STATIONARY SOLUTIONS TO A KELLER-SEGEL CHEMOTAXIS SYSTEM

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Abstract. We consider the following stationary Keller-Segel system from chemotaxis

$$\Delta u - au + u^p = 0,\ u > 0 \text{ in } \Omega,\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^2$ is a smooth and bounded domain. We show that given any two positive integers K, L, for p sufficiently large, there exists a solution concentrating in K interior points and L boundary points. The location of the blow-up points is related to the Green's function. The solutions are obtained as critical points of some finite dimensional reduced energy functional. No assumption on the symmetry, geometry nor topology of the domain is needed.

1. Introduction and statement of main results

Chemotaxis is the influence of chemical substances in the environment on the movement of mobile species (amoebae). This can lead to strictly oriented movement or to partially oriented and partially tumbling movement. A positive chemotaxis is a movement towards a higher concentration of the chemical substance while the movement towards regions of lower chemical concentration is called negative chemotactical movement. Chemotaxis is an important means for cellular communication. Communication by chemical signals determines how cells arrange and organize themselves, like for instance in development or in living tissues.

A basic model in chemotaxis was introduced by Keller and Segel [29]. They considered an advection-diffusion system consisting of two coupled parabolic equations for the concentration of the considered species and that of the chemical released, represented, respectively, by positive quantities $v(x,t)$ and $u(x,t)$ defined on a bounded, smooth domain in $\mathbb{R}^N$ under no-flux boundary conditions. The system reads as follows:

$$\begin{cases} v_t = D_1 \Delta v - \chi \nabla (v \nabla \phi(u)), & \text{in } \Omega \times (0,T) \\ u_t = D_2 \Delta u + k(u,v), & \text{in } \Omega \times (0,T) \\ \\ u, v > 0 \text{ in } \Omega, & \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where D_1, D_2, and χ are positive constants; ϕ is a smooth function such that $\phi^{'}(r) > 0$ for $r > 0$; k is a smooth function with $k_v \geq 0$ and $k_u \leq 0$; Ω

is a smooth and bounded domain in $\mathbb{R}^N$; ν denotes the outer unit normal to $\partial\Omega$. A commonly used $k(u,v)$ is $k(u,v) = -au + bv$ with $a, b > 0$. The function $\phi(u)$ is the so-called sensitivity function.

When $\phi(u) = u$, system (1.1) equals the most common formulation of the Keller-Segel model. One interesting question in connection with this version of the model is the possibility of "Chemotactic Collapse", i.e., solutions may become unbounded in finite or infinite time for $n \geq 2$. We refer the reader for instance to [4, 7, 8], [21]-[28], [34]-[37], [44]-[45].

The functional forms in the most common version of the Keller-Segel model are based on simplifying assumptions made by Nanjundiah in [33]. The original paper by Keller-Segel [29] allows more general functional forms. There have been several attempts to introduce certain reasonable effects in the Keller-Segel equations that might prevent blow-up. See [26] and [27]. The boundedness and blow-up of solutions for Keller-Segel system with general sensitivity functions are studied in [25].

In this paper, we are concerned with stationary solutions to Keller-Segel system with logarithmic sensitivity function

$$\phi(u) = \log u. \tag{1.2}$$

This point of view was first taken by Lin, Ni and Takagi [31].

Since $\int_\Omega v(x,t)dx = \int_\Omega v_0(x)dx$ for all $t > 0$ by virtue of the Neumann boundary condition, the steady-state problem (1.1) for positive functions v and u is reduced to a single equation for u:

$$\epsilon^2 \Delta u - au + u^p = 0 \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \tag{1.3}$$

for some constant $\epsilon = \epsilon(D_2, \bar{v})$, where $\bar{v}$ stands for the average of v, i.e., $\bar{v} = \frac{1}{|\Omega|}\int_\Omega v dx$, and

$$p = \frac{\chi}{D_1}. \tag{1.4}$$

In the last decade, a lot of works have been done to (1.3) in the case of *small* diffusion coefficient, i.e., $\epsilon << 1$, after the fundamental works of Ni and Takagi [38] and [39], in which they showed that the least energy solution has a boundary spike at the most curved part of $\partial\Omega$. See [3], [9], [17], [18], [30], [32], [40], [47] and the references therein. In particular, we mention the result of [17], in which they showed that for any two nonnegative integers $K, L \geq 0, K + L > 0$, (1.3) has a solution with K interior spikes and L boundary spikes, provided that ϵ is small and p is subcritical.

In this paper, we assume that $N = 2$ and ϵ is finite. (Without loss of generality, we let $\epsilon = 1$.) We consider another limit $p \to +\infty$. In particular, we show that for any two nonnegative integers $K, L, K + L > 0$, (1.3) has a solution with K interior spikes and L boundary spikes, provided that p is large. More precisely, we consider the following nonlinear problem

$$\Delta u - au + u^p = 0 \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \tag{1.5}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^2$.

Let $y \in \bar{\Omega}$. We define $G(x,y)$ to be Green function solving the following problem:

$$\Delta_x G(x,y) - aG(x,y) + \delta_y = 0 \text{ in } \Omega, \quad \frac{\partial G(x,y)}{\partial \nu_x} = 0 \text{ on } \partial\Omega. \tag{1.6}$$

Now we define the regular part of $G(x,y)$:

$$H(x,y) := \begin{cases} G(x,y) + \frac{1}{2\pi}\log|x-y|, & \text{if } y \in \Omega, \\ \\ G(x,y) + \frac{1}{\pi}\log|x-y|, & \text{if } y \in \partial\Omega. \end{cases} \tag{1.7}$$

In this way, the function $H(\cdot,y)$ is $C^{1,\alpha}$ in $\bar{\Omega}$.

For $d > 0$ sufficiently small and $m = K + L$, we define a configuration space as:

$$\mathcal{M}_d := \Big\{ \xi = (\xi_1, ..., \xi_m) \in \Omega^K \times (\partial\Omega)^L \Big| \min_{i=1,\dots,K} d(\xi_i, \partial\Omega) \geq d,$$

$$\min_{i \neq j} |\xi_i - \xi_j| \geq d \Big\}. \tag{1.8}$$

Let $\xi = (\xi_1, ..., \xi_m) \in \mathcal{M}_d$. We set

$$\varphi_m(\xi) := \sum_{k=1}^{m} c_k^2 H(\xi_k, \xi_k) + \sum_{i \neq j} c_i c_j G(\xi_i, \xi_j). \tag{1.9}$$

Here the constants c_i are defined as follows

$$c_i := \begin{cases} 8\pi, & \text{if } \; i = 1, \dots, K, \\ \\ 4\pi, & \text{if } \; i = K+1, \dots, m. \end{cases} \tag{1.10}$$

Our result is

Theorem 1. *Let Ω be a smooth and bounded domain in $\mathbb{R}^2$, and $K, L \in I\!N^* = I\!N \cup \{0\}$ be such that $m = K + L \geq 1$. There exists $p_{K,L} > 0$ such that for $p > p_{K,L}$, problem* (1.5) *has a positive solution u_p with the following property: u_p has $K + L$ local maximum points $\xi_i^p, i = 1, ..., K + L$ such that $\xi_i^p \in \Omega, i = 1, ..., K$ and $\xi_i^p \in \partial\Omega, i = K+1, ..., K+L$. The m-tuple $\xi^p = (\xi_1^p, \dots, \xi_m^p)$ converges (up to subsequence) to $\bar{\xi} = (\bar{\xi}_1, \dots, \bar{\xi}_m) \in \Omega^K \times (\partial\Omega)^L$, so that*

$$\varphi_m(\xi_1^p, ..., \xi_m^p) \to \varphi_m(\bar{\xi}) \equiv \min_{\xi \in \mathcal{M}_d} \varphi_m(\xi) \quad \text{as } p \to \infty.$$

Furthermore, for any $\delta > 0$

$$u_p \to 0 \quad \text{uniformly in} \quad \Omega \setminus \cup_{j=1}^m B_\delta(\xi_j^p)$$

and

$$\sup_{x \in B_\delta(\xi_j^p)} u_p(x) \to \sqrt{e}$$

as $p \to \infty$.

Remark 1.1. *The existence of a global minimum for the function* $\varphi_m(\xi)$ *in* $\mathcal{M}_d$ *follows from the properties of the Green function - see the proof of Lemma 6.1.*

Remark 1.2. *A biological interpretation of Theorem 1 is as follows: The Keller-Segel model with logarithmic sensitivity function suggests that it is possible for amoebae to form stationary aggregates if* $p = \frac{\chi}{D_1}$ *is large. An interesting question is the stability of such solutions. We believe that the one with* $K + L = 1$ *is stable. (The stability of spike solutions when* ϵ *is small has been studied in* [43].*)*

It is important to remark the analogy existing between our results and those known for the Dirichlet problem

$$\Delta u + u^p = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{1.11}$$

Asymptotic behavior of least energy solutions of (1.11) is well understood after the works [1, 19, 41, 42]: pu^p approaches a Dirac delta at the harmonic center of Ω. Construction of solutions when p is large has been achieved in [16], in which it is shown that problem (1.11) has solutions with K interior spikes if Ω is not simple connected. Note that our results here do not require any properties of Ω.

Our basic strategy is to connect problem (1.5) when p is large with the following nonlinear Neumann problem

$$\Delta u - au + \epsilon^2 e^u = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{1.12}$$

It has been shown in [14] that problem (1.12) admits solutions with K interior spikes and L boundary spikes when ϵ is small. A main difficulty in (1.5) is that the error term is only $O(\frac{1}{p^2}) = O(\frac{1}{|\log \epsilon|^2})$. On the other hand, the spectrum gap is of the order $O(\frac{1}{|\log \epsilon|})$ which makes this problem more difficult than (1.12). (For (1.12) the error is of the order $O(\epsilon)$.) A further expansion of the approximate solution and the errors up to the order $O(\frac{1}{p^4})$ is needed. Finally, we remark that related constructions for problems involving exponential nonlinearity have been performed in [2], [10], [11], [13] and [15].

The proof of our result is based on a Liapunov-Schmidt reduction. The scheme of the proof is the following: we introduce a first approximation for the solution (ansatz) and we reduce problem (1.5) to a fixed point problem. This is contained in Section 2. We solve the fixed point problem first in the orthogonal of a finite dimensional space. In order to do so we first need to study the invertibility of a certain linear operator, subject to suitable orthogonality conditions. This is done in Section 3. Section 4 is devoted to the resolution of the complete fixed point problem. In Section 5 we prove that finding a solution to (1.5) becomes at this point equivalent to finding a critical point of the function φ_m introduced in (1.9). The proof of Theorem 1 is contained in Section 6.

