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MULTIPLICITY AND SINGULAR SOLUTIONS FOR A LIOUVILLE-TYPE SYSTEM IN A BALL

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Abstract. We consider the Liouville system

$$-\Delta u = \lambda e^v, \quad -\Delta v = \mu e^u \quad \text{in } B$$

with $u = v = 0$ on ∂B, where B is the unit ball in $\mathbb{R}^N$, $N \geq 3$, and λ and μ are positive parameters. First we show that radial solutions in $B \setminus \{0\}$ are either regular or have a log-type singularity. Then, in dimensions $3 \leq N \leq 9$ we prove that there is an unbounded curve $\mathcal{S} \subset (0,\infty)^2$ such that for each $(\mu, \lambda) \in \mathcal{S}$ there exist infinitely many regular solutions. Moreover, the number of regular solutions tends to infinity as (μ, λ) approaches a fixed point in $\mathcal{S}$.

1. Introduction

We study radially symmetric solutions to the cooperative system

$$\begin{cases} -\Delta u = \lambda e^v & \text{in } B \\ -\Delta v = \mu e^u & \text{in } B \\ u = v = 0 & \text{on } \partial B \end{cases} \tag{1.1}$$

where B is the unit ball in $\mathbb{R}^N$, $N \geq 3$, and λ and μ are positive parameters.

In 2 dimensions, more general cooperative versions have been considered in [4, 5, 3, 20, 21]. In this article we investigate (1.1) for dimensions $N \geq 3$.

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All classical solutions to (1.1) are radially symmetric by a result of Troy [27]; see also [10]. This elliptic system is a natural generalization of the Liouville–Gelfand problem

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.2}$$

since for $\lambda = \mu$ and u and v classical solutions of (1.1), necessarily $u = v$, which can be seen by multiplying each equation by $u - v$ and integrating.

Concerning (1.2), classical solutions are radial by Gidas, Ni, Nirenberg [16]. Moreover, all of them can be found from one entire radial solution, which leads to a complete description of the bifurcation diagram of (1.2); see e.g. Joseph and Lundgren [19] and also [15]. In particular, there exists $\lambda^* = \lambda^*(N) > 0$ such that for $0 < \lambda < \lambda^*$, (1.2) has a minimal solution u_λ; for $\lambda = \lambda^*$, (1.2) has a unique solution u^* (possibly singular); and for $\lambda > \lambda^*$ (1.2) has no solution. Moreover, if $N = 1, 2$, then for $0 < \lambda < \lambda^*$, there are exactly two solutions; one of them is the minimal solution u_λ, and the other one has Morse index 1. If $3 \le N \le 9$, then $\lambda^* > 2(N-2)$. For $0 < \lambda < \lambda^*$, $\lambda \neq 2(N-2)$, (1.2) has finitely many solutions, and for $\lambda = 2(N-2)$, (1.2) has infinitely many solutions that converge to $-2\log|x|$, which is a singular solution. If $N \ge 10$, then $\lambda^* = 2(N-2)$ and $u_* = -2\log|x|$. Moreover, (1.2) has a unique solution for each $\lambda \in (0, \lambda^*)$.

For problem (1.1) Montenegro [24] showed that there is a non-empty open set $\mathcal{U} \subset (0,\infty)^2$ such that a minimal classical solution $(u_{\mu,\lambda}, v_{\mu,\lambda})$ exists if $(\mu,\lambda) \in \mathcal{U}$ and no solution exists if $(\mu,\lambda) \notin \overline{\mathcal{U}}$. Moreover, $\mathcal{C} = \partial\mathcal{U} \cap (0,\infty)^2$ can be described as a continuous curve, and for $(\mu,\lambda) \in \mathcal{C}$ the limit

$$\lim_{m\to 1^-} (u_{m\mu,m\lambda}, v_{m\mu,m\lambda})$$

is a weak solution, called the extremal solution. This suggests strong analogies between (1.1) and (1.2). In this direction, for general domains it was proved by Cowan [7] (with some restriction on λ, μ) and Dupaigne, Farina, and Sirakov [11] (without restrictions) that if $N < 10$ the extremal solution is bounded; and by Dávila and Goubet [9] that the singular set of the extremal solution has dimension at most $N - 10$ in general.

In this work we focus on the analysis of singular radial solutions and multiplicity in low dimensions.

We prove the following results.

Theorem 1.1. *Let $N \geq 3$. Suppose $u, v \in C^2(B_1 \setminus \{0\})$ is a radial solution of*

$$\begin{cases} -\Delta u = \lambda e^v \quad \text{in } B_1 \setminus \{0\} \\ -\Delta v = \mu e^u \quad \text{in } B_1 \setminus \{0\} \\ u, v > 0 \quad \text{in } B_1 \setminus \{0\}, \end{cases} \tag{1.3}$$

where $\lambda, \mu > 0$. Then either both u and v admit a smooth extension to B_1, or u and v are both singular and satisfy

$$\begin{cases} u(r) = -2\log r + \log(\dfrac{2(N-2)}{\mu}) + o(1), \qquad ru'(r) = -2 + o(1), \\ v(r) = -2\log r + \log(\dfrac{2(N-2)}{\lambda}) + o(1), \qquad rv'(r) = -2 + o(1), \end{cases} \tag{1.4}$$

as $r \to 0$.

Thanks to Theorem 1.1, any radial singular solution (u, v) of system (1.1) in $B_1(0) \setminus \{0\}$ can be extended as a distribution solution in $B_1(0)$. We will call such solutions just radial singular solutions.

Theorem 1.2. *Assume $N \geq 3$. There is a curve $\mathcal{S} \subset \overline{\mathcal{U}}$ described by $\lambda = \bar{h}(\mu)$, where $\bar{h} : (0, \infty) \to (0, \infty)$ is smooth and decreasing, with*

$$\lim_{\mu \to 0} \bar{h}(\mu) = \infty, \quad \lim_{\mu \to \infty} \bar{h}(\mu) = 0,$$

such that (1.1) has a radial singular solution (u, v) with parameters (μ, λ) if and only if $\lambda = \bar{h}(\mu)$. Moreover, the radial singular solution is unique.

Theorem 1.3. *Assume $3 \leq N \leq 9$. Then the curve $\mathcal{S}$ is contained in $\mathcal{U}$, and for each $(\mu, \lambda) \in \mathcal{S}$ there exist infinitely many regular solutions of (1.1). Moreover, the number of regular solutions tends to infinity as (μ, λ) approaches a fixed point in $\mathcal{S}$.*

In Figures 1 and 2 we have plotted the regions of existence computed numerically in dimensions 5 and 10 respectively. In both figures we have shown with a thick line the curve $\mathcal{C} = \partial\mathcal{U} \cap (0, \infty)^2$ and a dashed line $\lambda = \bar{h}(\mu)$, which is clearly visible in the case $N = 5$, while in the case $N = 10$ it is indistinguishable from $\mathcal{C}$. In the regions of existence, we have chosen to plot some curves obtained numerically from an initial-value problem; see Remark 3.1 for more details. From these numerical results and the analogy with the scalar equation (1.2) it is reasonable to conjecture that if $N \geq 10$ then the extremal curve for existence $\mathcal{C}$ coincides with the curve of singular solutions $\mathcal{S}$. Actually, on the diagonal $\mu = \lambda$ this is true, and in dimension

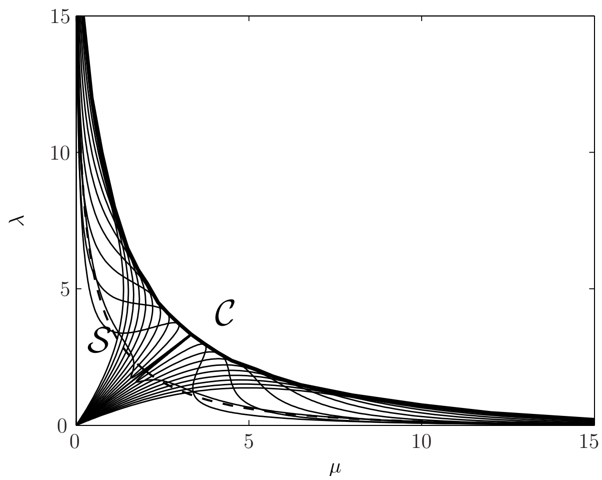

FIGURE 1. Region of existence for $N = 5$

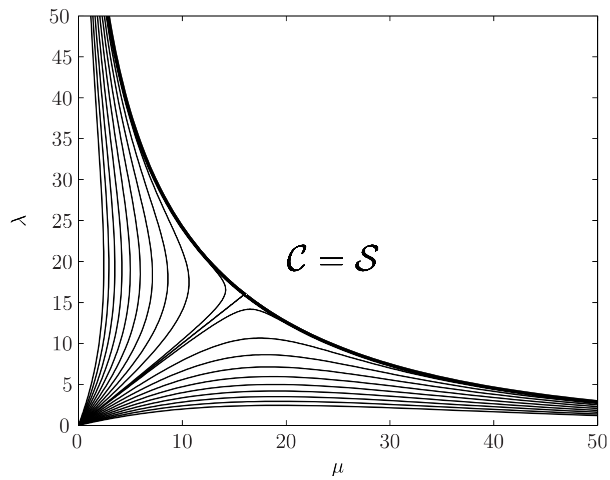

FIGURE 2. Region of existence for $N = 10$

$N \geq 11$ maybe one can prove that near the diagonal $\mathcal{C}$ and $\mathcal{S}$ coincide. Another related property that we conjecture in dimensions $N \geq 10$ is that for all (μ, λ) in $\mathcal{U}$ there is a unique solution.

