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Fujita type exponent for fully nonlinear parabolic equations and existence results

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ABSTRACT

In this paper, we prove that a class of parabolic equations involving a second order fully nonlinear uniformly elliptic operator has a Fujita type exponent. These exponents are related with an eigenvalue problem in all $\mathbb{R}^N$ and play the same role in blow-up theorems as the classical $p^* = 1 + 2/N$ introduced by Fujita for the Laplacian. We also obtain some associated existence results.

1. Introduction

In this article we study qualitative properties of fully nonlinear parabolic equations of the type

$$\partial_t u = F\left(D^2 u, x\right) + f(u), \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+, \tag{1.1}$$

where F is a general uniformly elliptic operator and f is a source term, the classical model for f is $f(s) = |s|^{p-1}s$ with $p > 1$. We focus on blow-up phenomena, global existence and related properties.

A pioneering work in the study of blow-up phenomena for parabolic equations is due to Fujita, see [21]. Fujita proved that there exists a critical exponent $p^* = 1 + 2/N$ for the equation with $F(\cdot, x) = \operatorname{tr}(\cdot)$ (the Laplacian) and $f(s) = |s|^{p-1}s$ such that for $1 < p < p^*$ there is blow-up in finite time for any non-trivial positive initial condition, and for $p > p^*$ global existence holds. After that work, Hayakawa [24], Sugitani [35], Kobayashi, Sirao and Tanaka [29], Aronson and Weinberger [3] and Weissler [39] proved that at the critical exponent the same conclusion holds. Then other related works appeared; for a more complete review on the subject see for example the books by Samarskii, Galaktionov, Kurdyumov and Mikhailov [34] and Quittner and Souplet [32], see also the articles by Levine [30], Deng and Levine [16], Kavian [28], Galaktionov and Vazquez [22].

In this work we prove the existence of a Fujita type exponent for the equation

$$\partial_t u = F\left(D^2 u\right) + u^p, \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+. \tag{1.2}$$

This exponent, that we call $p^*(F)$, depends on F through an eigenvalue problem in all $\mathbb{R}^N$ that involves F. Moreover, this exponent is optimal in the following sense: If $1 < p < p^*(F)$ then all viscosity solutions $u(x,t)$ of (1.2) with $0 \leqslant u(x,0) \neq 0$ blow up in finite time. On the other hand, if $p > p^*(F)$ there are initial conditions so that u globally exists. Finally, in case $p = p^*(F)$ there is a large class of initial conditions $u(x,0)$ so that u blows up in finite time. We also extend these

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qualitative results to more general nonlinearities f and with x dependent operators. To our knowledge, the results will be new when F is a general linear second order operator with continuous coefficient in non-divergence form.

We are assuming that

$(F1)$ F is a uniformly elliptic operator with ellipticity constant $0 < \lambda \leqslant \Lambda$, that is,

$$\lambda \operatorname{tr}(B) \leqslant F(M+B, x) - F(M, x) \leqslant \Lambda \operatorname{tr}(B),$$

for all $M, B \in \mathcal{S}_N$, $B \geqslant 0$.

$(F2)$ i) F is either convex or concave, the function $F(M, x)$ is continuous in $\mathbb{R}^N \times \mathcal{S}_{\mathcal{N}}$,
or
ii) there exists L_F such that

$$\left|F(M, x) - F(M, y)\right| \leqslant L_F\big(|x-y|\big(\|M\|+1\big)\big),$$

for all $M \in \mathcal{S}_{\mathcal{N}}$, and $x, y \in \mathbb{R}^N$.

$(F3)$ F is positively homogeneous of order one, that is,

$$F(\alpha M, x) = \alpha F(M, x), \quad \text{for all } \alpha \geqslant 0,\ M \in \mathcal{S}_{\mathcal{N}},\ x \in \mathbb{R}^N.$$

Here and in the rest of the paper, $\mathcal{S}_{\mathcal{N}}$ denotes the set of all $N \times N$ symmetric matrices.

We start first with a basic local existence proposition for Eq. (1.1). For that, $BUC(\mathbb{R}^N)$ denotes the set of all uniform continuous and bounded functions in $\mathbb{R}^N$, where we consider the initial conditions. By the notion of solution, we use the viscosity solution for parabolic equations defined by Juutinen in [26], which is equivalent in our setting to the classical viscosity solution since comparison holds, for more details see [26].

Proposition 1.1. *Let $u_0 \in BUC(\mathbb{R}^N)$ and assume that F satisfies $(F1)$, $(F2)$ and $f \in C^1(\mathbb{R})$. Then there exists a local viscosity solution of*

$$\partial_t u = F\big(D^2u, x\big) + f(u), \quad \text{for } x \in \mathbb{R}^N,\ t > 0, \tag{1.3}$$

with $u(x, 0) = u_0(x)$.

The existence result is based on finding appropriate sub- and super-solutions, a comparison theorem and then Perron's method. This method is standard but $(F2)$ii) is not the classical assumption (3.14) in Crandall, Ishii, Lions [10] and the super-solution is found thought an adaptation of the variation of constant formula.

In Section 2 we prove first a comparison theorem for this setting followed by Proposition 1.1. Notice that many different types of existence results are established for general parabolic equations in different but related settings, see for example the papers by Crandall, Lions [13], Giga, Goto, Ishii, Sato [23], Chen, Giga, Goto, [9] Evans, Spruck [18], Crandall, Ishii, Lions [10], Papi [31], Barles [5], Da Lio, Ley [14,15] and the references therein. Observe also that these types of parabolic equations are connected, for example, with problems of optimal control for stochastic differential equations and financial mathematics, see Papi [31], Barles [5], Da Lio, Ley [14,15].

Now we give one of our main results for Eq. (1.2) that gives the Fujita type exponent for F.

Theorem 1.1. *Assume $F(\cdot)$ satisfies $(F1)$, $(F2)$, $(F3)$, and let $u(x, t)$ be a viscosity solution of (1.2) with $u(x, 0) = u_0(x)$ and $0 \leqslant u_0(x) \not\equiv 0$. Then there exists a critical exponent $p^*(F)$ such that:*

i) *If $1 < p < p^*(F)$, then $u(x, t)$ blows up in finite time.*
ii) *If $p = p^*(F)$ and $u_0(x) > A \exp\{-\alpha|x|^2\}$ for some family of constants A, α, then $u(x, t)$ blows up in finite time.*
iii) *If $p^*(F) < p$ then there exist initial conditions $u_0(x) \in BUC(\mathbb{R}^N)$ such that $u(x, t)$ exists for all $t > 0$.*

The critical exponent $p^*(F)$ can be written as $p^*(F) = 1 + 2/(N(F))$, with $(N(F), \Phi)$ a solution of

$$\begin{cases} F\big(D^2\Phi\big) + \dfrac{1}{2} D\Phi \cdot y = -\dfrac{N(F)}{2}\Phi, & \text{in } \mathbb{R}^N, \\ \lim\limits_{|y|\to\infty} \Phi(y) = 0, \end{cases} \tag{1.4}$$

with Φ positive. This connection between a fully nonlinear eigenvalue problem and the critical exponent $p^*(F)$, seems to be new and we believe that it can be useful for other parabolic problems.

In Section 3 we prove, in particular, that if $F(\cdot)$ satisfies $(F1)$, $(F2)$ and $(F3)$, the problem (1.4) has a solution $(N(F)/2, \Phi)$. This solution gives a self-similar viscosity solution to

$$\partial_t v = F\big(D^2 v\big), \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+, \tag{1.5}$$

where $v(x,t) = C^*(t+\tau)^{-N(F)/2}\Phi(x/(\sqrt{t+\tau}))$ and $C^* > 0$, $\tau > 0$ are any positive constants. This function v is useful to get a sub-solution that blows up in finite time and a similar one to get the global super-solution.