Throughout the paper, without loss of generality, we will assume that the constant a which appears in Problem (1.5) is equal to 1. The letter C will always denote various generic constants which are independent of $p \geq 1$.

2. Ansatz

In this section we introduce the basic elements to build a solution to problem (1.5). The basic idea is to build a solution (see (2.13)) with error in the order of $O(\frac{1}{p^4})$ (see estimate (2.24)–this estimate is needed to control the spectrum gap, see Lemma 3.2).

Let us first introduce the standard single bubble solution. Given $\xi_j \in \bar{\Omega}$, $\mu_j > 0$ we call

$$\omega_{0j}(y) := \omega_j(|y - \xi_j'|),$$

where $\omega_j(r)$ is the radial solution to $\Delta u + e^u = 0$ in $\mathbb{R}^2$ with $\int e^u < \infty$ given by

$$\omega_j(r) = \log \frac{8\mu_j^2}{(\mu_j^2 + r^2)^2},$$

and

$$\xi_j' = e^{\frac{p}{4}} \xi_j.$$

In order to construct a first approximation for a solution to (1.5), we introduce the following functions

$$u_{0j}(x) := p + \omega_{0j}(e^{\frac{p}{4}} x), \quad x \in \Omega, \tag{2.1}$$

which has the explicit expression

$$u_{0j}(x) = \log \frac{8\mu_j^2}{(e^{-\frac{p}{2}} \mu_j^2 + |x - \xi_j|^2)^2}.$$

It turns out that u_{0j} is not good enough. We have to introduce the next two terms in the expansion. Thus, for $i = 1, 2$, we define

$$u_{ij}(x) := \omega_{ij}(e^{\frac{p}{4}} x), \quad x \in \Omega, \tag{2.2}$$

where the functions ω_{ij}, for $i = 1, 2$, introduced above, are defined as follows. Let ω_{1j} be the solution to

$$\Delta \omega_{1j} + e^{\omega_{0j}} \omega_{1j} = f_{1j}, \tag{2.3}$$

where f_{1j} is given by

$$f_{1j}(y) = \frac{1}{2} e^{\omega_{0j}} \omega_{0j}^2,$$

with the properties that

$$\omega_{1j}(y) = \omega_{1j}(|y - \xi_j'|)$$

and that

$$\omega_{1j}(y) = C_{1j} \log \frac{|y - \xi_j'|}{\mu_j} + O(\frac{1}{|y - \xi_j'|}) \text{ as } |y - \xi_j'| \to \infty,$$

where the constant C_{1j} in the above formula is explicit, namely

$$C_{1j} = 8\log\mu_j + 12 - 4\log 8. \tag{2.4}$$

Let us now define f_{2j} to be given by

$$f_{2j}(y) = f_{2j}(|y-\xi_j'|) = e^{\omega_{0j}}\left(\omega_{0j}\omega_{1j} - \frac{1}{3}\omega_{0j}^3 - \frac{1}{2}\omega_{1j}^2 - \frac{1}{8}\omega_{0j}^2 + \frac{1}{2}\omega_{0j}^2\omega_{1j}\right)$$

and ω_{2j} a solution of

$$\Delta\omega_{2j} + e^{\omega_{0j}}\omega_{2j} = f_{2j} \tag{2.5}$$

with the property that

$$\omega_{2j}(y) = \omega_{2j}(|y-\xi_j'|)$$

and

$$\omega_{2j}(y) = C_{2j}\log\frac{|y-\xi_j'|}{\mu_j} + O(\frac{1}{|y-\xi_j'|}) \text{ as } |y-\xi_j'| \to \infty \tag{2.6}$$

for some explicit constant C_{2j}, depending on μ_j. It is easy to prove the existence of ω_{1j} and ω_{2j}.

As we will see below, a direct computation shows that a proper multiple of the sum of u_{ij}, for $i = 0, 1, 2$, defined in (2.1) and (2.2), almost solves the equation in Problem (1.5). In order now that this first approximation fits the boundary condition, we need to introduce a further correction.

Let H_{ij} be the solution of

$$\begin{cases} -\Delta H_{ij} + H_{ij} = -u_{ij}, & \text{in } \Omega, \\ \frac{\partial H_{ij}}{\partial\nu} = -\frac{\partial u_{ij}}{\partial\nu}, & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

Using the asymptotic behaviors of the functions ω_{ij}, one can easily prove the following

Lemma 2.1. *For any* $0 < \alpha < 1$,

$$H_{0j}(x) = c_j H(x,\xi_j) - \log 8\mu_j^2 + O(e^{-\alpha\frac{p}{4}}) \tag{2.8}$$

and, for $i = 1, 2$

$$H_{ij}(x) = -\frac{c_j}{4}C_{ij}H(x,\xi_j) + C_{ij}\log\mu_j - \frac{C_{ij}}{4}p + O(e^{-\alpha\frac{p}{4}}) \tag{2.9}$$

uniformly in $\overline{\Omega}$, *where* H *is the regular part of the Green function defined* (1.7), *the constants* C_{ij} *are given by* (2.4) *and* (2.6), *while* c_j *is defined in* (1.10).

We postpone the proof of this Lemma to the end of this section.

We have now all the elements to define a first approximation for a solution to (1.5). Set

$$\tilde{u}_{ij}(x) := u_{ij}(x) + H_{ij}(x), \quad x \in \Omega. \tag{2.10}$$

A direct consequence of Lemma 2.1 is

$$\tilde{u}_{0j}(x) = c_j G(x, \xi_j) + O(e^{-\alpha \frac{p}{4}}) \tag{2.11}$$

and, for $i = 1, 2$,

$$\tilde{u}_{ij}(x) = -c_j \frac{C_{ij}}{4} G(x, \xi_j) + O(e^{-\alpha \frac{p}{4}}) \tag{2.12}$$

uniformly in C^1 sense over compacts of $\bar{\Omega} \setminus \{\xi_j\}$.

We will look for solutions to (1.5) whose main part is given by the function $U(x)$ defined as a proper multiple of the sum of the functions $\tilde{u}_{ij}$, namely

$$U(x) := \frac{e^{\frac{p}{2(p-1)}}}{p^{\frac{p}{p-1}}} \sum_{j=1}^{k} \left(\sum_{i=0,1,2} \frac{1}{p^i} \quad \tilde{u}_{ij}(x) \right). \tag{2.13}$$

In order to understand better the problem, it is now useful to perform the change of variables

$$v(y) = e^{-\frac{p}{2(p-1)}} u(e^{-\frac{p}{4}} y) \tag{2.14}$$

for $y \in \Omega_p \equiv e^{\frac{p}{4}} \Omega$. Observe that u is a solution of problem (1.5) if and only if the function v is a solution to

$$\Delta v - e^{-\frac{p}{2}} v + v^p = 0 \text{ in } \Omega_p,\ u > 0 \text{ in } \Omega_p,\ \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial \Omega_p. \tag{2.15}$$

In the expanded variable $y \in \Omega_p$, we define $V(y) = e^{-\frac{p}{2(p-1)}} U(e^{-\frac{p}{4}} y)$ whose explicit expression is given by

$$V(y) := \frac{1}{p^{\frac{p}{p-1}}} \sum_{j=1}^{k} \left(\sum_{i=0,1,2} \frac{1}{p^i} \quad \hat{u}_{ij}(y) \right) \tag{2.16}$$

where $\hat{u}_{ij}$ are defined as follows

$$\hat{u}_{ij}(y) := \tilde{u}_{ij}(e^{-\frac{p}{4}} y), \quad y \in \Omega_p. \tag{2.17}$$

We will seek for solution v of (2.15) of the form

$$v = V + \phi.$$

Problem (2.15) can be stated as to find ϕ a solution to

$$\begin{cases} -\Delta \phi + e^{-\frac{p}{2}} \phi - W \phi = R + N(\phi), & \text{in } \Omega_p, \\ \frac{\partial \phi}{\partial \nu} = 0, & \text{on } \partial \Omega_p, \end{cases} \tag{2.18}$$

where the “nonlinear term” $N(\phi)$ is given by

$$N(\phi) = (V + \phi)^p - V^p - p V^{p-1} \phi \tag{2.19}$$

and the “error term” R is given by

$$R = \Delta V - e^{-\frac{p}{2}} V + V^p. \tag{2.20}$$

Finally,

$$W(y) = pV^{p-1}(y). \tag{2.21}$$

At this point it is convenient to make a choice of the parameters μ_j, the main objective being to make the error term small. We assume the parameters μ_j to be given by the relation

$$\begin{aligned}\log 8\mu_j^2 \quad := \quad & c_j H(\xi_j,\xi_j) + \sum_{i\neq j} c_i G(\xi_i,\xi_j) - \frac{C_{1j}}{4} \\ & -\frac{1}{p}\left[\frac{C_{1j}}{4}\left(c_j H(\xi_j,\xi_j) + \sum_{i\neq j} c_i G(\xi_i,\xi_j) + 4\log\mu_j\right) + \frac{C_{2j}}{4}\right] \\ & -\frac{C_{2j}}{4p^2}\left[c_j H(\xi_j,\xi_j) + \sum_{i\neq j} c_i G(\xi_i,\xi_j) + 4\log\mu_j\right]. \end{aligned} \tag{2.22}$$

Taking into account the explicit expression (2.4) of the constant C_{1j}, one easily sees that μ_j bifurcates, as p gets large, by

$$\bar\mu_j = e^{-\frac{3}{4}} e^{\frac{1}{4}(c_j H(\xi_j,\xi_j) + \sum_{i\neq j} c_i G(\xi_i,\xi_j))},$$

solution of equation

$$\log 8\mu_j^2 = c_j H(\xi_j,\xi_j) + \sum_{i\neq j} c_i G(\xi_i,\xi_j) - \frac{C_{1j}}{4}.$$