Section 2 is devoted to the proof of Theorem 1.1, which is based on arguments similar to those used for some fourth-order problems such as

[1, 8, 13, 14]. In Section 3 we introduce a change of variables that transforms the first-order system of ODE (1.1) which we use later to prove the results on the existence of singular solutions and multiplicity. In this section we explain Figures 1 and 2 more. In Section 4 we prove Theorem 1.2, and in Section 5 we give the proof of Theorem 1.3.

2. Classification of singularities

This section is devoted to the proof of Theorem 1.1. In this argument we can assume that $\lambda = \mu = 2(N-2)$. Indeed, we can replace u and v by $\tilde{u}(r) = u(Rr) + \log \frac{\mu}{2(N-2)} + 2\log R$ and $\tilde{v}(r) = v(Rr) + \log \frac{\lambda}{2(N-2)} + 2\log R$. Then $\tilde{u}$ and $\tilde{v}$ satisfy system (1.3) in $B_\rho(0) \setminus \{0\}$ with $\lambda = \mu = 2(N-2)$ and $\rho = 1/R$. By choosing R large we can assume that $\tilde{u}$ and $\tilde{v}$ are positive in $B_\rho(0) \setminus \{0\}$, and thus we are left to study radial functions u and v which are $C^2(B_\rho \setminus \{0\})$ and satisfy

$$\begin{cases} -\Delta u = 2(N-2)e^v & \text{in } B_\rho \setminus \{0\} \\ -\Delta v = 2(N-2)e^u & \text{in } B_\rho \setminus \{0\} \\ u, v > 0 \quad \text{in } B_\rho \setminus \{0\}, \end{cases} \tag{2.1}$$

where $\rho > 0$. We define new variables

$$U(t) = u(r) + 2t, \quad V(t) = v(r) + 2t \quad \text{with } r = e^t \tag{2.2}$$

and obtain the system

$$\begin{cases} U'' + (N-2)U' + 2(N-2)(e^V - 1) = 0, \\ V'' + (N-2)V' + 2(N-2)(e^U - 1) = 0 \end{cases} \tag{2.3}$$

for t in $(-\infty, \log\rho)$. We observe that this system is autonomous, so we can assume that U and V solve the system in $(-\infty, 0)$. After this shift in time ($t = t_{old} - \log\rho$), from the positivity of $u(r)$ and $v(r)$, the functions U and V satisfy

$$U(t), V(t) \geq 2t - C \quad \forall t \leq 0, \tag{2.4}$$

where $C > 0$.

Lemma 2.1. *There is $T > 0$ such that $U < V$ or $V > U$ or $U \equiv V$ in $(-\infty, 0)$.*

Proof. Suppose $U \not\equiv V$ but that $U - V$ changes sign more than once in $(-\infty, 0)$. Let $t_0 < t_1 < 0$ be such that $U(t_0) = V(t_0)$ and $U(t_1) = V(t_1)$.

Subtracting both equations we find

$$(U-V)''+(N-2)(U-V)'+(e^V-e^U)=0.$$

Let $w=U-V$ and $a=2(N-2)\frac{e^V-e^U}{V-U}\geq 0$ (whenever $U\neq V$). Then

$$w''+(N-2)w'-2(N-2)aw=0 \quad \text{in } (-\infty,0).$$

Multiplying by w and integrating in (t_0,t_1), and using that $w(t_0)=w(t_1)=0$, we get

$$\int_{t_0}^{t_1}(w')^2+aw^2=0,$$

from which we deduce that $U\equiv V$ in $[t_0,t_1]$. By uniqueness of the solution to ODE's we obtain $U\equiv V$ in $(-\infty,0)$. □

The case $U\equiv V$ corresponds to a radial solution of the equation $-\Delta u = 2(N-2)e^u$ in $B_\rho(0)\setminus\{0\}$, and then we know that either $u(r)=-2\log r$ or u can be extended to 0 as a smooth function; see [23]. So it remains to study the case when the components are not identical. Therefore, thanks to Lemma 2.1 and shifting time, from here on we assume that

$$V<U \quad \text{in } (-\infty,0). \tag{2.5}$$

Notice that (2.4) is still valid after this shift in time.

Lemma 2.2. *We have*

$$\liminf_{t\to-\infty} U(t)\leq 0. \tag{2.6}$$

Proof. Suppose for the sake of contradiction that $U(t)\geq\delta>0$ for all $t\leq t_0$ where $t_0\leq 0$. Note that $2(N-2)(e^{U(t)}-1)\geq\tilde{\delta}>0$ for all $t\leq t_0$. Thus, by (2.3)

$$V''+(N-2)V'\leq-\tilde{\delta} \quad \text{for all } t\leq t_0.$$

Multiplying by $e^{(N-2)t}$ and integrating in $[s,t_1]$ with $s\leq t_1\leq t_0$, we find

$$e^{(N-2)t_1}V'(t_1)-e^{(N-2)s}V'(s)\leq-\tilde{\delta}\frac{e^{(N-2)t_1}-e^{(N-2)s}}{N-2}.$$

Suppose that for some $t_1\leq t_0$ we have $V'(t_1)\geq 0$. Then we obtain

$$\tilde{\delta}\frac{e^{(N-2)(t_1-s)}-1}{N-2}\leq V'(s) \quad \text{for all } s\leq t_1.$$

Let us simplify the notation, writing

$$\bar{\delta}e^{-(N-2)s}-C\leq V'(s) \quad \text{for all } s\leq t_1,$$

where $\bar\delta, C > 0$. Integrating in an interval $[t, t_1]$ with $t \le t_1$, we see that

$$V(t) \le V(t_1) + \frac{\bar\delta}{N-2} e^{-(N-2)t_1} - \frac{\bar\delta}{N-2} e^{-(N-2)t} + C(t_1 - t),$$

for all $t \le t_1$. But this contradicts (2.4).

Therefore it remains to do the analysis in the case $V'(t) \le 0$ for all $t \le t_0$, which implies in particular $V(t) \ge V(t_0)$ $\forall t \le t_0$. Here we follow an idea of [22]; see also [1, 12, 13]. By shifting time, we assume that

$$U(t) \ge \delta > 0, \quad V(t) \ge V(0) \quad \forall t \le 0.$$

Let $\phi \in C^\infty(\mathbb{R})$ be such that $0 \le \phi \le 1$, $\phi(t) = 0$ for $t \in (-\infty, -3] \cup [0, \infty)$, $\phi(t) > 0$ for $t \in (-3, 0)$, $\phi(t) = 1$ for $t \in [-2, -1]$, and for $i = 1, 2$

$$\int_{-3}^0 \frac{(\phi^{(i)})^2}{\phi}\, dt < +\infty.$$

Let $\tau > 1$ and $\phi_\tau(t) = \phi(t/\tau)$. Multiplying the second equation in (2.3) by ϕ_τ and integrating, we find

$$2(N-2)\int_{-3\tau}^0 (e^U - 1)\phi_\tau = \int_{-3\tau}^0 (-V\phi_\tau'' + (N-2)V\phi_\tau'). \tag{2.7}$$

Let $\varepsilon > 0$ be fixed later on. For all $t \in (-3\tau, 0)$ and $i = 1, 2$ we have

$$|V\phi_\tau^{(i)}| \le \varepsilon V^2 \phi_\tau + C_\varepsilon \frac{(\phi_\tau^{(i)})^2}{\phi_\tau}$$

so that from (2.7) we deduce that

$$\int_{-3\tau}^0 (e^U - 1)\phi_\tau \le C\varepsilon \int_{-3\tau}^0 V^2\phi_\tau + C_\varepsilon \sum_{i=1,2} \int_{-3\tau}^0 \frac{(\phi_\tau^{(i)})^2}{\phi_\tau}\, dt. \tag{2.8}$$

But

$$\int_{-3\tau}^0 \frac{(\phi_\tau^{(i)})^2}{\phi_\tau}\, dt = \tau^{1-2i} \int_{-3}^0 \frac{(\phi^{(i)})^2}{\phi}\, dt \le C\tau^{1-2i},$$

so from (2.8) we have

$$\int_{-3\tau}^0 (e^U - 1)\phi_\tau \le C\varepsilon \int_{-3\tau}^0 V^2\phi_\tau + C_\varepsilon \tau^{-1} \tag{2.9}$$

(assuming $\tau > 1$). Now we use that $V \le U$, $V \ge V(0)$, and $U \ge \delta$. From these inequalities we can deduce that

$$V^2 \le (1 + \frac{V(0)^2}{\delta^2})U^2.$$

Combining with (2.9) we obtain

$$\int_{-3\tau}^{0} (e^U - 1)\phi_\tau \le C\varepsilon(1 + \frac{V(0)^2}{\delta^2}) \int_{-3\tau}^{0} U^2\phi_\tau + C_\varepsilon \tau^{-1}. \tag{2.10}$$

We can select $\varepsilon > 0$ sufficiently small so that

$$e^u - 1 - C\varepsilon(1 + V(0)^2/\delta^2)u^2 \ge \delta/4 \quad \text{for } u \ge \delta.$$

From (2.10) we obtain then $\frac{\delta}{4}\tau \le C_\varepsilon \tau^{-1}$, which is not possible for $\tau > 1$ large. □

Lemma 2.3. *We have*

$$\limsup_{t\to-\infty} U(t) < +\infty.$$

Proof. We follow the idea of Lemma 1 in [12]. Assume for the sake of contradiction that $\limsup_{t\to-\infty} U(t) = +\infty$. Then, taking into account (2.6), we can find a sequence $t_k \to -\infty$ such that $U(t_k) \to +\infty$, and for all $k \ge 1$ we have $t_{k+1} + \log 2 < t_k$, $U(t_{k+1}) \ge U(t_k)$, $U'(t_k) = 0$, and $U''(t_k) \le 0$.