Before submitting this paper we were informed of the recent preprint by Armstrong and Trokhimtchouk [1], where they obtain, with a different method, these self-similar solutions. These authors study asymptotic behavior of solution to (1.5) with respect to the self-similar solutions. The first authors that obtain self-similar solution for parabolic equations of this type were Kamin, Peletier, and Vázquez [27], for the case $F(\cdot) = \max\{1/(1-\gamma)\operatorname{tr}(\cdot), 1/(1+\gamma)\operatorname{tr}(\cdot)\}$, the number $N(F)$ is called anomalous dimension for the Barenblatt equation of elasto-plastic filtration. In this paper we study the eigenvalue problem (1.4) with more general F operators and obtain bounds on the eigenvalue (and therefore for the Fujita type exponent), these bounds are better than those obtained in [1], for more details see Section 3.

Now, we are in position to give the precisely assumption of the nonlinearity f to state our second main theorem that deals with equation

$$\partial_t u = F\big(D^2 u\big) + f(u), \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+, \tag{1.6}$$

and generalize Theorem 1.1.

($f1$) $f : \mathbb{R} \to \mathbb{R}$, $f \in \mathcal{C}^1(\mathbb{R})$, with $f(0) = 0$, f non-decreasing convex in $(0,\infty)$.
($f2$) For $s > 0$, $\int_s^{+\infty} \frac{1}{f(y)}\,dy < +\infty$.

The typical models that satisfy ($f1$) and ($f2$) are $f(s) = |s|^{p-1}s$ and $f(s) = |s|^{p-1}\log(1+s)$, with $p > 1$.

Theorem 1.2. *Assume that f satisfies ($f1$), ($f2$) and the operator $F(\cdot)$ satisfies ($F1$), ($F2$), ($F3$). Let u be a viscosity solution of* (1.6) *with* $0 \leqslant u(x,0) = u_0(x) \not\equiv 0$.

Let $p^*(F) = 1 + 2/(N(F))$ and $k = \limsup_{s\downarrow 0}(s^{p^*(F)})/(f(s))$.

i) *If $k = 0$ then $u(x,t)$ blows up in finite time.*
ii) *If $0 < k < \infty$ and $u_0(x) > A\exp\{-\alpha|x|^2\}$ for some family of constants A, α, then $u(x,t)$ blows up in finite time.*
iii) *If there exists $\delta > 0$ such that $\liminf_{s\downarrow 0}(s^{p^*(F)+\delta})/(f(s)) = c$ for $c \in (0,+\infty]$, then there are initial conditions $u_0(x) \in BUC(\mathbb{R}^N)$ so that $u(x,t)$ globally exists.*

Finally, we also obtain a result for general F where two dimension-like numbers appear: $N^-(F) \leqslant N^+(F)$ (see the last section for the precise definition $N^-(F)$ and $N^+(F)$).

Corollary 1.1. *Suppose that f satisfies ($f1$), ($f2$), and $F(\cdot,x)$ satisfies ($F1$), ($F2$), ($F3$). Let u be a solution of* (1.1) *with* $0 \leqslant u(x,0) = u_0(x) \not\equiv 0$. *Then there exist dimension like numbers $N^-(F) \leqslant N^+(F)$ such that*:

i) *If $\lim_{s\downarrow 0}(s^{p^+(F)})/(f(s)) = 0$, with $p^+(F) = 1 + 2/(N^+(F))$, then $u(x,t)$ blows up in finite time.*
ii) *If $\lim_{s\downarrow 0}(s^{p^-(F)+\delta})/(f(s)) > 0$, for some $\delta > 0$, with $p^-(F) = 1 + 2/(N^-(F))$, then there are initial conditions $u_0(x) \in BUC(\mathbb{R}^N)$ so that $u(x,t)$ globally exists.*

Remark 1.1. The corollary when F is a linear operator in non-divergence form with continuous coefficient is new, up to our knowledge.

The paper is organized as follows. In Section 2 we prove our comparison and the local existence results. In Section 3 we establish the existence and properties of the eigenvalue problem in $\mathbb{R}^N$ that will give rise to the self-similar solutions. We also define a class of the extremal operators and the bounds for the dimension like numbers or eigenvalue. Finally, in Section 4 we prove our main theorems.

2. Comparison theorem and local existence result

We start this section with the comparison theorem needed for the main theorems and the local existence result. For sake of notation simplicity we will use:

$$\mathcal{P}(u) = \partial_t u - F\big(D^2 u, x\big) - f(u).$$

Theorem 2.1 *(Comparison theorem). Assume that F satisfies ($F1$), ($F2$)ii) and $f(s) \in \mathcal{C}^1(\mathbb{R})$. Let $\underline{u}$, $\overline{u}$ be continuous and bounded functions in $\mathbb{R}^N \times [0,T]$, such that $\mathcal{P}(\overline{u}) \geqslant 0 \geqslant \mathcal{P}(\underline{u})$ in $\mathbb{R}^N \times\,]0,T]$, in the viscosity sense. If $\overline{u}(x,0) \geqslant \underline{u}(x,0)$ in $\mathbb{R}^N$, then $\overline{u}(x,t) \geqslant \underline{u}(x,t)$ in $\mathbb{R}^N \times [0,T]$.*

Remark 2.1. i) This theorem is more or less standard but $(F2)$ii) is not the classical assumption (3.14) in [10]. Hence, we give the proof for completeness.

ii) In the case that $\underline{u}$ or $\overline{u}$ is regular, the theorem is direct. This fact is used when $(F2)$i) holds, since the regularity result of [11] can be used.

Within the proof of the theorem we use the following lemma that gives an equation for the function $W(x,t) := (\underline{u}-\overline{u})(x,t)$.

Lemma 2.1. *Let $\underline{u}$ and $\overline{u}$ be as in Theorem 2.1. The continuous function $W(x,t) = (\underline{u}-\overline{u})(x,t)$ satisfies*

$$\partial_t w \leqslant \mathcal{M}^+_{\lambda,\Lambda}\big(D^2 w\big) + c(x,t)w,$$

in the viscosity sense $\mathbb{R}^N \times]0,T]$, where $c(x,t) = \int_0^1 f'(\underline{u}(x,t) + s(\overline{u}-\underline{u})(x,t))\,ds$.

Here $\mathcal{M}^-_{\lambda,\Lambda}$, $\mathcal{M}^+_{\lambda,\Lambda}$ denote the Pucci extremal operators defined as

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = \lambda \sum_{\mu_j>0} \mu_j + \Lambda \sum_{\mu_j<0} \mu_j, \qquad \mathcal{M}^+_{\lambda,\Lambda}(M) = \Lambda \sum_{\mu_j>0} \mu_j + \lambda \sum_{\mu_j<0} \mu_j,$$

where the μ_j are the eigenvalues of $M \in \mathcal{S}_{\mathcal{N}}$.

After this comparison theorem, we can establish our local existence result Proposition 1.1. In the proof of Lemma 2.1 classical viscosity solution methods are used, similar to those used in [10,5,23].

Proof of Lemma 2.1. Consider $(\hat{x},\hat{t})$ in $\mathbb{R}^N \times]0,T]$, and ϑ in $\mathbb{R}^N$ to be a relative compact set, with $\hat{x}$ being an interior point of ϑ. Using the notation for the parabolic jets defined for example in [10] we have $-\mathcal{P}^{2,-}_{\vartheta}(\overline{u}(\hat{x},\hat{t})) = \mathcal{P}^{2,+}_{\vartheta}(-\overline{u})(\hat{x},\hat{t})$.

Given a test function $\varphi \in C^2$ such that $W-\varphi$ has a local maximum in $(\hat{x},\hat{t})$, we want to prove that

$$\partial_t \varphi \leqslant \mathcal{M}^+_{\lambda,\Lambda}\big(D^2_x\varphi(\hat{x},\hat{t})\big) + c(\hat{x},\hat{t})W(\hat{x},\hat{t}). \tag{2.7}$$

Notice that we can suppose that the maximum is strict, by a perturbation with a quadratic function.

