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Existence and non-existence of global solutions for uniformly parabolic equations

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Abstract. In this paper, we prove the existence of Fujita-type critical exponents for x-dependent fully nonlinear uniformly parabolic equations of the type

$$(*) \quad \partial_t u = F(D^2 u, x) + u^p \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+.$$

These exponents, which we denote by $p(F)$, determine two intervals for the p values: in $]1, p(F)[$, the positive solutions have finite-time blow-up, and in $]p(F), +\infty[$, global solutions exist. The exponent $p(F) = 1 + 1/\alpha(F)$ is characterized by the long-time behavior of the solutions of the equation without reaction terms

$$\partial_t u = F(D^2 u, x) \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+.$$

When F is a x-independent operator and p is the critical exponent, that is, $p = p(F)$. We prove as main result of this paper that any non-negative solution to $(*)$ has finite-time blow-up. With this more delicate critical situation together with the results of Meneses and Quaas (J Math Anal Appl 376:514–527, 2011), we completely extend the classical result for the semi-linear problem.

1. Introduction

In [10], we proved the existence of critical exponents of Fujita type for parabolic equations of the form

$$\partial_t u = F(D^2 u) + u^p \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+, \tag{1}$$

with the operator F uniformly elliptic and positively homogeneous. In that paper, we proved the existence of exponents $p = p(F)$ such that for $p \in]1, p(F)[$, positive solutions of (1) have finite-time blow-up, and for $p \in]p(F), +\infty[$, global solutions exist. Moreover, from the local existence results for Eq. (1) (proved in [10]), we proved that if $p \in]1, p(F)[$, then there exists $T_{\max}(u)$ such that

$$\lim_{t \uparrow T_{\max}(u)} u(x^*, t) = \infty, \quad \text{for some } x^* \in \mathbb{R}^N.$$

This last behavior is usually called finite-time blow-up.

Keywords: Fujita-type exponent, Blow-up in finite time, Viscosity solutions, Comparison arguments.

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The model problem that motivated our work is the classical semi-linear equation

$$\partial_t u = \Delta u + u^p \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+, \tag{2}$$

where $p_\Delta = 1 + 2/N$ is the critical exponent, presented by H. Fujita in [4]. These results are generalized and extended in [13] for the case $p = p_\Delta$, where (2) is a particular case. Based on the existence of p_Δ, it is possible to perform further studies on such features as the blow-up rate, the blow-up set, and the asymptotic behavior of global solutions. See, for example, [11] and [6] for more detail.

As the first result of this article, we prove the existence of Fujita-type exponents for uniformly parabolic equations

$$\partial_t u = F(D^2 u, x) + u^p \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+, \tag{3}$$

extending the results of [10] for x-dependent operators. In particular, the results of this paper cover linear elliptic operators with bounded Lipschitz coefficients.

As our second main result, we examine Eq. (1) and we prove that the critical value $p = p(F)$ always belongs to the blowup case.

In Eq. (3), we consider operators $F : \mathcal{S}_N \times \mathbb{R}^N \to \mathbb{R}$ satisfying the following conditions:

(F1) (F uniformly elliptic) There exist $0 < \lambda \leq \Lambda$, such that for all $x \in \mathbb{R}^N$ and $M \in \mathcal{S}_N$, we have

$$\lambda \operatorname{tr} N \leq F(M + N, x) - F(M, x) \leq \Lambda \operatorname{tr} M, \quad \text{with } N \geq 0 \text{ in } \mathcal{S}_N.$$

(F2) (F homogeneous) For $\alpha \geq 0$,

$$F(\alpha M, x) = \alpha F(M, x), \text{ for all } x \in \mathbb{R}^N \text{ and } M \in \mathcal{S}_N.$$

(F3)(i) $F(M, x)$ continuous operator in $\mathcal{S}_N \times \mathbb{R}^N$ and concave (convex) in M.

(ii) There exists L_F such that

$$|F(M, x) - F(M, y)| \leq L_F \|x - y\| (\|M\| + 1),$$

for all x, y in $\mathbb{R}^N$.

Here, $\mathcal{S}_N$ denotes the $N \times N$ symmetric matrix space.

For F satisfying the above conditions, we consider the Cauchy problems for (3) subject to a non-negative initial condition $u_0(x) \not\equiv 0$ that is bounded and uniformly continuous in $\mathbb{R}^N$, so that the local existence Theorem holds (see [10]). Moreover, for the solution $u(x, t)$, we can define the maximal existence time $T_{\max}(u)$. Here and in the rest of the paper, we suppose that $u_0(x)$ is bounded and uniformly continuous in $\mathbb{R}^N$. Our first main result for x-dependent F operators is presented below.