Let $M_k = U(t_k)$, $r_k = e^{t_k}$, and $\rho_k = \frac{r_{k+1}}{r_k}$. Define

$$u_k(r) = u(rr_k) - M_k + 2\log r_k, \quad v_k(r) = v(rr_k) + 2\log r_k,$$

where (u, v) is a solution of (2.1). Then

$$-\Delta u_k = 2(N-2)e^{v_k}, \quad -\Delta v_k = 2(N-2)e^{M_k}e^{u_k}$$

in $B_1 \setminus \{0\}$, and satisfy the conditions

$$u_k(1) = 0, \quad u_k(\rho_k) > 0, \quad v_k(1) \ge 0, \quad v_k(\rho_k) > 0. \tag{2.11}$$

The inequalities for v_k are obtained as follows. Since $U'(t_k) = 0$ and $U''(t_k) \le 0$, from the system (2.3) we get $V(t_k) \ge 0$. Then $v_k(1) = V(t_k) \ge 0$ and $v_k(t_{k+1}) = V(T_{k+1}) + 2\log\frac{r_k}{r_{k+1}} > 0$.

Consider the principal Dirichlet eigenvalue λ_k and eigenfunction $\phi_k > 0$ of $-\Delta$ in $B \setminus B_{\rho_k}$, namely,

$$-\Delta\phi_k = \lambda_k\phi_k \quad \text{in} \quad B_1 \setminus B_{\rho_k}$$
$$\phi_k = 0 \quad \text{on} \quad \partial(B_1 \setminus B_{\rho_k}),$$

where $\|\phi_k\|_{L^2} = 1$. Integration by parts, using (2.11), gives

$$\lambda_k \int_{B_1\setminus B_{\rho_k}} u_k\phi_k \ge 2(N-2)\int_{B_1\setminus B_{\rho_k}} e^{v_k}\phi_k$$

$$\lambda_k \int_{B_1\setminus B_{\rho_k}} v_k\phi_k \geq 2(N-2)e^{M_k}\int_{B_1\setminus B_{\rho_k}} e^{u_k}\phi_k.$$

Since u_k and v_k are positive, we have $e^{u_k} \geq u_k$ and $e^{v_k} \geq v_k$, and we conclude that

$$4(N-2)^2 e^{M_k} \leq \lambda_k^2.$$

Note that λ_k is uniformly bounded, since the annulus $B_1 \setminus B_{\rho_k}$ has a width that does not converge to zero; in fact, $0 < \rho_k \leq 1/2$. It follows that M_k remains bounded as $k \to \infty$, which is a contradiction. □

Lemma 2.4. *Suppose U and V solve* (2.3) *and $U \not\equiv 0$* (*equivalently $V \not\equiv 0$*). *If $t_0 < 0$ and $U'(t_0) = 0$, then U is strictly monotone in $(t_0 - \varepsilon, t_0)$ and on $(t_0, t_0 + \varepsilon)$ for some $\varepsilon > 0$.*

Proof. If $V(t_0) \neq 0$ this follows from (2.3) because then $U''(t_0) \neq 0$. Suppose $V(t_0) = 0$. If $V'(t_0) \neq 0$, then V has a sign on $(t_0 - \varepsilon, t_0)$ and on $(t_0, t_0 + \varepsilon)$, and then the conclusion still follows from the formula

$$U'(t) = -2(N-2)\int_{t_0}^{t} e^{(N-2)(s-t)}(e^{V(s)} - 1)\,ds,$$

which is obtained from (2.3) by integration. The same argument applies if $V'(t_0) = 0$ and $V''(t_0) \neq 0$. In the case $V'(t_0) = 0$ and $V''(t_0) = 0$ then by (2.3) $U(t_0) = 0$. But then $U \equiv 0$ and $V \equiv 0$ by uniqueness of the initial-value problem. □

Lemma 2.5. *If*

$$\liminf_{t\to-\infty} U(t) = -\infty,$$

then

$$\lim_{t\to-\infty} U(t) = -\infty.$$

We will see later that in the case $\lim_{t\to-\infty} U(t) = -\infty$, which implies $\lim_{t\to-\infty} V(t) = -\infty$ by (2.5), the original pair (u, v) has a removable singularity at 0.

Proof of Lemma 2.5. For the sake of contradiction, let us assume that

$$\liminf_{t\to-\infty} U(t) = -\infty \quad \text{and} \quad \limsup_{t\to-\infty} U(t) > -\infty. \tag{2.12}$$

We define

$$F(t) = U'(t)V'(t) + 2(N-2)(e^{U(t)} - U(t) + e^{V(t)} - V(t)). \tag{2.13}$$

By a calculation we have

$$F'(t) = -2(N-2)U'(t)V'(t).$$

The idea is that if (2.12) holds then U oscillates more and more as $t \to -\infty$. We will argue that it is possible to find a non-trivial interval $[a,b]$ where U and V are decreasing (hence F is decreasing in this interval) and $F(a)$ is bounded, while $F(b) >> 1$. But this contradicts that F is decreasing in this interval.

We start by fixing a local minimum $t_1 < 0$ of U such that $U(t_1) = -L$ with $L > 0$ large. Note that $V(t_1) < -L < 0$ by (2.5), so U is increasing in a small interval to the left of t_1 (by Lemma 2.4). Let $t_2 < t_1$ be the first local maximum of U. Then by (2.3) and (2.5),

$$0 \le V(t_2) \le U(t_2). \tag{2.14}$$

Note that V is either increasing or decreasing on some interval to the left of t_1 (by Lemma 2.4). If V is increasing in some interval to the left of t_1 we define $t_3 < t_1$ as the first local minimum of V. Otherwise we define $t_3 = t_1$. In any case $t_3 \ge t_2$ by (2.14), $V(t_3) < -L$, and either $U'(t_3) = 0$ or $V'(t_3) = 0$. It follows that $F(t_3) = 2(N-2)(e^{U(t_3)} - U(t_3) + e^{V(t_3)} - V(t_3))$, which is very large, if we take L large. Again V is decreasing to the left of t_3.

Let $t_4 \le t_3$ be the first local maximum of V. If $t_4 \le t_2$ we compare $F(t_2)$ with $F(t_3)$. We note that U and V are non-increasing in $[t_2, t_3]$, and hence F is non-increasing in this interval. Also $F(t_2) = 2(N-2)(e^{U(t_2)} - U(t_2) + e^{V(t_2)} - V(t_2))$ is bounded independently of L, because of (2.14) (the bound depends only on upper bounds on U and V). But $F(t_3)$ is very large, and this contradicts that F is non-increasing in $[t_2, t_3]$.

If $t_4 > t_2$ we compare $F(t_4)$ with $F(t_3)$. Again F is non-increasing in $[t_4, t_3]$ and $F(t_3)$ is large. We claim that $F(t_4)$ is bounded, and to prove this it is sufficient to show that $V(t_4) \ge 0$. First note that $U(t_4) \ge 0$ (because t_4 is a local maximum of V and (2.3)). If $V(t_4) < 0$, since $V(t_2) \ge 0$ and t_4 is a local maximum of V, we see that there is a local minimum s of V with $s \in (t_2, t_4)$ (we get $s < t_4$ from Lemma 2.4). This implies $U(s) \le 0$. But U is non-increasing in $[t_2, t_1]$, and it follows that $U \equiv 0$ in $[s, t_4]$. This leads to $U \equiv 0$ and $V \equiv 0$, a contradiction. □

Lemma 2.6. *If*

$$\liminf_{t \to -\infty} U(t) > -\infty,$$

then

$$\liminf_{t\to-\infty} V(t) > -\infty. \tag{2.15}$$

Proof. 1.- We prove first that

$$\limsup_{t\to-\infty} V(t) > -\infty. \tag{2.16}$$

For the sake of contradiction, assume that $\lim_{t\to-\infty} V(t) = -\infty$. Since $U(t)$ is bounded as $t \to -\infty$, we can find a sequence $t_k \to -\infty$ such that $U'(t_k)$ remains bounded as $k \to \infty$. For any $t_k < t < t$, by integration of (2.3) we get

$$U'(t) = -2(N-2)\int_{t_k}^{t} e^{(N-2)(s-t)}(e^{V(s)}-1)\,ds + e^{(N-2)(t_0-t)}U'(t_k). \tag{2.17}$$

Letting $k \to \infty$, we find for any $t < 0$

$$U'(t) = -2(N-2)\int_{-\infty}^{t} e^{(N-2)(s-t)}(e^{V(s)}-1)\,ds. \tag{2.18}$$

Under the assumption $\lim_{t\to-\infty} V(t) = -\infty$ we deduce that $\lim_{t\to-\infty} U'(t) = 2$, which implies that $\lim_{t\to-\infty} U(t) = -\infty$, a contradiction.

From now on we prove (2.15) by contradiction; that is, we assume

$$\liminf_{t\to-\infty} V(t) = -\infty. \tag{2.19}$$

2.- Now let us show that V' is bounded. Under the assumption (2.19) and knowing (2.16) we can find a sequence $t_k \to -\infty$ as $k \to \infty$ such that $V'(t_k)$ remains bounded. Similarly as (2.17) we have

$$V'(t) = -2(N-2)\int_{t_k}^{t} e^{(N-2)(s-t)}(e^{U(s)}-1)\,ds + e^{(N-2)(t_k-t)}V'(t_k)$$

for $t_k < t < 0$. Letting $k \to \infty$ we obtain

$$V'(t) = -2(N-2)\int_{-\infty}^{t} e^{(N-2)(s-t)}(e^{U(s)}-1)\,ds,$$

which shows that V' is bounded.

