J. Differential Equations 253 (2012) 3147–3157

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

A note on nonlinear biharmonic equations with negative exponents

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ARTICLE INFO

Article history:
Received 3 August 2009
Revised 22 March 2012
Available online 5 September 2012

ABSTRACT

In this paper we study radially symmetric entire solutions of

$$\Delta^2 u = -u^{-q}, \quad u > 0 \text{ in } \mathbb{R}^3.$$

We find the asymptotic behavior of these solutions for any $q > 1$ and prove that for any $q > 3$ there exists a solution with exactly linear growth.

1. Introduction

In this note we study the entire solutions of the problem

$$\Delta^2 u = -u^{-q}, \quad u > 0 \text{ in } \mathbb{R}^3, \tag{1}$$

with $q > 0$. This problem was studied in [2] and [7]. In both references the authors study the existence and properties of solutions of (1), but we found a few incorrect statements that we would like to clarify. First, in [2], it is stated that if $1 < q \leqslant 7$, and $u \in C^4$ is a positive solution of (1) with exactly linear growth uniformly at infinity, that is, $\lim_{x\to\infty} u(x)/|x| = \alpha_1$ uniformly for some constant $\alpha_1 > 0$, then $q = 7$. This result turns out to be incorrect as we shall see below. On the other hand, in [7], is stated that for $4 < q < 7$ no solution with linear growth exists, which is also not true. Basically the mistake is in the proof of certain properties that are related with the next term in the asymptotic behaviour at infinity of entire solutions with linear growth. We shall see that there is a change in the behavior of that term in $q = 7$ and $q = 4$. Note that $q = 7$ can be seen as the critical exponent for this problem, in fact $(N + 2)/(N - 2) = -7$ for $N = 3$.

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0022-0396/$ – see front matter
http://dx.doi.org/10.1016/j.jde.2012.08.037

We know from [2] and [7], that the following results hold:

Theorem 1.1. *(See [2].) Let $u \in C^4$ be a solution of (1). If u has exactly linear growth uniformly at infinity, then $q > 3$. Moreover if $q = 7$, u is given by*

$$u(x) = \sqrt{1/\sqrt{15} + |x|^2} \tag{2}$$

and is unique up to dilations and translations.

Theorem 1.2. *(See [2].) There are no entire solutions of (1) for $0 < q \leqslant 1$.*

Theorem 1.3. *(See [7].) For $q \geqslant 7$ there exists a unique radially symmetric entire solutions of (1) with linear growth. For $q = 7$ the solution is given by (2).*

The following theorem gives a complete description of entire *radially* symmetric solutions of (1) for $q > 1$.

Theorem 1.4. *We have the following cases.*

(a) *For $q > 3$ and given any $\alpha > 0$ there exists a unique radially symmetric solution u of (1) such that*

$$\lim_{|x| \to \infty} \frac{u(x)}{|x|} = \alpha.$$

(b) *For $q = 3$ there exists a unique radially symmetric solution u of (1) such that*

$$\lim_{|x| \to \infty} \frac{u(x)}{|x| \log(|x|)^{1/4}} = 2^{1/4}.$$

(c) *For $1 < q < 3$ there exists a unique radially symmetric solution u of (1) such that*

$$\lim_{|x| \to \infty} |x|^{-\frac{4}{q+1}} u(x) = K_q^{-\frac{1}{q+1}},$$

where

$$K_q = \tau(2 - \tau)(\tau + 1)(\tau - 1), \quad \tau = \frac{4}{q+1}.$$

We call the solutions from this theorem *typical growth* entire solutions. It was proved in [7], that for $q > 1$ there exist infinitely many entire solutions of (1) with quadratic growth, that is, $\lim_{x \to \infty} u(x)/|x|^2 = A > 0$. Thus a typical growth entire solution is in the borderline of entire solutions and compact support solutions.

In the case of exactly linear growth α can be computed by the formula

$$\alpha = \frac{1}{8\pi} \int_{\mathbb{R}^3} u^{-q}(x)\, dx. \tag{3}$$

Note that solutions in (a) are of the form $u(x) = \alpha^{-\frac{4}{q-3}} \bar{u}(\alpha^{\frac{q+1}{q-3}} x)$ where $\bar{u}(x)$ is the solution with $\alpha = 1$.

