Solutions to the Nonlinear Schrödinger Equation Carrying Momentum along a Curve

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Abstract

We prove existence of a special class of solutions to the (elliptic) nonlinear Schrödinger equation $-\varepsilon^2 \Delta\psi + V(x)\psi = |\psi|^{p-1}\psi$ on a manifold or in Euclidean space. Here V represents the potential, p an exponent greater than 1, and ε a small parameter corresponding to the Planck constant. As ε tends to 0 (in the semiclassical limit) we exhibit complex-valued solutions that concentrate along closed curves and whose phases are highly oscillatory. Physically these solutions carry quantum-mechanical momentum along the limit curves. © 2008 Wiley Periodicals, Inc.

1 Introduction

In this paper we are concerned with concentration phenomena for solutions of the singularly perturbed elliptic equation

$$(\mathrm{NLS}_\varepsilon) \qquad -\varepsilon^2 \Delta_g \psi + V(x)\psi = |\psi|^{p-1}\psi \quad \text{on } M,$$

where M is an n-dimensional compact manifold (or the flat Euclidean space $\mathbb{R}^n$), V a smooth, positive function on M satisfying the properties

$$(1.1) \qquad 0 < V_1 \le V \le V_2, \quad \|V\|_{C^3} \le V_3$$

(for some fixed constants V_1, V_2, and V_3), ψ a complex-valued function, $\varepsilon > 0$ a small parameter, and p an exponent greater than 1. Here Δ_g stands for the Laplace-Beltrami operator on (M, g).

Equation (NLS_ε) arises from the study of the focusing nonlinear Schrödinger equation

$$(1.2) \qquad i\hbar \frac{\partial \tilde\psi}{\partial t} = -\hbar^2 \Delta \tilde\psi + V(x)\tilde\psi - |\tilde\psi|^{p-1}\tilde\psi \quad \text{on } M \times [0, +\infty),$$

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where $\tilde{\psi} = \tilde{\psi}(x,t)$ is the *wave function*, $V(x)$ a potential, and $\hbar$ the *Planck constant*. A special class of solutions to (1.2) consists of functions whose dependence on the variables x and t are of the form $\tilde{\psi}(x,t) = e^{-i\omega t/\hbar}\psi(x)$. Such solutions are called *standing waves* and up to substituting $V(x)$ with $V(x) - \omega$, they give rise to solutions of (NLS_ε) for $\varepsilon = \hbar$.

An interesting case is the *semiclassical limit* $\varepsilon \to 0$, where one should expect to recover the Newton law of classical mechanics. In particular, near stationary points of the potential, one is led to search highly concentrated solutions, which could mimic point particles at rest.

In recent years, much attention has been devoted to the study of the above problem: one of the first results in this direction was due to Floer and Weinstein [21], where the case of $M = \mathbb{R}$ and $p = 3$ was considered, and where existence of solutions highly concentrated near critical points of V has been proved. This result has since been extended by Oh [48] to the case of $\mathbb{R}^n$ for arbitrary n, provided $1 < p < \frac{n+2}{n-2}$. The profile of these solutions is given by the *ground state* U_{x_0} (namely, the solution with minimal energy, which is real-valued, positive, and radial) of the *limit equation*

$$-\Delta u + V(x_0)u = u^p \quad \text{in } \mathbb{R}^n, \tag{1.3}$$

where x_0 is the concentration point. The solutions u_ε obtained in the aforementioned papers behave qualitatively like

$$u_\varepsilon(x) \simeq U_{x_0}\left(\frac{x - x_0}{\varepsilon}\right)$$

as ε tends to 0, and since U_{x_0} decays exponentially to 0 at infinity, u_ε vanishes rapidly away from x_0.

The above existence results have been generalized in several directions, including the construction of solutions with multiple peaks, the case of degenerate potentials, potentials tending to 0 at infinity, and more general nonlinearities. We refer the interested reader to the (incomplete) list of works [1, 2, 3, 6, 7, 8, 13, 17, 23, 28] and to the bibliographies therein.

We also mention the mathematical similarities between (NLS_ε) and problem

$$(P_\varepsilon) \qquad \begin{cases} -\varepsilon^2\Delta u + u = u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $p > 1$, and ν denotes the exterior unit normal vector to $\partial\Omega$. Problem (P_ε) arises in the study of some biological models; see, e.g., [44] and references therein, and as (NLS_ε), it exhibits concentration of solutions at some points of $\bar{\Omega}$. Since the last equation is homogeneous, the location of the concentration points is determined by the geometry of the domain. About this topic, we refer the reader to [14, 18, 24, 25, 26, 27, 31, 32, 33, 45, 46, 47, 52].

More recently new types of solutions to (NLS$_\varepsilon$) have been found, since when ε tends to 0 these solutions do not concentrate at points but instead at sets of higher dimension. Before stating our main result, it is convenient to recall the progress on this topic and to illustrate the new phenomena involved. Some first results in this direction were given in [9, 11] in the case of radial symmetry, and later improved in [4] (see also [5] for the problem in bounded domains), where necessary and sufficient conditions for the location of the concentration set have been given. Unlike the previous case, the limit set is not stationary for the potential V: indeed, from heuristic considerations, the *energy* of a solution concentrated near a sphere of radius r depends both on V and on its *volume*, proportional to εr^{n-1}.

Based on the above energy considerations, in [4] a conjecture is stated concerning concentration on k-dimensional manifolds for $k = 1, \dots, n-1$; it is indeed expected that, under suitable nondegeneracy assumptions, the limit set should satisfy the equation

$$\theta_k \nabla^N V = V\mathbf{H} \quad \text{with } \theta_k = \frac{p+1}{p-1} - \frac{1}{2}(n-k), \tag{1.4}$$

where ∇^N stands for the normal gradient and $\mathbf{H}$ for the curvature vector, and the profile of the solutions at a point x_0 in the limit set should be asymptotic, in the normal directions, to the ground state of

$$-\Delta u + V(x_0)u = u^p \quad \text{in } \mathbb{R}^{n-k}. \tag{1.5}$$

Since the Pohozaev identity implies $p < \frac{n-k+2}{n-k-2}$ for the existence of nontrivial solutions, the latter condition is expected to be a natural one for dealing with this phenomenon.

Actually, concerning (P_ε) another conjecture has been stated, asserting existence of solutions concentrating at sets of positive dimension. About the latter problem, starting from the paper [40], there has been some progress in the general setting (without symmetry assumptions), and after the works [34, 39, 41], existence is now known for arbitrary dimension and codimension. About problem (NLS$_\varepsilon$), the conjecture in [4] has been verified in [19] for $n = 2$ and $k = 1$. Some other (and related) results, under some reduced symmetry assumptions have been given in [10, 15, 43, 49].

It is worth pointing out a major difference between the symmetric and the nonsymmetric situation. In fact, since the ground states of (1.3) or (1.5) are of mountain-pass type (namely critical points of some Euler functional with Morse index equal to 1), equation (NLS$_\varepsilon$) becomes highly resonant. To explain this phenomenon, we consider a real-valued function ψ in $\mathbb{R}^2$ with a radial potential. We can begin by finding approximate (radial) solutions of the form $u_{\bar{r},\varepsilon}(r) \simeq U_{\bar{r}}(\frac{r-\bar{r}}{\varepsilon})$, where $U_{\bar{r}}$ is the solution of (1.3) for $n = 1$ corresponding to $V(\bar{r})$. Then, with a good choice of $\bar{r}$, we can try to linearize the equation and find true solutions via a local inversion. The linearized equation, taking ψ real for simplicity, becomes

$$-\varepsilon^2 \Delta\psi + V(r)\psi - p u_{\bar{r},\varepsilon}(r)^{p-1}\psi \quad \text{in } \mathbb{R}^2.$$

Using polar coordinates (r, ϑ) and a Fourier decomposition of ψ with respect to ϑ, $\psi(r, \vartheta) = \sum_j e^{ij\vartheta}\psi_j(r)$, we see that the following operator acts on each component ψ_j:

$$(1.6)\quad \underbrace{-\varepsilon^2\psi_j'' - \varepsilon^2\frac{1}{r}\psi_j' + V(r)\psi_j - pu_{\bar{r},\varepsilon}(r)^{p-1}\psi_j}_{L_{1,\varepsilon}\psi_j} + \frac{1}{r^2}\varepsilon^2 j^2\psi_j \qquad \text{on } [0, +\infty),$$

where $L_{1,\varepsilon}$ (apart from the term $\varepsilon^2\frac{1}{r}\psi_j'$, which is not relevant to the next discussion) represents the linearized equation of (NLS_ε) in one dimension near a soliton. Since one expects to deal with functions that are highly concentrated near $r = \bar{r}$, the last term in the above formula naively *increases* the eigenvalues by a quantity of order $\frac{1}{\bar{r}^2}\varepsilon^2 j^2$ compared to those of $L_{1,\varepsilon}$.

The operator $L_{1,\varepsilon}$ possesses a negative eigenvalue η_ε lying between two negative constants independent of ε (since $U_{\bar{r}}$ is of mountain-pass type, as explained before) and a (nearly) zero eigenvalue σ_ε, by the translation invariance of (1.3) in $\mathbb{R}^1$. As a consequence, the operator in (1.6) possesses two sequences of eigenvalues qualitatively of the form $\eta_{j,\varepsilon} \simeq \eta_\varepsilon + \varepsilon^2 j^2$ and $\sigma_{j,\varepsilon} \simeq \sigma_\varepsilon + \varepsilon^2 j^2$. This might generate two kinds of resonances: for small values of j, when $\sigma_{j,\varepsilon} \simeq 0$, and for j of order $\frac{1}{\varepsilon}$, when $\eta_{j,\varepsilon}$ could be close to 0.

A comment is in order on resonant modes, which can be roughly studied with a separation of variables as before. The ones relative to $\sigma_{j,\varepsilon}$ (for j small) oscillate slowly along the limit set, while the ones relative to the resonant $\eta_{j,\varepsilon}$'s oscillate quickly with the number of oscillations proportional to $k \simeq \frac{1}{\varepsilon}$.

The invertibility of the linearized operator will then be equivalent to having all the $\sigma_{j,\varepsilon}$'s and all the $\eta_{j,\varepsilon}$'s different from 0. A control on the resonant $\sigma_{j,\varepsilon}$'s can be obtained (via some careful expansions) from a suitable nondegeneracy condition involving the limit set and the potential V. On the other hand, the possible vanishing of some $\eta_{j,\varepsilon}$ is peculiar to this concentration behavior and more intrinsic, so invertibility can only be achieved by choosing suitable values of ε.

These formal considerations can also apply to the case of concentration near a general manifold (without symmetries) in higher dimension or codimension, and related phenomena appear in some geometric problems as well that deal with the construction of surfaces with constant mean curvature; see [38, 42]. When Ω is a radially symmetric domain and the potential V is radially symmetric, the problem is simpler, since working in spaces of invariant functions avoids most of the above resonances.

In this paper we construct a new type of solution, which concentrates along some curve γ, and which physically carries momentum along the limit set. Differently from those discussed before, these solutions are complex valued and their profile near any point x_0 in the image of γ is asymptotic to a solution to (1.3), which decays exponentially to 0 away from the x_n-axis of $\mathbb{R}^n$ and is periodic in x_n. More precisely, we consider profiles of the form

$$\phi(x', x_n) = e^{-i\hat{f}x_n}\hat{U}(x'), \quad x' = (x_1, \dots, x_{n-1}), \tag{1.7}$$

where $\hat{f}$ is some constant and $\hat{U}(x')$ a real function. With this choice of ϕ, the function $\hat{U}$ satisfies

$$-\Delta\hat{U} + (\hat{f}^2 + V(x_0))\hat{U} = |\hat{U}|^{p-1}\hat{U} \quad \text{in } \mathbb{R}^{n-1} \tag{1.8}$$

and decays to 0 at infinity. Solutions to (1.8) can be found by considering the (real) function U satisfying $-\Delta U + U = U^p$ in $\mathbb{R}^{n-1}$ (decaying to 0 at infinity), and by using the scaling

$$\begin{gathered} \hat{U}(x') = \hat{h}U(\hat{k}x'), \\ \hat{h} = (\hat{f}^2 + V(x_0))^{\frac{1}{p-1}}, \quad \hat{k} = (\hat{f}^2 + V(x_0))^{\frac{1}{2}}. \end{gathered} \tag{1.9}$$

In the above formulas $\hat{f}$ can be taken arbitrarily, and $\hat{h}$ and $\hat{k}$ have to be chosen accordingly, depending on $V(x_0)$. The constant $\hat{f}$ represents the speed of the phase oscillation and is physically related to the velocity of the quantum-mechanical particle described by the wave function.

Usually standing waves have zero angular momentum, and with the exception of the present paper we are aware of only one result in this direction, given in [16] (see also the comments there) where the case of an axially symmetric potential is considered. Our goal here is to treat this phenomenon in a generic situation, without any symmetry restriction. Some of the difficulties of such an extension were naively summarized in the above discussion, but some new ones arise due to the fact that the standing waves are complex valued and their phase is highly oscillatory; more comments on these issues are given later.

Before stating our main result, we discuss how to determine the limit set. If we look for a solution ψ to (NLS_ε) with the above profile, then it should qualitatively behave as

$$\psi(\bar{s}, \zeta) \simeq e^{-i\frac{f(\bar{s})}{\varepsilon}} h(\bar{s}) U\left(\frac{k(\bar{s})\zeta}{\varepsilon}\right), \tag{1.10}$$

where $\bar{s}$ is the arc length parameter of γ, and ζ a system of geodesic coordinates normal to γ. To obtain more flexibility, we choose the phase oscillation to depend on the point $\gamma(\bar{s})$, while $h(\bar{s})$ and $k(\bar{s})$ should satisfy

$$h(\bar{s}) = \left((f'(\bar{s}))^2 + V(\gamma(\bar{s}))\right)^{\frac{1}{p-1}}, \quad k(\bar{s}) = \left((f'(\bar{s}))^2 + V(\gamma(\bar{s}))\right)^{\frac{1}{2}}, \tag{1.11}$$

which is the counterpart of (1.9) for a variable potential.

The function $f(\bar{s})$ can be (heuristically) determined by using an expansion of (NLS_ε) at order ε; a computation performed in Subsection 2.3 (see in particular formula (2.8)) shows that

$$f'(\bar{s}) \simeq \mathcal{A}h^\sigma(\bar{s}) \quad \text{with } \sigma = \frac{(n-1)(p-1)}{2} - 2, \tag{1.12}$$

where $\mathcal{A}$ is an arbitrary constant. At this point, only the curve γ should be determined.

First of all, we notice that the phase should be a periodic function in the length of the curve, and therefore by (1.12) it is natural to work in the class of loops

$$\Gamma := \left\{ \gamma : \mathbb{R} \to M \text{ periodic} : \mathcal{A} \int_\gamma h(\bar{s})^\sigma \, d\bar{s} = \text{const} \right\}, \tag{1.13}$$

where $\bar{s}$ stands for the arc length parameter on γ. Problem (NLS_ε) has a variational structure, with the Euler-Lagrange functional given by

$$E_\varepsilon(\psi) = \frac{1}{2} \int_M (\varepsilon^2 |\nabla_g \psi|^2 + V(x)|\psi|^2) - \frac{1}{p+1} \int_M |\psi|^{p+1}.$$

For a function of the form (1.10), by a scaling argument (see (2.11)) we have

$$E_\varepsilon(\psi) \simeq \varepsilon^{n-1} \int_\gamma h(\bar{s})^\theta d\bar{s} \quad \text{with } \theta = p + 1 - \frac{1}{2}(p-1)(n-1); \tag{1.14}$$

therefore a limit curve γ should be a critical point of the functional $\int_\gamma h(\bar{s})^\theta \, d\bar{s}$ in the class Γ. With a direct computation (see Subsection 2.4), we can check that the extremality condition is

$$\nabla^N V = \mathbf{H} \left(\frac{p-1}{\theta} h^{p-1} - 2\mathcal{A}^2 h^{2\sigma} \right) \tag{1.15}$$

where, as before, $\nabla^N V$ represents the normal gradient of V and $\mathbf{H}$ the curvature vector of γ. Similarly, via some long but straightforward calculation, we can find a natural nondegeneracy condition for stationary points that is expressed by the invertibility of the operator in (2.22) acting on the normal sections to γ (we refer the reader to Section 2 for the notation used in the formula). We notice that, since formula (1.12) determines only the derivative of the phase, to obtain periodicity we need to introduce some nonlocal terms; see (2.14). After these preliminaries, we are in position to state our main result.

THEOREM 1.1 *Let M be a compact n-dimensional manifold, let $V : M \to \mathbb{R}$ be a smooth, positive function, and let $1 < p < \frac{n+1}{n-3}$. Let $L > 0$; then there exists a positive constant $\mathcal{A}_0$, depending on V, p, and L, for which the following holds: If $0 \le \mathcal{A} < \mathcal{A}_0$, if γ has length less than or equal to L and satisfies* (1.15)*, and the operator in* (2.22) *is invertible on the normal sections of γ, there is a sequence $\varepsilon_k \to 0$ such that problem* ($\mathrm{NLS}_{\varepsilon_k}$) *possesses solutions ψ_{ε_k} having the asymptotics in* (1.10)*, with f satisfying* (1.12).

Remark 1.2.

(i) The statement of Theorem 1.1 remains unchanged if we replace M by $\mathbb{R}^n$ (or with an open manifold asymptotically Euclidean at infinity), and we assume

V to be bounded between two positive constants and for which $\|\nabla^l V\| \leq C_l$, $l = 1, 2, 3$, for some positive constants C_l.

(ii) The restriction on the exponent p is natural, since (1.8) admits solitons if and only if p is subcritical with respect to dimension $n-1$.

(iii) The smallness requirement on $\mathcal{A}$ is technical, and we believe this condition can be relaxed. Anyway, for $\frac{n+2}{n-2} \leq p < \frac{n+1}{n-3}$, $\mathcal{A}$ should have an upper bound depending on V to have solvability for both (1.9) and (1.12). About this condition, see Remark 2.2 and Remark 4.7.

(iv) Apart from the assumption on $\mathcal{A}$, Theorem 1.1 improves the result in [16]. In fact, in addition to removing the symmetry condition (which is the main issue), the characterization of the limit set is explicit, the assumptions on V are purely local, and the upper bound on p is sharp.

(v) The existence of solutions to $(\mathrm{NLS}_\varepsilon)$ only for a suitable sequence $\varepsilon_k \to 0$ is related to the resonance phenomenon described above. The result can be extended to a sequence of intervals in the parameter ε approaching 0 but, at least with our proof, we do not expect to find existence for all epsilons.

Taking $\mathcal{A} = 0$ (hence $f' \equiv 0$), from (1.11) it follows that $V = h^{p-1}$ and that (1.15) is equivalent to (1.4), so as a consequence of our theorem, we can prove the conjecture in [4] for $k = 1$, extending the result in [19].

COROLLARY 1.3 *Let M be a compact Riemannian n-dimensional manifold with metric g, let $V : M \to \mathbb{R}$ be a function satisfying* (1.1)*, and let $1 < p < \frac{n+1}{n-3}$. Let γ be a simple closed curve that is a nondegenerate geodesic with respect to the weighted metric*

$$V^{\frac{p+1}{p-1} - \frac{n-1}{2}} g.$$

Then there exists $\varepsilon_k \to 0$ such that problem $(\mathrm{NLS}_{\varepsilon_k})$ *has real-valued solutions ψ_{ε_k} concentrating near γ as $j \to +\infty$ and having the asymptotics*

$$\psi_{\varepsilon_k}(\bar{s}, \zeta) \simeq V(\gamma(\bar{s}))^{\frac{1}{p-1}} U\left(\frac{V(\gamma(\bar{s}))^{\frac{1}{2}}}{\varepsilon_k} \zeta\right),$$

where $\bar{s}$ stands for the arc length parameter of γ, and ζ for a system geodesic coordinates normal to γ.

Corollary 1.3 also gives some criterion for the applicability of Theorem 1.1; in fact, starting from a nondegenerate geodesic in the weighted metric, via the implicit function theorem for $\mathcal{A}$ sufficiently small, we obtain a curve for which (1.15) and the invertibility of (2.22) hold. In particular, when V is constant, we can start with nondegenerate close geodesics on M in the ordinary sense.

To prove Theorem 1.1 we proceed as follows: We collect some preliminary material in Section 2, where we recall some geometric facts, and we study the functional in (1.14) constrained to the class of curves Γ, determining the Euler-Lagrange equation together with the nondegeneracy condition.

In Section 3 we derive some expansions (in powers of ε) of equation (NLS_ε) for ψ of the form (1.10). To do this, it is convenient to scale problem (NLS_ε) in the following way:

$$-\Delta_{g_\varepsilon}\psi + V(\varepsilon x)\psi = |\psi|^{p-1}\psi \quad \text{in } M_\varepsilon, \tag{1.16}$$

where M_ε denotes the manifold M endowed with the scaled metric $g_\varepsilon = (1/\varepsilon^2)g$ (with an abuse of notation we might often write $M_\varepsilon = \frac{1}{\varepsilon}M$, and if $x \in M_\varepsilon$, we write εx to indicate the corresponding point on M).

We are now looking for a solution concentrated near the dilated curve $\gamma_\varepsilon := \frac{1}{\varepsilon}\gamma$. We let s be the arc length parameter of γ_ε so that $\bar{s} = \varepsilon s$, and we let $(E_j)_{j=2,\dots,n}$ denote an orthonormal frame in $N\gamma$ (the normal bundle of γ) transported in parallel to the normal connection; see Section 2. We also let $(y_j)_j$ be a corresponding set of normal coordinates. Since we want to allow some flexibility both in the choice of the phase and of the curve of concentration, we define $\tilde{f}_0(\bar{s}) = f(\bar{s}) + \varepsilon f_1(\bar{s})$, and we set $z_j = y_j - \Phi_j(\bar{s})$, where $(\Phi_j)_{j=2,\dots,n}$ are the components (with respect to the coordinates y) of a section Φ in $N\gamma$. Then, with a formal expansion of ψ in powers of ε up to second order, in the coordinates (s, z) near γ_ε, we set

$$\psi_{2,\varepsilon}(s,z) = e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}\{h(\varepsilon s)U(k(\varepsilon s)z) + \varepsilon[w_r + i w_i] + \varepsilon^2[v_r + i v_i]\},$$

$s \in [0, \frac{L}{\varepsilon}]$, $z \in \mathbb{R}^{n-1}$, and $L = L(\gamma)$, the length of γ, for some corrections w_r, w_i, v_r, and v_i (which have to be determined) to the above approximate solutions.

In Subsection 3.2 we show that these terms satisfy equations of the form $\mathcal{L}_r w_r = \mathcal{F}_r$, $\mathcal{L}_i w_i = \mathcal{F}_i$, $\mathcal{L}_r v_r = \tilde{\mathcal{F}}_r$, and $\mathcal{L}_i v_i = \tilde{\mathcal{F}}_i$ where

$$\begin{cases} \mathcal{L}_r v = -\Delta_z v + V(\bar{s})v - p h(\bar{s})^{p-1} U(k(\bar{s})z)^{p-1} v \\ \mathcal{L}_i v = -\Delta_z v + V(\bar{s})v - h(\bar{s})^{p-1} U(k(\bar{s})z)^{p-1} v \end{cases} \quad \text{in } \mathbb{R}^{n-1}, \tag{1.17}$$

and where $\mathcal{F}_r$, $\mathcal{F}_i$, $\tilde{\mathcal{F}}_r$, and $\tilde{\mathcal{F}}_i$ are given data that depend on V, γ, $\bar{s}$, $\mathcal{A}$, Φ, and f_1. The operators $\mathcal{L}_r$ and $\mathcal{L}_i$ are Fredholm (and symmetric) from $H^2(\mathbb{R}^{n-1})$ into $L^2(\mathbb{R}^{n-1})$, and the above equations for the corrections can be solved provided the right-hand sides are orthogonal to the kernels. It is well-known (see, e.g., [30]) that $\mathcal{L}_r$ has a single negative eigenvalue, a kernel with multiplicity $n-1$ spanned by the functions $\partial_l U(k(\bar{s})z)$, $l = 2, \dots, n$ (the generators of the normal translations), while all the remaining eigenvalues are positive. The operator $\mathcal{L}_i$ instead has one zero eigenvalue with eigenfunction $U(k(\bar{s})z)$ (the generator of complex rotations) and all the remaining eigenvalues positive. As explained above, condition (1.15) and the nondegeneracy of the operator $\mathfrak{J}$ help us to determine w_r, w_i, v_r, and v_i (and f_1), respectively, namely to solve (1.16) at order ε first, and then at order ε^2.

As discussed before, some fast-oscillating functions (along γ) contribute to generate some resonance, but these aspects are treated in later sections. Even at this stage, however, there are new difficulties compared to the results in [19, 34, 39, 40].

In our case the solutions are complex valued, and this causes an extra degeneracy due to their invariance under multiplication by a phase factor. As a consequence, we have a further (infinite-dimensional) approximate kernel, corresponding roughly to a factor of ψ_ε in the form $e^{-if_1(\bar{s})}$ for f_1 arbitrary. The correction in the phase can also be determined by a formal expansion in ε and, as for f', we still obtain nonlocal terms. Also, when expanding formally the solutions in ε, the highly oscillatory behavior of solutions generates an increasing number of derivatives in $\bar{s}$.

In Section 4 then we set up the strategy to obtain true solutions from the approximate ones. First of all, since U (and its derivatives) decay fast at infinity like

$$U(r) \simeq e^{-r} r^{-\frac{n-2}{2}} \quad \text{as } r \to +\infty, \tag{1.18}$$

it is possible to localize the problem in a neighborhood of the scaled curve γ_ε; this step is inspired by [19] and worked out in Subsection 4.2. We then try to find a true solution of the form

$$e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\{h(\varepsilon s)U(k(\varepsilon s)z) + \varepsilon[w_r + iw_i] + \varepsilon^2[v_r + iv_i] + \tilde{w}\},$$

with $\tilde{w}$ suitably small and $\tilde{f}$ close to $\tilde{f}_0$, via some local inversion arguments. From a linearization of the equation near $\psi_{2,\varepsilon}$, the operator L_ε acting on $\tilde{w}$ in the coordinates (s, z) is then the following:

$$\begin{aligned} L_\varepsilon \tilde{w} := -\partial_{ss}^2 \tilde{w} - \Delta_z \tilde{w} + V(\varepsilon x) - |\psi_{2,\varepsilon}|^{p-1}\tilde{w} \\ - (p-1)|\psi_{2,\varepsilon}|^{p-3}\psi_{2,\varepsilon}\Re(\psi_{2,\varepsilon}\bar{\tilde{w}}). \end{aligned} \tag{1.19}$$

Here $\Re$ denotes the real part. Decomposing first $\tilde{w}$ into its real and imaginary parts, and then in Fourier modes with respect to the variable εs, we can write

$$\tilde{w} = \tilde{w}_r + i\tilde{w}_i = \sum_j \sin(j\varepsilon s)\tilde{w}_{r,j}(z) + i\sum_j \sin(j\varepsilon s)\tilde{w}_{i,j}(z)$$

(forgetting for simplicity about the cosine functions). If we take (as a model problem) $V \equiv 1$, then the operators (in the z-variables) acting on the real and imaginary components are, respectively, $\mathcal{L}_r + \varepsilon^2 j^2$ and $\mathcal{L}_i + \varepsilon^2 j^2$.

As for (1.6), the kernels of $\mathcal{L}_r$ and $\mathcal{L}_i$ produce a sequence of eigenvalues for L_ε that behave *qualitatively* like $\varepsilon^2 j^2$, and for small values of j these become resonant. With an accurate expansion of these eigenvalues, we find that the nondegeneracy assumption on (2.22) prevents each of them from vanishing. However, the fact that $\mathcal{L}_r$ possesses a negative eigenvalue as well generates an extra sequence of eigenvalues of L_ε, qualitatively of the form $-1 + \varepsilon^2 j^2$, $j \in \mathbb{N}$. As explained before, the only hope to get invertibility is to choose the values of ε appropriately. For the Neumann problem (P_ε) the resonance phenomenon was taken care of using a theorem by T. Kato [29, p. 445], which allows us to differentiate eigenvalues with respect to ε. In the aforementioned papers it was shown that when varying the parameter ε the spectral gaps near 0 shrink only slightly, and invertibility can be obtained for a large family of epsilons.

However, when the concentration set is one dimensional, the spectral gaps of the resonant eigenvalues (with fast-oscillating eigenfunctions) are relatively large, of order ε, and the profile of the corresponding eigenfunctions can be analyzed by means of a scalar function on $[0, L]$ (see below and in Subsection 6.2). This might allow us to bypass Kato's theorem and use a more direct approach, employed in [42] to exhibit constant mean curvature surfaces of cylindrical type embedded in manifolds, and in [19] for studying solutions of (NLS_ε) in $\mathbb{R}^2$. We can partially take advantage of these techniques (see the comments in Section 4), but some new difficulties arise due to the phase oscillations in (1.10).

From the above discussion, we expect to find three possible resonances: two of them for small values of the index j (with eigenvectors roughly of the form $e^{-i(f(\varepsilon s))/\varepsilon}\partial_l U(k(\bar{s})z)\sin(\varepsilon j s)$, $l = 2, \dots, n$, and $i e^{-i(f(\varepsilon s))/\varepsilon} U(k(\bar{s})z)\sin(\varepsilon j s)$, respectively), and a third one for j of order $\frac{1}{\varepsilon}$, precisely when $-\varepsilon^2 j^2$ coincides with the first eigenvalue of $\mathcal{L}_r$.

To understand this behavior, we first study the spectrum of a *model* operator similar to (1.19), where we assume $V \equiv \widehat{V} > 0$ and $\psi_{2,\varepsilon}$ to coincide with the function in (1.7). For this case we characterize completely the spectrum of the operator and the properties of the eigenfunctions; see Subsection 4.3 and in particular Proposition 4.5. The condition on the smallness of $\mathcal{A}$ appears precisely here (and only here) and is used to show that the resonant eigenvalues are only of the forms described above. Removing the smallness assumption might indeed lead to further resonance phenomena; see Remark 4.7 for related comments.

We next consider the case of nonconstant potential V. Since this has a slow dependence in s along γ_ε, one might guess that the approximate kernel of L_ε (see (1.19)) might be obtained from that for constant V, allowing a slow dependence in s of the profile of these functions. With this criterion, given a small positive parameter δ, we introduce a set K_δ (see (4.50) and the previous formulas) consisting of *candidate* approximate eigenfunctions on L_ε, once multiplied by the phase factor $e^{-i(f(\varepsilon s))/\varepsilon}$. More comments on the specific construction of this set can be found in Subsection 4.3, especially before (4.50).

In Proposition 4.9 we show that this guess is indeed correct; in fact, we prove that the operator L_ε is invertible provided we restrict ourselves to the subset $\bar{H}_\varepsilon$ of functions that are orthogonal to $e^{-i(f(\varepsilon s))/\varepsilon}K_\delta$. This property allows us to solve the equation up to a Lagrange multiplier in K_δ; see Proposition 4.14. For technical reasons, we prove invertibility of L_ε in suitable weighted norms, which are convenient to deal with functions decaying exponentially away from γ_ε.

Compared to the other papers that deal with this kind of resonance, the approximate kernel here depends *genuinely* on the variable s (in [34, 39, 40, 41, 42]; the problem is basically homogeneous along the limit set, while in [19] it can be made such through a change of variables). To deal with this feature, which is the main cause of difficulty in Proposition 4.9, we localize the problem in the variable s as well. Multiplying by a cutoff function in s, we show that orthogonality to K_δ

implies approximate orthogonality to the set $\hat{K}_\delta$; see (4.53) and the previous formulas, which is the counterpart of K_δ for a potential frozen at some point in γ_ε; once this is shown, we use the spectral analysis of Proposition 4.5.

Section 5 is devoted to choosing a family of approximate solutions to (1.16); since we have many small eigenvalues appearing, it is natural to try looking for functions that solve (1.16) as accurately as possible. Our final goal is to annihilate the Lagrange multiplier in Proposition 4.14, and to do this we choose approximate solutions $\tilde{\Psi}_{2,\varepsilon}$ (in the notation of Section 5), which depend on suitable parameters: a normal section Φ, a phase factor f_2, and a real function β. These parameters correspond to different components of K_δ and are related to the kernels of $\mathcal{L}_r(+\varepsilon^2 j^2)$ and $\mathcal{L}_i(+\varepsilon^2 j^2)$; see the above comments. The function β in particular is highly oscillatory and takes care of the resonances due to the fast Fourier modes.

We next need to derive rigorous estimates on the error terms and to study in particular their Lipschitz dependence on the data Φ, f_2, and β. Proposition 5.2 collects the final expression for $-\Delta_{g_\varepsilon}\psi + V(\varepsilon x)\psi - |\psi|^{p-1}\psi$ on the approximate solutions $\tilde{\Psi}_{2,\varepsilon}$; the error terms $\tilde{\mathfrak{A}}$ are listed (and estimated) before in that section, together with their Lipschitz dependence on the parameters.

Finally, after performing a Lyapunov-Schmidt reduction onto the set K_δ (see Proposition 6.1), we study the bifurcation equation. In doing this we crucially use the formal computations in Section 3 and the error estimates in Section 5. In particular, for Φ and f_2 we find as main terms, respectively, the operator $\mathfrak{J}$ in (2.22) and the one in the middle of (3.24), both appearing when we perform formal expansions. These operators are both invertible by our assumptions, and therefore we are able to determine Φ and f_2 without difficulty.

The operator acting on β instead is more delicate, since it is *qualitatively* of the form

$$-\varepsilon^2 \beta''(\bar{s}) + \lambda(\bar{s})\beta \quad \text{on } [0, L], \tag{1.20}$$

with periodic boundary conditions, where λ is a negative function. This operator is precisely the one related to the peculiar resonances described above. In particular, it is resonant at frequencies of order $\frac{1}{\varepsilon}$, and this requires us to choose a norm for β that is weighted in the Fourier modes; see (5.6) and Subsection 6.2. For operators like that in (1.20) there is in general a sequence of epsilons for which a nontrivial kernel exists. Using Kato's theorem though, as in [34, 38, 39, 40, 41], we provide estimates on the derivatives of the eigenvalues with respect to ε, showing that for several values of this parameter the operator acting on β is invertible. In this operation also the value of the constant $\mathcal{A}$ (see (1.12)) has to be suitably modified (depending on ε) in order to preserve the periodicity of our functions. Once we have this, we apply the contraction mapping theorem to solve the bifurcation equation as well.

The results in this paper were first written in two different preprints [35, 36]. Some lengthy proofs, consisting of many explicit computations, have been omitted here for reasons of brevity, but precise references to the preprints will be given. Theorem 1.1 was announced in the note [37].

Notation and Conventions

- Dealing with coordinates, capital letters like $A, B, \dots$ will vary between 1 and n while indices like $j, l, \dots$ will run between 2 and n. The symbol i will always stand for the imaginary unit.
- For summations, we use the standard convention of summing terms where repeated indices appear.
- We will choose coordinates $(x_1, \dots, x_n)$ near a curve γ, and we will parametrize γ by arc length letting $x_1 = \bar{s}$. Its dilation $\gamma_\varepsilon := \frac{1}{\varepsilon}\gamma$ will be parametrized by $s = \frac{1}{\varepsilon}\,\bar{s}$. The length of γ is denoted by L.
- For simplicity, a constant C is allowed to vary from one formula to another, and also within the same line.
- For a real positive variable r and integer m, $O(r^m)$ (respectively, $o(r^m)$) will denote a complex-valued quantity for which $|O(r^m)/r^m|$ remains bounded (respectively, $|o(r^m)/r^m|$ tends to 0) when r tends to 0. We might also write $o_\varepsilon(1)$ for a quantity that tends to 0 as ε tends to 0.

2 Study of the Reduced Functional

In this section we consider the functional in the right-hand side of (1.14) defined on the set Γ, representing the approximate energy E_ε of a function concentrated near γ with the profile (1.10). We first introduce a convenient set of coordinates near an arbitrary (smooth) closed curve in M. Then, using these coordinates, we write the Euler equation and the second variation formula at a stationary point.

2.1 Geometric Preliminaries

In this subsection we discuss some preliminary geometric facts, referring, for example, to [20, 50]. Given an arbitrary simple closed curve γ in M, we choose coordinates $x_1, \dots, x_n$ near γ, called *Fermi coordinates*, in the following way: We let x_1 parametrize the curve γ by arc length. At some point q in the image of γ, we consider an orthonormal $(n-1)$-tuple $(Y_2, \dots, Y_n)$, which forms a basis for $N_q\gamma$, the normal bundle of γ at q. We extend the Y_l's as vector fields along γ via parallel transport along the curve with respect to the normal connection ∇^N, namely, by the condition $\nabla^N_{\dot\gamma} Y_l = 0$ for $l = 2, \dots, n$.

Next we parametrize a point near γ using the following coordinates $(\bar{s}, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$,

$$(\bar{s}, y_2, \dots, y_n) \mapsto \exp_{\gamma(\bar{s})}(y_2 Y_2 + \cdots + y_n Y_n),$$

where $\exp_q$ is the exponential map in M through the point q. In this way, fixing $\bar{s}$, each curve $t \mapsto ty$, for $y \in \mathbb{R}^{n-1} \setminus \{0\}$ and t close to 0, is mapped into a geodesic in M passing through $\gamma(\bar{s})$.

Let us now define the vector fields $E_1 = \frac{\partial}{\partial \bar{s}}$ and $E_l = \frac{\partial}{\partial y_l}$ for $l = 2, \dots, n$. We notice that on γ each E_l coincides with Y_l, while E_1 on γ is nothing but $\dot{\gamma}$. By our choice of coordinates it follows that $\nabla_E E = 0$ on γ for any vector field E that is a linear combination (with coefficients depending only on $\bar{s}$) of the E_j's, $j = 2, \dots, n$. In particular, for any $l, j = 2, \dots, n$ and for any $\alpha \in \mathbb{R}$, we have $\nabla_{E_l + \alpha E_j}(E_l + \alpha E_j) = 0$ on γ, which implies $\nabla_{E_l} E_j + \nabla_{E_j} E_l = 0$ for every $l, j = 2, \dots, n$. Using the fact that E_A's are coordinate vectors for $A = 1, \dots, n$ and in particular $\nabla_{E_A} E_B = \nabla_{E_B} E_A$ for all $A, B = 1, \dots, n$, we obtain that $\nabla_{E_l} E_j = 0$ for every $l, j = 2, \dots, n$. This immediately yields

$$\begin{aligned}\partial_m g_{lj} &= E_m \langle E_l, E_j \rangle \\ &= \langle \nabla_{E_m} E_l, E_j \rangle + \langle E_l, \nabla_{E_m} E_j \rangle = 0 \quad \text{on } \gamma, \quad l, j, m = 2, \dots, n.\end{aligned}$$

Moreover, since the E_A's are coordinate vectors for $A = 1, \dots, n$, we obtain

$$\begin{aligned}\partial_m g_{1j} &= E_m \langle E_1, E_j \rangle \\ &= \langle \nabla_{E_m} E_1, E_j \rangle + \langle E_1, \nabla_{E_m} E_j \rangle \\ &= \langle \nabla_{E_1} E_m, E_j \rangle + \langle E_1, \nabla_{E_m} E_j \rangle = 0 \quad \text{on } \gamma, \quad m, j = 2, \dots, n.\end{aligned}$$

Here we used the fact that $\nabla^N_{E_1} E_m = 0$ on γ, namely, that $\nabla_{E_1} E_m$ has zero normal components.

If $\mathbf{H} = H^m E_m$ is the curvature vector of γ (which is normal to the curve), then we have $\langle \nabla_{E_1} E_m, E_1 \rangle = -H^m$ on γ, so we easily deduce that

$$\partial_m g_{11} = E_m \langle E_1, E_1 \rangle = 2 \langle \nabla_{E_1} E_m, E_1 \rangle = -2H^m \quad \text{on } \gamma. \tag{2.1}$$

We can also prove that the components R_{1m1j} of the curvature tensor are given by

$$R_{1m1j} = -\frac{1}{2}\, \partial^2_{jm} g_{11} + H^m H^j. \tag{2.2}$$

Indeed, we have

$$\begin{aligned}-R_{1m1j} &= \langle R(E_1, E_j) E_1, E_m \rangle = \langle \nabla_{E_1} \nabla_{E_j} E_1, E_m \rangle - \langle \nabla_{E_j} \nabla_{E_1} E_1, E_m \rangle \\ &= \langle \nabla_{E_1} \nabla_{E_j} E_1, E_m \rangle - E_j \langle \nabla_{E_1} E_1, E_m \rangle - \langle \nabla_{E_1} E_1, \nabla_{E_j} E_m \rangle \\ &= \langle \nabla_{E_1} \nabla_{E_j} E_1, E_m \rangle - E_j \langle \nabla_{E_1} E_1, E_m \rangle \\ &= \langle \nabla_{E_1} \nabla_{E_j} E_1, E_m \rangle - E_j E_1 \langle E_1, E_m \rangle + E_j \langle E_1, \nabla_{E_1} E_m \rangle \\ &= \langle \nabla_{E_1} \nabla_{E_j} E_1, E_m \rangle + E_j \langle E_1, \nabla_{E_m} E_1 \rangle \\ &= E_1 \langle \nabla_{E_j} E_1, E_m \rangle - \langle \nabla_{E_j} E_1, \nabla_{E_1} E_m \rangle + \tfrac{1}{2} E_j E_m \langle E_1, E_1 \rangle \\ &= \tfrac{1}{2} \partial^2_{jm} g_{11} - \tfrac{1}{4} \partial_m g_{11}\, \partial_j g_{11},\end{aligned}$$

where here we have used the above properties and the fact that

$$\nabla_{E_j} E_1 = \nabla_{E_1} E_j = \tfrac{1}{2} \partial_j g_{11} E_1.$$

Using (2.1) and (2.2), the above discussion can be summarized in the following result:

LEMMA 2.1 *In the coordinates* $(\bar{s}, y)$, *for* y *close to* 0 *the metric coefficients satisfy*

$$g_{11}(y) = 1 - 2\sum_{m=2}^{n} H^m y_m + \frac{1}{2}\sum_{m,l=2}^{n}(H^m H^j - R_{1m1j}|_\gamma) y_m y_l + O(|y|^3),$$

$$g_{1j}(y) = \frac{1}{2}\sum_{m,l=2}^{n} \partial^2_{ml} g_{1j}\big|_\gamma y_m y_l + O(|y|^3),$$

$$g_{kj}(y) = \delta_{kj} + \frac{1}{2}\sum_{m,l=2}^{n} \partial^2_{ml} g_{kj}\big|_\gamma y_m y_l + O(|y|^3).$$

The second derivatives $\partial^2_{ml} g_{1j}$ and $\partial^2_{ml} g_{kj}$ can be expressed in terms of the curvature tensor and the curvature of γ through reasoning as for (2.2). However, for our purposes it is not necessary to have such a formula, so we leave the expansion of these coefficients in a generic form.

2.2 First and Second Variations of the Length Functional

We next recall the formulas for the variations of the length of a curve with respect to normal displacements. We start with a regular closed curve γ in M of length L, which we parametrize by arc length, using a parameter $\bar{s} \in [0, L]$. Then we consider a two-parameter family of closed curves $\gamma_{t_1,t_2} : [0, L] \to M$ for t_1, t_2 in a neighborhood of 0 in $\mathbb{R}$ such that $\gamma_{0,0} \equiv \gamma$. The length $L(t_1, t_2)$ of γ_{t_1,t_2} is given by

$$L(t_1, t_2) = \int_{\gamma_{t_1,t_2}} dl = \int_0^L \langle \dot{\gamma}_{t_1,t_2}, \dot{\gamma}_{t_1,t_2} \rangle^{\frac{1}{2}} d\bar{s},$$

where dl is the arc length parameter and $\dot{\gamma}_{t_1,t_2}$ stands for $d\gamma_{t_1,t_2}/d\bar{s}$. We also define the vector fields $\mathcal{V}$ and $\mathcal{W}$ along γ_{t_1,t_2} as $\mathcal{V} = \partial\gamma_{t_1,t_2}/\partial t_1$ and $\mathcal{W} = \partial\gamma_{t_1,t_2}/\partial t_2$. In the above coordinates, the vector fields $\mathcal{V}$ and $\mathcal{W}$ along γ can be written as

$$\mathcal{V} = \sum_{j=2}^{n} \mathcal{V}^j(\bar{s}) E_j, \qquad \mathcal{W} = \sum_{m=2}^{n} \mathcal{W}^m(\bar{s}) E_m.$$

Differentiating $L(t_1, t_2)$ with respect to t_1 we find

$$\frac{\partial L(t_1, t_2)}{\partial t_1} = -\int_0^L \frac{\langle \nabla_{\mathcal{V}} \dot{\gamma}_{t_1,t_2}, \dot{\gamma}_{t_1,t_2} \rangle}{\langle \dot{\gamma}_{t_1,t_2}, \dot{\gamma}_{t_1,t_2} \rangle^{\frac{1}{2}}} d\bar{s}. \tag{2.3}$$

Using (2.1), at $(t_1, t_2) = (0, 0)$ we have

$$\langle \nabla_{\mathcal{V}} \dot{\gamma}_{t_1,t_2}, \dot{\gamma}_{t_1,t_2} \rangle = -\mathcal{V}^m H^m;$$

therefore we can write the variation of the length at γ in the following way:

$$\frac{\partial L(t_1, t_2)}{\partial t_1}\bigg|_{(t_1,t_2)=(0,0)} = -\int_0^L \mathcal{V}^m H^m \, d\bar{s} = -\int_0^L \langle \mathcal{V}, \mathbf{H} \rangle d\bar{s}. \tag{2.4}$$

Using (2.3) we can evaluate the second variation of the length as

$$\frac{\partial^2 L(t_1,t_2)}{\partial t_1 \partial t_2} = \int_0^L \left[\frac{\langle \nabla_{\mathcal{W}} \dot\gamma_{t_1,t_2}, \nabla_{\mathcal{V}} \dot\gamma_{t_1,t_2}\rangle + \langle \dot\gamma_{t_1,t_2}, \nabla_{\mathcal{W}} \nabla_{\mathcal{V}} \dot\gamma_{t_1,t_2}\rangle}{\langle \dot\gamma_{t_1,t_2}, \dot\gamma_{t_1,t_2}\rangle^{\frac{1}{2}}} - \frac{\langle \dot\gamma_{t_1,t_2}, \nabla_{\mathcal{V}} \dot\gamma_{t_1,t_2}\rangle \langle \dot\gamma_{t_1,t_2}, \nabla_{\mathcal{W}} \dot\gamma_{t_1,t_2}\rangle}{\langle \dot\gamma_{t_1,t_2}, \dot\gamma_{t_1,t_2}\rangle^{\frac{3}{2}}} \right] d\bar{s},$$

so at $(t_1, t_2) = (0, 0)$ we find

$$\left.\frac{\partial^2 L(t_1,t_2)}{\partial t_1 \partial t_2}\right|_{(t_1,t_2)=(0,0)} = \int_0^L [\langle \nabla_{\mathcal{W}} \dot\gamma, \nabla_{\mathcal{V}} \dot\gamma\rangle + \langle \dot\gamma, \nabla_{\mathcal{W}} \nabla_{\mathcal{V}} \dot\gamma\rangle - \langle \dot\gamma, \nabla_{\mathcal{V}} \dot\gamma\rangle \langle \dot\gamma, \nabla_{\mathcal{W}} \dot\gamma\rangle]\, d\bar{s}.$$

By using the definition of the Riemann tensor and the fact that $\mathcal{V}$ and $\mathcal{W}$ are coordinate vector fields (so that $[\mathcal{V}, \mathcal{W}] = 0$), the last formula yields

$$\begin{aligned}
&\left.\frac{\partial^2 L(t_1,t_2)}{\partial t_1 \partial t_2}\right|_{(0,0)} \\
&= \int_0^L \big[\langle \nabla_{\dot\gamma} \mathcal{W}, \nabla_{\dot\gamma} \mathcal{V}\rangle + \langle \dot\gamma, \nabla_{\mathcal{W}} \nabla_{\dot\gamma} \mathcal{V}\rangle - \langle \dot\gamma, \nabla_{\dot\gamma} \mathcal{V}\rangle \langle \dot\gamma, \nabla_{\dot\gamma} \mathcal{W}\rangle\big]\, d\bar{s} \\
&= \int_0^L \big[\langle \nabla_{\dot\gamma} \mathcal{W}, \nabla_{\dot\gamma} \mathcal{V}\rangle + \langle R(\mathcal{W}, \dot\gamma)\mathcal{V}, \dot\gamma\rangle - \langle \dot\gamma, \nabla_{\dot\gamma} \mathcal{V}\rangle \langle \dot\gamma, \nabla_{\dot\gamma} \mathcal{W}\rangle\big]\, d\bar{s} \\
&\quad - \int_0^L \langle \nabla_{\dot\gamma}\dot\gamma, \nabla_{\mathcal{W}} \mathcal{V}\rangle d\bar{s}.
\end{aligned}$$

Here, we have used the fact that

$$g(\dot\gamma, \nabla_{\mathcal{W}} \nabla_{\dot\gamma} \mathcal{V}) = \langle R(\mathcal{W}, \dot\gamma)\mathcal{V}, \dot\gamma\rangle + \dot\gamma \langle \nabla_{\mathcal{W}} \mathcal{V}, \dot\gamma\rangle - \langle \nabla_{\dot\gamma}\dot\gamma, \nabla_{\mathcal{W}} \mathcal{V}\rangle$$

and $\int_0^L \dot\gamma \langle \nabla_{\mathcal{W}} \mathcal{V}, \dot\gamma\rangle d\bar{s} = 0$. Since $\nabla_{E_l} E_j = 0$ on γ for $l, j = 2, \dots, n$, we have

$$\nabla_{\mathcal{W}} \mathcal{V} = \sum_{j,m=2,\dots,n} \mathcal{W}^m \mathcal{V}^j \nabla_{E_m} E_j \quad \Longrightarrow \quad \int_0^L \langle \nabla_{\dot\gamma}\dot\gamma, \nabla_{\mathcal{W}} \mathcal{V}\rangle d\bar{s} = 0.$$

Moreover, recalling (2.1), we obtain

$$\nabla_{\dot\gamma} \mathcal{V} = \sum_{j=2}^n \dot{\mathcal{V}}^j E_j + \sum_{j=2}^n \mathcal{V}^j \nabla_{E_1} E_j = \sum_{j=2}^n \dot{\mathcal{V}}^j E_j - \sum_{j=2}^n H^j \mathcal{V}^j E_1.$$

This implies, at γ,

$$\langle \nabla_{\dot\gamma} \mathcal{W}, \nabla_{\dot\gamma} \mathcal{V}\rangle + \langle R(\mathcal{W}, \dot\gamma)\mathcal{V}, \dot\gamma\rangle - \langle \dot\gamma, \nabla_{\dot\gamma} V\rangle \langle \dot\gamma, \nabla_{\dot\gamma} \mathcal{W}\rangle = \sum_{j=2}^n \dot{\mathcal{V}}^j \dot{\mathcal{W}}^j - \sum_{j,l=2}^n R_{1j1l} \mathcal{V}^j \mathcal{W}^l.$$

In this way the second variation of the length at γ becomes

$$\left.\frac{\partial^2 L(t_1,t_2)}{\partial t_1 \partial t_2}\right|_{(t_1,t_2)=(0,0)} = \int_0^L \Big(\sum_j^n \dot{\mathcal{V}}^j \dot{\mathcal{W}}^j - \sum_{j,l=2}^n R_{1j1l} \mathcal{V}^j \mathcal{W}^l \Big) d\bar{s}. \tag{2.5}$$

2.3 Determining the Phase Factor

In this section we formally derive the asymptotic profile of the solutions to (NLS_ε) that concentrate near some curve γ, and we determine some necessary conditions satisfied by the limit curve. For doing this, using the coordinates $(\bar{s}, y)$ introduced in Subsection 2.1, we look for approximate solutions $\psi(\bar{s}, y)$ of (NLS_ε) making the ansatz

$$\psi(\bar{s}, y) = e^{-i\frac{f(\bar{s})}{\varepsilon}} h(\bar{s}) U\left(\frac{k(\bar{s}) y}{\varepsilon}\right), \quad \bar{s} \in [0, L], \quad y \in \mathbb{R}^{n-1},$$

where the function U is the unique radial solution (see [12, 22, 30, 51]) of the problem

$$\begin{cases} -\Delta U + U = U^p & \text{in } \mathbb{R}^{n-1}, \\ U(y) \to 0 & \text{as } |y| \to +\infty, \\ U > 0 & \text{in } \mathbb{R}^{n-1}, \end{cases} \tag{2.6}$$

and where the functions f, h, and k are periodic on $[0, L]$ and have to be determined. With some easy computations we obtain

$$\begin{aligned}
\frac{\partial \psi}{\partial \bar{s}} &= -\frac{i f'(\bar{s})}{\varepsilon} h(\bar{s}) U\left(\frac{k(\bar{s}) y}{\varepsilon}\right) e^{-i\frac{f(\bar{s})}{\varepsilon}} + e^{-i\frac{f(\bar{s})}{\varepsilon}} h'(\bar{s}) U\left(\frac{k(\bar{s}) y}{\varepsilon}\right) \\
&\quad + e^{-i\frac{f(\bar{s})}{\varepsilon}} h(\bar{s}) k'(\bar{s}) \nabla_y U\left(\frac{k(\bar{s}) y}{\varepsilon}\right) \cdot \frac{y}{\varepsilon}, \\
\frac{\partial^2 \psi}{\partial \bar{s}^2} &= \Bigg[-i\frac{f''(\bar{s})}{\varepsilon} h(\bar{s}) U\left(\frac{k(\bar{s}) y}{\varepsilon}\right) - 2i\frac{f'(\bar{s})}{\varepsilon} h'(\bar{s}) U\left(\frac{k(\bar{s}) y}{\varepsilon}\right) \\
&\quad - 2i\frac{f'(\bar{s})}{\varepsilon} h(\bar{s}) k'(\bar{s}) \nabla_y U\left(\frac{k(\bar{s}) y}{\varepsilon}\right) \cdot \frac{y}{\varepsilon} - \frac{(f'(\bar{s}))^2}{\varepsilon^2} h(\bar{s}) U\left(\frac{k(\bar{s}) y}{\varepsilon}\right) \\
&\quad + 2h'(\bar{s}) k'(\bar{s}) \nabla U\left(\frac{k(\bar{s}) y}{\varepsilon}\right) \cdot \frac{y}{\varepsilon} + h(\bar{s}) k''(\bar{s}) \nabla_y U\left(\frac{k(\bar{s}) y}{\varepsilon}\right) \cdot \frac{y}{\varepsilon} \\
&\quad + h(\bar{s}) (k'(\bar{s}))^2 \nabla_y^2 U\left(\frac{k(\bar{s}) y}{\varepsilon}\right) \left[\frac{y}{\varepsilon}, \frac{y}{\varepsilon}\right] + h''(\bar{s}) U\left(\frac{k(\bar{s}) y}{\varepsilon}\right) \Bigg] e^{-i\frac{f(\bar{s})}{\varepsilon}},
\end{aligned}$$

and also

$$\Delta_y \psi(\bar{s}, y) = \frac{(k(\bar{s}))^2}{\varepsilon^2} \Delta_y U\left(\frac{k(\bar{s}) y}{\varepsilon}\right) e^{-i\frac{f(\bar{s})}{\varepsilon}} h(\bar{s}).$$

Since U decays to 0 at infinity (exponentially indeed, by the results in [22]), and since the function ψ is scaled of order ε near the curve γ, in a first approximation we can assume the metric g of M to be flat in the coordinates $(\bar{s}, y)$; see the expansions in Lemma 2.1.