3.- From (2.19) we can find $t_0 < 0$ such that $V(t_0) < -M$, $M > 0$ a large constant to be fixed. Since V' is uniformly bounded, we get $V(t) \le -M/2$ on an interval centered at t_0 of length of order M/C, where C is independent of M. Using this information together with (2.18), we see that $U'(t) \ge c > 0$, on an interval $[t_0 - M/C, t_0 + M/C]$. Here $c > 0$ can be chosen independently of M. If M is large enough, this contradicts that U is bounded as $t \to -\infty$. $\square$

Proof of Theorem 1.1. By (2.5) and Lemmas 2.5 and 2.6 we have only two possibilities: either $U(t), V(t) \to -\infty$ as $t \to -\infty$ or $U(t)$ and $V(t)$ remain bounded as $t \to -\infty$.

Let us assume that $U(t), V(t) \to -\infty$ as $t \to -\infty$. We claim that in this case the original u and v satisfying (2.1) have a removable singularity at 0. Indeed, by (2.4) and Lemma 2.3, which ensures that U and V are bounded above, we have

$$|U(t)| + |V(t)| \le C(1 + |t|) \quad \forall t \le 0.$$

This combined with the equations (2.3), the upper bound for U and V obtained in Lemma 2.3 and (2.5), yields that

$$|U'(t)| + |V'(t)| \le C(1 + |t|) \quad \forall t \le 0. \tag{2.20}$$

Integrating (2.3) we find for $s \le t \le 0$

$$e^{(N-2)s}U'(s) = e^{(N-2)t}U'(t) + 2(N-2)\int_s^t e^{(N-2)\tau}(e^{V(\tau)} - 1)\, d\tau.$$

Thanks to (2.20), $e^{(N-2)s}U'(s) \to 0$ as $s \to -\infty$, and we obtain

$$U'(t) = -2(N-2)\int_{-\infty}^t e^{(N-2)(\tau-t)}(e^{V(\tau)} - 1)\, d\tau \tag{2.21}$$

for all $t \le 0$. Using $V(t) \to -\infty$ as $t \to -\infty$ we find

$$\lim_{t\to-\infty} U'(t) = 2.$$

In a similar way, we find

$$\lim_{t\to-\infty} V'(t) = 2.$$

Going back to u and v by the change of variables (2.2), we obtain

$$\lim_{r\to 0} r u_r(r) = \lim_{r\to 0} r v_r(r) = 0.$$

This is enough to show that u and v admit smooth extensions to 0. Indeed, given $\varepsilon \in (0, 1/2)$, let $\delta > 0$ be so that $|ru_r(r)| + |rv_r(r)| \le \varepsilon$ for $r \in (0, \delta]$. Integrating once $-\Delta u = 2(N-2)e^v$ in $[r_0, r] \subset (0, \delta]$ and then letting $r_0 \to 0$, we get

$$u'(r) \ge -Cr^{1-\varepsilon}, \quad \forall 0 < r \le \delta.$$

The same estimate for $v'(r)$ is also valid. Integrating once again we see that u and v are bounded near the origin. Then by standard arguments u and v are smooth up to the origin.

Let us consider now the second case, i.e., $U(t)$ and $V(t)$ remain bounded as $t \to -\infty$. By the same argument as in the previous case we have the estimate (2.20) and also (2.21) and the corresponding one for V'. Since U and V remain bounded as $t \to -\infty$ we also deduce that U', V', U'', and V'' remain bounded as $t \to -\infty$. Let us recall F defined by (2.13) and that $F'(t) = -2(N-2)U'(t)V'(t)$. For $t_0 \le t_1 \le 0$ we then find

$$-2(N-2)\int_{t_0}^{t_1} U'(t)V'(t)\,dt = \int_{t_0}^{t_1} F'(t)\,dt = F(t_1) - F(t_0) = O(1) \quad (2.22)$$

as $t_0 \to -\infty$. Note however that $U'(t)V'(t)$ has no definite sign. Multiplying the equation for U in (2.3) by V and integrating in the interval $[t_0, t_1] \subset (-\infty, 0]$ we get

$$\int_{t_0}^{t_1} [U''V + (N-2)U'V + 2(N-2)(e^V - 1)V]\,dt = 0.$$

But

$$\int_{t_0}^{t_1} U''V\,dt = U'(t_1)V(t_1) - U'(t_0)V(t_0) - \int_{t_0}^{t_1} U'V'\,dt = O(1)$$

as $t_0 \to -\infty$, by (2.22) and since $U, V, U', V' = O(1)$ as $t \to -\infty$. Hence

$$\int_{t_0}^{t_1} U'V\,dt + 2\int_{t_0}^{t_1} (e^V - 1)V\,dt = O(1) \quad \text{as } t_0 \to -\infty. \quad (2.23)$$

In a similar way we can derive

$$\int_{t_0}^{t_1} UV'\,dt + 2\int_{t_0}^{t_1} (e^U - 1)U\,dt = O(1) \quad \text{as } t_0 \to -\infty, \quad (2.24)$$

and adding (2.23) and (2.24) we get

$$\int_{t_0}^{t_1} (e^U - 1)U + (e^V - 1)V\,dt = O(1) \quad \text{as } t_0 \to -\infty$$

since

$$\int_{t_0}^{t_1} (UV' + U'V)\,dt = O(1).$$

Since the integrand has a sign we may write

$$\int_{-\infty}^{0} (e^U - 1)U + (e^V - 1)V\,dt < \infty. \quad (2.25)$$

Since $U(t)$ and $V(t)$ are bounded as $t \to -\infty$, there is some uniform $\delta > 0$ so that $(e^{U(t)} - 1)U(t) \ge \delta U(t)^2$ for all $t \le 0$ and similarly for V. We deduce from (2.25) that

$$\int_{-\infty}^{0} U^2 + V^2 \, dt < \infty. \tag{2.26}$$

Observe that for $[t_0, t_1] \subset (-\infty, 0]$

$$\int_{t_0}^{t_1} U'' U \, dt = -\int_{t_0}^{t_1} (U')^2 \, dt + O(1) \tag{2.27}$$

$$\int_{t_0}^{t_1} U' U \, dt = \frac{1}{2}(U(t_1)^2 - U(t_0)^2) = O(1) \quad \text{as } t_0 \to -\infty, \tag{2.28}$$

and

$$\int_{-\infty}^{0} (e^V - 1) U \, dt \leq C \int_{-\infty}^{0} |VU| \, dt < \infty, \tag{2.29}$$

for some $C > 0$ since V remains uniformly bounded, and where the last statement follows from (2.26). Multiplying the equation for U in (2.3) by U and integrating on $[t_0, t_1] \subset (-\infty, 0]$, we obtain, using (2.27), (2.28), and (2.29),

$$\int_{-\infty}^{0} (U')^2 \, dt < \infty. \tag{2.30}$$

A similar calculation yields

$$\int_{-\infty}^{0} (V')^2 \, dt < \infty.$$

Using (2.3) and the L^2 estimates for U, U', V, and V' we also obtain

$$\int_{-\infty}^{0} ((U'')^2 + (V'')^2) \, dt < \infty. \tag{2.31}$$

Let us show now that $U'(t) \to 0$ as $t \to -\infty$. Indeed, thanks to (2.30) there is a decreasing sequence $t_n \to -\infty$ as $n \to \infty$ such that $t_n - t_{n+1} \to 0$ and $U'(t_n) \to 0$ as $n \to \infty$. For any $t \in [t_{n+1}, t_n]$ we have

$$|U'(t)| = |U'(t_{n+1}) + \int_{t_{n+1}}^{t} U''| \leq |U'(t_{n+1})| + C(t_n - t_{n+1})^{1/2}$$

by (2.31), and this shows $U'(t) \to 0$ as $t \to -\infty$. A similar argument applies to V'. Since $U'(t), V'(t) \to 0$ as $t \to -\infty$ and $U(t)$ and $V(t)$ remain bounded, by applying standard interpolation inequalities to equations obtained from (2.3) by differentiation, we obtain that $U^{(k)}(t), V^{(k)}(t) \to 0$ as $t \to -\infty$, for any integer $k \geq 1$. Then the equations (2.3) also yield $U(t), V(t) \to 0$ as $t \to -\infty$. Using the definition (2.2) we obtain the desired behavior (1.4). □

3. The dynamical system

We assume throughout that $N \geq 3$. If u, v is a radial solution of

$$\begin{cases} -\Delta u = \lambda e^v & \text{in } B_R \subset \mathbb{R}^N \\ -\Delta v = \mu e^u & \text{in } B_R \end{cases}$$

the functions

$$w_1 = \mu e^{2t+u}, \quad w_2 = r u_r, \quad w_3 = \lambda e^{2t+v}, \quad w_4 = r v_r, \quad r = e^t, \tag{3.1}$$

satisfy

$$\begin{cases} w_1' = w_1(2 + w_2), & w_2' = -w_3 - (N-2)w_2 \\ w_3' = w_3(2 + w_4), & w_4' = -w_1 - (N-2)w_4 \end{cases} \tag{3.2}$$

for $t \in (-\infty, \log(R))$.