Clearly from Theorem 1.4 it follows that Theorem 1.1 in [2], and Theorem 4.2 parts (b) and (c), cannot hold.

Next we continue examine further asymptotics of the typical growth entire solutions. The following result can be found in [2], see Lemma 4.4 and formula (4.42).

Theorem 1.5. *(See [2].) For $q > 4$ the solution u of (1) admits the integral representation*

$$u(x) = \frac{1}{8\pi} \int_{\mathbb{R}^3} |x-y| u(y)^{-q}\, dy + \gamma,$$

where

$$\gamma = \frac{q-7}{q-1} \frac{\int_{\mathbb{R}^3} u^{1-q}(x)\, dx}{\int_{\mathbb{R}^3} u^{-q}(x)\, dx}. \tag{4}$$

Moreover we have

$$\lim_{|x|\to\infty} \big(u(x) - \alpha|x|\big) = \gamma,$$

where α is given by (3).

Note that (4) is formula (4.42) in [2]. We point out that formula (4.42) was correctly derived in [2] inside the proof of Lemma 4.9, which is not valid as we shall explain later in Section 5.

Note that this theorem does not contradict the result of [9], see an extension in [6] and [10]. There, it was proved that if u is a C^4 positive solution of

$$u(x) = \frac{1}{8\pi} \int_{\mathbb{R}^3} |x-y| u(y)^{-q}\, dy \quad \text{in } \mathbb{R}^3$$

with $q > 0$, then $q = 7$ and up to translation a dilations u takes the form (2).

Finally to complement the above result we prove the following result.

Theorem 1.6. *We have the further asymptotics: for $q = 4$*

$$\lim_{|x|\to\infty} \frac{1}{\log(|x|)} \big(u(x) - \alpha|x|\big) = -\frac{1}{2\alpha^4}$$

and for $3 < q < 4$,

$$\lim_{|x|\to\infty} |x|^{q-4} \big(u(x) - \alpha|x|\big) = \bar{\gamma},$$

where

$$\bar{\gamma} = \frac{1}{2\alpha^q} \left(\frac{1}{3-q} - \frac{1}{4-q} + \frac{1}{3(5-q)} - \frac{1}{3(2-q)} \right).$$

The plan of the paper is as follows. In Section 2 an initial value problem is studied for radial solutions of (1). In Section 3 the asymptotic behavior of solutions is studied using the phase-space analysis and a change of variables introduced in [5]. In Section 4, we give the proof of the theorems and finally in Section 5 we make some comments on the mistakes of proofs found in [2], and [7].

2. Initial value problem

We consider the initial value problem

$$\begin{cases} \Delta^2 U = -U^{-q}, \quad U > 0, \qquad r \in \big(0, R_{\max}(\beta)\big), \\ U(0) = 1, \quad U'(0) = 0, \quad \Delta U(0) = \beta, \quad (\Delta U)'(0) = 0. \end{cases} \tag{5}$$

Here $[0, R_{\max}(\beta))$ is the interval of existence of the solution. The main result here is the following.

Proposition 2.1. *Assume $q > 1$. Then there is a unique $\beta^* > 0$ such that:*

(a) *If $\beta < \beta^*$ then $R_{\max}(\beta) < \infty$.*
(b) *If $\beta \geqslant \beta^*$ then $R_{\max}(\beta) = \infty$.*
(c) *If $\beta \geqslant \beta^*$ then and $\lim_{r\to\infty} \Delta U_\beta(r) \geqslant 0$.*
(d) *We have $\beta = \beta^*$ if and only if $\lim_{r\to\infty} \Delta U_\beta(r) = 0$.*

Proof. Parts (a) and (b) can be found in [7].