Thus, there exists $\hat{r}>0$ such that for all $(x,t) \in \overline{B_{\hat{r}}}(\hat{x}) \times [\hat{t}-\hat{r},\hat{t}] \subset \mathbb{R}^N \times]0,T]$, $(W-\varphi)(x,t) < (W-\varphi)(\hat{x},\hat{t})$.

Let $V = \overline{B_{\hat{r}}}(\hat{x}) \times [\hat{t}-\hat{r},\hat{t}] \subset \mathbb{R}^{N+1}$, $k \in \mathbb{N}$ and define

$$M_k = \sup_{\overline{B_{\hat{r}}}\times V} \big\{ v_k(x,y,t) \big\},$$

for $v_k(x,y,t) = \underline{u}(x,t) + (-\overline{u})(y,t) - \psi_k(x,y,t)$ with

$$\psi_k(x,y,t) = \varphi(x,t) + \frac{k}{2}|x-y|^2.$$

Let (x_k,y_k,t_k) be the point where this maximum is attained, then using for example Lemma 2.3 in [5], we get

$$\begin{cases} \text{(i)}\ \lim_{k\to\infty} k|x_k-y_k|^2 = 0, \\ \text{(ii)}\ \lim_{k\to\infty} M_k = \sup_V \big\{(W-\varphi)(x,t)\big\}. \end{cases} \tag{2.8}$$

Let $\vartheta = \overline{B_{\hat{r}}}(\hat{x})$. Thus we are in position to apply Theorem 8.3 in [10] and get for any $\epsilon>0$, the existence of X_k, Y_k in $\mathcal{S}_N$ such that

$$\begin{cases} \text{(i)}\ \big(a_k, D_x\psi_k(x_k,y_k,t_k), X_k\big) \in \overline{\mathcal{P}}^{2,+}_{\vartheta}\underline{u}(x_k,t_k), \\ \quad\ \big(b_k, D_y\psi_k(x_k,y_k,t_k), Y_k\big) \in \overline{\mathcal{P}}^{2,+}_{\vartheta}(-\overline{u})(y_k,t_k), \\ \text{(ii)}\ -\Big(\dfrac{1}{\epsilon} + \|A_k\|\Big) Id \leqslant \begin{pmatrix} X_k & 0 \\ 0 & Y_k \end{pmatrix} \leqslant A_k + \epsilon A_k^2, \\ \text{(iii)}\ a_k + b_k = \partial_t\psi_k(x_k,y_k,t_k) = \partial_t\varphi(x_k,t_k), \end{cases} \tag{2.9}$$

where $A_k = D^2\psi_k(x_k,y_k,t_k) = \begin{pmatrix} D^2_x\varphi(x_k,t_k)+k\,Id & -k\,Id \\ -k\,Id & k\,Id \end{pmatrix}$.

Therefore,

$$\partial_t\varphi(x_k,t_k) \leqslant F(X_k,x_k) - F(-Y_k,y_k) + \big(f\big(\underline{u}(x_k,t_k)\big) - f\big(\overline{u}(x_k,t_k)\big)\big), \tag{2.10}$$

and taking $\epsilon = \frac{1}{k^4}$ in (ii) of (2.9) we obtain

$$\langle X_k - D^2\varphi(t_k, x_k)\xi, \xi\rangle + \langle Y_k\eta, \eta\rangle \leqslant k|\xi - \eta|^2 + g(k)$$

with $g(k) \sim O(1/k^2)$. Then the same argument of Lemma 2.2 in [4] gives

$$X_k - D^2\varphi(t_k, x_k) + Y_k \leqslant -\frac{2}{k}Y_k^2 + O(1/k).$$

Now by $(F1)$ we have

$$P_k = F(X_k, x_k) - F(-Y_k, y_k) \leqslant \mathcal{M}^+_{\lambda,\Lambda}\big(X_k + Y_k - D^2\varphi(t_k, x_k) + D^2\varphi(t_k, x_k)\big) + F(-Y_k, x_k) - F(-Y_k, y_k).$$

So, by $(F2)$ii) we get

$$P_k \leqslant -\lambda\frac{2}{k}\operatorname{tr}\big(Y_k^2\big) + \mathcal{M}^+_{\lambda,\Lambda}\big(D^2\varphi(t_k, x_k)\big) + L_F|x_k - y_k|\big(\|Y_k\| + 1\big) + O(1/k).$$

Using Cauchy–Schwarz's inequality

$$L_F|x_k - y_k|\|Y_k\| \leqslant \lambda\frac{2}{k}\operatorname{tr}\big(Y^2\big) + O\big(k|x_k - y_k|^2\big).$$

Thus, by the above inequalities taking $k \to \infty$ from (2.10), we get

$$\partial_t\varphi(\hat{x}, \hat{t}) \leqslant \mathcal{M}^+_{\lambda,\Lambda}\big(D_x^2\varphi(\hat{x}, \hat{t})\big) + \big(f\big(\underline{u}(\hat{x}, \hat{t})\big) - f\big(\overline{u}(\hat{x}, \hat{t})\big)\big), \tag{2.11}$$

therefore we obtain (2.7). □

Proof of Theorem 2.1. For $c(x,t)$ given by Lemma 2.1 we define $C^+ = \sup_{\mathbb{R}^N\times[0,T]}|c(x,t)|$, $\tilde{c}(x,t) = c(x,t) - C^+ \leqslant 0$ in $\mathbb{R}^N \times [0,T]$ and the function $\tilde{W}(x,t) := e^{-C^+t}W(x,t)$. By Lemma 2.1 the function $\tilde{W}$ satisfies

$$\partial_t\tilde{W} \leqslant \mathcal{M}^+_{\lambda,\Lambda}\big(D^2\tilde{W}\big) + \tilde{c}(x,t)\tilde{W}, \quad \text{in } \mathbb{R}^N \times]0,T]. \tag{2.12}$$

Now we define $\tilde{W}^+(x,t) = \max\{0, \tilde{W}(x,t)\}$ and get that

$$\begin{cases} \partial_t\tilde{W}^+ \leqslant \mathcal{M}^+_{\lambda,\Lambda}\big(D^2\tilde{W}^+\big), & \text{in } \mathbb{R}^N \times]0,T], \\ \tilde{W}^+(x,0) = 0, & \text{in } \mathbb{R}^N, \end{cases} \tag{2.13}$$

in the viscosity sense. Finally using the comparison principle of [13] we get $\tilde{W}^+(x,t) \leqslant 0$ in all $\mathbb{R}^N \times]0,T]$. Thus $W(x,t) \leqslant 0$ in $\mathbb{R}^N \times]0,T]$. □

Now we prove the local existence proposition.

Proof of Proposition 1.1. In [13], Crandall and Lions proved that when $f = 0$, and $F = \mathcal{M}^+_{\lambda,\Lambda}$ in Eq. (1.3), there exists a semigroup such that $S(t)u_0$ is a viscosity solution of (1.3), and satisfies:

$$\big\|S(t)\varphi - S(t)\phi\big\|_\infty \leqslant \|\varphi - \phi\|_\infty, \quad \text{for any } \varphi, \phi \in BUC\big(\mathbb{R}^N\big),$$

and

$$\lim_{t\to 0}\big\|S(t)\varphi - \varphi\big\|_\infty = 0.$$

Moreover, since $C = \sup\|u_0\| < \infty$ is a solution of $\partial_t u - \mathcal{M}^+_{\lambda,\Lambda}(D^2u) = 0$, then by comparison $S(t)u_0 \leqslant C$ for all $t \in (0,\infty)$. So, $S(t)u_0 \in BUC(\mathbb{R}^N)$ for all $t \in (0,\infty)$. Furthermore, based on Wang's results [37,38] the solution is classical.