We look now at the leading terms in (NLS$_\varepsilon$), which are of order 1. Since $-\Delta_g \psi$ is multiplied by ε^2, we have to focus on the terms of order $1/\varepsilon^2$ in the Laplacian of ψ. In the above expressions of $\partial\psi/\partial\bar{s}$, $\partial^2\psi/\partial\bar{s}^2$, and $\Delta_y\psi$, we have that the function U and its derivatives are of order 1 when $|y| = O(\varepsilon)$; therefore when the variables y appear as factors in these expressions, we consider them to be of order ε. For example, $\nabla^2 U(\frac{k(\bar{s})y}{\varepsilon})[\frac{y}{\varepsilon}, \frac{y}{\varepsilon}]$ will be regarded as a term of order 1.

With these criteria, using the above computations and assumptions, and if we impose that the leading terms in (NLS$_\varepsilon$) vanish, we obtain

$$-k^2(\bar{s})h(\bar{s})\Delta_y U\left(\frac{k(\bar{s})y}{\varepsilon}\right) + h(\bar{s})\left[V(\bar{s}) + (f'(\bar{s}))^2\right] U\left(\frac{k(\bar{s})y}{\varepsilon}\right) = h(\bar{s})^p U\left(\frac{k(\bar{s})y}{\varepsilon}\right)^p.$$

From (2.6), we have the two relations

$$k^2(\bar{s}) = h(\bar{s})^{p-1} \quad \text{and} \quad [V(\bar{s}) + (f'(\bar{s}))^2] = k(\bar{s})^2 = h(\bar{s})^{p-1}. \tag{2.7}$$

We next obtain an equation for f, which is derived looking at the next-order expansion of (NLS$_\varepsilon$). The next coefficient arises from the terms of order $\frac{1}{\varepsilon}$ in $-\Delta_g \psi$, which are given by

$$i\left[\frac{f''(\bar{s})}{\varepsilon} h(\bar{s})U\left(\frac{k(\bar{s})y}{\varepsilon}\right) + 2\frac{f'(\bar{s})}{\varepsilon} h'(\bar{s})U\left(\frac{k(\bar{s})y}{\varepsilon}\right) + 2\frac{f'(\bar{s})}{\varepsilon} h(\bar{s})k'(\bar{s})\nabla_y U\left(\frac{k(\bar{s})y}{\varepsilon}\right)\cdot\frac{y}{\varepsilon}\right] e^{-i\frac{f(\bar{s})}{\varepsilon}}.$$

Multiplying this expression by $U(\frac{k(\bar{s})y}{\varepsilon})$, integrating in $y \in \mathbb{R}^{n-1}$, and if we impose that this integral vanishes as well, we get

$$0 = f''(\bar{s})h(\bar{s}) \int_{\mathbb{R}^{n-1}} U^2\left(\frac{k(\bar{s})y}{\varepsilon}\right) dy + 2h'(\bar{s})f'(\bar{s}) \int_{\mathbb{R}^{n-1}} U^2\left(\frac{k(\bar{s})y}{\varepsilon}\right) dy + 2f'(\bar{s})h(\bar{s})k'(\bar{s}) \int_{\mathbb{R}^{n-1}} U\left(\frac{k(\bar{s})y}{\varepsilon}\right)\nabla_y U\left(\frac{k(\bar{s})y}{\varepsilon}\right)\cdot\frac{y}{\varepsilon}\, dy.$$

Integrating by parts and reasoning as for the usual Pohozaev's identity, we obtain that f must satisfy

$$f''(\bar{s})h(\bar{s}) + 2f'(\bar{s})h'(\bar{s}) - (n-1)f'(\bar{s})h(\bar{s})\frac{k'(\bar{s})}{k(\bar{s})} = 0.$$

This is solvable in $f'(\bar{s})$ and gives, for an arbitrary constant $\mathcal{A}$,

$$f'(\bar{s}) = \mathcal{A}k(\bar{s})^{n-1}h(\bar{s})^{-2} = \mathcal{A}h(\bar{s})^{\frac{(n-1)(p-1)}{2}-2}, \tag{2.8}$$

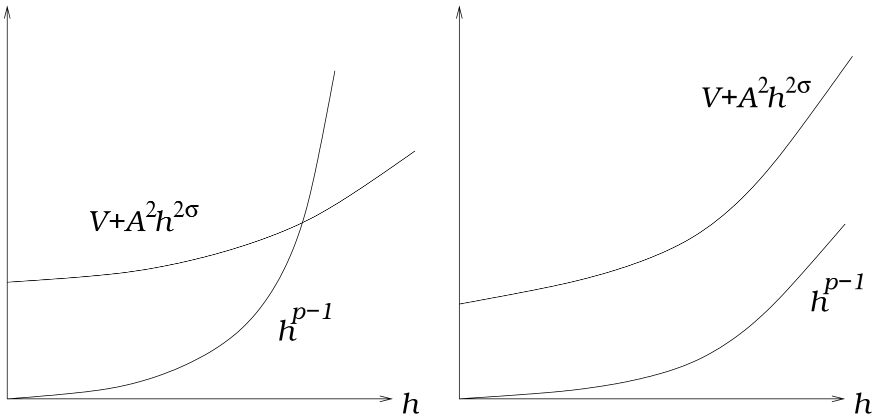

FIGURE 2.1. The graphs of $V + \mathcal{A}^2 h^{2\sigma}$ and h^{p-1} for $p < \frac{n+2}{n-2}$ and for $p = \frac{n+2}{n-2}$ with $\mathcal{A} < 1$.

where we have used equation (2.7) for k. Now we can solve the equation for $h(\bar{s})$ depending on the potential $V(\bar{s})$ and the above constant $\mathcal{A}$. In fact, we get that $h(\bar{s})$ should solve

$$V(\bar{s}) + \mathcal{A}^2 h(\bar{s})^{2\sigma} := V(\bar{s}) + \mathcal{A}^2 h(\bar{s})^{(n-1)(p-1)-4} = h(\bar{s})^{p-1}, \tag{2.9}$$

where we have set

$$\sigma = \frac{(n-1)(p-1)}{2} - 2. \tag{2.10}$$

Remark 2.2. We notice that, assuming $\mathcal{A}$ to be small enough (depending on V and p), the above equation is always solvable in $h(\bar{s})$. More precisely, when $p < \frac{n+2}{n-2}$ (and hence when $2\sigma < p - 1$), the solution is also unique. For $p \geq \frac{n+2}{n-2}$ there might be a second solution. In this case, we just consider the smallest one, which stays uniformly bounded (both from above and below) when $\mathcal{A}$ is small enough; see Figures 2.1 and 2.2.

Remark 2.3. In the above expansions, considering the terms of order ε, as already noticed, we considered the metric g to be flat near the curve γ, and we tacitly assumed the potential V to depend only on the variable $\bar{s}$. Indeed, expanding the Laplace-Beltrami operator and the potential V and taking the variables y into account, we obtain an extra term of order ε that does not affect our computations since it turns out to be odd in y, so it vanishes once multiplied by $U(k(\bar{s})y/\varepsilon)$ and integrated over $\mathbb{R}^{n-1}$. For more details, we refer to Section 3, where precise estimates are worked out (in a system of coordinates scaled in ε).

2.4 The Euler Equation

Using the heuristic considerations of the previous subsection, we now compute the energy of an approximate solution ψ concentrated near a closed curve γ, and

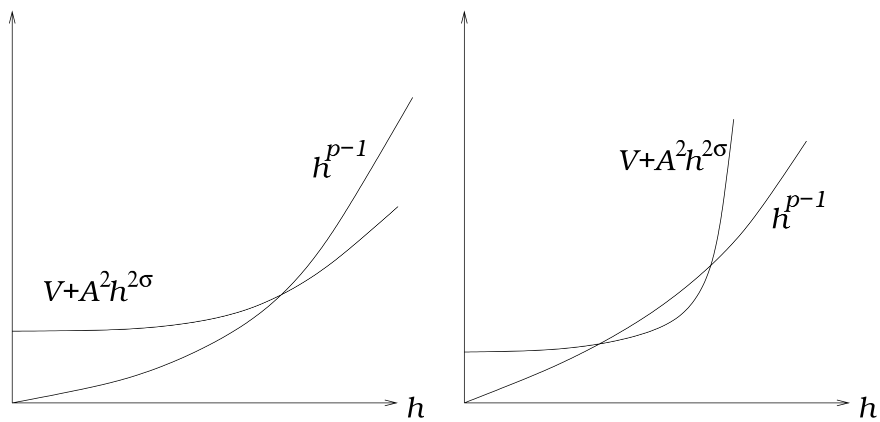

FIGURE 2.2. The graphs of $V + \mathcal{A}^2 h^{2\sigma}$ and h^{p-1} for $p = \frac{n+2}{n-2}$ with $\mathcal{A} \geq 1$ and for $p > \frac{n+2}{n-2}$ with $\mathcal{A}$ small.

then find the γ's for which this energy is stationary. We let $\psi_{\gamma,\mathcal{A}}$ denote the function constructed in Subsection 2.3. In order for the function $\psi_{\gamma,\mathcal{A}}$ to be globally well defined, we need to impose one more restriction, namely that $\psi_{\gamma,\mathcal{A}}$ is periodic in $\bar{s}$ with period L. This is equivalent to requiring that $\int_0^L f'(\bar{s}) d\bar{s}$ be an integer multiple of $2\pi\varepsilon$, since we have the phase factor $e^{-i(f(\bar{s})/\varepsilon)}$ in the expression of $\psi_{\gamma,\mathcal{A}}$. From (2.8), we then find that $\int_0^L h(\bar{s})^\sigma \, d\bar{s}$ is also an integer multiple of $2\pi\varepsilon$.

Multiplying (NLS$_\varepsilon$) by $\psi_{\gamma,\mathcal{A}}$ and integrating by parts, from the fact that $\psi_{\gamma,\mathcal{A}}$ is an approximate solution we find

$$E_\varepsilon(\psi_{\gamma,\mathcal{A}}) = \frac{1}{2} \int_M (\varepsilon^2 |\nabla_g \psi_{\gamma,\mathcal{A}}|^2 + V(x)|\psi_{\gamma,\mathcal{A}}|^2) dV_g - \frac{1}{p+1} \int_M |\psi_{\gamma,\mathcal{A}}|^{p+1}$$
$$\simeq \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_M |\psi_{\gamma,\mathcal{A}}|^{p+1} \, dV_g.$$

Since $\psi_{\gamma,\mathcal{A}}$ is highly concentrated near γ, using the coordinates $(\bar{s}, y)$ introduced in 2.1, we have that

$$\int_M |\psi_{\gamma,\mathcal{A}}|^{p+1} \, dV_g \simeq \int_0^L d\bar{s} \int_{\mathbb{R}^{n-1}} h(\bar{s})^{p+1} \left| U\left(\frac{k(\bar{s})y}{\varepsilon}\right)\right|^{p+1} dy.$$

Using a change of variables, the last two formulas, and (2.7), we find that

$$E_\varepsilon(\psi_{\gamma,\mathcal{A}}) \simeq \bar{C}_0 \varepsilon^{n-1} \int_\gamma h(\bar{s})^\theta \, d\bar{s}, \tag{2.11}$$

where

$$\bar{C}_0 = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int\limits_{\mathbb{R}^{n-1}} |U(y)|^{p+1}\, dy,$$

and where we have set

$$\theta = p + 1 - \frac{1}{2}(p-1)(n-1) = p - \sigma - 1. \tag{2.12}$$

Consider now a one-parameter family of closed curves $\gamma_t : [0, L] \to M$, where t belongs to a neighborhood of 0 in $\mathbb{R}$ and where $\gamma_0 \equiv \gamma$. We compute next the approximate value of the derivative in t of the corresponding energy defined by (2.11).

As in Subsection 2.2 we let $\mathcal{V}_t$ denote the vector field $\mathcal{V}_t(\bar{s}) = \frac{\partial \gamma_t}{\partial t}(\bar{s})$, and we assume that $\mathcal{V} := \mathcal{V}_0$ is normal to γ. For any t near 0, we let $k_t(\bar{s})$, $h_t(\bar{s})$, and $f_t(\bar{s})$ be defined by (2.7), replacing γ by γ_t and $V(\bar{s})$ by $V_t(\bar{s}) := V(\gamma_t(\bar{s}))$. Since we require periodicity of each curve γ_t in the variable $\bar{s}$, we also allow the constant $\mathcal{A}$ given in (2.8) to depend on t. Denoting this by $\mathcal{A}_t$, by the above considerations we choose $\mathcal{A}_t$ so that the following condition holds for every value of t:

$$\int_0^L \mathcal{A}_t h_t(\bar{s})^\sigma \, d\bar{s} = \int_0^L f_t'(\bar{s}) d\bar{s} = \text{const.} \tag{2.13}$$

Below, we let $\mathcal{A}_t' = \frac{d}{dt}\mathcal{A}_t$, and we will consider $h_t(\bar{s})$ to be a function of $\mathcal{A}_t$ while $V_t(\bar{s})$ is as implicitly defined in (2.9). From (2.4),

$$\left.\frac{\partial V_t(\bar{s})}{\partial t}\right|_{t=0} = \langle \nabla^N V(\bar{s}), \mathcal{V}(\bar{s})\rangle,$$

and differentiating (2.13) with respect to t at $t = 0$, we get

$$\int_0^L \mathcal{A}\sigma h^{\sigma-1} \frac{\partial h}{\partial V} \langle \nabla^N V, \mathcal{V}\rangle d\bar{s} - \mathcal{A} \int_0^L h^\sigma \langle \mathcal{V}, \mathbf{H}\rangle d\bar{s} + \mathcal{A}\mathcal{A}'\sigma \int_0^L h^{\sigma-1} \frac{\partial h}{\partial \mathcal{A}} \, d\bar{s} + \mathcal{A}' \int_0^L h^\sigma \, d\bar{s} = 0,$$

where we have set $\mathcal{A}' = \mathcal{A}_0'$ and where $\nabla^N V$ stands for the component of ∇V normal to γ. From this formula we obtain the following expression for $\mathcal{A}'$:

$$\mathcal{A}' = -\mathcal{A} \frac{\int_0^L \left(\sigma h^{\sigma-1} \frac{\partial h}{\partial V} \langle \nabla^N V, \mathcal{V}\rangle - h^\sigma \langle \mathcal{V}, \mathbf{H}\rangle\right) d\bar{s}}{\int_0^L \left(\mathcal{A}\sigma h^{\sigma-1} \frac{\partial h}{\partial \mathcal{A}} + h^\sigma\right) d\bar{s}}. \tag{2.14}$$

Similarly, computing the derivative of the (approximate) energy with respect to t, we find

$$\left.\frac{dE_\varepsilon(u_{\gamma_t, \mathcal{A}_t})}{dt}\right|_{t=0} = \int_0^L \left(\theta h^{\theta-1} \frac{\partial h}{\partial V} \langle \nabla^N V, \mathcal{V}\rangle - h^\theta \langle \mathcal{V}, \mathbf{H}\rangle + \theta \mathcal{A}' h^{\theta-1} \frac{\partial h}{\partial \mathcal{A}}\right) d\bar{s}.$$

Using (2.14), we deduce that the variation is given by

$$\left.\frac{dE_\varepsilon(u_{\gamma_t,\mathcal{A}_t})}{dt}\right|_{t=0} = \int_0^L \frac{\partial h}{\partial V}\langle \nabla^N V, \mathcal{V}\rangle \left[\theta h^{\theta-1} - \frac{\mathcal{A}\sigma h^{\sigma-1}\int_0^L \theta h^{\theta-1}\frac{\partial h}{\partial \mathcal{A}}\,d\bar{s}}{\int_0^L \left(\mathcal{A}\sigma h^{\sigma-1}\frac{\partial h}{\partial \mathcal{A}} + h^\sigma\right)d\bar{s}}\right]d\bar{s} - \int_0^L \langle \mathcal{V}, \mathbf{H}\rangle \left[h^\theta - \frac{\mathcal{A}\, h^\sigma \int_0^L \theta h^{\theta-1}\frac{\partial h}{\partial \mathcal{A}}\,d\bar{s}}{\int_0^L \left(\mathcal{A}\sigma h^{\sigma-1}\frac{\partial h}{\partial \mathcal{A}} + h^\sigma\right)d\bar{s}}\right]d\bar{s}.$$

Differentiating (2.9) with respect to $\mathcal{A}$ and V, we get

$$\frac{\partial h}{\partial \mathcal{A}} = \frac{2\mathcal{A}h^{2\sigma}}{(p-1)h^{p-2} - 2\sigma\mathcal{A}^2 h^{2\sigma-1}} = 2\mathcal{A}h^{2\sigma}\frac{\partial h}{\partial V}, \tag{2.15}$$

so it follows that

$$\mathcal{A}\sigma h^{\sigma-1}\frac{\partial h}{\partial \mathcal{A}} + h^\sigma = \frac{(p-1)h^{p-1}}{(p-1)h^\theta - 2\sigma\mathcal{A}^2 h^\sigma}. \tag{2.16}$$

Similarly, since $\theta = p - \sigma - 1$ (see (2.10) and (2.12)), we deduce that

$$h^{\theta-1}\frac{\partial h}{\partial \mathcal{A}} = \frac{2\mathcal{A}h^{p+\sigma-2}}{(p-1)h^{p-2} - 2\sigma\mathcal{A}^2 h^{2\sigma-1}}.$$

Therefore we also find

$$\frac{\theta h^{\theta-1}\frac{\partial h}{\partial \mathcal{A}}}{\mathcal{A}\sigma h^{\sigma-1}\frac{\partial h}{\partial \mathcal{A}} + h^\sigma} = \frac{2\mathcal{A}\theta}{p-1}. \tag{2.17}$$

Hence from the last formulas the variation of the energy becomes

$$\left.\frac{dE_\varepsilon(\psi_{\gamma_t,\mathcal{A}_t})}{dt}\right|_{t=0} = \int_0^L \langle \nabla^N V, \mathcal{V}\rangle \frac{\partial h}{\partial V}\left[\theta h^{\theta-1} - \frac{2\mathcal{A}^2\sigma\theta}{p-1}h^{\sigma-1}\right]d\bar{s} - \int_0^L \langle \mathcal{V}, \mathbf{H}\rangle \left[h^\theta - \frac{2\mathcal{A}^2\theta}{p-1}h^\sigma\right]d\bar{s}. \tag{2.18}$$

Also, from the second equality in (2.15), dividing by h^σ, multiplying by $\frac{\theta}{p-1}$, and using the identity $p - \sigma - 2 = \theta - 1$, we obtain

$$h^{-\sigma}\frac{\theta}{p-1} + 2\sigma\mathcal{A}^2 h^{\sigma-1}\frac{\partial h}{\partial V}\frac{\theta}{p-1} = \theta h^{p-\sigma-2}\frac{\partial h}{\partial V} = \theta h^{\theta-1}\frac{\partial h}{\partial V}.$$

Using (2.18) and the last formula, we get the following simplified expression:

$$\left.\frac{dE_\varepsilon(\psi_{\gamma_t,\mathcal{A}_t})}{dt}\right|_{t=0} = \int_0^L \frac{\theta}{p-1}h^{-\sigma}\left[\langle \nabla^N V, \mathcal{V}\rangle - \langle \mathcal{V}, \mathbf{H}\rangle\left(\frac{p-1}{\theta}h^{p-1} - 2\mathcal{A}^2 h^{2\sigma}\right)\right]d\bar{s}.$$

Therefore the stationarity condition for the energy (under the constraint (2.13)) becomes $\langle \nabla^N V, \mathcal{V}\rangle = \langle \mathcal{V}, \mathbf{H}\rangle(\frac{p-1}{\theta} h^{p-1} - 2\mathcal{A}^2 h^{2\sigma})$ for every normal section $\mathcal{V}$, namely,

$$\nabla^N V = \mathbf{H}\left(\frac{p-1}{\theta}\, h^{p-1} - 2\mathcal{A}^2 h^{2\sigma}\right). \tag{2.19}$$

We will see that this formula will be crucial to finding approximate solutions.

Remark 2.4. By (2.16), we have that

$$\frac{\partial}{\partial \mathcal{A}}(\mathcal{A}h^{\sigma}) = \frac{(p-1)h^{p-1}}{(p-1)h^{\theta} - 2\sigma\mathcal{A}^2 h^{\sigma}}.$$

If $\mathcal{A}$ is sufficiently small (depending on V and p), then $\frac{\partial}{\partial \mathcal{A}}(\mathcal{A}h^{\sigma}) > 0$. This will be used in the last section where, for a fixed ε, we will adjust the value of the constant $\mathcal{A}$ for obtaining periodicity of the function f.

2.5 Second Variation and Nondegeneracy Condition

We next evaluate the second variation of the Euler functional. As in Subsection 2.2 we consider a two-parameter family of closed curves γ_{t_1,t_2}, where t_1 and t_2 are two real numbers belonging to a small neighborhood of 0 in $\mathbb{R}$, and where $\gamma_{0,0} = \gamma$. As before, we require the constraint (2.13) along the whole two-dimensional family of curves, and we assume that the functions f, h, and k and the constant $\mathcal{A}$ depend on t_1 and t_2, and we will use the notation $\mathcal{A}_{t_1,t_2}$, etc. Keeping this in mind, we define the two vector fields

$$\mathcal{V}_{t_1,t_2} = \frac{\partial \gamma_{t_1,t_2}}{\partial t_1}, \qquad \mathcal{W}_{t_1,t_2} = \frac{\partial \gamma_{t_1,t_2}}{\partial t_2},$$

and we can assume that $\mathcal{V} := \mathcal{V}_{0,0}$ and $\mathcal{W} := \mathcal{W}_{0,0}$ are normal to the initial curve γ.

With some computations, which are worked out in [36, sec. 5.2], we find that, at $(t_1, t_2) = (0, 0)$,

$$\begin{aligned} \frac{\partial^2 E_\varepsilon(u_{\psi_{t_1,t_2},\mathcal{A}_{t_1,t_2}})}{\partial t_1 \partial t_2} &= \int_0^L \left[h^{\theta} - \frac{2\mathcal{A}^2\theta}{p-1}\, h^{\sigma}\right]\Big[\sum_j \dot{\mathcal{V}}^j \dot{\mathcal{W}}^j - \sum_{j,m} R_{1j1m}\mathcal{V}^j \mathcal{W}^m\Big] d\bar{s} \\ &\quad + \frac{\theta}{p-1}\int_0^L \big\{((\nabla^N)^2 V)[\mathcal{V}, \mathcal{W}] - \langle \nabla^N V, \mathcal{V}\rangle\langle \mathbf{H}, \mathcal{W}\rangle \\ &\qquad - \langle \nabla^N V, \mathcal{W}\rangle\langle \mathbf{H}, \mathcal{V}\rangle\big\} h^{-\sigma}\, d\bar{s} - \end{aligned} \tag{2.20}$$

$$- \frac{\sigma\theta}{p-1} \int_0^L h^{-\sigma-1} \frac{\partial h}{\partial V} \langle \nabla^N V, \mathcal{V} \rangle \langle \nabla^N V, \mathcal{W} \rangle d\bar{s}$$
$$+ \mathcal{A}_1' \mathcal{A}_2' \frac{2\theta}{p-1} \int_0^L \left(\mathcal{A}\sigma h^{\sigma-1} \frac{\partial h}{\partial \mathcal{A}} + h^\sigma \right) d\bar{s}.$$

Here

$$\mathcal{A}_l' = \frac{\partial \mathcal{A}_{ot_1,t_2}}{\partial t_l}\bigg|_{(t_1,t_2)=(0,0)}, \quad \dot{\mathcal{V}}^j = \frac{d\mathcal{V}^j}{d\bar{s}}, \quad \text{and} \quad \dot{\mathcal{W}}^j = \frac{d\mathcal{W}^j}{d\bar{s}},$$

where the $\mathcal{V}^j$ and $\mathcal{W}^j$ are the components of $\mathcal{V}$ and $\mathcal{W}$ with respect to the basis $(E_j)_j$ introduced in Subsection 2.1.

Integrating by parts and using (2.14), from the last formula we derive that the nondegeneracy condition is equivalent to the invertibility of the linear operator $\mathfrak{J} : \chi(N\gamma) \to \chi(N\gamma)$ (from the family of smooth sections of the normal bundle to γ into itself) whose components are defined by

$$(2.21) \quad \begin{aligned} (\mathfrak{J}\mathcal{V})^m = &- \left(h^\theta - \frac{2\mathcal{A}^2\theta}{p-1} h^\sigma \right) \Big[\ddot{\mathcal{V}}^m + \sum_j R_{1j1m} \mathcal{V}^j \Big] \\ &- \theta \left(h^{\theta-1} - \frac{2\mathcal{A}^2\sigma}{p-1} h^{\sigma-1} \right) h' \dot{\mathcal{V}}^m \\ &+ \frac{\theta}{p-1} h^{-\sigma} \big\{ ((\nabla^N)^2 V)(\mathcal{V}, E_m) - H^m \langle \nabla^N V, \mathcal{V} \rangle \\ &\qquad - \langle \mathbf{H}, \mathcal{V} \rangle \langle \nabla^N V, E_m \rangle \big\} \\ &- \frac{\sigma\theta}{p-1} h^{-(\sigma+1)} \frac{\partial h}{\partial V} \langle \nabla^N V, \mathcal{V} \rangle \langle \nabla^N V, E_m \rangle \\ &- \frac{2\theta}{p-1} \mathcal{A}\mathcal{A}_1' \left(\sigma h^{\sigma-1} \frac{\partial h}{\partial V} \langle \nabla^N V, E_m \rangle - h^\sigma H^m \right), \end{aligned}$$

where $h' = \frac{dh(\bar{s})}{d\bar{s}}$.

Using (2.19) and some other elementary computations (see subsection 2.5 in [36]), we also find

$$(2.22) \quad \begin{aligned} (\mathfrak{J}\mathcal{V})^m = &-\left(h^\theta - \frac{2\mathcal{A}^2\theta}{p-1} h^\sigma \right) \ddot{\mathcal{V}}^m - \theta \left(h^{\theta-1} - \frac{2\mathcal{A}^2\sigma}{p-1} h^{\sigma-1} \right) h' \dot{\mathcal{V}}^m \\ &+ \frac{\theta}{p-1} h^{-\sigma} ((\nabla^N)^2 V)[\mathcal{V}, E_m] + \frac{1}{2} \left(h^\theta - \frac{2\mathcal{A}^2\theta}{p-1} h^\sigma \right) \Big(\sum_j (\partial^2_{jm} g_{11}) \mathcal{V}^j \Big) \\ &- 2\mathcal{A}\mathcal{A}_1' \frac{(\theta-\sigma) h^{p-1}}{[(p-1)h^\theta - 2\sigma\mathcal{A}^2 h^\sigma]} H^m \\ &+ H^m \langle \mathbf{H}, \mathcal{V} \rangle \left[\frac{-(p-1)(3+\frac{\sigma}{\theta}) h^{2\theta} - \frac{16\sigma\theta\mathcal{A}^4}{p-1} h^{2\sigma} + 2\mathcal{A}^2(5\sigma+3\theta) h^{\theta+\sigma}}{(p-1)h^\theta - 2\mathcal{A}^2\sigma h^\sigma} \right]. \end{aligned}$$

The latter expression of $\mathfrak{J}\mathcal{V}$ is going to be useful later on. We summarize the results of this section in the following proposition.

PROPOSITION 2.5 *Consider the functional on curves* $\int_\gamma h^\theta(\bar{s})d\bar{s}$ *restricted to the set* Γ *in* (1.13). *Then the stationarity condition is* (2.19)*, and the nondegeneracy of a critical point is equivalent to the invertibility of the operator* $\mathfrak{J}$ *in* (2.22).

3 Some Preliminary Expansions

In this section we find a family of approximate solutions to the scaled equation (1.16). We consider a simple closed curve γ that is stationary within the class Γ, namely satisfying (2.19). First, we introduce some convenient coordinates near the scaled curve $\gamma_\varepsilon = \frac{1}{\varepsilon}\gamma$, expanding the Laplace-Beltrami operator with respect to the scaled metric in powers of ε. Then, using these expansions, we construct the approximate solutions formally solving (1.16) up to order ε.

3.1 Choice of Coordinates in M_ε and Expansion of the Metric Coefficients

Using the coordinates $(\bar{s}, y)$ of Section 2 defined near γ, for some smooth normal section $\Phi(\bar{s})$ in $N\gamma$, we define the following new coordinates (s, z) (here and below we use the notation $\bar{s} = \varepsilon s$) near $\frac{1}{\varepsilon}\gamma$

$$z = y - \Phi(\varepsilon s), \quad z \in \mathbb{R}^{n-1}. \tag{3.1}$$

In this choice we are motivated by the fact that in general we allow the approximate solutions to be *tilted* normally to γ_ε, where the tilting Φ depends (slowly) on the variable s; this allows some extra flexibility in the construction, as in [19, 34, 39]. As we will see, the choice of Φ is irrelevant for solving (1.16) up to order ε; on the other hand, the nondegeneracy assumption will be necessary to guarantee solvability of the equation up to higher orders.

We denote by $\tilde{g}_{AB}$ the metric coefficients in the new coordinates (s, z). Since $y = z + \Phi(\varepsilon s)$, it follows that

$$\tilde{g}_{CD} = \sum_{AB} g_{AB}\left(\frac{\partial y_A}{\partial z_C}\right)\left(\frac{\partial y_B}{\partial z_D}\right).$$

Explicitly, we then find

$$\tilde{g}_{11} = g_{11}\big|_{z+\Phi} + 2\varepsilon \sum_j \Phi'_j g_{1j}\big|_{z+\Phi} + \varepsilon^2 \sum_{j,m} \Phi'_j(\varepsilon s)\Phi'_m(\varepsilon s) g_{jm}\big|_{z+\Phi},$$

$$\tilde{g}_{1j} = g_{1j}\big|_{z+\Phi} + \varepsilon \sum_m \Phi'_m(\varepsilon s) g_{jm}\big|_{z+\Phi}, \quad \tilde{g}_{jm} = g_{jm}|_{z+\Phi}.$$

At this point, it is convenient to introduce some notation. For a positive integer q, we denote by $R_q(z)$, $R_q(z,\Phi)$, and $R_q(z,\Phi,\Phi')$ error terms that satisfy, respectively, the following bounds, for some positive constants C and d:

$$|R_q(z)| \le C\varepsilon^q(1+|z|^d),$$

$$\begin{cases} |R_q(z,\Phi)| \le C\varepsilon^q(1+|z|^d), \\ |R_q(z,\Phi) - R_q(z,\tilde{\Phi})| \le C\varepsilon^q(1+|z|^d)[\,|\Phi-\tilde{\Phi}|], \end{cases}$$

and

$$\begin{cases} |R_q(z,\Phi,\Phi')| \le C\varepsilon^q(1+|z|^d), \\ |R_q(z,\Phi,\Phi') - R_q(z,\tilde{\Phi},\tilde{\Phi}')| \le C\varepsilon^q(1+|z|^d)[|\Phi-\tilde{\Phi}|+|\Phi'-\tilde{\Phi}'|]. \end{cases}$$

We also introduce error terms involving second derivatives of Φ, $R_q(z,\Phi,\Phi',\Phi'')$, which satisfy

$$|R_q(z,\Phi,\Phi',\Phi'')| \le C\varepsilon^q(1+|z|^d) + C\varepsilon^{q+1}(1+|z|^d)|\Phi''|$$

and

$$\begin{aligned} &|R_q(z,\Phi,\Phi',\Phi'') - R_q(z,\tilde{\Phi},\tilde{\Phi}',\tilde{\Phi}'')| \\ &\quad \le C\varepsilon^q(1+|z|^d)[|\Phi-\tilde{\Phi}|+|\Phi'-\tilde{\Phi}'|](1+\varepsilon(|\Phi''|+|\tilde{\Phi}''|)) \\ &\qquad + C\varepsilon^{q+1}(1+|z|^d)|\Phi''-\tilde{\Phi}''|. \end{aligned}$$

Using the expansion of the metric coefficients g_{AB} in Lemma 2.1 and this notation, we then obtain

$$\begin{aligned} \tilde{g}_{11} &= 1 - 2\varepsilon \sum_{m=2}^{n} H^m(z_m+\Phi_m) \\ &\quad + \frac{1}{2}\varepsilon^2 \sum_{m,l=2}^{n} \partial^2_{ml} g_{11}\big|_\gamma (z_m+\Phi_m)(z_l+\Phi_l) \\ &\quad + \varepsilon^2|\Phi'|^2 + R_3(z,\Phi,\Phi'), \end{aligned} \tag{3.2}$$

$$\begin{aligned} \tilde{g}_{1j} &= \varepsilon\Phi'_j + \frac{1}{2}\varepsilon^2 \sum_{m,l=2}^{n} \partial^2_{ml} g_{1j}\big|_\gamma (z_m+\Phi_m)(z_l+\Phi_l) + R_3(z,\Phi,\Phi'), \\ \tilde{g}_{kj} &= \delta_{kj} + \frac{1}{2}\varepsilon^2 \sum_{m,l=2}^{n} \partial^2_{ml} g_{kj}\big|_\gamma (z_m+\Phi_m)(z_l+\Phi_l) + R_3(z,\Phi,\Phi'). \end{aligned} \tag{3.3}$$

Next we compute the inverse metric coefficients. Recall that, given a formal expansion of a matrix as $M = 1 + \varepsilon A + \varepsilon^2 B$, we have

$$M^{-1} = 1 - \varepsilon A + \varepsilon^2 A^2 - \varepsilon^2 B.$$

In our specific case the matrix A is the following:

$$A = \begin{pmatrix} -2\sum_{m=2}^{n} H^m(z_m+\Phi_m) & \Phi'_j \\ \Phi'_j & 0 \end{pmatrix}, \tag{3.4}$$

and the elements of its square are given by

$$(A^2)_{11} = 4\Big(\sum_{m=2}^{n} H^m(z_m+\Phi_m)\Big)^2 + \sum_j (\Phi'_j)^2,$$

$$(A^2)_{1j} = -2\Big(\sum_{m=2}^{n} H^m(z_m+\Phi_m)\Big)(\Phi'_j), \quad (A^2)_{lj} = (\Phi'_l)(\Phi'_j).$$

Therefore, using the above formula, the inverse coefficients are

$$\begin{aligned}\tilde{g}^{11} = 1 &+ 2\varepsilon \sum_{m=2}^{n} H^m(z_m+\Phi_m) \\ &- \frac{1}{2}\varepsilon^2 \sum_{m,l=2}^{n} \partial^2_{ml} g_{11}\big|_\gamma (z_m+\Phi_m)(z_l+\Phi_l) \\ &+ 4\varepsilon^2\Big(\sum_{m=2}^{n} H^m(z_m+\Phi_m)\Big)^2 + R_3(z,\Phi,\Phi').\end{aligned}$$

We also get

$$\begin{aligned}\tilde{g}^{1j} &= -\varepsilon\Phi'_j - \frac{1}{2}\varepsilon^2 \sum_{m,l=2}^{n} \partial^2_{ml} g_{1j}\big|_\gamma (z_m+\Phi_m)(z_l+\Phi_l) \\ &\quad - 2\varepsilon^2\Big(\sum_{m=2}^{n} H^m(z_m+\Phi_m)\Big)\Phi'_j + R_3(z,\Phi,\Phi').\end{aligned}$$

Moreover,

$$\partial_j(\tilde{g}^{1j}) = -\varepsilon^2 \sum_{l=2}^{n} \partial^2_{lj} g_{1j}\big|_\gamma (z_l+\Phi_l) - 2\varepsilon^2 H^j \Phi'_j + R_3(z,\Phi,\Phi').$$

Similarly, with some simple calculations we also find

$$\begin{aligned}\partial_1(\tilde{g}^{11}) = 2\varepsilon^2 &\sum_{m=2}^{n} (H^m)'(z_m+\Phi_m) \\ &+ 2\varepsilon^2 \sum_{m=2}^{n} H^m \Phi'_m + R_3(z,\Phi,\Phi',\Phi'').\end{aligned}$$

Differentiating now $\tilde{g}^{1j}$ with respect to the first variable, we obtain

$$\partial_1(\tilde{g}^{1j}) = -\varepsilon^2\Phi_j'' - 2\varepsilon^3\Big(\sum_{m=2}^{n} H^m(z_m + \Phi_m)\Big)\Phi_j'' + R_3(z, \Phi, \Phi', \Phi'').$$

Analogously, we get

$$\begin{aligned}\tilde{g}^{kj} &= \delta_{kj} - \frac{1}{2}\varepsilon^2 \sum_{m,l=2}^{n} \partial^2_{ml} g_{kj}\big|_\gamma (z_m + \Phi_m)(z_l + \Phi_l)\\ &\quad + \varepsilon^2\Phi_k'\Phi_j' + R_3(z, \Phi, \Phi'),\\ \partial_k(\tilde{g}^{kj}) &= -\varepsilon^2 \sum_{l=2}^{n} \partial^2_{kl} g_{kj}\big|_\gamma (z_l + \Phi_l) + R_3(z, \Phi, \Phi').\end{aligned}$$

Finally, using the formal expansion $\tilde{g}_{CD} = \delta_{CD} + \varepsilon A_{CD} + \varepsilon^2 B_{CD} + o(\varepsilon^2)$ and carefully analyzing the error terms, we obtain

$$\begin{aligned}\sqrt{\det \tilde{g}} &= 1 + \frac{1}{2}\,\varepsilon\,\mathrm{tr}(A) + \varepsilon^2\left(\frac{1}{8}(\mathrm{tr}(A))^2 - \frac{1}{4}\,\mathrm{tr}(A^2)\right)\\ &\quad + \frac{1}{2}\,\varepsilon^2\,\mathrm{tr}(B) + O(\varepsilon^3).\end{aligned}$$

From the above expressions in (3.2) and (3.3) we deduce that

$$\begin{aligned}\sqrt{\det \tilde{g}} &= 1 - \varepsilon \sum_m H^m(z_m + \Phi_m)\\ &\quad + \varepsilon^2\bigg[\frac{1}{4}\sum_{m,l} \partial^2_{ml} g_{11}(z_m + \Phi_m)(z_l + \Phi_l)\\ &\qquad - \frac{1}{2}\Big(\sum_{m=2}^{n} H^m(z_m + \Phi_m)\Big)^2\bigg] + R_3(z, \Phi, \Phi'),\\ \partial_m\sqrt{\det \tilde{g}} &= -\varepsilon H^m + \varepsilon^2\bigg[\frac{1}{2}\sum_l \partial^2_{ml} g_{11}(z_l + \Phi_l)\\ &\qquad - H^m\Big(\sum_l H^l(z_l + \Phi_l)\Big)\bigg] + R_3(z, \Phi, \Phi').\end{aligned}$$

Moreover,

$$\begin{aligned}\partial_1\sqrt{\det \tilde{g}} &= -\varepsilon^2 \sum_m (H^m)'(z_m + \Phi_m) - \varepsilon^2 \sum_m H^m \Phi_m'\\ &\quad + R_3(z, \Phi, \Phi', \Phi'').\end{aligned}$$

The Laplacian of a smooth function u in coordinates (s,z) has the following expression:

$$-\Delta_{\tilde{g}} u = -\sum_{A,B} \tilde{g}^{AB} \partial^2_{AB} u - \sum_{A,B} \partial_B (\tilde{g}^{AB}) \partial_A u - \frac{1}{\sqrt{\det \tilde{g}}} \sum_{A,B} \tilde{g}^{AB} \big(\partial_B \sqrt{\det \tilde{g}}\big) \partial_A u.$$

We are going to expand next each of these terms. First, we consider the determinant of $\tilde{g}$. Recall that for a matrix of the form $1 + \varepsilon A + \varepsilon^2 B$, the square root of the determinant admits the formal expansion

$$\sqrt{\det g} = 1 + \frac{\varepsilon}{2} \operatorname{tr} A + \varepsilon^2 \left(\frac{1}{8} (\operatorname{tr} A)^2 - \frac{1}{4} \operatorname{tr}(A^2) + \frac{1}{2} \operatorname{tr} B \right) + o(\varepsilon^2). \tag{3.5}$$

LEMMA 3.1 *Let u be a smooth function. Then in the above coordinates (s,z), we have that*

$$\begin{aligned}
\Delta_{\tilde{g}} u = {} & \partial^2_{ss} u + \Delta_z u - \varepsilon \sum_j H^j \partial_j u - 2\varepsilon \sum_j \Phi'_j \partial^2_{sj} u + 2\varepsilon \langle \mathbf{H}, z + \Phi \rangle \partial^2_{ss} u \\
& - \varepsilon^2 \langle \mathbf{H}, z + \Phi \rangle \sum_{m,j} H^j \partial_j u - \frac{1}{2} \varepsilon^2 \partial^2_{ml} g_{11} (z_m + \Phi_m)(z_l + \Phi_l) \partial^2_{ss} u \\
& + 4\varepsilon^2 \langle \mathbf{H}, z + \Phi \rangle^2 \partial^2_{ss} u - \varepsilon^2 \partial^2_{ml} g_{1j} (z_m + \Phi_m)(z_l + \Phi_l) \partial^2_{sj} u \\
& - 4\varepsilon^2 \langle \mathbf{H}, z + \Phi \rangle \sum_j \Phi'_j \partial^2_{sj} u + \varepsilon^2 \sum_{t,j} \Phi'_t \Phi'_j \partial^2_{tj} u \\
& - \frac{1}{2} \varepsilon^2 \sum_{m,l} \partial^2_{ml} g_{tj} (z_m + \Phi_m)(z_l + \Phi_l) \partial^2_{tj} u + \varepsilon^2 \langle \mathbf{H}', z + \Phi \rangle \partial_s u \\
& - \varepsilon^2 \sum_{l,j} \partial^2_{lj} g_{1j} (z_l + \Phi_l) \partial_s u - \varepsilon^2 \sum_j \Phi''_j \partial_j u \\
& - \varepsilon^2 \sum_{t,j,l} \partial^2_{tl} g_{tj} (z_l + \Phi_l) \partial_j u - 2\varepsilon^3 \langle \mathbf{H}, z + \Phi \rangle \sum_j \Phi''_j \partial_j u \\
& + R_3(z, \Phi, \Phi') \partial^2_{ss} u + R_3(z, \Phi, \Phi') \partial^2_{sj} u \\
& + R_3(z, \Phi, \Phi') \partial^2_{lj} u + R_3(z, \Phi, \Phi', \Phi'')(\partial_s u + \partial_j u).
\end{aligned}$$

Moreover, given two smooth normal sections Φ and $\tilde{\Phi}$ and defining the corresponding coordinates

$$(s, y - \Phi(\varepsilon s)) \quad \textit{and} \quad (s, y - \tilde{\Phi}(\varepsilon s)),$$

we set $u_{\Phi}(s,y) := u(s, y-\Phi(\varepsilon s))$, $u_{\tilde\Phi}(s,y) := u(s, y-\tilde\Phi(\varepsilon s))$. *We then have*

$$
\begin{aligned}
&\Delta_{\tilde g} u_{\Phi} - \Delta_{\tilde g} u_{\tilde\Phi} \\
&\quad = -2\varepsilon \sum_j (\Phi_j' - \tilde\Phi_j')\partial^2_{sj} u + 2\varepsilon \langle \mathbf{H}, \Phi - \tilde\Phi\rangle \partial^2_{ss} u + \varepsilon^2 \sum_{t,j} (\Phi_t'\Phi_j' - \tilde\Phi_t'\tilde\Phi_j')\partial^2_{tj} u \\
&\qquad - \frac12 \varepsilon^2 \sum_{m,l} \partial^2_{ml} g_{tj}\big[2z_m(\Phi_l - \tilde\Phi_l) + \Phi_l(\Phi_m - \tilde\Phi_m) + \tilde\Phi_l(\Phi_m - \tilde\Phi_m)\big]\partial^2_{tj} u \\
&\qquad - \varepsilon^2 \sum_{m,l} \partial^2_{ml} g_{1j}\big[2z_m(\Phi_l - \tilde\Phi_l) + \Phi_l(\Phi_m - \tilde\Phi_m) + \tilde\Phi_l(\Phi_m - \tilde\Phi_m)\big]\partial^2_{sj} u \\
&\qquad - \frac12 \varepsilon^2 \sum_{m,l} \partial^2_{ml} g_{11}\big[2z_m(\Phi_l - \tilde\Phi_l) + \Phi_l(\Phi_m - \tilde\Phi_m) + \tilde\Phi_l(\Phi_m - \tilde\Phi_m)\big]\partial^2_{ss} u \\
&\qquad - 2\varepsilon^2 \sum_l H^j\big[z_l(\Phi_l' - \tilde\Phi_l') + \Phi_l(\Phi_l' - \tilde\Phi_l') + \tilde\Phi_l'(\Phi_l - \tilde\Phi_l)\big]\partial^2_{sj} u \\
&\qquad + 4\varepsilon^2 \sum_{m,l} H^m H^l\big[2z_m(\Phi_l - \tilde\Phi_l) + \Phi_l(\Phi_m - \tilde\Phi_m) + \tilde\Phi_l(\Phi_m - \tilde\Phi_m)\big]\partial^2_{ss} u \\
&\qquad - \varepsilon^2 \sum_j (\Phi_j'' - \tilde\Phi_j'')\partial_j u - \varepsilon^2 \sum_{t,j,l} \partial^2_{tl} g_{tj}(\Phi_l - \tilde\Phi_l)\partial_j u - \varepsilon^2 \langle \mathbf{H}, \Phi - \tilde\Phi\rangle \sum_j H^j \partial_j u \\
&\qquad + \varepsilon^2 \langle \mathbf{H}', \Phi - \tilde\Phi\rangle \partial_s u - \varepsilon^2 \sum_{l,j} \partial^2_{lj} g_{1j}(\Phi_l - \tilde\Phi_l)\partial_s u \\
&\qquad - 2\varepsilon^3 \sum_{mj} H^m\big[(z_m + \Phi_m)(\Phi_j'' - \tilde\Phi_j'') + \tilde\Phi_j''(\Phi_m - \tilde\Phi_m)\big]\partial_j u \\
&\qquad + O(1+|z|^d)\big[\varepsilon^4(|\Phi - \tilde\Phi| + |\Phi' - \tilde\Phi'|)|\partial^2_{ss} u| \\
&\qquad\qquad + \varepsilon^3(|\Phi - \tilde\Phi| + |\Phi' - \tilde\Phi'|)(|\partial^2_{sj} u| + |\partial^2_{lj} u|)\big] \\
&\qquad + O(1+|z|^d)\big[\varepsilon^3(|\Phi - \tilde\Phi| + |\Phi' - \tilde\Phi'|) \\
&\qquad\qquad + \varepsilon^4(|\Phi''||\Phi - \tilde\Phi| + |\Phi'' - \tilde\Phi''|)\big](|\partial_s u| + |\partial_j u|).
\end{aligned}
$$

PROOF: The proof is based on the Taylor expansion of the metric coefficients given above. We recall that the Laplace-Beltrami operator is given by

$$
\Delta_{\tilde g} = \sum_{A,B} \frac{1}{\sqrt{\det \tilde g}}\, \partial_A\big(\sqrt{\det \tilde g}\,(g_\varepsilon)^{AB}\partial_B\big),
$$

where indices A and B run between 1 and n. We can also write

$$
\Delta_{\tilde g} = \sum_{A,B}\left(\tilde g^{AB}\partial^2_{AB} + (\partial_A \tilde g^{AB})\partial_B + \frac{1}{\sqrt{\det \tilde g}}\,\tilde g^{AB}(\partial_B\sqrt{\det \tilde g})\partial_A\right).
$$

Using the expansion of the metric coefficients determined above and (3.5), we can easily prove that

$$
\begin{aligned}
&\sum_{AB} \tilde{g}^{AB} \partial^2_{AB} u \\
&\quad = \Delta_z u + \partial^2_{ss} u - 2\varepsilon \sum_j \Phi'_j \partial^2_{sj} u + 2\varepsilon \langle \mathbf{H}, z + \Phi \rangle \partial^2_{ss} u \\
&\qquad + \varepsilon^2 \sum_{l,j} \Phi'_l \Phi'_j \partial^2_{lj} u + 4\varepsilon^2 \langle \mathbf{H}, z + \Phi \rangle^2 \partial^2_{ss} u \\
&\qquad - \frac{1}{2} \varepsilon^2 \sum_{m,l} \partial^2_{ml} g_{kj} (z_m + \Phi_m)(z_l + \Phi_l) \partial^2_{kj} u \\
&\qquad - \frac{1}{2} \varepsilon^2 \sum_{m,l} \partial^2_{ml} g_{1j} (z_m + \Phi_m)(z_l + \Phi_l) \partial^2_{sj} u \\
&\qquad - 4\varepsilon^2 \langle \mathbf{H}, z + \Phi \rangle \Phi'_j \partial^2_{sj} u \\
&\qquad - \frac{1}{2} \varepsilon^2 \sum_{m,l} \partial^2_{ml} g_{11} (z_m + \Phi_m)(z_l + \Phi_l) \partial^2_{ss} u \\
&\qquad + R_3(z, \Phi, \Phi')(\partial^2_{ss} u + \partial^2_{sj} u + \partial^2_{lj} u),
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{A,B} \partial_A \tilde{g}^{AB} \partial_B u \\
&\quad = -\varepsilon^2 \sum_j \Phi''_j \partial_j u - \varepsilon^2 \sum_{i,j,l} \partial^2_{kl} g_{kj} (z_l + \Phi_l) \partial_j u \\
&\qquad - 2\varepsilon^3 \langle \mathbf{H}, z + \Phi \rangle \sum_j \Phi''_j \partial_j u + 2\varepsilon^2 \langle \mathbf{H}', z + \Phi \rangle \partial_s u \\
&\qquad - \varepsilon^2 \sum_{l,j} \partial^2_{lj} g_{1j} (z_l + \Phi_l) \partial_s u + R_3(z, \Phi, \Phi', \Phi'')(\partial_s u + \partial_j u),
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{A,B} \frac{1}{\sqrt{\det \tilde{g}}} \tilde{g}^{AB} \big(\partial_B \sqrt{\det \tilde{g}}\big) \partial_A u \\
&\quad = -\varepsilon \sum_j H^j \partial_j u - \varepsilon^2 \langle \mathbf{H}, z + \Phi \rangle \sum_j H^j \partial_j u - \varepsilon^2 \langle \mathbf{H}', z + \Phi \rangle \partial_s u \\
&\qquad + \frac{1}{2} \sum_j \partial^2_{lj} g_{11} (z_l + \Phi_l) \partial_j u + R_3(z, \Phi, \Phi')(\partial_s u + \partial_j u).
\end{aligned}
$$

The result then follows by collecting these three terms. □

3.2 Expansion at First Order in ε

In this subsection we solve equation (1.16) up to order ε, discarding the terms that turn out to be of order ε^2 and higher. Here and in the next two subsections we will display *formal expansions* only: we will assume that all the data are smooth and write $O(\varepsilon^k)$ for terms that appear at the k^{th}-order in a formal expansion. Since all the functions we are dealing with decay exponentially in z, the error terms do also: precise statements are given in Lemma 3.2 and Proposition 3.3 below.