Part (c). For any $\beta > 0$, from the formula

$$\frac{d\Delta U_\beta}{dr}(r) = -r^{-2} \int_0^r s^2 U_\beta(s)^{-q}\, ds < 0 \quad \text{for } 0 \leqslant r < R_{\max}(\beta) \tag{6}$$

we deduce that $\Delta U_\beta(r)$ is decreasing on $[0, R_{\max}(\beta))$. If $R_{\max}(\beta) = \infty$ it follows that

$$\lim_{r\to\infty} \Delta U_\beta(r) \text{ exists.}$$

Now if $\lim_{r\to\infty} \Delta U_\beta(r) < 0$ integrating twice we deduce $U_\beta(r) \leqslant -C_1 r^2 + C_2$ for all $r \geqslant 0$ with $C_1, C_2 > 0$, which is impossible, so part (c) follows.

Part (d). Assume now that $\lim_{r\to\infty} \Delta U_\beta(r) > 0$ then

$$U_\beta(r) \geqslant cr^2 \quad \text{and} \quad U'_\beta(r) \geqslant cr \quad \text{for all } r \geqslant 0 \tag{7}$$

for some $c > 0$. Also, integrating once Eq. (5) we see that for all $r \geqslant 2$:

$$(\Delta U_\beta)'(r) \geqslant -c \begin{cases} r^{-2} & \text{if } 3 < 2q, \\ r^{-2}\log r & \text{if } 3 = 2q, \\ r^{1-2q} & \text{if } 3 > 2q. \end{cases} \tag{8}$$

In [7], it was found that $w(r) = (1 + (br)^2)^{1-\varepsilon/2}$ satisfies

$$\Delta^2 w \leqslant -w^{-q} \quad \text{for all } r \geqslant 0,$$

whenever $\varepsilon < 2(q-1)/(q+1)$ and b is chosen sufficiently large.

Now from (7) and (8) there exists $r_0 > 0$ such that $U_\beta(r_0) > w(r_0)$, $U'_\beta(r_0) > w'(r_0)$, $\Delta U_\beta(r_0) > \Delta w(r_0)$ and $(\Delta U_\beta)'(r_0) > (\Delta w)'(r_0)$. By the continuous dependence of the solution to (5) there is $\beta_1 < \beta$ such that

$$U_{\beta_1}(r_0) > w(r_0), \qquad U'_{\beta_1}(r_0) > w'(r_0)$$

and

$$\Delta U_{\beta_1}(r_0) > \Delta w(r_0), \qquad (\Delta U_{\beta_1})'(r_0) > (\Delta w)'(r_0).$$

Using the comparison lemma in [7, Lemma 3.2], we deduce that $U_{\beta_1} \geqslant w$ for all $r \geqslant r_0$. This shows that u_{β_1} is defined for all $r \geqslant 0$ and hence $\beta_1 \geqslant \beta^*$. We deduce that $\beta > \beta^*$. The uniqueness of β^* follows as in [4]. □

Lemma 2.2. *(See [2,7].) Let $q > 1$. We have $U''_\beta(r) > 0$ for all $r \geqslant 0$ and $\beta \geqslant \beta^*$.*

This lemma implies that

$$\lim_{r\to\infty} U''_{\beta^*}(r) = 0 \quad \text{and} \quad \lim_{r\to\infty} \frac{U'_{\beta^*}(r)}{r} = 0. \tag{9}$$

In addition Lemma 2.2 implies that $U'_{\beta^*}(r) > U'_{\beta^*}(1) > 0$ for all $r > 1$ and

$$U_{\beta^*}(r) > U'_{\beta^*}(1)(r-1) + U_{\beta^*}(1). \tag{10}$$

Lemma 2.3. *If $q = 3$ then*

$$\lim_{r\to\infty} \frac{U_{\beta^*}(r)}{r} = \infty.$$

Proof. By (10), we have that $U_{\beta^*}(r) \geqslant ar + b$, for $r > 1$. By contradiction, we assume that

$$U_{\beta^*}(r) \leqslant Ar + B \quad \text{for } r \text{ large}.$$

This implies that $\bar{a} \leqslant m(r) := U_{\beta^*}(r)/r \leqslant \bar{A}$ for r large. Applying the L'Hôpital's rule and using (9), we have

$$\lim_{r\to\infty} m'(r) = \lim_{r\to\infty} \frac{rU'_{\beta^*}(r) - U_{\beta^*}(r)}{r^2} = \lim_{r\to\infty} \frac{U'_{\beta^*}(r)}{2r} = 0.$$