Observe that, with this choice of the parameters μ_j, we get

$$p^{\frac{p}{p-1}} V(y) = p + \omega_{0j}(y) + \frac{1}{p}\omega_{1j}(y) + \frac{1}{p^2}\omega_{2j}(y) + O(e^{-\alpha\frac{p}{4}}) + O(e^{-\frac{p}{4}}y) \tag{2.23}$$

uniformly in the region $|y-\xi_j'| < \delta e^{-\frac{p}{4}}$, for any fixed and small $\delta > 0$. Indeed, using (2.8), (2.9), (2.11), (2.12) and taking into account (2.17) and (2.10), we get

$$\begin{aligned} & p^{\frac{p}{p-1}} V(y) - p - \omega_{0j}(y) - \frac{1}{p}\omega_{1j}(y) - \frac{1}{p^2}\omega_{2j}(y) \\ = \quad & \sum_{l=0,1,2} \frac{1}{p^l} H_{lj}(e^{-\frac{p}{4}}y) + \sum_{i\neq j}\left[\sum_{l=0,1,2}\frac{1}{p^l}\hat u_{li}(y)\right] \\ = \quad & c_j H(\xi_j,\xi_j) - \log 8\mu_j^2 + \sum_{i\neq j} c_i G(\xi_i,\xi_j) \\ + \quad & \frac{C_{1j}}{p}\left[-\frac{c_j}{4}H(\xi_i,\xi_j) + \log\mu_j - \frac{p}{4} - \sum_{i\neq j}\frac{c_i}{4}G(\xi_j,\xi_i)\right] \\ + \quad & \frac{C_{2j}}{p^2}\left[-\frac{c_j}{4}H(\xi_i,\xi_j) + \log\mu_j - \frac{p}{4} - \sum_{i\neq j}\frac{c_i}{4}G(\xi_j,\xi_i)\right] + O(e^{-\alpha\frac{p}{4}}) + O(e^{-\frac{p}{4}}y) \end{aligned}$$

in the considered region. From (2.22), estimate (2.23) follows.

We claim that, with the previous choice (2.22) of the parameters μ_j, we achieve the following behavior for R and for W: there exists a constant $C>0$, independent of p, such that, for any $y\in\Omega_p$,

$$|R(y)|\le C\ \ \frac{1}{p^4}\sum_j\frac{1}{(1+|y-\xi_j'|^3)}, \tag{2.24}$$

and

$$W(y)=\sum_j e^{\omega_{0j}(y)}\left[1+\theta_p(y)\right] \tag{2.25}$$

where

$$|\theta_p(y)|\le C\frac{1}{p}\sum_j\Big[1+|y-\xi_j'|\Big] \tag{2.26}$$

for any p large enough.

Proof of (2.24)**.** Directly from (2.10) and (2.17) we argue that, for all j,

$$\begin{cases} -\Delta\hat u_{0j}+e^{-\frac{p}{2}}\hat u_{0j}=e^{\omega_{0j}}, & \text{in } \Omega_p,\\ \frac{\partial\hat u_{0j}}{\partial\nu}=0, & \text{on } \partial\Omega_p, \end{cases} \tag{2.27}$$

and, for $i=1,2$,

$$\begin{cases} -\Delta\hat u_{ij}+e^{-\frac{p}{2}}\hat u_{ij}=e^{\omega_{0j}}\omega_{ij}-f_{ij}, & \text{in } \Omega_p,\\ \frac{\partial\hat u_{ij}}{\partial\nu}=0, & \text{on } \partial\Omega_p. \end{cases} \tag{2.28}$$

Hence one gets directly that

$$p^{\frac{p}{p-1}}\Big[\Delta V-e^{-\frac{p}{2}}V\Big]=\sum_{j=1}^k\left[\sum_{i=0,1,2}\frac{1}{p^i}(\Delta\hat u_{ij}-e^{-\frac{p}{2}}\hat u_{ij})\right]$$

$$=\sum_{j=1}^k\left[-e^{\omega_{0j}}-\frac{1}{p}(e^{\omega_{0j}}\omega_{1j}-f_{1j})-\frac{1}{p^2}(e^{\omega_{0j}}\omega_{2j}-f_{2j})\right]. \tag{2.29}$$

Fix $\delta>0$ small. From the explicit expression of ω_{0j} and the asymptotic behavior of ω_{ij}, $i=1,2$, we get easily that in the region $|y-\xi_j'|>\delta e^{\frac{p}{4}}$ for all j,

$$\left[\sum_j(1+|y-\xi_j'|^3)^{-1}\right]^{-1}|\Delta V(y)-e^{-\frac{p}{2}}V(y)|=O(p^{-1}e^{-\frac{p}{4}}).$$

In the same region, by using that $(1+\frac{s}{p})^p \le e^s$, we obtain

$$\left[\sum_j (1+|y-\xi'_j|^3)^{-1}\right]^{-1} |V^p(y)| = O(e^{-\frac{p}{4}}).$$

Let us fix now j and the region $|y-\xi'_j| < \delta e^{\frac{p}{4}}$.

From (2.23) we get

$$V^p(y) = \frac{1}{p^{\frac{p^2}{p-1}}}\left[p+\omega_{0j}(y)+\frac{1}{p}\omega_{1j}+\frac{1}{p^2}\omega_{2j}+O(e^{-\frac{p}{4}})+O(e^{-\frac{p}{4}}|y|)\right]^p$$

$$= \frac{1}{p^{\frac{p}{p-1}}}\left[1+\frac{1}{p}\omega_{0j}(y)+\frac{1}{p^2}\omega_{1j}+\frac{1}{p^3}\omega_{2j}+O(p^{-1}e^{-\frac{p}{4}})+O(p^{-1}e^{-\frac{p}{4}}|y|)\right]^p.$$

Let us restrict our attention to the region $|y-\xi'_j| < \delta e^{\frac{p}{8}}$. Here we use Taylor expansion and we obtain

$$V^p(y) = \frac{1}{p^{\frac{p}{p-1}}}\Bigg[e^{\omega_{0j}}+\frac{1}{p}(e^{\omega_{0j}}\omega_{1j}-f_{1j})+\frac{1}{p^2}(e^{\omega_{0j}}\omega_{2j}-f_{2j})$$

$$+e^{\omega_{0j}}O\left(\frac{\log^6(2+|y|)}{p^3}+p^{-4}e^{-\frac{p}{4}}+p^{-4}e^{-\frac{p}{4}}|y|\right)\Bigg]. \tag{2.30}$$

Joining together (2.29) and (2.30), we get that in the region $|y-\xi'_j| < \delta e^{\frac{p}{8}}$ the following estimate holds true

$$\left[\sum_j (1+|y-\xi'_j|^3)^{-1}\right]^{-1} |\Delta V(y) - e^{-\frac{p}{2}}V(y) + V^p(y)|$$

$$\le Cp^{-4}(1+|y-\xi'_j|^3)\frac{\log^6(2+|y|)}{(1+|y-\xi'_j|^2)^2} \le p^{-4}.$$

Finally, in the region $\delta e^{\frac{p}{8}} < |y-\xi'_j| < \delta e^{\frac{p}{4}}$, we have

$$\left[\sum_j (1+|y-\xi'_j|^3)^{-1}\right]^{-1} |\Delta V(y) - e^{-\frac{p}{2}}V(y) + V^p(y)|$$

$$\le C(p^{-1}+p^{-p})\frac{(1+|y-\xi'_j|^3)}{(1+|y-\xi'_j|^2)^2} \le Cp^{-1}e^{\frac{p}{8}}.$$

Thus we get estimate (2.24).

Proof of (2.25)-(2.26)**.** Fix a δ small and positive. First observe that, if $|y-\xi'_j| > \delta e^{\frac{p}{4}}$ for all j, then

$$|pV^{p-1}(y)| \le Ce^{-p}.$$

Fix now j and assume $|y-\xi_j'|<\delta e^{\frac{p}{4}}$. Using (2.23) and Taylor expansion, one gets

$$pV^{p-1}(y)=\left[1+\frac{\omega_{0j}(y)}{p}+\frac{\omega_{1j}(y)}{p^2}+\frac{\omega_{2j}(y)}{p^3}+O(e^{-\alpha\frac{p}{4}})+O(e^{-\frac{p}{4}}y)\right]^{p-1}$$
$$=e^{\omega_{0j}(y)}\left[1+\frac{1}{p}O((1+|y-\xi_j'|))\right],$$

from which estimates (2.25)-(2.26) follow.

Proof of Lemma 2.1. A direct computation shows that, for $\xi_j\in\Omega$, we have

$$\frac{\partial H_{0j}}{\partial\nu}=4\frac{(x-\xi_j)\cdot\nu(x)}{|x-\xi_j|^2}+O(e^{-\frac{p}{2}})\ \text{ on }\partial\Omega$$

and, if $i=1,2$,

$$\frac{\partial H_{ij}}{\partial\nu}=-C_{ij}\frac{(x-\xi_j)\cdot\nu(x)}{|x-\xi_j|^2}+O(e^{-\frac{p}{2}})\ \text{ on }\partial\Omega.$$

On the other hand, for $\xi_j\in\partial\Omega$, we have, for all $x\in\partial\Omega\setminus\{\xi_j\}$

$$\lim_{p\to\infty}\frac{\partial H_{0j}}{\partial\nu}(x)=4\frac{(x-\xi_j)\cdot\nu(x)}{|x-\xi_j|^2}$$

and, if $i=1,2$,

$$\lim_{p\to\infty}\frac{\partial H_{ij}}{\partial\nu}(x)=-C_{ij}\frac{(x-\xi_j)\cdot\nu(x)}{|x-\xi_j|^2}.$$

Recall that the regular part of Green's function $H(x,y)$ satisfies the following equations: if $y\in\Omega$,

$$\begin{cases}-\Delta_x H(x,y)+H(x,y)=-\frac{1}{2\pi}\log\frac{1}{|x-y|}, & x\in\Omega,\\ \\ \frac{\partial H}{\partial\nu_x}(x,y)=\frac{1}{2\pi}\frac{(x-y)\cdot\nu(x)}{|x-y|^2} & x\in\partial\Omega;\end{cases}$$

while, if $y\in\partial\Omega$,

$$\begin{cases}-\Delta_x H(x,y)+H(x,y)=-\frac{1}{\pi}\log\frac{1}{|x-y|}, & x\in\Omega,\\ \\ \frac{\partial H}{\partial\nu_x}(x,y)=\frac{1}{\pi}\frac{(x-y)\cdot\nu(x)}{|x-y|^2} & x\in\partial\Omega.\end{cases}$$