To study radial solutions of (1.1) it is convenient to consider the initial-value problem

$$\begin{cases} -\Delta u = e^v, \qquad -\Delta v = e^u \quad \text{in } \mathbb{R}^N \\ u(0) = \alpha, \quad v(0) = -\alpha, \quad u'(0) = v'(0) = 0, \end{cases} \tag{3.3}$$

where $\alpha \in \mathbb{R}$ is a parameter. We write as $u_\alpha(r), v_\alpha(r)$ the unique radial solution to this problem. This solution is defined on a maximal interval which turns out to be $[0, \infty)$, because u_α and v_α are decreasing, and hence one can replace the nonlinearity e^s by a globally Lipschitz one that the coincides with e^s for $s \leq |\alpha|$. We shall write $w_i(t; \alpha)$, $i = 1, \ldots, 4$, the functions obtained applying the transformations (3.1) with $\lambda = \mu = 1$ to u_α and v_α. They are solutions of (3.2). In the case $\alpha = 0$ we have that $u_0 = v_0$ is the radial solution of the scalar equation

$$-\Delta u_0 = e^{u_0} \quad \text{in } \mathbb{R}^N, \quad u_0(0) = 0, \tag{3.4}$$

and it is known that it has the behavior

$$u_0(r) = -2\log(r) + \log(2(N-2)) + o(1) \quad \text{as } r \to \infty. \tag{3.5}$$

The only stationary points of the system (3.2) are

$$\begin{cases} P_1 = (0,0,0,0) \\ P_2 = (2(N-2), -2, 2(N-2), -2)\,. \end{cases} \tag{3.6}$$

A smooth radial solution of (1.1) or (3.3) produces an orbit that emanates from P_1; in other words, the orbits $(w_1(\cdot\,;\alpha),\dots,w_4(\cdot\,;\alpha))$ are contained in $W^u(P_1)$. They do not exhaust $W^u(P_1)$, however, because $w_1, w_3 > 0$ and $w_2, w_4 < 0$. The boundary conditions in (1.1) imply that a radial solution of this system will also cross the hyperplanes $w_3 = \lambda$ and $w_1 = \mu$.

The usefulness of the solutions u_α and v_α of (3.3) and the associated functions $w_i(t;\alpha)$ is that the curves $(w_1(t;\alpha), w_3(t;\alpha))$, $t \in \mathbb{R}$, describe points (μ,λ) for which the original system (1.1) has a classical radial solution. Thus the region of existence

$$\mathcal{U} = \{(\mu,\lambda) \in (0,\infty)^2 : \text{system (1.1) has a classical soluion}\}$$

is precisely $\{(w_1(t;\alpha), w_3(t;\alpha)) : t \in \mathbb{R},\ \alpha \in \mathbb{R}\}$.

Remark 3.1. In Figures 1 and 2 we have plotted the components w_1 (horizontal axis) and w_3 (vertical axis) of the transformation (3.1) obtained from the numerical solution of (3.3) for different values of $\alpha \in \mathbb{R}$. This gives an idea of the region of existence $\mathcal{U}$.

The linearization of (3.2) around the point P_1 is given by $Z' = \bar{M}Z$, where

$$\bar{M} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -(N-2) & -1 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & -(N-2) \end{bmatrix}.$$

The eigenvalues of this matrix are $-(N-2)$ and 2, with multiplicity two. Then P_1 is hyperbolic, has 2-dimensional unstable manifold $W^u(P_1)$, and a 2-dimensional stable manifold $W^s(P_1)$.

The linearization of (3.2) around P_2 is given by $Z' = MZ$, where

$$M = \begin{bmatrix} 0 & 2(N-2) & 0 & 0 \\ 0 & -(N-2) & -1 & 0 \\ 0 & 0 & 0 & 2(N-2) \\ -1 & 0 & 0 & -(N-2) \end{bmatrix}. \tag{3.7}$$

The eigenvalues of M are given by

$$\begin{cases} \nu_1 = \frac{1}{2}\left(2 - N + \sqrt{(N+6)(N-2)}\right) \\ \nu_2 = \frac{1}{2}\left(2 - N - \sqrt{(N+6)(N-2)}\right) \\ \nu_3 = \frac{1}{2}\left(2 - N + \sqrt{(N-10)(N-2)}\right) \\ \nu_4 = \frac{1}{2}\left(2 - N - \sqrt{(N-10)(N-2)}\right). \end{cases} \tag{3.8}$$

Note that for $N \geq 3$ we have $\nu_2 < 0 < \nu_1$. If $3 \leq N \leq 9$, then ν_3 and ν_4 are complex conjugate with nonzero imaginary part and negative real part. More precisely, we have

$$\nu_2 < \mathrm{Re}(\nu_4) = \mathrm{Re}(\nu_3) < 0 < \nu_1.$$

If $N \geq 11$,

$$\nu_2 < \nu_4 < \nu_3 < 0 < \nu_1,$$

and if $N = 10$,

$$\nu_2 < \nu_4 = \nu_3 < 0 < \nu_1.$$

Concerning the eigenvectors of M we have the following:

Lemma 3.2. *The vector*

$$v^{(k)} = [4(N-2)^2,\, 2(N-2)\nu_k,\, -2(N-2)(\nu_k + N - 2)\nu_k,\, -(\nu_k + N - 2)\nu_k^2] \tag{3.9}$$

is the eigenvector of M associated to ν_k, $k = 1, \dots, 4$. We have that $v^{(1)}$ and $v^{(2)}$ are always real, and $v^{(3)}$ and $v^{(4)}$ are complex conjugate if $3 \leq N \leq 9$. Let us write $v^{(i)} = (v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, v_4^{(i)})$, $i = 1, \dots, 4$; then

$$v_1^{(1)} > 0, \quad v_2^{(1)} > 0, \quad v_3^{(1)} < 0, \quad v_4^{(1)} < 0, \tag{3.10}$$

and

$$v_1^{(2)} > 0, \quad v_2^{(2)} < 0, \quad v_3^{(2)} < 0, \quad v_4^{(2)} > 0.$$

Proof. Use that $\nu_2 + \nu_1 = 2 - N$. □

Proposition 3.3. *There exists a heteroclinic orbit connecting P_1 and P_2.*

The proof is to consider the solution of (3.3) with $\alpha = 0$, in which case $u_0 = v_0$ and the system (1.1) reduces to the equation (3.4). This solution is studied in [19], and provides the desired heteroclinic orbit.

4. Curve of singular solutions

Let P_1 and P_2 be the stationary points of the system (3.2) defined in (3.6). Then P_1 has a 2-dimensional unstable manifold $W^u(P_1)$, while P_2 has a 1-dimensional unstable manifold $W^u(P_2)$ and a 3-dimensional stable manifold $W^s(P_2)$.

Lemma 4.1. *Let $V = (w_1, \dots, w_4) : (-\infty, T) \to \mathbb{R}^4$ be the trajectory in $W^u(P_2)$ such that $\langle V'(t), v^{(1)} \rangle > 0$ for t near $-\infty$, where T is the maximal time of existence. Then*

$$w_1' > 0, \quad w_2' > 0, \quad w_3' < 0, \quad w_4' < 0 \quad \text{for all } t < T. \tag{4.1}$$

Proof. By (3.10) and the hypothesis $\langle V'(t), v^{(1)}\rangle > 0$ for $t \to -\infty$, we have

$$w_1'(t) > 0, \quad w_2'(t) > 0, \quad w_3'(t) < 0, \quad w_4'(t) < 0$$

for t near $-\infty$. For the sake of contradiction, suppose that $w_1' = 0$ at $t = t_0$; then $w_2 = -2$ and $w_2' \le 0$ at $t = t_0$. This implies that $w_3 \ge 2(N-2)$ at $t = t_0$. Consequently there exists $t_1 < t_0$ such that $w_3' = 0$ and $w_3 < 2(N-2)$, and so $w_4 = -2$ and $w_4' \ge 0$ at $t = t_1$. Then $w_1 \le 0$, but $w_1 > 0$ at $t = t_1$. Then

$$w_1' > 0 \quad \text{for all } t < T.$$

Next let us see that $w_4' > 0$ for all $t < T$. If not, there is a first t_1 such that $w_4'(t_1) = 0$. Then $w_4''(t_1) \ge 0$. But from $w_4'' = -w_1' - (N-2)w_4'$, we see that $w_4''(t_1) < 0$, a contradiction. Then $w_4' < 0$ for all $t < T$.

Similarly we have $w_3' < 0$ and $w_2' > 0$. □

Lemma 4.2. *Let $V = (w_1, \dots, w_4) : (-\infty, T) \to \mathbb{R}^4$ be the trajectory in $W^u(P_2)$ such that $\langle V'(t), v^{(1)}\rangle > 0$ for t near $-\infty$, where T is the maximal time of existence. Then $T = \infty$ and*

$$\begin{aligned} &w_1(t) \to \infty, \quad w_3(t) \to 0, \quad w_2(t) \to 0, \quad w_4(t) \to -\infty \\ &\frac{w_4(t)}{w_1(t)} \to -\frac{1}{N} \quad \text{as } t \to \infty. \end{aligned} \tag{4.2}$$

Proof. We first observe that w_1 and w_3 remain always positive, since this is true for $t \to -\infty$ and if one of them vanished for some time, it would be identically zero.

Let us show that $T = \infty$. Indeed, assume the maximal time of existence T is finite. Then from the equation for w_2' in (3.2), $w_2(t) \le e^{(N-2)(t_0 - t)} w_2(t_0)$ for any $t_0, t < T$. Fixing t_0 this gives an upper bound for w_2 as $t \uparrow T$. It follows that also w_1 has an upper bound as $t \uparrow T$. The same argument shows that w_4 is bounded as $t \uparrow T$. Next, since w_4 is decreasing and equal to -2 at $t = -\infty$, we get $w_4 + 2 < 0$ for all t. Then the equation for w_3 implies that w_3 remains bounded as $t \uparrow T$. Therefore all components remain bounded as $t \uparrow T$, which contradicts the maximality of T.