For the approximate solution as in (1.10), we make a more precise ansatz of the following form:

$$\psi_{1,\varepsilon}(s,z) = e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}\{h(\varepsilon s)U(k(\varepsilon s)z) + \varepsilon[w_r + i w_i]\}, \qquad s \in [0, 2\pi], \quad y \in \mathbb{R}^{n-1}, \tag{3.6}$$

where $\tilde{f}_0(\varepsilon s) = f(\varepsilon s) + \varepsilon f_1(\varepsilon s)$. By direct computation, the first and second derivatives of $\psi_{1,\varepsilon}$ satisfy

$$\begin{aligned}
\partial_s \psi_{1,\varepsilon} = {} & e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}\big[-i\tilde{f}_0'(\varepsilon s)h(\varepsilon s)U(k(\varepsilon s)z) + \varepsilon h'(\varepsilon s)U(k(\varepsilon s)z) \\
& \qquad + \varepsilon h(\varepsilon s)k'(\varepsilon s)\nabla U(k(\varepsilon s)z)\cdot z\big] \\
& + e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}[-i\varepsilon f' w_r + \varepsilon f' w_i] + O(\varepsilon^2),
\end{aligned}$$

$$\begin{aligned}
\partial_i \psi_{1,\varepsilon} = {} & e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}[h(\varepsilon s)k(\varepsilon s)\partial_i U(k(\varepsilon s)z) + \varepsilon\partial_i w_r + i\varepsilon\partial_i w_i], \\
\partial^2_{ss}\psi_{1,\varepsilon} = {} & -(\tilde{f}_0')^2 hU(kz)e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}} \\
& - i\varepsilon e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}[f''hU(kz) + 2f'h'U(kz) + 2f'hk'\nabla U(kz)\cdot z] \\
& - \varepsilon f'^2 e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}[w_r + i w_i] + O(\varepsilon^2),
\end{aligned}$$

$$\partial^2_{lj}\psi_{1,\varepsilon} = e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}\big[h(\varepsilon s)k^2(\varepsilon s)\partial^2_{lj}U(k(\varepsilon s)z) + \varepsilon\partial^2_{lj}w_r + i\varepsilon\partial^2_{lj}w_i\big],$$

$$\begin{aligned}
\partial^2_{sj}\psi_{1,\varepsilon} = {} & e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}\big[-i\tilde{f}_0'(\varepsilon s)h(\varepsilon s) + \varepsilon h'(\varepsilon s)\big]k(\varepsilon s)\partial_j U(kz) \\
& + \varepsilon e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}} h(\varepsilon s)k'(\varepsilon s)\Big[k\sum_l \partial^2_{lj}U(kz)z_l + \partial_j U(kz)\Big] \\
& - i\varepsilon f'(\varepsilon s)\partial_j w_r(\varepsilon s, z)e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}} \\
& + \varepsilon f'(\varepsilon s)\partial_j w_i(\varepsilon s, z)e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}} + O(\varepsilon^2).
\end{aligned}$$

Similarly, the potential V satisfies

$$V(\varepsilon x) = V(\varepsilon s) + \varepsilon\langle\nabla^N V, z + \Phi\rangle + O(\varepsilon^2).$$

Expanding (1.16) in powers of ε, we obtain

$$e^{i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}(-\Delta_g \psi_{1,\varepsilon} + V(\varepsilon x)\psi_{1,\varepsilon} - |\psi_{1,\varepsilon}|^{p-1}\psi_{1,\varepsilon}) = \varepsilon \mathcal{R}_r + i\varepsilon \mathcal{R}_i + O(\varepsilon^2),$$

with

$$\begin{aligned} \mathcal{R}_r &= \mathcal{L}_r w_r + 2f' f_1' hU + 2f'^2 hU(kz)\langle \mathbf{H}, z + \Phi\rangle + hk\langle \mathbf{H}, \nabla U(kz)\rangle \\ &\quad + \langle \nabla^N V, z + \Phi\rangle hU(kz), \end{aligned} \tag{3.7}$$

$$\begin{aligned} \mathcal{R}_i &= \mathcal{L}_i w_i + [f'' hU(kz) + 2f'h'U(kz) + 2f'hk'\nabla U(kz)\cdot z] \\ &\quad - 2\sum_j [\Phi_j' f' hk \partial_j U(kz)], \end{aligned} \tag{3.8}$$

and where we have defined the two operators $\mathcal{L}_r$ and $\mathcal{L}_i$ as

$$\mathcal{L}_r w = -\Delta_z w + (V + f'^2)w - ph^{p-1}U(kz)^{p-1}w,$$
$$\mathcal{L}_i w = -\Delta_z w + (V + f'^2)w - h^{p-1}U(kz)^{p-1}w.$$

It is well-known (see, e.g., [48]) that the kernel of $\mathcal{L}_r$ is generated by the $(n-1)$-tuple of functions $\partial_2 U(k\cdot), \dots, \partial_n U(k\cdot)$, while that of $\mathcal{L}_i$ is one-dimensional and generated by $U(k\cdot)$.

We choose the functions w_r and w_i in such a way that $\mathcal{R}_r$ and $\mathcal{R}_i$ vanish. Since $\mathcal{L}_r$ is Fredholm, the solvability condition for w_r is that the right-hand side of this equation is orthogonal in $L^2(\mathbb{R}^{n-1})$ to $\partial_2 U(k\cdot), \dots, \partial_n U(k\cdot)$. Therefore, to get solvability, we should multiply the right-hand side by each of these functions and get 0. The same holds true for w_i, but by replacing the functions $\partial_{z_j} U(k\cdot)$ by $U(k\cdot)$.

We discuss the solvability in w_i first. Writing this equation as $\mathcal{L}_i w_i = \mathfrak{f}$, we can multiply it by $U(k\cdot)$ and use the self-adjointness of $\mathcal{L}_i$ to get

$$0 = \int_{\mathbb{R}^{n-1}} w_i \mathcal{L}_i U(k\cdot) = \int_{\mathbb{R}^{n-1}} U(k\cdot)\mathcal{L}_i w_i = \int_{\mathbb{R}^{n-1}} \mathfrak{f} U(k\cdot).$$

Following the computations of Subsection 2.3, this condition yields

$$f'' hk^{-(n-1)} + 2f'h'k^{-(n-1)} = (n-1)hk'k^{-n}f',$$

which implies

$$f' = \mathcal{A}\frac{k^{n-1}}{h^2} = \mathcal{A}h^{\sigma}.$$

This equation is nothing but (2.8), and hence the solvability is guaranteed. Since $\mathcal{L}_i$ clearly preserves the parity in z, we can decompose w_i in its even and odd parts as

$$w_i = w_{i,e} + w_{i,o},$$

with $w_{i,e}$ and $w_{i,o}$ solving, respectively, the equations

$$\mathcal{L}_i w_{i,e} = -[f''hU(kz) + 2f'h'U(kz) + 2f'hk'\nabla U(kz)\cdot z],$$
$$\mathcal{L}_i w_{i,o} = 2\sum_j [\Phi_j' f'hk\partial_j U(kz)],$$

where the right-hand sides are, respectively, the even and odd parts of the datum in (3.8). We notice that, since the kernel of $\mathcal{L}_i$ consists of even functions, only the even part of the equation plays a role in the solvability, since the product with the odd part vanishes by oddness.

Indeed, (3.7) and (3.8) can be solved explicitly, and the solutions are given by

$$w_{i,e} = \frac{p-1}{4} f'h'|z|^2 U(kz), \quad w_{i,o} = -\sum_j \Phi_j' f'hz_j U(kz). \tag{3.9}$$

In fact, as we can easily check, we have the following relations:

$$\mathcal{L}_i(z_j U(kz)) = -2k\partial_j U(kz),$$
$$\mathcal{L}_i(|z|^2 U(kz)) = -2(n-1)U(kz) - 4k\nabla U(kz)\cdot z,$$

which imply the above claim (here we also used (2.7) and some manipulations).

Turning to w_r, if we multiply by $\partial_j U$, integrate by parts, and use some scaling, we find that the following condition holds true, for $j = 2, \dots, n$:

$$2H^j\left((f')^2 \int_{\mathbb{R}^{n-1}} U^2\,dz - \frac{k^2}{n-1}\int_{\mathbb{R}^{n-1}} |\nabla U|^2\,dz\right) + \langle \nabla^N V, E_j\rangle \int_{\mathbb{R}^{n-1}} U^2\,dz = 0.$$

Using (2.7), we get equivalently, for $j = 2, \dots, n$,

$$2H^j\left(\mathcal{A}^2 h^{2\sigma} \int_{\mathbb{R}^{n-1}} U^2\,dz - \frac{h^{p-1}}{n-1}\int_{\mathbb{R}^{n-1}} |\nabla U|^2\,dz\right) + \langle \nabla^N V, E_j\rangle \int_{\mathbb{R}^{n-1}} U^2\,dz = 0.$$

From a Pohozaev-type identity (playing with (2.6) and integrating by parts) we find

$$\int_{\mathbb{R}^{n-1}} |\nabla U(z)|^2\,dz = \frac{(n-1)(p-1)}{(3-n)(p+1)+2(n-1)}\int_{\mathbb{R}^{n-1}} U(z)^2\,dz = \frac{(n-1)(p-1)}{2\theta}\int_{\mathbb{R}^{n-1}} U(z)^2\,dz. \tag{3.10}$$

By using this formula, the solvability condition then becomes

$$H^j\left((p-1)\frac{h^{p-1}}{\theta}-2\mathcal{A}^2h^{2\sigma}\right)=\langle\nabla^N V, E_j\rangle,\quad j=2,\dots,n,$$

which is nothing but the stationary condition (2.19). Therefore, since we are indeed assuming this condition, the solvability for w_r is also guaranteed. As for w_i, we can decompose w_r in its even and odd parts as

$$w_r = w_{r,e}+w_{r,o},$$

where $w_{r,e}$ and $w_{r,o}$ solve, respectively,

$$\mathcal{L}_r w_{r,e} = -2f'f_1'hU-2(f')^2hU(kz)\langle\mathbf{H},\Phi\rangle-\langle\nabla^N V,\Phi\rangle hU(kz), \tag{3.11}$$

$$\begin{aligned}\mathcal{L}_r w_{r,o} &= -2(f')^2hU(kz)\langle\mathbf{H},z\rangle - hk\sum_j H^j\partial_j U(kz)\\ &\quad -\langle\nabla^N V,z\rangle hU(kz).\end{aligned} \tag{3.12}$$

Using the Euler equation, we get

$$\mathcal{L}_r w_{r,o} = -h\sum_j H^j\left(k\partial_j U + h^{p-1}\,\frac{p-1}{\theta}\,z_j U\right).$$

It is also convenient to have the explicit expression of w_r. We notice first that

$$\mathcal{L}_r\left(-\frac{1}{(p-1)h^{p-1}}\,U(kz)-\frac{1}{2k}\nabla U(kz)\cdot z\right)=U(kz).$$

Hence it follows

$$\begin{aligned}w_{r,e} &= \left[h\langle\nabla^N V+2(f')^2\mathbf{H},\Phi\rangle+2f'f_1'h\right]\\ &\quad\times\left(\frac{1}{(p-1)h^{p-1}}U(kz)+\frac{1}{2k}\nabla U(kz)\cdot z\right).\end{aligned} \tag{3.13}$$

Using (2.19) we finally find

$$w_{r,e}=\left[\frac{p-1}{\theta}\,h^p\langle\mathbf{H},\Phi\rangle+2f'f_1'h\right]\left(\frac{1}{(p-1)h^{p-1}}\,U(kz)+\frac{1}{2k}\,\nabla U(kz)\cdot z\right).$$

By the above computations (and the comments at the beginning of this subsection) we obtain the following result:

LEMMA 3.2 *Suppose* $h(\bar{s})$ *and* $f(\bar{s})$ *satisfy* (1.11) *and* (1.12) *for some* $\mathcal{A}>0$; *assume also that the curve* γ *satisfies* (1.15). *Then there exist two smooth functions* $w_r(\bar{s},z)$ *and* $w_i(\bar{s},z)$ *for which the terms* $\mathcal{R}_r$ *and* $\mathcal{R}_i$ *in* (3.7)–(3.8) *vanish identically. Therefore, the function* $\psi_{1,\varepsilon}$ *in* (3.6) *satisfies* (1.16) *up to an error of the form* $R_2(z)e^{-k(\varepsilon s)|z|}$.

3.3 Expansions at Second Order in ε

Next we compute the terms of order ε^2 in the above expression. Adding a correction $\varepsilon^2[v_r + i v_i]$ to the function in (3.6), we define an approximate solution of the form

$$\psi_{2,\varepsilon}(s,z) = e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}\{h(\varepsilon s)U(k(\varepsilon s)z) + \varepsilon[w_r + i w_i] + \varepsilon^2[v_r + i v_i]\} \tag{3.14}$$

with $s \in [0, 2\pi]$ and $y \in \mathbb{R}^{n-1}$, where $\tilde{f}_0 = f(\varepsilon s) + \varepsilon f_1(\varepsilon s)$. The first and second derivatives of $\psi_{2,\varepsilon}$ are given by

$$\begin{aligned}
e^{i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}\partial_s\psi_{2,\varepsilon}
&= \big[-i\tilde{f}_0'(\varepsilon s)h(\varepsilon s)U(k(\varepsilon s)z) + \varepsilon h'(\varepsilon s)U(k(\varepsilon s)z)\\
&\qquad + \varepsilon h(\varepsilon s)k'(\varepsilon s)\nabla U(k(\varepsilon s)z)\cdot z\big]\\
&\quad + \big[-i\varepsilon\tilde{f}_0' w_r + \varepsilon\tilde{f}_0' w_i\big] + \big[-i\varepsilon^2\tilde{f}_0' v_r + \varepsilon^2\tilde{f}_0' v_i\big]\\
&\quad + \varepsilon^2(\partial_s w_r + i w_i) + O(\varepsilon^3),
\end{aligned}$$

$$\begin{aligned}
e^{i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}\partial_j\psi_{2,\varepsilon}
&= \big[h(\varepsilon s)k(\varepsilon s)\partial_j U(k(\varepsilon s)z) + \varepsilon\partial_j w_r + i\varepsilon\partial_j w_i + \varepsilon^2\partial_j v_r + i\varepsilon^2\partial_j v_i\big],
\end{aligned}$$

$$\begin{aligned}
e^{i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}\frac{\partial^2\psi_{2,\varepsilon}}{\partial s^2}
&= -(\tilde{f}_0')^2 hU(kz) - i\varepsilon[\tilde{f}_0'' hU(kz) + 2\tilde{f}_0' h' U(kz) + 2\tilde{f}_0' hk'\nabla U(kz)\cdot z]\\
&\quad - \varepsilon\tilde{f}_0'^2[w_r + i w_i] - \varepsilon^2\tilde{f}_0'^2[v_r + i v_i]\\
&\quad + \varepsilon^2\big[2\tilde{f}_0'\partial_s w_i + h'' U(kz) + 2h'k'\nabla U\cdot z + hk''\nabla U\cdot z\\
&\qquad\quad + hk'^2\nabla^2 U(kz)[z,z] + \tilde{f}_0'' w_i\big] - i\varepsilon^2[2\tilde{f}_0'\partial_s w_r + \tilde{f}_0'' w_r] + O(\varepsilon^3),
\end{aligned}$$

$$\begin{aligned}
e^{i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}\partial^2_{lj}\psi_{1,\varepsilon}
&= \big[h(\varepsilon s)k^2(\varepsilon s)\partial^2_{lj} U(k(\varepsilon s)z) + \varepsilon\partial^2_{lj} w_r + i\varepsilon\partial^2_{lj} w_i + \varepsilon^2\partial^2_{lj} v_r + i\varepsilon^2\partial^2_{lj} v_i\big],
\end{aligned}$$

$$\begin{aligned}
e^{i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}\partial^2_{sj}\psi_{2,\varepsilon}
&= \big[-i\tilde{f}_0'(\varepsilon s)h(\varepsilon s)k(\varepsilon s) + \varepsilon h'(\varepsilon s)k(\varepsilon s) + \varepsilon h(\varepsilon s)k'(\varepsilon s\big]\partial_j U(kz)\\
&\quad + \varepsilon h(\varepsilon s)k(\varepsilon s)k'(\varepsilon s)\sum_l \partial^2_{lj} U(kz)z_l + \partial_j U(kz) - i\varepsilon\tilde{f}_0'\partial_j w_r\\
&\quad + \varepsilon\tilde{f}_0'(\varepsilon s)\partial_j w_i - i\varepsilon^2\tilde{f}_0'\partial_j v_r + \varepsilon^2\tilde{f}_0'(\varepsilon s)\partial_j v_i + \varepsilon^2\partial^2_{sj} w_r + i\varepsilon^2\partial^2_{sj} w_r.
\end{aligned}$$

We also have the formal expansion

$$e^{i\frac{\tilde f_0(\varepsilon s)}{\varepsilon}}|\psi_{2,\varepsilon}|^{p-1}\psi_{2,\varepsilon}$$

$$= h^p|U|^{p-1}U + p\varepsilon h^{p-1}|U|^{p-1}w_r + i\varepsilon h^{p-1}|U|^{p-1}w_i + \frac{1}{2}p(p-1)\varepsilon^2 h^{p-2}|U|^{p-3}Uw_r^2 + \frac{1}{2}(p-1)\varepsilon^2 h^{p-2}|U|^{p-3}Uw_i^2 + i(p-1)\varepsilon^2 h^{p-2}|U|^{p-3}Uw_r w_i + p\varepsilon^2 h^{p-1}|U|^{p-1}v_r + i\varepsilon^2 h^{p-1}|U|^{p-1}v_i + O(\varepsilon^3).$$

Similarly, expanding V up to order ε^2, we have

$$V(\varepsilon x) = V(\varepsilon s) + \varepsilon\langle\nabla^N V, z+\Phi\rangle + \frac{1}{2}\varepsilon^2(\nabla^N)^2 V[z+\Phi, z+\Phi] + R_3(z,\Phi).$$

Using the expansions of Subsection 2.3, we obtain

$$e^{i\frac{\tilde f_0(\varepsilon s)}{\varepsilon}}(-\Delta_g\psi_{2,\varepsilon} + V(\varepsilon x)\psi_{2,\varepsilon} - |\psi_{2,\varepsilon}|^{p-1}\psi_{2,\varepsilon})$$
$$= \varepsilon^2(\tilde{\mathcal{R}}_r + i\tilde{\mathcal{R}}_i)$$
$$= \varepsilon^2(\tilde R_{r,e} + \tilde R_{r,o}) + \varepsilon^2 i(\tilde R_{i,e} + \tilde R_{i,o}) + \varepsilon^2(\tilde R_{r,e,f_1} + \tilde R_{r,o,f_1}) + \varepsilon^2 i(\tilde R_{i,e,f_1} + \tilde R_{i,o,f_1}) + \varepsilon^2\mathcal{L}_r v_r + \varepsilon^2 i\mathcal{L}_i v_i + O(\varepsilon^3),$$

where

$$\tilde R_{r,e} = -\frac{1}{2}(f')^2 hU(kz)\sum_{l,m}\partial^2_{lm}g_{11}(z_m z_l + \Phi_m\Phi_l) + 2(f')^2\langle\mathbf{H}, w_{r,e}\Phi + w_{r,o}z\rangle + 4(f')^2 hU(kz)[\langle\mathbf{H},z\rangle^2 + \langle\mathbf{H},\Phi\rangle^2] + 2f'\partial_s w_{i,e} + f''w_{i,e} - \big[h''U(kz) + 2h'k'\nabla U(kz)\cdot z + hk''\nabla U(kz)\cdot z + h(k')^2\nabla^2 U(kz)[z,z]\big] + 2\Phi'_j f'\partial_j w_{i,o} + \left[\frac{1}{2}\sum_{l,m}\partial^2_{lm}g_{tj}(z_m z_l + \Phi_m\Phi_l) - \Phi'_t\Phi'_j\right]hk^2\partial^2_{tj}U(kz) + hk\sum_{l,m,j}\partial^2_{lm}g_{mj}z_l\partial_j U(kz) + \sum_m H^m\partial_m w_{r,o} + \tag{3.15}$$

$$
\begin{aligned}
&+ hk\langle \mathbf{H}, z\rangle \sum_m H^m \partial_m U(kz)\\
&+ kh \sum_m \Big[\langle \mathbf{H}, z\rangle H^m - \frac{1}{2}\sum_l \partial^2_{ml} g_{11} z_l \Big] \partial_m U(kz)\\
&- \frac{1}{2}\, p(p-1) h^{p-2} U(kz)^{p-2} (w^2_{r,e} + w^2_{r,o})\\
&- \frac{1}{2}(p-1) h^{p-2} U(kz)^{p-2} (w^2_{i,e} + w^2_{i,o})\\
&+ \langle \nabla^N V, w_{r,o} z + w_{r,e}\Phi\rangle\\
&+ \frac{1}{2}\sum_{m,j} \partial^2_{mj} V (z_m z_j + \Phi_m \Phi_j) h U(kz),
\end{aligned}
$$

$$
\begin{aligned}
\tilde{R}_{r,o} = {} & -(f')^2 h U(kz) \sum_{l,m} \partial^2_{lm} g_{11} z_m \Phi_l + 8(f')^2 h U(kz) \langle \mathbf{H}, z\rangle \langle \mathbf{H}, \Phi\rangle\\
&+ 2(f')^2 \langle \mathbf{H}, w_{r,e} z + w_{r,o}\Phi\rangle - 2f'\partial_s w_{i,o} - f'' w_{i,o}\\
&+ 2h'k \sum_j \Phi'_j \partial_j U(kz)\\
&+ 2h'k \sum_j \Phi'_j \Big[k \sum_l \partial^2_{lj} U(kz) z_l + \partial_j U(kz) \Big]\\
&+ 2f' \sum_j \Phi'_j \partial_j w_{i,e} + hk \langle \mathbf{H}, \Phi\rangle \sum_m H^m \partial_m U(kz)\\
&+ \Big[\frac{1}{2}\sum_{l,m} \partial^2_{lm} g_{tj} (z_m \Phi_l + z_l \Phi_m) \Big] h k^2 \partial^2_{tj} U(kz)\\
&+ hk \sum \phi''_j \partial_j U(kz) + \sum_m H^m \partial_m w_{r,e}\\
&+ hk \sum_m \Big[\langle \mathbf{H}, \Phi\rangle H^m - \frac{1}{2}\sum_l \partial^2_{ml} g_{11} \Phi_l \Big] \partial_m U(kz)\\
&- p(p-1) h^{p-2} U(kz)^{p-2} w_{r,e} w_{r,o}\\
&- (p-1) h^{p-2} U(kz)^{p-2} w_{i,e} w_{i,o} + \langle \nabla^N V, w_{r,e} z + w_{r,o}\Phi\rangle\\
&+ \sum_{j,l} \partial^2_{jl} V z_j \Phi_l h U(kz) + \Big(\sum_{j,l,m} \partial^2_{lm} g_{mj} \Phi_l \Big) hk \partial_j U(kz),
\end{aligned} \tag{3.16}
$$

$$\begin{aligned}\tilde{R}_{i,e} &= 2\big[f''hU(kz) + 2f'h'U(kz) + 2f'hk'\nabla U(kz)\cdot z\big]\langle \mathbf{H}, \Phi\rangle \\ &\quad + 2(f')^2\langle \mathbf{H}, w_{i,e}\Phi + w_{i,o}z\rangle + 2f'\partial_s w_{r,e} + f'' w_{r,e} \\ &\quad - 2f'\sum_j \Phi_j' \partial_j w_{r,o} \\ &\quad - 2f'hk\sum_j \partial_j U(kz)\Big[2\langle \mathbf{H}, z\rangle \Phi_j' \\ &\qquad\qquad + \frac{1}{2}\sum_{l,m}\partial^2_{lm} g_{1j}(z_m\Phi_l + z_l\Phi_m)\Big] \\ &\quad - f'hU(kz)\Big(\sum_m(\partial^2_{1m} g_{11}\Phi_m - 2\langle \mathbf{H}, \Phi'\rangle)\Big) \\ &\quad - f'h\Big[2\langle \mathbf{H}, \Phi'\rangle + \sum_{j,l}\partial^2_{lj} g_{1j}\Phi_l\Big]U(kz) \\ &\quad + \frac{1}{2}f'h\Big(\sum_l \partial_{1l} g_{11}\Phi_l\Big)U(kz) + \sum_j H^j\partial_j w_{i,o} \\ &\quad - (p-1)h^{p-2}U(kz)^{p-2}(w_{r,e}w_{i,e} + w_{r,o}w_{i,o}) \\ &\quad + \langle \nabla^N V, w_{i,o}z + w_{i,e}\Phi\rangle,\end{aligned} \tag{3.17}$$

$$\begin{aligned}\tilde{R}_{i,o} &= 2\big[f''hU(kz) + 2f'h'U(kz) + 2f'hk'\nabla U(kz)\cdot z\big]\langle \mathbf{H}, z\rangle \\ &\quad + \sum_i H^j\partial_j w_{i,e} + 2(f')^2\langle \mathbf{H}, w_{i,e}z + w_{i,o}\Phi\rangle + 2f'\partial_s w_{r,o} \\ &\quad + f''w_{r,o} - 2f'\sum_j \Phi_j'\partial_j w_{r,e} \\ &\quad - 2f'hk\sum_j \partial_j U(kz)\Big[2\langle \mathbf{H}, \Phi\rangle\Phi_j' \\ &\qquad\qquad + \frac{1}{2}\sum_{l,m}\partial^2_{lm} g_{1j}(z_m z_l + \Phi_l\Phi_m)\Big] \\ &\quad - f'hU(kz)\Big(\sum_m \partial^2_{1m} g_{11} z_m\Big) - f'h\Big(\sum_{j,l}\partial^2_{lj} g_{1j} z_l\Big)U(kz) \\ &\quad + \frac{1}{2}f'h\Big(\sum_l \partial^2_{1l} g_{11} z_l\Big)U(kz) -\end{aligned} \tag{3.18}$$

$$- (p-1)h^{p-2}U(kz)^{p-2}(w_{r,e}w_{i,o} + w_{r,o}w_{i,e})$$

$$+ \langle \nabla^N V, w_{i,e}z + w_{i,o}\Phi\rangle.$$

We used the notation $\tilde{R}_{r,e,f_1}$, $\tilde{R}_{r,o,f_1}$, $\tilde{R}_{i,e,f_1}$, and $\tilde{R}_{i,o,f_1}$ for the terms involving f_1, namely,

$$\begin{aligned} \tilde{R}_{r,e,f_1} &= (f_1')^2 hU + 2f'f_1'w_{r,e} + 4\langle \mathbf{H}, \Phi\rangle f'f_1'hU \\ &\quad - 2p(p-1)h^{p-2}|U|^{p-2}h^2 f'^2 f_1'^2 \tilde{U}^2 \\ &\quad + 2f'f_1'h\langle \nabla^N V, \Phi\rangle \tilde{U} + 4\langle \mathbf{H}, \Phi\rangle (f')^3 f_1' h\tilde{U} \\ &\quad - 2p\,\frac{(p-1)^2}{\theta}\, h^{2p-1} f' f_1' \langle \mathbf{H}, \Phi\rangle U^{p-2}\tilde{U}^2, \end{aligned} \tag{3.19}$$

$$\begin{aligned} \tilde{R}_{r,o,f_1} &= 2f'f_1'w_{r,o} + 4\langle \mathbf{H}, z\rangle f'f_1'hU \\ &\quad - 2p(p-1)f'f_1'h^{p-1}U^{p-2}\tilde{U}w_{r,o} \\ &\quad + 2f'f_1'h\langle \nabla^N V, z\rangle \tilde{U} + 2H^j f'f_1'hk\partial_j\tilde{U} \\ &\quad + 4\langle \mathbf{H}, z\rangle (f')^2 f'f_1'h\tilde{U}, \end{aligned} \tag{3.20}$$

$$\begin{aligned} \tilde{R}_{i,e,f_1} &= 2h'f_1'U + 2hf_1'k'\nabla U \cdot z + 2f'f_1'w_{i,e} + f_1''hU \\ &\quad + 4f'\partial_s(hf'f_1'\tilde{U}) + 2f''hf'f_1'\tilde{U} \\ &\quad - 2(p-1)h^{p-1}|U|^{p-2}f'f_1'\tilde{U}w_{i,e}, \end{aligned} \tag{3.21}$$

$$\begin{aligned} \tilde{R}_{i,o,f_1} &= 2f'f_1'w_{i,o} - 2(p-1)h^{p-1}|U|^{p-2}f'f_1'\tilde{U}w_{i,o} \\ &\quad - 4(f')^2hkf_1'\Phi_j'\partial_j\tilde{U} - 2hkf_1'\Phi_j'\partial_j U, \end{aligned} \tag{3.22}$$

where we wrote for brevity

$$\tilde{U} = \frac{1}{(p-1)h^{p-1}}\,U(kz) + \frac{1}{2k}\nabla U(kz)\cdot z.$$

Again, we collect the results of this section in one proposition.

PROPOSITION 3.3 *Suppose Φ and f_1 are smooth functions on* $[0, L]$. *Let z be the normal coordinates given in* (3.1)*, and w_i and w_r be as in Lemma* 3.2*. Then, if*

$\psi_{2,\varepsilon}$ *is as in* (3.14)*, in the coordinates* (s, z) *we have*

$$
\begin{aligned}
&-\Delta_{\tilde{g}_\varepsilon}\psi_{2,\varepsilon} + V(\varepsilon x)\psi_{2,\varepsilon} - |\psi_{2,\varepsilon}|^{p-1}\psi_{2,\varepsilon} \\
&\quad = \varepsilon^2(\mathcal{L}_r v_r + i\mathcal{L}_i v_i) \\
&\qquad + \varepsilon^2\big(\tilde{R}_{r,e} + \tilde{R}_{r,o} + i\tilde{R}_{i,e} + i\tilde{R}_{i,o} + \tilde{R}_{r,e,f_1} \\
&\qquad\qquad + \tilde{R}_{r,o,f_1} + i\tilde{R}_{i,e,f_1} + i\tilde{R}_{i,o,f_1}\big) \\
&\qquad + R_3(z)e^{-k(\varepsilon s)|z|},
\end{aligned} \tag{3.23}
$$

where the above error terms are given, respectively, in (3.15)–(3.22).

It is useful next to evaluate the projections of the errors on the kernels of the operators $\mathcal{L}_i$ and $\mathcal{L}_r$ to see what the effect of Φ and f_1 is. Concerning $\mathcal{L}_i$, the kernel is spanned by $iU(k(\varepsilon s)\cdot)$. To compute this projection, for parity reasons, we need to multiply $\tilde{R}_{i,e}$ and $\tilde{R}_{i,e,f_1}$ by $U(k(\varepsilon s)\cdot)$ and to integrate over $\mathbb{R}^{n-1}$. For brevity, we only display the final results, referring to [36, sec. 4] for complete proofs.

Contribution of $\tilde{R}_{i,e}$:

$$
\int_{\mathbb{R}^{n-1}} \tilde{R}_{i,e}U(kz) = -\frac{2\mathcal{A}}{h}\left(\frac{p-1}{2\theta} - 1\right)\int_{\mathbb{R}^{n-1}} U^2\partial_s\langle \mathbf{H}, \Phi\rangle.
$$

Contribution of $\tilde{R}_{i,e,f_1}$:

$$
\begin{aligned}
\frac{h}{C_0}\int_{\mathbb{R}^{n-1}} \tilde{R}_{i,e,f_1}U(kz) &= \partial_{\bar{s}}\left(\frac{h^2 f_1'}{k^{n-1}}\left[1 - 2f'^2\frac{\sigma}{(p-1)k^2}\right]\right) \\
&= \partial_{\bar{s}}\left(\frac{h^2 f_1'}{(p-1)k^{n+1}}[(p-1)h^{p-1} - 2\sigma\mathcal{A}^2h^{2\sigma}]\right).
\end{aligned}
$$

To annihilate this projection, we should find f_1 such that

$$
\int_{\mathbb{R}^{n-1}} \tilde{R}_{i,e,f_1}U(kz) + \int_{\mathbb{R}^{n-1}} \tilde{R}_{i,e}U(kz) = 0.
$$

This is equivalent to

$$
\begin{aligned}
Tf_1 &:= \partial_{\bar{s}}\left(\frac{h^2 f_1'}{(p-1)k^{n+1}}[(p-1)h^{p-1} - 2\sigma\mathcal{A}^2h^{2\sigma}]\right) \\
&= 2\mathcal{A}\left(\frac{p-1}{2\theta} - 1\right)\partial_{\bar{s}}\langle \mathbf{H}, \Phi\rangle.
\end{aligned} \tag{3.24}
$$

Hence it is sufficient to set

$$f_1' = \frac{2\mathcal{A}(p-1)k^{n+1}}{(p-1)h^{p+1} - 2\sigma\mathcal{A}^2 h^{2\sigma+2}}\left(\frac{p-1}{2\theta} - 1\right)\langle \mathbf{H}, \Phi\rangle + c\,\frac{(p-1)k^{n+1}}{(p-1)h^{p+1} - 2\sigma\mathcal{A}^2 h^{2\sigma+2}},$$

where c is a constant to be chosen so that $\int_0^L f_1'\,ds = 0$. By (2.16), we have

$$\frac{(p-1)k^{n+1}}{(p-1)h^{p+1} - 2\sigma\mathcal{A}^2 h^{2\sigma+2}} = A\sigma h^{\sigma-1}\,\frac{\partial h}{\partial \mathcal{A}} + h^{\sigma},$$

and so the required condition becomes

$$c = -\left(\frac{p-1}{2\theta} - 1\right)2\mathcal{A}(p-1)\frac{\int_0^L \frac{k^{n+1}}{(p-1)h^{p+1} - 2\sigma\mathcal{A}^2 h^{2\sigma+2}}\langle \mathbf{H}, \Phi\rangle ds}{\int_0^L A\sigma h^{\sigma-1}\frac{\partial h}{\partial \mathcal{A}} + h^{\sigma}\,ds}.$$

As we can easily check from (2.14) and (2.19), c coincides with $\mathcal{A}'$ and therefore we have in conclusion

$$f_1' = \frac{2\mathcal{A}(p-1)k^{n+1}}{(p-1)h^{p+1} - 2\sigma\mathcal{A}^2 h^{2\sigma+2}}\left(\frac{p-1}{2\theta} - 1\right)\langle \mathbf{H}, \Phi\rangle + \mathcal{A}'\frac{(p-1)k^{n+1}}{(p-1)h^{p+1} - 2\sigma\mathcal{A}^2 h^{2\sigma+2}}. \tag{3.25}$$

We evaluate next the projection of the $\tilde{R}$'s in (3.23) onto the kernel of $\mathcal{L}_r$. This corresponds to multiplying the error terms by $\partial_m U(k\cdot)$, $m = 1, \dots, n-1$, integrating over $\mathbb{R}^{n-1}$, and taking the real part. As before, we are left to consider only two terms: $\tilde{R}_{r,o}$ and $\tilde{R}_{r,o,f_1}$. The final result (proven in section 4 of [36]) is the following:

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} (\tilde{R}_{r,o} + \tilde{R}_{r,o,f_1})\partial_m U(k(\varepsilon s)z)dz \\
&= -\frac{p-1}{2\theta}\frac{1}{hk}C_0\Bigg\{-\left(h^{\theta} - \frac{2\mathcal{A}^2\theta}{p-1}h^{\sigma}\right)\Phi_m'' \\
&\quad - \theta\left(h^{\theta-1} - \frac{2\mathcal{A}^2\sigma}{p-1}h^{\sigma-1}\right)h'\Phi_m' + \frac{\theta}{p-1}h^{-\sigma}((\nabla^N)^2 V)\Phi_m \\
&\quad + \frac{1}{2}\left(h^{\theta} - \frac{2\mathcal{A}^2\theta}{p-1}h^{\sigma}\right)\Big(\sum_j (\partial^2_{jm} g_{11})\Phi_j\Big) \\
&\quad - 2\mathcal{A}\mathcal{A}_1'\frac{(\theta-\sigma)h^{p-1}}{[(p-1)h^{\theta} - 2\sigma\mathcal{A}^2 h^{\sigma}]}H^m \\
&\quad + H^m\langle \mathbf{H}, \Phi\rangle\left[\frac{-(p-1)(3+\frac{\sigma}{\theta})h^{2\theta} - \frac{16\sigma\theta\mathcal{A}^4}{p-1}h^{2\sigma} + 2\mathcal{A}^2(5\sigma+3\theta)h^{\theta+\sigma}}{(p-1)h^{\theta} - 2\mathcal{A}^2\sigma h^{\sigma}}\right]\Bigg\}.
\end{aligned} \tag{3.26}$$

We notice that the operator between brackets coincides precisely with the one in (2.22), corresponding to the second variation of the reduced functional that we determined in Subsection 2.5. This is going to be useful in the last section to get full solvability.

Remark 3.4. According to the considerations in Subsection 2.4, to every normal variation of γ there corresponds some variation in the phase due to both the variation of position and the variation of the constant $\mathcal{A}$. Recall that the phase of the approximate solution is the following:

$$F_\varepsilon = \frac{1}{\varepsilon} f(\varepsilon s) = \frac{1}{\varepsilon} \int_0^{\varepsilon s} f' \, dl.$$

Differentiating with respect to a variation ν (see (1.12)) we obtain

$$\frac{\partial}{\partial \nu} F_\varepsilon = \frac{1}{\varepsilon} \int_0^{\varepsilon s} \mathcal{A}_\nu h^\sigma + \frac{1}{\varepsilon} \int_0^{\varepsilon s} \mathcal{A} \sigma h^{\sigma - 1} \left(\frac{\partial h}{\partial \mathcal{A}} \mathcal{A}_\nu + \frac{\partial h}{\partial V} \frac{\partial V}{\partial \nu} \right) + \frac{1}{2} \frac{1}{\varepsilon} \int_0^{\varepsilon s} \partial_\nu g_{11} \mathcal{A} h^\sigma.$$

Recalling formula (2.17), we find

$$\frac{\partial}{\partial \nu} F_\varepsilon = \frac{1}{\varepsilon} \frac{p-1}{2\mathcal{A}} \mathcal{A}_\nu \int_0^{\varepsilon s} h^{\theta - 1} \frac{\partial h}{\partial \mathcal{A}} + \frac{1}{\varepsilon} \int_0^{\varepsilon s} \mathcal{A} \sigma h^{\sigma - 1} \frac{\partial h}{\partial V} \frac{\partial V}{\partial \nu} + \frac{1}{2} \frac{1}{\varepsilon} \int_0^{\varepsilon s} \partial_\nu g_{11} \mathcal{A} h^\sigma.$$

Therefore, when we take a variation ν_2 of γ, this also corresponds to a variation of the phase of $\frac{\partial}{\partial \nu_2} F_\varepsilon$. Notice that multiplying the horizontal part by $h \partial_m U$ corresponds to adding a variation of $-\frac{\nu_2}{k}$.

Hence, integrating by parts, we get

$$\mathcal{A} \left(\frac{p-1}{2\theta} - 1 \right) \int \left(\sum_m \Phi_m \partial_m g_{11} \right) \left[\frac{p-1}{2\mathcal{A}} \mathcal{A}_{\nu_2} h^{\theta - 1} \frac{\partial h}{\partial \mathcal{A}} - \mathcal{A} \sigma h^{\sigma - 1} \frac{\partial h}{\partial V} \frac{\partial V}{\partial \nu_2} - \frac{1}{2} \mathcal{A} h^\sigma \partial_{\nu_2} g_{11} \right].$$

Remark 3.5. If we multiply the operators $\mathfrak{J}$ and T (see (2.22) and (3.24)) by $h(\bar{s}) k(\bar{s})$ and $h(\bar{s})$, respectively, they become self-adjoint. This fact will become crucial below; see in particular Subsection 4.3.

4 Lyapunov-Schmidt Reduction of the Problem

In this section we show how to reduce problem (1.16) to a system of three ordinary (integro)differential equations on $\mathbb{R}/[0, L]$. We first introduce a metric on the normal bundle $N\gamma_\varepsilon$ of γ_ε and then study operators that mimic the properties of the linearization of (1.16) near an approximate solution. Next, we turn to the reduction procedure: this follows basically from a localization method, since the functions we are dealing with have an exponential decay away from γ_ε. We introduce a set K_δ consisting of approximate (resonant) eigenfunctions of the linearized operator

L_ε: calling $\bar{H}_\varepsilon$ the orthogonal complement of this set (which has to be multiplied by a phase factor close to $e^{-i(f(\varepsilon s))/\varepsilon}$), we show in Proposition 4.14 that L_ε is invertible on the projection onto this set once suitable weighted norms are introduced.

4.1 A Metric Structure on $N\gamma_\varepsilon$

In this subsection we define a metric $\widehat{g}_\varepsilon$ on $N\gamma_\varepsilon$, the normal bundle to γ_ε, and then introduce some basic tools that are useful for working in local coordinates on this set.

First of all, we choose a local orthonormal frame $(E_i)_i$ in $N\gamma$ and, using the notation of [34, subsection 2.2], we set $\nabla^N_{\partial_{\bar{s}}} E_j = \beta^l_j(\partial_{\bar{s}})E_l$, $j, l = 1, \dots, n-1$. If we impose that the E_j's are transported in parallel via the normal connection ∇^N, as in Subsection 2.1, we find that $\beta^l_j(\partial_{\bar{s}}) \equiv 0$ for all j, l. As a consequence (see formula (18) in [34]), we have that if $(V^j)_j$, $j = 1, \dots, n-1$, is a normal section to γ, then the components of the normal Laplacian $\Delta^N V$ are simply given by

$$(\Delta^N V)^j = \Delta_\gamma(V^j) = \partial^2_{\bar{s}\bar{s}} V^j, \quad j = 1, \dots, n-1. \tag{4.1}$$

We next define a metric $\widehat{g}$ on $N\gamma$ as follows. Given $v \in N\gamma$, a tangent vector $W \in T_v N\gamma$ can be identified with the velocity of a curve $w(t)$ in $N\gamma$ that is equal to v at time $t = 0$. The metric $\widehat{g}$ on $N\gamma$ acts on an arbitrary couple $(W, \tilde{W}) \in (T_v N\gamma)^2$ in the following way (see [20, p. 79])

$$\widehat{g}(W, \tilde{W}) = g(\pi_* W, \pi_* \tilde{W}) + \left\langle \frac{D^N w}{dt}\bigg|_{t=0}, \frac{D^N \tilde{w}}{dt}\bigg|_{t=0} \right\rangle_N.$$

In this formula π denotes the natural projection from $N\gamma$ onto γ, $D^N w/dt$ the (normal) covariant derivative of the vector field $w(t)$ along the curve $\pi w(t)$, and $\tilde{w}(t)$ a curve in $N\gamma$ with initial value v and initial velocity equal to $\tilde{W}$.

Following the notation in Subsection 2.1 we have that, if $w(t) = w^j(t)E_j(t)$, then

$$\frac{D^N w}{dt} = \frac{dw^j(t)}{dt} E_j(t).$$

Therefore, if we choose a system of coordinates $(\bar{s}, \bar{y})$ on $N\gamma$ defined by

$$(\bar{s}, \bar{y}) \in \mathbb{R} \times \mathbb{R}^{n-1} \mapsto \bar{y}_j E_j(\gamma(\bar{s})),$$

we get that

$$\widehat{g}_{11}(\bar{s}, \bar{y}) = g_{11}(\bar{s}) + \bar{y}_l \bar{y}_j \left\langle \nabla^N_{E_1} E_l, \nabla^N_{E_1} E_j \right\rangle_N = g_{11}(\bar{s}) \equiv 1$$

and

$$\widehat{g}_{1\bar{l}}(\bar{s}, \bar{y}) \equiv 0, \quad \widehat{g}_{\bar{l}\bar{j}}(\bar{s}, \bar{y}) = \delta_{\bar{l}\bar{j}},$$

where we have set $\partial_{\bar{l}} = \partial/\partial \bar{y}_l$. We also notice that the following co-area type formula holds for any smooth compactly supported function $f : N\gamma \to \mathbb{R}$:

$$\int_{N\gamma} f \, dV_{\widehat{g}} = \int_{\gamma} \left(\int_{N\gamma(\bar{s})} f(\bar{y}) d\bar{y} \right) d\bar{s}. \tag{4.2}$$

This follows immediately from the fact that $\det \widehat{g} = \det g$ and by our choice of $(\bar{s}, \bar{y})$.