Define the following differences: $z_0(x):=H_{0j}(x)+\log 8\mu_j^2-c_jH(x,\xi_j)$ and, if $i=1,2$, $z_i(x):=H_{ij}(x)+\frac{C_{ij}p}{4}-C_{ij}\log\mu_j+\frac{C_{ij}c_j}{4}H(x,\xi_j)$. Then we have

$$\begin{cases}-\Delta z_i+z_i=f_i & \text{in }\Omega\\ \\ \frac{\partial z_i}{\partial\nu}=g_i & \text{on }\partial\Omega,\end{cases}$$

where

$$f_0=-\log\frac{1}{(e^{-\frac{p}{2}}\mu_j^2+|x-\xi_j|^2)^2}+\log\frac{1}{|x-\xi_j|^4},\quad g_0=\frac{\partial H_{0j}}{\partial\nu}-4\frac{(x-\xi_j)\cdot\nu(x)}{|x-\xi_j|^2}$$

while, if $i = 1, 2$,

$$f_i = -u_{ij} + \frac{C_{ij}p}{4} - C_{ij}\log\mu_j - C_{ij}\log\frac{1}{|x-\xi_j|}, \quad g_i = \frac{\partial H_{ij}}{\partial\nu} + C_{ij}\frac{(x-\xi_j)\cdot\nu(x)}{|x-\xi_j|^2}.$$

A direct computation shows that, for any $i = 0, 1, 2$, for any $1 < q < 2$ there exists $C > 0$ such that

$$\Big\|g_i\Big\|_{L^q(\partial\Omega)} \le Ce^{-p/4q}, \quad \Big\|f_i\Big\|_{L^q(\Omega)} \le Ce^{-p/4}.$$

By L^q theory

$$\|z_i\|_{W^{1+s,q}(\Omega)} \le C\Big(\|\frac{\partial z_i}{\partial\nu}\|_{L^q(\partial\Omega)} + \|\Delta z_i\|_{L^q(\Omega)}\Big) \le Ce^{-p/4q}$$

for any $0 < s < \frac{1}{q}$. By the Morrey embedding we obtain

$$\|z_i\|_{C^\gamma(\overline{\Omega})} \le Ce^{-p/4q}$$

for any $0 < \gamma < \frac{1}{2} + \frac{1}{q}$. This proves the result (with $\alpha = \frac{1}{q}$). $\square$

3. Solvability of a linear equation

The main result of this section is the solvability of the following linear problem: given h find ϕ, c_{ij} such that

$$\begin{cases} -\Delta\phi + e^{-\frac{p}{2}}\phi - W\phi = h + \sum_{j=1,\dots,m}\sum_{i=1,J_j} c_{ij}\chi_j Z_{ij}, & \text{in } \Omega_p, \\ \frac{\partial\phi}{\partial\nu} = 0, \quad \text{on } \partial\Omega_p, \\ \int_{\Omega_p}\chi_j Z_{ij}\phi = 0 \quad \forall j = 1,\dots,m, \ i = 1, J_j \end{cases} \tag{3.31}$$

where $m = K + L$, W is defined in (2.21) and satisfies (2.25), $h \in L^\infty(\Omega_p)$, $J_j = 1$ if $j = K+1, ..., K+L$ while $J_j = 2$ if $j = 1, ..., K$ and Z_{ij}, χ_j are defined as follows.

Let z_{ij} be

$$z_{0j} := \frac{1}{\mu_j} - 2\frac{\mu_j}{\mu_j^2 + y^2}, \quad z_{ij} := \frac{y_i}{\mu_j^2 + y^2} \ i = 1, 2. \tag{3.32}$$

The following fact are very well known:

- any solution to
$$\Delta\phi + e^{\omega_j(|y|)}\phi = 0 \text{ in } \mathbb{R}^2, \quad |\phi| \le C(1+|y|)^\sigma \tag{3.33}$$
is a linear combination of $z_{ij}, i = 0, 1, 2$;
- any solution to
$$\Delta\phi + e^{\omega_j(|y|)}\phi = 0 \text{ in } \mathbb{R}^2_+, \quad |\phi| \le C(1+|y|)^\sigma \tag{3.34}$$
where $\mathbb{R}^2_+ = \{(y_1, y_2) : y_2 > 0\}$, is a linear combination of $z_{ij}, i = 0, 1$ (see [6]).

Next we choose a large but fixed number R_0 and nonnegative smooth function $\chi : \mathbb{R} \to \mathbb{R}$ so that $\chi(r) = 1$ for $r \le R_0$ and $\chi(r) = 0$ for $r \ge R_0+1$, $0 \le \chi \le 1$.

For $j = 1, ..., K$ (corresponding to interior bubble case), we define

$$\chi_j(y) = \chi(|y - \xi_j^{'}|), \quad Z_{ij}(y) = z_{ij}(y), \quad i = 0, 1, 2. \tag{3.35}$$

For $j = K + 1, ..., K + L$ (corresponding to boundary bubble case), we have to strength the boundary first. More precisely, at the boundary point $\xi_j \in \partial\Omega$, we assume that $\xi_j = 0$ and the unit outward normal at ξ_j is $-\mathbf{e}_2 = (0, -1)$. Let $G(x_1)$ be the defining function for the boundary $\partial\Omega$ in a neighborhood $B_\rho(\xi_j)$ of ξ_j, that is, $\Omega \cap B_\rho(\xi_j) = \{(x_1, x_2)|x_2 > G(x_1), (x_1, x_2) \in B_\rho(\xi_j)\}$. Then, let $F_j : B_\rho(\xi_j) \cap \Omega \to \mathbb{R}^2$ be defined by $F_j = (F_{j,1}, F_{j,2})$ where

$$F_{j,1} = x_1 + \frac{x_2 - G(x_1)}{1 + |G^{'}(x_1)|^2} G^{'}(x_1), \quad F_{j,2} = x_2 - G(x_1).$$

Then we set

$$F_j^p(y) = e^{\frac{p}{4}} F_j(e^{-\frac{p}{4}} y). \tag{3.36}$$

Note that F_j preserves the Neumann boundary condition. Define, for $j = K + 1, \dots, K + L$

$$\chi_j(y) = \chi(|F_j^p(y)|), \qquad Z_{ij}(y) = z_{ij}(F_j^p(y)) \qquad i = 0, 1.$$

The functions Z_{ij} satisfy the Neumann boundary condition (since F_j preserves the Neumann boundary condition).

It is important to note that

$$\Delta Z_{0j} + e^{\omega_{0j}} Z_{0j} = O(e^{-\frac{p}{4}}(1 + |y - \xi_j^{'}|)^{-3}) \tag{3.37}$$

since

$$\nabla z_{0j} = O\left((1 + |y - \xi_j^{'}|)^{-3}\right).$$

Equation (3.31) will be solved for $h \in L^\infty(\Omega_p)$ but we will be able to estimate the size of the solution in terms of the following norm

$$\|h\|_\infty = \sup_{y \in \Omega_p} |h(y)|, \quad \|h\|_* = \sup_{y \in \Omega_p} \frac{|h(y)|}{e^{-\frac{p}{2}} + \sum_{j=1}^m (1 + |y - \xi_j'|)^{-2-\sigma}}, \tag{3.38}$$

where we fix $0 < \sigma < 1$ although the precise choice will be made later on.

Proposition 3.1. *Let $d > 0$ and m a positive integer. Then there exist p_0, $C > 0$ such that for any $p > p_0$, any family of points $(\xi_1, \dots, \xi_m)$ such that*

$$\min_{i \neq j} |\xi_i - \xi_j| \ge d, \qquad \min_{j=1,\dots,K} dist(\xi_j, \partial\Omega) \ge d \tag{3.39}$$

and any $h \in L^\infty(\Omega_p)$ there exists a unique solution $\phi \in L^\infty(\Omega_p)$, $c_{ij} \in \mathbb{R}$ to (3.31). *Moreover*

$$\|\phi\|_\infty \le Cp\|h\|_*.$$

We begin by stating an a-priori estimate for solutions of (3.31) satisfying orthogonality conditions with respect to $Z_{ij}, i = 0, J_j, j = 1..., m$.

Lemma 3.1. *There exist $R_0 > 0$ and $p_0 > 0$ so that for $p > p_0$ and any solution ϕ of* (3.31) *with the orthogonality conditions*

$$\int_{\Omega_p} Z_{ij}\chi_j\,\phi = 0 \qquad \forall i = 0, ..., J_j \quad \forall j = 1, \dots, m \tag{3.40}$$

we have

$$\|\phi\|_\infty \le C\|h\|_*$$

where C is independent of p.

Proof of Lemma 3.1. We divide the proof into three steps.
Step 1. We first construct a suitable barrier. Namely, we show that, for p large enough there exist $R_1 > 0$, and

$$\psi : \Omega_p \setminus \cup_{j=1}^m B_{R_1}(\xi_j') \to \mathbb{R}$$

smooth and positive so that

$$-\Delta\psi + e^{-\frac{p}{2}}\psi - W\psi \ge \sum_{j=1}^m \frac{1}{|y-\xi_j'|^{2+\sigma}} + e^{-\frac{p}{2}}, \quad \text{in } \Omega_p \setminus \mathop{\cup}_{j=1}^m B_{R_1}(\xi_j'),$$

$$\frac{\partial\psi}{\partial\nu} \ge 0, \quad \text{on } \partial\Omega_p \setminus \mathop{\cup}_{j=1}^m B_{R_1}(\xi_j'),$$

$$\psi > 0, \quad \text{in } \Omega_p \setminus \mathop{\cup}_{j=1}^m B_{R_1}(\xi_j'),$$

$$\psi \ge 1, \quad \text{on } \Omega_p \cap \Big(\mathop{\cup}_{j=1}^m \partial B_{R_1}(\xi_j')\Big).$$

Take

$$\psi_{1j}(r) = 1 - \frac{1}{r^\sigma}, \quad \text{where } r = |y - \xi_j^{'}|.$$

Then

$$-\Delta\psi_{1j} = \sigma^2\frac{1}{r^{2+\sigma}},$$

and, for $|y-\xi_j^{'}| > R$,

$$-\Delta\psi_{1j} + e^{-\frac{p}{2}}\psi_{1j} - W\psi_{1j} \ge \sigma^2\frac{1}{r^{2+\sigma}} - W\psi_{1j} \ge \frac{\sigma^2}{2}\frac{1}{r^{2+\sigma}} \tag{3.41}$$

since

$$W \le \sum_{j=1}^m \frac{1}{1+|y-\xi_j'|^4}.$$

If $\xi_j^{'} \in \Omega_p$, then we have

$$\frac{\partial\psi_{1j}}{\partial\nu} = o(e^{-\frac{p}{4}}) \text{ on } \partial\Omega_p.$$

If $\xi_j' \in \partial\Omega_p$ and $|y-\xi_j'|>R$, we have

$$\frac{\partial\psi_{1j}}{\partial\nu}=\sigma\frac{< y-\xi_j',\nu >}{r^{2+\sigma}}$$

As before, we write $\partial\Omega_p$ near ξ_j' as the graph $\{(y_1,y_2): y_2=e^{\frac{p}{4}}G(e^{-\frac{p}{4}}y_1)\}$ with $G(0)=0$ and $G'(0)=0$.