Lemma 3.1 from the next section implies that $m'(r)$ has one sign for all $r > r_0 \geqslant 0$, and then

$$\lim_{r\to\infty} m(r) = \hat{a} \in [\bar{a}, \bar{A}].$$

But according to Lemma 4.1 in [2] this is not possible when $q = 3$. □

3. Asymptotic behaviour

For the study of the asymptotic behaviour, we write $u = U_\beta$ with $\beta \geqslant \beta^*$ which satisfies the system

$$\Delta u = v \quad \text{in } \mathbb{R}^3, \tag{11}$$

$$-\Delta v = u^{-q} \quad \text{in } \mathbb{R}^3. \tag{12}$$

Following the ideas in [5], we now set

$$x(t) = \frac{ru'}{u}, \qquad y(t) = \frac{rv'}{v}, \qquad z(t) = \frac{r^2 v}{u}, \qquad w(t) = \frac{r^2 u^{-q}}{v}, \quad t = \log(r). \tag{13}$$

Since u is positive and increasing, see [2,7], we have $x > 0$ for all $t > -\infty$. By Proposition 2.1, we found that v is positive and decreasing, so $y < 0$ for all $t > -\infty$. We also have that $z > 0$ and $w > 0$ for all $t > -\infty$. In addition by Lemma 2.2 a simple calculation gives $z > 2x$ for all $t \in (-\infty, \infty)$.

The system is transformed in a 4-dimensional quadratic system of the form

$$x' = x(-1-x) + z, \tag{14}$$

$$y' = y(-1-y) - w, \tag{15}$$

$$z' = z(2-x+y), \tag{16}$$

$$w' = w(2-qx-y), \tag{17}$$

where $' = d/dt$. The critical points of this system are

$$\begin{aligned}
&p_0 = (0,0,0,0), \qquad p_1 = (1,-1,2,0), \qquad p_2 = (2,0,6,0),\\
&p_3 = \bigl(a, a-2, a(a+1), (2-a)(a-1)\bigr), \qquad p_4 = (0,2,0,-6),\\
&p_5 = (0,-1,0,0), \qquad p_6 = (-1,0,0,0),\\
&p_7 = (-1,-1,0,0), \quad \text{and} \quad p_8 = \bigl(-1, q+2, 0, -(q+2)(q+3)\bigr),
\end{aligned}$$

where $a = 4/(q+1)$.

The entire solutions from Proposition 2.1 are in this quadratic system heteroclinic orbits. In fact an entire solution emanates from p_0 and lies on the two-dimensional unstable manifold. For the point where the heteroclinic orbits end, we have the following cases:

(i) for $q \geqslant 3$ the typical growth entire solution ends at p_1,
(ii) for $1 < q < 3$ the typical growth entire solution ends at p_3,
(iii) entire solutions with quadratic growth end at p_2.

The following is an important property of the heteroclinic orbits.

Lemma 3.1. *Let $q > 1$. The entire solution satisfies either $0 < x < 1$ for all $t \in \mathbb{R}$, and then $\lim_{t\to\infty} x(t) = 1$, or there exists t_0 such that $x > 1$ for any $t > t_0$.*

Proof. Using that $z > 2x$, we have that $x' > x(1-x)$, so when $x < 1$, x increases and if the first alternative holds then $\lim_{t\to\infty} x(t) = 1$. If not there exists t_0 such that $x(t_0) = 1$ and $x'(t_0) > 0$. Arguing by contradiction, if a time $t_1 > t_0$ such that $x(t_1) = 1$ exists, this point must satisfy $x'(t_1) \leqslant 0$, which is impossible by the inequality for x' at the beginning of the proof. $\square$

Note that if $q > 3$ then $0 < a < 1$ and so p_3 cannot be the end point of an entire solution since variable w is negative. For the same reason p_4 cannot be an end point of an entire solution. We can also discard by the sign of the first component the points p_6, p_7, p_8. On the other hand, the point p_5 cannot be an end point of an entire solution, since the first component cannot be zero by Lemma 3.1. Note also that for $q = 3$, we have $p_3 = p_1$.