Now, define $G : BUC([0,T] \times \mathbb{R}^N) \to BUC([0,T] \times \mathbb{R}^N)$ by

$$G(v) = S(t)u_0 + \int_0^t S(t-s)f(v)\,ds.$$

Then, by the above properties, G is a contraction for small T. Thus, by Banach's fix point theorem, there exists a local solution of $G(v) = v$. Therefore,

$$\partial_t v = \mathcal{M}^+_{\lambda,\Lambda}\big(D^2\big(S(t)u_0\big)\big) + \int_0^t \mathcal{M}^+_{\lambda,\Lambda}\big(D^2\big(S(t-s)f(v)\big)\big)\,ds + f(v).$$

Since $\mathcal{M}^+_{\lambda,\Lambda}(M) + \mathcal{M}^+_{\lambda,\Lambda}(N) \geqslant \mathcal{M}^+_{\lambda,\Lambda}(M+N)$ for any $M, N \in \mathcal{S}_N$, we deduce

$$\partial_t v \geqslant \mathcal{M}^+_{\lambda,\Lambda}\left(D^2\left[S(t)u_0 + \int_0^t \big(S(t-s)f(v)\big)\,ds\right]\right) + f(v).$$

Thus, using $(F1)$ and $(F3)$ we have that v is a local super-solution of (1.3). On the other hand, the semigroup $\tilde{S}(t)$ of $\partial_t u - \mathcal{M}^-_{\lambda,\Lambda}(D^2u) = 0$ together with $(F1)$, $(F3)$, and the positivity of f imply that $\tilde{S}(t)u_0$ is a sub-solution of (1.3). Therefore by Perron's method, there exists a local solution of (1.3). □

3. Eigenvalue problem, extremal operator and self-similar solution

As mentioned in the introduction, a starting point in the proof of our main theorems is to find a self-similar solution of

$$\partial_t v = F\big(D^2 v\big), \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+, \tag{3.14}$$

with F satisfying $(F1)$, $(F2)$ and $(F3)$. This self-similar solution is obtained in terms of the solution of the eigenvalue problem

$$F\big(D^2\Phi\big) + \frac{1}{2}D\Phi \cdot y = -\lambda\Phi, \quad y \in \mathbb{R}^N. \tag{3.15}$$

In this section we consider more general eigenvalue problems, so that (3.15) is a particular case. These eigenvalue problems are of the form

$$\begin{cases} H(\Phi) := \mathcal{F}\big(D^2\Phi, D\Phi, y\big) + \frac{1}{2}D\Phi \cdot y = -\lambda\Phi, & \text{in } \mathbb{R}^N, \\ \lim\limits_{|y|\to\infty} \Phi(y) = 0, \end{cases} \tag{3.16}$$

where $\mathcal{F} : \mathcal{S}_\mathcal{N} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and satisfies the following assumptions:

$(H0)$ $\mathcal{F}$ is positively homogeneous of degree 1, that is, for all $\alpha \geqslant 0$ and for all $(M, p, x) \in \mathcal{S}_\mathcal{N} \times \mathbb{R}^N \times \mathbb{R}^N$

$$\mathcal{F}(\alpha M, \alpha p, x) = \alpha\mathcal{F}(M, p, x).$$

$(H1)$ There exists $\gamma > 0$ such that for all $M, N \in \mathcal{S}_\mathcal{N}$, $p, q \in \mathbb{R}^N$, $x \in \mathbb{R}^N$

$$\mathcal{M}^-_{\lambda,\Lambda}(M-N) - \gamma\big(|p-q|\big) \leqslant \mathcal{F}(M, p, x),$$
$$-\mathcal{F}(N, q, x) \leqslant \mathcal{M}^+_{\lambda,\Lambda}(M-N) + \gamma\big(|p-q|\big).$$

$(H2)$ The function has *continuous coefficients*, that is, $\mathcal{F}(M, 0, x)$ is continuous in $\mathcal{S}_\mathcal{N} \times \mathbb{R}^N$.

(DF) If we denote $G(M, p, x) = -\mathcal{F}(-M, -p, x)$ then

$$\begin{aligned} G(M-N, p-q, x) &\leqslant \mathcal{F}(M, p, x) - \mathcal{F}(N, q, x) \\ &\leqslant \mathcal{F}(M-N, p-q, x). \end{aligned}$$

The assumptions (DF) and $(H0)$ imply that $\mathcal{F}(M, p, \cdot)$ is convex in $\mathcal{S}_\mathcal{N} \times \mathbb{R}^N$. This fact was proved in [33, Lemma 1.1].

Alternatively, we can also obtain the existence of an eigenvalue for the non-convex operator, but then we need more regularity on x for $\mathcal{F}$. More precisely:

$(\overline{DF})$ For each $K > 0$, there exist an increasing continuous function $w_K : [0,\infty) \to [0,\infty)$ for which $w_K(0) = 0$, and a positive constant $1/2 < \sigma \leqslant 1$, depending on K, such that

$$\big|\mathcal{F}(M, p, x) - \mathcal{F}(M, p, y)\big| \leqslant w_K\big(|x-y|^\sigma\big(|M|+1\big)\big),$$

for all $M \in \mathcal{S}_\mathcal{N}$, $x, y \in \mathbb{R}^N$ and $|p| \leqslant K$.

Now we are in position to state one of our main theorems of this section.

Theorem 3.1. *Assume that $\mathcal{F}$ satisfies $(H0)$–$(H2)$ and (DF) or $(H0)$, $(H1)$ and $(\overline{DF})$. Then there exists a positive viscosity solution (λ^+, Φ) to* (3.16)*.*

In addition, $\lambda^+(\mathcal{F})$ is characterized by

$$\lambda^+(\mathcal{F}) := \sup\{\lambda \in \mathbb{R} \mid \text{there exists } \phi \in \mathcal{C}(\mathbb{R}^N),\ \phi > 0,\ H(\phi) + \lambda u \leqslant 0,\ \text{in } \mathbb{R}^N\}. \tag{3.17}$$

Furthermore, $\Phi \in C^{1,\alpha}(\mathbb{R}^N)$.

Remark 3.1. i) These assumptions on $\mathcal{F}$ are in order to use the existence result for balls that we need for our proof, see [33] for the case of $(H0)$–$(H2)$ and (DF) and [2] or [25] for $(H0)$, $(H1)$ and $(\overline{DF})$. See also [17] for a simpler ODE proof with classical solution when the operator is radially invariant.

ii) Conditions $(H0)$, $(H1)$, $(H2)$, (DF), $(\overline{DF})$ are equivalent to $(F1)$, $(F2)$, $(F3)$ when $\mathcal{F}(\cdot, p, x) = F(\cdot, x)$.

iii) This theorem is obtained, with a different method, in [1] in the case $\mathcal{F}(M, p, x) = F(M)$.

iv) For radial invariant operator, the solution is radially symmetric.

v) The basic ideas of the existence and characterization are based on ideas in [7], see also [6] for the case of unbounded domains.

Proof of Theorem 3.1. We prove the existence by a limit procedure. Using the existence results quoted in the above Remark 3.1i), we find a positive viscosity solution to

$$\begin{cases} H(u) = -\lambda_R^+ u, & \text{in } B_R(0), \\ u(x) = 0, & \text{on } \partial B_R(0), \end{cases} \tag{3.18}$$

for all $R > 0$. Observe that in the case of operators that are radially invariant, we obtain a radial solution that follows just by simplicity or by the result in [17].

Now we take a sequence of solutions (λ_n, φ_n) of (3.18) with $R = n$. By the characterization of λ_n as the supremum like in (3.17) but in bounded domain, see [33] and [2], the sequence of eigenvalues λ_n is monotone decreasing and converges to $\lambda^* = \lim_{n\to\infty} \lambda_n$. We may assume by $(H0)$, that $\varphi_n(0) = 1$. Using Harnack inequality in $B_R(0)$ (see for example Theorem 3.6 in [33]) we get a constant $C(R)$ such that

$$\sup_{B_R(0)} \varphi_n \leqslant C(R) \inf_{B_R(0)} \varphi_n \leqslant C(R).$$

Then, using a compactness argument (see for example Proposition 4.2 in [12]) we find that φ_n converges in any fixed ball. So, after a diagonal extraction method and using Theorem 3.8 of [8] we find a viscosity solution $\Phi \geqslant 0$ to

$$H(\Phi) = -\lambda^* \Phi, \quad \text{in } \mathbb{R}^N.$$

Moreover, by the strong maximum principle $\Phi > 0$. This implies that $\lambda^* \leqslant \lambda^+$. Again by the characterization in bounded domains we have $\lambda^+ \leqslant \lambda_n$ for all $n \in \mathbb{N}$, then $\lambda^+ \leqslant \lambda^*$. Hence, both the existence and characterization follow. Using the regularity result of Trudinger [36] we find that $\Phi \in C^{1,\alpha}(\mathbb{R}^N)$.

