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RESONANCE PHENOMENA FOR SECOND-ORDER STOCHASTIC CONTROL EQUATIONS*

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Abstract. We study the existence and the properties of solutions to the Dirichlet problem for uniformly elliptic second-order Hamilton–Jacobi–Bellman operators, depending on the principal eigenvalues of the operator.

Key words. Bellman operator, Dirichlet problem, principal eigenvalue, resonance

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1. Introduction.

In this article we consider the Dirichlet problem

$$\left\{ \begin{aligned} F(D^2u, Du, u, x) &= f(x) \quad \text{in} \quad \Omega, \\ u &= 0 \qquad \text{on} \quad \Omega, \end{aligned} \right. \tag{1.1}$$

where the second-order differential operator F is of Hamilton–Jacobi–Bellman (HJB) type and $\Omega \subset \mathbb{R}^N$ is a bounded domain. These equations—see the book [17] and the surveys [20], [29], and [9], as well as [21] (various other references will be given below)—have been very widely studied because of their connection with the general problem of optimal control for stochastic differential equations (SDEs). We recall that a powerful approach to this problem is the so-called dynamic programming method, initiated by R. Bellman, which indicates that the optimal cost (value) function of a controlled SDE should be a solution of a PDE like (1.1). More precisely, let us have a stochastic process X_t satisfying

$$dX_t = b^{\alpha_t}(X_t)dt + \sigma^{\alpha_t}(X_t)dW_t\,,$$

with $X_0 = x$ for some $x \in \Omega$, and the cost function

$$J(x,\alpha) = \mathbb{E}\int_0^{\tau_x} f(X_t)\exp\left\{\int_0^t c^{\alpha_s}(X_s)ds\right\} dt,$$

where τ_x is the first exit time from Ω of X_t, and α_t is an index (control) process with values in a set $\mathcal{A}$. Then the optimal cost function $v(x) = \inf_{\alpha\in\mathcal{A}} J(x,\alpha)$ is such that $-v$ is a solution of (1.1), which is in the form

$$\left\{ \begin{aligned} \sup_{\alpha\in\mathcal{A}}\{\mathrm{tr}(A^\alpha(x)D^2u) + b^\alpha(x).Du + c^\alpha(x)u\} &= f(x) \quad \text{in} \quad \Omega, \\ u &= 0 \qquad \text{on} \quad \partial\Omega. \end{aligned} \right. \tag{1.2}$$

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We are going to study this boundary value problem under the following hypotheses, which will be kept throughout the paper: for some constants $0 < \lambda \le \Lambda$, $\gamma \ge 0$, $\delta \ge 0$, we assume $A^\alpha(x) := \sigma^\alpha(x)^T \sigma^\alpha(x) \in C(\overline{\Omega})$, $\lambda I \le A^\alpha(x) \le \Lambda I$, $|b^\alpha(x)| \le \gamma$, $|c^\alpha(x)| \le \delta$ for almost all $x \in \Omega$ and all $\alpha \in \mathcal{A}$, and $f \in L^p(\Omega)$ for some $p > N$. We stress, however, that all our results are new even for operators with smooth coefficients.

Our main statements on resonance, applied to this setting, imply in particular that for some A, b, c, the optimal cost becomes arbitrarily large or small, depending on the function f which stays bounded. We give conditions under which (1.2) is solvable or not and describe properties of its solutions.

The majority of works on HJB equations concern *proper* equations, that is, cases when F is monotone in the variable u ($c^\alpha \le 0$), in which no resonance phenomena can arise. It was shown in the well-known papers [15], [16], and [22] that a proper equation of type (1.2) has a unique strong solution, which is classical, if the coefficients are smooth. Uniqueness in the viscosity sense was proved in [19], [14], [12], and [30].

Two of the authors recently showed in [24] that existence and uniqueness of viscosity solutions hold for a larger class of operators, including nonproper operators whose principal eigenvalues—defined below—are positive. This had been proved much earlier for HJB operators with smooth coefficients, in [21], through a mix of probabilistic and analytic techniques. Very recently, existence, nonexistence, and multiplicity results for cases when the eigenvalues are negative or have different signs, but are different from zero, appeared in [1] and [27].

Thus, the only situations which remain completely unstudied are the cases when (1.2) is "at resonance," that is, when one of the principal eigenvalues of F is zero. The present paper is devoted to this problem. We also obtain a number of new results for cases without resonance.

We shall make essential use of the work [24], where the properties of the eigenvalues are studied. In particular, based on the definition for the linear case in [4], it is shown in [24] that the numbers

$$\lambda_1^+(F,\Omega) = \sup\{\lambda \,|\, \Psi^+(F,\Omega,\lambda) \neq \emptyset\}, \qquad \lambda_1^-(F,\Omega) = \sup\{\lambda \,|\, \Psi^-(F,\Omega,\lambda) \neq \emptyset\},$$

where the sets $\Psi^+(F,\Omega,\lambda)$ and $\Psi^-(F,\Omega,\lambda)$ are defined as

$$\Psi^\pm(F,\Omega,\lambda) = \{\psi \in C(\overline{\Omega}) \,|\, \pm\left(F(D^2\psi, D\psi, \psi, x) + \lambda\psi\right) \le 0,\ \pm\psi > 0 \text{ in } \Omega\},$$