THEOREM 1.1. *Let $u(x, t)$ be the viscosity solution of (3), satisfying $u(x, 0) = u_0(x)$ in $\mathbb{R}^N$. There exists a unique exponent $p(F)$ such that*

(i) If $1 < p < p(F)$*, then* $T_{\max}(u) < \infty$.
(ii) If $p(F) < p$*, then there exists* u_0 *such that* $T_{\max}(u) = \infty$.
Furthermore, if $p \in]1, p(F)[$*, then the solutions have finite-time blow-up.*

Those results are obtained using comparison arguments between the viscosity sub- and super-solutions of (3). For the construction of sub- and super-solutions, some of the ideas are taken from [15] and [9].

The exponent $p(F)$ is characterized by studying the long-time asymptotical behavior of the solutions of

$$\partial_t u = F(D^2 u, x) \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+, \tag{4}$$

for a class of initial condition $u(x, 0) = u_0(x)$, with some exponential spatial decay. More precisely, we prove the existence of a unique $\alpha(F) > 0$ such that for all $u(x, t)$, the solution of (4) satisfies

$$\lim_{t\to\infty} t^{\alpha(F)} \|u(\cdot, t)\|_\infty < \infty \tag{5}$$

and for each $\alpha > \alpha(F)$, we have

$$\lim_{t\to\infty} t^{\alpha} \|u(\cdot, t)\|_\infty = \infty. \tag{6}$$

This $\alpha(F)$ is already found in [2] for the case of x-independent operator. For this $\alpha(F)$, we characterize the Fujita-type exponent as

$$p(F) = 1 + \frac{1}{\alpha(F)}.$$

For the particular case $F(\cdot, x) = \mathrm{tr}(\cdot)$ and $u_0(x)$ satisfying the above conditions, we have $\alpha(\Delta) = N/2$ and $p(\Delta) = p_\Delta$. Therefore, the result given in Theorem 1.1 is a generalization of the result presented by Fujita in [4].

As our second and main result, we consider (1) for the critical and more delicate case $p = p(F)$.

THEOREM 1.2. *Assume* $F(\cdot)$ *is a* x*-independent operator satisfying (F1), (F2). Let* $u(x, t)$ *be a viscosity solution of* (1) *satisfying* $u(x, 0) = u_0(x)$*. If* $p = p(F)$*, then* $u(x, t)$ *has finite-time blow-up.*

With this Theorem and the results of [10], we have a complete generalization of the results for Eq. (2). Therefore, both results together can be seen as a sort of elliptic analog of results of [1]. Some ideas for our approach can be found in [12] and [5].

Finally, we use the result of Theorem 1.2 to obtain qualitative results for equations with more general reaction terms.

This article is organized as follows. In Sect. 1, we present some preliminaries defining the concept of viscosity solutions, and we prove the long-term asymptotic behavior of solutions to (4). Section 3 is devoted to the proof of the two main Theorem. In Sect. 4, we state and prove a more general version of our results with more general reaction terms.

2. Preliminaries

We begin by recalling the concept of solution we will use. For the equation we consider, the natural concept is the viscosity solution. More precisely, we use the definition of the parabolic viscosity solution given in [8]. Below, we present the definition used in this work. For this purpose, we will say that $\mathcal{V} \subset \mathbb{R}^N \times \mathbb{R}_0^+$ is a parabolic neighborhood of (x_0, t_0) if it has the form $V \times]t_0 - \epsilon, t_0]$, where V is a neighborhood of x_0.

DEFINITION 2.1. *An upper semi-continuous function* $u{:}\Omega \to \mathbb{R}$ *is a parabolic viscosity sub solution of equation* $\partial_t u = G(x, t, u, D^2u)$ *in* Ω *if*

$$\partial_t \varphi(x_0, t_0) \leq G(x_0, t_0, \varphi(x_0, t_0), D^2\varphi(x_0, t_0)) \tag{7}$$

for any $(x_0, t_0) \in \Omega$ *and* $\varphi \in \mathcal{C}^2(\Omega)$, *such that* $u(x_0, t_0) = \varphi(x_0, t_0)$ *and*

$$u(x, t) \leq \varphi(x, t) \quad \text{for all } (x, t) \in \mathcal{V}, \tag{8}$$

where $\mathcal{V}$ *is a parabolic neighborhood of* (x_0, t_0). *The functions* $\varphi(x_0, t_0)$ *are called test functions of u in* (x_0, t_0).

Similarly, we define parabolic viscosity supersolutions considering $u : \Omega \to \mathbb{R}$ as a lower semi-continuous function, satisfying (7) and (8) and switching "$\leq$" for "$\geq$".

Finally, a continuous function $u : \Omega \to \mathbb{R}$ is a parabolic viscosity solution of $\partial_t u = G(x, t, u, D^2u)$ if it is both a parabolic viscosity subsolution and a parabolic viscosity supersolution. In this paper, of course, $G(x, t, s, X) = F(X, x) + f(u)$. For the proofs, we used fundamental comparison arguments between viscosity sub- and super-solutions. Below, we use the comparison criterion presented in [10]; see Theorem 2.1. We continue now by presenting Pucci's extremal operators.