Finally, notice that in the case of radially invariant operator we obtain a radial classical solution (limit of radial classical solution). □

For $(\lambda^+(\mathcal{F}), \Phi)$ a solution to (3.16) we will denote by $\lambda^+(\mathcal{F}) = N(\mathcal{F})/2$.

As mentioned above, for the proof of our main theorems we consider $\mathcal{F}(\cdot, p, x) = F(\cdot)$, with $F(\cdot)$ satisfying $(F1)$, $(F2)$ and $(F3)$. Using Theorem 3.1 we obtain the following lemma concerning self-similar solution.

Lemma 3.1. *Let $(N(F)/2, \Phi)$ be a solution to* (3.16), *with $F(\cdot)$ satisfying $(F1)$, $(F2)$ and $(F3)$, then for all positive constants C^*, τ the function*

$$v(x,t) = C^*(t+\tau)^{\frac{-N(F)}{2}} \Phi\left(\frac{x}{\sqrt{t+\tau}}\right) \tag{3.19}$$

is a viscosity solution of (3.14).

Proof. Let $\Psi \in C^2$ be a test function such that $v - \Psi$ has a local maximum that is zero at (x_0, t_0). Since Φ is $C^{1,\alpha}$ we have $v_t(x_0, t_0) = \Psi_t(x_0, t_0)$ and $Dv(x_0, t_0) = D\Psi(x_0, t_0)$. Therefore,

$$\Psi_t(x_0, t_0) = (t_0 + \tau)^{-1}\left(\frac{-N(F)}{2}\Psi(x_0, t_0) - \frac{1}{2} D\Psi(x_0, t_0) \cdot x_0\right). \tag{3.20}$$

On the other hand, $\Phi - \tilde{\Psi}$ has a local maximum at $\xi_0 = x_0/\sqrt{t_0 + \tau}$ where

$$\tilde{\Psi}(\xi_0) := \frac{(t_0 + \tau)^{\frac{N(F)}{2}}}{C^*} \Psi(\sqrt{t_0 + \tau}\,\xi_0, t_0).$$

Thus, by the definition of viscosity solution

$$F\big(D^2\tilde{\Psi}(\xi_0)\big) + \frac{1}{2}D\tilde{\Psi}(\xi_0)\cdot\xi_0 \geqslant -\frac{N(F)}{2}\tilde{\Psi}(\xi_0),$$

and, by the definition of $\tilde{\Psi}$ and (3.20) we find

$$\Psi_t(x_0,t_0) \leqslant F\big(D^2\Psi(\xi_0)\big),$$

that means that v is sub-solution.

The other inequality in the case of local minimum is similar. □

Now we want to study the asymptotic behavior of the eigenfunction Φ and bounds for the eigenvalues. For that purpose we define a general class of extremal operators as in [20] or [19]. To start with the definition of these operators, let $\mathcal{J}$ denote the set of subsets of $[\lambda,\Lambda]^N$ which are invariant with respect to permutations of coordinates, and $\mathcal{A}_{\lambda,\Lambda}\subset\mathcal{S}_{\mathcal{N}}$ be the set of matrices whose eigenvalues are in $[\lambda,\Lambda]$. First, there is a one-to-one correspondence between elements of $\mathcal{J}$ and $\mathcal{A}\subset\mathcal{A}_{\lambda,\Lambda}$, such that $P\mathcal{A}P^T=\mathcal{A}$, for each orthogonal matrix P. Namely, for each such $\mathcal{J}$ we can take $\mathcal{A}$ to be the set of matrices whose eigenvalues are in $\mathcal{J}$, and for each such $\mathcal{A}$ we can take $\mathcal{J}$ to contain the eigenvalues of all matrices in $\mathcal{A}$. So we can indifferently write

$$\begin{aligned}\mathcal{M}_J^+(M) &= \sup_{\sigma(A)\in J}\operatorname{tr}(AM) = \mathcal{M}_{\mathcal{A}}^+(M) = \sup_{A\in\mathcal{A}}\operatorname{tr}(AM),\\ \mathcal{M}_J^-(M) &= \inf_{\sigma(A)\in J}\operatorname{tr}(AM) = \mathcal{M}_{\mathcal{A}}^-(M) = \inf_{A\in\mathcal{A}}\operatorname{tr}(AM),\end{aligned}\tag{3.21}$$

where $\sigma(A)$ denotes the eigenvalue set of A. Therefore if we take $J=[\lambda,\Lambda]^N$ we have $\mathcal{M}_J^-=\mathcal{M}_{\lambda,\Lambda}^-$, $\mathcal{M}_J^+=\mathcal{M}_{\lambda,\Lambda}^+$. On the other hand, given $F(\cdot,x)$ satisfying $(F1)$ and $(F3)$ there exist $J_1(F)$, $J_2(F)$ in $\mathcal{J}$ such that

$$\mathcal{M}_{J_1(F)}^-(M) \leqslant F(M,x) \leqslant \mathcal{M}_{J_2(F)}^+(M),\tag{3.22}$$

for all $M\in\mathcal{S}_{\mathcal{N}}$.

Moreover, we also have that for $\mathcal{F}$ satisfying $(H0)$ and $(H1)$ there exist $J_1(\mathcal{F})$, $J_2(\mathcal{F})$ in $\mathcal{J}$ such that

$$\mathcal{M}_{J_1(\mathcal{F})}^-(M) - \gamma|p| \leqslant \mathcal{F}(M,p,x) \leqslant \mathcal{M}_{J_2(\mathcal{F})}^+(M) + \gamma|p|.\tag{3.23}$$

Remark 3.2. i) For the operators $\mathcal{F}(\cdot,p,x)=\mathcal{M}_J^-(\cdot)$ (resp. $\mathcal{M}_J^+(\cdot)$) in (3.16), with $\mathcal{M}_J^-,\mathcal{M}_J^+(\cdot)$ defined in (3.21), we use the notation $\lambda^+(\mathcal{F})=N_J^-/2$ (resp. $N_J^+/2$).

ii) From (3.22) the dimension like numbers N_J^-, N_J^+ will give bounds for the eigenvalues in (3.15) with $F(\cdot)$ satisfying $(F1)$, $(F2)$ and $(F3)$.

The next proposition describes the asymptotic behavior of the eigenfunction found in Theorem 3.1 and also bounds for N_J^- and N_J^+.

Proposition 3.1. *Let Φ be the eigenfunction found in Theorem* 3.1. *Then there exist constants c^+, C^+ a^+ and b^+ such that*

$$c^+\exp\left\{-\frac{r^2}{b^+}\right\} \leqslant \Phi(x) \leqslant C^+\exp\left\{-\frac{r^2}{a^+}\right\},\quad |x|=r.$$

In addition

$$\begin{aligned}&\underline{N}_J^- \leqslant N_J^- \leqslant \overline{N}_J^-,\\ &\underline{N}_J^+ \leqslant N_J^+ \leqslant \overline{N}_J^+,\end{aligned}\tag{3.24}$$

where $\underline{N}_J^-$, $\underline{N}_J^+$, $\overline{N}_J^-$, $\overline{N}_J^+$ are given by

$$\overline{N}_J^+ = \min_{x\in J}\frac{\sum_{i=1}^N x_i}{\max x_i} \leqslant \max_{x\in J}\frac{\sum_{i=1}^N x_i}{\min x_i} = \underline{N}_J^-,\tag{3.25}$$

$$\underline{N}_J^+ = \min_{x\in J}\frac{\sum_{i=1}^N x_i}{\max_{x\in J} x_i} \leqslant \max_{x\in J}\frac{\sum_{i=1}^N x_i}{\min_{x\in J} x_i} = \overline{N}_J^-.\tag{3.26}$$

Remark 3.3. These bounds on the eigenvalue were obtained in [1] only in the case of the Pucci extremal operators.