That $w_1 \to \infty$ follows from the system equation for w_1' in (3.2), since fixing any $t_0 \in \mathbb{R}$, we have $w_2(t) + 2 \ge w_2(t_0) + 2 > 0$ for all $t \ge t_0$ by Lemma 4.1, and then $w_1'(t) \ge (w_2(t_0) + 2) w_1(t)$ for all $t \ge t_0$.

Next let us see that $w_3(t) \to 0$ as $t \to \infty$. Otherwise, since w_3 is positive and decreasing, we would have $w_3(t) \to \bar{w}_3 > 0$ as $t \to \infty$. Then the equation for w_2' in (3.2) would imply that $w_2(t) \to -\infty$ as $t \to \infty$. This is not possible because w_2 is increasing by Lemma 4.1. Using that $w_3(t) \to 0$ as $t \to \infty$

and the second equation in (3.2), we can deduce that $w_2(t) \to 0$ as $t \to \infty$. Similarly, using the fourth equation and $w_1(t) \to \infty$ as $t \to \infty$ we can obtain that $w_4(t) \to -\infty$ as $t \to \infty$.

L'Hopital's rule gives for

$$L = \lim_{t\to\infty} \frac{w_4(t)}{w_1(t)}$$

the equation $L = -1/2 - L(N-2)/2$, and we obtain (4.2). □

Proof of Theorem 1.2. Consider the trajectory $V = (w_1, \dots, w_4) : (-\infty, \infty) \to \mathbb{R}^4$ in $W^u(P_2)$ such that $\langle V'(t), v^{(1)} \rangle > 0$ for t near $-\infty$, where $v^{(1)}$ is given in Lemma 3.9. By Lemmas 4.1 and 4.2 we can define w_3 as a function of w_1:

$$w_3 = \bar{h}(w_1)$$

for $w_1 \in [2(N-2), \infty)$. This function is smooth monotone decreasing, and $\bar{h}(w_1) \to 0$ as $w_1 \to \infty$. By symmetry we define

$$\bar{h}(w_1) = \bar{h}^{-1}(w_1),$$

where $\bar{h}^{-1}$ is the inverse of $\bar{h}$.

We see that for $\lambda = \bar{h}(\mu)$ there exists a radial singular solution of (1.1). On the other hand, suppose that (u, v) is a radial singular solution associated to parameters (μ, λ). We can assume that $\mu \geq \lambda$ by symmetry. Then by Theorem 1.1, after the change of variables (3.1) we have that $(w_1, \dots, w_4) \to P_2$ as $t \to -\infty$ (this is contained in the proof of Theorem 1.1). Since the unstable manifold of P_2 is one-dimensional, the trajectory $(w_1, \dots, w_4)$ is unique and $\lambda = \bar{h}(\mu)$. This shows that on $\mathcal{S} = \{(\mu, \bar{h}(\mu) : \mu \in (0, \infty)\}$ we find singular solutions and that the singular solution is unique. □

5. Multiplicity in dimensions $3 \leq N \leq 9$

Let $V_0 : \mathbb{R} \to \mathbb{R}^4$ be the heteroclinic connection from P_1 to P_2 of Proposition 3.3 and $\hat{V}_0 = V_0(-\infty, \infty)$. Then $\hat{V}_0$ is contained in both $W^u(P_1)$ and $W^s(P_2)$.

Lemma 5.1. *Assume $N \geq 3$. $W^u(P_1)$ and $W^s(P_2)$ intersect transversally on points of $\hat{V}_0$. More precisely, for points $Q \in \hat{V}_0$ sufficiently close to P_2 there are directions in the tangent plane to $W^u(P_1)$ which are almost parallel to $v^{(1)}$, the tangent vector to $W^u(P_2)$ at P_2.*

Proof. Let u_α, v_α be the solution of (3.3) with $\alpha > 0$, and let $W = (w_1, \dots, w_4)$ be defined by (3.1) with $\lambda = \mu = 1$. Then, from the system we get $v_\alpha' < v_0'$. Integrating,

$$v_\alpha(r) \le -\alpha + v_0(r).$$

Then

$$-\Delta(u_\alpha - u_0) = e^{v_\alpha} - e^{v_0} < e^{v_0}(e^{-\alpha} - 1).$$

By the asymptotic formula (3.5), $e^{v_0(r)} \sim r^{-2}$ as $r \to \infty$, and therefore, integrating we get

$$u_\alpha'(r) - u_0'(r) > (1 - e^{-\alpha}) r^{-1}$$

for all $r \ge 1$. Therefore

$$w_2(r, \alpha) - w_2(r, 0) \ge c\alpha$$

for some $c > 0$. We deduce that

$$\frac{\partial w_2}{\partial \alpha}(r, 0) \ge c > 0 \tag{5.1}$$

for all $r > 0$ large. Let $Z = \frac{\partial W}{\partial \alpha}|_{\alpha=0}$. Then $Z = (z_1, \dots, z_4)$ satisfies

$$Z' = (M + R(t))Z,$$

where M is the matrix defined in (3.7) and

$$R(t) = \begin{bmatrix} (2 + w_2) & (w_1 - 2(N-2)) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (2 + w_4) & (w_3 - 2(N-2)) \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Recall that $V(t) \to P_2$ as $t \to \infty$. Moreover, the convergence is exponential; that is, there are $C, \sigma > 0$ such that $|V(t) - P_2| \le Ce^{-\sigma t}$ for all $t \ge 0$. This follows from the Hartman–Grobman theorem (see Theorem 7.1 in [18] or Theorem 1.1.3 in [17]), which shows that the system (3.2) is C^0-conjugate to its linearization near P_2. Recall that the eigenvalues of M are $\nu_1 > 0 > \nu_2$ and ν_3 and ν_4, which have negative real part and nonzero imaginary part. Let $v^{(i)} \in \mathbb{C}^4$ denote an eigenvector associated to ν_i. By Theorem 8.1 in [6, Chapter 3] there are solutions φ_k to

$$\varphi_k' = (M + R(t))\varphi_k, \quad t > 0$$

such that $\lim_{t\to\infty} \varphi_k(t) e^{-\nu_k t} = v^{(k)}$. It follows from this that $Z = \sum_{i=1}^4 c_i \varphi_i$ for some constants $c_1, \dots, c_4 \in \mathbb{C}$. The condition (5.1) imply that $|z_2(t)| \ge c$

for some $c > 0$ and all $t \geq 0$, so $|Z(t)| \geq c$ for t large. Since $\nu_1 > 0$ and ν_2, ν_3, and ν_4 have negative real part, we conclude that $c_1 \neq 0$ and

$$Z(t) = c_1 v^{(1)} e^{\nu_1 t} + o(e^{\nu_1 t}) \quad \text{ as } t \to \infty.$$

Since $v^{(1)}$ is the tangent vector to $W^u(P_2)$, we have that $\frac{\partial W}{\partial \alpha}|_{\alpha=0}$ is not tangent to $W^s(P_2)$ for t large. On the other hand, $\frac{\partial W}{\partial \alpha}|_{\alpha=0}$ is tangent to $W^u(P_1)$ by construction. This shows that $W^s(P_2)$ and $W^u(P_1)$ intersect transversally on points of $\hat{V}_0$ close to P_2. By the invertibility of the flow away from the stationary points, $W^s(P_2)$ and $W^u(P_1)$ intersect transversally on all points of $\hat{V}_0$ □

Let $v^{(j)}$ denote the eigenvectors of the linearization of (3.2) at P_2 with corresponding eigenvalue ν_j, given explicitly in (3.9). Then $W^u(P_2)$ is one-dimensional and tangent to $v^{(1)}$ at P_2. Hence, if $V = (v_1, \ldots, v_4) : (-\infty, T) \to \mathbb{R}^4$ is any trajectory in $W^u(P_2)$ there are 2 cases: $\langle V'(t), v^{(1)} \rangle < 0$ or $\langle V'(t), v^{(1)} \rangle > 0$ for t near $-\infty$.

Lemma 5.2. *The system* (3.2) *is C^1-conjugate to its linearization around P_2 in a neighborhood of this point.*

Proof. This follows from a result of Belickiĭ (see [2] or [26, p. 25]), which says that the system (3.2) is C^1-conjugate to its linearization around the point P_2 under the non-resonance condition

$$\text{Re}(\nu_i) \neq \text{Re}(\nu_j) + \text{Re}(\nu_k) \quad \text{when } \text{Re}(\nu_j) < 0 < \text{Re}(\nu_k),$$

where $\nu_1, \ldots, \nu_4$ are the eigenvalues of M defined in (3.7). For $3 \leq N \leq 9$ we have

$$\nu_2 < \text{Re}(\nu_4) = \text{Re}(\nu_3) = \frac{2-N}{2} < 0 < \nu_1.$$

Considering the pair $\nu_2 < 0 < \nu_1$ we see that $\text{Re}(\nu_2) + \text{Re}(\nu_1) = 2 - N$, which is different from $\text{Re}(\nu_3)$ and $\text{Re}(\nu_4)$. The only case left is $\text{Re}(\nu_3) < 0 < \nu_1$, and we need to verify that

$$\text{Re}(\nu_3) + \nu_1 \neq \nu_2, \quad \text{Re}(\nu_3) + \nu_1 \neq \text{Re}(\nu_4).$$

Both relations hold for all integer $N \geq 3$. □

Proof of Theorem 1.3. We will write generic points in the phase space $\mathbb{R}^4$ as (w_1, w_2, w_3, w_4). Let $\{e_j : j = 1, \ldots, 4\}$ denote the canonical basis of $\mathbb{R}^4$.