Let $[[\lambda, \Lambda]] \subset \mathcal{S}_N$ be the matrix space with eigenvalues in $[\lambda, \Lambda]$. For $M \in \mathcal{S}_N$, Pucci's extremal operators are defined as

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = \inf_{A\in[[\lambda,\Lambda]]} (\operatorname{tr} AM), \quad \mathcal{M}^+_{\lambda,\Lambda}(M) = \sup_{A\in[[\lambda,\Lambda]]} (\operatorname{tr} AM). \tag{9}$$

They satisfy (F1), (F2) directly by their definition. Moreover, for F satisfying (F1), (F2), the following equation holds:

$$\mathcal{M}^-_{\lambda,\Lambda}(M) \leq F(M, x) \leq \mathcal{M}^+_{\lambda,\Lambda}(M) \quad \text{for all } M \in \mathcal{S}_N,\ x \in \mathbb{R}^N. \tag{10}$$

Now, we want to find the asymptotic behaviors given in (5), (6).

We continue to assume that F satisfies (F1), (F2), (F3), and the initial condition $u_0(x) \not\equiv 0$ non-negative function in $BUC(\mathbb{R}^N)$, the space of bounded uniformly continuous functions in $\mathbb{R}^N$. We consider the Cauchy problem

$$\begin{cases} \partial_t u = F(D^2u, x) & \text{in } \mathbb{R}^N \times \mathbb{R}^+ \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{11}$$

We begin by considering the Eq. (11) with Pucci's operators. Given the convexity and concavity of these operators, using the existence result of [3], there is a global solution for this case. For F satisfying (F1), (F2), (F3), from (10), using comparison arguments and the Perron method, the problem (11) also has global solutions.

To obtain the asymptotic behavior for the solution of (11), we consider $\mathcal{A}(\mathbb{R}^N) \subset BUC(\mathbb{R}^N)$, a set of functions $w(x)$ such that there exist A_0, β_0 positive constants such that $w(x) \le A_0 \exp\{-\beta_0 \|x\|^2\}$.

PROPOSITION 2.1. *Let $u(x,t)$ be a global viscosity solution of* (11). *If $u_0(x) \in \mathcal{A}(\mathbb{R}^N)$, then there exists a unique $\alpha = \alpha(F)$ such that* (5), (6) *hold.*

For the proof, we need the following lemmas.

LEMMA 2.1. *Let $u(x,t)$ be the global solution for* (11). *If $u_0(x) \in BUC(\mathbb{R}^N)$, then for all $\beta > 0$, there exist t_0, C^* positive constants such that $u(x,t_0) \ge C^* \exp\left\{-\beta\|x\|^2\right\}$.*

Proof. By the relation given in (10) and comparison arguments, the result is presented with $F(\cdot, x) = \mathcal{M}^-_{\lambda,\Lambda}(\cdot)$.

From the concavity of the $\mathcal{M}^-_{\lambda,\Lambda}(\cdot)$ operator, there exists $w(x,t)$, classically satisfying

$$\partial_t w = \mathcal{M}^-_{\lambda,\Lambda}(D^2 w) \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+$$

and the initial condition $u(x,0) = u_0(x)$ in $\mathbb{R}^N$. For the existence and the regularity of $u(x,t)$, we use [3] and [14], respectively. By the strong maximum principle, we can also assume that $w_0(x) > 0$ in $\mathbb{R}^N$.

Let t_0, α, γ, R_0 be positive constants with $R_0 = 2\sqrt{\gamma/\alpha}$. Define $\Omega = \mathbb{R}^N \setminus \overline{B_{R_0}(0)}$, and let $\phi(x,t) \in \mathcal{C}^\infty(\Omega \times [t_0/2, \infty[)$ be given by

$$\phi(x,t) = \begin{cases} \exp\left\{-\frac{\alpha|x|^2 - \gamma}{t - t_0/2}\right\}, & x \in \Omega,\ t > t_0/2, \\ 0, & x \in \Omega,\ t = t_0/2. \end{cases}$$

Taking $\alpha = 1/(4\lambda)$, $(2\alpha)/\beta = t_0$ and $\gamma > (\Lambda t_0 N)/(4\lambda)$, we have

$$\partial_t \phi - \mathcal{M}^-_{\lambda,\Lambda}(D^2\phi) < 0, \quad \text{in} \quad \Omega \times]t_0/2, t_0[.$$

Let $E = \Omega \times]t_0/2, t_0]$, and $\partial_p E = \Omega \times \{t_0/2\} \cup \partial\Omega \times [t_0/2, t_0]$ be the parabolic boundary of E.