Proof of Proposition 3.1. For the bound on N_J^+, we first observe that $u = e^{-\frac{r^2}{4a^+}}$ with $a^+ = \max_{x\in J} x_1$ satisfies

$$\mathcal{M}_J^+\big(D^2u\big) + \frac{1}{2}Du\cdot x = \left(\frac{a(r)+b(r)(N-1)}{2a^+} + r^2\left\{\frac{a^+ - a(r)}{4(a^+)^2}\right\}\right)(-u) \leqslant -\frac{\underline{N}_J^+}{2}u,$$

for $r\in(0,\infty)$, some measurable function $a(r) = x_1(r)$ and $b(r) = \sum_{i=2}^N x_i(r)$ with $x(r)\in J$ (for more details see [20]). From this point, abusing notation, the functions represent both the function in $\mathbb{R}^N$ and the radially symmetric function ($r = |x|$). Now we use the characterization of $N_J^+/2$ to obtain $N_J^+/2 \geqslant \underline{N}_J^+/2 > 0$.

At this point, we establish the asymptotic behavior of Φ. We first observe that $u_1 := e^{-\frac{r^2}{4(a^+ +\varepsilon)}}$, for $\varepsilon > 0$ and $a^+ = \max_{x\in J_2(\mathcal{F})} x_1$ satisfies

$$H(u_1) \leqslant a_1(r)u_1'' + b_1(r)\frac{(N-1)}{r}u_1' + \gamma\big|u_1'\big| + \frac{1}{2}ru_1' < -\lambda_1 u_1, \quad \text{for } r\in[r_0,+\infty),$$

for some $r_0 > 0$ large, where a_1 and b_1 are functions as a and b but with $x(r)\in J_2(\mathcal{F})$ and φ_n and λ_n are denoting the sequences of Theorem 3.1. Since $\lambda_1 > \lambda_n$

$$H(u_1) < -\lambda_n u_1, \quad \text{for } r\in[r_0,+\infty) \text{ and } n\in\mathbb{N}.$$

Now, we choose c independent of n such that $w_n = \varphi_n - cu_1 \leqslant 0$ on $\partial B_{r_0}(0)$ (φ_n is bounded in compact sets) and notice that $w_n < 0$ in $\partial B_n(0)$. Since u_1 is regular, using $(H0)$ and $(H1)$, we have that w_n satisfies in the viscosity sense

$$\mathcal{M}_{\lambda,\Lambda}^+\big(D^2w_n\big) + \gamma|Dw_n| + \frac{1}{2}Dw_n\cdot x + \lambda_n w_n \geqslant 0, \quad \text{in } B_n(0)\setminus B_{r_0}(0).$$

Let ϕ be the positive eigenvalue of the operator $\mathcal{M}_{\lambda,\Lambda}^+(D^2\cdot) + \gamma|D\cdot| + \frac{1}{2}D\cdot\cdot x$ in $B_n(0)\setminus B_{r_0}(0)$. Then ϕ satisfies

$$0 = \mathcal{M}_{\lambda,\Lambda}^+\big(D^2\phi\big) + \gamma|D\phi| + \frac{1}{2}D\phi\cdot x + \lambda_1^+\phi \geqslant \mathcal{M}_{\lambda,\Lambda}^+\big(D^2\phi\big) + \gamma|D\phi| + \frac{1}{2}D\phi\cdot x + \lambda_n\phi.$$

In the last inequality we have used the monotonicity of the eigenvalue with respect to the domain. To conclude, we apply a strong simplicity result in $B_n(0)\setminus B_{r_0}(0)$, see Theorem 4.1 in [33], so we get that if there exists $x_0\in B_n(0)\setminus B_{r_0}(0)$ such that $w_n(x_0) > 0$, then $w_n = t\phi$ which is a contradiction. Therefore, $w_n \leqslant 0$ in $B_n(0)\setminus B_{r_0}(0)$ which implies that

$$\Phi \leqslant cu_1,$$

when $n\to\infty$.

For the lower bound, let $u_2 = e^{-\frac{r^2}{4(a^- -\varepsilon)}}$, $\varepsilon > 0$ small and $a^- = \min_{x\in J_1(\mathcal{F})} x_1$ we observe that

$$H(u_2) \geqslant \tilde{a}(r)u_2'' + \tilde{b}(r)\frac{(N-1)}{r}u_2' + \frac{1}{2}ru_2' + \gamma u_2' > -\lambda^+(\mathcal{F})u_2, \quad \text{in } B_{\hat{r}_0}(0)^c,$$

for some large $\hat{r}_0$. So, by $(H1)$

$$\mathcal{M}_{\lambda,\Lambda}^+\big(D^2(cu_2-\Phi)\big) + \frac{1}{2}D(cu_2-\Phi)\cdot x + \gamma\big|D(cu_2-\Phi)\big| > -\lambda^+(\mathcal{F})(cu_2-\Phi), \quad \text{in } B_{\hat{r}_0}(0)^c.$$

Observe now that $v(r) := e^{-r}$ satisfies

$$\tilde{a}(r)(v)'' + \tilde{b}(r)\frac{(N-1)}{r}(v)' + \frac{1}{2}r(v)' - \gamma v' < -\lambda^+(\mathcal{F})(v), \quad \text{in } [r_0,+\infty),$$

for some $r_0 \geqslant \hat{r}_0$, where $\mathcal{M}_{\lambda,\Lambda}^+(D^2v) = \tilde{a}(r)(v)'' + (\tilde{b}(r)(N-1)/r)(v)'$. Take c small such that $\Phi(x) > cu_2(r_0)$ with $|x| = r_0$ and suppose by contradiction that there exists $|x_0| > r_0$ such that $\Phi(x_0) < cu_2(|x_0|)$. Then the function $w_s = cu_2 - \Phi - sv$ is negative in $B_{r_0}(0)^c$ for large s and since $\Phi(x_0) < cu(x_0)$ there exists $\bar{s}$ such that $w_{\bar{s}} \leqslant 0$ in $B_{r_0}(0)^c$ and a maximum point $w_{\bar{s}}(x^*) = 0$ for some $x^*\in B_{r_0}(0)^c$. But this is a contradiction using $\bar{s}v$ as a test function for $cu_2 - \Phi$ at x^*. Thus, $cu_2 \leqslant \Phi$ in $(B_{r_0}(0))^c$. Therefore, the bounds for Φ are finished.

For the bound $N_J^+ \leqslant \overline{N}_J^+$, define $u_3 = e^{-\frac{r^2}{4(a_J^+)}}$ where a_J^+ is the point such that $\overline{N}_J^+ - 1 = b_J^+(N-1)/a_J^+$, then u satisfies

$$L(u_3) = a_J^+u_3'' + b_J^+\frac{(N-1)}{r}u_3' + \frac{1}{2}ru_3' = -\frac{\overline{N}_J^+}{2}u_3.$$

If we suppose, by contradiction, that $N_J^+/2 > \overline{N}_J^+/2$, then by the above type argument, there exists $c > 0$ such that $cu_3 \leqslant \Phi$.

Define now

$$t^* = \inf_{r\in[0,+\infty)} \frac{\Phi}{u_3} > 0.$$

So $w = tu_3 - \Phi$ for $t = \frac{t^* N_J^+/2}{\overline{N}_J^+/2}$ satisfies

$$Lw \geqslant 0$$

and $\lim_{r\to\infty} w(r) = 0$. Since $\{\rho^+ w'\}' \geqslant 0$ for the weight function

$$\rho^+(r) = \exp^{\int_{r_0}^r \frac{b^+(N-1)}{a^+ s} + \frac{s}{2a^+} ds},$$

we obtain $w \leqslant 0$. By the definition of t^* we get $\overline{N}_J^+/2 \geqslant N_J^+/2$, getting a contradiction. Therefore $N_J^+/2 \leqslant \overline{N}_J^+/2$.