Since in the above coordinates the metric $\widehat{g}$ is diagonal, the Laplacian of any (real- or complex-valued) function ϕ defined on $N\gamma$ with respect to this metric is

$$\Delta_{\widehat{g}} \phi = \partial^2_{\bar{s}\bar{s}} \phi + \partial^2_{\bar{j}\bar{j}} \phi \quad \text{in } N\gamma.$$

We next endow $N\gamma_\varepsilon$ with a natural metric, inherited by $\widehat{g}$ through a scaling. If T_ε denotes the dilation $x \mapsto \varepsilon x$, we define a metric $\widehat{g}_\varepsilon$ on $N\gamma_\varepsilon$ simply by

$$\widehat{g}_\varepsilon = \frac{1}{\varepsilon^2} [(T_\varepsilon)_* \widehat{g}].$$

In particular, choosing coordinates (s, y) on $N\gamma_\varepsilon$ via the scaling $(\bar{s}, \bar{y}) = \varepsilon(s, y)$, we easily check that the components of $\widehat{g}_\varepsilon$ are given by

$$(\widehat{g}_\varepsilon)_{11}(s, y) = g_{11}(\bar{s}) \equiv 1, \quad (\widehat{g}_\varepsilon)_{1l}(s, y) \equiv 0, \quad (\widehat{g}_\varepsilon)_{lj}(s, y) = \delta_{lj}.$$

Therefore, if ψ is a smooth function in $N\gamma_\varepsilon$, it follows that in the above coordinates (s, y)

$$\Delta_{\widehat{g}_\varepsilon} \psi = \partial^2_{ss} \psi + \partial^2_{jj} \psi \quad \text{in } N\gamma_\varepsilon.$$

In the case $\psi(s, y) = e^{-i\widehat{f}s} u(s, y)$, for $\widehat{f} = \mathcal{A}\widehat{h}^\sigma$ (see (1.9)) and for u real, we clearly have that

$$\Delta_{\widehat{g}_\varepsilon} \psi = e^{-i\widehat{f}s} \partial^2_{ss} u - 2i\widehat{f} e^{-i\widehat{f}s} \partial_s u - \widehat{f}^2 e^{-i\widehat{f}s} u + e^{-i\widehat{f}s} \partial^2_{jj} u.$$

Similarly to (4.2), we easily find that

$$\int_{N\gamma_\varepsilon} f \, dV_{\widehat{g}_\varepsilon} = \int_{\gamma_\varepsilon} \left(\int_{N\gamma_\varepsilon(s)} f(y) dy \right) ds. \tag{4.3}$$

4.2 Localizing the Problem to a Subset of the Normal Bundle $N\gamma_\varepsilon$

We next exploit the exponential decay of solutions (or approximate solutions) away from γ_ε to reduce (1.16) from the whole scaled manifold M_ε to the normal bundle $N\gamma_\varepsilon$. This step of the proof closely follows a procedure in [19]. We first define a smooth, nonincreasing cutoff function $\bar{\eta} : \mathbb{R} \to \mathbb{R}$ satisfying

$$\begin{cases} \bar{\eta}(t) = 1 & \text{for } t \leq 0, \\ \bar{\eta}(t) = 0 & \text{for } t \geq 1, \\ \bar{\eta}(t) \in [0, 1] & \text{for every } t \in \mathbb{R}. \end{cases}$$

Next, if (s, y) are the coordinates introduced above in $N\gamma_\varepsilon$, and if $\Phi(\varepsilon s)$ is a section of $N\gamma$, using the notation of Subsection 3.1, we define

$$z = y - \Phi(\varepsilon s).$$

We will assume throughout the paper that Φ satisfies the following bounds:

$$\|\Phi\|_\infty + \|\Phi'\|_\infty + \varepsilon\|\Phi''\|_\infty \leq C\varepsilon \tag{4.4}$$

for some fixed constant $C > 0$. Next, for a small $\bar{\delta} > 0$ and for a smooth function $K(\varepsilon s) > 0$, both to be determined below, and for $h, k : [0, L] \to \mathbb{R}$ as defined in the introduction, we set

$$\begin{aligned} \tilde{\psi}_{0,\varepsilon} &= \bar{\eta}_\varepsilon(s, z)\psi_{0,\varepsilon} \\ &:= \bar{\eta}\left(K(\varepsilon s)\left(|z| - \frac{\varepsilon^{-\bar{\delta}}}{K(\varepsilon s)}\right)\right)e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}} h(\varepsilon s)U(k(\varepsilon s)z), \end{aligned} \tag{4.5}$$

where $\tilde{f}$ (to be defined later) is close to the function f (also defined in the introduction). For $\tau \in (0, 1)$, we let $S_\varepsilon : C^{2,\tau}(M_\varepsilon) \to C^\tau(M_\varepsilon)$ be the operator

$$S_\varepsilon(\psi) = -\Delta_{g_\varepsilon}\psi + V(\varepsilon x)\psi - |\psi|^{p-1}\psi \quad \text{in } M_\varepsilon. \tag{4.6}$$

If we let $\tilde{\psi}_\varepsilon$ denote an approximate solution of (1.16) (we will later take $\tilde{\psi}_\varepsilon$ equal to $\tilde{\psi}_{0,\varepsilon}$ with some small correction), then setting $\psi = \tilde{\psi}_\varepsilon + \tilde{\phi}$, we have $S_\varepsilon(\psi) = 0$ if and only if

$$L_\varepsilon(\tilde{\phi}) = S_\varepsilon(\tilde{\psi}_\varepsilon) + N_\varepsilon(\tilde{\phi}) \quad \text{in } M_\varepsilon,$$

where $L_\varepsilon(\tilde{\phi})$ stands for the linear correction in $\tilde{\phi}$, namely,

$$\begin{aligned} L_\varepsilon(\tilde{\phi}) &= -\Delta_{g_\varepsilon}\tilde{\phi} + V(\varepsilon x)\tilde{\phi} - |\tilde{\psi}_\varepsilon|^{p-1}\tilde{\phi} \\ &\quad - (p-1)|\tilde{\psi}_\varepsilon|^{p-3}\tilde{\psi}_\varepsilon\Re(\tilde{\psi}_\varepsilon\bar{\tilde{\phi}}) \quad \text{in } M_\varepsilon, \end{aligned} \tag{4.7}$$

and where the nonlinear operator $N_\varepsilon(\tilde{\phi})$ is defined as

$$\begin{aligned} N_\varepsilon(\tilde{\phi}) &= |\tilde{\psi}_\varepsilon + \tilde{\phi}|^{p-1}(\tilde{\psi}_\varepsilon + \tilde{\phi}) - |\tilde{\psi}_\varepsilon|^{p-1}\tilde{\psi}_\varepsilon - |\tilde{\psi}_\varepsilon|^{p-1}\tilde{\phi} \\ &\quad - (p-1)|\tilde{\psi}_\varepsilon|^{p-3}\tilde{\psi}_\varepsilon\Re(\tilde{\psi}_\varepsilon\bar{\tilde{\phi}}). \end{aligned} \tag{4.8}$$

Then, in the coordinates (s, z), we can write

$$\tilde{\phi} = \bar{\eta}_\varepsilon(z)\phi + \varphi$$

where, with an abuse of notation, we assume ϕ defined on $N\gamma_\varepsilon$ (through the exponential map normal to γ_ε) and where the correction φ is defined on the whole M_ε. In this way we need to solve the equation

$$L_\varepsilon(\bar{\eta}_\varepsilon(z)\phi) + L_\varepsilon(\varphi) = S_\varepsilon(\tilde{\psi}_\varepsilon) + N_\varepsilon(\bar{\eta}_\varepsilon(z)\phi + \varphi) \quad \text{in } M_\varepsilon. \tag{4.9}$$

We will require ϕ to be supported in a cylindrically shaped region in $N\gamma_\varepsilon$ centered around the zero section. For technical reasons, convenient for proving the results in the next subsection, we define

$$\tilde{D}_\varepsilon = \left\{ (s, z) \in N\gamma_\varepsilon : |z| \le \frac{\varepsilon^{-\bar{\delta}} + 1}{K(\varepsilon s)} \right\}, \tag{4.10}$$

and then the subspace of functions in $N\gamma_\varepsilon$ as

$$H_{\tilde{D}_\varepsilon} = \{ u \in L^2(N\gamma_\varepsilon; \mathbb{C}) : u \text{ is supported in } \tilde{D}_\varepsilon \}.$$

Using elementary computations, we see that (4.9) is satisfied if (tautologically) the following two conditions are imposed:

$$\begin{aligned} L_\varepsilon(\phi) = {} & \left[S_\varepsilon(\tilde{\psi}_\varepsilon) + N_\varepsilon(\bar{\eta}_\varepsilon(z)\phi + \varphi) \right] + |\tilde{\psi}_\varepsilon|^{p-1}\varphi \\ & + (p-1)|\tilde{\psi}_\varepsilon|^{p-3}\tilde{\psi}_\varepsilon \Re(\tilde{\psi}_\varepsilon \bar{\varphi}) \quad \text{in } \tilde{D}_\varepsilon,\ \phi \in H_{\tilde{D}_\varepsilon}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} \mathcal{L}_{\tilde{\psi}_\varepsilon}\varphi = {} & (1 - \bar{\eta}_\varepsilon(z)) \left[S_\varepsilon(\tilde{\psi}_\varepsilon) + N_\varepsilon(\bar{\eta}_\varepsilon(z)\phi + \varphi) \right] \\ & + 2\nabla_{g_\varepsilon}\bar{\eta}_\varepsilon(z) \cdot \nabla_{g_\varepsilon}\phi + \Delta_{g_\varepsilon}\bar{\eta}_\varepsilon(z)\phi \quad \text{in } M_\varepsilon, \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} \mathcal{L}_{\tilde{\psi}_\varepsilon}\varphi = {} & -\Delta_{g_\varepsilon}\varphi + V(\varepsilon x)\varphi \\ & - (1 - \bar{\eta}_\varepsilon(z)) \left[|\tilde{\psi}_\varepsilon|^{p-1}\varphi + (p-1)|\tilde{\psi}_\varepsilon|^{p-2}\tilde{\psi}_\varepsilon \Re(\tilde{\psi}_\varepsilon \bar{\varphi}) \right]. \end{aligned} \tag{4.13}$$

We next have an existence result for equation (4.12). In order to state it, we need to introduce some notation. For a regular periodic function $\mathfrak{p} : [0, L] \to \mathbb{R}$, for $m \in \mathbb{N}$ and $\tau \in (0, 1)$, we define the weighted norms

$$\|\varphi\|_{C^{m,\tau}_{\mathfrak{p}}} = \sup_{x \in \tilde{D}_\varepsilon} [e^{\mathfrak{p}(\varepsilon s)|z|} \|\varphi\|_{C^{m,\tau}(B_1(x))}], \quad x = (s, z). \tag{4.14}$$

We also recall the definition of $k(\varepsilon s)$ in (1.11).

PROPOSITION 4.1 *Let $k_2(\bar{s}) < k_1(\bar{s}) < k_0(\bar{s})$, $K(\bar{s})$ be smooth positive L-periodic functions in $\bar{s}$, and $\tau \in (0, 1)$. Then, if $V(\bar{s}), K^2(\bar{s}) > k_2^2(\bar{s})$ and if $\|\tilde{\psi}_\varepsilon\|_{C^\tau_{k_0}}, \|S_\varepsilon(\tilde{\psi}_\varepsilon)\|_{C^\tau_{k_0}} \le 1$, there exists a positive constant C depending on $\bar{\delta}$, τ, k, k_0, k_1, and k_2 such that given any ϕ with $\|\phi\|_{C^{1,\tau}_{k_1}} \le 1$, problem (4.12) has a unique solution $\varphi(\phi)$ whose restriction to $\tilde{D}_\varepsilon$ satisfies*

$$\begin{aligned} \|\varphi(\phi)\|_{C^\tau_{-k_2}} \le C \Big(& e^{-\varepsilon^{-\bar{\delta}} \inf \frac{k_2 + k_0}{K}} \|S_\varepsilon(\tilde{\psi}_\varepsilon)\|_{C^\tau_{k_0}} \\ & + e^{-\varepsilon^{-\bar{\delta}} \inf \frac{k_1 + k_2}{K}} \|\phi\|_{C^{1,\tau}_{k_1}} \Big). \end{aligned} \tag{4.15}$$

Moreover, if $\tilde{\psi}^1_\varepsilon, \tilde{\psi}^2_\varepsilon$ satisfy $\|S_\varepsilon(\tilde{\psi}^j_\varepsilon)\|_{C^\tau_{k_0}} \le 1$, $j = 1, 2$, if $\|\phi_j\|_{C^{1,\tau}_{k_1}} \le 1$, $j = 1, 2$, and if $\varphi_j(\phi_j)$, $j = 1, 2$, are the corresponding solutions, for the restrictions *to $\tilde{D}_\varepsilon$, we also have*

$$\|\varphi(\phi_1) - \varphi(\phi_2)\|_{C^\tau_{-k_2}} \le C\Big(e^{-\varepsilon^{-\bar{\delta}} \inf \frac{k_2+k_0}{K}} \big\| S_\varepsilon(\tilde{\psi}^1_\varepsilon) - S_\varepsilon(\tilde{\psi}^2_\varepsilon)\big\|_{C^\tau_{k_0}} + e^{-\varepsilon^{-\bar{\delta}} \inf \frac{k_1+k_2}{K}} \|\phi_1 - \phi_2\|_{C^{1,\tau}_{k_1}}\Big). \tag{4.16}$$

Remark 4.2.

(1) The choice of the norm in (4.14) is done for considering functions that grow at most like $e^{-\mathfrak{p}(\varepsilon s)|z|}$, and in particular functions that decay at infinity if $\mathfrak{p}$ is positive. In the left-hand side of (4.15) we have a negative exponent, representing the fact that φ can grow as $|z|$ increases. However (we will later take k_0, k_1, k_2, and K very close), the coefficients in the right-hand side are so tiny that φ is everywhere small in $\tilde{D}_\varepsilon$, and indeed with an even smaller bound for $|z|$ close to 0. This reflects the fact that the support of the right-hand side in (4.12) is

$$\frac{\varepsilon^{-\bar{\delta}}}{K(\varepsilon s)} < |z| < \frac{\varepsilon^{-\bar{\delta}} + 1}{K(\varepsilon s)},$$

so φ should decay away from this set.

(2) We introduced the functions k_0, k_1, and k_2 for technical reasons, since we want to allow some flexibility for the (exponential) decay rate in $|z|$.

PROOF: We prove the result only when the manifold M in (NLS$_\varepsilon$) is compact. For the modifications needed for $M = \mathbb{R}^n$, see Remark 4.3(2).

Consider a smooth, nondecreasing cutoff function $\chi : [0,1] \to [0,1]$ satisfying

$$\begin{cases} \chi(t) = 0 & \text{for } t \le \frac{1}{4}, \\ \chi(t) = t & \text{for } t \ge \frac{3}{4}, \\ 0 \le \chi'(t) \le 4 & \text{for all } t, \\ 0 \le \chi''(t) \le 16 & \text{for all } t. \end{cases}$$

Next, given a large constant $\mathcal{B}$ (to be specified later) depending only on V and k_2, we define $\tilde{\chi}(\bar{s}, |z|)$ as

$$\tilde{\chi}(\bar{s},|z|) = \begin{cases} \mathcal{B}\chi\big(\frac{|z|}{\mathcal{B}}\big) & \text{for } |z| \le \mathcal{B}, \\ |z| & \text{for } \mathcal{B} \le |z| \le \frac{\varepsilon^{-\bar{\delta}}}{k_2(\bar{s})} - 1, \\ |z| - \frac{\varepsilon^{-\bar{\delta}}}{k_2(\bar{s})} - \frac{1}{2} & \\ \qquad - 2\chi\big(|z| - \frac{\varepsilon^{-\bar{\delta}}}{k_2(\bar{s})} - \frac{1}{2}\big) & \text{for } \frac{\varepsilon^{-\bar{\delta}}}{k_2(\bar{s})} - 1 \le |z| \le \frac{\varepsilon^{-\bar{\delta}}}{k_2(\bar{s})} + 1, \\ 2\frac{\varepsilon^{-\bar{\delta}}}{k_2(\bar{s})} - |z| & \text{for } \frac{\varepsilon^{-\bar{\delta}}}{k_2(\bar{s})} + 1 \le |z| \le 2\frac{\varepsilon^{-\bar{\delta}}}{k_2(\bar{s})} - \mathcal{B}, \\ \mathcal{B}\chi\big(\frac{2\varepsilon^{-\bar{\delta}}/k_2(\bar{s}) - |z|}{\mathcal{B}}\big) & \text{for } 2\frac{\varepsilon^{-\bar{\delta}}}{k_2(\bar{s})} - \mathcal{B} \le |z| \le 2\frac{\varepsilon^{-\bar{\delta}}}{k_2(\bar{s})}, \\ 0 & \text{for } |z| \ge 2\frac{\varepsilon^{-\bar{\delta}}}{k_2(\bar{s})}. \end{cases}$$

By our choice of χ, the function $\tilde{\chi}$ satisfies the following inequalities (where, here, the gradient and the Laplacian are taken with respect to the Euclidean metric)

$$|\nabla_z \tilde{\chi}| \le 1, \quad \Delta_z \tilde{\chi} \le \frac{16 + 4(n-2)}{\mathcal{B}}.$$

Using the above coordinates (s, z), we next define the barrier function $\mathfrak{u} : M_\varepsilon \to \mathbb{R}$ as

$$\mathfrak{u}(s, z) = e^{k_2(\varepsilon s)\tilde{\chi}(\varepsilon s, z)} \quad \text{for } |z| \le 2\,\frac{\varepsilon^{-\bar{\delta}}}{k_2(\varepsilon s)},$$

and we extend $\mathfrak{u}$ identically equal to 1 elsewhere. By our choice of $\tilde{\chi}$, this function is indeed smooth and strictly positive on the whole M_ε. We consider next the linear equation (motivated by (4.13))

$$\mathcal{L}_{\tilde{\psi}_\varepsilon} \varphi = \vartheta \quad \text{on } M_\varepsilon,$$

where $\vartheta : M_\varepsilon \to \mathbb{R}$ is Hölder-continuous (with $\operatorname{supp}(\vartheta) \Subset \tilde{D}_\varepsilon$; see (4.18) below). Since the operator $\mathcal{L}_{\tilde{\psi}_\varepsilon}$ is uniformly elliptic, the latter equation is (uniquely) solvable, and we would next like to derive some pointwise estimates on its solutions. To this aim we define

$$v(x) = \frac{\varphi(x)}{\mathfrak{u}(x)}, \quad x \in M_\varepsilon.$$

With this notation, we have that

$$\mathfrak{u}\mathcal{L}_{\tilde{\psi}_\varepsilon} v - v\Delta_{g_\varepsilon}\mathfrak{u} - 2\nabla_{g_\varepsilon} v \cdot \nabla_{g_\varepsilon}\mathfrak{u} = \vartheta \quad \text{on } M_\varepsilon.$$

Using the expression of the metric coefficients in the coordinates (s, z) (see Lemma 2.1), (4.4), and the properties of the cutoff function $\tilde{\chi}$, we easily check that

$$\Delta_{g_\varepsilon}\mathfrak{u} \begin{cases} \le (k_2(\bar{s})^2 + o_{\mathcal{B}}(1) + o_\varepsilon(1))\mathfrak{u} & \text{for } |z| \le \frac{\varepsilon^{-\bar{\delta}}}{k_2(\bar{s})}, \\ = 0 & \text{elsewhere,} \end{cases}$$

where $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$ and $o_{\mathcal{B}}(1) \to 0$ as $\mathcal{B} \to +\infty$. Therefore we obtain that the function v satisfies

$$\begin{cases} \left|\left(\mathcal{L}_{\tilde{\psi}_\varepsilon} - k_2^2(\bar{s}) + o_{\mathcal{B}}(1) + o_\varepsilon(1)\right)v\right| \le \frac{|\vartheta|}{\mathfrak{u}} & \text{for } |z| \le 2\frac{\varepsilon^{-\bar{\delta}}}{k_2(\bar{s})}, \\ \mathcal{L}_{\tilde{\psi}_\varepsilon} v = \frac{\vartheta}{\mathfrak{u}} & \text{elsewhere.} \end{cases}$$

Since we assumed $V(\bar{s}) > k_2(\bar{s})^2$, we obtain that $V - k_2(\bar{s})^2 + o_{\mathcal{B}}(1) + o_\varepsilon(1)$ is strictly positive (provided $\mathcal{B}$ is sufficiently large and ε sufficiently small) for $|z| \le 2(\varepsilon^{-\bar{\delta}}/k_2(\bar{s}))$, and hence the function v satisfies a uniformly elliptic equation with a nonnegative coefficient in the zeroth-order term with right-hand side given by $\frac{\vartheta}{\mathfrak{u}}$. Therefore from the maximum principle we derive the estimate

$$\max_{M_\varepsilon} |v| \le C \max_{M_\varepsilon} \frac{|\vartheta|}{\mathfrak{u}},$$

where C depends on the uniform lower bound of the above coefficient. The latter formula clearly implies

$$|\varphi(x)| \leq C\,\mathfrak{u}(x) \max_{M_\varepsilon} \frac{\vartheta}{\mathfrak{u}} \quad \text{for every } x \in M_\varepsilon.$$

We next define the weighted norm

$$\|\varphi\|_{m,\tau,\mathfrak{u}} := \sup_{x\in M_\varepsilon} \left\| \frac{\varphi}{\mathfrak{u}} \right\|_{C^{m,\tau}(B_1(x))},$$

which is equivalent (with constants depending on $\mathscr{B}$ only) to $\|\cdot\|_{C^{m,\tau}_{-k_2}}$ on the set $\tilde{D}_\varepsilon$. Using the explicit form of the function $\mathfrak{u}$ and standard elliptic regularity results, we can improve the latter inequality to

$$\|\varphi\|_{2,\tau,\mathfrak{u}} \leq C\,\|\vartheta\|_{0,\tau,\mathfrak{u}}. \tag{4.17}$$

The proof of the proposition now follows from this linear estimate and the contraction mapping theorem; in fact, defining

$$\begin{aligned} G_{\phi,\varepsilon}(\varphi) = {} & (1-\bar{\eta}_\varepsilon(s,z))[S_\varepsilon(\tilde{\psi}_\varepsilon) + N_\varepsilon(\bar{\eta}_\varepsilon(z)\phi + \varphi)] \\ & + 2\nabla_{g_\varepsilon}\bar{\eta}_\varepsilon(z)\cdot\nabla_{g_\varepsilon}\phi + \Delta_{g_\varepsilon}\bar{\eta}_\varepsilon(z)\phi, \end{aligned}$$

equation (4.12) is equivalent to

$$\varphi = \mathscr{L}^{-1}_{\tilde{\psi}_\varepsilon} G_{\phi,\varepsilon}(\varphi). \tag{4.18}$$

First of all, notice that $\mathscr{L}_{\tilde{\psi}_\varepsilon}$ is invertible since we are assuming M (and hence M_ε) to be compact (see the beginning of the proof). Second, to apply (4.17), we need to estimate $\|G_{\phi,\varepsilon}(\varphi)\|_{0,\tau,\mathfrak{u}}$, together with its Lipschitz dependence in φ.

Let us consider, for instance, the term $(1-\bar{\eta}_\varepsilon(s,z))S_\varepsilon(\tilde{\psi}_\varepsilon)$. Using the fact that $(1-\bar{\eta}_\varepsilon)$ is 0 for $|z| \leq \varepsilon^{-\bar{\delta}}/K(\varepsilon s)$, that $S_\varepsilon(\tilde{\psi}_\varepsilon)$ is 0 for $|z| \geq (\varepsilon^{-\bar{\delta}}+1)/K(\varepsilon s)$, and that $k_2 < K$, we obtain

$$\begin{aligned} \|(1-\bar{\eta}_\varepsilon)S_\varepsilon(\tilde{\psi}_\varepsilon)\|_{0,\tau,\mathfrak{u}} & \leq C \sup_{\left\{\frac{\varepsilon^{-\bar{\delta}}}{K(\varepsilon s)} \leq |z| \leq \frac{\varepsilon^{-\bar{\delta}}+1}{K(\varepsilon s)}\right\}} \left\| \frac{S_\varepsilon(\tilde{\psi}_\varepsilon)}{\mathfrak{u}} \right\|_{C^\tau(B_1(z))} \\ & \leq C e^{-\varepsilon^{-\bar{\delta}} \inf \frac{k_0+k_2}{K}} \|S_\varepsilon(\tilde{\psi}_\varepsilon)\|_{C^\tau_{k_0}}. \end{aligned} \tag{4.19}$$

Now to estimate the remaining terms of $G_{\phi,\varepsilon}$, we notice that

$$|N_\varepsilon(\bar{\eta}_\varepsilon(z)\phi+\varphi)| \leq \begin{cases} C|\tilde{\psi}_\varepsilon|^{p-2}|\bar{\eta}_\varepsilon(z)\phi+\varphi|^2 & \text{if } |\bar{\eta}_\varepsilon(z)\phi+\varphi| \leq |\tilde{\psi}_\varepsilon|, \\ |\bar{\eta}_\varepsilon(z)\phi+\varphi|^p & \text{otherwise.} \end{cases}$$

Since $p > 1$, we can find a number $\zeta \in (0,1)$ such that $p - 2 + 1 - \zeta > 0$, so the last formula implies

$$|N_\varepsilon(\bar\eta_\varepsilon(z)\phi + \varphi)| \le C\big(|\tilde\psi_\varepsilon|^{p-1-\zeta}(|\bar\eta_\varepsilon(z)\phi|^\zeta + |\varphi|^\zeta)(|\bar\eta_\varepsilon(z)\phi| + |\varphi|) + |\bar\eta_\varepsilon(z)\phi|^p + |\varphi|^p\big). \tag{4.20}$$

Using the fact that $\|\tilde\psi_\varepsilon\|_{C^\tau_{k_0}}, \|\phi\|_{C^{1,\tau}_{k_1}} \le 1$ and reasoning as for (4.19), after some computations we deduce (assuming $\|\varphi\|_\infty \le 1$, which will be verified later)

$$\begin{aligned} &\|G_{\phi,\varepsilon}(\varphi)\|_{0,\tau,\mathrm{u}} \\ &\quad\le C\Big(e^{-\varepsilon^{-\bar\delta}\inf\frac{k_2+k_0}{K}}\|S_\varepsilon(\tilde\psi_\varepsilon)\|_{C^\tau_{k_0}} + e^{-\varepsilon^{-\bar\delta}\inf\frac{k_2+k_1}{K}}\|\phi\|_{C^{1,\tau}_{k_1}} \\ &\qquad + e^{-\varepsilon^{-\bar\delta}\inf\frac{pk_1+k_2}{K}}\|\phi\|_{C^{0,\tau}_{k_1}} + e^{-\varepsilon^{-\bar\delta}\inf\frac{(p-1)k_0+k_1+k_2}{K}}\|\phi\|_{C^{0,\tau}_{k_1}}\Big) \\ &\qquad + C\big(e^{-\varepsilon^{-\bar\delta}(p-1-\zeta)\inf\frac{k_0}{K}} + \|\varphi\|_\infty^{p-1}\big)\|\varphi\|_{0,\tau,\mathrm{u}}. \end{aligned} \tag{4.21}$$

Similarly, for two functions φ_1 and φ_2 with $\|\varphi_1\|_\infty, \|\varphi_2\|_\infty \le 1$ and with finite $\|\cdot\|_{0,\tau,\mathrm{u}}$ norm, we have

$$\begin{aligned} &\|G_{\phi,\varepsilon}(\varphi_1) - G_{\phi,\varepsilon}(\varphi_2)\|_{0,\tau,\mathrm{u}} \le \\ &\qquad \big(e^{-\varepsilon^{-\bar\delta}(p-1-\zeta)\inf\frac{k_0}{K}} + \|\varphi_1\|_\infty^{p-1} + \|\varphi_2\|_\infty^{p-1}\big)\|\varphi_1 - \varphi_2\|_{0,\tau,\mathrm{u}}. \end{aligned} \tag{4.22}$$

We now consider the map $\varphi \mapsto G_{\phi,\varepsilon}(\varphi)$ in the set

$$\mathfrak{B} = \Big\{\varphi : \|\varphi\|_{0,\tau,\mathrm{u}} \le C_1\big(e^{-\varepsilon^{-\bar\delta}\inf\frac{k_2+k_0}{K}}\|S_\varepsilon(\tilde\psi_\varepsilon)\|_{C^\tau_{k_0}} + e^{-\varepsilon^{-\bar\delta}\inf\frac{k_2+k_1}{K}}\|\phi\|_{C^{1,\tau}_{k_1}}\big)\Big\},$$

where C_1 is a sufficiently large positive constant. Notice that if $\varphi \in \mathfrak{B}$, then $\|\varphi\|_\infty = o_\varepsilon(1)$. From (4.21) and (4.22), it then follows that this map is a contraction from $\mathfrak{B}$ into itself, endowed with the above norm, and therefore a solution φ exists as a fixed point of $G_{\phi,\varepsilon}$. The fact that $k_2 < K$ implies that the norm $\|\cdot\|_{C^\tau_{-k_2}}$ is equivalent to $\|\cdot\|_{0,\tau,\mathrm{u}}$ in $\tilde D_\varepsilon$ (see also the comments in Remark 4.2), so we obtain (4.15). A similar reasoning, still based on regularity theory and elementary inequalities, also yields (4.16). □

Remark 4.3.

(1) From elliptic regularity theory it follows that in (4.15)–(4.16) the norm $\|\cdot\|_{C^\tau_{-k_2}}$ can be replaced by the stronger $\|\cdot\|_{C^{2,\tau}_{-k_2}}$, yielding an estimate in the norm $\|\cdot\|_{C^{\tau'}_{-k_2}}$ for any $\tau' \in (\tau, 1)$.

(2) In the case $M = \mathbb{R}^n$, the above proof needs to be slightly modified: in fact, the invertibility of $\mathcal{L}_{\tilde\psi_\varepsilon}$ will be guaranteed provided we work in an appropriate class of functions Y decaying exponentially at infinity. To guarantee this

condition, we can vary the form of the barrier function u in order that it both remains a supersolution of $\mathcal{L}_{\tilde{\psi}_\varepsilon} = 0$ and decays exponentially to 0 at infinity. This is indeed possible using the uniform positive lower bound on V; see (1.1). We omit the details of this construction.

As a consequence of Proposition 4.1, we obtain that the solvability of (NLS$_\varepsilon$) is equivalent to that of (4.11).

PROPOSITION 4.4 *Suppose the assumptions of Proposition* 4.1 *hold, and consider the corresponding* $\varphi = \varphi(\phi)$. *Then* $\psi = \tilde{\psi}_\varepsilon + \bar{\eta}_\varepsilon(z)\phi + \varphi(\phi)$ *solves* (1.16) *if and only if* $\phi \in H_{\tilde{D}_\varepsilon}$ *satisfies*

$$L_\varepsilon(\phi) = \tilde{S}_\varepsilon(\phi) \quad \text{in } \tilde{D}_\varepsilon, \tag{4.23}$$

with

$$\begin{aligned}\tilde{S}_\varepsilon(\phi) = {} & S_\varepsilon(\tilde{\psi}_\varepsilon) + N_\varepsilon(\bar{\eta}_\varepsilon(z)\phi + \varphi(\phi)) + |\tilde{\psi}_\varepsilon|^{p-1}\varphi(\phi) \\ & + (p-1)|\tilde{\psi}_\varepsilon|^{p-3}\tilde{\psi}_\varepsilon \Re(\tilde{\psi}_\varepsilon \bar{\varphi(\phi)}) \quad \text{in } \tilde{D}_\varepsilon,\end{aligned} \tag{4.24}$$

where $\bar{\eta}_\varepsilon$, S_ε, *and* N_ε *are given in* (4.5), (4.6), *and* (4.8), *respectively.*

4.3 Construction of an Approximate Kernel for L_ε

We perform here some preliminary analysis useful to understand the spectral properties of L_ε. More precisely, we consider a *model* case, when the domain $\tilde{D}_\varepsilon$ (see (4.10)) is replaced by $[0, \frac{L}{\varepsilon}] \times \mathbb{R}^{n-1}$ and the profile of approximate solutions is independent of the variable s (only the phase varies, periodically in s). As in formula (1.9), we consider positive constants $\widehat{V}$, $\widehat{h}$, and $\widehat{k}$ satisfying

$$\widehat{h} = (\widehat{f}^2 + \widehat{V})^{\frac{1}{p-1}}, \quad \widehat{k} = (\widehat{f}^2 + \widehat{V})^{\frac{1}{2}}. \tag{4.25}$$

Our goal is to study the following eigenvalue problem, which models our linearized equation:

$$\begin{aligned}&\widehat{L}_\varepsilon u = \lambda u \quad \text{in } [0, \tfrac{L}{\varepsilon}] \times \mathbb{R}^{n-1}, \\ &\widehat{L}_\varepsilon u = -\Delta_{\widehat{g}_\varepsilon} u + \widehat{V} u - \widehat{h}^{p-1} U(\widehat{k} y)^{p-1} u \\ &\qquad - (p-1)\widehat{h}^{p-1} U(\widehat{k} y)^{p-1} e^{-i\widehat{f}s} \Re(e^{-i\widehat{f}s} \bar{u}),\end{aligned} \tag{4.26}$$

and in particular we would like to characterize the small eigenvalues and the corresponding eigenfunctions.

First of all, we can write u as

$$u = e^{-i\widehat{f}s}(u_r + i u_i)$$

for some real u_r and u_i. With this notation, we are reduced to studying the coupled system

$$\begin{cases} -\Delta_{\widehat{g}_\varepsilon} u_r + (\widehat{V} + \widehat{f}^2) u_r \\ \quad - p\widehat{h}^{p-1} U(\widehat{k} y)^{p-1} u_r - 2\widehat{f} \frac{\partial u_i}{\partial s} = \lambda u_r & \text{in } [0, \frac{L}{\varepsilon}] \times \mathbb{R}^{n-1}, \\ -\Delta_{\widehat{g}_\varepsilon} u_i + (\widehat{V} + \widehat{f}^2) u_i \\ \quad - \widehat{h}^{p-1} U(\widehat{k} y)^{p-1} u_i + 2\widehat{f} \frac{\partial u_r}{\partial s} = \lambda u_i & \text{in } [0, \frac{L}{\varepsilon}] \times \mathbb{R}^{n-1}. \end{cases}$$

Making the change of variables $y \mapsto \widehat{k} y$ and using (4.25), we are reduced to

$$\begin{cases} -\frac{1}{\widehat{k}^2} \frac{\partial^2 u_r}{\partial s^2} - \Delta_y u_r + u_r \\ \quad - pU(y)^{p-1} u_r - \frac{2\widehat{f}}{\widehat{k}^2} \frac{\partial u_i}{\partial s} = \frac{\lambda}{\widehat{k}^2} u_r & \text{in } [0, \frac{L}{\varepsilon}] \times \mathbb{R}^{n-1}, \\ -\frac{1}{\widehat{k}^2} \frac{\partial^2 u_i}{\partial s^2} - \Delta_y u_i + u_i \\ \quad - U(y)^{p-1} u_i + \frac{2\widehat{f}}{\widehat{k}^2} \frac{\partial u_r}{\partial s} = \frac{\lambda}{\widehat{k}^2} u_i & \text{in } [0, \frac{L}{\varepsilon}] \times \mathbb{R}^{n-1}. \end{cases} \tag{4.27}$$

It is now convenient to use a Fourier decomposition in s of u_r and u_i, writing

$$u_r = \sum_j \left(\cos\left(\frac{2\pi\varepsilon j s}{L} \right) u_{r,c,j}(y) + \sin\left(\frac{2\pi\varepsilon j s}{L} \right) u_{r,s,j}(y) \right),$$
$$u_i = \sum_j \left(\cos\left(\frac{2\pi\varepsilon j s}{L} \right) u_{i,c,j}(y) + \sin\left(\frac{2\pi\varepsilon j s}{L} \right) u_{i,s,j}(y) \right),$$

where $s \in [0, \frac{L}{\varepsilon}]$ and $y \in \mathbb{R}^{n-1}$. In this way the functions $u_{r,c,j}, u_{r,s,j}, u_{i,c,j}$, and $u_{i,s,j}$ satisfy the following systems of equations:

$$\begin{cases} -\Delta_y u_{r,c,j} + \left(1 + \frac{4\pi^2 \varepsilon^2 j^2}{L^2 \widehat{k}^2} \right) u_{r,c,j} \\ \quad - pU(y)^{p-1} u_{r,c,j} - \frac{4\pi \widehat{f} \varepsilon j}{L\widehat{k}^2} u_{i,s,j} = \frac{\lambda}{\widehat{k}^2} u_{r,c,j} & \text{in } \mathbb{R}^{n-1}, \\ -\Delta_y u_{i,s,j} + \left(1 + \frac{4\pi^2 \varepsilon^2 j^2}{L^2 \widehat{k}^2} \right) u_{i,s,j} \\ \quad - U(y)^{p-1} u_{i,s,j} - \frac{4\pi \widehat{f} \varepsilon j}{L\widehat{k}^2} u_{r,c,j} = \frac{\lambda}{\widehat{k}^2} u_{i,s,j} & \text{in } \mathbb{R}^{n-1}, \end{cases}$$

$$\begin{cases} -\Delta_y u_{r,s,j} + \left(1 + \frac{4\pi^2 \varepsilon^2 j^2}{L^2 \widehat{k}^2} \right) u_{r,s,j} \\ \quad - pU(y)^{p-1} u_{r,s,j} + \frac{4\pi \widehat{f} \varepsilon j}{L\widehat{k}^2} u_{i,c,j} = \frac{\lambda}{\widehat{k}^2} u_{r,s,j} & \text{in } \mathbb{R}^{n-1}, \\ -\Delta_y u_{i,c,j} + \left(1 + \frac{4\pi^2 \varepsilon^2 j^2}{L^2 \widehat{k}^2} \right) u_{i,c,j} \\ \quad - U(y)^{p-1} u_{i,c,j} + \frac{4\pi \widehat{f} \varepsilon j}{L\widehat{k}^2} u_{r,s,j} = \frac{\lambda}{\widehat{k}^2} u_{i,c,j} & \text{in } \mathbb{R}^{n-1}. \end{cases}$$

If we set

$$\frac{2\pi\varepsilon j}{L\widehat{k}} = \alpha, \quad \frac{2\widehat{f}}{\widehat{k}} = \mu, \quad \text{and} \quad \tilde{\lambda} = \frac{\lambda}{\widehat{k}^2},$$

then the latter two systems are equivalent to the following:

$$\begin{cases} -\Delta_y u + (1+\alpha^2)u - pU(y)^{p-1}u + \mu\alpha v = \tilde{\lambda} u & \text{in } \mathbb{R}^{n-1}, \\ -\Delta_y v + (1+\alpha^2)v - U(y)^{p-1}v + \mu\alpha u = \tilde{\lambda} v & \text{in } \mathbb{R}^{n-1}. \end{cases} \tag{4.28}$$

The equivalence with the second system is obvious: for the first one it is sufficient to switch the sign of the second component. We characterize the spectrum of the last system in the next proposition: the value of μ is fixed, while α is allowed to vary. We remark that it is irrelevant for our purposes to take α positive or negative, since we can still switch the sign of one of the two components.

PROPOSITION 4.5 *Let $\eta_\alpha, \sigma_\alpha$, and τ_α denote the first three eigenvalues of* (4.28). *Then there exists $\mu_0 > 0$ such that for $\mu \in [0, \mu_0]$ the following properties hold*:

(i) *There exists $\alpha_0 > 0$ such that η_α is simple, increasing, and differentiable in α for $\alpha \in [0, \alpha_0]$, $\frac{\partial \eta_\alpha}{\partial \alpha} > 0$ for $\alpha \in (0, \alpha_0]$, $\eta_0 < 0$, and $\eta_{\alpha_0} > 0$.*
(ii) *The eigenvalue σ_α is 0 for $\alpha = 0$ with multiplicity n, it satisfies $\frac{\partial \sigma_\alpha}{\partial \alpha} > 0$ for α small and positive, and stays uniformly bounded away from 0 if α stays bounded away from 0.*
(iii) *τ_α is strictly positive and stays uniformly bounded away from 0 for all α's.*
(iv) *The eigenfunction u_α corresponding to η_α is simple, radial in y, and radially decreasing and depends smoothly on α; for $\alpha = 0$ the eigenfunction of* (4.28) *corresponding to $\eta_0 < 0$ is of the form $(\tilde{Z}, 0)$ with $\tilde{Z}$ radial and radially decreasing, while those corresponding to $\sigma_0 = 0$ are linear combinations of $(\nabla_{y_j} U, 0)$, $j = 1, \dots, n-1$, and $(0, U)$.*
(v) *Let $\bar{\alpha}$ be the unique α for which $\eta_{\bar{\alpha}} = 0$ (see* (i)); *then the corresponding eigenfunction is of the form (Z, W) for some radial functions Z and W satisfying the decay $|Z| + |W| \leq Ce^{-(1+\widehat{\eta})|y|}$ for some constants $C, \widehat{\eta} > 0$.*

PROOF: This result is known for $\mu = 0$; see, for example, [34, prop. 4.2] and [39, prop. 2.9].

For $\mu \neq 0$ sufficiently small, the functions $\alpha \mapsto \eta_\alpha$, $\alpha \mapsto \sigma_\alpha$, and $\alpha \mapsto \tau_\alpha$ will be C^1-close to those corresponding to $\mu = 0$; therefore, to prove (i)–(iv), it is sufficient to show that η_α and σ_α are twice-differentiable in α for α small, that $\partial\eta_\alpha/\partial\alpha = \partial\sigma_\alpha/\partial\alpha = 0$, and that $\partial^2\eta_\alpha/\partial\alpha^2, \partial^2\sigma_\alpha/\partial\alpha^2 > 0$.

We prove this statement only heuristically, but a rigorous proof can easily be derived. Differentiating

$$\begin{cases} -\Delta_y u_\alpha + (1+\alpha^2)u_\alpha - pU(y)^{p-1}u_\alpha + \mu\alpha v_\alpha = \eta_\alpha u_\alpha & \text{in } \mathbb{R}^{n-1}, \\ -\Delta_y v_\alpha + (1+\alpha^2)v_\alpha - U(y)^{p-1}v_\alpha + \mu\alpha u_\alpha = \eta_\alpha v_\alpha & \text{in } \mathbb{R}^{n-1}, \end{cases} \tag{4.29}$$

with respect to α, we find

$$\begin{cases} -\Delta_y \frac{\partial u_\alpha}{\partial \alpha} + (1+\alpha^2)\frac{\partial u_\alpha}{\partial \alpha} - pU(y)^{p-1}\frac{\partial u_\alpha}{\partial \alpha} \\ \quad + \mu\alpha\frac{\partial v_\alpha}{\partial \alpha} + 2\alpha u_\alpha + \mu v_\alpha = \eta_\alpha \frac{\partial u_\alpha}{\partial \alpha} + \frac{\partial \eta_\alpha}{\partial \alpha} u_\alpha & \text{in } \mathbb{R}^{n-1}, \\ -\Delta_y \frac{\partial v_\alpha}{\partial \alpha} + (1+\alpha^2)\frac{\partial v_\alpha}{\partial \alpha} - U(y)^{p-1}\frac{\partial v_\alpha}{\partial \alpha} \\ \quad + \mu\alpha\frac{\partial u_\alpha}{\partial \alpha} + 2\alpha v_\alpha + \mu u_\alpha = \eta_\alpha \frac{\partial v_\alpha}{\partial \alpha} + \frac{\partial \eta_\alpha}{\partial \alpha} v_\alpha & \text{in } \mathbb{R}^{n-1}. \end{cases} \tag{4.30}$$

To compute $\frac{\partial \eta_\alpha}{\partial \alpha}$ at $\alpha = 0$ it is sufficient to multiply the first equation by u_α and the second by v_α to take the sum and integrate; if we choose $\frac{\partial u_\alpha}{\partial \alpha}$ and $\frac{\partial v_\alpha}{\partial \alpha}$ so that $\int_{\mathbb{R}^{n-1}} u_\alpha \frac{\partial u_\alpha}{\partial \alpha} + v_\alpha \frac{\partial v_\alpha}{\partial \alpha} = 0$ (choosing, e.g., $\int(u_\alpha^2 + v_\alpha^2) = 1$ for all α's), then with an integration by parts we find that

$$\frac{\partial \eta_\alpha}{\partial \alpha}\bigg|_{\alpha=0} \int_{\mathbb{R}^{n-1}} u_\alpha^2 + v_\alpha^2 = 2\mu \int_{\mathbb{R}^{n-1}} u_0 v_0.$$

Using the fact that $v_0 = 0$ (see (iv)), we then obtain $\frac{\partial \eta_\alpha}{\partial \alpha}|_{\alpha=0} = 0$. The same argument applies for evaluating $\frac{\partial \sigma_\alpha}{\partial \alpha}|_{\alpha=0}$, since the eigenfunctions corresponding to $\sigma_0 = 0$ always have one component vanishing.

To compute the second derivative with respect to α, we differentiate (4.30) once more at $\alpha = 0$, obtaining

$$\begin{cases} -\Delta_y \frac{\partial^2 u_\alpha}{\partial \alpha^2} + \frac{\partial^2 u_\alpha}{\partial \alpha^2} \\ \quad - pU(y)^{p-1}\frac{\partial^2 u_\alpha}{\partial \alpha^2} + 2\mu\frac{\partial v_\alpha}{\partial \alpha} + 2u_0 = \frac{\partial^2 \eta_\alpha}{\partial \alpha^2} u_0 & \text{in } \mathbb{R}^{n-1}, \\ -\Delta_y \frac{\partial^2 v_\alpha}{\partial \alpha^2} + \frac{\partial^2 v_\alpha}{\partial \alpha^2} \\ \quad - U(y)^{p-1}\frac{\partial^2 v_\alpha}{\partial \alpha^2} + 2\mu\frac{\partial u_\alpha}{\partial \alpha} + 2v_0 = \frac{\partial^2 \eta_\alpha}{\partial \alpha^2} v_0 & \text{in } \mathbb{R}^{n-1}. \end{cases}$$

As for the previous case we get

$$\frac{\partial^2 \eta_\alpha}{\partial \alpha^2}\bigg|_{\alpha=0} = 2 + 2\mu \int_{\mathbb{R}^{n-1}} \left(u_0 \frac{\partial v_\alpha}{\partial \alpha}\bigg|_{\alpha=0} + v_0 \frac{\partial u_\alpha}{\partial \alpha}\bigg|_{\alpha=0} \right);$$

so, using the smallness of μ, the claim follows.

For the second derivative of σ_α the procedure is similar, but notice that in this case we might obtain a multivalued function, due to the multiplicity (n) of σ_0 (see (ii)). However, if in the last formula we plug in the corresponding eigenfunctions (see (iv)), we still obtain a sign condition for each of the two branches of σ_α (one of them will have multiplicity $n-1$ by the rotation invariance of the equations). □

Remark 4.6. Using the same argument in the previous proof, we can show that

$$\frac{\partial \eta}{\partial \alpha}\bigg|_{\alpha=\bar{\alpha}} = 2\bar{\alpha} + 2\mu \int_{\mathbb{R}^{n-1}} Z_{\bar{\alpha}} W_{\bar{\alpha}}.$$

Remark 4.7. Proposition 4.5 is the only result where the smallness of the constant $\mathcal{A}$ is used; see Theorem 1.1. We remark that $V \equiv \widehat{V}$ implies $\widehat{f} = \mathcal{A}\widehat{h}^\sigma$, and that $\mu = 2(\widehat{f}/\widehat{k})$, so the smallness of $\mathcal{A}$ is equivalent to that of μ. Notice that by (1.1) and (1.9), when $\mathcal{A} \to 0$, $\widehat{h}$ and $\widehat{k}$ stay uniformly bounded and bounded away from 0.

We believe that dropping this smallness condition might lead to further resonance phenomena in addition to those encountered here (see the introduction and the last section).

Remark 4.8. Considering (4.30) with σ_α replacing η_α and for $\alpha = 0$, we find that $\mathcal{L}_r^0 \frac{\partial u_\alpha}{\partial \alpha}|_{\alpha=0} = -\mu v_0$ and $\mathcal{L}_i^0 \frac{\partial v_\alpha}{\partial \alpha}|_{\alpha=0} = -\mu u_0$, where

$$\mathcal{L}_r^0 v = -\Delta_y v + v - pU(y)^{p-1} v,$$
$$\mathcal{L}_i^0 v = -\Delta_y v + v - U(y)^{p-1} v.$$

Since for $\alpha = 0$ we have $(u_0, v_0) = (\partial_j U, 0)$ or $(u_0, v_0) = (0, U)$ (see (iv)) and

$$\mathcal{L}_r^0\left(-\frac{1}{p-1}U - \frac{1}{2}\nabla U(y)\cdot y\right) = U, \quad \mathcal{L}_i^0(y_j U(y)) = -2\partial_j U, \tag{4.31}$$

see Subsection 3.2, we find that

$$\left.\frac{\partial v_\alpha}{\partial \alpha}\right|_{\alpha=0} = \frac{\mu}{2}\, y_j U(y), \quad \left.\frac{\partial u_\alpha}{\partial \alpha}\right|_{\alpha=0} = \mu\left(\frac{1}{p-1}\,U + \frac{1}{2}\nabla U(y)\cdot y\right).$$

These expressions, together with (3.26) and some integration by parts, allow us to compute explicitly $\partial^2\sigma_\alpha/\partial\alpha^2$, whose values along the two branches are

$$\frac{\partial^2\sigma_\alpha}{\partial\alpha^2} = \frac{2}{(p-1)}\big((p-1) - 2\mathcal{A}^2\theta\widehat{h}^{2\sigma-p+1}\big),$$
$$\frac{\partial^2\sigma_\alpha}{\partial\alpha^2} = \frac{2}{(p-1)}\big((p-1) - 2\mathcal{A}^2\sigma\widehat{h}^{2\sigma-p+1}\big).$$

Therefore, we find that the second derivatives of the eigenfunctions satisfy, respectively, the equations

$$\begin{aligned}\mathcal{L}_r^0 \frac{\partial^2 u_\alpha}{\partial\alpha^2} = {} & \frac{2}{p-1}\big((p-1) - 2\mathcal{A}^2\theta\widehat{h}^{2\sigma-p+1}\big)\nabla_j U \\ & - 2\nabla_j U - 4\mathcal{A}^2\widehat{h}^{2\sigma-p+1} y_j U,\end{aligned} \tag{4.32}$$

$$\begin{aligned}\mathcal{L}_i^0 \frac{\partial^2 v_\alpha}{\partial\alpha^2} = {} & \frac{2}{p-1}\big((p-1) - 2\mathcal{A}^2\sigma\widehat{h}^{2\sigma-p+1}\big) U \\ & - 2U - 8\mathcal{A}^2\widehat{h}^{2\sigma-p+1}\tilde{U}.\end{aligned} \tag{4.33}$$

These formulas will be crucial later on. Below, we will denote for brevity

$$\widehat{\mathfrak{V}}_j := \frac{1}{2}\frac{\partial^2 u_\alpha}{\partial\alpha^2}, \quad j = 1, \dots, n-1, \qquad \widehat{\mathfrak{W}} := \frac{1}{2}\frac{\partial^2 v_\alpha}{\partial\alpha^2}. \tag{4.34}$$

The factor $\frac{1}{2}$ arises in the Taylor expansion of the eigenfunctions in α, and j is the index in (4.32).

We next consider the case of variable coefficients, which can be reduced to the previous one through a localization argument in s. To have a more accurate model for L_ε, the constants $\widehat{k}$ and $\widehat{f}$ in (4.27) have to be substituted with the functions $k(\varepsilon s)$ and $f(\varepsilon s)$ satisfying (1.11). Specifically, in $N\gamma_\varepsilon$ we define

$$(4.35)\qquad \begin{aligned} L^1_\varepsilon u &= -\Delta_{\widehat{g}_\varepsilon} u + V(\varepsilon s)u - h(\varepsilon s)^{p-1}U(k(\varepsilon s)y)^{p-1}u \\ &\quad - (p-1)h(\varepsilon s)^{p-1}U(k(\varepsilon s)y)^{p-1}e^{-i\frac{f(\varepsilon s)}{\varepsilon}}\Re(e^{-i\frac{f(\varepsilon s)}{\varepsilon}}\bar{u}) \end{aligned}$$

(recall the definition of $\widehat{g}_\varepsilon$ in Subsection 4.1; in particular, working with the coordinates (s, y) integrals will be computed using the co-area formula (4.3)).

Before proving rigorous results, we first discuss heuristically what the approximate kernel of L^1_ε should look like. Using Fourier expansions as above (freezing the coefficients at some $\bar{s}$), the profile of the functions that lie in an approximate kernel of L^1_ε will be given by the solution of (recall (4.28))

$$(4.36)\qquad \begin{cases} -\Delta_y u + (1+\alpha^2)u - pU(y)^{p-1}u + 2\frac{f'(\bar{s})}{k(\bar{s})}\alpha v = \tilde{\lambda} u & \text{in } \mathbb{R}^{n-1}, \\ -\Delta_y v + (1+\alpha^2)v - U(y)^{p-1}v + 2\frac{f'(\bar{s})}{k(\bar{s})}\alpha u = \tilde{\lambda} v & \text{in } \mathbb{R}^{n-1}, \end{cases}$$

where $\tilde{\lambda}$ is close to 0. For α small (low Fourier modes), Proposition 4.5(iv) gives the profile $\nabla_y U(k(\bar{s})y)$ or $iU(k(\bar{s})y)$ (recall also the scaling in y before (4.27)). The remaining part of the approximate kernel is the counterpart of that given in Proposition 4.5(v); for variable coefficients it is a uniquely defined function $\alpha(\bar{s})$ such that

$$(4.37)\qquad \eta_{\alpha(\bar{s})} = 0,$$

where η_α here stands for the first eigenvalue of (4.36). We denote by

$$\big(Z_{\alpha(\bar{s})}(k(\bar{s})y), W_{\alpha(\bar{s})}(k(\bar{s})y)\big)$$

the components of the relative eigenfunction.

We next consider two bases of eigenfunctions for the weighted eigenvalue problems (the operators $\mathfrak{J}$ and T are defined in (2.22) and (6.18) and are self-adjoint)

$$(4.38)\qquad \mathfrak{J}\varphi_j(\bar{s}) = h(\bar{s})^\theta \lambda_j \varphi_j(\bar{s}), \qquad T\omega_j = h(\bar{s})^{-\sigma}\rho_j\omega_j.$$

Because of the weights on the right-hand sides, we can choose these eigenfunctions to be normalized so that $\int_0^L h^\theta \varphi_j\varphi_l = \delta_{jl}$ and $\int_0^L h^{-\sigma}\omega_j\omega_l = \delta_{jl}$; this choice will be useful in Subsection 6.2.

These heuristic arguments suggest that the following subspaces $K_{1,\delta}$ and $K_{2,\delta}$, where δ is a small positive constant, once multiplied by $e^{-if(\varepsilon s)/\varepsilon}$, consist of approximate eigenfunctions for L^1_ε with eigenvalues close to 0 (this will be verified below, in the proof of Proposition 4.9; see also Remark 4.11):

$$\begin{aligned} K_{1,\delta} = \operatorname{span}\biggl\{ h(\varepsilon s)^{\frac{p+1}{4}} \biggl(\langle \varphi_j(\varepsilon s), \nabla_y U(ky)\rangle + i\varepsilon \langle \varphi_j'(\varepsilon s), y\rangle \frac{f'}{k} U(ky) \\ - \frac{\varepsilon^2}{k^2} \langle \varphi_j''(\varepsilon s), \mathfrak{V}(ky)\rangle \biggr) \biggr\}, \end{aligned} \tag{4.39}$$

$$\begin{aligned} K_{2,\delta} = \operatorname{span}\biggl\{ h(\varepsilon s)^{\frac{1}{2}} \biggl(\omega_j(\varepsilon s) i U(ky) + \frac{2\varepsilon f'(\varepsilon s)}{k} \omega_j'(\varepsilon s) \tilde{U}(y) \\ - \frac{\varepsilon^2}{k^2} \omega_j''(\varepsilon s) \mathfrak{W}(ky) \biggr) \biggr\}, \end{aligned} \tag{4.40}$$

$j = 0, \dots, \frac{\delta}{\varepsilon}$, where

$$\tilde{U} = \biggl(\frac{1}{h^{p-1}(p-1)} U(ky) + \frac{1}{2k} \nabla U(ky) \cdot y \biggr).$$

Here $\mathfrak{V} = (\mathfrak{V}_j)_{j=1,\dots,n-1}$ is the counterpart of $\widehat{\mathfrak{V}}$ in (4.34) substituting $\widehat{h}$ with $h(\bar{s})$ (the same holds for $\mathfrak{W}$). The choice of the weights (as powers of h) in (4.38) and in $K_{1,\delta}$ and $K_{2,\delta}$ are again done for technical reasons, and will be convenient below; see, in particular, Subsection 6.2.