Then

$$\begin{aligned}\frac{\partial\psi_{1j}}{\partial\nu} &= \frac{\sigma}{r^{2+\sigma}}\frac{1}{\sqrt{G'(e^{-\frac{p}{4}}y_1)^2+1}}\Big(-y_1G'(e^{-\frac{p}{4}}y_1),e^{\frac{p}{4}}G(e^{-\frac{p}{4}}(y_1)\Big)\\ &= \frac{\sigma}{r^{2+\sigma}}\frac{1}{\sqrt{O(\delta^2)+1}}O(e^{-\frac{p}{4}}r^2) \qquad \forall\ \ R_1<r<\delta e^{\frac{p}{4}}\\ &= O(\frac{e^{-\frac{p}{4}}}{r^\sigma}) \qquad \forall\ \ R_1<r<\delta e^{\frac{p}{4}}. \end{aligned} \tag{3.42}$$

Combining together the previous estimates, we see that

$$\frac{\partial\psi_{1j}}{\partial\nu}=o(e^{-\frac{p}{4}}) \text{ on } \partial\Omega_p. \tag{3.43}$$

Now let ψ_0 be the unique solution of

$$\Delta\psi_0-e^{-\frac{p}{2}}\psi_0+e^{-\frac{p}{2}}=0 \text{ in } \Omega_p,\ \frac{\partial\psi_0}{\partial\nu}=e^{-\frac{p}{4}} \text{ on } \partial\Omega_p.$$

Set

$$\psi:=\sum_{j=1}^m\psi_{1j}+C\psi_0. \tag{3.44}$$

It is easy to see that ψ is the function we are looking for. Observe that the constants $R_1>0$ can be chosen independently of p and that ψ is uniformly bounded, that is

$$0<\psi\le C \quad \text{in } \Omega_p\setminus\cup_{j=1}^m B_{R_1}(\xi_j')$$

with a positive constant C independent of p.

Step 2. We take $R_0=2R_1$. Thanks to the barrier ψ of the previous step we deduce that the following maximum principle holds in $\Omega_p\setminus\cup_{j=1}^m B_{R_1}(\xi_j')$: if $\phi\in H^1(\Omega_p\setminus\cup_{j=1}^m B_{R_1}(\xi_j'))$ satisfies

$$\begin{cases} -\Delta\phi+e^{-\frac{p}{2}}\phi\ge W\phi, \quad \text{in } \Omega_p\setminus\overset{m}{\underset{j=1}{\cup}}B_{R_1}(\xi_j'),\\ \\ \frac{\partial\phi}{\partial\nu}=0, \quad \text{on } \partial\Omega_p\setminus\overset{m}{\underset{j=1}{\cup}}B_{R_1}(\xi_j')\\ \\ \phi\ge0 \quad \text{on } \Omega_p\cap\Big(\overset{m}{\underset{j=1}{\cup}}\partial B_{R_1}(\xi_j')\Big)\end{cases}$$

then $\phi\ge0$ in $\Omega_p\setminus\cup_{j=1}^m B_{R_1}(\xi_j')$.

Let h be bounded and ϕ a solution to (3.31) satisfying (3.40). We first claim that $\|\phi\|_\infty$ can be controlled in terms of $\|h\|_*$ and the following inner norm of ϕ

$$\|\phi\|_{IN} = \sup_{\Omega_p \cap \left(\cup_{j=1}^m B_{R_1}(\xi_j')\right)} |\phi|.$$

Indeed, set

$$\widetilde{\phi} = C_1 \psi \left(\|\phi\|_{IN} + \|f\|_* \right),$$

with C_1 a constant independent of p. By the above maximum principle we have $\phi \le \widetilde{\phi}$ and $-\phi \le \widetilde{\phi}$ in $\Omega_p \setminus \cup_{j=1}^m B_{R_1}(\xi_j')$. Since ψ is uniformly bounded we deduce

$$\|\phi\|_\infty \le C \left(\|\phi\|_{IN} + \|f\|_* \right), \tag{3.45}$$

for some constant C independent of ϕ and p.

Step 3. We prove the lemma by contradiction. Assume that there exist a sequence $p_n \to \infty$, points $(\xi_1^n, \dots \xi_m^n)$ satisfying (3.39) and functions ϕ_n, f_n and h_n with $\|\phi_n\|_\infty = 1$ and $\|h_n\|_* \to 0$ so that for each n ϕ_n solves (3.31) and satisfies (3.40). By (3.45) we see that $\|\phi_n\|_{IN}$ stays away from zero. For one of the indices, say j, we can assume that $\sup_{B_{R_1}(\xi_j')} |\phi_n| \ge c > 0$ for all n. Consider $\hat{\phi}_n(z) = \phi_n(z - (\xi_j^n)')$. We may assume, up to subsequence, that ξ_j^n converges to ξ_j. Let us translate and rotate Ω_{p_n} so that $\xi_j' = 0$ and Ω_{p_n} approaches the entire plane $\mathbb{R}^2$ if $j = 1, \dots, K$ or the upper half plane $\mathbb{R}^2_+$ if $j = K+1, \dots, m$. Then by elliptic estimates $\hat{\phi}_n$ converges uniformly on compact sets to a nontrivial solution of

$$\Delta \phi + e^{\omega_j(|y|)} \phi = 0, |\phi| \le C.$$

Thus $\hat{\phi}$ is a linear combination of $z_{ij}, i = 0, ..., J_j$ (see (3.33) and (3.34)). On the other hand we can take the limit in the orthogonality relations (3.40), observing that limits of the functions Z_{ij} are just rotations and translations of z_{ij}, and we find $\int_{\mathbb{R}^2_+} \chi \hat{\phi} z_{ij} = 0$ for $i = 0, J_j$. This contradicts the fact that $\hat{\phi} \not\equiv 0$. $\square$

We will establish next an a-priori estimate for solutions to problem (3.31) that satisfy orthogonality conditions with respect to $Z_{ij}, i = 1, J_j$ only.

Lemma 3.2. *For p sufficiently large, if ϕ solves*

$$\begin{cases} -\Delta \phi + e^{-\frac{p}{2}} \phi - W \phi = h & \text{in} \quad \Omega_p \\ \frac{\partial \phi}{\partial \nu} = 0 & \text{on} \quad \partial \Omega_p \end{cases} \tag{3.46}$$

and satisfies

$$\int_{\Omega_p} Z_{ij} \chi_j \, \phi = 0 \qquad \forall j = 1, \dots, m, i = 1, J_j \tag{3.47}$$

then

$$\|\phi\|_\infty \le Cp\|h\|_* \tag{3.48}$$

where C is independent of p.

Proof. Let ϕ satisfy (3.46) and (3.47). We will modify ϕ to satisfy all orthogonality relations in (3.40) and for this purpose we consider modifications with compact support of the functions Z_{0j}. Let $R > R_0 + 1$ be large and fixed.

Let

$$a_{0j} := \frac{1}{\mu_j(\frac{4}{c_j}\log\frac{e^{\frac{p}{4}}}{R} + H(\xi_j, \xi_j))}. \tag{3.49}$$

Set

$$\widehat{Z}_{0j}(y) := Z_{0j}(y) - \frac{1}{\mu_j} + a_{0j}G(\xi_j, e^{-\frac{p}{4}}y). \tag{3.50}$$

Note that by our definition, $\widehat{Z}_{0,j}$ satisfies the Neumann boundary condition.

Let η be radial smooth cut-off function on $\mathbb{R}^2$ so that

$$0 \le \eta \le 1, \quad |\nabla\eta| \le C \quad \text{in} \quad \mathbb{R}^2$$

$$\eta \equiv 1 \quad \text{in} \quad B_R(0), \quad \eta \equiv 0 \quad \text{in} \quad \mathbb{R}^2 \setminus B_{R+1}(0).$$

If $\xi_j \in \Omega$, we write

$$\eta_j(y) = \eta(|y - \xi_j'|).$$

If $\xi_j \in \partial\Omega$, we write

$$\eta_j(y) = \eta(|F_j^p(y)|). \tag{3.51}$$

Now define

$$\widetilde{Z}_{0j} := \eta_j Z_{0j} + (1 - \eta_j)\widehat{Z}_{0j}. \tag{3.52}$$

Given ϕ satisfying (3.46) and (3.47) let

$$\widetilde{\phi} := \phi + \sum_{j=1}^m d_j \widetilde{Z}_{0j}, \qquad \text{where } d_j := -\frac{\int_{\Omega_p} Z_{0j}\chi_j\,\phi}{\int_{\Omega_p} Z_{0j}^2\chi_j}.$$

Estimate (3.48) is a direct consequence of

$$|d_j| \le Cp\|h\|_* \quad \forall j = 1, \dots, m. \tag{3.53}$$

We start proving this by observing, using the notation $L = -\Delta + e^{-\frac{p}{2}} - W$, that

$$L(\widetilde{\phi}) = h + \sum_{j=1}^m d_j L(\widetilde{Z}_{0j}) \quad \text{in } \Omega_p, \tag{3.54}$$

and

$$\frac{\partial\widetilde{\phi}}{\partial\nu} = 0 \quad \text{on } \partial\Omega_p. \tag{3.55}$$

Thus by Lemma 3.1 we have

$$\|\widetilde{\phi}\|_\infty \le C\sum_{j=1}^m |d_j|\|L(\widetilde{Z}_{0j})\|_* + C\|h\|_*. \tag{3.56}$$

Multiplying equation (3.54) by $\widetilde{Z}_{0k}$, integrating by parts and using (3.55) we find

$$\begin{aligned}\sum_{j=1}^m d_j\int_{\Omega_p} L(\widetilde{Z}_{0j})\widetilde{Z}_{0k} \qquad &\le C\|h\|_*[1+\sum_{j=1}^m \|L(\widetilde{Z}_{0j})\|_*] \\ &+C\sum_{j=1}^m |d_j|\|L(\widetilde{Z}_{0j})\|_*^2. \end{aligned}\tag{3.57}$$

We now measure the size of $\|L(\widetilde{Z}_{0j})\|_*$. To this end, we have for $|y-\xi_j'|>R$, according to (3.37)

$$\begin{aligned}L(\widehat{Z}_{0j}) \qquad &= -e^{\omega_{0j}}Z_{0j} - W\widehat{Z}_{0j} + O(e^{-\frac{p}{4}}(1+|y-\xi_j'|)^{-3}) \\ &= e^{\omega_{0j}}(a_{0j}G(\xi_j, e^{-\frac{p}{4}}y) - \frac{1}{\mu_j}) \\ &+O(e^{-\frac{p}{4}}(1+|y-\xi_j'|)^{-3} + e^{-\frac{(2+\alpha)p}{4}}).\end{aligned}\tag{3.58}$$

Thus

$$\|(1-\eta_j)L(\widehat{Z}_{0j})\|_* \le \frac{C}{p}$$

where the number C depends in principle of the chosen large constant R.