In the following we prove (i). The proof of (ii) can be found in [3, proof of Proposition 8] and (iii) is similar to (i) and is left to the reader. Since we are in case (i), we assume from now on that $u = U_{\beta^*}$. Since $-\Delta v = u^{-q}$ with $v'(0) = 0$, we have

$$v(r) = v(0) - \int_0^r t u^{-q}\,dt + \frac{1}{r}\int_0^r t^2 u^{-q}\,dt.$$

Using (10), we have $u(r) \geqslant ar + b$, $r > 1$ and $q \geqslant 3$ we find that $t^2 u^{-q} \to 0$ as $t \to \infty$ and then $\frac{1}{r}\int_0^r t^2 u^{-q} \to 0$. The condition $\lim_{t\to\infty} v(t) = 0$ in Proposition 2.1(d) implies

$$v(0) = \int_0^\infty t u^{-q}\, dt < \infty.$$

Finally we have

$$v(r) = \int_r^\infty t u^{-q}\, dt + \frac{1}{r}\int_0^r t^2 u^{-q}\, dt. \tag{18}$$

We compute now the inverse of y,

$$\frac{v(r)}{r v'(r)} = -\frac{r\int_r^\infty t u^{-q}\, dt}{\int_0^r t^2 u^{-q}\, dt} - 1.$$

If $q > 3$ then $\lim_{r\to\infty} r\int_r^\infty t u^{-q}\, dt = \lim_{r\to\infty} r^3 u^{-q}(r) = 0$ by using that $u(r) \geqslant ar + b$ see (10). If $q = 3$ then by Lemma 2.3, we have $\lim_{r\to\infty} u(r)/r = \infty$. As before $\lim_{r\to\infty} r\int_r^\infty t u^{-q}\, dt = 0$. Consequently

$$\lim_{r\to\infty} \frac{v(r)}{r v'(r)} = -1. \tag{19}$$

Also if $q \geqslant 3$, by the same arguments

$$\lim_{r\to\infty} \frac{r^2 u^{-q}}{v} = \lim_{r\to\infty} \frac{r^2 u^{-q}}{r v'(r)} \lim_{r\to\infty} \frac{r v'(r)}{v(r)} = -\lim_{r\to\infty} \frac{r^3 u^{-q}}{\int_0^r t^2 u^{-q}\, dt} = 0.$$

Now from $\Delta u = v$ and $u'(0) = 0$, we have

$$u(r) = u(0) + \int_0^r t v(t)\, dt - \frac{1}{r}\int_0^r t^2 v(t)\, dt. \tag{20}$$

We can write

$$\frac{u(r)}{r^2 v(r)} = \frac{u(0)}{r^2 v(r)} + \frac{1}{r^2 v(r)}\int_0^r t v(t)\, dt - \frac{1}{r^3 v(r)}\int_0^r t^2 v(t)\, dt.$$

Clearly $\lim_{r\to\infty} r\int_0^r t^2 u^{-q} = \infty$, and so by (18) we have that $\lim_{r\to\infty} r^2 v(r) = \infty$. Using the L'Hôpital rule and (19), we find

$$\lim_{r\to\infty} \frac{1}{r^2 v(r)}\int_0^r t v(t)\, dt = \lim_{r\to\infty} \frac{r v(r)}{2 r v(r) + r^2 v'(r)} = 1$$

and

$$\lim_{r\to\infty}\frac{1}{r^3 v(r)}\int_0^r t^2 v(t)\,dt=\lim_{r\to\infty}\frac{r^2 v(r)}{3r^2 v(r)+r^3 v'(r)}=\frac{1}{2}$$