We also need to construct an approximate kernel with the profile (Z, W); see Proposition 4.5(v). To this aim we introduce the functions (recall (4.37))

$$\begin{aligned} Q_{1,\alpha}(\bar{s}) = \int_{\mathbb{R}^{n-1}} Z^2_{\alpha(\bar{s})}, \qquad Q_{2,\alpha}(\bar{s}) = \int_{\mathbb{R}^{n-1}} W^2_{\alpha(\bar{s})}, \\ Q_{3,\alpha}(\bar{s}) = \int_{\mathbb{R}^{n-1}} Z_{\alpha(\bar{s})} W_{\alpha(\bar{s})}, \end{aligned} \tag{4.41}$$

and consider the following eigenvalue problem (with periodic boundary conditions)

$$-\varepsilon^2 \xi'' - k^2 \alpha^2 \xi = \frac{\tilde{\nu}}{1 + 2f' \frac{Q_{3,\alpha}}{k\alpha}} \xi \quad \text{on } [0, L].$$

By Weyl's asymptotic formula we have that the eigenvalues $\tilde{\nu}_j$ (counted with multiplicity) have the qualitative behavior $\tilde{\nu}_j \simeq -1 + \varepsilon^2 j^2$ as $j \to +\infty$. Hence, there is a first index j_ε (of order $\frac{1}{\varepsilon}$) for which $\tilde{\nu}_{j_\varepsilon} \geq 0$. Setting $\nu_j = \tilde{\nu}_{j_\varepsilon + j}$ and denoting by ξ_j the eigenfunctions corresponding to ν_j, we then have

$$(4.42)\qquad \begin{aligned} -\varepsilon^2\xi_j'' - k^2\alpha^2\xi_j &= \frac{\nu_j}{1+2f'\frac{Q_{3,\alpha}}{k\alpha}}\xi_j, \\ \nu_j &= \hat{C}_0\varepsilon j + O(\varepsilon^2 j^2) + O(\varepsilon) \quad \text{if } |j| \leq \frac{\delta^2}{\varepsilon}, \end{aligned}$$

where $\delta > 0$ is any given positive (small) constant. Notice that the family $(\xi_j)_j$ can be chosen normalized in L^2 with the weight

$$\frac{1}{1+2f'\frac{Q_{3,\alpha}}{k\alpha}}$$

(this follows from (4.42) and the Courant-Fischer formula). Next we set

$$(4.43)\qquad \beta_j = -\frac{1}{k\alpha}\left(1 - \frac{Q_{1,\alpha}}{k^2\alpha^2 + 2f'k\alpha Q_{3,\alpha}}\nu_j\right)\varepsilon\xi_j'.$$

By our choices, the functions β_j and ξ_j satisfy (this system will be useful in Subsection 6.3) for $|j| \leq \delta^2/\varepsilon$

$$(4.44)\qquad \begin{cases} -\varepsilon^2\beta_j'' - k^2\alpha^2\beta_j - 2f'\frac{Q_{3,\alpha}}{Q_{1,\alpha}}(\varepsilon\xi_j' + k\alpha\beta_j) = \\ \quad \nu_j\beta_j + (O(\nu_j^2) + O(\varepsilon))\beta_j, \\ -\varepsilon^2\xi_j'' - k^2\alpha^2\xi_j + 2f'\frac{Q_{3,\alpha}}{Q_{2,\alpha}}(\varepsilon\beta_j' - k\alpha\xi_j) = \\ \quad \nu_j\xi_j + (O(\nu_j^2) + O(\varepsilon))\xi_j. \end{cases}$$

Our next goal is to introduce a family of approximate eigenfunctions of L_ε^1 that have the profile (from now on, we might denote $(Z_{\alpha(\bar{s})}, W_{\alpha(\bar{s})})$, see (4.37), simply by (Z_α, W_α))

$$(4.45)\qquad v_{3,j} := (\beta_j + q_j)Z_\alpha + \gamma_j\frac{\partial Z_\alpha}{\partial\alpha} + i\xi_j W_\alpha + i\kappa_j\frac{\partial W_\alpha}{\partial\alpha};$$

the functions β_j and ξ_j are as in (4.42)–(4.43), while q_j, γ_j, and κ_j are small corrections to be chosen properly so that $L_\varepsilon^1(e^{-i[f(\varepsilon s)/\varepsilon]}v_{3,j}) = \nu_j e^{-i[f(\varepsilon s)/\varepsilon]}v_{3,j}$, up to an error $o(\nu_j) + O(\varepsilon)$.

With simple computations, using (4.29), (4.30), Remark 4.6, and (4.44), we find that

$$\begin{aligned} &e^{i\frac{f(\varepsilon s)}{\varepsilon}}\left(L_\varepsilon^1(e^{-i\frac{f(\varepsilon s)}{\varepsilon}}v_{3,j}) - \nu_j e^{-i\frac{f(\varepsilon s)}{\varepsilon}}v_{3,j}\right) \\ &\quad = Z_\alpha 2f'\frac{Q_{3,\alpha}}{Q_{1,\alpha}}[2Q_{1,\alpha}\gamma_j k + (\varepsilon\xi_j' + k\alpha\beta_j)] \\ &\qquad + W_\alpha 2f'(-\gamma_j k - (\varepsilon\xi_j' + \beta_j k\alpha) - k\alpha q_j) \\ &\qquad + \frac{\partial Z_\alpha}{\partial\alpha}(-\varepsilon^2\gamma_j'' - \alpha^2k^2\gamma_j) + \frac{\partial W_\alpha}{\partial\alpha}2f'(-\varepsilon\kappa_j' - k\alpha\gamma_j) + \end{aligned}$$

$$+ iZ_\alpha 2f'(-\kappa_j k - k\alpha\xi_j + \varepsilon\beta_j' + \varepsilon q_j')$$
$$+ iW_\alpha 2f' \frac{Q_{3,\alpha}}{Q_{2,\alpha}}(2Q_{2,\alpha}k\kappa_j + k\alpha\xi_j - \varepsilon\beta_j')$$
$$+ i\frac{\partial Z_\alpha}{\partial\alpha} 2f'(-k\alpha\kappa_j + \varepsilon\gamma_j') + i\frac{\partial W_\alpha}{\partial\alpha}(-\varepsilon^2\kappa_j'' - \alpha^2k^2\kappa_j)$$
$$+ Z_\alpha(-\varepsilon^2 q_j'' - \alpha^2k^2 q_j) + (O(\nu_j^2) + O(\varepsilon))\xi_j + (O(\nu_j^2) + O(\varepsilon))\beta_j.$$

To make the coefficients of the terms Z_α, W_α, and iW_α in the second and fourth lines vanish, we choose

$$\gamma_j = -\frac{1}{2kQ_{1,\alpha}}(\varepsilon\xi_j' + k\alpha\beta_j), \qquad \kappa_j = -\frac{1}{2kQ_{2,\alpha}}(k\alpha\xi_j - \varepsilon\beta_j'),$$
$$q_j = -\frac{1}{k\alpha}(\varepsilon\xi_j' + \beta_j k\alpha + k\gamma_j).$$

Using (4.43) we get

$$\gamma_j = -\varepsilon\xi_j' \frac{1}{2k}\frac{1}{k^2\alpha^2 + 2f'k\alpha Q_{3,\alpha}}\nu_j, \tag{4.46}$$
$$\kappa_j = \frac{1}{2k}\frac{\nu_j}{1 + 2f'\frac{Q_{3,\alpha}}{k\alpha}}\xi_j + O(\nu_j^2)\xi_j.$$

These equations and (4.42) imply the relations $-\varepsilon^2\gamma_j'' - \alpha^2k^2\gamma_j = O(\nu_j^2)\beta_j$, $-\varepsilon\kappa_j' - k\alpha\gamma_j = O(\nu_j^2)\beta_j$, $-k\alpha\kappa_j + \varepsilon\gamma_j' = O(\nu_j^2)\xi_j$, and $-\varepsilon^2\kappa_j'' - \alpha^2k^2\kappa_j = O(\nu_j^2)\xi_j$. Similarly, we find

$$q_j = \xi_j \frac{1}{2k}\frac{1}{k\alpha + 2f'Q_{3,\alpha}}\nu_j. \tag{4.47}$$

This also yields $-\gamma_j k - (\varepsilon\xi_j' + \beta_j k\alpha) - k\alpha q_j = O(\nu_j^2)\beta_j$, $-\kappa_j k - k\alpha\xi_j + \varepsilon\beta_j' + \varepsilon q_j' = O(\nu_j^2)\xi_j$ and $-\varepsilon^2 q_j'' - \alpha^2k^2 q_j = O(\nu_j^2)\beta_j$, so we obtain

$$L_\varepsilon^1(e^{-i\frac{f(\varepsilon s)}{\varepsilon}}v_{3,j}) = \nu_j e^{-i\frac{f(\varepsilon s)}{\varepsilon}}v_{3,j} + (O(\nu_j^2) \tag{4.48}$$
$$+ O(\varepsilon))\xi_j + (O(\nu_j^2) + O(\varepsilon))\beta_j \quad \text{for } |j| \le \frac{\delta^2}{\varepsilon},$$

which was our claim. We next define

$$K_{3,\delta} = \operatorname{span}\left\{(\beta_j + q_j)Z_\alpha + \gamma_j\frac{\partial Z_\alpha}{\partial\alpha} + i\xi_j W_j + i\kappa_j\frac{\partial W_\alpha}{\partial\alpha} : \right. \tag{4.49}$$
$$\left. j = -\frac{\delta^2}{\varepsilon}, \dots, \frac{\delta^2}{\varepsilon}\right\}.$$

In the $K_{l,\delta}$'s we added some corrections to the approximate eigenfunctions that take into account the variation of the profile with the frequency; see the derivation of (4.28) and Remark 4.8. In $K_{1,\delta}$ and $K_{2,\delta}$ the corrections are up to the second order (in εj), while in $K_{3,\delta}$ only up to the first; the reason is that the corresponding eigenvalues have a *quadratic* dependence in εj for $K_{1,\delta}$ and $K_{2,\delta}$ (they correspond to η_α in Proposition 4.5), and an *affine* dependence in εj for $K_{3,\delta}$ (corresponding to μ_α in Proposition 4.5). Since the former dependence is more delicate in the indices, we need a more accurate expansion of the eigenfunctions. We finally set

$$K_\delta = \operatorname{span}\{K_{1,\delta}, K_{2,\delta}, K_{3,\delta}\}. \tag{4.50}$$

4.4 Invertibility of L_ε in the Orthogonal Complement of K_δ

Since K_δ (multiplied by $e^{-i(f(\varepsilon s))/\varepsilon}$) is a good candidate for the span of the eigenfunctions of L_ε^1 with small eigenvalues, it seems plausible to invert L_ε^1 on the orthogonal complement to $e^{-i(f(\varepsilon s))/\varepsilon} K_\delta$; this is the content of the next result. We recall the definition of the constant $\mathcal{A}$ in the introduction.

PROPOSITION 4.9 *There exists $\mathcal{A}_0$ sufficiently small such that for any $\mathcal{A} \in [0, \mathcal{A}_0]$ the following property holds: For $\delta > 0$ small enough there exists $C > 0$ (independent of δ) such that if*

$$\Re \int_{N\gamma_\varepsilon} e^{-i\frac{f(\varepsilon s)}{\varepsilon}} v \bar{\phi} \, dV_{\hat{g}_\varepsilon} = 0 \quad \text{for all } v \in K_\delta, \tag{4.51}$$

we have $\|\Pi_\varepsilon L_\varepsilon^1(\phi)\|_{L^2(N\gamma_\varepsilon)} \geq C^{-1}\delta^2 \|\phi\|_{L^2(N\gamma_\varepsilon)}$. Here Π_ε denotes the projection in $L^2(N\gamma_\varepsilon)$ onto the orthogonal complement of the set $\{e^{-i(f(\varepsilon s))/\varepsilon} v : v \in K_\delta\}$.

Before starting the proof, which relies on a localization argument and the spectral analysis of Proposition 4.5, we introduce some notation and a preliminary lemma. We fix $\hat{s} \in [0, L]$ and denote by $\hat{f}$, $\hat{h}$, $\hat{k}$, and $\hat{\alpha}$ the values of $f'(\hat{s})$, $h(\hat{s})$, $k(\hat{s})$, and $\alpha(\hat{s})$, respectively, so that the counterpart of (1.11) holds true. For a large constant $\tilde{C}_0$ to be fixed later, we also define

$$\begin{aligned}
\hat{K}_{1,\delta} = \operatorname{span}\Big\{ & \langle \hat{\varphi}_j(\varepsilon s), \nabla_y U(\hat{k} y)\rangle \\
& + i\varepsilon \langle \hat{\varphi}_j'(\varepsilon s), y\rangle \frac{\hat{f}}{\hat{k}} U(\hat{k} y) - \frac{\varepsilon^2}{k^2} \langle \varphi_j''(\varepsilon s), \hat{\mathfrak{V}}(\hat{k} y)\rangle \Big\}, \\
\hat{K}_{2,\delta} = \operatorname{span}\Big\{ \Big(& \hat{\omega}_j(\varepsilon s) i U(\hat{k} y) \\
& + \frac{2\varepsilon f'(\varepsilon s)}{\hat{k}} \hat{\omega}_j'(\varepsilon s) \hat{\tilde{U}}(y) - \frac{\varepsilon^2}{\hat{k}^2} \tilde{\omega}_j''(\varepsilon s) \hat{\mathfrak{W}}(\hat{k} y) \Big) \Big\},
\end{aligned}$$

$j = 0, \dots, \delta/(\tilde{C}_0\varepsilon)$, and

$$\widehat{K}_{3,1,\delta} = \operatorname{span}\Big\{\cos(\widehat{\alpha}_j s) Z_{\widehat{\alpha}_j}(\widehat{k} y) - i \sin(\widehat{\alpha}_j s) W_{\widehat{\alpha}_j}(\widehat{k} y) : j = -\frac{\delta^2}{\tilde{C}_0\varepsilon}, \dots, \frac{\delta^2}{\tilde{C}_0\varepsilon}\Big\},$$

$$\widehat{K}_{3,2,\delta} = \operatorname{span}\Big\{\sin(\widehat{\alpha}_j s) Z_{\widehat{\alpha}_j}(\widehat{k} y) + i \cos(\widehat{\alpha}_j s) W_{\widehat{\alpha}_j}(\widehat{k} y) : j = -\frac{\delta^2}{\tilde{C}_0\varepsilon}, \dots, \frac{\delta^2}{\tilde{C}_0\varepsilon}\Big\},$$

where

$$\begin{aligned} \widehat{\tilde{U}} &= \left(\frac{1}{\widehat{h}^{p-1}(p-1)} U(\widehat{k} y) + \frac{1}{2\widehat{k}} \nabla U(\widehat{k} y) \cdot y\right), \\ \widehat{\alpha}_j &= \left[\frac{\widehat{\alpha}\widehat{k} L}{2\pi\varepsilon} + j\right] \frac{2\pi\varepsilon}{L} \end{aligned} \tag{4.52}$$

(again, the latter square bracket stands for the integer part, and this choice makes the functions $\frac{L}{\varepsilon}$-periodic). In the above formulas $(\widehat{\varphi}_j)_j$ are the eigenfunctions of the normal Laplacian with the flat metric on γ, and $\widehat{\omega}_j$ those of $\partial^2_{\bar{s}\bar{s}}$ on $[0, L]$; the symbols $Z_{\widehat{\alpha}_j}$ and $W_{\widehat{\alpha}_j}$ stand for the components of the eigenfunctions of (4.28) corresponding to $\eta_{\widehat{\alpha}_j}$.

In analogy with (4.50), we also define

$$\widehat{K}_\delta = \operatorname{span}\left\{\widehat{K}_{1,\delta}, \widehat{K}_{2,\delta}, \widehat{K}_{3,1,\delta}, \widehat{K}_{3,2,\delta}\right\}. \tag{4.53}$$

Given a small constant $\eta > 0$ to be chosen later (of order $\sqrt{\varepsilon}$), we consider then a smooth cutoff function χ_η (depending only on s) with support near $\frac{\widehat{s}}{\varepsilon}$ and with length of order $\frac{\eta}{\varepsilon}$. For example, we can take $\chi_\eta(s) = \chi(\frac{\varepsilon}{\eta}(s - \frac{\widehat{s}}{\varepsilon}))$ for a fixed, compactly supported cutoff χ that is 1 in a neighborhood of 0. The next result uses Fourier cancellation and is related to lemma 2.7 in [39].

LEMMA 4.10 *There exists $\tilde{C}_0$ sufficiently large (depending only on V, L, and $\mathcal{A}_0$) with the following property: for any given integer number m there exists $C_m > 0$ depending on m and χ_η such that for $|j| \le \delta^2/(\tilde{C}_0\varepsilon)$ and for $|l| \ge \delta^2/\varepsilon$ we have*

$$\left|\int \chi_\eta(s)\xi_l(s)\cos(\widehat{\alpha}_j s)ds\right| + \left|\int \chi_\eta(s)\xi_l(s)\sin(\widehat{\alpha}_j s)ds\right| \le \frac{1}{\varepsilon}\frac{C_m}{|\nu_l|^m}\left[\eta(1+|\nu_l|) + \frac{\varepsilon}{\eta}\right]^m.$$

PROOF: We clearly have that $(\cos(\widehat{\alpha}_j s))'' = -\widehat{\alpha}_j^2 \cos(\widehat{\alpha}_j s)$; therefore, integrating by parts, after some manipulation we obtain that

$$\begin{aligned}
&\int \chi_\eta \xi_l(s)\cos(\widehat{\alpha}_j s)ds \\
&\quad = \frac{1}{\widehat{\alpha}_j^2 - \widehat{k}^2\widehat{\alpha}^2 + \frac{\nu_l}{1+2\widehat{f}\frac{Q_{3,\widehat{\alpha}}}{\widehat{k}\widehat{\alpha}}}} \\
&\qquad \times \left\{ \int \chi_\eta \xi_l(s)\cos(\widehat{\alpha}_j s)\left(k^2\alpha^2 - \widehat{k}^2\widehat{\alpha}^2 + \frac{\nu_l}{1+2\widehat{f}\frac{Q_{3,\widehat{\alpha}}}{\widehat{k}\widehat{\alpha}}} - \frac{\nu_l}{1+2f'\frac{Q_{3,\alpha}}{k\alpha}}\right) \right. \\
&\qquad\quad \left. - \int \cos(\widehat{\alpha}_j s)[\chi_\eta'' \xi_l(s) + 2\chi_\eta' \xi_l'(s)] \right\}.
\end{aligned} \tag{4.54}$$

By (4.52) the numbers $\widehat{\alpha}_j$ satisfy $\widehat{\alpha}_j \simeq \widehat{k}\widehat{\alpha} + \frac{2\pi\varepsilon j}{L}$ for $|j| \leq \delta^2/(\tilde{C}_0\varepsilon)$, while by (4.42) $|\nu_l| \geq \frac{1}{2}\widehat{C}_0\delta^2$ for $|l| \geq \delta^2/\varepsilon$. Notice also that $1 + 2f'\frac{Q_{3,\alpha}}{k\alpha}$ is uniformly bounded above and below by positive constants when $\mathcal{A}_0$ tends to 0 (see, for example, the comments in Remark 4.7). By these facts and the properties of χ_η we find

$$\left| \int \chi_\eta \xi_l(s)\cos(\widehat{\alpha}_j s)ds \right| \leq \frac{1}{\varepsilon}\frac{C}{|\nu_l|}\left[\eta(1+|\nu_l|) + \frac{\varepsilon}{\eta}\right],$$

which yields the statement for $m = 1$ (similar computations can be performed to deal with the sine function). The factor $\frac{1}{\varepsilon}$ inside the brackets arises from the fact that we are integrating over the interval $[0, \frac{L}{\varepsilon}]$ and from the normalization of ξ_j (see the comments before (4.42)). To obtain the statement for general m, it is sufficient to iterate the procedure for (4.54) m times and integrate by parts. □

PROOF OF PROPOSITION 4.9: The proof relies mainly on a localization argument and Lemma 4.10. If $\eta = \sqrt{\varepsilon}$ and χ_η is as in Lemma 4.10, we show next that the function $\chi_\eta \phi$ is *almost* orthogonal to $e^{-i\widehat{f}s}\widehat{K}_\delta$ if ε is sufficiently small. We consider, for example, a function $\widehat{v} \in \widehat{K}_{3,1,\delta}$ of the form

$$\widehat{v} = \sum_{j=-\delta^2/(\tilde{C}_0\varepsilon)}^{\delta^2/(\tilde{C}_0\varepsilon)} \widehat{b}_j \big[\cos(\widehat{\alpha}_j s) Z_{\widehat{\alpha}_j}(\widehat{k}y) - i\,\sin(\widehat{\alpha}_j s) W_{\widehat{\alpha}_j}(\widehat{k}y)\big],$$

for some arbitrary coefficients $(\widehat{b}_j)_j$, and we also set

$$\tilde{v} = \sum_{j=-\delta^2/(\tilde{C}_0\varepsilon)}^{\delta^2/(\tilde{C}_0\varepsilon)} \widehat{b}_j \big[\cos(\widehat{\alpha}_j s) Z_{\alpha(\varepsilon s)}(k(\bar{s})y) - i\,\sin(\widehat{\alpha}_j s) W_{\alpha(\varepsilon s)}(k(\bar{s})y)\big].$$

We are going to evaluate the real part of the integral

$$\int_{N\gamma_\varepsilon} e^{-i\hat{f}s}\hat{v}\chi_\eta\bar{\phi}.$$

First of all, since $|k(\bar{s})-\hat{k}| \le C\eta$ and $|\hat{\alpha}_j - \alpha(\bar{s})| \le C(\eta+\delta^2)$ on the support of χ_η, we have that

$$\|Z_{\alpha(\varepsilon s)}(k(\bar{s})y) - Z_{\hat{\alpha}_j}(\hat{k}y)\|_{L^2(\mathbb{R}^{n-1})} = O(\eta+\delta^2) \quad \text{in } \operatorname{supp}(\chi_\eta),$$

so, clearly

$$\Re\int_{N\gamma_\varepsilon} e^{-i\hat{f}s}\hat{v}\chi_\eta\bar{\phi} = \Re\int_{N\gamma_\varepsilon} e^{-i\hat{f}s}\tilde{v}\chi_\eta\bar{\phi} + O(\eta+\delta^2)\|\chi_\eta\phi\|_{L^2(N\gamma_\varepsilon)}\,\|\hat{v}\|_{L^2(N\gamma_\varepsilon)}. \tag{4.55}$$

Next we write

$$\hat{w}(s) = \chi_\eta(s)\sum_{j=-\delta^2/(\tilde{C}_0\varepsilon)}^{\delta^2/(\tilde{C}_0\varepsilon)} \hat{b}_j \sin(\hat{\alpha}_j s),$$

and notice that

$$\begin{aligned}\hat{w}'(s) &= \chi_\eta'(s)\sum_{j=-\delta^2/(\tilde{C}_0\varepsilon)}^{\delta^2/(\tilde{C}_0\varepsilon)} \hat{b}_j \sin(\hat{\alpha}_j s) + \chi_\eta(s)\sum_{j=-\delta^2/(\tilde{C}_0\varepsilon)}^{\delta^2/(\tilde{C}_0\varepsilon)} \hat{b}_j\hat{\alpha}_j \cos(\hat{\alpha}_j s)\\ &= k\alpha\chi_\eta(s)\sum_{j=-\delta^2/(\tilde{C}_0\varepsilon)}^{\delta^2/(\tilde{C}_0\varepsilon)} \hat{b}_j \cos(\hat{\alpha}_j s) + \chi_\eta'(s)\sum_{j=-\delta^2/(\tilde{C}_0\varepsilon)}^{\delta^2/(\tilde{C}_0\varepsilon)} \hat{b}_j \sin(\hat{\alpha}_j s)\\ &\quad - \chi_\eta(s)\sum_{j=-\delta^2/(\tilde{C}_0\varepsilon)}^{\delta^2/(\tilde{C}_0\varepsilon)} \hat{b}_j(k\alpha-\hat{\alpha}_j)\sin(\hat{\alpha}_j s).\end{aligned} \tag{4.56}$$

Using this formula and the same argument as for (4.55), we get (recall our notation before (4.45))

$$\chi_\eta\tilde{v} = \frac{1}{k\alpha}\hat{w}'(s)Z_\alpha - i\,\hat{w}(s)W_\alpha + O(\eta+\delta^2)\|\chi_\eta\hat{v}\|_{L^2(N\gamma_\varepsilon)}. \tag{4.57}$$

Notice that, by the explicit form of $\hat{w}$ and $\hat{\alpha}_j$, for any integer m we have

$$\|\hat{w}\|^2_{H^m([0,L/\varepsilon])} \le C_m\|\hat{w}\|^2_{L^2([0,L/\varepsilon])};$$

therefore, if we write $\hat{w}$ with respect to the basis ξ_l as (notice the shift of index before (4.42))

$$\hat{w}(s) = \sum_{l=-j_\varepsilon}^{\infty} \check{b}_l \xi_l(\varepsilon s), \tag{4.58}$$

we also find that

$$\begin{aligned} \sum_{l=-j_\varepsilon}^{\infty} (1+|\nu_l|)^m \check{b}_l^2 \leq C_m \|\hat{w}\|^2_{H^m([0,L/\varepsilon])} &\leq C_m \|\hat{w}\|^2_{L^2([0,L/\varepsilon])} \\ &\leq C_m \|\hat{v}\|^2_{L^2(N\gamma_\varepsilon)}. \end{aligned} \tag{4.59}$$

Differentiating (4.58) with respect to s and using the definition of ξ_j together with (4.43), we find that

$$\hat{w}'(s) = \sum_{l=-j_\varepsilon}^{\infty} \check{b}_l \varepsilon \xi_l'(\varepsilon s) = \sum_{l=-j_\varepsilon}^{\infty} \check{b}_l (-k\alpha + O(\nu_l))\beta_l(\varepsilon s).$$

The last formula and (4.57) imply

$$\begin{aligned} \chi_\eta \tilde{v} &= -\sum_{l=-j_\varepsilon}^{\infty} \check{b}_l (\beta_l Z_\alpha + i\xi_l W_\alpha) \\ &\quad + \sum_{l=-j_\varepsilon}^{\infty} \check{b}_l O(\nu_l)\beta_l Z_\alpha + O(\eta+\delta^2)\|\chi_\eta \hat{v}\|_{L^2(N\gamma_\varepsilon)} \\ &= -\sum_{l=-j_\varepsilon}^{\infty} \check{b}_l v_{3,l} + \sum_{l=-j_\varepsilon}^{\infty} \check{b}_l (v_{3,l} - \beta_l Z_\alpha - i\xi_l W_\alpha) \\ &\quad + \sum_{l=-j_\varepsilon}^{\infty} \check{b}_l O(\nu_l)\beta_l Z_\alpha + O(\eta+\delta^2)\|\chi_\eta \hat{v}\|_{L^2(N\gamma_\varepsilon)}. \end{aligned} \tag{4.60}$$

In the support of χ_η there exists $\hat{\theta} \in \mathbb{R}$ such that $(f(\varepsilon s))/\varepsilon = \hat{f}s + \hat{\theta} + O(\eta)$, which yields

$$\int_{N\gamma_\varepsilon} \tilde{v} e^{-i\hat{f}s} \chi_\eta \bar{\phi} = \int_{N\gamma_\varepsilon} \tilde{v} e^{-i\frac{f(\varepsilon s)}{\varepsilon}} \chi_\eta \bar{\phi} + O(\eta)\|\phi\|_{L^2(\mathrm{supp}(\chi_\eta))} \|\hat{v}\|_{L^2(N\gamma_\varepsilon)}.$$

Now, recalling that $\eta = \sqrt{\varepsilon}$ and that ϕ and $e^{-if(\varepsilon s)/\varepsilon} K_\delta$ are orthogonal, from the last two formulas we obtain

$$\begin{aligned} \int_{N\gamma_\varepsilon} \tilde{v} e^{-i\hat{f}s} \chi_\eta \bar{\phi} &= A_1 + A_2 + A_3 \\ &\quad + O(\eta+\delta^2)\|\phi\|_{L^2(\mathrm{supp}(\chi_\eta))} \|\hat{v}\|_{L^2(N\gamma_\varepsilon)} \end{aligned} \tag{4.61}$$

where

$$A_1 = -\int_{\mathrm{supp}(\chi_\eta)} e^{-i\hat{f}s}\bar{\phi} \sum_{|l|\geq\frac{\delta^2}{\varepsilon}} \check{b}_l v_{3,l},$$

$$A_2 = \int_{\mathrm{supp}(\chi_\eta)} e^{-i\hat{f}s}\bar{\phi} \sum_{l=-j_\varepsilon}^{\infty} \check{b}_l (v_{3,l} - \beta_l Z_\alpha - i\xi_l W_\alpha),$$

$$A_3 = \int_{\mathrm{supp}(\chi_\eta)} e^{-i\hat{f}s}\bar{\phi} \sum_{l=-j_\varepsilon}^{\infty} \check{b}_l O(\nu_l)\beta_l Z_\alpha.$$

To estimate these terms we first notice that, by the normalization of ξ_j before (4.43), the coefficients $\check{b}_l$ in (4.58) can be computed as

$$\begin{aligned}
[\check{b}_l &= \varepsilon \int_0^{L/\varepsilon} \frac{\widehat{w}(s)}{1 + 2f'\frac{Q_{3,\alpha}}{k\alpha}} \xi_l(\varepsilon s) ds \\
&= \varepsilon \sum_{j=-\delta^2/(\tilde{C}_0\varepsilon)}^{\delta^2/(\tilde{C}_0\varepsilon)} \widehat{b}_j \int_0^{L/\varepsilon} \chi_\eta(s) \frac{\sin(\widehat{\alpha}_j s)}{1 + 2f'\frac{Q_{3,\alpha}}{k\alpha}} \xi_l(\varepsilon s) ds.
\end{aligned}$$

Using this formula, Lemma 4.10, and the Hölder inequality, we find that for any integer m

$$\begin{aligned}
\check{b}_l^2 &\leq C_m \varepsilon^{2m+2} \Big(\sum_{j=-\delta^2/(\tilde{C}_0\varepsilon)}^{\delta^2/(\tilde{C}_0\varepsilon)} |\widehat{b}_j| \Big)^2 \\
&\leq C_m \varepsilon^{2m+1} \sum_{j=-\delta^2/(\tilde{C}_0\varepsilon)}^{\delta^2/(\tilde{C}_0\varepsilon)} \widehat{b}_j^2 \\
&\leq C_m \varepsilon^{2m+2} \|\widehat{v}\|^2_{L^2(N\gamma_\varepsilon)}, \quad |l| \geq \frac{\delta^2}{\varepsilon}.
\end{aligned} \tag{4.62}$$

From the explicit expression for the functions $v_{3,\delta}$ the above term A_1 can be estimated as

$$A_1 \leq C\left(\frac{1}{\varepsilon} \sum_{|l|\geq\delta^2/\varepsilon} (1 + |\nu_l|^2)^2 \check{b}_l^2\right)^{\frac{1}{2}} \|\phi\|_{L^2(\mathrm{supp}(\chi_\eta))}.$$

As before, the factor $\frac{1}{\varepsilon}$ inside the brackets arises from the fact that we are integrating over $[0, \frac{L}{\varepsilon}]$.

Using the fact that $C^{-1}|\varepsilon l| \leq |\nu_l| \leq C(|\varepsilon l| + \varepsilon^2 l^2)$ for $|l| \geq \delta^2/\varepsilon$ (which follows from Weyl's asymptotic formula for the eigenvalue problem in (4.42)), (4.59), and (4.62), we find that for any large integer m and any $d \in (1, \frac{m}{8})$

$$\sum_{\frac{\delta^2}{\varepsilon} \leq |l| \leq \varepsilon^{-d}} (1 + |\nu_l|^2)^2 \check{b}_l^2 \leq C_m \varepsilon^{11+2m-8d} \|\hat{v}\|^2_{L^2(N\gamma_\varepsilon)},$$

$$\sum_{|l| \geq \varepsilon^{-d}} (1 + |\nu_l|^2)^2 \check{b}_l^2 \leq C_m \varepsilon^{(d-1)(m-4)} \|\hat{v}\|^2_{L^2(N\gamma_\varepsilon)}.$$

By the arbitrariness of m it follows that for any $m' \in \mathbb{N}$

$$|A_1| \leq C_{m'} \varepsilon^{m'} \|\hat{v}\|_{L^2(N\gamma_\varepsilon)} \|\phi\|_{L^2(\mathrm{supp}(\chi_\eta))}.$$

Dividing the set of indices l into $\{|l| \leq \delta^2/\varepsilon\}$ and $\{|l| \geq \delta^2/\varepsilon\}$ and using similar arguments (also taking into account (4.46) and (4.47)), we get

$$\begin{aligned} |A_2| + |A_3| &\leq C \|\phi\|_{L^2(\mathrm{supp}(\chi_\eta))} \left(\frac{1}{\varepsilon} \sum_{l=-j_\varepsilon}^{\infty} (\nu_l^2 + \nu_l^4) \check{b}_l^2 \right)^{\frac{1}{2}} \\ &\leq C \delta^2 \|\phi\|_{L^2(\mathrm{supp}(\chi_\eta))} \|\hat{v}\|_{L^2(N\gamma_\varepsilon)}. \end{aligned}$$

Therefore, using (4.55) and (4.61) we find

$$\Re \int_{N\gamma_\varepsilon} e^{-i\hat{f}s} \hat{v} \chi_\eta \bar{\phi} = O(\eta + \delta^2) \|\phi\|_{L^2(\mathrm{supp}(\chi_\eta))} \|\hat{v}\|_{L^2(N\gamma_\varepsilon)}, \quad \hat{v} \in \hat{K}_{3,1,\delta}.$$

Similar estimates hold for $\hat{v} \in \mathrm{span}\{\hat{K}_{1,\delta}, \hat{K}_{2,\delta}, \hat{K}_{3,2,\delta}\}$, so we obtain

$$\int_{N\gamma_\varepsilon} e^{-i\hat{f}s} \hat{v} \chi_\eta \bar{\phi} = O(\delta^2 + \eta) \|\phi\|_{L^2(\mathrm{supp}(\chi_\eta))} \|\hat{v}\|_{L^2(N\gamma_\varepsilon)} \quad \text{for every } \hat{v} \in \hat{K}_\delta. \tag{4.63}$$

Next we let $\hat{L}_\varepsilon$ denote the operator in (4.26) with coefficients frozen at $\hat{s}$. Since $e^{-i\hat{f}s}\hat{K}_\delta$ consist of all the eigenfunctions of $\hat{L}_\varepsilon$ (up to an error $o(\delta^2)$) with eigenvalues smaller in absolute value than δ^2 (see Proposition 4.5 and Remark 4.8), from (4.63) we then deduce

$$\|\hat{L}_\varepsilon(e^{-i\hat{f}s}\chi_\eta\phi)\|_{L^2(N\gamma_\varepsilon)} \geq \frac{\delta^2}{C} \|\chi_\eta\phi\|_{L^2(N\gamma_\varepsilon)} + O(\delta^4 + \delta^2\eta) \|\phi\|_{L^2(\mathrm{supp}(\chi_\eta))} \tag{4.64}$$

for some fixed constant C independent of δ.

It is now possible to choose the cutoff function χ (see the comments before Lemma 4.10) so that it is even and compactly supported in $[-2, 2]$, $\chi \equiv 1$ in $[-1, 1]$, and $\chi(2 - t) + \chi(t) \equiv 1$ for $t \in [1, 2]$. With this choice, we can find a partition of unity $(\chi_{\eta,j})_j$ of $[0, \frac{L}{\varepsilon}]$ consisting of translates of χ_η (plus a negligible scaling), with j running between 1 and a number of order $L/\sqrt{\varepsilon}$. For each index

j we choose a point $\widehat{s}_j$ in the support of $\chi_{\eta,j}$, and we denote by $\widehat{L}_j$ the operator corresponding to (4.26) with coefficients frozen at $\widehat{s}_j$. Then, using (4.64), with easy computations we find

$$\begin{aligned}\|L^1_\varepsilon \phi\|^2_{L^2(N\gamma_\varepsilon)} &= \Big\| L^1_\varepsilon \sum_j \chi_{\eta,j}\phi\Big\|^2_{L^2(N\gamma_\varepsilon)} \\ &= \Big\| \sum_j \widehat{L}_j(\chi_{\eta,j}\phi)\Big\|^2_{L^2(N\gamma_\varepsilon)} + O(\sqrt{\varepsilon})\|\phi\|^2_{L^2(N\gamma_\varepsilon)} \\ &\geq \frac{\delta^4}{C}\sum_j \big\|\chi_{\eta,j}\phi\big\|^2_{L^2(N\gamma_\varepsilon)} + O(\sqrt{\varepsilon})\|\phi\|^2_{L^2(N\gamma_\varepsilon)} \\ &= \frac{\delta^4}{C}\|\phi\|^2_{L^2(N\gamma_\varepsilon)} + O(\sqrt{\varepsilon})\|\phi\|^2_{L^2(N\gamma_\varepsilon)}\end{aligned}$$

for some C independent of δ. To complete the proof, we need to bound from below the norm of $\Pi_\varepsilon L^1_\varepsilon \phi$, showing that

$$\|\Pi_\varepsilon L^1_\varepsilon \phi\|^2_{L^2(N\gamma_\varepsilon)} \geq \frac{\delta^2}{C}\|\phi\|^2_{L^2(N\gamma_\varepsilon)}.$$

To see this, by the last formula it is sufficient to have

$$(4.65)\quad (L^1_\varepsilon \phi, e^{-i\frac{f(\varepsilon s)}{\varepsilon}}v)_{L^2(N\gamma_\varepsilon)} = o(\delta^2)\|\phi\|_{L^2(N\gamma_\varepsilon)}\|v\|_{L^2(N\gamma_\varepsilon)} \quad \text{for any } v \in K_\delta.$$

We prove this claim for $v \in K_{1,\delta}$ only; for the other $K_{j,\delta}$'s the arguments are similar (see Remark 4.11 below for more details). Setting $v = v_r + iv_i$ we find (see (1.17))

$$\begin{aligned}L^1_\varepsilon\big(e^{-\frac{if(\varepsilon s)}{\varepsilon}}v\big) &= e^{-\frac{if(\varepsilon s)}{\varepsilon}}(\mathfrak{L}_r v_r + i\mathfrak{L}_i v_i) - e^{-\frac{if(\varepsilon s)}{\varepsilon}}\left(\frac{\partial^2 v_r}{\partial s^2} + i\frac{\partial^2 v_i}{\partial s^2}\right) \\ &\quad + 2if' e^{-\frac{if(\varepsilon s)}{\varepsilon}}\left(\frac{\partial v_r}{\partial s} + i\frac{\partial v_i}{\partial s}\right).\end{aligned}$$

When differentiating v with respect to s, we either hit the functions φ_j's (and their derivatives) or other functions like $k(\varepsilon s)$ or $f'(\varepsilon s)$ (see the definition of $K_{1,\delta}$ above). The latter ones have a *slow* dependence in s, and therefore these terms can be collected within an error of the form $O(\varepsilon)\|v\|_{L^2(N\gamma_\varepsilon)}$.

However, by our choices of the second and third parts of the elements in $K_{1,\delta}$ (see Remark 4.8, in particular formulas (4.31), (4.32), and (4.33)), terms containing zeroth- or first-order derivatives of φ_j will have coefficients bounded by ε, while the only term containing second derivatives of φ_j will be a linear combination (in j) of the expressions

$$-\varepsilon^2 h(\varepsilon s)^{\frac{p+1}{4}}\left(1 - \frac{2\mathcal{A}^2\theta}{p-1}h(\bar{s})^{\sigma-\theta}\right)\langle \varphi''_j(\varepsilon s), \nabla_y U(k(\bar{s})y)\rangle, \quad j = 0, \ldots, \frac{\delta}{\varepsilon}.$$

The remaining terms will contain third and the fourth derivatives of φ_j only (multiplied, respectively, by ε^3 and ε^4). Therefore, if we set

$$v_{1,j} = h(\varepsilon s)^{\frac{p+1}{4}} \bigg(\langle \varphi_j(\varepsilon s), \nabla_y U(ky) \rangle + i\varepsilon \langle \varphi_j'(\varepsilon s), y \rangle \frac{f'}{k} U(ky) - \frac{\varepsilon^2}{k^2} \langle \varphi_j''(\varepsilon s), \mathfrak{V}(ky) \rangle \bigg),$$

by the above comments and the fact that $\mathfrak{J}\varphi_j = h(\varepsilon s)^{\theta} \lambda_j \varphi_j$ (see (4.38)) we have

$$\begin{aligned} L_\varepsilon^1 \Big(e^{-i\frac{f(\varepsilon s)}{\varepsilon}} \sum_{j=0}^{\delta/\varepsilon} a_j v_{1,j} \Big) &= e^{-i\frac{f(\varepsilon s)}{\varepsilon}} \sum_{j=0}^{\delta/\varepsilon} \lambda_j a_j v_{1,j} + R(v), \\ v &= \sum_{j=0}^{\delta/\varepsilon} a_j v_{1,j}, \end{aligned} \tag{4.66}$$

where $R(v)$ contains terms of order ε or linear combinations of third and fourth derivatives of $\varphi_j(\varepsilon s)$. Thus, using Fourier analysis, we can derive the estimate

$$\begin{aligned} \|R(v)\|_{L^2(N\gamma_\varepsilon)} &\le C \left(\frac{1}{\varepsilon} \sum_{j=0}^{\delta/\varepsilon} a_j^2 (\varepsilon + \varepsilon^3 j^3)^2 \right)^{\frac{1}{2}} \\ &\le C(\varepsilon + \delta^3) \|v\|_{L^2(N\gamma_\varepsilon)} \end{aligned} \tag{4.67}$$

for some constant $C > 0$. Therefore, using (4.66) and (4.67), we obtain

$$\big(L_\varepsilon^1 \phi, e^{-i\frac{f(\varepsilon s)}{\varepsilon}} v \big)_{L^2(N\gamma_\varepsilon)} = O(\varepsilon + \delta^3) \|\phi\|_{L^2(N\gamma_\varepsilon)} \|v\|_{L^2(N\gamma_\varepsilon)},$$

which yields (4.65) and concludes the proof. □

Remark 4.11. The last step in the proof of Proposition 4.9 is nearly identical for $v \in K_{2,\delta}$ except that, still by the computations in Remark 4.8, in the counterpart of (4.66) we will obtain ρ_j instead of λ_j (see (4.38)). When considering $K_{3,\delta}$, setting $v = \sum_{j=-\delta^2/\varepsilon}^{\delta^2/\varepsilon} a_j v_{3,j}$ (see (4.48)), we find

$$\begin{aligned} L_\varepsilon^1 \big(e^{-i\frac{f(\varepsilon s)}{\varepsilon}} v \big) &= e^{-i\frac{f(\varepsilon s)}{\varepsilon}} \sum_{j=-\delta^2/\varepsilon}^{\delta^2/\varepsilon} \nu_j a_j v_{3,j} + \tilde{R}(v), \\ \|\tilde{R}(v)\|_{L^2} &\le C \left(\frac{1}{\varepsilon} \sum_{j=-\delta^2/\varepsilon}^{\delta^2/\varepsilon} a_j^2 (\varepsilon + \varepsilon^2 j^2)^2 \right)^{\frac{1}{2}} \\ &\le C(\varepsilon + \delta^4) \|v\|_{L^2(N\gamma_\varepsilon)}. \end{aligned} \tag{4.68}$$

4.5 Invertibility of L_ε in Weighed Spaces

Our goal is to show that the linearized operator L_ε (see (4.7)) at approximate solutions is invertible on spaces of functions satisfying suitable constraints. We begin with some preliminary notation and lemmas; we first collect decay properties of Green's kernels in Euclidean space. Let us consider the equation

$$-\Delta u + u = f \quad \text{in } \mathbb{R}^{n-1}, \tag{4.69}$$

where f decays to 0 at infinity. The solution of the above equation can be represented as

$$u(x) = \int_{\mathbb{R}^{n-1}} G_0(|x-y|) f(y) dy,$$

where $G_0 : \mathbb{R}_+ \to \mathbb{R}_+$ is a function singular at 0 that decays exponentially to 0 at infinity. Using the notation of Subsection 4.2 and standard elliptic regularity theory, we can prove the following result (the choice $\alpha \geq \frac{1}{2}$ for the Hölder exponent is technical and is used in the proof of Lemma 4.13).

LEMMA 4.12 *Let $\bar{Q} > 0$, $\alpha \geq \frac{1}{2}$, $0 < \tau < 1$, $0 < \varsigma < 1$, and $f \in C^\alpha_\tau$. Then equation* (4.69) *has a* (*unique*) *solution u of class $C^{2,\alpha}_{\varsigma\tau}$ that vanishes on $\partial B_{\bar{Q}}(0)$. Moreover, there exist $\varsigma_0 > 0$ sufficiently close to* 1 *and $\check{C}_0$ sufficiently large* (*depending only on n, α,* $\min\{\bar{Q}, 1\}$*, and* ς) *such that for $\varsigma_0 \leq \varsigma < 1$*

$$\|u\|_{C^{2,\alpha}_{\varsigma\tau}} \leq \check{C}_0 \|f\|_{C^\alpha_\varsigma}.$$

Let now $\tau, \varsigma \in (0,1)$ (to be fixed later). For any integer m we let $\bar{C}^{m,\tau}_{\varsigma,}$ denote the weighted Hölder space

$$\bar{C}^{m,\tau}_\varsigma = \left\{u : \mathbb{R}^{n-1} \to \mathbb{C} : \sup_{y \in \mathbb{R}^{n-1}} e^{\varsigma|y|} \|u\|_{C^{m,\tau}(B_1(y))} < +\infty\right\}. \tag{4.70}$$

We also consider the following set of functions $\frac{L}{\varepsilon}$-periodic in s:

$$\begin{aligned} \bar{L}^2(\bar{C}^{m,\tau}_\varsigma) = \Big\{&u : [0, \tfrac{L}{\varepsilon}] \times \mathbb{R}^{n-1} \to \mathbb{C} : \\ &s \mapsto u(s,\cdot) \in L^2\big([0, \tfrac{L}{\varepsilon}]; \bar{C}^{m,\tau}_\varsigma\big)\Big\}, \end{aligned} \tag{4.71}$$

and for $l \in \mathbb{N}$, we similarly define the functional space

$$\begin{aligned} \bar{H}^l(\bar{C}^{m,\tau}_\varsigma) = \Big\{&u : [0, \tfrac{L}{\varepsilon}] \times \mathbb{R}^{n-1} \to \mathbb{C} : \\ &s \mapsto u(s,\cdot) \in H^l([0, \tfrac{L}{\varepsilon}]; C^{m,\tau}_\varsigma)\Big\}. \end{aligned} \tag{4.72}$$

The weights here are suited for studying functions that decay in y like $e^{-|y|}$, as the fundamental solution of $-\Delta_{\mathbb{R}^{n-1}} u + u = 0$. The parameter $\varsigma < 1$ has been introduced to allow some flexibility in the decay rate. When dealing with functions belonging to the above three spaces, the symbols

$$\|\cdot\|_{\bar{C}^{m,\tau}_\varsigma}, \quad \|\cdot\|_{\bar{L}^2(\bar{C}^{m,\tau}_\varsigma)}, \quad \text{and} \quad \|\cdot\|_{\bar{H}^l(\bar{C}^{m,\tau}_\varsigma)}$$

will denote norms induced by formulas (4.70), (4.71), and (4.72). Also, we keep the same notation for the norms when considering functions defined on subsets of $[0, \frac{L}{\varepsilon}] \times \mathbb{R}^{n-1}$.

We next consider some positive constants $\widehat{V}$, $\widehat{f}$, $\widehat{h}$, and $\widehat{k}$ that satisfy the relations in (1.9). If δ and $\widehat{K}_\delta$ are as in the previous subsection and $\bar{\delta}$ as in Section 4, letting

$$D_{L,\varepsilon} = [0, \tfrac{L}{\varepsilon}] \times B_{\varepsilon^{-\bar{\delta}}+1}(0) \subseteq [0, \tfrac{L}{\varepsilon}] \times \mathbb{R}^{n-1},$$

we define the space of functions

$$\widehat{H}_\varepsilon := \left\{ \phi : \Re \int_{D_{L,\varepsilon}} \bar{\phi}(s,y) e^{-i\widehat{f}s} v\big(s, y/\sqrt{\widehat{V}}\big) = 0 \quad \text{for all } v \in \widehat{K}_\delta \right\}.$$

This condition represents, basically, orthogonality with respect to $\widehat{K}_\delta$ (multiplied by the phase factor), when the function ϕ is scaled in y by $\sqrt{V}$. This is a choice made for technical reasons, which will be helpful in Proposition 4.14. We next have the following result, related to Proposition 4.9 once we scale y.

LEMMA 4.13 *Let $\frac{1}{2} \leq \tau < 1$ and $\varsigma \in (0,1)$. Then, for δ small there exists a positive constant C, depending only on p, τ, ς, L, $\widehat{V}$, and $\widehat{f}$, such that the following property holds: for $\widehat{f}$ small, for $\varepsilon \to 0$, and for any function $\mathfrak{b} \in \bar{L}^2(\bar{C}_\varsigma^\tau)$ there exist $u \in \widehat{H}_\varepsilon$ and $\underline{v} \in \widehat{K}_\delta$ such that, in $D_{L,\varepsilon}$,*

$$\begin{cases} -\frac{1}{\widehat{V}}\partial_{ss}^2 u - \Delta_y u + u - \frac{\widehat{h}^{p-1}}{\widehat{V}} U(y\widehat{k}/\sqrt{\widehat{V}})^{p-1} u \\ \quad - (p-1)\frac{\widehat{h}^{p-1}}{\widehat{V}} U(y\widehat{k}/\sqrt{\widehat{V}})^{p-1} e^{-i\widehat{f}s} \Re(e^{-i\widehat{f}s}\bar{u}) = \mathfrak{b} + e^{-i\widehat{f}s}\underline{v}, \\ u = 0 \quad \text{on } \partial D_{L,\varepsilon} \end{cases} \tag{4.73}$$

(notice that $\underline{v}$ above is intended to be scaled in y) and such that we have the estimates

$$\|u\|_{\bar{L}^2(\bar{C}_\varsigma^{2,\tau})} + \|u\|_{\bar{H}^1(\bar{C}_\varsigma^{1,\tau})} + \|u\|_{\bar{H}^2(\bar{C}_\varsigma^\tau)} \leq \frac{C}{\delta^2} \inf_{v \in \widehat{K}_\delta} \|\mathfrak{b} + e^{-i\widehat{f}s} v\|_{\bar{L}^2(\bar{C}_\varsigma^\tau)}, \tag{4.74}$$

$$\|\underline{v}\|_{\bar{L}^2(\bar{C}_\varsigma^\tau)} \leq C\|\mathfrak{b}\|_{\bar{L}^2(\bar{C}_\varsigma^\tau)}. \tag{4.75}$$

PROOF: First of all we observe that a solution to (4.73) of class L^2 exists. In fact, by replacing $D_{L,\varepsilon}$ with $[0, \frac{L}{\varepsilon}] \times \mathbb{R}^{n-1}$, this would simply follow from Proposition 4.9 with $V \equiv \widehat{V}$. However, since the functions in $\widehat{K}_\delta$ decay exponentially to 0 as $|y| \to +\infty$, the Dirichlet boundary conditions do not affect the solvability property; for more details, see, for example, [40, lemma 5.5]. Notice that indeed, by (1.18) and Proposition 4.5(v), the elements of K_δ decay at the rate $e^{-\widehat{k}|y|}$, and by (1.9), $\widehat{k} > \sqrt{\widehat{V}}$. In particular, $\|\underline{v}\|_{\bar{L}^2(\bar{C}_\varsigma^\tau)}$ is finite and (4.75) holds. We also have (4.74) replacing the left-hand side by the L^2 norm of u. We divide the rest of the proof into two steps.

