k - 1$ distinct positive solutions.

(3) P_ϵ has at least k^2 nodal solutions having exactly two nodal regions, that is both the supports of the positive and the negative part of u are connected subsets of Ω (see [2]). Moreover the obtained solutions $u_{\epsilon,i,j}$, for all $i, j \in \{1, \dots, k\}$, have the following properties:

$$\lim_{\epsilon \to 0} \int_\Omega |Du_{\epsilon,i,j}|^2 \, dx = 0,$$

$$\lim_{\epsilon \to 0} \Big(\int_\Omega |Du_{\epsilon,i,j}^+|^2 \, dx \Big)^{-1} \int_{\Omega_i} |Du_{\epsilon,i,j}^+|^2 \, dx = 1$$

and

$$\lim_{\epsilon \to 0} \Big(\int_\Omega |Du_{\epsilon,i,j}^-|^2 \, dx \Big)^{-1} \int_{\Omega_j} |Du_{\epsilon,i,j}^-|^2 \, dx = 1.$$

If $\sup_{x \in \Omega} g_t(x, 0) < 0$, the concentration phenomena are accentuated and the number of solutions can increase: we can have more than one solution even if the degeneration set is connected (i.e. $k = 1$), provided that it is topologically complex.

In fact, if we assume that

$$\lim_{\epsilon\to 0}\Big[\frac{1}{\epsilon}\sup_{x\in\mathcal{D}}\Lambda_2(\epsilon,x)-\frac{1}{\epsilon}\inf_{x\in\Omega}\Lambda_1(\epsilon,x)\Big]=0,$$

if $\mathcal{D}$ is not contractible in itself and if we denote by m its Ljusternik-Schnirelman category, then there exists $\bar{\epsilon}>0$ such that, for all $\epsilon\in]0,\bar{\epsilon}[$, P_ϵ has at least $m+1$ distinct positive solutions $u_\epsilon^1,\dots,u_\epsilon^m,u_\epsilon^{m+1}$, which converge to zero in $H_0^{1,2}(\Omega)$ as $\epsilon\to 0$.
Moreover, for all $i=1,\dots,m$, the function $(\int_\Omega |Du_\epsilon^i|^2\,dx)^{-1}|Du_\epsilon^i(x)|^2$ concentrates, as $\epsilon\to 0$, like Dirac mass near points of $\mathcal{D}$ (see [4]), while $(\int_\Omega |Du_\epsilon^{m+1}|^2\,dx)^{-1}|Du_\epsilon^{m+1}(x)|^2$ can be decomposed as sum of two functions, which concentrate, as $\epsilon\to 0$, like Dirac mass near points of $\mathcal{D}$.

2. NONEXISTENCE RESULT

In this section we assume that $g_t(x,0)>0$ somewhere in $\mathcal{D}$ and prove a nonexistence result, which implies, in particular, that P_ϵ has no positive solutions for $\epsilon>0$ small enough.
On the other hand, under symmetry assumptions on g, well known topological methods of the Calculus of Variations guarantee the existence of infinitely many solutions of P_ϵ for all $\epsilon>0$; because of next theorem, these solutions must change sign in every subset of $\mathcal{D}$, where $g_t(x,0)>0$, for $\epsilon>0$ small enough.

Theorem 2.1. Assume $(a.1)$, $(g.1)$ and $(g.2)$.
Moreover assume that:
(a.2)' there exists $\tilde{\epsilon}>0$ such that

$$\inf_{x\in\Omega}\Lambda_1(\epsilon,x)>0\quad\forall\epsilon\in]0,\tilde{\epsilon}[,$$

(a.3)' there exists a smooth domain $\mathcal{D}\subset\Omega$ such that

$$\lim_{\epsilon\to 0}\sup_{x\in\mathcal{D}}\Lambda_2(\epsilon,x)=0$$

(notice that $(a.2)$ and $(a.3)$ imply $(a.2)'$ and $(a.3)'$ respectively).
Then, for every open subset $S\subseteq\mathcal{D}$, with $S\neq\emptyset$, such that

$$\inf_{x\in S}g_t(x,0)>0, \tag{2.1}$$

there exists $\bar{\epsilon}>0$ such that, for all $\epsilon\in]0,\bar{\epsilon}[$, P_ϵ has no solutions having constant sign in S.

Proof. Suppose, contrary to our claim, that there exist an open subset S, with $S \neq \emptyset$, and a sequence $(\epsilon_n)_{n\geq 1}$ of positive numbers such that $\lim\limits_{n\to\infty} \epsilon_n = 0$ and each one of P_{ϵ_n} has a solution u_{ϵ_n} which is, for instance, positive in S.
Since u_{ϵ_n} is a solution of P_{ϵ_n}, then $u_{\epsilon_n} \in H_0^{1,2}(\Omega)$ satisfies

$$\int_\Omega \sum_{i,j=1}^N a_{\epsilon_n}^{i,j}(x)\partial_{x_i}u_{\epsilon_n}\partial_{x_j}v\,dx - \int_\Omega g(x,u_{\epsilon_n})v\,dx = 0 \quad \forall\, v \in H_0^{1,2}(\Omega). \tag{2.2}$$

Let us define

$$\mu_{\epsilon_n} := \inf\{Q_{\epsilon_n}(u)\ :\ u \in H_0^{1,2}(S),\ \int_S |u|^2\,dx = 1\} \quad \forall\, n \geq 1 \tag{2.3}$$

where, for all $n \geq 1$, $Q_{\epsilon_n} : H_0^{1,2}(S) \to \mathbf{R}^+$ is defined by

$$Q_{\epsilon_n}(u) := \int_S \sum_{i,j=1}^N a_{\epsilon_n}^{i,j}(x)\partial_{x_i}u\partial_{x_j}u\,dx.$$