So

$$\begin{aligned}L(\widetilde{Z}_{0j}) \qquad &= \eta_j L(Z_{0j}) + (1-\eta_j)L(\widehat{Z}_{0j}) + 2\nabla\eta_j\nabla(Z_{0j}-\widehat{Z}_{0j}) + \Delta\eta_j(Z_{0j}-\widehat{Z}_{0j}) \\ &= O(e^{-\frac{(2+\alpha)p}{4}}) + (1-\eta_j)e^{\omega_{0j}}(a_{0j}G(\xi_j, e^{-\frac{p}{4}}y) - \frac{1}{\mu_j}) \\ &+2\nabla\eta_j\nabla(Z_{0j}-\widehat{Z}_{0j}) + \Delta\eta_j(Z_{0j}-\widehat{Z}_{0j}).\end{aligned}\tag{3.59}$$

Note that for $r=|y-\xi_j'|\in(R,R+1)$, we have

$$\begin{aligned}\widehat{Z}_{0j}-Z_{0j} &= a_{0j}G(\xi_j, e^{-\frac{p}{4}}y) - \tfrac{1}{\mu_j} \\ &= a_{0j}\left(\tfrac{4}{c_j}\log\tfrac{1}{e^{-\frac{p}{4}}|\xi_j'-y|} + H(\xi_j, e^{-\frac{p}{4}}y)\right) - \tfrac{1}{\mu_j}\end{aligned}$$

Hence we derive that for $r\in(R,R+1)$,

$$\widehat{Z}_{0j}-Z_{0j} = \frac{C}{p}\log\frac{1}{r} + O(\frac{e^{-\frac{\alpha p}{4}}}{p}),\ \nabla(\widehat{Z}_{0j}-Z_{0j}) = -\frac{C}{p}\frac{1}{r} + O(\frac{e^{-\frac{\alpha p}{4}}}{p}). \tag{3.60}$$

From (3.59) and (3.60), we conclude that

$$\|L(\widetilde{Z}_{0j})\|_* \le \frac{C}{p}.$$

Now we estimate the left hand side integral of (3.57). From (3.59), we see that for $j \neq k$,

$$\int_{\Omega_p} L(\widetilde{Z}_{0j})\widetilde{Z}_{0k} = O(e^{-\frac{\alpha p}{4}}) + \int_{\Omega_p} O(\frac{1}{p}(|\eta_j^{'}| + |\Delta\eta_j|))\widetilde{Z}_{0,k} = O(\frac{1}{p^2}).$$

For $j = k$, we decompose

$$\int_{\Omega_p} L(\widetilde{Z}_{0k})\widetilde{Z}_{0k} = I + II + O(e^{-\frac{p}{4}})$$

where

$$\begin{aligned} II &= \int_{\Omega_p} O(e^{-\frac{(2+\alpha)p}{4}}) + (1-\eta_k)e^{\omega_{0j}}(a_{0k}G(\xi_k, e^{-\frac{p}{4}}y) - \frac{1}{\mu_k})\widetilde{Z}_{0k} \\ &= O(e^{-\frac{\alpha p}{4}}) + O(\frac{1}{pR}) \end{aligned}$$

and

$$I = \int_{\Omega_p} (2\nabla\eta_k\nabla(Z_{0k} - \widehat{Z}_{0k}) + \Delta\eta_k(Z_{0k} - \widehat{Z}_{0k}))\widetilde{Z}_{0k}.$$

Thus integrating by parts we find

$$I = \int \nabla\eta\nabla(Z_{0k} - \widehat{Z}_{0k})\widehat{Z}_{0k} - \int \nabla\eta_j(Z_{0k} - \widehat{Z}_{0k})\nabla\widehat{Z}_{0k} + O(e^{-\frac{p}{4}})$$

Now, we observe that in the considered region, $r \in (R, R+1)$ with $r = |y - \xi_k^{'}|$, $|\widehat{Z}_{0k} - Z_{0k}| \leq \frac{C}{p}$ while $|\nabla Z_{0k}^{'}| \leq \frac{1}{R^3} + \frac{1}{p}$. Thus

$$|\int \nabla\eta_j(Z_{0k} - \widehat{Z}_{0k})\nabla\widehat{Z}_{0k}| \leq \frac{D}{pR^3}$$

where D may be chosen independent of R. Now

$$\begin{aligned} \int \nabla\eta_k\nabla(Z_{0k} - \widehat{Z}_{0k})\widehat{Z}_{0k} &= \int_R^{R+1} \eta^{'}(a_{0k}\frac{1}{r} + O(e^{-\frac{p}{4}}))\widehat{Z}_{0k} r dr \\ &= a_{0k}\int_R^{R+1} \eta^{'}(1 + O(e^{-\frac{p}{4}}) + O(R^{-1})) \\ &= -\frac{E}{p}[1 + O(R^{-1})] \end{aligned} \tag{3.61}$$

where E is a positive constant independent of p. Thus we conclude, choosing R large enough, that $I \sim -\frac{E}{p}$. Combining this and the estimate for II we find

$$\int_{\Omega_p} L(\widetilde{Z}_{0k})\widetilde{Z}_{0k} = -\frac{E}{p}[1 + O(R^{-1})], \int_{\Omega_p} L(\widetilde{Z}_{0j})\widetilde{Z}_{0k} = O((pR)^{-1}) \text{ for } j \neq k \tag{3.62}$$

This, combined with (3.57), proves the lemma.

□

Proof of Proposition 3.1.

First we prove that for any ϕ, c_{ij} solution to (3.31) the bound

$$\|\phi\|_\infty \leq Cp\|h\|_* \tag{3.63}$$

holds.

The previous lemma yields

$$\|\phi\|_\infty \le Cp(\|h\|_* + \sum_{j=1}^{m} \sum_{i=1}^{J_j} |c_{ij}|). \tag{3.64}$$

So it suffices to estimate the values of the constants c_{ij}. We show that

$$|c_{ij}| \le Cp\|h\|_* \tag{3.65}$$

To this end, we multiple (3.31) by Z_{ij} and integrate to find

$$\int_{\Omega_p} L(\phi)(Z_{ij}) = \int_{\Omega_p} hZ_{ij} + c_{ij} \int_{\Omega_p} \chi_j |Z_{ij}|^2 \tag{3.66}$$

Note that for $i \neq 0$

$$Z_{ij} = O((1 + |y - \xi_j|)^{-1})$$

So

$$\int_{\Omega_p} hZ_{ij} = O(\|h\|_*) \tag{3.67}$$

and

$$\int_{\Omega_p} L(\phi)Z_{ij} = \int_{\Omega_p} L(Z_{ij})\phi = O(e^{-\frac{p}{4}}\|\phi\|_\infty) \tag{3.68}$$

Substituting (3.67) and (3.68) into (3.66), we obtain (3.65).

Now consider the Hilbert space

$$H = \left\{ \phi \in H^1(\Omega_p) : \frac{\partial \phi}{\partial \nu}\Big|_{\partial\Omega_p} = 0, \quad \int_{\Omega_p} \chi_j Z_{ij} \phi = 0 \quad \forall j = 1, \dots, m, i = 1, J_j \right\}$$

with the norm $\|\phi\|_{H^1}^2 = \int_{\Omega_p} |\nabla \phi|^2 + e^{-\frac{p}{2}}\phi^2$. Equation (3.31) is equivalent to find $\phi \in H$ such that

$$\int_{\Omega_p} (\nabla\phi\nabla\psi + e^{-\frac{p}{2}}\phi\psi) - \int_{\Omega_p} W\phi\psi = \int_{\Omega_p} h\psi \qquad \forall \psi \in H.$$

By Fredholm's alternative this is equivalent to the uniqueness of solutions to this problem, which is guaranteed by (3.63).

□

Remark 3.1. *Given $h \in L^\infty(\Omega_p)$ with $\|h\|_* < \infty$, let ϕ be the solution to* (3.31) *given by Proposition 3.1. Multiplying the first equation in* (3.31) *against ϕ and integrating by parts, we get*

$$\|\phi\|_{H^1}^2 \equiv \int_{\Omega_p} |\nabla\phi|^2 + e^{-\frac{p}{2}}\phi^2 = \int_{\Omega_p} W\phi - \int_{\Omega_p} h\phi.$$

Taking into account (2.25) *we conclude that*

$$\|\phi\|_{H^1} \le C(\|\phi\|_\infty + \|h\|_*).$$

4. The nonlinear problem

Consider the nonlinear equation

$$\begin{cases} -\Delta\phi + e^{-\frac{p}{4}}\phi - W\phi = R + N(\phi) + \sum_{ij} c_{ij}\chi_j Z_{ij} \quad \text{in} \quad \Omega_p \\ \frac{\partial\phi}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega_p \\ \int_{\Omega_p} \chi_j Z_{ij}\phi = 0 \quad \forall j = 1,\dots,m,\ i = 1, J_j \end{cases} \tag{4.69}$$

where W is as in (2.25) and N, R are defined in (2.19) and (2.20) respectively.