Define

$$s_0 = \min_{\substack{|x| \le R_0 \\ t_0/2 \le t \le t_0}} w(x,t) > 0,$$

and $\phi_0(x,t) = s_0 \phi(x,t)$. Because $\phi_0(x, t_0/2) - w(x, t_0/2) = -w(x, t_0/2) \le 0$ for $x \in \bar{\Omega}$, and $\phi_0(x,t) - w(x,t) \le 0$ for $|x| = R_0$, $t_0/2 \le t \le t_0$, using the comparison principle of [7], we find $\phi_0(x,t) \le w(x,t)$ in all E. Hence, the result follows because $(2\alpha)/t_0 = \beta$. □

LEMMA 2.2. *Let $u(x,t)$ be a global solution of problem* (11) *with $u_0 \in \mathcal{A}(\mathbb{R}^N)$. Then,*

$$\alpha(F,u_0) := \sup\{\alpha > 0 : \limsup_{t\to\infty} t^{\alpha}\|u_i(\cdot,t)\|_{\infty} < \infty\}, \tag{12}$$

is well defined.

Proof. Because $u_0 \in \mathcal{A}(\mathbb{R}^N)$, there exist A, β positive constants such that $u_0(x) \le A\exp\{-\beta\|x\|^2\}$.

Consider now

$$\overline{u}(x,t) = \exp\{-\varphi(t)\|x\|^2 + \psi(t)\}, \tag{13}$$

with the sufficiently regular functions $\varphi, \psi : [0,\infty[\to [0,\infty[$, to be determined.

For $v(r,t)$ with $v(\|x\|,t) = \overline{u}(x,t)$, from the radial representation of Pucci's extremal operators, we have

$$\partial_t \overline{u} - \mathcal{M}^+_{\lambda,\Lambda}(D^2\overline{u}) \ge (-\|x\|^2(\dot{\varphi} + 4\Lambda\varphi^2) + \dot{\psi} + 2N\lambda\varphi)\overline{u}. \tag{14}$$

Now, we consider $\varphi(t),\ \psi(t)$ such that $\dot{\varphi} + 4\Lambda\varphi^2 = 0$, $\dot{\psi} + 2N\lambda\varphi = 0$. From (14), we have $\partial_t\overline{u} - \mathcal{M}^+_{\lambda,\Lambda}(D^2\overline{u}) \ge 0$ in $\mathbb{R}^N \times \mathbb{R}^+$. By substitution in (13), we obtain

$$\overline{u}(x,t) = \exp\{\psi(0)\}(1 + 4\Lambda\varphi(0)t)^{-\frac{N\lambda}{2\Lambda}} \exp\left\{-\frac{\varphi(0)}{1 + 4\varphi(0)\Lambda t}\|x\|^2\right\}.$$

Taking $\varphi(0) = \beta$ and $\exp\{\psi(0)\} = A$, it follows directly that $u_0(x) \le A\exp\{-\beta\|x\|^2\} = \overline{u}(x,0)$. Using (10) and comparison arguments, we obtain $u(x,t) \le \overline{u}(x,t)$ in $\mathbb{R}^N \times \mathbb{R}^+_0$.

On the other hand, from Lemma 2.1, given β, there exist $t_0,\ C^*$ positive constants such that $u(x,t_0) \ge C^*\exp\left\{-\beta\|x\|^2\right\}$.

As in the above proof,

$$\underline{u}(x,t) = C^*(1 + 4\lambda\beta t)^{-\frac{N\Lambda}{2\lambda}} \exp\left\{-\frac{\beta}{1 + 4\beta\lambda t}\|x\|^2\right\}$$

classically satisfies $\partial_t\underline{u} \le \mathcal{M}^-_{\lambda,\Lambda}(D^2\underline{u})$ in $\mathbb{R}^N \times \mathbb{R}^+$. So, by comparison, we have

$$\underline{u}(x,t) \le u(x,t+t_0) \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+_0. \tag{15}$$

From here and $u(x,t) \le \overline{u}(x,t)$, we determine that $\alpha(F,u_0)$ is well defined and

$$\frac{N\lambda}{2\Lambda} \le \alpha(F,u_0) \le \frac{N\Lambda}{2\lambda}.$$

□

Now, we present the proof of Proposition 2.1.

Proof Proposition 2.1:
Let $u_0^1, u_0^2 \in \mathcal{A}(\mathbb{R}^N)$. We need to prove that $\alpha(F, u_0^1) = \alpha(F, u_0^2)$, where $\alpha(F, u_0^i)$ is defined in Lemma 2.2. Let $u^i(x,t)$ be the solutions of (11) with the respective initial conditions u_0^i, $i = 1, 2$.

Note that by assumption, there exist A_1 and β_1 such that $u_0^1(x) \le A_1 \exp\left\{-\beta_1 \|x\|^2\right\}$. From Lemma 2.1, given β_1, there exist t_0, C^* positive constants such that $u^2(x, t_0) \ge C^* \exp\left\{-\beta_1 \|x\|^2\right\}$. Thus,

$$u^2(x, t_0) \le \frac{C^*}{A_1} u_0^1(x),$$

and by comparison,

$$u^2(x, t_0 + t) \le \frac{C^*}{A_1} u^1(x, t).$$

Hence, we obtain $\alpha(F, u_0^1) \ge \alpha(F, u_0^2)$. In the same way, we obtain the reverse inequality: therefore, $\alpha(F, u_0^1) = \alpha(F, u_0^2)$. □