and so we have

$$\lim_{r\to\infty}\frac{u(r)}{r^2 v(r)}=\frac{1}{2}.$$

On the other hand, we write

$$\frac{u(r)}{ru'(r)}=\frac{u(0)+\int_0^r t v(t)\,dt-\frac{1}{r}\int_0^r t^2 v(t)\,dt}{\frac{1}{r}\int_0^r t^2 v(t)\,dt}=\frac{u(0)}{\frac{1}{r}\int_0^r t^2 v(t)\,dt}+\frac{\frac{1}{r^2 v(t)}\int_0^r t v(t)\,dt}{\frac{1}{r^3 v(t)}\int_0^r t^2 v(t)\,dt}-1$$

and using again (19), we get

$$\lim_{r\to\infty}\frac{u(r)}{ru'(r)}=1.$$

This concludes that for $q\geqslant 3$ the entire solution is a heteroclinic orbit connecting $(0,0,0,0)$ and $(1,-1,2,0)$. As a consequence the asymptotic behaviour is obtained analyzing the behaviour of the solution about $(1,-1,2,0)$. For $q>3$ the linearization of the reduced system around this point gives a matrix with eigenvalues $\lambda_1=3-q$, $\lambda_2=-2$, $\lambda_3=-1$, and $\lambda_4=1$, and so there exists a constant c_q such that for $q>4$

$$\frac{ru'(r)}{u(r)}=1+c_q e^{-t}+o\big(e^{-t}\big)\quad\text{as } t\to\infty, \tag{21}$$

for $q=4$

$$\frac{ru'(r)}{u(r)}=1+c_q te^{-t}+o\big(te^{-t}\big)\quad\text{as } t\to\infty \tag{22}$$

and for $3<q<4$

$$\frac{ru'(r)}{u(r)}=1+c_q e^{-(q-3)t}+o\big(e^{-(q-3)t}\big)\quad\text{as } t\to\infty. \tag{23}$$

For $q=3$ we have to use the Center Manifold Theory [1,8], to find the asymptotic behavior of the entire solution. Set $X=x-1$, $Y=y+1$, $Z=z-2$ and $W=w$.

$$\begin{aligned}
X'&=-X^2-3X+Z,\\
Y'&=-Y^2+Y-W,\\
Z'&=(Z+2)(Y-X),\\
W'&=-W(3X+Y).
\end{aligned}\tag{24}$$

Now the eigenvalues of the matrix given by the linearization are $\lambda_1 = 0$, $\lambda_2 = -2$, $\lambda_3 = -1$, and $\lambda_4 = 1$, and the corresponding eigenvectors are the columns of the matrix

$$V = \begin{bmatrix} 1 & 1 & 1/2 & 1/4 \\ 1 & 0 & 0 & 3/4 \\ 3 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

To proceed we make the following change of variables

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \\ \bar{w} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 2/3 & -1 & 1/3 \\ -2 & -2 & 2 & -2 \\ 0 & 4/3 & 0 & -4/3 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix} = V \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \\ \bar{w} \end{pmatrix}.$$

The system (24) is transformed into

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \\ \bar{w} \end{pmatrix}' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \\ \bar{w} \end{pmatrix} + \begin{pmatrix} -R \\ -2X^2 - \frac{2}{3}Y^2 - Z(Y - X) - \frac{1}{3}R \\ 2X^2 + 2Y^2 + 2Z(Y - X) + 2R \\ -\frac{4}{3}Y^2 + \frac{4}{3}R \end{pmatrix},$$

where $R = W(3X + Y)$. By Theorem 2.1.1 in [8], the center manifold can be locally represented by

$$W^c(0) = \left\{ (\bar{x}, \bar{y}, \bar{z}, \bar{w}) \in \mathbb{R}^4 \;\middle|\; \bar{y} = h_1(\bar{x}),\ \bar{z} = h_2(\bar{x}),\ \bar{w} = h_3(\bar{x}),\ h_i(0) = h_i'(0) = 0,\ i = 1, \dots, 3 \right\}$$

for $\bar{x}$ sufficiently small. In addition Theorem 2.1.3 in [8] allows us to consider the power series expansions $h_i(x) = a_i x^2 + o(x^2)$ as $x \to 0$. We have

$$X = \bar{x} + (a_1 + a_2/2 + a_3/4)(\bar{x})^2 + O\big((\bar{x})^3\big), \qquad Y = \bar{x} + 3a_3/4(\bar{x})^2 + O\big((\bar{x})^3\big),$$
$$Z = 3\bar{x} + (a_1 + a_2 + a_3)(\bar{x})^2 + O\big((\bar{x})^3\big) \quad \text{and} \quad W = \bar{x}$$

and this gives

$$R = W(3X + Y) = 4\bar{x}^2 + (3a_1 + 3a_2/2 + 3a_3/2)(\bar{x})^3 + O\big((\bar{x})^4\big).$$