First we see that

$$\lim_{n\to\infty} \mu_{\epsilon_n} = 0. \tag{2.4}$$

In fact, let us fix $\varphi \in H_0^{1,2}(S)$ with $\int_S |\varphi|^2\,dx = 1$; from $(a.3)'$ we get

$$0 \leq \limsup_{n\to\infty} Q_{\epsilon_n}(\varphi) \leq \lim_{n\to\infty} \sup_{x\in S} \Lambda_2(\epsilon_n, x) \int_S |D\varphi|^2\,dx = 0.$$

Since $Q_{\epsilon_n}(\varphi) \geq \mu_{\epsilon_n} \geq 0$ for all $n \geq 1$, (2.4) follows.
Since S is a smooth bounded domain in $\mathbf{R}^N$ and $(a.2)'$ holds, μ_{ϵ_n} is achieved for n large enough; let φ_{ϵ_n} be a minimizing function for μ_{ϵ_n}. We can assume $\varphi_{\epsilon_n} \geq 0$, since otherwise we replace φ_{ϵ_n} by $|\varphi_{\epsilon_n}|$; moreover $\varphi_{\epsilon_n} \not\equiv 0$.
Since φ_{ϵ_n} is a minimum point for Q_{ϵ_n} on $\{u \in H_0^{1,2}(S)\ :\ \int_S |u|^2\,dx = 1\}$, which is a C^1-manifold in $H_0^{1,2}(S)$, the Lagrange multiplier's rule implies that

$$\frac{1}{2}Q'_{\epsilon_n}(\varphi_{\epsilon_n})[v] = \mu_{\epsilon_n} \int_S \varphi_{\epsilon_n} v\,dx \quad \forall\, v \in H_0^{1,2}(S). \tag{2.5}$$

Now let us consider the function $\overline{u}_{\epsilon_n} \in H_0^{1,2}(\Omega)$ such that $\overline{u}_{\epsilon_n} \equiv u_{\epsilon_n}$ in $\Omega \setminus S$ and $Q'_{\epsilon_n}(\overline{u}_{\epsilon_n})[v] = 0\ \forall v \in H_0^{1,2}(S)$.

Since $u_{\epsilon_n} \geq 0$ in S, also $\overline{u}_{\epsilon_n} \geq 0$ in S . As $Q'_{\epsilon_n}(\cdot)[\cdot]$ is a symmetric quadratic form in $H_0^{1,2}(S)$ and $(g.2)$, (2.2) and (2.5) hold, we obtain

$$\begin{aligned}
0 &= \int_\Omega \sum_{i,j=1}^N a^{i,j}_{\epsilon_n}(x)\partial_{x_i}u_{\epsilon_n}\partial_{x_j}\varphi_{\epsilon_n}\,dx - \int_\Omega g(x,u_{\epsilon_n})\varphi_{\epsilon_n}\,dx \\
&= \frac{1}{2}Q'_{\epsilon_n}(u_{\epsilon_n})[\varphi_{\epsilon_n}] - \int_S g(x,u_{\epsilon_n})\varphi_{\epsilon_n}\,dx \\
&= \frac{1}{2}Q'_{\epsilon_n}(u_{\epsilon_n}-\overline{u}_{\epsilon_n})[\varphi_{\epsilon_n}] + \frac{1}{2}Q'_{\epsilon_n}(\overline{u}_{\epsilon_n})[\varphi_{\epsilon_n}] - \int_S g(x,u_{\epsilon_n})\varphi_{\epsilon_n}\,dx \\
&= \frac{1}{2}Q'_{\epsilon_n}(\varphi_{\epsilon_n})[u_{\epsilon_n}-\overline{u}_{\epsilon_n}] - \int_S g_t(x,0)u_{\epsilon_n}\varphi_{\epsilon_n}\,dx - \int_S [g(x,u_{\epsilon_n}) - g_t(x,0)u_{\epsilon_n}]\varphi_{\epsilon_n}\,dx \\
&= \mu_{\epsilon_n}\int_S \varphi_{\epsilon_n}(u_{\epsilon_n}-\overline{u}_{\epsilon_n})\,dx - \int_S g_t(x,0)u_{\epsilon_n}\varphi_{\epsilon_n}\,dx - \int_S [g(x,u_{\epsilon_n}) - g_t(x,0)u_{\epsilon_n}]\varphi_{\epsilon_n}\,dx \\
&\leq [\mu_{\epsilon_n} - \inf_{x\in S} g_t(x,0)]\int_S \varphi_{\epsilon_n}u_{\epsilon_n}\,dx - \int_S [\mu_{\epsilon_n}\overline{u}_{\epsilon_n} + g(x,u_{\epsilon_n}) - g_t(x,0)u_{\epsilon_n}]\varphi_{\epsilon_n}\,dx \\
&\leq [\mu_{\epsilon_n} - \inf_{x\in S} g_t(x,0)]\int_S \varphi_{\epsilon_n}u_{\epsilon_n}\,dx. \qquad (2.6)
\end{aligned}$$

Taking into account that (2.4) holds, that $\inf_{x\in S} g_t(x,0) > 0$ and that $\int_S u_{\epsilon_n}\varphi_{\epsilon_n}\,dx > 0$ if $u_{\epsilon_n} \not\equiv 0$ in S, from (2.6) we obtain

$$0 \leq [\mu_{\epsilon_n} - \inf_{x\in S} g_t(x,0)]\int_S \varphi_{\epsilon_n}u_{\epsilon_n}\,dx < 0 \quad \text{for all } n \text{ large enough},$$

which is a contradiction.

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