3. Proofs of the main results

As mentioned in the Introduction, the results given in Theorem 1.1 are obtained using comparison arguments between the viscosity sub- and super-solutions. To construct those functions, we use the $v(x,t)$ solution of (11) and the asymptotical results presented in Proposition 2.1. We mainly follow the ideas presented in [9].
Proof of Theorem 1.1:

Part (i) Let $p(F) = 1 + 1/\alpha(F)$, with $\alpha(F)$ given in Proposition 2.1. For all $1 < p < p(F)$, the following holds: $\alpha(F) < \dfrac{1}{p-1}$. From the asymptotic results given in (5), (6), for all $v(x,t)$ satisfying (11), it holds that $\limsup_{t\to\infty} t^{1/(p-1)} \|v(\cdot, t)\|_\infty = \infty$. Therefore, there exist $T^* > 0$ and $x^* \in \mathbb{R}^N$ such that

$$\lim_{t \uparrow T^*} (1-(p-1)t(v(x^*,t))^{p-1}) = 0, \text{ and } \; 1-(p-1)t(v(x,t))^{p-1} > 0 \text{ in } \mathbb{R}^N \times [0, T^*[. \tag{16}$$

Let $\underline{u}(x,t) = v(x,t)/(1-(p-1)t(v(x,t))^{p-1})^{\frac{1}{p-1}}$. It is not hard to prove (see Lemma 4.1 of [10]) that $\underline{u}$ satisfies $\partial_t \underline{u} \le F(D^2\underline{u}, x) + (\underline{u})^p$ in $\mathbb{R}^N \times]0, T^*[$. Moreover, from (16), we have $\underline{u}$ blow-up at T^*.

On the other hand, let $u(x,t)$ be a solution of (3) satisfying the initial condition $u_0(x)$. As $\underline{u}(x, 0) = u_0(x)$, by comparison, we have $\underline{u}(x,t) \le u(x,t)$ in $\mathbb{R}^N \times [0, T^*[$. Therefore $T_{\max}(u) < \infty$, which is the result in the first case.

Part (ii) Let $v(x,t)$ be a solution of (11). To obtain a global super solution, we begin by presenting the following result. Using Proposition 2.1, there exists M_1 such that $t^{\alpha(F)}\|v(\cdot,t)\|_\infty \le M_1$. If $p > p(F)$, then we have $\alpha(F)(p-1) > 1$ directly. Therefore, there exists $t_0 > 0$ such that

$$\int_{t_0}^{\infty} \|v(\cdot,s)\|_\infty^{p-1}\,\mathrm{d}s \le (M_1)^{p-1}\int_{t_0}^{\infty}\frac{1}{s^{(p-1)\alpha(F)}}\,\mathrm{d}s < \infty.$$

By rescaling, there exists $u_0 \in BUC(\mathbb{R}^N)$ such that $\int_0^\infty \|v(\cdot,s)\|_\infty^{p-1}\,\mathrm{d}s < \dfrac{1}{p-1}$.

Under the condition above, for $h(t) = \left(1-(p-1)\int_0^t \|v(\cdot,s)\|^{p-1}\,\mathrm{d}s\right)^{-1/(p-1)}$, consider the function $\overline{u}(x,t) = h(t)v(x,t)$ defined in all $\mathbb{R}^N \times \mathbb{R}^+$.

Given $(x_0,t_0) \in \mathbb{R}^N \times \mathbb{R}^+$, let $\varphi(x,t)$ be sufficiently regular such that $(\overline{u}-\varphi)(x,t) \ge (\overline{u}-\varphi)(x_0,t_0) = 0$ in $\mathcal{V}$, a parabolic neighborhood of (x_0,t_0). From the definition of $\overline{u}(x,t)$, for $\psi(x,t) = \varphi(x,t)/h(t)$, we have $(v-\psi)(x,t) \ge (v-\psi)(x_0,t_0) = 0$ in $\mathcal{V}$. Thus, ψ is a test function for $v(x,t)$ in (x_0,t_0). Then, it holds that $\partial_t\psi(x_0,t_0) \ge F(D^2\psi(x_0,t_0))$. From the regularity of functions $h(t)$, $\psi(x,t)$ the following holds:

$$\begin{aligned}\partial_t\varphi(x_0,t_0) &= h(t_0)\partial_t\psi(x_0,t_0) + \psi(x_0,t_0)\dot{h}(t_0)\\ &\ge h(t_0)F(D^2\psi(x_0,t_0),x_0) + \psi(x_0,t_0)\dot{h}(t_0)\\ &\ge F(D^2\varphi(x_0,t_0),x_0) + \psi(x_0,t_0)\dot{h}(t_0).\end{aligned} \tag{17}$$