The bounds for $N_J^-/2$ are similar. □

4. Proof of the main theorems

We start this section with some preliminaries that we need for the proofs of Theorems 1.1 and 1.2. Notice that some of the ideas are based on the method found in [34, Chapter IV, Section 7.1].

Lemma 4.1. *Assume F and f satisfy $(F1)$, $(F2)$, $(F3)$ and $(f1)$, $(f2)$ respectively, let $v(x,t) = C^*(t+\tau)^{-\frac{N(F)}{2}}\Phi^+(x/\sqrt{t+\tau})$, with $(N(F)/2, \Phi^+)$ solution to (1.4) and C^*, τ positive constants. Then for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$ such that $Q(v(v,t)) - t > 0$, the function*

$$\underline{u}(x,t) = Q^{-1}\big(Q\big(v(x,t)\big) - t\big)$$

is well defined and is viscosity sub-solution of

$$\partial_t \underline{u} \leqslant F\big(D^2\underline{u}\big) + f(\underline{u}). \tag{4.27}$$

Proof. Notice that by $(f2)$ and the strict positivity of $f(s)$ for $s > 0$ given by $(f1)$, the function

$$Q(s) = \int_s^\infty \frac{1}{f(\eta)}\, d\eta$$

is well defined and is strictly decreasing for $s > 0$. Therefore, we define $Q^{-1}(s)$ for all $s > 0$ and by $(f2)$ we have $\lim_{s\downarrow 0} Q^{-1}(s) = \infty$.

Given $T_1 > 0$ we define $M = \max_{(x,t)\in\mathbb{R}\times[0,T_1]} v(x,t) > 0$. Then $Q(v(x,t)) \geqslant Q(M)$, so there exists $T_2 < T_1$ such that $Q(M) - T_2 > 0$ and the function

$$\underline{u}(x,t) = Q^{-1}\big(Q\big(v(x,t)\big) - t\big)$$

is well defined in $\mathbb{R}^N \times [0, T_2]$.

Take now $\phi \in \mathcal{C}^2$ a test function such that $\underline{u} - \phi$ has a local maximum that is zero at $(x_0, t_0) \in \mathbb{R}^N \times]0, T_2]$.

Therefore, there exists a neighborhood of (x_0, t_0), denoted by $\Omega \subset \mathbb{R}^N \times]0, \underline{T}_1]$ such that

$$\underline{u}(x,t) \leqslant \phi(x,t), \quad \text{in } \Omega.$$

Using that the functions Q, Q^{-1} are decreasing, we get

$$v(x,t) \leqslant Q^{-1}\big(Q\big(v(x,t)\big) + t\big), \quad \text{in } \Omega.$$

Since $v(x_0,t_0) - Q^{-1}(Q(\phi(x_0,t_0)) + t_0) = 0$, then the function $v(x,t) - Q^{-1}(Q(\phi(x,t)) + t)$ has a local maximum in (x_0, t_0). Hence, we can use $\psi(x,t) = Q^{-1}(Q(\phi(x,t)) + t)$ as a test function for $v(x,t)$.

Thus, by Lemma 3.1

$$\partial_t \psi \leqslant F\big(D^2\psi\big), \quad \text{at } (x_0, t_0). \tag{4.28}$$

But $\partial_t \psi = \frac{f(\psi)}{f(\phi)}\partial_t \phi - f(\psi)$, so

$$\partial_t \phi \leqslant \frac{f(\phi)}{f(\psi)} F\big(D^2\psi\big) + f(\phi), \quad \text{at } (x_0, t_0). \tag{4.29}$$

On the other hand

$$D^2\psi = \frac{f(\psi)}{f(\phi)} D^2\phi + \frac{f(\psi)}{(f(\phi))^2}\big(f'(\psi) - f'(\phi)\big) D\phi \otimes D\phi. \tag{4.30}$$

Notice now that $Q(\psi(x,t)) = Q(\phi(x,t)) + t$ so $Q(\psi(x,t)) \geqslant Q(\phi(x,t))$ in Ω and then $\psi(x,t) \leqslant \phi(x,t)$ in Ω. So, by the convexity of f, $f'(\psi) - f'(\phi) < 0$ in Ω. Therefore, the second matrix in (4.30) is negative-definite in Ω. Hence

$$D^2\psi \leqslant \frac{f(\psi)}{f(\phi)} D^2\phi, \quad \text{in } \Omega.$$

By $(F1)$ and $(F3)$ we find

$$F\big(D^2\psi\big) \leqslant \frac{f(\psi)}{f(\phi)} F\big(D^2\phi\big).$$

Finally from (4.29) we get

$$\partial_t \phi \leqslant F\big(D^2\phi\big) + f(\phi), \quad \text{at } (x_0, t_0).$$

So, $\underline{u}(x,t)$ satisfies, in the viscosity sense, (4.27) in $\mathbb{R}^N \times]0, T_1[$. □

Our next lemma is for using only exponential initial conditions, more precisely, we can assume that $u_0(x)$ is of the form $A\exp\{-\alpha|x|^2\}$, with α, A positive constant.

Lemma 4.2. *Assume F satisfies $(F1)$ and $(F3)$. Let $w(x,t)$ be a viscosity solution of*

$$\partial_t w = F\big(D^2 w\big), \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+ \tag{4.31}$$

with $0 \leqslant w(x,0) = w_0(x) \neq 0$ in $BUC(\mathbb{R}^N)$. Given $t_0 > 0$ there exist A and $\alpha > 0$ such that

$$w(x, t_0) \geqslant A\exp\big\{-\alpha|x|^2\big\}, \quad \text{for } x \in \mathbb{R}^N. \tag{4.32}$$

Proof. Notice first that by the comparison principle we need to prove the result only in the case $F(\cdot) = \mathcal{M}^-_{\lambda,\Lambda}(\cdot)$.

From the existence result of [13], and regularity of [37,38] there exists $w(x,t)$ a smooth solution of

$$(p) \quad \begin{cases} \partial_t w = \mathcal{M}^-_{\lambda,\Lambda}\big(D^2 w\big), & \text{in } \mathbb{R}^N \times \mathbb{R}^+, \\ w(x,0) = w_0(x) \geqslant 0, & \text{in } \mathbb{R}^N. \end{cases}$$

By the strong maximum principle we can also assume that $w_0(x) > 0$ in $\mathbb{R}^N$. Given t_0, α, γ, R_0, η, positive constants with $R_0 = 2\sqrt{\gamma/\alpha}$ and $0 < \eta < t_0$, define $\Omega = \mathbb{R}^N \setminus \overline{B_{R_0}(0)}$ and let $\phi(x,t) \in \mathcal{C}^\infty(\Omega \times [t_0 - \eta, \infty[)$ be given by

$$\phi(x,t) = \begin{cases} \exp\{-\frac{\beta|x|^2 - \gamma}{t - t_0 + \eta}\}, & x \in \Omega,\ t > t_0 - \eta, \\ 0, & x \in \Omega,\ t = t_0 - \eta. \end{cases}$$

For all $(x,t) \in \Omega \times]t_0 - \eta, \infty[$, we have

$$\partial_t \phi - \mathcal{M}^-_{\lambda,\Lambda}\big(D^2\phi\big) = \left[\frac{\beta|x|^2}{(t - t_0 + \eta)^2}\big(1 - 4\alpha a\big(|x|\big)\big) + \frac{1}{(t - t_0 + \eta)}\left(2\beta\big(a\big(|x|\big) + (N-1)\Lambda\big) - \frac{\gamma}{(t - t_0 + \eta)}\right)\phi\right],$$

where $a(|x|) = \lambda$ or $a(|x|) = \Lambda$ depending on the sign of $\partial^2_{x/|x|}\phi$. Taking $\alpha = 1/(4\lambda)$, and $\gamma > \Lambda\eta N/(2\lambda)$, we have

$$\partial_t \phi - \mathcal{M}^-_{\lambda,\Lambda}\big(D^2\phi\big) < 0, \quad \text{in } \Omega \times]t_0 - \eta, \infty[.$$

Let $E = \Omega \times]t_0 - \eta, t_0]$, and $\partial_p E = \Omega \times \{t_0 - \eta\} \cup \partial\Omega \times [t_0 - \eta, t_0]$ the parabolic boundary of E.