Step 1. $u \in \bar{L}^2(\bar{C}_\varsigma^\tau)$ AND $\|u\|_{\bar{L}^2(\bar{C}_\varsigma^\tau)} \leq \frac{C}{\delta^2}\|\mathfrak{b}\|_{\bar{L}^2(\bar{C}_\varsigma^\tau)}$. We set $u = e^{-i\hat{f}s}v$ and $\mathfrak{c} = e^{-i\hat{f}s}(\mathfrak{b} + \underline{v})$, so v satisfies

$$\begin{cases} -\Delta v + \left(1 + \frac{\hat{f}^2}{\hat{V}}\right)v + 2i\frac{\hat{f}}{\hat{V}}\partial_s v - \frac{\hat{h}^{p-1}}{\hat{V}}U(y\hat{k}/\sqrt{\hat{V}})^{p-1}v & \\ \quad - (p-1)\frac{\hat{h}^{p-1}}{\hat{V}}U(y\hat{k}/\sqrt{\hat{V}})^{p-1}\Re(\bar{v}) = \mathfrak{c} & \text{in } B_{\varepsilon^{-\bar{\delta}}+1}(0), \\ v = 0 & \text{on } \partial B_{\varepsilon^{-\bar{\delta}}+1}(0). \end{cases}$$

We now use a Fourier decomposition in the variable s; setting

$$\mathfrak{c}(s,y) = \sum_j \mathfrak{c}_j(y)e^{ij\varepsilon s}, \quad v(s,y) = \sum_j v_j(y)e^{ij\varepsilon s}$$

(here we are assuming for simplicity that $L = 2\pi$), we see that each $\mathfrak{c}_j$ belongs to $C^\tau_{\varsigma,\hat{V}}$, that

$$\begin{aligned} &\sum_j \|\mathfrak{c}_j\|^2_{\bar{C}_\varsigma^\tau} = \frac{1}{\varepsilon}\|\mathfrak{c}\|^2_{\bar{L}^2(\bar{C}_\varsigma^\tau)}, \\ &\sum_j \|v_j\|^2_{\bar{C}_\varsigma^\tau} = \frac{1}{\varepsilon}\|v\|^2_{\bar{L}^2(\bar{C}_\varsigma^\tau)} \leq \frac{C}{\varepsilon\delta^2}\|\mathfrak{b}\|_{\bar{L}^2(\bar{C}_\varsigma^\tau)}, \end{aligned} \tag{4.76}$$

and that each v_j solves

$$\begin{cases} -\Delta_y v_j + \left(1 + \frac{\hat{f}^2+\varepsilon^2 j^2 - 2\hat{f}\varepsilon j}{\hat{V}}\right)v_j & \\ \quad - \frac{\hat{h}^{p-1}}{\hat{V}}U(y\hat{k}/\sqrt{\hat{V}})^{p-1}v_j & \\ \quad - (p-1)\frac{\hat{h}^{p-1}}{\hat{V}}\,U(y\hat{k}/\sqrt{\hat{V}})^{p-1}\Re(\bar{v}_j) = \mathfrak{c}_j & \text{in } B_{\varepsilon^{-\bar{\delta}}+1}(0), \\ v_j = 0 & \text{on } \partial B_{\varepsilon^{-\bar{\delta}}+1}(0). \end{cases} \tag{4.77}$$

From elliptic regularity theory, we find that for any $R > 0$ there exists a constant C depending only on R, p, and τ such that

$$\|v_j\|_{C^\tau(B_R)} \leq C\|\mathfrak{c}_j\|_{\bar{C}_\varsigma^\tau} + C\|v_j\|_{L^2}.$$

Now we choose R (depending on p and ς) so large that

$$p\,\frac{\hat{h}^{p-1}}{\hat{V}}\,U^{p-1}\left(\frac{y\hat{k}}{\sqrt{\hat{V}}}\right) < \frac{1}{4}(1-\varsigma) \quad \text{for } |y| \geq \frac{R}{2},$$

and a smooth radial cutoff function $\widehat{\chi}$ such that $\widehat{\chi}(y) = 1$ for $|y| \leq \frac{R}{2}$, and $\widehat{\chi}(y) = 0$ for $|y| \geq R$. Next, we write equation (4.77) as

$$\begin{cases} -\Delta_y v_j + \big(1 + \frac{\widehat{f}^2 + \varepsilon^2 j^2 - 2\widehat{f}\varepsilon j}{\widehat{V}}\big) v_j - (1 - \widehat{\chi}) p \frac{\widehat{h}^{p-1}}{\widehat{V}} U(y\widehat{k}/\sqrt{\widehat{V}})^{p-1} v_j = \\ \quad \tilde{\mathfrak{c}}_j + \widehat{\chi} p \, \frac{\widehat{h}^{p-1}}{\widehat{V}} \, U(y\widehat{k}/\sqrt{\widehat{V}})^{p-1} v_j, \\ v_j = 0 \quad \text{on } \partial B_{\varepsilon^{-\bar{\delta}}+1}(0). \end{cases}$$

We notice that the first linear coefficient of v_j is bounded below (uniformly in j) by 1. Therefore, using Green's representation formula, the maximum principle, and our choice of R (see Lemma 4.12) for any $\varsigma' < \varsigma$, we have the estimate

$$\|v_j\|_{\bar{C}^{\tau}_{\varsigma'}} \leq C(\|\mathfrak{c}_j\|_{\bar{C}^{\tau}_{\varsigma}} + \|v_j\|_{L^2})$$

for some fixed constant C depending only on p, ς, and τ. Taking the square and summing over j, we get

$$\|u\|^2_{\bar{L}^2(\bar{C}^{\tau}_{\varsigma'})} = \|v\|^2_{\bar{L}^2(\bar{C}^{\tau}_{\varsigma'})} \leq C\|\mathfrak{b}\|^2_{\bar{L}^2(\bar{C}^{\tau}_{\varsigma})} + C\|v\|^2_{L^2} \leq \|\mathfrak{b}\|^2_{\bar{L}^2(\bar{C}^{\tau}_{\varsigma})}.$$

We next want to replace in the last formula ς' with ς. Rewrite (4.73) as

$$\begin{cases} -\frac{1}{\widehat{V}} \partial^2_{ss} u - \Delta_y u + u = \widehat{\mathfrak{c}} := \frac{\widehat{h}^{p-1}}{\widehat{V}} \, U(y\widehat{k}/\sqrt{\widehat{V}})^{p-1} u \\ \quad + (p-1) \frac{\widehat{h}^{p-1}}{\widehat{V}} \, U(y\widehat{k}/\sqrt{\widehat{V}})^{p-1} e^{-i\widehat{f}s} \Re(e^{-i\widehat{f}s} \bar{u}) + \mathfrak{b} + e^{-i\widehat{f}s} \underline{v}, \\ u = 0 \quad \text{on } \partial D_{L,\varepsilon}. \end{cases}$$

Using the same procedure as above, write $\widehat{\mathfrak{c}}(s, y) = \sum_j \widehat{\mathfrak{c}}_j(y) e^{ij\varepsilon s}$ and $u(s, y) = \sum_j u_j(y) e^{ij\varepsilon s}$.

We now consider the function $U^{p-1} u_j$: by (1.18), if we choose

$$\varsigma' + \frac{(p-1)\widehat{k}}{\sqrt{\widehat{V}}} > \varsigma,$$

it follows from the above estimates that $\|\widehat{\mathfrak{c}}\|_{\bar{L}^2(\bar{C}^{\tau}_{\varsigma})}$ is finite and that

$$\sum_j \|\widehat{\mathfrak{c}}_j\|^2_{\bar{C}^{\tau}_{\varsigma}} \leq \frac{C}{\varepsilon} \|\mathfrak{b}\|^2_{\bar{L}^2(\bar{C}^{\tau}_{\varsigma})}.$$

Moreover, u_j satisfies

$$\begin{cases} -\Delta u_j + \big(1 + \frac{\varepsilon^2 j^2}{\widehat{V}}\big) u_j = \widehat{\mathfrak{c}}_j & \text{in } B_{\varepsilon^{-\bar{\delta}}+1}(0), \\ u_j = 0 & \text{on } \partial B_{\varepsilon^{-\bar{\delta}}+1}(0). \end{cases}$$

Also, it is easy to show

$$\|u\|^2_{\bar L^2(\bar C^{2,\tau}_\varsigma)} + \|u\|^2_{\bar H^1(\bar C^{1,\tau}_\varsigma)} + \|u\|^2_{\bar H^2(\bar C^{\tau}_\varsigma)} \\ = \frac{1}{\varepsilon}\sum_j \Big[\|u_j\|^2_{\bar C^{2,\tau}_\varsigma} + (1+\varepsilon^2 j^2)\|u_j\|^2_{\bar C^{1,\tau}_\varsigma} \\ + (1+\varepsilon^2 j^2 + \varepsilon^4 j^4)\|u_j\|^2_{\bar C^{\tau}_\varsigma}\Big], \tag{4.78}$$

and therefore we are reduced to finding the estimates

$$\|u_j\|^2_{\bar C^{2,\tau}_\varsigma}, \quad \|u_j\|^2_{\bar C^{1,\tau}_\varsigma}, \quad \|u_j\|^2_{\bar C^{\tau}_\varsigma},$$

done in the next step.

Step 2. CONCLUSION OF PROOF. We now set $a_j = 1 + \varepsilon^2 j^2/\widehat V$ and $v_j(y) = u_j(y/\sqrt{a_j})$. Then, from a change of variables we have the equation

$$\begin{cases} -\Delta v_j(y) + v_j(y) = \widehat F_j(y) := \frac{1}{a_j}\hat{\mathfrak c}_j\big(\frac{y}{\sqrt{a_j}}\big) & \text{in } B_{\sqrt{a_j}(\varepsilon^{-\bar\delta}+1)}(0),\\ v_j = 0 & \text{on } \partial B_{\sqrt{a_j}(\varepsilon^{-\bar\delta}+1)}(0). \end{cases}$$

Notice that $a_j > 0$ stays bounded from below independently of j, and therefore by a scaling argument (and some elementary inequalities), we find

$$\sup_{y,z\in B_1(x)} |\widehat F_j(y) - \widehat F_j(z)| \\ = \frac{1}{a_j}\sup_{y,z\in B_{1/\sqrt{a_j}}(x)}\left|\hat{\mathfrak c}_j\left(\frac{y}{\sqrt{a_j}}\right) - \hat{\mathfrak c}_j\left(\frac{z}{\sqrt{a_j}}\right)\right| \\ \le \frac{C}{a_j}\sup_{y,z\in B_1(x)}\left|\hat{\mathfrak c}_j\left(\frac{y}{\sqrt{a_j}}\right) - \hat{\mathfrak c}_j\left(\frac{z}{\sqrt{a_j}}\right)\right| \\ \le \frac{C\|\hat{\mathfrak c}_j\|_{\bar C^{\tau}_\varsigma}}{a_j^{1+\frac{\tau}{2}}}\sup_{y,z\in B_1(x)} |y-z|^\tau e^{-\frac{\varsigma|x|}{\sqrt{a_j}}} \tag{4.79}$$

where C depends only on τ, and hence we get

$$\|\widehat F_j\|_{\bar C^{\tau}_{\varsigma/\sqrt{a_j}}} \le \frac{C}{a_j^{1+\tau/2}}\|\hat{\mathfrak c}_j\|_{\bar C^{\tau}_\varsigma}. \tag{4.80}$$

Now Lemma 4.12 implies that

$$\|v_j\|_{\bar C^{2,\tau}_{\varsigma/\sqrt{a_j}}} \le \frac{C}{a_j^{1+\tau/2}}\|\hat{\mathfrak c}_j\|_{\bar C^{\tau}_\varsigma}.$$

From this estimate, we will obtain some control on u_j by scaling back the variables.

We consider an arbitrary $x \in \mathbb{R}^{n-1}$. As before, we have

$$\sup_{y,z \in B_1(x)} \frac{|u_j(y) - u_j(z)|}{|y-z|^\tau} = \sup_{y,z \in B_1(x)} \frac{|v_j(\sqrt{a_j}\, y) - v_j(\sqrt{a_j}\, z)|}{|y-z|^\tau}.$$

Since a_j can be arbitrarily large, we cannot evaluate the difference $v_j(\sqrt{a_j}\, y) - v_j(\sqrt{a_j}\, z)$ directly using the weighted norm in the definition (4.70) (as we did for the first inequality in (4.79)), since the two points $\sqrt{a_j}\, y$ and $\sqrt{a_j}\, z$ might not belong to the same unit ball. We avoid this problem by choosing $[\sqrt{a_j}]$ (the integer part of $\sqrt{a_j}$) points $(y^l)_l$ lying on the segment $[\sqrt{a_j}\, y, \sqrt{a_j}\, z]$ at equal distance from each other and using the triangle inequality. Now the distance of two consecutive points y^l and y^{l+1} will stay uniformly bounded from above, and the minimal norm of the y^l's is bounded from below by $C^{-1}\sqrt{a_j}(|x|-1)$. Therefore, adding $[\sqrt{a_j}]$ times the inequality and using (4.80), we obtain

$$\begin{aligned}\sup_{y,z \in B_1(x)} \frac{|u_j(y) - u_j(z)|}{|y-z|^\tau} &\leq \frac{C\sqrt{a_j}}{|y-z|^\tau} \frac{C}{a_j^{1+\tau/2}} \left| \frac{y-z}{\sqrt{a_j}} \right|^\tau e^{-\sqrt{a_j}\varsigma \frac{(|x|-1)}{\sqrt{a_j}}} \|\hat{\mathfrak{c}}_j\|_{\bar{C}^\tau_\varsigma} \\ &\leq \frac{C}{a_j^{1+\tau-1/2}} e^{-\varsigma|x|} \|\hat{\mathfrak{c}}_j\|_{\bar{C}^\tau_\varsigma} \leq \frac{C}{a_j} e^{-\varsigma|x|} \|\hat{\mathfrak{c}}_j\|_{\bar{C}^\tau_\varsigma} \leq \frac{C}{a_j} e^{-\varsigma|x|} \|\tilde{\mathfrak{c}}_j\|_{\bar{C}^\tau_\varsigma},\end{aligned}$$

since we chose $\tau \geq \frac{1}{2}$ and since a_j is uniformly bounded from below.

Similarly, taking first and second derivatives, we find that

$$\sup_{y,z \in B_1(x)} \frac{|\nabla u_j(y) - \nabla u_j(z)|}{|y-z|^\tau} \leq \frac{C}{\sqrt{a_j}} e^{-\varsigma|x|} \|\tilde{\mathfrak{c}}_j\|_{\bar{C}^\tau_\varsigma},$$
$$\sup_{y,z \in B_1(x)} \frac{|\nabla^2 u_j(y) - \nabla^2 u_j(z)|}{|y-z|^\tau} \leq C e^{-\varsigma|x|} \|\tilde{\mathfrak{c}}_j\|_{\bar{C}^\tau_\varsigma},$$

where, again, C depends only on τ. Recalling that $a_j = \widehat{V} + \varepsilon^2 j^2$, we have in this way proved that

$$\|u_j\|^2_{\bar{C}^{2,\tau}_\varsigma} \leq C \|\tilde{\mathfrak{c}}_j\|^2_{\bar{C}^\tau_\varsigma}, \qquad \|u_j\|^2_{\bar{C}^{1,\tau}_\varsigma} \leq \frac{C}{1+\varepsilon^2 j^2} \|\tilde{\mathfrak{c}}_j\|^2_{\bar{C}^\tau_\varsigma},$$
$$\|u_j\|^2_{\bar{C}^\tau_\varsigma} \leq \frac{C}{(1+\varepsilon^2 j^2)^2} \|\tilde{\mathfrak{c}}_j\|^2_{\bar{C}^\tau_\varsigma}.$$

Now the conclusion follows from (4.76), (4.78), the last formula, and the fact that

$$\|\tilde{\mathfrak{c}}\|_{\bar{L}^2(\bar{C}^\tau_\varsigma)} \leq C \inf_{v \in \hat{K}_\delta} \|\mathfrak{b} + v\|_{\bar{L}^2(\bar{C}^\tau_\varsigma)};$$

see the beginning of Step 1. □

We next consider the operator L_ε in $\tilde{D}_\varepsilon$ (see (4.7)), acting on a suitable subset of $H_{\tilde{D}_\varepsilon}$ (verifying an orthogonality condition similar to (4.51)). We want to allow some flexibility in the choice of approximate solutions: to do this we consider a normal section Φ to γ that satisfies the following two conditions:

$$\Phi \in \operatorname{span}\left\{h^{\frac{p+1}{4}}\varphi_j : j = 0, \dots, \frac{\delta}{\varepsilon}\right\} \quad \text{and} \quad \|\Phi\|_{H^2(0,L)} \le c_1\varepsilon. \tag{4.81}$$

Here $(\varphi_j)_j$ are as in (4.38), while c_1 is a large constant to be determined later. Notice that by (4.81) Φ has, naively, only Fourier modes with index bounded by $\frac{1}{\varepsilon}$. This yields estimates of the type $\|\Phi^{(k+1)}\|_{L^2} \le C_{k,\delta}\frac{1}{\epsilon}\|\Phi^{(k)}\|_{L^2}$. We have therefore $\|\Phi'''\|_{L^2[0,L]} \le C$, which implies $\|\Phi''\|_{L^\infty} \le C$, so also (4.4) holds true. This will allow us, in the next section, to apply Proposition 4.1.

Next, consider the variables z defined in (3.1). In the coordinates (s, z), we will consider the approximate solution

$$\tilde{\psi}_\varepsilon = e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\eta_\varepsilon\big(h(\varepsilon s)U(k(\varepsilon s)z) + U_1(s,z)\big) := \tilde{\psi}_{0,\varepsilon} + \widehat{\psi}_\varepsilon, \tag{4.82}$$

where $\varepsilon s = \bar{s}$, and where $\tilde{f}$ and U_1 satisfy, for some fixed $C > 0$ and $\tau \in (0, 1)$,

$$\|\tilde{f} - f\|_{H^2([0,L])} \le C\varepsilon^2, \quad |U_1|(s,z) \le C\varepsilon(1 + |z|^C)e^{-k(\bar{s})|z|}, \tag{4.83}$$

$$\||hU(k\cdot) + U_1|^{p-1} - |hU(k\cdot)|^{p-1}\|_{C^\tau} \le C\varepsilon \quad \text{in } \tilde{D}_\varepsilon. \tag{4.84}$$

With this choice of $\tilde{\psi}_\varepsilon$, we are going to study the analogue of Lemma 4.13 for L_ε (see (4.7)), using a perturbation method.

To state our final result, we need to introduce some more notation. Recalling the definition in (4.70) and still using the coordinates (s, z) for $\tau \in (0, 1)$ and $\varsigma > 0$, we define the function space

$$L^2(C^{m,\tau}_{\varsigma,V}) = \left\{u : \tilde{D}_\varepsilon \to \mathbb{C} : s \mapsto u\left(s, \frac{\cdot}{\sqrt{V(\varepsilon s)}}\right) \in L^2\big([0, \tfrac{L}{\varepsilon}]; C^{m,\tau}_{\varsigma,1}\big)\right\}. \tag{4.85}$$

Also, for $m \in \mathbb{N}$, we similarly define

$$H^l(C^{m,\tau}_{\varsigma,V}) = \left\{u : \tilde{D}_\varepsilon \to \mathbb{C} : s \mapsto u\left(s, \frac{\cdot}{\sqrt{V(\varepsilon s)}}\right) \in H^l\big([0, \tfrac{L}{\varepsilon}]; C^{m,\tau}_{\varsigma,1}\big)\right\}. \tag{4.86}$$

We next let $\tilde{K}_\delta$ be the counterpart of K_δ (see (4.50)), when we replace the coordinates y by z. Finally, we denote by $\bar{H}_\varepsilon$ the following subspace of functions:

$$\bar{H}_\varepsilon := \left\{\phi \in H_{\tilde{D}_\varepsilon} : \Re \int_{\tilde{D}_\varepsilon} e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}} v\,\bar{\phi} = 0 \text{ for all } v \in \tilde{K}_\delta\right\}. \tag{4.87}$$

Defining

$$\|\cdot\|_{\varsigma,V} := \|\cdot\|_{L^2(C^{2,\tau}_{\varsigma,V})} + \|\cdot\|_{H^1(C^{1,\tau}_{\varsigma,V})} + \|\cdot\|_{H^2(C^{\tau}_{\varsigma,V})}, \tag{4.88}$$

we then have the following result (recall the definition of $\tilde{D}_\varepsilon$ in (4.10)):

PROPOSITION 4.14 *Suppose $0 < \varsigma < 1$ and $\frac{1}{2} \le \tau < 1$. Suppose $\tilde{\psi}_\varepsilon$ is as in* (4.82)*, with $\tilde{f}$ and U_1 satisfying* (4.83)*. Then, if $K^2(\varepsilon s) = V(\varepsilon s)$ and if $\mathcal{A}$ and δ are sufficiently small, in the limit $\varepsilon \to 0$ the following property holds: for any function $b \in L^2(C^\tau_{\varsigma,V})$ there exist $\tilde{u} \in \bar{H}_\varepsilon$ and $\underline{\tilde{v}} \in \tilde{K}_\delta$ such that*

$$\begin{cases} -\Delta_{g_\varepsilon}\tilde{u} + V(\varepsilon x)\tilde{u} - |\tilde{\psi}_\varepsilon|^{p-1}\tilde{u} \\ \quad -(p-1)|\tilde{\psi}_\varepsilon|^{p-3}\tilde{\psi}_\varepsilon \Re(\tilde{\psi}_\varepsilon \bar{\tilde{u}}) = b + e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\underline{\tilde{v}} & \text{in } \tilde{D}_\varepsilon, \\ \tilde{u} = 0 & \text{on } \partial\tilde{D}_\varepsilon, \end{cases} \tag{4.89}$$

is solvable, and such that for every $\varsigma' < \varsigma$ there exists some $C > 0$ for which we have the estimates

$$\begin{aligned} \|\tilde{u}\|_{\varsigma',V} &\le \frac{C}{\delta^2} \inf_{\tilde{v}\in\tilde{K}_\delta} \|b + e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\tilde{v}\|_{L^2(C^\tau_{\varsigma,V})} \\ \|\underline{\tilde{v}}\|_{L^2(C^\tau_{\varsigma,V})} &\le C\|b\|_{L^2(C^\tau_{\varsigma,V})}. \end{aligned} \tag{4.90}$$

PROOF: We divide the proof into two steps.

Step 1. SOLVABILITY OF (4.89). First of all, we notice that, from Proposition 4.9 and from elliptic regularity results, if H_ε denotes the subspace of function in $H^2(N\gamma_\varepsilon)$ satisfying (4.51), then the operator L^1_ε is invertible from $(H_\varepsilon, \|\cdot\|_{H^2(N\gamma_\varepsilon)})$ onto $(\Pi_\varepsilon L^2(N\gamma_\varepsilon), \|\cdot\|_{L^2(N\gamma_\varepsilon)})$; moreover, the norm of the inverse operator is bounded by C/δ^2.

From the comments at the beginning of the proof of Lemma 4.13, we also deduce the following property: Given $b \in L^2(\{|y| \le (\varepsilon^{-\bar{\delta}} + 1)/K(\varepsilon s)\})$ there exist $u \in H^2(\{|y| \le (\varepsilon^{-\bar{\delta}} + 1)/K(\varepsilon s)\})$ and $\underline{v} \in K_\delta$ such that

$$\begin{aligned} \tilde{L}_\varepsilon u &:= -\Delta_{\hat{g}_\varepsilon} u + V(\varepsilon s)u - h(\varepsilon s)^{p-1}U(k(\varepsilon s)y)^{p-1}u \\ &\qquad - (p-1)h(\varepsilon s)^{p-1}U(k(\varepsilon s)y)^{p-1}e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\Re(e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\bar{u}) \\ &= b + e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\underline{v} \quad \text{in } \{|y| \le (\varepsilon^{-\bar{\delta}} + 1)/K\}, \end{aligned} \tag{4.91}$$

$$\Re \int_{\{|y|\le(\varepsilon^{-\bar{\delta}}+1)/K\}} \bar{u}e^{-i\frac{f(\varepsilon s)}{\varepsilon}} v\, dV_{\hat{g}_\varepsilon} = 0 \quad \text{for every } v \in K_\delta.$$

Again, we have the estimates

$$\begin{aligned} \|u\|_{H^2(\{|y|\le(\varepsilon^{-\bar{\delta}}+1)/K\})} &\le \frac{C}{\delta^2}\|b\|_{L^2(\{|y|\le(\varepsilon^{-\bar{\delta}}+1)/K\})}, \\ \|\underline{v}\|_{L^2(\{|y|\le(\varepsilon^{-\bar{\delta}}+1)/K\})} &\le C\|b\|_{L^2(\{|y|\le(\varepsilon^{-\bar{\delta}}+1)/K\})}. \end{aligned} \tag{4.92}$$

Using a perturbative argument, we show that we can recover the same invertibility result for (4.89) where, compared to (4.91), we need to substitute y with z, $\Delta_{\hat{g}_\varepsilon}$ with $\Delta_{\tilde{g}_\varepsilon}$, f with $\tilde{f}$, and $e^{-i(f(\varepsilon s)/\varepsilon}hU$ with $\tilde{\psi}_\varepsilon$.

In fact, let us denote by Π_y and Π_z the orthogonal projections in L^2 onto the orthogonal complements of the sets $\{e^{-i(f(\varepsilon s))/\varepsilon} v \,:\, v \in K_\delta\}$, $\{e^{-i\tilde{f}/(\varepsilon s)\varepsilon} v \,:\, v \in \tilde{K}_\delta\}$ with respect to the scalar products induced by the metrics $\widehat{g}_\varepsilon$ and $\tilde{g}_\varepsilon$, respectively. By (4.81), Lemma 3.1, and (4.83) for every $u \in H^2(\tilde{D}_\varepsilon)$ and every $b \in L^2(\tilde{D}_\varepsilon)$, we have

$$\|L_\varepsilon u - \tilde{L}_\varepsilon u\|_{L^2(\tilde{D}_\varepsilon)} \leq C(c_1)\varepsilon \|u\|_{H^2(\tilde{D}_\varepsilon)},$$
$$\|\Pi_y b - \Pi_z b\|_{L^2(\tilde{D}_\varepsilon)} \leq C(c_1)\varepsilon \|b\|_{L^2(\tilde{D}_\varepsilon)},$$

where $C(c_1)$ is a positive constant which depends on γ, V, and the constant c_1 in (4.81).

From (4.92) and the last formula we deduce the solvability of (4.89), together with the estimates $\|\tilde{u}\|_{H^2(\tilde{D}_\varepsilon)} \leq (C/\delta^2)\|b\|_{L^2(\tilde{D}_\varepsilon)}$ and $\|\underline{\tilde{v}}\|_{L^2(\tilde{D}_\varepsilon)} \leq C\|b\|_{L^2(\tilde{D}_\varepsilon)}$.

Step 2. PROOF OF (4.90). Recall that the coordinates y (see the beginning of this section) are not global, since they are defined locally in s by normal parallel transport; the same holds, of course, for the coordinates z. Therefore, if we prolong the z's along γ_ε, there will be a discontinuity between 0 and $\frac{L}{\varepsilon}$.

To reduce ourselves to the periodic case, as in Lemma 4.13, we apply a rotation $R_\varepsilon = R_\varepsilon(\varepsilon s)$ to the z-axes that makes the coordinates $\tilde{z} := R(\varepsilon s)z$ periodic in s. To compute the Laplace-Beltrami operator in the new coordinates $\tilde{z}$, we should apply the chain rule in this way:

$$\partial_{z_j} u = (R_\varepsilon)_{jl}\partial_{\tilde{z}_l} u, \quad \partial^2_{s z_j} u = \varepsilon \partial_{\bar{s}}(R_\varepsilon)_{jl}\partial_{\tilde{z}_l} u + (R_\varepsilon)_{jl}\partial^2_{s\tilde{z}_l} u,$$
$$\partial^2_{z_j z_l} u = R_{mj} R_{tl} \partial^2_{\tilde{z}_m \tilde{z}_t} u.$$

In particular, since R_ε is orthogonal,

$$\partial^2_{z_j z_j} u = R_{mj} R_{tj} \partial^2_{\tilde{z}_m \tilde{z}_t} u = (R_\varepsilon)_{mj}(R_\varepsilon^{-1})_{jt}\partial^2_{\tilde{z}_m \tilde{z}_t} u$$
$$= \partial^2_{\tilde{z}_m \tilde{z}_m} u,$$

namely, the main term in the Laplacian stays invariant. Taking into account Lemma 3.1 and the last formulas, for $\varsigma'' \in (\varsigma', \varsigma)$ we find

$$\|\Delta^{\tilde{z}}_{\tilde{g}_\varepsilon} u - \Delta^{z}_{\tilde{g}_\varepsilon} u\|_{L^2(C^\tau_{\varsigma'',k})} \leq C(c_1)\varepsilon \|u\|_{\varsigma'',V}. \tag{4.93}$$

We next use a localization argument as in the proof of Proposition 4.9. If $\widehat{s}_j$ and $\chi_{\eta,j}$ are as in that proof, by (4.83) we can find $\widehat{\theta}_j \in \mathbb{R}$ such that $\tilde{f}(\varepsilon s)/\varepsilon - \widehat{f}_j s - \widehat{\theta}_j = O(\sqrt{\varepsilon})$ in the support of $\chi_{\eta,j}$. If we set $\Delta^{(s,\tilde{z})}_{\mathbb{R}^n} = \Delta^{\tilde{z}}_{\mathbb{R}^{n-1}} + \partial^2_{ss}$, and if we scale the $\tilde{z}$-variables by $K(\varepsilon s) = \sqrt{V(\varepsilon s)}$, the function $\chi_{\eta,j}(s)u(s,\tilde{z})$ (which is now periodic in s) satisfies the equation

$$\begin{cases} -\frac{1}{\widehat{V}_j}\partial^2_{ss}\chi_{\eta,j}u - \Delta^{\tilde{z}}_{\mathbb{R}^{n-1}}(\chi_{\eta,j}u) + \chi_{\eta,j}u - \frac{\widehat{h}_j^{p-1}}{\widehat{V}_j}U\big(\tilde{z}\widehat{k}_j/\sqrt{\widehat{V}_j}\big)^{p-1}\chi_{\eta,j}u \\ \quad -(p-1)\frac{\widehat{h}_j^{p-1}}{\widehat{V}_j}U\big(\tilde{z}\widehat{k}_j/\sqrt{\widehat{V}_j}\big)^{p-1}e^{-i(\widehat{f}s+\widehat{\theta}_j)}\Re(e^{-i(\widehat{f}s+\widehat{\theta}_j)}\chi_{\eta,j}\bar{u}) \\ \qquad = \mathcal{F}_j \quad \text{in } \big\{|\tilde{z}| \leq \frac{\varepsilon^{-\bar{\delta}}+1}{K}\big\}, \\ \chi_{\eta,j}u = 0 \quad \text{on } \big\{|\tilde{z}| = \frac{\varepsilon^{-\bar{\delta}}+1}{K}\big\}, \end{cases}$$

where

$$\begin{aligned} \mathcal{F}_j = {} & \frac{1}{V(\bar{s})}\chi_{\eta,j}e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}(b+\underline{v}) + \frac{1}{V(\bar{s})}\big(\Delta^{\tilde{z}}_{\tilde{g}_\varepsilon} - \Delta^{(s,\tilde{z})}_{\mathbb{R}^n}\big)\chi_{\eta,j}u \\ & - \frac{1}{V(\bar{s})}\big(2\nabla^{\tilde{z}}_{\tilde{g}_\varepsilon}u\cdot\nabla^{\tilde{z}}_{\tilde{g}_\varepsilon}\chi_{\eta,j} + u\Delta^{\tilde{z}}_{\tilde{g}_\varepsilon}\chi_{\eta,j}\big) + \frac{1}{V(\bar{s})}(\widehat{V}_j - V)\chi_{\eta,j}u \\ & + \frac{1}{V(\bar{s})}|\tilde{\psi}_\varepsilon|^{p-1}\chi_{\eta,j}u + \frac{1}{V(\bar{s})}(p-1)|\tilde{\psi}_\varepsilon|^{p-2}\tilde{\psi}_\varepsilon\Re(\tilde{\psi}_\varepsilon\chi_{\eta,j}\bar{u}) \\ & - \frac{\widehat{h}_j^{p-1}}{\widehat{V}_j}U\big(\tilde{z}\widehat{k}_j/\sqrt{\widehat{V}_j}\big)^{p-1}\chi_{\eta,j}u \\ & - (p-1)\frac{\widehat{h}_j^{p-1}}{\widehat{V}_j}U\big(\tilde{z}\widehat{k}_j/\sqrt{\widehat{V}_j}\big)^{p-1}e^{-i(\widehat{f}s+\widehat{\theta}_j)}\Re(e^{-i(\widehat{f}s+\widehat{\theta}_j)}\chi_{\eta,j}\bar{u}). \end{aligned}$$

In the last formula, the functions b, $\underline{v}$, V, and $\tilde{\psi}_\varepsilon$ are intended to be scaled in $\tilde{z}$ by $\sqrt{V(\varepsilon s)}$. Reasoning as for (4.63), from (4.87) we find that

$$\int_{\tilde{D}_\varepsilon} e^{-i\widehat{f}_j s - \widehat{\theta}_j}\,\widehat{v}\chi_{\eta,j}\bar{\phi} = O(\delta^2 + \sqrt{\varepsilon})\|\phi\|_{L^2(\mathrm{supp}(\chi_{\eta,j}))}\|\widehat{v}\|_{L^2(\tilde{D}_\varepsilon)}$$

for every $\widehat{v} \in \widehat{K}_\delta$. Moreover, as for (4.93) we can show that

$$\begin{aligned} & \|(\Delta^{\tilde{z}}_{\tilde{g}_\varepsilon} - \Delta^{(s,\tilde{z})}_{\mathbb{R}^n})\chi_{\eta,j}u\|_{\bar{L}^2(\bar{C}^{\tau}_{\varsigma''})} \leq \\ & \qquad C(c_1)\varepsilon\big(\|\chi_{\eta,j}u\|_{\bar{L}^2(\bar{C}^{2,\tau}_{\varsigma''})} + \|\chi_{\eta,j}u\|_{\bar{H}^1(\bar{C}^{1,\tau}_{\varsigma''})} + \|\chi_{\eta,j}u\|_{\bar{H}^2(\bar{C}^{\tau}_{\varsigma''})}\big). \end{aligned}$$

Therefore, using Lemma 4.13, (4.83), and (4.84) we obtain the estimate

$$
\begin{aligned}
&\|\chi_{\eta,j}u\|_{\bar L^2(\bar C^{2,\tau}_{\varsigma''})} + \|\chi_{\eta,j}u\|_{\bar H^1(\bar C^{1,\tau}_{\varsigma''})} + \|\chi_{\eta,j}u\|_{\bar H^2(\bar C^{\tau}_{\varsigma''})}\\
&\quad\le \frac{C}{\delta^2}\|\mathcal{F}_j\|_{\bar L^2(\bar C^{\tau}_{\varsigma''})}\\
&\quad\le \frac{C}{\delta^2}\|\chi_{\eta,j}b\|_{\bar L^2(\bar C^{\tau}_{\varsigma''})} + \frac{C}{\delta^2}\|\chi_{\eta,j}\underline{v}\|_{\bar L^2(\bar C^{\tau}_{\varsigma''})}\\
&\qquad + C\sqrt{\varepsilon}\Big(\|u\|_{\bar L^2(\bar C^{2,\tau}_{\varsigma''},\mathrm{supp}(\chi_{\eta,j}))} + \|u\|_{\bar H^1(\bar C^{1,\tau}_{\varsigma''},\mathrm{supp}(\chi_{\eta,j}))}\\
&\qquad\qquad + \|u\|_{\bar H^2(\bar C^{\tau}_{\varsigma''},\mathrm{supp}(\chi_{\eta,j}))}\Big),
\end{aligned}
\tag{4.94}
$$

where the last symbols denote the restrictions of the weighted norms to $\mathrm{supp}(\chi_{\eta,j})$. Recall that the functions in the previous formula have been scaled in $\tilde z$ by $\sqrt{V(\varepsilon s)}$; therefore, from the uniform continuity of $V(\bar s)$, for some $C > 0$ we have (recall that $\varsigma'' \in (\varsigma', \varsigma)$)

$$
\begin{aligned}
\frac{1}{C}\|\chi_{\eta,j}u\|_{\varsigma',V} \le{}& \|\chi_{\eta,j}u(\cdot,\sqrt{V(\bar s)}\,\cdot)\|_{\bar L^2(\bar C^{2,\tau}_{\varsigma''})} + \|\chi_{\eta,j}u(\cdot,\sqrt{V(\bar s)}\,\cdot)\|_{\bar H^1(\bar C^{1,\tau}_{\varsigma''})}\\
&+ \|\chi_{\eta,j}u(\cdot,\sqrt{V(\bar s)}\,\cdot)\|_{\bar H^2(\bar C^{\tau}_{\varsigma''})}.
\end{aligned}
$$

A similar inequality holds for the restriction of u to the support of $\chi_{\eta,j}$, together with

$$
\begin{aligned}
&\|\chi_{\eta,j}b(\cdot,\sqrt{V(\bar s)}\,\cdot)\|_{\bar L^2(\bar C^{\tau}_{\varsigma''})} + \|\chi_{\eta,j}\underline{v}(\cdot,\sqrt{V(\bar s)}\,\cdot)\|_{\bar L^2(\bar C^{\tau}_{\varsigma''})} \le\\
&\qquad\qquad C\big(\|\chi_{\eta,j}b\|_{L^2(C^{\tau}_{\varsigma,V})} + \|\chi_{\eta,j}\underline{v}\|_{L^2(C^{\tau}_{\varsigma,V})}\big).
\end{aligned}
$$

Using the last two inequalities, taking the square of (4.94), and summing over j, we can bring the last term in the right-hand side to the left, so we get (4.90). □

5 Approximate Solutions

In this section we construct some approximate solutions to (1.16) that depend on suitable parameters and find estimates on the error terms. As in the previous subsection, we let y be a system of Fermi coordinates in $N\gamma_\varepsilon$, and for a normal section Φ of $N\gamma_\varepsilon$ of class H^2 we define the coordinates (see (3.1))

$$
z = y - \Phi(\varepsilon s), \quad z \in \mathbb{R}^{n-1}.
$$

By the results in Subsection 4.2, we will restrict our attention to the set $\tilde D_\varepsilon$.

Remark 5.1. In the spirit of Proposition 4.4, we will work with approximate solutions $\tilde\psi_\varepsilon$ supported in $\tilde D_\varepsilon$. Therefore, using the above coordinates, $\tilde\psi_\varepsilon(s,z)$ has to vanish for $|z|$ sufficiently large. This can be achieved by formally defining $\tilde\psi_\varepsilon(s,z)$ on $N\gamma_\varepsilon$ and multiplying it by a cutoff function η_ε as in Subsection 4.2. However, since the functions we are dealing with decay exponentially to 0 as $|z| \to \infty$, the effect of this cutoff on the expansions below is exponentially small in ε, and it will turn out to be negligible for our purposes. Therefore, for reasons of brevity and clarity, we will tacitly assume that $\tilde{\psi}_\varepsilon(s,z)$ is multiplied by such a cutoff without writing it explicitly.

Recall that in (4.6) we defined

$$S_\varepsilon(\psi) = -\Delta_g \psi + V(\varepsilon x)\psi - |\psi|^{p-1}\psi.$$

We set $\tilde{f}_0(\bar{s}) = f(\bar{s}) + \varepsilon f_1(\bar{s})$, where f is given in (1.12) and f_1 in (3.25) (we refer to Subsection 2.4 for the definition of $\mathcal{A}'$). If w_r and w_i are smooth functions of $\bar{s}$, we saw in Subsection 3.2 that

$$S_\varepsilon\big(e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}(hU(kz) + \varepsilon(w_r + i w_i))\big) = e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}(\varepsilon(\mathcal{R}_r + i\mathcal{R}_i)) + R_2(z)e^{-k(\varepsilon s)|z|}$$

for some quantities $\mathcal{R}_r$ and $\mathcal{R}_i$. We choose $w_r = w_{r,e} + w_{r,o}$ and $w_i = w_{i,e} + w_{i,o}$ (see (3.13), (3.9), and (3.12)) to make $\mathcal{R}_r$ and $\mathcal{R}_i$ vanish, using the stationarity condition (1.15) (see Lemma 3.2).

From Proposition 3.3 we also have that

$$\begin{aligned} & e^{i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}} S_\varepsilon(\psi_{1,\varepsilon}) \\ & = \varepsilon^2(\tilde{R}_{r,e} + \tilde{R}_{r,o}) + \varepsilon^2(\tilde{R}_{r,e,f_1} + \tilde{R}_{r,o,f_1}) \\ & \quad + \varepsilon^2 i(\tilde{R}_{i,e} + \tilde{R}_{i,o}) + \varepsilon^2 i(\tilde{R}_{i,e,f_1} + \tilde{R}_{i,o,f_1}) + R_3(z)e^{-k(\varepsilon s)|z|}. \end{aligned} \tag{5.1}$$

Here we want to prove error estimates when Φ and the phase satisfy some precise conditions in terms of Fourier analysis and Sobolev norms, and we add further correction terms.

To allow more flexibility in the choice of approximate solutions, we substitute the phase $\tilde{f}_0$ with the function $\tilde{f} = f + \varepsilon f_1 + \varepsilon^2 f_2$, where f_2 is some function of class H^2. On Φ and f_2 we assume the following conditions for some constants c_1 and c_2 to be determined later:

$$\|\Phi\|_{H^2} \le c_1\varepsilon, \qquad \|f_2\|_{H^2} \le c_2. \tag{5.2}$$

Moreover, letting δ be as in Subsection 4.3 and φ_j and ω_j as in (4.38), we also assume that

$$\begin{aligned} & \Phi \in \operatorname{span}\left\{h^{\frac{p+1}{4}}\varphi_j : j = 0, \ldots, \frac{\delta}{\varepsilon}\right\}, \\ & f_2 \in \operatorname{span}\left\{h^{\frac{1}{2}}\omega_j : j = 0, \ldots, \frac{\delta}{\varepsilon}\right\}. \end{aligned} \tag{5.3}$$

Notice that for f_2 we have similar observations to those made for Φ after (4.81). Since, again, the Fourier modes of f_2 have index bounded by $\frac{1}{\varepsilon}$, we get estimates of the type $\|f_2^{(k+1)}\|_{L^2} \le C_{k,\delta}\frac{1}{\epsilon}\|f_2^{(k)}\|_{L^2}$. This allows us to control terms involving f_2''', $f_2^{(4)}$, etc. To deal with the resonance phenomenon mentioned in the introduction, related to the components in $K_{3,\delta}$ of the approximate kernel, we add to the approximate solutions a function v_δ like

$$v_\delta = \beta(\varepsilon s) Z_{\alpha(\varepsilon s)} + i \xi(\varepsilon s) W_{\alpha(\varepsilon s)} \tag{5.4}$$

(see (4.37) and the lines after), with β and ξ given by

$$\beta(\varepsilon s) = \sum_{j=-\delta^2/\varepsilon}^{\delta^2/\varepsilon} b_j \beta_j(\varepsilon s), \qquad \xi = \sum_{j=-\delta^2/\varepsilon}^{\delta^2/\varepsilon} b_j \xi_j(\varepsilon s), \tag{5.5}$$

where, we recall, ξ_j solves (4.42) and is related to β_j by (4.43). Below we will regard β as an independent variable, and ξ as a function of β. Introducing the norm

$$\|\beta\|_\sharp := \Big(\sum_{j=-\delta^2/\varepsilon}^{\delta^2/\varepsilon} b_j^2 (1 + |j|)^2 \Big)^{\frac{1}{2}}, \tag{5.6}$$

we will assume later on that

$$\|\beta\|_\sharp \leq c_3 \varepsilon^2 \tag{5.7}$$

for some constant $c_3 > 0$ to be specified below.

We will look for approximate solutions of the form

$$\tilde{\Psi}_{2,\varepsilon}(s,z) := e^{-i \frac{\tilde{f}(\varepsilon s)}{\varepsilon}} \big\{ h(\varepsilon s) U(k(\varepsilon s) z) + \varepsilon [w_r + i w_i] + \varepsilon^2 \tilde{v} + \varepsilon^2 v_0 + v_\delta \big\}. \tag{5.8}$$

In this formula $\tilde{f}$ is as above, while $\tilde{v}$ and v_0 are corrections whose choice is given below in order to improve the accuracy of the approximate solutions.