Because $\overline{u}(x_0,t_0) = \varphi(x_0,t_0)$ and from the definition of function $h(t)$, we have

$$\begin{aligned}\psi(x_0,t_0)\dot{h}(t_0) &= \psi(x_0,t_0)\|v(\cdot,t_0)\|_\infty^{p-1}h(t_0)^p\\ &\ge \psi(x_0,t_0)(v(x_0,t_0))^{p-1}h(t_0)^p\\ &\ge (\psi(x_0,t_0))^p h(t_0)^p\\ &\ge (\varphi(x_0,t_0))^p = (\overline{u}(x_0,t_0))^p.\end{aligned}$$

By substitution in (18), we have

$$\partial_t\varphi(x_0,t_0) \ge F(D^2\varphi(x_0,t_0),x_0) + (\overline{u}(x_0,t_0))^p.$$

Because the choice of (x_0,t_0) and $\varphi(x,t)$ was arbitrary, function $\overline{u}(x,t)$ satisfies

$$\partial_t\overline{u} \ge F(D^2\overline{u},x) + (\overline{u})^p \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+.$$

Finally, taking $\underline{u}(x,t) = v(x,t)$ as a subsolution and using the Perron method, there exists $u(x,t)$, a global solution of (3), satisfying $\underline{u}(x,t) \le u(x,t) \le \overline{u}(x,t)$ in $\mathbb{R}^N \times \mathbb{R}_0^+$. □

Based only on the asymptotic result, it is not possible to conclude directly that the solutions are unbounded for the case $p = p(F)$. Generally, behaviors in the critical cases are obtained using other types of tools.

In our second main result, we present the unbounded behavior of the solution of Eq. (1) in the critical case. In the proof, we use the construction of self-similar solutions presented in [10]. The proof presented is completely independent of the proofs in the semi-linear case. The idea of the proof is to show that the positive solutions of equations of the form $\partial_t u = F(D^2 u) + 2u^{p(F)}$ blow-up in finite time. Then, we obtain the result by rescaling.

Proof of Theorem 1.2: For $F(\cdot, x) = F(\cdot)$, let $\Psi(x,t) = (t+\tau)^{-\alpha(F)}\Phi(x/\sqrt{t+\tau})$ be a solution of (4) where Φ is a positive solution of

$$F(D^2\Phi) + \frac{1}{2} y \cdot D\Phi = -\alpha(F)\Phi \quad \text{in} \quad \mathbb{R}^N,$$

found in [10] (see also [2]). Moreover, we have $\Psi(\cdot, t) \in \mathcal{A}(\mathbb{R}^N)$ for all $t > 0$, and we can assume $\|\Phi\|_\infty = \Phi(0)$.

Now, we want to find $w(x,t)$ satisfying

$$\partial_t w = F(D^2 w) + (\Psi(x,t))^{p(F)-1} w \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+, \text{ and } \quad w(x,0) = u_0(x) \text{ in } \mathbb{R}^N.$$

For this purpose, we use a comparison principle (see Theorem 2.1 in [10]) and the Perron method with the supersolution $\bar{w} = ve^{Ct}$, where v is a solution of (11) and $C = \Phi(0)^{p-1}$, and the subsolution $\underline{w} = \Psi(x,t)$. Note that $\Psi(x,t) \le w(x,t)$, and therefore, w satisfies $\partial_t w \le F(D^2 w) + w^{p(F)}$ in $\mathbb{R}^N \times \mathbb{R}^+$.

Let $\underline{u}_2(x,t) = w(x,t)(1 - (p(F)-1)t(w(x,t))^{p(F)-1})^{\frac{-1}{p(F)-1}}$. Then, by Lemma 4.1 of [10], we have

$$\partial_t \underline{u}_2 \le F(D^2 \underline{u}_2) + 2(\underline{u}_2)^{p(F)} \quad \text{in } \mathbb{R}^N \times]0,T], \text{ and } \quad \underline{u}_2(x,0) = u_0(x) \text{ in } \mathbb{R}^N,$$

for each $T < T_{\max}(\underline{u}_2)$. As in the previous proof, the idea is to obtain a result like (16) for $w(x,t)$. For this purpose, it suffices to work in the neighborhood of the point where function $w(x,t)$ reaches its maximum for a given t.

For $k \in \mathbb{N}$, there exist $(\alpha_k(F), \Phi_k) \in \mathbb{R}^+ \times \mathcal{C}(\mathbb{R}^N)$ satisfying

$$\begin{cases} F(D^2\Phi_k) + \frac{1}{2} y \cdot D\Phi_k = -\alpha_k(F)\Phi_k & \text{in } B_k(0) \\ \Phi_k(y) > 0 & \text{in } B_k(0) \\ \Phi_k(y) = 0 & \text{in } \partial B_k(0), \end{cases} \tag{18}$$

with

$$\alpha_k(F) = \sup\{\mu > 0 : \text{ exists } \phi > 0, \ H(\phi) \le -\mu\phi \text{ en } B_k(0), \ \phi(y) = 0 \text{ in } \partial B_k(0)\},$$

holding $\alpha_k(F) \searrow \alpha(F)$. See [10] for the proof. Here, $B_k(0)$ denotes the ball of radius k centered in $x = 0$.