Lemma 4.1. *Let $m > 0$, $d > 0$. Then there exist $p_0 > 0$, $C > 0$ such that for $p > p_0$ and any $(\xi_1,\dots,\xi_m)$ satisfying constraint (3.39) the problem (4.69) admits a unique solution ϕ, c_{ij} such that*

$$\|\phi\|_\infty \le Cp^{-3} \tag{4.70}$$

and

$$\sum_{i,j} |c_{ij}| \le \frac{C}{p^4}\ ,\ \ \|\phi\|_{H^1} \le \frac{C}{p^3}. \tag{4.71}$$

Proof. Let us denote by $\mathcal{C}_*$ the function space $C(\bar\Omega)$ endowed with the norm $\|\cdot\|_*$. Proposition 3.1 implies that the unique solution $\phi = T(h)$ of (3.31) defines a continuous linear map from the Banach space $\mathcal{C}_*$ into $C_0(\bar\Omega)$, with norm bounded by a multiple of p. Problem (4.69) becomes

$$\phi = A(\phi) := -T(R + N(\phi)).$$

For a given number $\gamma > 0$, let us consider the region

$$\mathcal{F}_\gamma := \{\phi \in C_0(\bar\Omega)\ :\ ||\phi||_\infty \le \frac{\gamma}{p^3}\}.$$

We have the following estimates

$$\begin{cases} \|N(\phi)\|_* \le Cp\|\phi\|_\infty^2 \\ \|N(\phi_1) - N(\phi_2)\|_* \le Cp\left(\max_{i=1,2}\|\phi_i\|_\infty\right)\|\phi_1 - \phi_2\|_\infty, \end{cases} \tag{4.72}$$

for any $\phi, \phi_1, \phi_2 \in \mathcal{F}_\gamma$. In fact, by Lagrange theorem we have that

$$|N(\phi)| \le p(p-1)\left(V + O(\frac{1}{p^3})\right)^{p-2}\phi^2$$

and

$$|N(\phi_1) - N(\phi_2)| \le p(p-1)\left(V + O(\frac{1}{p^3})\right)^{p-2}\left(\max_{i=1,2}|\phi_i|\right)|\phi_1 - \phi_2|$$

for any $x \in \Omega$, and hence, we get (4.72) since $\|\sum_{j=1}^m e^{U_j}\|_* = O(1)$. By (4.72), Proposition 3.1 and (2.24) imply that

$$\|A(\phi)\|_\infty \le D'p\ (\|N(\phi)\|_* + \|R\|_*) \le O(p^2\|\phi\|_\infty^2) + \frac{D}{p^3}$$

and

$$\|A(\phi_1)-A(\phi_2)\|_\infty \le C'p\,\|N(\phi_1)-N(\phi_2)\|_* \le Cp^2\left(\max_{i=1,2}\|\phi_i\|_\infty\right)\|\phi_1-\phi_2\|_\infty$$

for any $\phi, \phi_1, \phi_2 \in \mathcal{F}_\gamma$, where D is independent of γ. Hence, if $\|\phi\|_\infty \le \frac{2D}{p^3}$, we have that

$$\|A(\phi)\|_\infty = O(\frac{1}{p}\|\phi\|_\infty) + \frac{D}{p^3} \le \frac{2D}{p^3}.$$

Choose $\gamma = 2D$. Then, A is a contraction mapping of $\mathcal{F}_\gamma$ since

$$\|A(\phi_1) - A(\phi_2)\|_\infty \le \frac{1}{2}\|\phi_1 - \phi_2\|_\infty,$$

for any $\phi_1, \phi_2 \in \mathcal{F}_\gamma$. Therefore, a unique fixed point ϕ of A exists in $\mathcal{F}_\gamma$. By (3.65), we get that

$$\sum_{i,j}|c_{ij}(\xi)| = O\left(\|N(\phi)\|_* + \|R\|_* + \frac{1}{p}\|\phi\|_\infty\right) \le \frac{C}{p^4}$$

and by Remark 3.1 we deduce that

$$\|\phi\|_{H^1} = O\left(\|\phi\|_\infty + \|N(\phi)\|_* + \|R\|_*\right) \le \frac{C}{p^3}.$$

□

Remark 4.1. *The function $V+\phi$, where ϕ is given by Lemma 4.1, is positive in Ω_p. In fact, we observe first that V is positive. Indeed, from* (2.23) *we argue that, in the region $|y-\xi_j'| < \delta e^{\frac{p}{4}}$, V is positive. Outside this region, we may conclude the same from* (2.11), (2.12). *Finally observe that $p|\phi| \to 0$ uniformly over compacts of $\bar\Omega_p$.*

5. Variational reduction

In view of Lemma 4.1, given $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{M}_d$, we define $\phi(\xi)$ and $c_{ij}(\xi)$ to be the unique solution to (4.69) satisfying the bound (4.70). Define the following function of ξ,

$$F_p(\xi) = J_p(U[\xi] + \tilde\phi[\xi]) \tag{5.73}$$

where J_p is the functional defined by

$$J_p(v) = \frac{1}{2}\int_\Omega(|\nabla v|^2 + v^2) - \frac{1}{p+1}\int_\Omega v^{p+1}, \tag{5.74}$$

$U[\xi](x)$ is the function defined in (2.13) (with this notation we just want to stress the dependence on ξ) and

$$\tilde\phi[\xi](x) = e^{\frac{p}{2(p-1)}}\phi[\xi](e^{\frac{p}{4}}x), \quad x \in \Omega. \tag{5.75}$$

Lemma 5.1. *Let p be large. The function F_p is of class C^1. If $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{M}_d$ is a critical point of F_p then $u(x) = U[\xi](x) + \tilde\phi[\xi](x)$ is a critical point of J_p, that is, a solution to* (1.5).

Proof. Fix p large. Since all the terms in (4.69) depends C^1 on ξ, the map $\xi \to \phi$ in a C^1-function in $L^\infty(\Omega_p)$, as consequence of the Implicit Function Theorem. From Remark 3.1 one gets that $F_p \in C^1$.

Let

$$I_p(v) = \frac{1}{2}\int_{\Omega_p} |\nabla v|^2 + e^{-\frac{p}{2}} v^2 - \frac{1}{p+1}\int_{\Omega_p} v^{p+1}.$$

Then, performing a change of variable, $F_q(\xi) = J_q(U[\xi] + \tilde{\phi}[\xi]) = e^{\frac{p}{p-1}} I_p(V[\xi'] + \phi[\xi'])$ where $\xi' = e^{\frac{p}{4}}\xi$ and V is given by (2.16). Therefore

$$e^{-\frac{p}{p-1}} \frac{\partial F_p}{\partial \xi_{k,l}} = e^{\frac{p}{4}} DI_p(V[\xi'] + \phi[\xi'])\Big[\frac{\partial V[\xi']}{\partial \xi'_{k,l}} + \frac{\partial \phi[\xi']}{\partial \xi'_{k,l}}\Big].$$

Since $v = V[\xi'] + \phi[\xi']$ solves (4.69)

$$e^{-\frac{p}{4}} e^{-\frac{p}{p-1}} \frac{\partial F_p}{\partial \xi_{k,l}} = \sum_{i=1,J_j,j=1,\dots,m} c_{ij} \int_{\Omega_p} \chi_j Z_{ij} \Big[\frac{\partial V[\xi']}{\partial \xi'_{k,l}} + \frac{\partial \phi[\xi']}{\partial \xi'_{k,l}}\Big].$$

Let us assume that $DF(\xi) = 0$. From the previous equation and the orthogonality condition $\int_{\Omega_p} \chi_j Z_{ij} \phi = 0$, we conclude that, for all $k = 1, \dots, m$, $l = 1, J_k$

$$\sum_{j=1,\dots,m} \sum_{i=1,J_j} c_{ij} \left[\int_{\Omega_p} \chi_j Z_{ij} \frac{\partial V[\xi']}{\partial \xi'_{k,l}} + \int_{\Omega_p} \phi[\xi'] \frac{\partial(\chi_j Z_{ij})}{\partial \xi'_{k,l}}\right] = 0.$$

Now direct computation shows that $\frac{\partial V[\xi']}{\partial \xi'_{k,l}} = \pm Z_{kl} + o(1)$ where $o(1)$ is in the L^∞ norm and that

$$\int_{\Omega_p} \phi[\xi'] \frac{\partial(\chi_j Z_{ij})}{\partial \xi'_{k,l}} = o(1) \int_{\Omega_p} \chi_j Z_{ij} \frac{\partial V[\xi']}{\partial \xi'_{k,l}}.$$

Hence, equation $DF(\xi) = 0$ is reformulated into

$$\sum_{j=1,\dots,m} \sum_{i=1,J_j} c_{ij} \int_{\Omega_p} \chi_j Z_{ij} (\pm Z_{kl} + o(1)) = 0 \qquad \forall k = 1, \dots, m.$$

This is a strictly diagonal dominant system. We thus get that $c_{ij} = 0$ $\forall j = 1, \dots, m, i = 1, J_j$. Positivity of the function u follows from Remark 4.1.

$\square$

Next Lemma shows that the leading part of $F_p(\xi)$ is given by φ_m.