Our goal is to estimate with some accuracy the quantity $S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})$: for simplicity, to treat separately some terms in this expression, we will write $\tilde{\Psi}_{2,\varepsilon}$ as

$$\tilde{\Psi}_{2,\varepsilon}(s,z) = \tilde{\Psi}_{1,\varepsilon}(s,z) + E(s,z) + F(s,z) + G(s,z), \tag{5.9}$$

where $\tilde{\Psi}_{1,\varepsilon}$, E, F, and G are, respectively, defined by

$$\tilde{\Psi}_{1,\varepsilon}(s,z) := e^{-i \frac{\tilde{f}(\varepsilon s)}{\varepsilon}} \{ h(\varepsilon s) U(k(\varepsilon s) z) + \varepsilon [w_r + i w_i] \}$$
$$:= e^{-i \frac{\varepsilon^2 f_2(\varepsilon s))}{\varepsilon}} \tilde{\psi}_{1,\varepsilon},$$

$$E(s,z) := \varepsilon^2 e^{-i \frac{\tilde{f}(\varepsilon s)}{\varepsilon}} \tilde{v}, \qquad F(s,z) := \varepsilon^2 e^{-i \frac{\tilde{f}(\varepsilon s)}{\varepsilon}} v_0,$$
$$G(s,z) := e^{-i \frac{\tilde{f}(\varepsilon s)}{\varepsilon}} v_\delta.$$

To expand $S_\varepsilon(\Psi_{2,\varepsilon})$ conveniently, we can write

$$S_\varepsilon(\tilde{\Psi}_{2,\varepsilon}) = S_\varepsilon(\tilde{\Psi}_{1,\varepsilon}) + \mathfrak{A}_3 + \mathfrak{A}_4 + \mathfrak{A}_5 + \mathfrak{A}_6,$$

where $\mathfrak{A}_3, \dots, \mathfrak{A}_6$ are, respectively, the linear terms in the equation that involve E, F, and G (see (5.9)):

$$\begin{aligned} \mathfrak{A}_3 = {} & -\Delta_g E + V(\varepsilon x)E - |\tilde{\Psi}_{1,\varepsilon}|^{p-1} E \\ & - (p-1)|\tilde{\Psi}_{1,\varepsilon}|^{p-3}\tilde{\Psi}_{1,\varepsilon}\Re(\tilde{\Psi}_{1,\varepsilon}\bar{E}), \end{aligned} \tag{5.10}$$

$$\begin{aligned} \mathfrak{A}_4 = {} & -\Delta_g F + V(\varepsilon x)F - |\tilde{\Psi}_{1,\varepsilon}|^{p-1} F \\ & - (p-1)|\tilde{\Psi}_{1,\varepsilon}|^{p-3}\tilde{\psi}_{1,\varepsilon}\Re(\tilde{\Psi}_{1,\varepsilon}\bar{F}), \end{aligned} \tag{5.11}$$

$$\begin{aligned} \mathfrak{A}_5 = {} & -\Delta_g G + V(\varepsilon x)G - |\tilde{\Psi}_{1,\varepsilon}|^{p-1} G \\ & - (p-1)|\tilde{\Psi}_{1,\varepsilon}|^{p-3}\tilde{\Psi}_{1,\varepsilon}\Re(\tilde{\Psi}_{1,\varepsilon}\bar{G}), \end{aligned} \tag{5.12}$$

and where $\mathfrak{A}_6$ contains the contribution of the nonlinear part

$$\begin{aligned} \mathfrak{A}_6 = {} & -|\tilde{\Psi}_{2,\varepsilon}|^{p-1}\tilde{\Psi}_{2,\varepsilon} + |\tilde{\Psi}_{1,\varepsilon}|^{p-1}(E+F+G) \\ & + (p-1)|\tilde{\Psi}_{1,\varepsilon}|^{p-3}\tilde{\Psi}_{1,\varepsilon}\Re(\tilde{\Psi}_{1,\varepsilon}(\bar{E}+\bar{F}+\bar{G})). \end{aligned} \tag{5.13}$$

Next we write (tautologically)

$$\begin{aligned} S_\varepsilon(\tilde{\Psi}_{1,\varepsilon}) &= e^{-i\frac{\varepsilon^2 f_2(\varepsilon s)}{\varepsilon}} S_\varepsilon(\tilde{\psi}_{1,\varepsilon}) + \mathfrak{A}_1, \\ \mathfrak{A}_1 &= S_\varepsilon(\tilde{\Psi}_{1,\varepsilon}) - e^{-i\frac{\varepsilon^2 f_2(\varepsilon s)}{\varepsilon}} S_\varepsilon(\tilde{\psi}_{1,\varepsilon}), \end{aligned} \tag{5.14}$$

and set

$$\begin{aligned} \mathfrak{A}_2 = e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}} \Big(& e^{i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}} S_\varepsilon(\tilde{\psi}_{1,\varepsilon}) \\ & - \varepsilon^2(\tilde{R}_{r,o} + \tilde{R}_{r,e}) - \varepsilon^2(\tilde{R}_{r,o,f_1} + \tilde{R}_{r,e,f_1}) \\ & - \varepsilon^2 i(\tilde{R}_{i,e} + \tilde{R}_{i,o}) - \varepsilon^2 i(\tilde{R}_{i,e,f_1} + \tilde{R}_{i,o,f_1})\Big), \end{aligned} \tag{5.15}$$

so that $\mathfrak{A}_2$ represents the terms that are formally of order ε^3 and higher in $S_\varepsilon(\tilde{\psi}_{1,\varepsilon})$ (multiplied by a phase factor). Therefore, from definitions (5.10)–(5.15), we find that

$$\begin{aligned} S_\varepsilon(\tilde{\Psi}_{2,\varepsilon}) = e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}} \Big(& \varepsilon^2(\tilde{R}_{r,o} + \tilde{R}_{r,e}) + \varepsilon^2(\tilde{R}_{r,o,f_1} + \tilde{R}_{r,e,f_1}) \\ & + \varepsilon^2 i(\tilde{R}_{i,e} + \tilde{R}_{i,o}) + \varepsilon^2 i(\tilde{R}_{i,e,f_1} + \tilde{R}_{i,o,f_1})\Big) \\ & + \mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{A}_3 + \mathfrak{A}_4 + \mathfrak{A}_5 + \mathfrak{A}_6. \end{aligned} \tag{5.16}$$

To estimate rigorously the $\mathfrak{A}_i$'s, we display the first- and second-order derivatives of $\widetilde{\Psi}_{2,\varepsilon}$:

$$
\begin{aligned}
\partial_s \tilde{\Psi}_{2,\varepsilon} &= -i\,\widetilde{f}'(\varepsilon s)e^{-i\frac{\widetilde{f}(\varepsilon s)}{\varepsilon}}\big[h(\varepsilon s)U(k(\varepsilon s)z) + \varepsilon[w_r + iw_i] \\
&\qquad + \varepsilon^2\widetilde{v} + \varepsilon^2 v_0 + v_\delta\big] \\
&\quad + e^{-i\frac{\widetilde{f}(\varepsilon s)}{\varepsilon}}\big[\varepsilon h' U(kz) + \varepsilon h k' \nabla U\cdot z + \varepsilon^2\partial_s w_r + i\varepsilon^2\partial_s w_i \\
&\qquad + \varepsilon^3\partial_s\widetilde{v} + \varepsilon^3\partial_s v_0 + \partial_s v_\delta\big],\\
\partial_j \tilde{\Psi}_{2,\varepsilon} &= e^{-i\frac{\widetilde{f}(\varepsilon s)}{\varepsilon}}\big[hk\partial_j U(k(\varepsilon s)z) + \varepsilon[\partial_j w_r + i\partial_j w_i] + \varepsilon^2\partial_j\widetilde{v} \\
&\qquad + \varepsilon^2\partial_j v_0 + \partial_j v_\delta\big],\\
\partial^2_{ss} \tilde{\Psi}_{2,\varepsilon} &= (-\widetilde{f}'^2 - i\varepsilon\widetilde{f}'')(\varepsilon s)e^{-i\frac{\widetilde{f}(\varepsilon s)}{\varepsilon}}\big[h(\varepsilon s)U(k(\varepsilon s)z) + \varepsilon[w_r + iw_i] + \varepsilon^2\widetilde{v} \\
&\qquad + \varepsilon^2 v_0 + a(\varepsilon s)Z(kz)\big] \\
&\quad + e^{-i\frac{\widetilde{f}(\varepsilon s)}{\varepsilon}}\big[\varepsilon^2 h'' U(kz) + 2\varepsilon^2 h'k'\nabla U\cdot z + \varepsilon^2 hk''\nabla U\cdot z \\
&\qquad + \varepsilon^2 hk'^2\nabla^2 U[z,z] + \varepsilon^3\partial^2_{ss}w_r + i\varepsilon^3\partial^2_{ss}w_i + \varepsilon^4\partial^2_{ss}\widetilde{v} \\
&\qquad + \varepsilon^4\partial^2_{ss}v_0 + \partial^2_{ss}v_\delta\big] \\
&\quad - 2i\,\widetilde{f}'(\varepsilon s)e^{-i\frac{\widetilde{f}(\varepsilon s)}{\varepsilon}}\big[\varepsilon h' U(kz) + \varepsilon hk'\nabla U\cdot z + \varepsilon^2\partial_s w_r \\
&\qquad + i\varepsilon^2\partial_s w_i + \varepsilon^3\partial_s\widetilde{v} + \varepsilon^3\partial_s v_0 + \partial_s v_\delta\big],\\
\partial^2_{jl} \tilde{\Psi}_{2,\varepsilon} &= e^{-i\frac{\widetilde{f}(\varepsilon s)}{\varepsilon}}\big[h(\varepsilon s)k^2\partial^2_{jl}U(k(\varepsilon s)z) + \varepsilon[\partial^2_{jl}w_r + i\partial^2_{jl}w_i] + \varepsilon^2\partial^2_{jl}\widetilde{v} \\
&\qquad + \varepsilon^2\partial^2_{jl}v_0 + \partial^2_{jl}v_\delta\big],\\
\partial^2_{sj} \tilde{\Psi}_{2,\varepsilon} &= -i\,\widetilde{f}'(\varepsilon s)e^{-i\frac{\widetilde{f}(\varepsilon s)}{\varepsilon}}\big[h(\varepsilon s)k\partial_j U(k(\varepsilon s)z) + \varepsilon[\partial_j w_r + i\partial_j w_i] + \varepsilon^2\partial_j\widetilde{v} \\
&\qquad + \varepsilon^2\partial_j v_0 + a(\varepsilon s)k\partial_j Z(kz)\big] \\
&\quad + e^{-i\frac{\widetilde{f}(\varepsilon s)}{\varepsilon}}\big[\varepsilon h' k\partial_j U(kz) + \varepsilon hk'\partial_j U + \varepsilon hkk' z_l\partial^2_{jl}U + \varepsilon^2\partial^2_{sj}w_r \\
&\qquad + i\varepsilon^2\partial^2_{sj}w_i + \varepsilon^3\partial^2_{sj}\widetilde{v} + \varepsilon^3\partial^2_{sj}v_0 + \partial^2_{sj}v_\delta\big].
\end{aligned}
$$

To simplify the expressions of the error terms, we introduce some convenient notation. For any positive integer q, the two symbols $\mathfrak{R}_q(\Phi,\Phi')$ and $\mathfrak{R}_q(\Phi,\Phi',\Phi'')$ will denote error terms satisfying the following bounds for some fixed constants C and d (which depend on q, c_1, c_2, and c_3 but not on ε, s, or δ)

$$
\begin{cases}
|\mathfrak{R}_q(\Phi,\Phi')| \le C\varepsilon^q(1+|z|^d)e^{-k|z|},\\
|\mathfrak{R}_q(\Phi,\Phi') - \mathfrak{R}_q(\tilde{\Phi},\tilde{\Phi}')| \le C\varepsilon^q(1+|z|^d)[|\Phi-\tilde{\Phi}| + |\Phi'-\tilde{\Phi}'|]e^{-k|z|},
\end{cases}
$$

while the term $\mathfrak{R}_q(\Phi, \Phi', \Phi'')$ (which also involves second derivatives of Φ) stands for a quantity for which

$$|\mathfrak{R}_q(\Phi, \Phi', \Phi'')| \le C\varepsilon^q(1+|z|^d)e^{-k|z|} + C\varepsilon^{q+1}(1+|z|^d)e^{-k|z|}|\Phi''|,$$

$$\begin{aligned}|\mathfrak{R}_q(\Phi, \Phi', \Phi'') &- \mathfrak{R}_q(\tilde\Phi, \tilde\Phi', \tilde\Phi'')|\\ &\le C\varepsilon^q(1+|z|^d)[|\Phi-\tilde\Phi| + |\Phi'-\tilde\Phi'|]e^{-k|z|}\\ &\quad + C\varepsilon^{q+1}(1+|z|^d)\big(|\Phi''+\tilde\Phi''|(|\Phi-\tilde\Phi| + |\Phi'-\tilde\Phi'|)\\ &\qquad + |\Phi''-\tilde\Phi''|\big)e^{-k|z|}.\end{aligned}$$

Similarly, we will let $\mathfrak{R}_q(\bar s)$ denote a quantity (depending only on $\bar s$ and z) such that

$$|\mathfrak{R}_q(\bar s)| \le C\varepsilon^q(1+|z|^d)e^{-k|z|},$$

and which depends smoothly on $\bar s$.

In the estimates below, assumptions (5.2)–(5.3) will be used. On one hand, by (5.2) we have L^∞ estimates on Φ, f_2, and their first derivatives; on the other, by (5.3) we have L^2 estimates on the higher-order derivatives, of the type $\|\Phi^{(l)}\|_{L^2} \le C_l(\delta^l/\varepsilon^l)\|\Phi\|_{L^2}$, for $l \in \mathbb{N}$.

We will also use notation like $\Phi\mathfrak{R}_q(\Phi, \Phi')$, $f_2''\mathfrak{R}_q(\Phi, \Phi')$, etc., to denote error terms that are products of functions of $\bar s$, like Φ or f_2', and the above $\mathfrak{R}_q$'s. Having defined this notation, we can compute (and estimate) $S_\varepsilon(\tilde\Psi_{2,\varepsilon})$ term by term.

5.1 Estimate of $\mathfrak{A}_1$

From the expression of the Laplace-Beltrami operator (see Subsection 3.1) it follows that

$$\begin{aligned}&e^{i\frac{\varepsilon^2 f_2(\varepsilon s)}{\varepsilon}} S_\varepsilon(\tilde\Psi_{1,\varepsilon}) - S_\varepsilon(\tilde\psi_{1,\varepsilon})\\ &\quad = \tilde g^{11}\big[\varepsilon^4(f_2')^2\tilde\psi_{1,\varepsilon} + i\varepsilon^3 f_2''\tilde\psi_{1,\varepsilon} + 2i\varepsilon^2 f_2'\partial_s\tilde\psi_{1,\varepsilon}\big]\\ &\qquad + 2i\sum_l \tilde g^{1l}\varepsilon^2 f_2'\partial_l\tilde\psi_{1,\varepsilon} + \frac{i}{\sqrt{\det\tilde g}}\partial_A\big(g^{A1}\sqrt{\det\tilde g}\big)\varepsilon^2 f_2'\tilde\psi_{1,\varepsilon}.\end{aligned}$$

Using the expressions of w_r, w_i and the expansions of the metric coefficients in Subsection 3.1 and multiplying the last equation by $e^{i(\tilde f_0(\varepsilon s))/\varepsilon}$, we obtain

$$\begin{aligned}e^{i\frac{\tilde f(\varepsilon s)}{\varepsilon}}\mathfrak{A}_1 &= e^{i\frac{\tilde f_0(\varepsilon s)}{\varepsilon}}\Big(e^{i\frac{\varepsilon^2 f_2(\varepsilon s)}{\varepsilon}} S_\varepsilon(\tilde\Psi_{1,\varepsilon}) - S_\varepsilon(\tilde\psi_{1,\varepsilon})\Big)\\ &= e^{i\frac{\tilde f(\varepsilon s)}{\varepsilon}} S_\varepsilon(\tilde\Psi_{1,\varepsilon}) - e^{i\frac{\tilde f_0(\varepsilon s)}{\varepsilon}} S_\varepsilon(\tilde\psi_{1,\varepsilon})\\ &= \mathfrak{A}_{1,0} + \tilde{\mathfrak{A}}_1\\ &:= \mathfrak{A}_{1,0} + \mathfrak{A}_{1,r,e} + \mathfrak{A}_{1,r,o} + \mathfrak{A}_{1,i,e} + \mathfrak{A}_{1,i,o} + \mathfrak{A}_{1,1},\end{aligned} \tag{5.17}$$

where

$$
\begin{aligned}
\mathfrak{A}_{1,0} &= 2\varepsilon^2 f' f_2' hU, \\
\mathfrak{A}_{1,r,e} &= \varepsilon^3 f_2'[2f_1'hU + 4\langle \mathbf{H}, \Phi\rangle f'hU + 2f'w_{r,e}], \\
\mathfrak{A}_{1,r,o} &= \varepsilon^3 f_2'[4\langle \mathbf{H}, z\rangle f'hU + 2f'w_{r,o}], \\
\mathfrak{A}_{1,i,e} &= i\varepsilon^3 f_2''hU + 2i\varepsilon^3 f_2'[f'w_{i,e} + h'U + hk'\nabla U \cdot z], \\
\mathfrak{A}_{1,i,o} &= 2i\varepsilon^3 f' f_2' w_{i,o}, \\
\mathfrak{A}_{1,1} &= (f_2')^2 \mathfrak{R}_4(\Phi, \Phi') + f_2''\mathfrak{R}_5(\Phi, \Phi') + f_2'\Phi''\mathfrak{R}_4(\Phi, \Phi') \\
&\quad + f_2'\mathfrak{R}_4(\Phi, \Phi').
\end{aligned}
\tag{5.18}
$$

5.2 Estimate of $\mathfrak{A}_2$

Reasoning as for the previous estimate and collecting the terms of order ε^3 and higher in $S_\varepsilon(\tilde{\psi}_{1,\varepsilon})$, we obtain

$$
e^{i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\mathfrak{A}_2 = \mathfrak{A}_{2,0} + \tilde{\mathfrak{A}}_2 := \mathfrak{A}_{2,r,e} + \mathfrak{A}_{2,r,o} + \mathfrak{A}_{2,i,e} + \mathfrak{A}_{2,i,o} + \mathfrak{A}_{2,1},
\tag{5.19}
$$

where $\mathfrak{A}_{2,0} = 0$ and where the remaining terms are given by

$$
\begin{aligned}
\mathfrak{A}_{2,r,e} &= \varepsilon^3 \Phi'' F_e(\bar{s}), \\
\mathfrak{A}_{2,r,o} &= 2\varepsilon^3 hk\langle \mathbf{H}, \Phi\rangle \sum_j \Phi_j''\partial_j U + \varepsilon^3 f'hf_1' \sum_j z_j \Phi_j'' U \\
&\quad + 2\varepsilon^3 f'^2 h\langle \mathbf{H}, \Phi\rangle \sum_j \Phi_j'' z_j U, \\
\mathfrak{A}_{2,i,e} &= -2i\varepsilon^3 f'h \sum_{j,l} \Phi_l'\Phi_j''\,\partial_l(z_j U) + 2i\varepsilon^4 f'h\langle \mathbf{H}, z\rangle \sum_j \Phi_j''' z_j U, \\
\mathfrak{A}_{2,i,o} &= i\varepsilon^3 \sum_j \Phi_j'' z_j (f''hU + f'h'U + f'hk'\nabla U \cdot z) + i\varepsilon^3 \Phi''' F_o(\bar{s}) \\
&\quad + 2i\varepsilon^4 f'h\langle \mathbf{H}, \Phi\rangle \sum_j \Phi_j''' z_j U, \\
\mathfrak{A}_{2,1} &= \mathfrak{R}_3(\bar{s}) + (\Phi + \Phi')\mathfrak{R}_3(\Phi, \Phi', \Phi'') + \mathfrak{R}_4(\Phi, \Phi', \Phi''),
\end{aligned}
$$

where $F_e(\bar{s})$ and $F_o(\bar{s})$ are, respectively, an even real function and an odd real function in the variables z, with smooth coefficients in $\bar{s} = \varepsilon s$, and satisfying the decay property $|F_e(\bar{s})| + |F_o(\bar{s})| \le C(1 + |z|^d)e^{-k|z|}$.

5.3 Choice of $\tilde{v}$ and Estimate of $\mathfrak{A}_3$

We choose the function $\tilde{v}$ in such a way to annihilate (roughly) one of the main terms in (5.17), namely $2\varepsilon^2 f' f_2' hU(kz)$. Hence we define $\tilde{v}$ so that it solves

$$
\mathfrak{L}_r \tilde{v} = -2f' f_2' hU(kz).
\tag{5.20}
$$

Reasoning as for the definition of w_r (see Subsection 3.2), we can explicitly determin $\widetilde{v}$ as

$$\widetilde{v} = 2f' f_2' h \widetilde{U}(kz).$$

With this definition, using the above estimates on the metric coefficients and the expressions of error terms, the linear terms involving E in $S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})$ can be written as

$$\begin{aligned} e^{i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}} \mathfrak{A}_3 &= \mathfrak{A}_{3,0} + \tilde{\mathfrak{A}}_3 \\ &:= \mathfrak{A}_{3,0} + \mathfrak{A}_{3,r,e} + \mathfrak{A}_{3,r,o} + \mathfrak{A}_{3,i,e} + \mathfrak{A}_{3,i,o} + \mathfrak{A}_{3,1}, \end{aligned} \tag{5.21}$$

where

$$\mathfrak{A}_{3,0} = \varepsilon^2 \mathfrak{L}_r \tilde{v}, \tag{5.22}$$

$$\begin{aligned} \mathfrak{A}_{3,r,e} &= 2\varepsilon^3 f' h f_2' \big(2(f')^2 \langle \mathbf{H}, \Phi \rangle + \langle \nabla^N V, \Phi \rangle \\ &\qquad - p(p-1) h^{p-2} U^{p-2} w_{r,e} \big) \tilde{U} \\ &\quad + 4\varepsilon^3 (f')^2 f_1' h f_2' \tilde{U}(kz) - 2\varepsilon^4 f' f_2''' h \widetilde{U}, \\ \mathfrak{A}_{3,r,o} &= 2\varepsilon^3 f' h f_2' \big(2(f')^2 \langle \mathbf{H}, z \rangle + \langle \nabla^N V, z \rangle \\ &\qquad - p(p-1) h^{p-2} U^{p-2} w_{r,o} \big) \tilde{U} \\ &\quad + 2\varepsilon^3 f' f_2' h k \sum_j H^j \partial_j \tilde{U}(kz) \\ &\quad - 2\varepsilon^4 f' h k f_2' \sum_j \Phi_j'' \partial_j \widetilde{U}, \\ \mathfrak{A}_{3,i,e} &= 4 i f' \varepsilon^3 \partial_{\bar{s}} (h f' f_2' \tilde{U}) \\ &\quad + 2 i \varepsilon^3 f' f_2' \big(f'' h - (p-1) h^{p-1} U^{p-1} w_{i,e} \big) \tilde{U}, \\ \mathfrak{A}_{3,i,o} &= -2(p-1) i \varepsilon^3 f' f_2' h^{p-1} U^{p-1} w_{i,o} \tilde{U} \\ &\quad - 4 i \varepsilon^3 (f')^2 f_2' h k \sum_j \Phi_j' \partial_j \tilde{U}, \end{aligned}$$

$$
\begin{aligned}
\mathfrak{A}_{3,1} = {} & f_2'[\mathfrak{R}_4(\Phi,\Phi',\Phi'')] + f_2''[\mathfrak{R}_4(\Phi,\Phi') + \mathfrak{R}_6(\Phi,\Phi',\Phi'')] \\
& + \varepsilon^4 f_2'''[\mathfrak{R}_1(\Phi,\Phi')] + \mathfrak{R}_4(\Phi,\Phi') f_2'[f_2'(1+\varepsilon^2 f_2') + \varepsilon f_2''] \\
& + \mathfrak{R}_5(\Phi,\Phi') f_2'' f_2' + \mathfrak{R}_6(\Phi,\Phi',\Phi'')(f_2')^2.
\end{aligned}
$$

5.4 Choice of v_0 and Estimate of $\mathfrak{A}_4$

In order to make the approximate solution as accurate as possible, we add a correction $\varepsilon^2 v_0$ in such a way to compensate (most of) the terms $\varepsilon^2(\tilde{R}_{r,e} + i\tilde{R}_{i,o})$; see Subsection 3.3. We notice that these terms contain parts that are independent of Φ, which we denote by $\tilde{R}^0_{r,e}$ and $\tilde{R}^0_{i,o}$, and parts that are quadratic in Φ or its derivatives $\tilde{R}^\Phi_{r,e}$ and $\tilde{R}^\Phi_{i,o}$, respectively. Since we will take Φ of order ε, we regard the latter ones as higher-order terms, and we add corrections to cancel $\tilde{R}^0_{r,e}$ and $\tilde{R}^0_{i,o}$.

Specifically, we define $v^0_{r,e}$ and $v^0_{i,o}$ by

$$
\begin{aligned}
(5.23)\qquad -\mathfrak{L}_r v^0_{r,e} = {} & -\frac{1}{2}(f')^2 hU(kz)\sum_{l,m}\partial^2_{lm} g_{11} z_m z_l + 2(f')^2\langle \mathbf{H}, +w_{r,o} z\rangle \\
& + 4(f')^2 hU(kz)\langle \mathbf{H}, z\rangle^2 + 2f'\partial_s w_{i,e} + f'' w_{i,e} \\
& - \big[h''U(kz) + 2h'k'\nabla U(kz)\cdot z + hk''\nabla U(kz)\cdot z \\
& \quad + h(k')^2\nabla^2 U(kz)[z,z]\big] \\
& + \frac{1}{2}\sum_{l,m}\partial^2_{lm} g_{tj} z_m z_l hk^2 \partial^2_{tj} U(kz) + kh[\langle \mathbf{H}, z\rangle H^m \\
& - \frac{1}{2}\sum_{l}\partial^2_{ml} g_{11} z_l]\partial_m U(kz) + \\
& + hk\sum_{l}\partial^2_{lm} g_{mj} z_l \partial_j U(kz) + \sum_{l} H^l \partial_l w_{r,o} \\
& + hk\langle \mathbf{H}, z\rangle H^l \partial_l U(kz) + \langle \nabla^N V, w_{r,o} z\rangle \\
& - \frac{1}{2}(p-1)h^{p-2} U(kz)^{p-2} w^2_{i,e} \\
& - \frac{1}{2}p(p-1)h^{p-2} U(kz)^{p-2} w^2_{r,o} \\
& + \frac{1}{2}\sum_{m,j}\partial^2_{mj} V z_m z_j hU(kz),
\end{aligned}
$$

$$
\begin{aligned}
-\mathcal{L}_i v_{i,o}^0 = {} & 2\big[f''hU(kz) + 2f'h'U(kz) + 2f'hk'\nabla U(kz)\cdot z\big]\langle \mathbf{H}, z\rangle \\
& + \sum_i H^j \partial_j w_{i,e} + 2(f')^2\langle \mathbf{H}, w_{i,e} z\rangle + 2f'\partial_s w_{r,o} \\
& + f'' w_{r,o} - f'hk \sum_j \partial_j U(kz) \sum_{l,m} \partial^2_{lm} g_{1j} z_m z_l \\
& - f'hU(kz)\Big(\sum_m \partial^2_{1m} g_{11} z_m\Big) - f'h\Big(\sum_{j,l} \partial^2_{lj} g_{1j} z_l\Big) U(kz) \\
& + \frac{1}{2} f'h\Big(\sum_l \partial^2_{1l} g_{11} z_l\Big) U(kz) \\
& - (p-1)h^{p-2} U(kz)^{p-2} w_{r,o} w_{i,e} + \langle \nabla^N V, w_{i,e} z\rangle.
\end{aligned} \tag{5.24}
$$

We notice that the right-hand side of (5.23) is even in z, and hence orthogonal to the kernel of $\mathcal{L}_r$. As a consequence, the equation is indeed solvable in $v_{r,e}^0$; see the comments after (1.17). The same comment applies to (5.24), where the right-hand side is odd in z. Furthermore, the right-hand sides decay at infinity at most like $(1+|z|^d)e^{-k|z|}$ for some integer d, so the same holds true for $v_{r,e}^0$ and $v_{i,o}^0$.

In conclusion, after some computations we find

$$
e^{i\frac{\tilde f(\varepsilon s)}{\varepsilon}} \mathfrak{A}_4 = \mathfrak{A}_{4,0} + \tilde{\mathfrak{A}}_4 := \mathfrak{A}_{4,0} + \mathfrak{A}_{4,r,e} + \mathfrak{A}_{4,r,o} + \mathfrak{A}_{4,i,e} + \mathfrak{A}_{4,i,o} + \mathfrak{A}_{4,1},
$$

where

$$
\begin{aligned}
\mathfrak{A}_{4,0} &= \varepsilon^2 \mathcal{L}_r v_{r,e}^0 + i\varepsilon^2 \mathcal{L}_i v_{i,o}^0, \\
\mathfrak{A}_{4,r,e} &= \varepsilon^3 F_{4,r,e}(\bar s), \quad \mathfrak{A}_{4,r,o} = \varepsilon^3 F_{4,r,o}(\bar s), \\
\mathfrak{A}_{4,i,e} &= \varepsilon^3 F_{4,i,e}(\bar s), \quad \mathfrak{A}_{4,i,o} = \varepsilon^3 F_{4,i,o}(\bar s),
\end{aligned} \tag{5.25}
$$

$$
\begin{aligned}
\mathfrak{A}_{4,1} = {} & \mathfrak{R}_4(\Phi, \Phi') + (\Phi + \Phi')(1 + f_2')\mathfrak{R}_3(\Phi, \Phi') + f_2''\mathfrak{R}_5(\Phi, \Phi') \\
& + (f_2')^2 \mathfrak{R}_6(\bar s) + \Phi''\mathfrak{R}_4(\Phi, \Phi').
\end{aligned} \tag{5.26}
$$

As for $F_e(\bar s)$ and $F_o(\bar s)$ in $\mathfrak{A}_2$, the F_4's depend only on V, γ, and M and are bounded above by $C(1+|z|^d)e^{-k|z|}$.

5.5 Estimate of $\mathfrak{A}_5$

The term involving v_δ in $S_\varepsilon(\tilde\Psi_{2,\varepsilon})$ is given by

$$
\mathfrak{A}_5 = \mathfrak{A}_{5,0} + \tilde{\mathfrak{A}}_5 := \mathfrak{A}_{5,0} + \mathfrak{A}_{5,r,e} + \mathfrak{A}_{5,r,o} + \mathfrak{A}_{5,i,e} + \mathfrak{A}_{5,i,o} + \mathfrak{A}_{5,1} \tag{5.27}
$$

where

$$
\begin{aligned}
\mathfrak{A}_{5,0} &= \beta \mathfrak{L}_r Z_\alpha(kz) - \varepsilon^2 \beta'' Z_\alpha(kz) - 2\varepsilon \xi' W_\alpha f' + i\xi \mathfrak{L}_i W_\alpha \\
&\quad - i\varepsilon^2 \xi'' W_\alpha + 2i\varepsilon\beta' f' Z_\alpha, \\
\mathfrak{A}_{5,r,e} &= -\varepsilon f'' \xi W_\alpha - 2\varepsilon^2 \beta' \left(\frac{\partial Z_\alpha}{\partial \alpha} \alpha' + k' \nabla Z_\alpha(kz) \cdot z \right) \\
&\quad - 2\varepsilon f' \xi \left(\frac{\partial W_\alpha}{\partial \alpha} \alpha' + k' \nabla W_\alpha(kz) \cdot z \right) \\
&\quad - (p-1)\varepsilon h^{p-2} U^{p-2} \xi w_{i,e} W_\alpha, \\
\mathfrak{A}_{5,r,o} &= \varepsilon\beta \sum_j H^j \partial_j Z_\alpha \\
&\quad + 2\varepsilon \langle \mathbf{H}, z\rangle [(f')^2 \beta Z_\alpha + \varepsilon^2 \beta'' Z_\alpha - 2\varepsilon \xi' f' W_\alpha] \\
&\quad + \varepsilon \langle \nabla^N V, z\rangle \beta Z_\alpha - p(p-1)\varepsilon h^{p-2} U^{p-2} w_{r,o} \beta Z_\alpha, \\
\mathfrak{A}_{5,i,e} &= \varepsilon f'' \beta Z_\alpha - 2\varepsilon^2 \xi' \left(\frac{\partial W_\alpha}{\partial \alpha} \alpha' + k' \nabla W_\alpha(kz) \cdot z \right) \\
&\quad + 2\varepsilon f' \beta \left(\frac{\partial Z_\alpha}{\partial \alpha} \alpha' + k' \nabla Z_\alpha(kz) \cdot z \right) \\
&\quad - (p-1)\varepsilon h^{p-2} U^{p-2} \beta w_{i,e} Z_\alpha, \\
\mathfrak{A}_{5,i,o} &= \varepsilon\xi \sum_j H^j \partial_j W_\alpha \\
&\quad + 2\varepsilon \langle \mathbf{H}, z\rangle [(f')^2 \xi W_\alpha + \varepsilon^2 \xi'' W_\alpha + 2\varepsilon \beta' f' Z_\alpha] \\
&\quad + \varepsilon \langle \nabla^N V, z\rangle \xi W_\alpha - (p-1)\varepsilon h^{p-2} U^{p-2} w_{r,o} \xi W_\alpha.
\end{aligned} \tag{5.28}
$$

The error term $\mathfrak{A}_{5,1} = \mathfrak{A}_{5,1}(\beta, \Phi, f_2)$ satisfies the following estimates:

$$
\begin{aligned}
|\mathfrak{A}_{5,1}(\beta, \Phi, f_2)| \le{}& C(\varepsilon^2 + \varepsilon^2 |f_2'| + \varepsilon^3 |f_2''| + \varepsilon^3 |\Phi''|)(1 + |z|^d) e^{-k|z|} \\
&\times (|\beta| + \varepsilon|\beta'| + \varepsilon^2 |\beta''| + \varepsilon^3 |\beta'''|),
\end{aligned}
$$

$$
\begin{aligned}
&|\mathfrak{A}_{5,1}(\beta, \Phi, f_2) - \mathfrak{A}_{5,1}(\tilde{\beta}, \tilde{\Phi}, \tilde{f}_2)| \\
&\quad \leq C\big(\varepsilon|\Phi - \tilde{\Phi}| + \varepsilon|\Phi' - \tilde{\Phi}'| + \varepsilon^3|\Phi'' - \tilde{\Phi}''| \\
&\qquad + \varepsilon^2|f_2' - \tilde{f}_2'| + \varepsilon^3|f_2'' - \tilde{f}_2''|\big) \\
&\qquad \times (1 + |z|^d)e^{-k|z|}(|\beta| + \varepsilon|\beta'| + \varepsilon^2|\beta''| + \varepsilon^3|\beta'''|) + \\
&\qquad + C(\varepsilon^2 + \varepsilon^2|f_2'| + \varepsilon^3|f_2''| + \varepsilon^3|\Phi''|) \\
&\qquad \times (1 + |z|^d)e^{-k|z|}\big(|\beta - \tilde{\beta}| + \varepsilon|\beta' - \tilde{\beta}'| + \varepsilon^2|\beta'' - \tilde{\beta}''| + \varepsilon^3|\beta''' - \tilde{\beta}'''|\big).
\end{aligned}
$$

By the form of the function β (see (4.42), (4.43), and (5.5)), its Fourier modes are mainly concentrated around indices of order $\frac{1}{\varepsilon}$. As a consequence, L^2 norms of functions like $\varepsilon\beta$, $\varepsilon^2\beta''$, $\varepsilon^3\beta'''$, etc., can be controlled with the L^2 norm of β; see also the comments before (4.58).

5.6 Estimate of $\mathfrak{A}_6$

First of all, we notice that we are taking Φ' and f_2' in $H^1([0, L])$, and hence they belong to $L^\infty([0, L])$. As a consequence, since we have the bound

$$
\begin{aligned}
\|\beta\|_{L^\infty([0,L])} + \varepsilon\|\beta'\|_{L^\infty([0,L])} &+ \varepsilon^2\|\beta''\|_{L^\infty([0,L])} \\
&+\varepsilon^3\|\beta'''\|_{L^\infty([0,L])} \leq C\varepsilon^2
\end{aligned}
$$

(which follows from (5.7) and the above comments), we have the estimate

$$
|E| + |F| + |G| \leq C\varepsilon^2(1 + |z|^d)e^{-k|z|}. \tag{5.29}
$$

If we then choose δ sufficiently small (also recall the expressions of w_r and w_i and (5.9)), we deduce that

$$
|\tilde{\Psi}_{2,\varepsilon} - \tilde{\Psi}_{1,\varepsilon}| \leq |\tilde{\Psi}_{1,\varepsilon}| \quad \text{in } \tilde{D}_\varepsilon.
$$

This estimate implies that $\mathfrak{A}_6$ admits a uniform quadratic Taylor expansion in $|\tilde{\Psi}_{2,1} - \tilde{\Psi}_{1,\varepsilon}|$ and is bounded by $|\tilde{\Psi}_{1,\varepsilon}|^{p-2}|\tilde{\Psi}_{2,\varepsilon} - \tilde{\Psi}_{1,\varepsilon}|^2$.

Specifically, we can write

$$
\mathfrak{A}_6 = \mathfrak{A}_{6,0} + \tilde{\mathfrak{A}}_6 := \mathfrak{A}_{6,0} + \mathfrak{A}_{6,r,e} + \mathfrak{A}_{6,r,o} + \mathfrak{A}_{6,i,e} + \mathfrak{A}_{6,i,o} + \mathfrak{A}_{6,1}, \tag{5.30}
$$

where

$$
\begin{aligned}
\mathfrak{A}_{6,0} &= \mathfrak{A}_{6,r,e} = \mathfrak{A}_{6,r,o} = \mathfrak{A}_{6,i,e} = \mathfrak{A}_{6,i,o} = 0, \\
\mathfrak{A}_{6,1} &= R_4(f_2', \Phi, \Phi', \beta),
\end{aligned} \tag{5.31}
$$

where $R_4(f_2', \Phi, \Phi', \beta)$ is a quantity satisfying the estimates

$$
\begin{aligned}
&|R_4(f_2', \Phi, \Phi', \beta)| \leq \\
&\qquad C\big[\varepsilon^4 + (\varepsilon^2 + \|\beta\|_{L^\infty} + \varepsilon\|\beta'\|_{L^\infty})(|\beta| + \varepsilon|\beta'|)\big](1 + |z|^d)e^{-k|z|},
\end{aligned}
$$

$$|R_4(f_2', \Phi, \Phi', \beta) - R_4(\tilde{f}_2', \tilde{\Phi}, \tilde{\Phi}', \tilde{\beta})| \\ \leq C(\varepsilon^2 + |\beta| + \varepsilon|\beta'| + |\tilde{\beta}| + \varepsilon|\tilde{\beta}'|)(1 + |z|^d)e^{-k|z|} \\ \times \big(\varepsilon|\Phi - \tilde{\Phi}| + \varepsilon|\Phi' - \tilde{\Phi}'| + \varepsilon^2|f_2' - \tilde{f}_2'| + |\beta - \tilde{\beta}| + \varepsilon|\beta' - \tilde{\beta}'|\big).$$

5.7 Final Estimate of $S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})$

By (5.16), in the above notation we have

$$e^{i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}} S_\varepsilon(\tilde{\Psi}_{2,\varepsilon}) = \varepsilon^2(\tilde{R}_{r,o} + \tilde{R}_{r,e}) + \varepsilon^2(\tilde{R}_{r,o,f_1} + \tilde{R}_{r,e,f_1}) \\ + \varepsilon^2 i(\tilde{R}_{i,e} + \tilde{R}_{i,o}) + \varepsilon^2 i(\tilde{R}_{i,e,f_1} + \tilde{R}_{i,o,f_1}) \\ + \sum_{i=1}^{6} \mathfrak{A}_{i,0} + \sum_{i=1}^{6} \tilde{\mathfrak{A}}_i.$$

Recalling the choices of $\tilde{v}$, $v_{r,e}^0$, and $v_{i,o}^0$ in (5.20), (5.23), and (5.24) (and recalling the notation for the R's after (5.1)), we finally obtain the following result:

PROPOSITION 5.2 *Suppose Φ, f_2, and β satisfy* (5.2), (5.3), *and* (5.7) *for some $c_1, c_2, c_3 > 0$. Let $\tilde{f} = f + \varepsilon f_1 + \varepsilon^2 f_2$, where f is given in* (1.12) *and f_1 in* (3.25). *Let also $w_r = w_{r,e} + w_{r,o}$, with $w_{r,e}$ and $w_{r,o}$ given, respectively, in* (3.13) *and* (3.12), *and $w_i = w_{i,e} + w_{i,o}$, where $w_{i,e}$ and $w_{i,o}$ are given in* (3.9). *Let $\tilde{\Psi}_{2,\varepsilon}$ be defined in* (5.8). *Then, as ε tends to* 0, *we have*

$$e^{i\frac{\tilde{f}}{\varepsilon}} S_\varepsilon(\tilde{\Psi}_{2,\varepsilon}) = \varepsilon^2(\tilde{R}_{r,e}^{\Phi} + \tilde{R}_{r,o} + \tilde{R}_{r,e,f_1} + \tilde{R}_{r,o,f_1}) \\ + \varepsilon^2 i(\tilde{R}_{i,e} + \tilde{R}_{i,o}^{\Phi} + \tilde{R}_{i,e,f_1} + \tilde{R}_{i,o,f_1}) \\ + \beta \mathfrak{L}_r Z_\alpha(kz) + \varepsilon^2 \beta'' Z_\alpha(kz) - 2\varepsilon\xi' W_\alpha f' \\ + i\xi \mathfrak{L}_i W_\alpha + i\varepsilon^2 \xi'' W_\alpha + 2i\varepsilon\beta' f' Z_\alpha + e^{i\frac{\tilde{f}}{\varepsilon}} \sum_{j=1}^{6} \tilde{\mathfrak{A}}_j, \tag{5.32}$$

where the R's are as in (5.1), *where $\tilde{R}_{r,e}^{\Phi}$ and $\tilde{R}_{i,o}^{\Phi}$ are the terms quadratic in Φ and Φ' within $\tilde{R}_{r,e}$ and $\tilde{R}_{i,o}$, and where the latter error terms are given in* (5.17), (5.19), (5.21), (5.26), (5.27), *and* (5.30).

Remark 5.3. In some of the error terms listed above, we sometimes see derivatives of order higher than 2 appearing on Φ, f_2, and β. However, we are not only assuming H^2 bounds on these functions, but also that they are linear combinations of eigenfunctions corresponding to suitable eigenvalues. This fact then allows us to derive bounds on higher-order norms; see the comments after (4.81) and (5.3) and the last comments in the part concerning $\mathfrak{A}_5$.

6 Proof of Theorem 1.1

In this section we prove our main theorem. First we solve the equation in the $\bar{H}_\varepsilon$ components (see (4.87)), using a Lyapunov-Schmidt reduction. Then we turn to the components in $\tilde{K}_\delta$ and solve the bifurcation equation as well; this last step crucially depends on the nondegeneracy assumption on γ and an accurate choice for the values of the parameter ε.

6.1 Solvability in the Component of $\bar{H}_\varepsilon$

In Proposition 4.4 we showed that problem (1.16) is reduced to finding a solution of $L_\varepsilon(\phi) = \tilde{S}_\varepsilon(\phi)$ in $\tilde{D}_\varepsilon$ (see (4.7), (4.23), and (4.24)) if we take $K^2(\varepsilon s) = V(\varepsilon s)$. Choosing $\tilde{\psi}_\varepsilon = \tilde{\Psi}_{2,\varepsilon}$ (the function constructed in the previous subsection) as an approximation to the solution of Proposition 4.14, we have the following result where, as usual, δ is sufficiently small. We recall Proposition 4.4, formulas (4.85)–(4.88), and the definition of $\tilde{K}_\delta$ after (4.86). Also, we denote by $\tilde{\Pi}_\varepsilon$ the orthogonal projection onto the set $\{e^{-i(\tilde{f}(\varepsilon s)/\varepsilon)}\tilde{v} : \tilde{v} \in \tilde{K}_\delta\}$.

PROPOSITION 6.1 *Let $\tilde{\Psi}_{2,\varepsilon}$ be as in Proposition 5.2. Then there exists $\check{v}_\delta \in \tilde{K}_\delta$, depending on the parameters Φ, f_2, and β, such that the following problem admits a solution*:

$$
\begin{cases}
-\Delta_{g_\varepsilon}\widehat{\phi} + V(\varepsilon x)\widehat{\phi} - |\tilde{\Psi}_{2,\varepsilon}|^{p-1}\widehat{\phi} - (p-1)|\tilde{\Psi}_{2,\varepsilon}|^{p-3}\tilde{\Psi}_{2,\varepsilon}\Re(\tilde{\Psi}_{2,\varepsilon}\bar{\widehat{\phi}}) \\
\quad = \tilde{S}_\varepsilon(\widehat{\phi}) + e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\check{v}, \\
\widehat{\phi} \in \bar{H}_\varepsilon, \quad \check{v}_\delta \in \tilde{K}_\delta.
\end{cases} \tag{6.1}
$$

Furthermore, if $m \in N$ and if $\tilde{\tilde{\Psi}}_{2,\varepsilon}$ is an approximate solution corresponding to different Φ, f_2, and β, for a fixed constant C independent of ε and δ, for $\tau = \frac{1}{2}$ and $0 < \varsigma' < \varsigma < 1$ sufficiently small, we have

$$
\begin{aligned}
\|\widehat{\phi}\|_{\varsigma',V} &\leq \frac{C}{\delta^2}\|\tilde{\Pi}_\varepsilon S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})\|_{L^2(C^\tau_{\varsigma,V})} + C\varepsilon^m, \\
\|\check{v}\|_{L^2(C^\tau_{\varsigma,V})} &\leq C\|S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})\|_{L^2(C^\tau_{\varsigma,V})},
\end{aligned} \tag{6.2}
$$

$$
\|\widehat{\phi} - \hat{\hat{\phi}}\|_{\varsigma',V} \leq \frac{C}{\delta^2}\left\|\tilde{\Pi}_\varepsilon(S_\varepsilon(\tilde{\Psi}_{2,\varepsilon}) - S_\varepsilon(\tilde{\tilde{\Psi}}_{2,\varepsilon}))\right\|_{L^2(C^\tau_{\varsigma,V})}. \tag{6.3}
$$

PROOF: The proof relies on Proposition 4.1, Proposition 4.14, and the contraction mapping theorem. By Proposition 4.14, the operator L_ε (see (4.7)) is invertible from $(\bar{H}_\varepsilon, \|\cdot\|_{\varsigma',V})$ into $L^2(C^\tau_{\varsigma,V})$, and the norm of the inverse is uniformly bounded by C/δ^2. By this invertibility, (6.1) is satisfied if and only if $\widehat{\phi}$ is a fixed point of the operator $\check{F}_\varepsilon : (\bar{H}_\varepsilon, \|\cdot\|_{\varsigma',V}) \to (\bar{H}_\varepsilon, \|\cdot\|_{\varsigma',V})$ defined by

$$
\begin{aligned}
\check{F}_\varepsilon(\widehat{\phi}) &= L_\varepsilon^{-1}[\tilde{\Pi}_\varepsilon(\tilde{S}_\varepsilon(\widehat{\phi}))] \\
&:= L_\varepsilon^{-1}\big[\tilde{\Pi}_\varepsilon\big(S_\varepsilon(\tilde{\Psi}_{2,\varepsilon}) + N_\varepsilon(\eta_\varepsilon\widehat{\phi} + \varphi(\widehat{\phi})) + |\tilde{\Psi}_{2,\varepsilon}|^{p-1}\varphi(\widehat{\phi}) \\
&\qquad + (p-1)|\tilde{\Psi}_{2,\varepsilon}|^{p-3}\tilde{\Psi}_{2,\varepsilon}\Re(\tilde{\Psi}_{2,\varepsilon}\overline{\varphi(\widehat{\phi})})\big)\big].
\end{aligned}
$$

We recall that, in the last formula, $\varphi(\widehat{\phi})$ is given by Proposition 4.1, while N_ε is defined in (4.8).

Our next goal is to show that $\check{F}_\varepsilon$ is a contraction on a metric ball (in the $\|\cdot\|_{\varsigma',V}$ norm) of radius $\frac{C}{\delta^2}\|\tilde{\Pi}_\varepsilon S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})\|_{L^2(C^\tau_{\varsigma,V})} + C\varepsilon^m$ for C large enough and m an arbitrary integer. Setting for simplicity

$$\begin{aligned}\check{G}_\varepsilon(\widehat{\phi}) = {} & N_\varepsilon(\eta_\varepsilon\widehat{\phi} + \varphi(\widehat{\phi})) + |\tilde{\Psi}_{2,\varepsilon}|^{p-1}\varphi(\widehat{\phi}) \\ & + (p-1)|\tilde{\Psi}_{2,\varepsilon}|^{p-3}\tilde{\Psi}_{2,\varepsilon}\Re(\tilde{\Psi}_{2,\varepsilon}\overline{\varphi(\widehat{\phi})}),\end{aligned}$$

by the above invertibility we clearly find

$$\begin{cases}\|\check{F}_\varepsilon(\widehat{\phi})\|_{\varsigma',V} \le \frac{C}{\delta^2}\big(\|\tilde{\Pi}_\varepsilon S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})\|_{L^2(C^\tau_{\varsigma,V})} + \|\check{G}_\varepsilon(\widehat{\phi})\|_{L^2(C^\tau_{\varsigma,V})}\big), \\ \|\check{F}_\varepsilon(\widehat{\phi}_1) - \check{F}_\varepsilon(\widehat{\phi}_2)\|_{\varsigma',V} \le \frac{C}{\delta^2}\|\check{G}_\varepsilon(\widehat{\phi}_1) - \check{G}_\varepsilon(\widehat{\phi}_2)\|_{L^2(C^\tau_{\varsigma,V})}.\end{cases} \tag{6.4}$$

We next evaluate $\|\check{G}_\varepsilon(\widehat{\phi})\|_{L^2(C^\tau_{\varsigma,V})}$ and show that it is *superlinear* in $\|\widehat{\phi}\|_{L^2(C^\tau_{\varsigma',V})}$ up to negligible terms. We first make the following claim:

Claim. In the notation of (4.14), letting $k_1(\bar{s}) = (\varsigma')^2\sqrt{V(\bar{s})}$ we have

$$\|\widehat{\phi}\|_{C^{1,1/2}_{k_1}} \le C\|\widehat{\phi}\|_{\varsigma',V}$$

for some $C > 0$. Assuming the claim true and choosing $\varsigma'' < (\varsigma')^2$, we can apply Proposition 4.1 with $\tau = \frac{1}{2}$, $k_0(\bar{s}) = \varsigma\sqrt{V(\bar{s})}$, $k_1(\bar{s}) = (\varsigma')^2\sqrt{V(\bar{s})}$, and $k_2(\bar{s}) = \varsigma''\sqrt{V(\bar{s})}$ to find

$$\begin{aligned}\|\varphi(\widehat{\phi})\|_{C^{1/2}_{-k_2}} \le C\big(& e^{-\inf\frac{k_2+k_0}{K}\varepsilon^{-\bar{\delta}}}\|S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})\|_{C^{1/2}_{k_0}} \\ & + e^{-\inf\frac{k_2+k_1}{K}\varepsilon^{-\bar{\delta}}}\|\widehat{\phi}\|_{C^{1,1/2}_{k_1}}\big).\end{aligned} \tag{6.5}$$

From the expressions for w_r, w_i, $\tilde{v}$, and v_0 and formula (5.29), we can deduce that $|\tilde{\Psi}_{2,\varepsilon}| \le Ce^{-k_0|z|}$; moreover, from the estimates in the proof of Proposition 5.2 we also find that $\|S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})\|_{L^2(C^\tau_{\varsigma,V})} \to 0$ as $\varepsilon \to 0$. By (4.20) (recall that $\zeta > 0$), the latter bounds on $\tilde{\Psi}_{2,\varepsilon}$, the previous claim, and (6.5), if m is an arbitrary integer and if ς'' is sufficiently close to 1 after some elementary computations, we deduce

$$\|\check{G}_\varepsilon(\widehat{\phi})\|_{L^2(C^\tau_{\varsigma,V})} \le C\big(\|\widehat{\phi}\|^{1+\zeta}_{L^2(C^\tau_{\varsigma,V})} + \|\widehat{\phi}\|^p_{L^2(C^\tau_{\varsigma,V})} + \varepsilon^m\big(1 + \|\widehat{\phi}\|_{L^2(C^\tau_{\varsigma,V})}\big)\big).$$

Similarly, if $\|\widehat{\phi}_1\|_{L^2(C^\tau_{\varsigma,V})}$ and $\|\widehat{\phi}_2\|_{L^2(C^\tau_{\varsigma,V})}$ are finite, we also find

$$\begin{aligned}&\|\check{G}_\varepsilon(\widehat{\phi}_1) - \check{G}_\varepsilon(\widehat{\phi}_2)\|_{L^2(C^\tau_{\varsigma,V})} \le \\ &\qquad C\Big[\max_{l=1,2}\big\{\|\widehat{\phi}_l\|^{\zeta\wedge(p-1)}_{L^2(C^\tau_{\varsigma,V})}\big\} + \varepsilon^m\Big]\|\widehat{\phi}_1 - \widehat{\phi}_2\|_{L^2(C^\tau_{\varsigma,V})},\end{aligned}$$

where the symbol $\widehat{\ }$ stands for the minimum. The last inequality and (6.4) show that $\check{F}_\varepsilon$ is a contraction, and we obtain (6.2); (6.3) follows similarly.

To prove the claim, we note that according to our previous notation, the norm $\|\cdot\|_{\varsigma',V}$ is evaluated using the variables (s,z), where the z's are defined in (3.1). If we want to estimate the $\|\cdot\|_{C^{1,1/2}_{k_1}}$ norm instead, we should use Lipschitz with respect to s and y.

Given $s_1, s_2 \in \mathbb{R}$ and $y_1, y_2 \in \mathbb{R}^{n-1}$, we want to consider the difference $\nabla\widehat{\phi}(s_1,y_1) - \nabla\widehat{\phi}(s_2,y_2)$. Recalling (3.1), we can write that

$$\begin{aligned}\partial_s\widehat{\phi}(s_1,y_1) - \partial_s\widehat{\phi}(s_2,y_2) &= \partial_s\widehat{\phi}(s_1, z_1 + \Phi(\varepsilon s_1)) - \partial_s\widehat{\phi}(s_2, z_1 + \Phi(\varepsilon s_1)) \\ &\quad + \partial_s\widehat{\phi}(s_2, z_1 + \Phi(\varepsilon s_1)) - \partial_s\widehat{\phi}(s_2, z_2 + \Phi(\varepsilon s_2)).\end{aligned}$$

By the definition of $\|\cdot\|_{\varsigma',V}$, $\partial_s\widehat{\phi} \in H^1(C^\tau_{\varsigma',V}) \subseteq C^{\frac{1}{2}}(C^\tau_{\varsigma',V})$. This fact, the smoothness of $V(\bar{s})$, and $\|\Phi\|_\infty + \|\Phi'\|_\infty \le C(c_1)\varepsilon$ (which follows from (5.2)) imply that if $(s_1,y_1),(s_2,y_2) \in B_1(s,y)$, then

$$\begin{aligned}&e^{(\varsigma')^2\sqrt{V(\bar{s})}|z|}|\partial_s\widehat{\phi}(s_1,y_1) - \partial_s\widehat{\phi}(s_2,y_2)| \le \\ &\qquad C(c_1)\|\widehat{\phi}\|_{\varsigma',V}(|s_1-s_2|^{\frac{1}{2}} + |z_1-z_2|^{\frac{1}{2}} + \varepsilon|s_1-s_2|).\end{aligned}$$

A similar estimate holds for the derivatives of $\widehat{\phi}$ with respect to y, so from (4.14) we get the conclusion. □

To apply Proposition 6.1, we establish explicit estimates on $\tilde{\Pi}_\varepsilon S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})$ and $\tilde{\Pi}_\varepsilon(S_\varepsilon(\tilde{\Psi}_{2,\varepsilon}) - S_\varepsilon(\tilde{\tilde{\Psi}}_{2,\varepsilon}))$. Specifically, assuming from now on $\tau = \frac{1}{2}$, we have the following result:

PROPOSITION 6.2 *Assume Φ, f_2, β, $\tilde{\Phi}$, $\tilde{f}_2$, and $\tilde{\beta}$ satisfy conditions* (5.2), (5.3), *and* (5.7). *Then, if $\widehat{\phi}$ is defined as in Proposition* 6.1, *we have the estimates*

$$\sqrt{\varepsilon}\delta^2\|\widehat{\phi}(\beta,\Phi,f_2)\|_{\varsigma',V} \le C(c_1,c_2,c_3)\varepsilon^3, \tag{6.6}$$

$$\begin{aligned}&\sqrt{\varepsilon}\delta^2\|\widehat{\phi}(\beta,\Phi,f_2) - \widehat{\phi}(\tilde{\beta},\tilde{\Phi},\tilde{f}_2)\|_{\varsigma',V} \le \\ &\qquad C(c_1,c_2,c_3)[\varepsilon^2\|\Phi-\tilde{\Phi}\|_{H^2} + \varepsilon^3\|f_2-\tilde{f}_2\|_{H^2} + \varepsilon\|\beta-\tilde{\beta}\|_\sharp],\end{aligned} \tag{6.7}$$

where $C(c_1,c_2,c_3)$ is a positive constant depending on c_1, c_2, and c_3 but independent of ε and δ.

PROOF: We prove (6.6) only; (6.7) will follow from similar considerations. To show (6.6) we use Proposition 6.1, so we are reduced to estimating

$$\|\tilde{\Pi}_\varepsilon S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})\|_{L^2(C^\tau_{\varsigma,V})},$$

for which we can employ (5.32).