Moreover, for $\Psi_k(x,t) = (t+\tau)^{-\alpha_k(F)}\Phi_k(x/\sqrt{t+\tau})$, we have $\partial_t \Psi_k = F(D^2\Psi_k)$ in $B_k(0) \times \mathbb{R}^+$.

Take $\epsilon > 0$ such that $\Phi(0) > 2\,(\epsilon)^{1/(p(F)-1)}$. Let $N = N(\epsilon)$ such that $0 \le \alpha_k(F) - \alpha(F) < \epsilon$ for all $k \ge N$. For this N, there exists $t_0 > 0$ such that

$$\Phi(x/\sqrt{t+\tau}) \ge \Phi(x/\sqrt{t_0+\tau}) > (\epsilon)^{1/(p(F)-1)}, \quad \text{for each } \|x\| \le N, \text{ and } t \ge t_0.$$

Then, $(\Psi(x,t))^{p-1} \geq \dfrac{\epsilon}{\tau + t}$ for $\|x\| < N$ and $t \geq t_0$, and therefore, we have

$$\partial_t w \geq F(D^2 w) + \frac{\epsilon}{t+\tau} w \quad \text{in } B_N(0) \times]t_0 \infty[,$$

holding $w(x,t) > 0$ in $\partial B_N(0) \times [t_0, \infty[$.

Taking $v(x,t) = (t+\tau)^{-\epsilon} w(x,t)$, we have $\partial_t v \geq F(D^2 v)$ in $B_N(0) \times]t_0, \infty[$. Based on comparison arguments in bounded domains, by rescaling, we can assume $\Psi_N(x,t) \leq (t+\tau)^{-\epsilon} w(x,t)$ in $B_N(0) \times [t_0, \infty[$.

From the previous result, and using $\epsilon - (\alpha_N(F) - \alpha(F)) > 0$, we have

$$\lim_{t\to\infty} (\tau + t)^{\epsilon - (\alpha_N(F) - \alpha(F))} \Phi_N^+(0) \leq \lim_{t\to\infty} t^{\alpha(F)} \|w(\cdot, t)\|_\infty = \infty,$$

and therefore, there exist $T^* < \infty$ and $x^* \in \mathbb{R}^N$ such that

$$\lim_{t\uparrow T^*} (1 - (p(F) - 1)t(w(x^*, t))^{p(F)-1}) = 0, \text{ and}$$
$$1 - (p(F) - 1)t(w(x,t))^{p(F)-1} > 0 \text{ in } \mathbb{R}^N \times [0, T^*[.$$

Then, the solutions of problems $\partial_t u_2 = F(D^2 u_2) + 2(u_2)^{p(F)}$ in $\mathbb{R}^N \times \mathbb{R}^+$ and $u_2(x,0) = u_0(x)$ in $\mathbb{R}^N$ have finite-time blow-up.

Finally, if we suppose that $u(x,t)$ is a global solution of (3), taking $u_2(x,t) = (1/2)^{1/(p(F)-1)} u(x,t)$, we arrive at a contradiction of the previous result. □

4. Comments

In [10], we present finite-time blow-up existence conditions for

$$\partial_t u = F(D^2 u) + f(u) \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+, \tag{19}$$

with function f, a regular function, where $f(u) = |u|^{p-1}u$ is a particular case. Under certain conditions, we can associate the behaviors of $f(s)$ for $s \to 0$ with the behaviors of the positive solutions of (19).

Let $f : [0, \infty[\to [0, \infty[$ satisfy

(f1) $f \in \mathcal{C}^1(\mathbb{R}_0^+)$, convex and strictly non-decreasing, such that $f(0) = 0$.

(f2) $\displaystyle\int_s^\infty \frac{1}{f(s)} \mathrm{d}s < \infty$ for all $s > 0$.

The results in [10] are as follows:

(i) If $\displaystyle\limsup_{s\downarrow 0} \frac{s^{p(F)}}{f(s)} = 0$, then the solutions of (19) have blow-up in finite time.

(ii) If $\displaystyle\liminf_{s\downarrow 0} \frac{s^{p(F)+\delta}}{f(s)} > 0$ for some $\delta > 0$, then (19) has non-trivial global solutions.

From Theorem 1.2, we have the following corollary:

COROLLARY 4.1. *Let f satisfy the conditions above. If* $\limsup_{s\downarrow 0} \frac{s^{p(F)}}{f(s)} < \infty$*, the positive solutions of* (19) *have finite-time blow-up.*

Proof. Let $z(t; z_0)$ satisfy $\dot{z} = f(z)$ and $z(0; z_0) = z_0$. Taking $Q(s) = \int_s^\infty \frac{1}{f(y)}\,\mathrm{d}y$, we have

$$z(t; z_0) = \begin{cases} Q^{-1}(Q(z_0) - t) & \text{if } 0 \le t < Q(z_0) \\ 0 & \text{if } z_0 = 0. \end{cases}$$

Moreover, for $w(x,t)$ a subsolution of (19), the function $\underline{u}_2(x,t) = Q^{-1}(Q(w(x,t)) - t)$ satisfies (again by Lemma 4.1 of [10]) $\partial_t \underline{u}_2 \le F(D^2\underline{u}_2) + 2f(\underline{u}_2)$ in $\mathbb{R}^N \times]0, T]$ for all $0 < T < T_{\max}(\underline{u}_2)$.