For $\mu \geq 2(N-2)$, by Lemmas 4.1 and 4.2, $W^u(P_2) \cap \{w_1 = \mu\}$ is a single point, which we call $P^*(\mu) = (P_1^*(\mu), P_2^*(\mu), P_3^*(\mu), P_4^*(\mu))$. Note that $\bar{h}(\mu) = P_3^*(\mu)$.

For $\alpha \in \mathbb{R}$, let u_α, v_α be the solution of (3.3) and let $W(t;\alpha) = (w_1, \dots, w_4)$ be defined by (3.1) with $\lambda = \mu = 1$. Define

$$\widetilde{W}^u(P_1) = \{W(t;\alpha) : \alpha \in \mathbb{R}, t \in \mathbb{R}\},$$

which is the part of $W^u(P_1)$ giving rise to smooth solutions of (1.1). Let $\mathcal{E} = \widetilde{W}^u(P_1) \cap \{w_1 = \mu\}$. We will prove Theorem 1.3 by showing that $\mathcal{E}$ contains a curve $\mathcal{S}$ which spirals around $P^*(\mu)$. By this we mean that there exist linearly independent vectors $S_1, S_2 \in \mathbb{R}^4$ and numbers $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $\mathcal{S}$ can be parametrized by

$$t \in [0,\infty) \mapsto P^*(\mu) + e^{-\alpha t}\cos(\beta t) S_1 + e^{-\alpha t}\sin(\beta t) S_2 + o(e^{-\alpha t}) \qquad (5.2)$$

as $t \to \infty$. Actually we will obtain $\alpha = -\operatorname{Re}(\nu_3) = \frac{N-2}{2}$ and $\beta = \operatorname{Im}(\nu_3)$, with ν_3 given in (3.8). In this setting we define the tangent plane to $\mathcal{S}$ at $P^*(\mu)$ as the plane generated by S_1 and S_2. An important property that we will prove later is that this tangent plane is transversal to the plane $\{w_3 = 0\}$.

Let us proceed with the construction of $\mathcal{S}$. Let X_t denote the flow generated by (3.2). Let M_D be the matrix

$$M_D = \begin{bmatrix} \nu_1 & 0 & 0 & 0 \\ 0 & \nu_2 & 0 & 0 \\ 0 & 0 & \operatorname{Re}(\nu_3) & -\operatorname{Im}(\nu_3) \\ 0 & 0 & \operatorname{Im}(\nu_3) & \operatorname{Re}(\nu_3) \end{bmatrix}. \qquad (5.3)$$

By Lemma 5.2 there is an open neighborhood N_{P_2} of P_2 and a C^1 diffeomorphism $H : N_{P_2} \to N_0$, where N_0 is an open neighborhood of 0, such that $H \circ X_t \circ H^{-1} = L_t$, where $L_t = e^{M_D t}$ is the flow generated by M_D, and the formula holds in some neighborhood of the origin.

Let

$$D = \{w = (w_1, \dots, w_4) : w_1 = \mu,\ |w - P^*(\mu)| < 1\}.$$

Then by Lemma 4.1 D is a 3-dimensional disk transversal to $W^u(P_2)$. Next we apply the λ-lemma of Palis [25], which says that there is an open neighborhood B^s of P_2 relative to $W^s(P_2)$ and an open neighborhood $\mathcal{N}$ of P_2, both of them contained in N_{P_2}, such that given $\varepsilon > 0$, the connected component of $X_{-t_0}(D) \cap \mathcal{N}$ that contains $X_t(P^*(\mu))$ is ε C^1-close to B^s if $t_0 > 0$ is sufficiently large. Let us write $\mathcal{M}$ for the connected component of $X_{-t_0}(D) \cap \mathcal{N}$ that contains $X_{-t_0}(P^*(\mu))$.

Choose some point $Q \in \hat{V}_0$ such that $Q \in N_{P_2}$. By Lemma (5.1) we may choose a C^1 curve contained in $\widetilde{W}^u(P_1)$, say $\Gamma = \{\gamma(s) : |s| < \delta\}$ with $\gamma : (-\delta, \delta) \to \mathbb{R}^4$ a C^1 function such that $\gamma(0) = Q$ and $\gamma'(0)$ not tangent to

$W^s(P_2)$ at Q. This curve can be taken to be of the form $\gamma(s) = W(t_1, s)$, where $W(t, \alpha) = (w_1, \dots, w_4)$ is defined by (3.1) with $\lambda = \mu = 1$ starting with u_α, v_α the solution of (3.3) with $\alpha \in \mathbb{R}$. We take t_1 large so that $\gamma(0) = W(t_1, 0)$ meets the requirements of being close to P_0 and $\gamma'(0)$ very close to the tangent to $W^u(P_2)$. We can assume also that this curve is contained in N_{P_2}. Choosing ε small we can assume that Γ intersects $\mathcal{M}$.

To describe the structure of $X_t(\Gamma) \cap \mathcal{M}$, thanks to the conjugation H, we assume that P_2 is at the origin and that near the origin the flow is given by $L_t = e^{M_D t}$ given in (5.3). In particular, after this change of variables, the local unstable manifold of P_2 is contained in the axis $e_1 = (1, 0, 0, 0)$ and the local stable manifold is contained in the space $\{(y_1, \dots, y_4) : y_1 = 0\}$. We may further assume that $B^s = \{(y_1, \dots, y_4) : y_1 = 0,\ |y| < \delta\}$ for some $\delta > 0$ and that the heteroclinic orbit V_0 near the origin in the new variables is given by

$$V_0(t) = (0, c_2 e^{\nu_2 t},\ e^{\nu_3 t}(c_3 + i c_4)), \quad t \geq 0 \tag{5.4}$$

for some constants $c_2, c_3, c_4 \in \mathbb{R}$, where in the last two components we are using complex notation. Note that the curve V_0 cannot have a tangent vector that becomes parallel to $e_2 = (0, 1, 0, 0)$ as $t \to \infty$, that is, $c_3 \neq 0$ or $c_4 \neq 0$ (recall that $\nu_2 < \mathrm{Re}(\nu_3) < 0$ by (3.8)). By choosing ε small, we can assume that the normal vector to $\mathcal{M}$ near P_2 is almost parallel to $e_1 = (1, 0, 0, 0)$. Thus by passing to a subset of $\mathcal{M}$ we may assume that $\mathcal{M}$ is a C^1 graph over the variables (y_2, y_3, y_4); that is, there exists a C^1 function $\psi : \{y' = (y_2, y_3, y_4) \in \mathbb{R}^3,\ |y'| < \delta\} \to \mathbb{R}$ with $\psi(0) > 0$ such that

$$\mathcal{M} = \{(\psi(y'), y') : y' \in \mathbb{R}^3,\ |y'| < \delta\}.$$

By Lemma 5.1 the tangent plane to $W^u(P_1)$ at points close to the origin (i.e., P_2 after the change of variables) contains vectors almost parallel to $e_1 = (1, 0, 0, 0)$, and hence $\gamma_1'(0) \neq 0$. We may assume that $\gamma_1'(0) > 0$. We claim that for all $t > 0$ large there is a unique small s such that $L_t(\gamma(s)) \in \mathcal{M}$. Indeed, this condition is equivalent to

$$e^{\nu_1 t} \gamma_1(s) = \psi(e^{\nu_2 t} \gamma_2(s), e^{\nu_3 t}(\gamma_3(s) + i \gamma_4(s))).$$

Write $\tau = 1/t > 0$ and

$$F(\tau, s) = \gamma_1(s) - e^{-\nu_1 t} \psi(e^{\nu_2 t} \gamma_2(s), e^{\nu_3 t}(\gamma_3(s) + i \gamma_4(s))). \tag{5.5}$$

Then $F(\tau, s)$ is well defined in C^1 in a set of the form $(0, \delta_0) \times (-\delta_0, \delta_0)$ for some $\delta_0 > 0$, and one can verify that it admits a C^1 extension to $\tau = 0$ with

$$F(0, s) = \gamma_1(s), \quad \frac{\partial F}{\partial s}(0, s) = \gamma_1'(s), \quad \frac{\partial F}{\partial \tau}(0, s) = 0.$$

Since $F(0,0)=0$ and $\frac{\partial F}{\partial s}(0,0)=\gamma_1'(0)\neq 0$, by the implicit function theorem, given $\tau>0$ small we can find a unique small s such that $F(\tau,s)=0$. This defines a function $s=s(t)$ defined for $t>0$ large such that $L_t(\gamma(s(t)))\in\mathcal{M}$. Moreover, from (5.5) we get

$$\gamma_1'(0)s+o(s)=e^{-\nu_1 t}(\psi(0)+O(e^{-\operatorname{Re}(\nu_3)t})),$$

and hence we find the expansion

$$s(t)=\frac{e^{-\nu_1 t}\psi(0)}{\gamma_1'(0)}(1+O(e^{-\operatorname{Re}(\nu_3)t}))\quad \text{as } t\to\infty.$$

The point of intersection $L_t(\gamma(s(t)))$ can be written then in the form

$$\begin{aligned}L_t(\gamma(s(t)))=(\psi(0),0,0,0)&+e^{\operatorname{Re}(\nu_3)t}\cos(\operatorname{Im}(\nu_3)t)\tilde S_1\\&+e^{\operatorname{Re}(\nu_3)t}\sin(\operatorname{Im}(\nu_3)t)\tilde S_2+o(e^{\operatorname{Re}(\nu_3)t})\quad\text{as } t\to\infty,\end{aligned}$$

where

$$\tilde S_1=(a\gamma_3(0)+b\gamma_4(0),0,\gamma_3(0),\gamma_4(0)) \tag{5.6}$$

$$\tilde S_2=(-a\gamma_4(0)+b\gamma_3(0),0,\gamma_3(0),\gamma_4(0)) \tag{5.7}$$

$$a=\frac{\partial\psi}{\partial y_3}(0),\quad b=\frac{\partial\psi}{\partial y_4}(0). \tag{5.8}$$

Thus the curve $\{L_t(\gamma(s(t))),\ t>0 \text{ large}\}$ defines a spiral contained in $\mathcal{M}$. Applying the conjugation H^{-1} and the flow X_{t_0} we see that

$$\mathcal{S}=\{X_{t+t_0}(\gamma(s(t))):t\geq t_1\}$$

with $t_1>0$ large has the structure of a spiral (5.2) with $\alpha=-\operatorname{Re}(\nu_3)=\frac{N-2}{2}$ and $\beta=\operatorname{Im}(\nu_3)$. By construction $\mathcal{S}$ is contained in $\mathcal{E}=\widetilde{W}^u(P_1)\cap\{w_1=\mu\}$.