Using that the center manifold is invariant under the dynamics generated by (24), we find that $a_1 = -2$, $a_2 = 12$ and $a_3 = -4$. This is done by solving a system of differential equations for $h_i(\bar{x})$, $i = 1, \dots, 3$, see for example Section 2.1c in [8]. So we have

$$\bar{x}' = -4\bar{x}^2 - 6\bar{x}^3 + O\big(|\bar{x}|^3\big) \quad \text{as } |\bar{x}| \to 0.$$

In the original variables, we get

$$\frac{ru'(r)}{u(r)} = 1 + \frac{1}{4}t^{-1} + O\big(t^{-1}\big) \quad \text{as } t \to \infty. \tag{25}$$

4. Proofs of the theorems

Before to give the proofs we write, combining (20) and (18), the formula

$$u(r)=\frac{r}{2}\int_0^r t^2u^{-q}(t)\,dt-\frac{1}{2}\int_0^r t^3u^{-q}(t)\,dt+\frac{1}{6r}\int_0^r t^4u^{-q}(t)\,dt+\frac{r^2}{6}\int_r^\infty tu^{-q}(t)\,dt+u(0). \quad (26)$$

This expression is valid for $q>1$ and for u a typical growth entire solution. Note that in the case $1<q<3$ the behavior of the typical growth entire solution is given by Theorem 1.4(c) as it was proven in [3, Proposition 8], and so the integral $\int_r^\infty tu^{-q}(t)\,dt$ is well defined.

Proof of Theorem 1.4. We have only to show (a) and (b), since (c) was proved in [3, Proposition 8] for $1<q<3$, and see also [4] for $q=2$. Part (a) follows from (21), (22), and (23). In each case we easily deduce that the asymptotic behavior is linear for u. Now using (26), for $q>3$ we have

$$\lim_{r\to\infty}\frac{u(r)}{r}=\alpha=\frac{1}{2}\int_0^\infty t^2u^{-q}(t)\,dt. \quad (27)$$

In case (b), we have (25) and so $\lim_{r\to\infty}u(r)/(r\log(r)^{1/4})=C>0$. Using this and (26), we conclude that $C=2^{1/4}$ from the contribution of the first integral in (26). □

Proof of Theorem 1.6. Using the asymptotics (22) and (23), the linear growth formula (27) and the representation (26), we obtain the result. When $q=4$ the constant is found computing the contribution of the second integral in (26), and when $3<q<4$ is found by the contribution of the four integrals. □

5. Comments on the papers [2] and [7]

The proof of Theorem 1.2 in [2] is based in Lemma 4.7 which is not valid. In fact, in the proof of Lemma 4.7 formula (4.24) is correct, so $\liminf_{|x|\to\infty}\{-\Delta(u^{-1}(x))\}\geqslant 0$, but if we have equality then is not true in general that $\liminf_{|x|\to\infty}|x|^5\{-\Delta(u^{-1})(x)\}\geqslant 0$ since $|x|^5\to\infty$. So (4.26) cannot be concluded and Lemma 4.7 is not valid. We observe also that Lemma 4.9 in [2] uses Lemma 4.7 to conclude, so Lemma 4.9 is also not true.

In [7], Lemma 4.7 says that $x\leqslant 1$ when $q>4$. But the conclusion in case (a) of the proof is not correct. So this lemma is not valid in general. In this note we showed Lemma 3.1 instead.

Acknowledgments

I.G. was partially supported by Fondecyt 1090470 and CAPDE-Anillo ACT-125.

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