Define

$$s_0 = \min_{\substack{|x| \leqslant R_0 \\ t_0 - \eta \leqslant t \leqslant t_0}} w(x,t) > 0,$$

and $\phi_0(x,t) = s_0\phi(x,t)$. Since $\phi_0(x, t_0 - \eta) - w(x, t_0 - \eta) = -w(x, t_0 - \eta) \leqslant 0$, for $x \in \bar{\Omega}$, and $\phi_0(x,t) - w(x,t) \leqslant 0$, for $|x| = R_0$, $t_0 - \eta \leqslant t \leqslant t_0$, using the comparison principle of [23], we find $\phi_0(x,t) \leqslant w(x,t)$ in all E. From here the result follows. □

Now we are in position to prove our main theorems.

Proof of Theorem 1.2, parts i) and ii). Let $u(x,t)$ the solution of

$$\partial_t u = F\big(D^2u\big) + f(u), \qquad u(x,0) = A\exp\big\{-\alpha|x|^2\big\} > 0, \quad \text{in } \mathbb{R}^N, \tag{4.33}$$

given by Proposition 1.1 in $\mathbb{R}^N \times [0, T_1]$, for some $T_1 > 0$. Let now $v(x,t) = C^*(t+\tau)^{\frac{-N(F)}{2}}\Phi(x/\sqrt{t+\tau})$ as in Lemma 4.1. From Proposition 3.1 we can choose C^* and τ such that $v(x,0) \leqslant u_0(x)$ in $\mathbb{R}^N$. Consider now $\underline{u}(x,t)$ defined in Lemma 4.1. Since $\underline{u}(x,0) \leqslant u(x,0)$, by comparison we only need to prove that $\underline{u}$ blows up in finite time.

By definition of $\underline{u}$, if there exists $T_1^* > 0$ such that $Q(v(0,T_1^*)) - T_1^* = 0$, then $\lim_{t\uparrow T_1^*}\|\underline{u}(\cdot,t)\|_\infty = \infty$. To prove that there exists $T_1^* > 0$ such that $Q(v(0,T_1^*)) - T_1^* = 0$. By the L'Hopital rule, we compute

$$\ell = \lim_{t\to\infty}\frac{Q(v(0,t))}{t} = \lim_{t\to\infty}\frac{\int_{v(0,t)}^{\infty}\frac{ds}{f(s)}}{t} = \lim_{t\to\infty}\frac{N(F)C^*(t)^{-\frac{N(F)+2}{2}}}{2f(C^*t^{-\frac{N(F)}{2}})}.$$

So, using the change of variables $s = C^*t^{-\frac{N(F)}{2}}$ and the assumption on k

$$\ell = \frac{kN(F)(C^*)^{-\frac{2}{N(F)}}}{2}.$$

If $\ell < 1$ the existence of a $T_1^* > 0$ is guaranteed.

When $k = 0$, $\ell = 0$, so T_1^* exists, and then $\underline{u}$ blows up in finite time.

In the case ii), since $k \in (0,\infty)$ if C^* is large, then $\ell < 1$ and so the same conclusion holds. □

Proof of Theorem 1.1. Parts i) and ii) follow by the previous theorem as a particular case. For part iii) let $p = p^*(F) + \delta$, with $\delta > 0$, and define

$$\bar{u}(x,t) = C^*\left(\frac{1}{t+\tau}\right)^{\frac{1}{p-1}}\Phi\left(\frac{x}{\sqrt{t+\tau}}\right), \tag{4.34}$$

with C^*, τ positive constants.

By the argument given in Lemma 3.1 we have that $\bar{u}$ satisfies

$$\partial_t\bar{u} = F\big(D^2\bar{u}\big) + \mu(t+\tau)^{-\frac{p}{p-1}}C^*\Phi, \quad \text{for } x \in \mathbb{R}^N,\ t > 0,$$

where $\mu = (N(F)/2 - 1/(p-1))$. Since we may assume by $(F3)$ that $\Phi \leqslant 1$, taking $C^* < \mu^{\frac{1}{p-1}}$ we get that $\bar{u}$ is global super-solution, that is,

$$\partial_t\bar{u} \geqslant F\big(D^2\bar{u}\big) + \bar{u}^p, \quad \text{for } x \in \mathbb{R}^N,\ t > 0.$$

Finally, by the comparison theorem, our local solution is bounded for above by the global super-solution, therefore it can be extended to a global solution. □

Proof of Theorem 1.2, part iii). By assumption let $\delta > 0$ be such that

$$0 \leqslant \liminf_{s\downarrow 0}\frac{s^{p^*(F)+\delta}}{f(s)} < \infty. \tag{4.35}$$

Then there exist k and ε positive number such that

$$f(s) \leqslant ks^{p^*(F)+\delta}, \quad \text{for } 0 < s < \epsilon. \tag{4.36}$$

Defining $\bar{u}(x,t)$ as in (4.34), with $p = p^*(F) + \delta$ we find that, as in the previous proof, for small C^*, $\bar{u}(x,t)$ satisfies

$$\partial_t\bar{u} \geqslant F\big(D^2\bar{u}\big) + f(\bar{u}), \quad \text{for } x \in \mathbb{R}^N,\ t > 0.$$

From here we get the conclusion. □

Finally, we define the numbers $N^{\pm}(F)$ which appear in Corollary 1.1. We define the sets

$$\mathcal{J}_F^{\pm} = \big\{J \in \mathcal{J}\colon \mathcal{M}_J^{\pm}(M) \leqslant F(M,x), \text{ for all } M \in \mathcal{S}_N,\ x \in \mathbb{R}^N\big\}.$$

Note that $\mathcal{J}_F^+$ can be empty but $\mathcal{J}_F^-$ is not empty and we set

$$N^-(F) := \min\Big\{\min_{J\in\mathcal{J}_F^+} N_J^+,\ \min_{J\in\mathcal{J}_F^-} N_J^-\Big\}.$$

Similarly we set

$$\tilde{\mathcal{J}}_F^{\pm} = \{ J \in \mathcal{J}\colon\ \mathcal{M}_J^{\pm}(M) \geqslant F(M,x),\ \text{for all } M \in \mathcal{S}_N,\ x \in \mathbb{R}^N \}.$$

Note that $\tilde{\mathcal{J}}_F^-$ can be empty but $\tilde{\mathcal{J}}_F^+$ is not empty and we set

$$N^+(F) := \max\Big\{ \max_{J \in \tilde{\mathcal{J}}_F^+} N_J^+,\ \max_{J \in \tilde{\mathcal{J}}_F^-} N_J^- \Big\}.$$

Proof of Corollary 1.1. By the definitions of $N^-(F)$ and $N^+(F)$ that are attained by the operators $\mathcal{M}_1$ and $\mathcal{M}_2$ satisfying

$$\mathcal{M}_1(\cdot) \leqslant F(\cdot, x) \leqslant \mathcal{M}_2(\cdot),$$

the result follows by using the sub-solution for $\mathcal{M}_1$ and global super-solution $\mathcal{M}_2$ found in Theorem 1.2. □

5. Conclusions

The main results in this paper are the starting point for research in some directions, for example the extension of well-known results for Eq. (1.2) with the Laplace operator to more general operators as the considered in this paper. Some of them are:

i) Give sufficient condition on the initial data so that blow-up occurs for Eq. (1.2) and $p > p_F$ and the connections with the elliptic critical exponent found in [19].
ii) Profile of the blow-up when it occurs.
iii) Continuation after blow-up.

Another interesting problem is to find the exact Fujita type exponent for x-dependent operator.

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