We now prove the following statement:

$$\text{the tangent plane to } \mathcal{S} \text{ at } P^*(\mu) \text{ is transversal to the plane } \{w_3=0\}. \tag{5.9}$$

Recall that by definition this plane is the one generated by S_1 and S_2 appearing in (5.2). Since $\mathcal{S}$ is contained in $\{w_1=\mu\}$, it is sufficient to show that inside the space $\{w_1=\mu\}$ the plane generated by e_2 and e_4 is transversal to the tangent plane to $\mathcal{S}$ at $P^*(\mu)$. Let $V=(w_1,\dots,w_4):(-\infty,\infty)\to\mathbb{R}^4$ denote the trajectory in $W^u(P_2)$ such that $\langle V'(t),v^{(1)}\rangle>0$ for t near $-\infty$, where $v^{(1)}$ is given in Lemma 3.9. To prove our claim we need to transport the plane generated by e_2 and e_4 back along V, and this is accomplished by solving the linearized equation around V. More precisely, let

$Z, \tilde{Z} : (-\infty, 0] \to \mathbb{R}^4$ be solutions to the linearization of (3.2) around V; that is, $Z = (z_1, z_2, z_3, z_4)$ satisfies for $t < 0$

$$\begin{cases} z_1' = (2 + w_2)z_1 + w_1 z_2, & z_2' = -(N-2)z_2 - z_3, \\ z_3' = (2 + w_4)z_3 + w_3 z_4, & z_4' = -(N-2)z_4 - z_1, \end{cases} \tag{5.10}$$

and similarly for $\tilde{Z} = (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4)$. As final conditions we take $Z(0) = e_2$ and $\tilde{Z}(0) = e_4$.

By Theorem 8.1 in [6, Chapter 3] there are solutions $\psi_k : (-\infty, 0] \to \mathbb{C}^4$ to (5.10) such that

$$\lim_{t \to -\infty} \psi_k(t) e^{-\nu_k t} = v^{(k)}, \tag{5.11}$$

where $v^{(1)}, \ldots, v^{(4)}$ are the eigenvectors of M. Recall that $v^{(1)}$ and $v^{(2)}$ are real, and $v^{(3)}$ and $v^{(4)}$ are complex conjugate. Thus one can assume that ψ_1 and ψ_2 are real, and ψ_3 and ψ_4 are complex conjugate. Let

$$\begin{cases} \varphi_i = \psi_i, \quad i = 1, 2 \\ \varphi_3 = \mathrm{Re}(\psi_3), \quad \varphi_4 = \mathrm{Im}(\psi_3), \end{cases} \tag{5.12}$$

so that now φ_i, $i = 1, \ldots, 4$ is a fundamental system of real-valued solutions of (5.10). Then we can write

$$Z(t) = \sum_{i=1}^{4} c_i \varphi_i(t), \quad \text{and} \quad \tilde{Z}(t) = \sum_{i=1}^{4} \tilde{c}_i \varphi_i(t)$$

for some constants $c_1, \ldots, c_4, \tilde{c}_1, \ldots, \tilde{c}_4 \in \mathbb{R}$. We remark that V' is a solution of (5.10), and therefore it can be written as a linear combination of the φ_i. But $V'(t) \to 0$ as $t \to -\infty$, and since the only function of the φ_i that tends to 0 as $t \to -\infty$ is φ_1 by (5.11) we must have that $V' = c_0 \varphi_1$ for some nonzero constant $c_0 \in \mathbb{R}$.

We claim that

$$c_2 \neq 0 \quad \text{or} \quad \tilde{c}_2 \neq 0. \tag{5.13}$$

Assume, for the sake of contradiction, that $c_2 = 0$ and $\tilde{c}_2 = 0$. Define $\forall t \leq 0$

$$f(t) = e^{(N-2)t} \left(\frac{z_4(t)\tilde{z}_1(t)}{w_1} - \frac{z_3(t)\tilde{z}_2(t)}{w_3} + \frac{z_2(t)\tilde{z}_3(t)}{w_3} - \frac{z_1(t)\tilde{z}_4(t)}{w_1} \right).$$

A calculation using (5.10) shows that f is constant. Using the final conditions for Z and $\tilde{Z}$ we see that $f(0) = 0$, and hence $f(t) = 0$ $\forall t \leq 0$. We can compute $\lim f(t)$ as $t \to -\infty$. Indeed, using the asymptotic behavior (5.11), the relations (5.12), the formulas for the eigenvectors (3.9), the behavior of w_1 and w_3 given by

$$w_1(t) = 2(N-2) + O(e^{\nu_1 t}), \quad w_3(t) = 2(N-2) + O(e^{\nu_1 t})$$

as $t \to -\infty$, and the assumption $c_2 = 0$ and $\tilde{c}_2 = 0$, we get

$$\lim_{t\to-\infty} f(t) = (c_3\tilde{c}_4 - \tilde{c}_3 c_4)B,$$

where $B = -(N-2)^2\sqrt{(10-N)(N-2)}$. Thus $B \neq 0$, and we conclude that $(c_3\tilde{c}_4 - \tilde{c}_3 c_4) = 0$. This means that there exists $\lambda \in \mathbb{R}$ such that $\tilde{c}_k = \lambda c_k$, $k = 3, 4$. Using $Z(0) = e_2$ and $\tilde{Z}(0) = e_4$ we see that

$$(\tilde{c}_1 - \lambda c_1)\varphi_1(0) = e_4 - \lambda e_2.$$

But, as remarked before, $\varphi_1 = c_0 V'$, for some constant $c_0 \in \mathbb{R}$, $c_0 \neq 0$. By Lemma 4.1 all components of $V'(0)$ are non-zero, which implies that $\tilde{c}_1 - \lambda c_1 = 0$, leading to $\tilde{Z}(0) = \lambda Z(0)$, a contradiction.

The condition (5.13) implies the assertion (5.9). Indeed, let us recall that we defined $\mathcal{M}$ as the connected component of $X_{-t_0}(D) \cap \mathcal{N}$ that contains $Q_{t_0} \equiv X_{-t_0}(P^*(\mu))$ with $t_0 > 0$ large. Using a C^1 conjugation that allows us to assume that near P_2 the system is linear, we saw that $\mathcal{M} \cap \widetilde{W}^u(P_1)$ contains a spiral $\tilde{\mathcal{S}}$ around the point Q_{t_0}. $\mathcal{S}$ was defined as X_{t_0} applied to $\tilde{\mathcal{S}}$. The tangent vectors to $\tilde{\mathcal{S}}$ at Q_{t_0} after the conjugation are $\tilde{S}_1$ and $\tilde{S}_2$ given in (5.6)–(5.8). Since the derivatives in (5.8) can be assumed to be small, we see that $\tilde{S}_1$ and $\tilde{S}_2$ are almost contained in the plane generated by e_3 and e_4, which by the conjugation correspond to $\mathrm{Re}(v^{(3)})$ and $\mathrm{Im}(v^{(3)})$. Therefore the tangent plane to $\tilde{\mathcal{S}}$ at Q_{t_0} is almost parallel to the plane generated by the eigenvectors $\mathrm{Re}(v^{(3)})$ and $\mathrm{Im}(v^{(3)})$. Since either $c_2 \neq 0$ or $\tilde{c}_2 \neq 0$, for t_0 large at least one of the vectors $Z(t_0)$ or $\tilde{Z}(t_0)$ is transversal to the tangent plane to $\tilde{\mathcal{S}}$ at Q_{t_0}.

Finally, once we have shown that $\widetilde{W}^u(P_1) \cap \{w_1 = \mu\}$ contains a spiral $\mathcal{S}$ centered around $P^*(\mu)$, using the transversality property (5.9) one can show that for $\lambda = \bar{h}(\mu)$ there are infinitely many intersections of $\mathcal{S}$ with the hyperplane $\{w_3 = \lambda\}$ and that for λ close to $\bar{h}(\mu)$ there is a large number of such intersections. Each intersection yields a regular solution of (1.1) with parameters (μ, λ).

We remark that the argument given above proves a slightly weaker statement than the one in Theorem 1.3, in the sense that we consider μ fixed and let λ approach $\bar{h}(\mu)$ to obtain a large number of solutions. The argument above can be adapted to prove the version stated in the theorem. □

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