By our assumptions on Φ, f_2, and β and by the estimates of the previous subsection, it is easy to see that

$$\Big\|\varepsilon^2(\tilde{R}^{\Phi}_{r,e}+\tilde{R}_{r,o}+\tilde{R}_{r,e,f_1}+\tilde{R}_{r,o,f_1}) + \varepsilon^2 i(\tilde{R}_{i,e}+\tilde{R}^{\Phi}_{i,o}+\tilde{R}_{i,e,f_1}+\tilde{R}_{i,o,f_1})\Big\|_{L^2(C^{\tau}_{\varsigma,V})} \le \frac{C(c_1,c_2,c_3)\varepsilon^3}{\sqrt{\varepsilon}},$$

$$\Big\|e^{i\frac{\tilde{f}}{\varepsilon}}\sum_{i=1}^{6}\tilde{\mathfrak{A}}_i\Big\|_{L^2(C^{\tau}_{\varsigma,V})} \le \frac{C(c_1,c_2,c_3)\varepsilon^3}{\sqrt{\varepsilon}}.$$

Recall that, by the above choices of $v^0_{r,e}$ and $v^0_{i,o}$ in (5.23) and (5.24), we have corrected all the terms in the equation of order up to ε^2, so we are left with terms of order ε^3 and higher. The factor $\sqrt{\varepsilon}$ in the denominator arises from the fact that the length of γ_ε is $\frac{L}{\varepsilon}$; this gives a factor $\frac{1}{\varepsilon}$ when computing the L^2 norm squared, and we then need to take the square root. For the estimates in $\tilde{\mathfrak{A}}_6$, which also require the L^∞ norm of β, we can use the interpolation inequalities

$$\|\beta\|_{L^\infty([0,L])} \le C\|\beta\|^{\frac12}_{L^2([0,L])}\|\beta'\|^{\frac12}_{L^2([0,L])} \le C\varepsilon^{\frac32},$$
$$\|\beta'\|_{L^\infty([0,L])} \le C\|\beta'\|^{\frac12}_{L^2([0,L])}\|\beta''\|^{\frac12}_{L^2([0,L])} \le C\varepsilon^{\frac12}.$$

It now remains to consider the other terms in the right-hand side of (5.32) involving the functions Z_α and W_α. Let us call $\tilde{L}^1_\varepsilon$ the operator obtained from L^1_ε (see (4.35)) by replacing the variables y with z and f with $\tilde{f}$. Let us first notice that the terms under interest, with this notation, are nothing but $\tilde{\Pi}_\varepsilon\tilde{L}^1_\varepsilon v_\delta$.

Let us now recall the expression of β in (5.5) and v_δ in (5.4); if $\tilde{v}_{3,j}$ stands for the functions in $K_{3,\delta}$ (see (4.49)) replacing y with z, we define the function

$$\tilde{v}_\delta = \sum_{j=-\delta^2/\varepsilon}^{\delta^2/\varepsilon} b_j\tilde{v}_{3,j}.$$

From the expression of $\tilde{v}_{3,j}$ (see (4.45)), we find that

$$\|v_\delta-\tilde{v}_\delta\|_{\varsigma,V} \le \frac{C}{\sqrt{\varepsilon}}\Big(\sum_{j=-\delta^2/\varepsilon}^{\delta^2/\varepsilon} b_j^2\varepsilon^2(1+j^2)\Big)^{\frac12}, \tag{6.8}$$

$$\tilde{L}^1_\varepsilon\big(e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}v_\delta\big) = \tilde{L}^1_\varepsilon(e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\tilde{v}_\delta) + O\Big(\frac{1}{\sqrt{\varepsilon}}\Big)\Big(\sum_{j=-\delta^2/\varepsilon}^{\delta^2/\varepsilon} b_j^2\varepsilon^2(1+j^2)\Big)^{\frac12} \tag{6.9}$$

(in the $\|\cdot\|_{L^2(C^{\tau}_{\varsigma,V})}$norm).

Similarly to (4.68), recalling the asymptotic of ν_j (see (4.42) and the lines before) we find that

$$\tilde{L}^1_\varepsilon(e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\tilde{v}_\delta) = e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}} \sum_{j=-\delta^2/\varepsilon}^{\delta^2/\varepsilon} \nu_j b_j \tilde{v}_{3,j} + R_1, \tag{6.10}$$

where

$$\|R_1\|_{L^2(C^\tau_{\varsigma,V})} \le \frac{C}{\sqrt{\varepsilon}}\Big(\sum_{j=-\delta^2/\varepsilon}^{\delta^2/\varepsilon} b_j^2\varepsilon^2(1+j^2)\Big)^{\frac{1}{2}} \le C\sqrt{\varepsilon}\|\beta\|_\sharp.$$

This implies the conclusion, by (5.7). □

6.2 Projections onto $\tilde{K}_\delta$

In this section we estimate the projections of the equation onto the components of $\tilde{K}_\delta$. We first estimate their size and their Lipschitz dependence in the data Φ, f_2, and β. Then we use the contraction mapping theorem to annihilate the function $\check{v}$ in Proposition 6.1, which implies the solvability of (1.16).

Projection onto $\tilde{K}_{1,\delta}$

We want to evaluate the $\tilde{K}_{1,\delta}$ component of the function $\check{v}_\delta$ in (6.1). To do this we consider a normal section $\underline{\Phi}$ to γ that satisfies the first relation in (5.3), and the function

$$v_{\underline{\Phi}} := h(\varepsilon s)^{\frac{p+1}{4}}\bigg(\langle\underline{\Phi}(\varepsilon s), \nabla_z U(kz)\rangle + i\varepsilon\langle\underline{\Phi}'(\varepsilon s), z\rangle\frac{f'}{k}U(kz) - \frac{\varepsilon^2}{k^2}\langle\underline{\Phi}''(\varepsilon s), \mathfrak{V}(kz)\rangle\bigg).$$

We then multiply both the left-hand side of (6.1) and $\tilde{S}_\varepsilon(\widehat{\phi})$ (see (4.24)) by the conjugate of $e^{-i(\tilde{f}(\varepsilon s))/\varepsilon}v_{\underline{\Phi}}$, integrate over $\tilde{D}_\varepsilon$, and take the real part. When multiplying the left-hand side, we can integrate by parts and let the operator L_ε act on $e^{-i(\tilde{f}(\varepsilon s))/\varepsilon}v_{\underline{\Phi}}$. Using the arguments in the proofs of Proposition 4.9 (see in particular (4.66) and (4.67)) and of Proposition 6.2, we find that

$$L_\varepsilon\big(e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}v_{\underline{\Phi}}\big) = e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\tilde{v}_{\underline{\Phi}} + R(v_{\underline{\Phi}}),$$

where $\tilde{v}_{\underline{\Phi}} \in \tilde{K}_{1,\delta}$ and where

$$\|R(v_{\underline{\Phi}})\|_{L^2(C^\tau_{\varsigma,V})} \le C(\varepsilon+\delta^3)\|v_{\underline{\Phi}}\|_{L^2(C^\tau_{\varsigma,V})} \le \frac{C}{\sqrt{\varepsilon}}(\varepsilon+\delta^3)\|\underline{\Phi}\|_{L^2([0,L])}.$$

Therefore, since $\widehat{\phi}$ is orthogonal to $\tilde{K}_\delta$, from (6.6) we deduce that

$$\left|\Re \int_{\tilde{D}_\varepsilon} e^{i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}} \overline{v_{\underline{\Phi}}} L_\varepsilon \widehat{\phi}\right| dV_{\tilde{g}_\varepsilon} \le \frac{C}{\sqrt{\varepsilon}}(\varepsilon + \delta^3)\|\underline{\Phi}\|_{L^2([0,L])}\|\widehat{\phi}\|_{L^2(C^\tau_{\varsigma,V})} \le C(c_1, c_2, c_3)\delta\varepsilon^2\|\underline{\Phi}\|_{L^2([0,L])}. \tag{6.11}$$

We next have to consider $\tilde{S}_\varepsilon(\widehat{\phi})$, whose main term is $S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})$. For this we use formula (5.32). Here we have three kinds of terms: the $\tilde{R}$'s, those involving Z_α and W_α (which coincide with $\mathfrak{A}_{5,0}$, with our notation in (5.28)), and the $\tilde{\mathfrak{A}}$'s.

For the $\tilde{R}$'s, since $v_{\underline{\Phi}}$ is odd in z, the products with the even terms will vanish. The products of the odd terms (notice that the two phases cancel and we use the change of variables $s \mapsto \varepsilon s$) instead give us

$$\varepsilon^2 \Re \int_{\tilde{D}_\varepsilon} (\tilde{R}_{r,o} + \tilde{R}_{r,o,f_1})\overline{v_{\underline{\Phi}}} dV_{\tilde{g}_\varepsilon} + \varepsilon^2 \Re \int_{\tilde{D}_\varepsilon} i(\tilde{R}^{\Phi}_{i,o} + \tilde{R}_{i,o,f_1})\overline{v_{\underline{\Phi}}} dV_{\tilde{g}_\varepsilon}$$
$$= -\varepsilon \frac{p-1}{2\theta} C_0 \int_0^L \langle \mathfrak{J}(\Phi), \underline{\Phi}\rangle \, d\bar{s} + \tilde{R}_0,$$

where $C_0 = \int_{\mathbb{R}^{n-1}} U(y)^2 \, dy$ and $|\tilde{R}_0| \le C\delta\varepsilon\|\underline{\Phi}\|_{L^2([0,L])}$. To explain why this estimate holds, we notice first that $-\frac{p-1}{2\theta} C_0 \langle \mathfrak{J}(\Phi), \underline{\Phi}\rangle$ is exactly the first term of $v_{\underline{\Phi}}$ multiplied by $\tilde{R}_{r,o} + \tilde{R}_{r,o,f_1}$, as shown in Subsection 3.3 (the factor $h^{(p+1)/4}$ in (4.39) is needed precisely to cancel the factor $\frac{1}{hk}$ in (3.26)). The remaining terms in the last equation are given either by products of the imaginary part of $v_{\underline{\Phi}}$ and the imaginary $\tilde{R}$'s or that of $\tilde{R}_{r,o} + \tilde{R}_{r,o,f_1}$ and the last term in $v_{\underline{\Phi}}$. In the latter case, for example (see the comments after (4.81)), we obtain a quantity bounded by

$$C\varepsilon^2 \int_0^{L/\varepsilon} (|\Phi| + |\Phi'| + |\Phi''|)\varepsilon^2 \underline{\Phi}'' \, ds \le C\varepsilon^2\delta^2\|\underline{\Phi}\|_{L^2([0,L])}.$$

The last inequality follows from (5.2) and the fact that $\underline{\Phi}$ satisfies the first condition in (5.3). On the other hand, the terms involving $\underline{\Phi}'$ once integrated will be bounded by $C\varepsilon^2\delta\|\underline{\Phi}\|_{L^2([0,L])}$, still by (5.3).

Concerning $\mathfrak{A}_{5,0}$, we next claim that for any $m \in \mathbb{N}$ we have

$$\left|\Re \int_{\tilde{D}_\varepsilon} \overline{v_{\underline{\Phi}}} \mathfrak{A}_{5,0} \, dV_{\tilde{g}_\varepsilon}\right| \le C\varepsilon^m \|\underline{\Phi}\|_{L^2([0,L])} \quad \text{as } \varepsilon \to 0. \tag{6.12}$$

To see this, notice that Φ satisfies (5.3), while $\mathfrak{A}_{5,0}$ arises from functions involving v_δ (in particular β, see (5.5)); since j ranges between $-\delta^2/\varepsilon$ and δ^2/ε, the main modes of β are much higher than those for Φ. Hence, using Fourier cancellation as in Lemma 4.10, we can deduce (6.12). It is also easy to see that

$$\left| \Re \int_{\tilde{D}_\varepsilon} \overline{v_{\underline{\Phi}}} \sum_{j=1}^{6} \tilde{\mathfrak{A}}_j \, dV_{\tilde{g}_\varepsilon} \right| \leq C(c_1, c_2, c_3)\varepsilon^2 \|\underline{\Phi}\|_{L^2([0,L])}. \tag{6.13}$$

Finally, it remains to consider the product of $v_{\underline{\Phi}}$ and the last three terms in (4.24). Indeed, since these are either superlinear in $\hat{\phi}$ (see (4.20)) or contain $\varphi(\hat{\phi})$ (see (4.15)), they are of lower order compared to (6.11).

Using (6.11)–(6.13) and the above arguments, we finally obtain that, if $\check{v}$ is as in Proposition 6.1, then

$$\int_{\tilde{D}_\varepsilon} \check{v}\overline{v_{\underline{\Phi}}} \, dV_{\tilde{g}_\varepsilon} = -\varepsilon \frac{p-1}{2\theta} C_0 \int_0^L \langle \mathfrak{J}(\Phi), \underline{\Phi} \rangle d\bar{s} + R_1,$$

$$|R_1| \leq C(c_1, c_2, c_3)\varepsilon^2 \|\underline{\Phi}\|_{L^2([0,L])}.$$

Similarly, using the estimates in Section 5, we find that if $\tilde{\check{v}}$ corresponds to the triple $(\tilde{\Phi}, \tilde{f}_2, \tilde{\beta})$, then

$$\int_{\tilde{D}_\varepsilon} (\check{v} - \tilde{\check{v}})\overline{v_{\underline{\Phi}}} \, dV_{\tilde{g}_\varepsilon} = -\varepsilon \, \frac{p-1}{2\theta} \, C_0 \int_0^L \langle \mathfrak{J}(\Phi - \tilde{\Phi}), \underline{\Phi} \rangle d\bar{s} + \tilde{R}_1, \tag{6.14}$$

where $\tilde{R}_1$ satisfies

$$\begin{aligned} |\tilde{R}_1| \leq C(c_1, c_2, c_3)\big(&\delta\varepsilon \|\Phi - \tilde{\Phi}\|_{H^2([0,L])} \\ &+ \varepsilon^2 \|f_2 - \tilde{f}_2\|_{H^2([0,L])} + \delta \|\beta - \tilde{\beta}\|_\sharp\big) \|\underline{\Phi}\|_{L^2}. \end{aligned} \tag{6.15}$$

Projection onto $\tilde{K}_{2,\delta}$

For this projection we will be sketchier since most of the arguments of the previous projection can be applied. If $\underline{f}_2$ satisfies the second condition in (5.3), we consider the function

$$v_{\underline{f}_2} = h(\varepsilon s)^{\frac{1}{2}} \left(i \underline{f}_2(\varepsilon s) U(kz) + 2\varepsilon \, \frac{f' \underline{f}_2'(\varepsilon s)}{k} \, \tilde{U}(kz) - i\varepsilon^2 \, \frac{\underline{f}_2''(\varepsilon s)}{k^2} \, \mathfrak{W}(kz) \right).$$

As for the previous case, the main contribution to the projection is given by the product of the first term in $v_{\underline{f}_2}$ and the imaginary parts of $S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})$ listed in (5.32), which are even in z.

We denote by $\widehat{R}_{i,e,f_2}$ the sum of all imaginary even terms of order ε^3 appearing in the equation, namely $\mathfrak{A}_{1,i,e}$, $\mathfrak{A}_{3,i,e}$, and $\mathfrak{A}_{4,i,e} = F_{4,i,e}(\bar{s})$ (see (5.18), (5.22), and (5.25))

$$
\begin{aligned}
\widehat{R}_{i,e,f_2} &= 2h' f_2' U + 2h f_2' k' \nabla U \cdot z + 2 f' f_2' w_{i,e} + f_2'' h U \\
&\quad + 4 f' \partial_s (h f' f_2' \tilde{U}) + 2 f'' h f' f_2' \tilde{U} \\
&\quad - 2(p-1) h^{p-1} |U|^{p-2} f' f_2' \tilde{U} w_{i,e} + F_{4,i,e}(\bar{s}) \\
&:= \tilde{R}_{i,e,f_2} + F_{4,i,e}(\bar{s}).
\end{aligned} \tag{6.16}
$$

Notice that $\tilde{R}_{i,e,f_2}$ coincides with the function $\tilde{R}_{i,e,f_1}$ in (5.1) (see Subsection 3.3 for the precise expression) if we replace f_1 with f_2. Therefore, from estimates similar to the previous ones (which mainly use the computations in subsection 4.1 in [36]), we find

$$
\begin{aligned}
\int_{\tilde{D}_\varepsilon} \check{v} \overline{v_{\underline{f}_2}} \, dV_{\tilde{g}_\varepsilon} &= \varepsilon^2 C_0 \int_0^L T(f_2) \underline{f}_2 \, d\bar{s} \\
&\quad + \varepsilon^2 \int_0^L \left(\int_{\mathbb{R}^{n-1}} F_{4,i,e} U(k(\bar{s})) \right) \underline{f}_2 \, d\bar{s} + R_2,
\end{aligned} \tag{6.17}
$$

where $C_0 = \int_{\mathbb{R}^{n-1}} U(y)^2 \, dy$,

$$
T(f_2) = \partial_{\bar{s}} \left(\frac{h^2 f_2'}{(p-1) k^{n+1}} [(p-1) h^{p-1} - 2\sigma \mathcal{A}^2 h^{2\sigma}] \right), \tag{6.18}
$$

and R_2 satisfies

$$
|R_2| \le C(c_1, c_2, c_3) \delta \varepsilon^2 \| \underline{f}_2 \|_{L^2([0,L])}. \tag{6.19}
$$

Moreover, if $\tilde{\check{v}}$ corresponds to the triple $(\tilde{\Phi}, \tilde{f}_2, \tilde{\beta})$, then

$$
\int_{\tilde{D}_\varepsilon} (\check{v} - \tilde{\check{v}}) \overline{v_{\underline{f}_2}} \, dV_{\tilde{g}_\varepsilon} = \varepsilon^2 C_0 \int_0^L T(f_2 - \tilde{f}_2) \underline{f}_2 \, d\bar{s} + \tilde{R}_2, \tag{6.20}
$$

with

$$
\begin{aligned}
|\tilde{R}_2| &\le C \big(\delta \varepsilon^2 \| f_2 - \tilde{f}_2 \|_{H^2([0,L])} \\
&\quad + \delta \varepsilon \| \Phi - \tilde{\Phi} \|_{H^2([0,L])} + \delta \| \beta - \tilde{\beta} \|_{\sharp} \big) \| \underline{f}_2 \|_{L^2([0,L])}
\end{aligned} \tag{6.21}
$$

with $C = C(c_1, c_2, c_3)$.

Projection onto $\tilde{K}_{3,\delta}$

To compute the last components of the projection, we recall our notation in Subsection 4.3 and define

$$\underline{\beta}(\varepsilon s) = \sum_{j=-\delta^2/\varepsilon}^{\delta^2/\varepsilon} \underline{b}_j \beta_j(\varepsilon s), \qquad v_{\underline{\beta}} = \sum_{j=-\delta^2/\varepsilon}^{\delta^2/\varepsilon} \underline{b}_j \tilde{v}_{3,j}.$$

As for the previous cases, the main contribution to the projection here still comes from $S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})$. In particular, following the arguments for $\tilde{K}_{1,\delta}$, when testing on $v_{\underline{\beta}}$, by Fourier cancellation and parity the major terms are indeed $\mathfrak{A}_{5,0}$, $\mathfrak{A}_{5,r,e}$, and $\mathfrak{A}_{5,i,e}$. With straightforward computations, we find that

$$\int_{\tilde{D}_\varepsilon} \check{v}\overline{v_{\underline{\beta}}}\, dV_{\tilde{g}_\varepsilon} = \frac{1}{\varepsilon}\int_0^L \Lambda(\beta, \xi, \underline{\beta}, \underline{\xi}) d\bar{s} + \hat{R}_3, \tag{6.22}$$

where

$$\begin{aligned}\Lambda(\beta, \xi, \underline{\beta}, \underline{\xi}) = {} & \beta\underline{\beta} Q_{4,\alpha} - \varepsilon^2 \beta'' \underline{\beta} Q_{1,\alpha} - 2\varepsilon \xi' f' \underline{\beta} Q_{3,\alpha} + \xi\underline{\xi} Q_{5,\alpha} \\ & - \varepsilon^2 \xi'' \underline{\xi} Q_{2,\alpha} - \varepsilon f''(\xi\underline{\beta} - \underline{\xi}\beta) Q_{3,\alpha} + 2\varepsilon \beta' f' \underline{\xi} Q_{3,\alpha} \\ & - 2\varepsilon^2 \alpha'(\beta' \underline{\beta} Q_{6,\alpha} + \underline{\xi}\xi' Q_{7,\alpha}) - 2\varepsilon^2 k'(\beta' \underline{\beta} Q_{10,\alpha} + \underline{\xi}\xi' Q_{11,\alpha}) \\ & - 2\varepsilon f' k'(\xi\underline{\beta} Q_{12,\alpha} - \beta\underline{\xi} Q_{13,\alpha}) - 2\varepsilon f' \alpha'(\xi\underline{\beta} Q_{8,\alpha} - \underline{\xi}\beta Q_{9,\alpha}) \\ & - (p-1)\varepsilon h^{p-2}(\underline{\xi}\beta + \xi\underline{\beta}) Q_{14,\alpha},\end{aligned}$$

$$\begin{aligned}
Q_{4,\alpha}(\bar{s}) &= \int_{\mathbb{R}^{n-1}} Z_{\alpha(\bar{s})} \mathcal{L}_r Z_{\alpha(\bar{s})}, & Q_{5,\alpha}(\bar{s}) &= \int_{\mathbb{R}^{n-1}} W_{\alpha(\bar{s})} \mathcal{L}_i W_{\alpha(\bar{s})}, \\
Q_{6,\alpha}(\bar{s}) &= \int_{\mathbb{R}^{n-1}} Z_{\alpha(\bar{s})} \frac{\partial Z_{\alpha(\bar{s})}}{\partial \alpha}, & Q_{7,\alpha}(\bar{s}) &= \int_{\mathbb{R}^{n-1}} W_{\alpha(\bar{s})} \frac{\partial W_{\alpha(\bar{s})}}{\partial \alpha}, \\
Q_{8,\alpha}(\bar{s}) &= \int_{\mathbb{R}^{n-1}} Z_{\alpha(\bar{s})} \frac{\partial W_{\alpha(\bar{s})}}{\partial \alpha}, & Q_{9,\alpha}(\bar{s}) &= \int_{\mathbb{R}^{n-1}} W_{\alpha(\bar{s})} \frac{\partial Z_{\alpha(\bar{s})}}{\partial \alpha}, \\
Q_{10,\alpha}(\bar{s}) &= \int_{\mathbb{R}^{n-1}} Z_{\alpha(\bar{s})} \nabla_z Z_{\alpha(\bar{s})} \cdot z, & Q_{11,\alpha}(\bar{s}) &= \int_{\mathbb{R}^{n-1}} W_{\alpha(\bar{s})} \nabla_z W_{\alpha(\bar{s})} \cdot z, \\
Q_{12,\alpha}(\bar{s}) &= \int_{\mathbb{R}^{n-1}} Z_{\alpha(\bar{s})} \nabla_z W_{\alpha(\bar{s})} \cdot z, & Q_{13,\alpha}(\bar{s}) &= \int_{\mathbb{R}^{n-1}} W_{\alpha(\bar{s})} \nabla_z Z_{\alpha(\bar{s})} \cdot z,
\end{aligned}$$

and

$$Q_{14,\alpha}(\bar{s}) = \int_{\mathbb{R}^{n-1}} U(kz)^{p-2} w_{i,e} W_{\alpha(\bar{s})} Z_{\alpha(\bar{s})},$$

$$|\hat{R}_3| \le C(c_1, c_2, c_3)\delta\varepsilon^2 \|\underline{\beta}\|_{L^2([0,L])}. \tag{6.23}$$

After some manipulation using the fact that (Z_α, W_α) solves (4.29) with $\eta_\alpha = 0$, the normalization $\int_{\mathbb{R}^{n-1}}(Z_\alpha^2 + W_\alpha^2) = 1$, and some integration by parts in z, we find that

$$\frac{1}{\varepsilon}\int_0^L \Lambda(\beta, \xi, \underline{\beta}, \underline{\xi}) d\bar{s} = \frac{1}{\varepsilon}\int_0^L \Lambda_0(\beta, \xi, \underline{\beta}, \underline{\xi}) d\bar{s} + \int_0^L \Lambda_1(\beta, \xi, \underline{\beta}, \underline{\xi}) d\bar{s}, \tag{6.24}$$

where

$$\begin{aligned} \Lambda_0(\beta, \xi, \underline{\beta}, \underline{\xi}) = {} & Q_{1,\alpha}(\varepsilon^2\beta'\underline{\beta}' - k^2\alpha^2\beta\underline{\beta}) + Q_{2,\alpha}(\varepsilon^2\xi'\underline{\xi}' - \alpha^2k^2\xi\underline{\xi}) \\ & + 2f' Q_{3,\alpha}(\varepsilon\beta'\underline{\xi} - \varepsilon\xi'\underline{\beta} - k\alpha\beta\underline{\beta} - k\alpha\xi\underline{\xi}) \end{aligned} \tag{6.25}$$

and

$$\Lambda_1 = (\beta\underline{\xi} + \xi\underline{\beta})\mathfrak{g}(\bar{s})$$

with

$$\mathfrak{g}(\bar{s}) = [f'' Q_{3,\alpha} + 2f'k' Q_{13,\alpha} + 2f'\alpha' Q_{9,\alpha} - (p-1)h^{p-2} Q_{14,\alpha}].$$

Now we notice that, by (4.43), we have

$$\begin{aligned} \beta\underline{\xi} + \xi\underline{\beta} = -\frac{\varepsilon}{k\alpha}\Big[& \sum_{j,l=-\delta^2/\varepsilon}^{\delta^2/\varepsilon} b_j \underline{b}_l (\xi_j' \xi_l + \xi_j \xi_l') \\ & - F_1 \sum_{j,l=-\delta^2/\varepsilon}^{\delta^2/\varepsilon} b_j \underline{b}_l (\nu_j \xi_l \xi_j' + \nu_l \xi_j \xi_l') \Big], \end{aligned}$$

where

$$F_1 = \frac{Q_{1,\alpha}}{k^2\alpha^2 + 2f'k\alpha Q_{3,\alpha}}.$$

Integrating by parts in $\bar{s}$ and using (4.42), we find that

$$\int_0^L \Lambda_1(\beta, \xi, \underline{\beta}, \underline{\xi}) d\bar{s} = \varepsilon \int_0^L \xi\underline{\xi}\left(\frac{\mathfrak{g}}{k\alpha}\right)'(\bar{s}) d\bar{s} + O(\delta^2)\|\beta\|_{L^2([0,L])}\|\underline{\beta}\|_{L^2([0,L])}. \tag{6.26}$$

Finally, combining (6.22), (6.23), (6.24), and (6.26), we deduce

$$\int_{\tilde{D}_\varepsilon} \check{v}\overline{v_{\underline{\beta}}}\, dV_{\tilde{g}_\varepsilon} = \frac{1}{\varepsilon}\int_0^L \Lambda_0(\beta,\xi,\underline{\beta},\underline{\xi})d\bar{s} + R_3, \tag{6.27}$$

where

$$|R_3| \le C(c_1,c_2,c_3)(\delta\varepsilon^2 + (\varepsilon+\delta^2)\|\beta\|_{L^2([0,L])})\|\underline{\beta}\|_{L^2([0,L])}. \tag{6.28}$$

Analogously we obtain

$$\int_{\tilde{D}_\varepsilon} (\check{v}-\tilde{\check{v}})\overline{v_{\underline{\beta}}}\, dV_{\tilde{g}_\varepsilon} = \frac{1}{\varepsilon}\int_0^L \Lambda_0(\beta-\tilde{\beta},\xi-\tilde{\xi},\underline{\beta},\underline{\xi}) + \tilde{R}_3, \tag{6.29}$$

where $\tilde{R}_3$ satisfies

$$\begin{aligned}|\tilde{R}_3| \le C\delta\big(&\varepsilon\|\Phi-\tilde{\Phi}\|_{H^2([0,L])}\\ &+\varepsilon^2\|f_2-\tilde{f}_2\|_{H^2([0,L])} + \|\beta-\tilde{\beta}\|_\sharp\big)\|\underline{\beta}\|_{L^2([0,L])}\end{aligned} \tag{6.30}$$

with $C = C(c_1,c_2,c_3)$.

Remark 6.3. Let us consider the eigenvalue problem in (β,ξ)

$$\int_0^L \Lambda_0(\beta,\xi,\underline{\beta},\underline{\xi}) = \nu\int_0^L (Q_{1,\alpha}\beta\underline{\beta} + Q_{2,\alpha}\xi\underline{\xi}) \quad \text{for all } (\underline{\beta},\underline{\xi})$$

where Q_1 and Q_2 are defined in (4.41). Then the eigenvalue equation is the following:

$$\begin{cases}-\varepsilon^2\frac{(Q_{1,\alpha}\beta')'}{Q_{1,\alpha}} - k^2\alpha^2\beta - 2f'\frac{Q_{3,\alpha}}{Q_{1,\alpha}}(\varepsilon\xi'+k\alpha\beta) = \nu\beta,\\ -\varepsilon^2\frac{(Q_{2,\alpha}\xi')'}{Q_{2,\alpha}} - k^2\alpha^2\xi + 2f'\frac{Q_{3,\alpha}}{Q_{2,\alpha}}(\varepsilon\beta'-k\alpha\xi) = \nu\xi.\end{cases} \tag{6.31}$$

By (4.44), the couple of functions (β_j,ξ_j) constructed in Subsection 4.3 represents a family of approximate eigenfunctions corresponding to $\nu = \nu_j$.

6.3 The Contraction Argument

The usual procedure in performing a fixed-point argument is to apply to the equation an invertible linear operator first. In the expansions in the last subsection, we showed that the main terms in the projections onto $\tilde{K}_\delta$ are the operators $\mathfrak{J}$, T, and Λ_0 (the last is identified by duality with the associated quadratic form); see (6.2), (6.17), and (6.27). By our nondegeneracy assumption on γ, $\mathfrak{J}$ is invertible and the same holds also for T, since it is coercive (and in divergence form). It remains then to invert Λ_0, which is the content of the next result. Before stating it we introduce some notation. Using the symbology of Subsection 4.3 we define the spaces

$$X_{1,\delta} = \operatorname{span}\Big\{\varphi_j : j = 0,\dots,\frac{\delta}{\varepsilon}\Big\}, \qquad X_{2,\delta} = \operatorname{span}\Big\{\omega_j : j = 0,\dots,\frac{\delta}{\varepsilon}\Big\},$$

$$X_{3,\delta} = \operatorname{span}\left\{\beta_j : j = -\frac{\delta^2}{\varepsilon}, \dots, \frac{\delta^2}{\varepsilon}\right\},$$

with $X_{1,\delta}$ and $X_{2,\delta}$ endowed with the H^2 norm on $[0, L]$, and $X_{3,\delta}$ with the $\|\cdot\|_\sharp$ norm.

We also call $Y_{1,\delta}$, $Y_{2,\delta}$, and $Y_{3,\delta}$ the same spaces of functions, but endowed with weighted L^2 norms: by the normalization after (4.38), it is natural to put the weights h^θ and $h^{-\sigma}$ on $Y_{1,\delta}$ and $Y_{2,\delta}$, respectively. Concerning $Y_{3,\delta}$, by Remark 6.3, we will endow it with the product $(\beta, \underline{\beta})_{Y_{3,\delta}} = \int_0^L (Q_{1,\alpha}\beta\underline{\beta} + Q_{2,\alpha}\xi\underline{\xi}) d\bar{s}$ where, as above, ξ is related to β by (4.43) and (5.5). Notice that by (4.38) $\mathfrak{J}$ and T are exactly diagonal from $X_{1,\delta}$ to $Y_{1,\delta}$ and from $X_{2,\delta}$ to $Y_{2,\delta}$, respectively, while Λ_0 is nearly diagonal (see also (4.44)).

LEMMA 6.4 *Letting $\Pi_{Y_{3,\delta}}$ denote the orthogonal projection onto $Y_{3,\delta}$, there exists a sequence $\varepsilon_k \to 0$ such that Λ_0 is invertible from $X_{3,\delta}$ into $Y_{3,\delta}$ and such that its inverse satisfies $\|(\Pi_{Y_{3,\delta}}\Lambda_0)^{-1}\| \le \frac{C}{\varepsilon_k}$ for some fixed constant C.*

PROOF: First of all, we show that there exists $\varepsilon_k \to 0$ such that $\Pi_{Y_{3,\delta}}\Lambda_0$ cannot have eigenvalues in $Y_{3,\delta}$ smaller in absolute value than $C^{-1}\varepsilon_k$; after this, we estimate the (stronger) $X_{3,\delta}$ norm of its inverse.

To prove the claim, we apply Kato's theorem (see [29, p. 445]), which allows us to compute the derivative of an eigenvalue $\nu(\varepsilon)$ of $\Pi_{Y_{3,\delta}}\Lambda_0$ with respect to ε. The (possibly multiple) value of this derivative is given by the eigenvalues of $\Pi_{Y_{3,\delta}}\partial_\varepsilon\Lambda_0$ restricted to the $\nu(\varepsilon)$-eigenspace of $\Pi_{Y_{3,\delta}}\Lambda_0$.

Suppose that β satisfies the eigenvalue equation $\Pi_{Y_{3,\delta}}\Lambda_0\beta = \nu\beta$, which is equivalent to

$$\int_0^L \Lambda_0(\beta, \xi, \underline{\beta}, \underline{\xi}) = \nu \int_0^L (Q_{1,\alpha}\beta\underline{\beta} + Q_{2,\alpha}\xi\underline{\xi}) \quad \text{for all } (\underline{\beta}, \underline{\xi}) \text{ with } \underline{\beta} \in Y_{3,\delta}. \tag{6.32}$$

Looking at the powers of ε in Λ_0 (see (6.25)), we write $\Lambda_0 = \Lambda_{0,0} + \varepsilon\Lambda_{0,1} + \varepsilon^2\Lambda_{0,2}$; notice that $\Lambda_{0,0}$ is negative definite and $\Lambda_{0,2}$ positive definite. We also point out that, since f' satisfies (1.12), for $f(\varepsilon s)/\varepsilon$ to be L/ε-periodic, when we vary ε, $\mathcal{A}$ also needs to be adjusted. Specifically, since the total variation of phase in (1.10) is

$$\mathcal{A}\int_0^{L/\varepsilon} h(\varepsilon s)^\sigma ds = \frac{\mathcal{A}}{\varepsilon}\int_0^L h(\bar{s}) d\bar{s} = \text{const},$$

when differentiating with respect to ε we find that $\frac{\partial\mathcal{A}}{\partial\varepsilon} = \frac{\mathcal{A}}{\varepsilon}$. Hence, applying Kato's theorem, we find

$$\frac{\partial\nu}{\partial\varepsilon} \in \Big[\min_{\beta_1,\beta_2\neq 0}\Theta(\beta_1,\beta_2), \max_{\beta_1,\beta_2\neq 0}\Theta(\beta_1,\beta_2)\Big], \tag{6.33}$$

where

$$\Theta(\beta_1, \beta_2) = \frac{\int_0^L (\Lambda_{0,1} + 2\varepsilon\Lambda_{0,2})(\beta_1, \xi_1, \beta_2, \xi_2)}{\int_0^L (Q_{1,\alpha}\beta_1\beta_2 + Q_{2,\alpha}\xi_1\xi_2)} + \frac{1}{\varepsilon}\frac{\int_0^L (2f' Q_{3,\alpha}(\varepsilon\beta_1'\xi_2 - \varepsilon\xi_1'\beta_2 - k\alpha\beta_1\beta_2 - k\alpha\xi_1\xi_2)}{\int_0^L (Q_{1,\alpha}\beta_1\beta_2 + Q_{2,\alpha}\xi_1\xi_2)},$$

and where $(\beta_1, \xi_1), (\beta_2, \xi_2)$ are functions satisfying (6.32). By using this, we can write $\Theta(\beta_1, \beta_2)$ as

$$\begin{aligned}
&\frac{\frac{1}{\varepsilon}\int_0^L (\Lambda_0 - \Lambda_{0,0})(\beta_1, \xi_1, \beta_2, \xi_2) + \varepsilon\int_0^L \Lambda_{0,2}(\beta_1, \xi_1, \beta_2, \xi_2)}{\int_0^L (Q_{1,\alpha}\beta_1\beta_2 + Q_{2,\alpha}\xi_1\xi_2)} \\
&\quad + \frac{1}{\varepsilon}\frac{\int_0^L (2f' Q_{3,\alpha}(\varepsilon\beta_1'\xi_2 - \varepsilon\xi_1'\beta_2 - k\alpha\beta_1\beta_2 - k\alpha\xi_1\xi_2)}{\int_0^L (Q_{1,\alpha}\beta_1\beta_2 + Q_{2,\alpha}\xi_1\xi_2)} \\
&\quad = \frac{\nu}{\varepsilon} + \frac{\varepsilon\int_0^L (Q_{1,\alpha}\beta_1'\beta_2' + Q_{2,\alpha}\xi_1'\xi_2')}{\int_0^L (Q_{1,\alpha}\beta_1\beta_2 + Q_{2,\alpha}\xi_1\xi_2)} \\
&\qquad + \frac{1}{\varepsilon}\frac{\int_0^L [k^2\alpha^2(Q_{1,\alpha}\beta_1\beta_2 + Q_{2,\alpha}\xi_1\xi_2) + 2f'k\alpha Q_{3,\alpha}(\beta_1\beta_2 + \xi_1\xi_2)]}{\int_0^L (Q_{1,\alpha}\beta_1\beta_2 + Q_{2,\alpha}\xi_1\xi_2)} \\
&\qquad + \frac{1}{\varepsilon}\frac{\int_0^L (2f' Q_{3,\alpha}(\varepsilon\beta_1'\xi_2 - \varepsilon\xi_1'\beta_2 - k\alpha\beta_1\beta_2 - k\alpha\xi_1\xi_2)}{\int_0^L (Q_{1,\alpha}\beta_1\beta_2 + Q_{2,\alpha}\xi_1\xi_2)}.
\end{aligned}$$

Applying (4.42), (4.43), and $Q_{1,\alpha} + Q_{2,\alpha} = 1$ (see (4.41) and the lines after (4.30)), the last expression simplifies to

$$\frac{\nu}{\varepsilon} + \frac{1}{\varepsilon}\frac{\int_0^L (2\alpha^2k^2 + 4f'\alpha k Q_{3,\alpha})\xi_1\xi_2}{\int_0^L \xi_1\xi_2} + O(\delta^2)\frac{1}{\varepsilon}.$$

Since the numerator is symmetric in ξ_1 and ξ_2, the infimum of the above ratio is realized by some ξ_0, so by (6.33) and the latter formula we find

$$\begin{aligned}
\frac{\partial\nu}{\partial\varepsilon} &\geq \frac{\nu}{\varepsilon} + \frac{1}{\varepsilon}\frac{\int_0^L (2\alpha^2k^2 + 4f'\alpha k Q_{3,\alpha})\xi_0^2}{\int_0^L \xi_0^2} + O(\delta^2)\frac{1}{\varepsilon} \\
&\geq \frac{1}{\varepsilon}[\nu + \inf_{[0,L]}(2\alpha^2k^2 + 4f'\alpha k Q_{3,\alpha}) - C\delta^2].
\end{aligned} \tag{6.34}$$

Notice that for ν and δ sufficiently small, the coefficient of $\frac{1}{\varepsilon}$ in the above formula is positive and uniformly bounded away from 0. From (4.44) and the asymptotics in (4.42) (which follows from Weyl's formula), we can show that $\Pi_{Y_{3,\delta}}\Lambda_0$ has a number of negative eigenvalues of order δ^2/ε. This fact and (6.34) yield the desired claim, which can be obtained as in [41, prop. 4.5]; since the argument is quite similar, we omit the details. The above claim provides invertibility of $\Pi_{Y_{3,\delta}}\Lambda_0$ in $Y_{3,\delta}$ and gives

$$\|(\Pi_{Y_{3,\delta}}\Lambda_0)^{-1}\beta\|_{Y_{3,\delta}} \leq \frac{C}{\varepsilon}\|\beta\|_{Y_{3,\delta}} \quad \text{for any } \beta \in Y_{3,\delta}. \tag{6.35}$$

We next want to estimate the $X_{3,\delta}$ norm of $(\Pi_{Y_{3,\delta}}\Lambda_0)^{-1}\beta$. Let

$$\beta = \sum_{j=-\delta^2/\varepsilon}^{\delta^2/\varepsilon} b_j \beta_j$$

and suppose $\widehat{\beta} = \sum_{j=-\delta^2/\varepsilon}^{\delta^2/\varepsilon} \widehat{b}_j \beta_j$ is such that $\Pi_{Y_{3,\delta}}\Lambda_0\widehat{\beta} = \beta$ in the sense that

$$\int_0^L \Lambda_0(\widehat{\beta}, \widehat{\xi}, \underline{\beta}, \underline{\xi}) = \int_0^L (Q_{1,\alpha}\beta\underline{\beta} + Q_{2,\alpha}\xi\underline{\xi}) \quad \text{for all } (\underline{\beta}, \underline{\xi}) \text{ with } \underline{\beta} \in Y_{3,\delta}.$$

If $\underline{\beta} = \sum_{j=-\delta^2/\varepsilon}^{\delta^2/\varepsilon} \underline{b}_j \beta_j$, then by (4.44), we find by integrating

$$\sum_{j=-\delta^2/\varepsilon}^{\delta^2/\varepsilon} \nu_j \widehat{b}_j \underline{b}_j + O\biggl(\Bigl(\sum_{j=-\delta^2/\varepsilon}^{\delta^2/\varepsilon} (\nu_j^2 + \varepsilon)^2 \widehat{b}_j^2\Bigr)^{\frac{1}{2}} \Bigl(\sum_{l=-\delta^2/\varepsilon}^{\delta^2/\varepsilon} \underline{b}_l^2\Bigr)^{\frac{1}{2}}\biggr) \leq C\Bigl(\sum_{j=-\delta^2/\varepsilon}^{\delta^2/\varepsilon} \widehat{b}_j^2\Bigr)^{\frac{1}{2}} \Bigl(\sum_{l=-\delta^2/\varepsilon}^{\delta^2/\varepsilon} \underline{b}_l^2\Bigr)^{\frac{1}{2}}.$$

Choosing $\underline{b}_j = \widehat{b}_j$ for $j > 0$ and $\underline{b}_j = -\widehat{b}_j$ for $j < 0$, from the asymptotics of ν_j in (4.42), we obtain for $\tilde{C}_1 > 0$ sufficiently large that

$$\varepsilon \sum_{\tilde{C}_1 \leq j \leq \delta^2/\varepsilon} |j| \widehat{b}_j^2 \leq C \sum_{j=-\delta^2/\varepsilon}^{\delta^2/\varepsilon} \widehat{b}_j^2 \leq C\|\widehat{\beta}\|_{Y_{3,\delta}}^2.$$

By (6.35) we have $\|\widehat{\beta}\|_{Y_{3,\delta}} \leq \frac{C}{\varepsilon}\|\beta\|_{Y_{3,\delta}}$, so recalling (5.6), we get

$$\|\widehat{\beta}\|_{X_{3,\delta}}^2 := \|\widehat{\beta}\|_\sharp^2 \leq C\|\widehat{\beta}\|_{Y_{3,\delta}}^2 \leq \frac{C^2}{\varepsilon^2}\|\beta\|_{Y_{3,\delta}}^2,$$

which yields the conclusion. □

PROOF OF THEOREM 1.1: Let us introduce the operators

$$G_l : X_{1,\delta} \times X_{2,\delta} \times X_{3,\delta} \to Y_{l,\delta}, \quad l = 1, 2, 3,$$

defined by duality as

$$(G_1(\Phi, f_2, \beta), \underline{\Phi})_{Y_{1,\delta}} = \int_{\tilde{D}_\varepsilon} \check{v}\overline{v_{\underline{\Phi}}}\, dV_{\tilde{g}_\varepsilon},$$

$$(G_2(\Phi, f_2, \beta), \underline{f}_2)_{Y_{2,\delta}} = \int_{\tilde{D}_\varepsilon} \check{v}\overline{v_{\underline{f}_2}}\, dV_{\tilde{g}_\varepsilon},$$

$$(G_3(\Phi, f_2, \beta), \underline{\beta})_{Y_{3,\delta}} = \int_{\tilde{D}_\varepsilon} \check{v}\overline{v_{\underline{\beta}}}\, dV_{\tilde{g}_\varepsilon},$$

where $\check{v} = \check{v}(\Phi, f_2, \beta)$ is the function appearing in Proposition 6.1.

By Proposition 4.4, equation (1.16) (or (NLS_ε)) is solved if and only if $\check{v} = 0$. In the above notation, this is equivalent to finding (Φ, f_2, β) such that $G_l(\Phi, f_2, \beta) = 0$ for every $l = 1, 2, 3$. If ε_k is the sequence given in Lemma 6.4, then Λ_0 is invertible, and the condition $\check{v} = 0$ is equivalent to the system (we set $\varepsilon = \varepsilon_k$)

$$\begin{cases} \Phi = \mathfrak{G}_1(\Phi, f_2, \beta) := -\frac{1}{\varepsilon}\tilde{\mathfrak{J}}^{-1}[G_1(\Phi, f_2, \beta) - \varepsilon\tilde{\mathfrak{J}}(\Phi)], \\ f_2 - \check{f}_2 \ = \mathfrak{G}_2(\Phi, f_2, \beta) \\ \qquad := -\frac{1}{\varepsilon^2}\tilde{T}^{-1}\big[G_2(\Phi, f_2, \beta) - \varepsilon^2\tilde{T} f_2 - \varepsilon^2 \int_{\mathbb{R}^{n-1}} F_{4,i,e} U(k(\bar{s})z)dz\big], \\ \beta = \mathfrak{G}_3(\Phi, f_2, \beta) := -\varepsilon(\Pi_{Y_{3,\delta}}\Lambda_0)^{-1}\big[G_3(\Phi, f_2, \beta) - \frac{1}{\varepsilon}\Pi_{Y_{3,\delta}}\Lambda_0\beta\big], \end{cases} \tag{6.36}$$

where

$$\tilde{\mathfrak{J}} = -\frac{p-1}{2\theta}C_0\mathfrak{J}, \quad \tilde{T} = C_0 T,$$

with ($C_0 = \int_{\mathbb{R}^{n-1}} U(y)^2 dy$) and where

$$\check{f}_2 = -\tilde{T}^{-1}\bigg(\int_{\mathbb{R}^{n-1}} F_{4,i,e} U(k(\bar{s})z)dz\bigg).$$

By (6.2)–(6.21), (6.27)–(6.29), and (6.30) we find

$$\|\mathfrak{G}_1(0,0,0)\|_{X_{1,\delta}} \le C\varepsilon, \quad \|\mathfrak{G}_2(0,0,0)\|_{X_{2,\delta}} \le C\delta, \quad \|\mathfrak{G}_3(0,0,0)\|_{X_{3,\delta}} \le C\delta\varepsilon^2;$$

moreover, if Φ, f_2, and β satisfy the bounds (5.2) and (5.7), then

$$\|\mathfrak{G}_1(\Phi, f_2, \beta) - \mathfrak{G}_1(\tilde{\Phi}, \tilde{f}_2, \tilde{\beta})\|_{X_{1,\delta}} \le C(c_1, c_2, c_3)\bigg(\delta\|\Phi - \tilde{\Phi}\|_{X_{1,\delta}} + \varepsilon\|f_2 - \tilde{f}_2\|_{X_{2,\delta}} + \frac{\delta}{\varepsilon}\|\beta - \tilde{\beta}\|_{X_{3,\delta}}\bigg),$$

$$\|\mathfrak{G}_2(\Phi, f_2, \beta) - \mathfrak{G}_2(\tilde{\Phi}, \tilde{f}_2, \tilde{\beta})\|_{X_{2,\delta}} \le C(c_1, c_2, c_3)\bigg(\frac{\delta}{\varepsilon}\|\Phi - \tilde{\Phi}\|_{X_{1,\delta}} + \delta\|f_2 - \tilde{f}_2\|_{X_{2,\delta}} + \frac{\delta}{\varepsilon^2}\|\beta - \tilde{\beta}\|_{X_{3,\delta}}\bigg),$$

$$\|\mathfrak{G}_3(\Phi, f_2, \beta) - \mathfrak{G}_3(\tilde{\Phi}, \tilde{f}_2, \tilde{\beta})\|_{X_{3,\delta}} \le C(c_1, c_2, c_3)\big(\delta\varepsilon\|\Phi - \tilde{\Phi}\|_{X_{1,\delta}} + \delta\varepsilon^2\|f_2 - \tilde{f}_2\|_{X_{2,\delta}} + \delta\|\beta - \tilde{\beta}\|_{X_{3,\delta}}\big).$$

We now consider the scaled norms

$$\varepsilon \|\cdot\|_{\hat{X}_{1,\delta}} = \|\cdot\|_{X_{1,\delta}}, \quad \delta^{\frac{1}{2}} \|\cdot\|_{\hat{X}_{2,\delta}} = \|\cdot\|_{X_{2,\delta}}, \quad \varepsilon^2 \|\cdot\|_{\hat{X}_{3,\delta}} = \|\cdot\|_{X_{3,\delta}}.$$

With this new notation the last formulas become

$$\|\mathfrak{G}_1(0,0,0)\|_{\hat{X}_{1,\delta}} \le C, \qquad \|\mathfrak{G}_2(0,0,0)\|_{\hat{X}_{2,\delta}} \le C\delta^{\frac{1}{2}}, \\ \|\mathfrak{G}_3(0,0,0)\|_{\hat{X}_{3,\delta}} \le C\delta, \tag{6.37}$$

$$\|\mathfrak{G}_1(\Phi, f_2, \beta) - \mathfrak{G}_1(\tilde{\Phi}, \tilde{f_2}, \tilde{\beta})\|_{\hat{X}_{1,\delta}} \le \\ C(c_1, c_2, c_3)\big(\delta \|\Phi - \tilde{\Phi}\|_{\hat{X}_{1,\delta}} + \delta^{\frac{1}{2}} \|f_2 - \tilde{f_2}\|_{\hat{X}_{2,\delta}} + \delta \|\beta - \tilde{\beta}\|_{\hat{X}_{3,\delta}}\big),$$

$$\|\mathfrak{G}_2(\Phi, f_2, \beta) - \mathfrak{G}_2(\tilde{\Phi}, \tilde{f_2}, \tilde{\beta})\|_{\hat{X}_{2,\delta}} \le \\ C(c_1, c_2, c_3)\big(\delta^{\frac{1}{2}} \|\Phi - \tilde{\Phi}\|_{\hat{X}_{1,\delta}} + \delta \|f_2 - \tilde{f_2}\|_{\hat{X}_{2,\delta}} + \delta^{\frac{1}{2}} \|\beta - \tilde{\beta}\|_{\hat{X}_{3,\delta}}\big),$$

$$\|\mathfrak{G}_3(\Phi, f_2, \beta) - \mathfrak{G}_3(\tilde{\Phi}, \tilde{f_2}, \tilde{\beta})\|_{\hat{X}_{3,\delta}} \le \\ C(c_1, c_2, c_3)\big(\delta \|\Phi - \tilde{\Phi}\|_{\hat{X}_{1,\delta}} + \delta^{\frac{3}{2}} \|f_2 - \tilde{f_2}\|_{\hat{X}_{2,\delta}} + \delta \|\beta - \tilde{\beta}\|_{\hat{X}_{3,\delta}}\big).$$

If C is the constant appearing in (6.37), from the last four formulas we deduce that if δ is sufficiently small then $(\mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3)$ has a fixed point in

$$\{\|\cdot\|_{\hat{X}_{1,\delta}} \le 2C\} \cap \{\|\cdot - \check{f_2}\|_{\hat{X}_{2,\delta}} \le 2C\delta^{\frac{1}{2}}\} \cap \{\|\cdot\|_{\hat{X}_{3,\delta}} \le 2C\delta\}.$$

This, by the comments before (6.36), leads to a solution of $(\mathrm{NLS}_\varepsilon)$ with the desired asymptotics. □

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