Lemma 5.2. *Let $d > 0$ be small and μ_j be given by* (2.22)*. Then*

$$\begin{aligned} F_p(\xi) \quad &= 8\pi(K + \frac{L}{2}) e^{\frac{p}{p-1}} p^{-\frac{p+1}{p-1}} + (C - \frac{p}{p+1} 8\pi)(K + \frac{L}{2}) e^{\frac{p}{p-1}} p^{-\frac{2p}{p-1}} \\ &-\frac{1}{2}\sum_{j=1}^m c_j \Big[c_j H(\xi_j, \xi_j) + \sum_{i \neq j} c_i G(\xi_i, \xi_j)\Big] e^{\frac{p}{p-1}} p^{-\frac{2p}{p-1}} \\ &+ \frac{1}{p^3}\Theta_p(\xi) \end{aligned} \tag{5.76}$$

where $\Theta_p(\xi)$ *is uniformly bounded in* $\mathcal{M}_d$ *as* $p \to \infty$ *and* C *is a universal constant.*

Proof. Observe first that

$$\int_\Omega (U+\tilde\phi)^{p+1} = e^{\frac{p}{p-1}} \int_{\Omega_p} (V+\phi)^{p+1}$$

$$= e^{\frac{p}{p-1}} \left[\int_{\Omega_p} [|\nabla (V+\phi)|^2 + e^{-\frac{p}{2}} (V+\phi)^2] + \sum_{i,j} c_{ij} \int_{\Omega_p} \chi_j Z_{ij} V \right]$$

$$= \int_\Omega [|\nabla (U+\tilde\phi)|^2 + (U+\tilde\phi)^2] + O(\tfrac{1}{p^4})$$

since (4.71) holds true. Hence

$$F_p(\xi) = (\frac{1}{2} - \frac{1}{p+1}) \int_\Omega [|\nabla (U+\tilde\phi)|^2 + (U+\tilde\phi)^2] + O(\frac{1}{p^4}). \tag{5.77}$$

Now,

$$\int_\Omega [|\nabla (U+\tilde\phi)|^2 + (U+\tilde\phi)^2] = \int_\Omega [|\nabla U|^2 + U^2] + 2 \int_\Omega [\nabla U \nabla \tilde\phi + U \tilde\phi]$$

$$+ \int_\Omega [|\nabla \tilde\phi|^2 + \tilde\phi^2]$$

$$= e^{\frac{p}{p-1}} \int_{\Omega_p} [|\nabla V|^2 + e^{-\frac{p}{2}} V^2] + 2 e^{\frac{p}{p-1}} \int_{\Omega_p} [\nabla V \nabla \phi + e^{-\frac{p}{2}} V \phi]$$

$$+ e^{\frac{p}{p-1}} \int_{\Omega_p} [|\nabla \phi|^2 + e^{-\frac{p}{2}} \phi^2]$$

$$= e^{\frac{p}{p-1}} [A+B+C].$$

We observe first that, as a consequence of (4.71)

$$B + C = O(\frac{1}{p^3}) \tag{5.78}$$

uniformly for $\xi \in \mathcal{M}$ as $p \to \infty$.

We will devote the rest of this proof to estimating A. Define

$$V_j(y) = \sum_{i=0,1,2} p^{-i} \hat u_{ij}(y) \quad y \in \Omega_p$$

(see (2.10)) so we may rewrite (2.16) in equivalent form $V(y) = \frac{1}{p^{\frac{p}{p-1}}} \sum_{j=1}^m V_j(y)$. Note that V_j satisfies

$$-\Delta V_j + e^{-\frac{p}{2}} V_j = e^{\omega_{0j}} + \frac{1}{p}(-\Delta \omega_{1j} + e^{-\frac{p}{2}} \omega_{1j}) + \frac{1}{p^2}(-\Delta \omega_{1j} + e^{-\frac{p}{2}} \omega_{1j}) \quad \text{in } \Omega_p,$$

$$\frac{\partial V_j}{\partial \nu} = 0 \text{ on } \partial \Omega_p.$$

Then

$$\begin{aligned} p^{\frac{2p}{p-1}}A &= \int_{\Omega_p}(-\Delta V + e^{-\frac{p}{2}}V)V \\ &= \sum_j \int_{B(\xi_j',\delta e^{\frac{p}{4}})}(-\Delta V + e^{-\frac{p}{2}}V)V + o(e^{-\frac{p}{4}}). \end{aligned} \tag{5.79}$$

Let us fix j. We have

$$\begin{aligned} &\int_{B(\xi_j',\delta e^{\frac{p}{4}})}(-\Delta V + e^{-\frac{p}{2}}V)V \\ &= \int_{B(\xi_j',\delta e^{\frac{p}{4}})}\Big(e^{\omega_{0j}} + \frac{1}{p}(-\Delta\omega_{1j} + e^{-\frac{p}{2}}\omega_{1j}) + \frac{1}{p^2}(-\Delta\omega_{2j} + e^{-\frac{p}{2}}\omega_{2j})\Big) \times \\ &\Big(p + \omega_{0j} + \frac{1}{p}\omega_{1j} + \frac{1}{p^2}\omega_{2j}\Big) + o(e^{-\frac{p}{4}}) \\ &= p\int_{B(\xi_j',\delta e^{\frac{p}{4}})} e^{\omega_{0j}} + \int_{B(\xi_j',\delta e^{\frac{p}{4}})}\Big(e^{\omega_{0j}}\omega_{0j} - \Delta\omega_{1j} + e^{-\frac{p}{2}}\omega_{1j}\Big) \\ &+O(\frac{1}{p}) \end{aligned} \tag{5.80}$$

Define η_j to be equal to 1 if $j = 1,\dots,K$ and equal to $\frac{1}{2}$ if $j = K+1,\dots,K+L$. We have

$$\int_{B(\xi_j',\delta e^{\frac{p}{4}})} e^{\omega_{0j}} = 8\pi\eta_j + O(e^{-\frac{3p}{4}}); \tag{5.81}$$

if we call $\beta = \int_{\mathbb{R}^2}\frac{8}{(1+|z|^2)^2}\log\frac{8}{(1+|z|^2)^2}$,

$$\begin{aligned} \int_{B(\xi_j',\delta e^{\frac{p}{4}})} e^{\omega_{0j}}\omega_{0j} &= \beta\eta_j - 2\eta_j\log\mu_j\int_{\mathbb{R}^2}\frac{8}{(1+|z|^2)^2} + o(e^{-\frac{p}{4}}) \\ &= \beta\eta_j - 16\eta_j\pi\log\mu_j + o(e^{-\frac{p}{4}}) \\ &= \beta\eta_j + 12\pi\eta_j - \frac{c_j}{2}[c_jH(\xi_j,\xi_j) - \sum_{i\neq j}c_iG(\xi_i,\xi_j)] \\ &+O(\frac{1}{p}) \end{aligned} \tag{5.82}$$

because of (2.22). Since $\omega_{1j}(y) = \tilde{\omega}_1(\frac{|y-\xi_j|}{\mu_j})$ for a certain real function $\tilde{\omega}_1$, we get

$$\int_{B(\xi_j',\delta e^{\frac{p}{4}})} \Delta\omega_{1j} = \alpha\eta_j + o(e^{-\frac{p}{4}}) \tag{5.83}$$

for a certain universal constant α, and

$$e^{-\frac{p}{2}}\int_{B(\xi_j',\delta e^{\frac{p}{4}})}\omega_{1j} = O(e^{-\frac{p}{2}}). \tag{5.84}$$

Summing up all the information contained in (5.79)–(5.84), we obtain

$$\begin{aligned} A &= \frac{8\pi}{p^{\frac{2p}{p-1}}}(K+\frac{L}{2})p + \frac{\beta-\alpha+12\pi}{p^{\frac{2p}{p-1}}}(K+\frac{L}{2}) \\ &- \frac{1}{2p^{\frac{2p}{p-1}}}\left[\sum_j [c_j^2 H(\xi_j,\xi_j) + \sum_{i\neq j} c_i c_j G(\xi_i,\xi_j)]\right] + O(\frac{1}{p^3}). \quad (5.85) \end{aligned}$$

The expansion (5.76) thus follows from (5.77), (5.78) and (5.85). □

6. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. We first need the following

Lemma 6.1. *We have*

$$\min_{\xi\in\partial\mathcal{M}_d} \varphi_m(\xi) \to +\infty \text{ as } \delta \to 0. \quad (6.86)$$

Proof: The proof of this Lemma is similar to [14], we briefly reproduce it here for completeness. Let $\xi = (\xi_1, ..., \xi_m) \in \partial\mathcal{M}_d$. There are two possibilities: either there exists $j_0 \le K$ such that $d(\xi_{j_0}, \partial\Omega) = d$, or there exists $i_0 \neq j_0$, $|\xi_{i_0} - \xi_{j_0}| = d$.

In the first case, a consequence of the properties of the Green's function is that for all $\xi \in \Omega$

$$H(\xi,\xi) \ge C \log \frac{1}{d(\xi,\partial\Omega)}. \quad (6.87)$$

In the second case, we may assume that there exists a fixed constant C such that $d(\xi_i, \partial\Omega) \ge C, i = 1, ..., K$, as otherwise it follows into the first case. But then it is easy to see that

$$G(\xi_i,\xi_j) \ge C \log \frac{1}{|\xi_i - \xi_j|}. \quad (6.88)$$

The statement of the Lemma follows from (6.87) and (6.88). □

We are now ready to give the

Proof of Theorem 1: According to Lemma 5.1, the function $U[\xi] + \tilde\phi[\xi]$, where U and $\tilde\phi$ are defined respectively by (2.13) and (5.75), is a solution of Problem (1.5) if we adjust ξ so that it is a critical point of $F_p(\xi) = J_p(U[\xi] + \tilde\phi[\xi])$ defined by (5.73). This is obviously equivalent to finding a critical point of

$$\tilde F_p(\xi) = -2e^{-\frac{p}{p-1}} p^{\frac{2p}{p-1}} \left(F_p(\xi) + Ae^{\frac{p}{p-1}} p^{-\frac{p+1}{p-1}} + Be^{\frac{p}{p-1}} p^{-\frac{2p}{p-1}} \right)$$

for suitable constants A and B. On the other hand, from Lemma 5.2, we have that for $\xi \in \mathcal{M}_d$,

$$\tilde F_p(\xi) = \varphi_m(\xi) + O(p^{-1})\Theta_p(\xi) \quad (6.89)$$

where φ_m is give by (1.9), Θ_p is uniformly bounded in $\mathcal{M}_d$ as $p \to \infty$.

From the above Lemma, the function φ_m is C^1, bounded from below in $\mathcal{M}_d$ and such that

$$\min_{\xi \in \partial \mathcal{M}_d} \varphi_m(\xi_1, \dots, \xi_m) \to +\infty \text{ as } d \to 0$$

Hence, for d is arbitrarily small, φ_m has an absolute minimum M in $\mathcal{M}_d$. This implies that $\tilde{F}_p(\xi)$ also has an absolute minimum $(\xi_1^p, ..., \xi_m^p) \in \mathcal{M}_d$ such that

$$\varphi_m(\xi_1^p, ..., \xi_m^p) \to \min_{\xi \in \mathcal{M}_d} \varphi_m(\xi) \text{ as } p \to \infty. \tag{6.90}$$

Hence Lemma 5.1 guarantees the existence of a solution u_p for (1.5). The qualitative properties of the solution follow directly from the ansatz (2.13). □

Remark 6.1. *By using Ljusternik-Schnirelmann theory, one can get another distinct solution satisfying Theorem 1. The proof is similar to* [10].

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