We show first that the solutions of

$$\partial_t u = F(D^2u) + 2f(u) \text{ in } \mathbb{R}^N \times \mathbb{R}^+ \tag{20}$$

blow-up in finite time, and then we obtain the result by rescaling.

Now, we want to find a subsolution $w(x,t)$ for (19) such that there exist $0 < T^* < \infty$ and $x^* \in \mathbb{R}^N$ satisfying $Q(w(x^*, T^*)) - T^* = 0$.

For this purpose, we begin with $\limsup_{s\downarrow 0} \frac{s^{p(F)}}{f(s)} < \infty$. From the continuity of functions $s^{p(F)}$, $f(s)$, there exist $\epsilon > 0$ and $c_\epsilon > 0$ such that

$$c_\epsilon s^{p(F)} \le f(s) \quad \text{for} \quad s < 2\epsilon. \tag{21}$$

Assume $\|u_0\|_\infty < \frac{1}{2}\epsilon$ and consider $w(x,t)$, global solutions of

$$\partial_t w = F(D^2w) + c_\epsilon w^{p(F)}(\epsilon - w) \text{ in } \mathbb{R}^N \times \mathbb{R}^+,$$

satisfying $w(x,0) = u_0(x)$ in $\mathbb{R}^N$. This solution exists by the comparison principle (see Theorem 2.1 in [10]) and the Perron method together with the subsolution $\underline{w}$ a solution of (11). For the super solution for $t \ge 0$, consider $g(t)$ such that $\frac{\mathrm{d}g}{\mathrm{d}t} = c_\epsilon g^{p(F)}(\epsilon - g)$ and $g(0) = \|u_0\|_\infty$. Because $\lim_{t\to\infty} g(t) = \epsilon$, $\overline{w}(x,t) = g(t)$ is a supersolution. Then, by comparison, we have $w(x,t) \le \epsilon$ in $\mathbb{R}^N \times \mathbb{R}_0^+$. Moreover, if $\lim_{t\to\infty} \|w(\cdot,t)\| < \epsilon$, there exists $\delta > 0$ such that $w(x,t) \le \epsilon - \delta$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$. Taking $w_\delta(x,t) = \delta^{1/(p(F)-1)} w(x,t)$, we have $\partial_t w_\delta \ge F(D^2 w_\delta) + c_\epsilon w_\delta^{p(F)}$ in $\mathbb{R}^N \times \mathbb{R}^+$, which contradicts Theorem 1.2. Then, the following holds:

$$\lim_{t\to\infty} \|w(\cdot,t)\|_\infty = \epsilon. \tag{22}$$

From (21), we have $c_\epsilon (w(x,t))^{p(F)}(\epsilon - w(x,t)) \le c_\epsilon (w(x,t))^{p(F)} \le f(w(x,t))$. Then,

$$\partial_t w \le F(D^2w) + f(w) \text{ en } \mathbb{R}^N \times \mathbb{R}^+.$$

From (22) and using condition (f2), we have $\lim_{t\uparrow\infty} Q(\|w(\cdot,t)\|_\infty) < \infty$, and therefore,

$$\lim_{t\uparrow\infty} Q(\|w(\cdot,t)\|_\infty) - t = -\infty.$$

Then, equation $Q(w(x,t)) - t = 0$ has a solution; therefore, function $\underline{u}_2(x,t) = Q^{-1}(Q(w(x,t)) - t)$ satisfies (see Lemma 4.1 of [10]) $\partial_t \underline{u}_2 \leq F(D^2\underline{u}_2) + 2f(\underline{u}_2)$ in $\mathbb{R}^N\times]0,T]$ for all $0 < T < T_{\max}(\underline{u}_2)$ and $T_{\max}(\underline{u}_2) < \infty$. By comparison, the solutions of (20) have finite-time blow-up.

Finally, if we suppose that there exists $u(x,t)$ a global solution of (19), taking $u_2(x,t) = u(\sqrt{2}x, 2t)$, we reach a contradiction with the previous result. □

The model function without power form is $f(s) = (1+s)\ln^p(1+s)$. From the previous result and the results in [10] for the Cauchy problem,

$$\begin{cases} \partial_t u = F(D^2u) + (1+u)\ln^p(1+u) & \text{in } \mathbb{R}^N \times \mathbb{R}^+ \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{23}$$

we have:

If $1 < p \leq p(F)$, then $T_{\max}(u) < \infty$.
If $p > p(F)$, there exists u_0 such that $T_{\max}(u) = \infty$.

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