are simple and isolated eigenvalues of F, associated with positive and negative eigenfunctions φ_1^+, $\varphi_1^- \in W^{2,q}(\Omega)$, $q < \infty$, and that their positivity guarantees the validity of one-sided Alexandrov–Bakelman–Pucci (ABP)-type estimates; see the review in the next section. From the optimal control point of view, λ_1^+ can be seen as the fastest exponential rate at which paths can exit the domain, and λ_1^- is the slowest one; we refer to the exact formulae given in equalities (30)–(31) of [21]. For extensions and related results on eigenvalues for fully nonlinear operators, we refer to [18] and [1], where Isaacs operators are studied, and to [5] and [6], where more general singular fully nonlinear elliptic operators are considered. When no confusion arises, we write $\lambda_1^\pm$ or $\lambda_1^\pm(F)$, and we always suppose that $\lambda_1^+ < \lambda_1^-$ — note it easily follows from the results in [24] that $\lambda_1^+ = \lambda_1^-$ can only happen if all linear operators which appear in (1.2) have the same principal eigenvalues *and* eigenfunctions. For simplicity, we assume that Ω is regular, in the sense that it satisfies an uniform interior ball condition, even though many of the results can be extended to general bounded domains.

We make the convention that all (in)equalities in the paper are meant to hold in the L^p-viscosity sense, as defined and studied in [12]. Note, however, that it is known that any viscosity solution of (1.2) is in $W^{2,p}(\Omega)$ and that any $W^{2,p}$-function which satisfies (1.2) in the viscosity sense is also a strong solution (that is, it satisfies (1.2) a.e. in Ω); see [11], [12], [30], and [32]. All constants in the estimates will be allowed to depend on $N,\ \lambda,\ \Lambda,\ \gamma,\ \delta$, and Ω.

Given a fixed function $h \in L^p(\Omega)$ which is not a multiple of the principal eigenfunction φ_1^+, everywhere in the paper we write

$$f = t\varphi_1^+ + h, \quad t \in \mathbb{R}, \tag{1.3}$$

and consider t as a parameter. We note that all results and proofs below hold without modifications if the function φ_1^+ in (1.3) is replaced by any other positive function, which vanishes on $\partial\Omega$ and whose interior normal derivative on the boundary is strictly positive. We visualize the set S of solutions of (1.2) in the space $C(\overline{\Omega}) \times \mathbb{R}$ as follows: $(u,t) \in S$ if and only if u is a solution of (1.2) with $f = t\varphi_1^+ + h$. The following notation will be useful: given a subset $A \subset C(\overline{\Omega}) \times \mathbb{R}$ and $t \in \mathbb{R}$, we define $A_t = \{u \in C(\overline{\Omega}) \,|\, (u,t) \in A\}$ and $A_I = \cup_{t\in I} A_t$ if I is an interval.

Our purpose is to describe the set $S = \{(u_t,t) \,|\, t \in \mathbb{R}\}$. When $\lambda_1^+(F) > 0$, this can be done in a rather precise way thanks to the results in [21] and [24].

THEOREM 1.1. *Assume $\lambda_1^+(F) > 0$. Then the following apply.*

1. (*See* [24].) *For every $t \in \mathbb{R}$ (1.2) possesses exactly one solution $u = u_t$. In addition, if $f = t\varphi_1^+ + h \neq 0$ and $f \le (\ge) 0$, then $u > (<) 0$ in Ω. If $t < s$, then $u_t > u_s$ in Ω.*
2. *The set S is a Lipschitz continuous curve such that $t \to u_t(x)$ is convex for each $x \in \Omega$. There exist numbers $t^\pm = t^\pm(h)$ such that if $t \ge t^+$ ($t \le t^-$), then $u_t < (>) 0$ in Ω. Moreover, for each compact $K \subset\subset \Omega$,*

$$\lim_{t\to-\infty} \min_{x\in K} u_t(x) = +\infty \qquad \text{and} \qquad \lim_{t\to+\infty} \max_{x\in K} u_t(x) = -\infty.$$

Next, we state our first main theorem, which describes the set S when the first eigenvalue is zero. In this case the set of solutions is again a unique continuous curve, but it exists only on a half-line with respect to t, and becomes unbounded when t is close or equal to a critical number t_+^*; see Figure 1 below. Note the picture is very different from the one we obtain in the linear case—if L is a linear operator, then the Fredholm alternative for $Lu + \lambda_1(L)u = t\varphi_1(L) + h$ says this equation has a solution only for one value of t, and then any two solutions differ by a multiple of $\varphi_1(L)$.

THEOREM 1.2. *Assume $\lambda_1^+(F) = 0$. Then the following apply.*

1. *There exists a number $t_+^* = t_+^*(h)$ such that if $t < t_+^*$, then there is no solution of (1.2), while for $t > t_+^*$, (1.2) has a solution.*
2. *The set S is a continuous curve such that S_t is a singleton for all $t > t_+^*$; that is, solutions are unique for $t > t_+^*$. If $t_+^* \le t < s$ and $(u_t,t),(u_s,s) \in S$, then $u_t > u_s$ in Ω. The map $t \to u_t(x)$ is convex for each $x \in \Omega$.*
3. *There exists $t^+ = t^+(h) > t_+^*$ such that if $t \ge t^+$, then $u_t < 0$ in Ω, and for every compact $K \subset\subset \Omega$, we have $\lim_{t\to+\infty} \max_{x\in K} u_t(x) = -\infty$.*
4. *If $t = t_+^*$, then either*
   (i) (1.2) *does not have a solution (that is, S_{t^*} is empty), $\lim_{t\searrow t_+^*} \min_{x\in K} u_t = +\infty$ for every compact $K \subset\subset \Omega$, and there exists $\epsilon = \epsilon(h) > 0$ such that if $t \in (t_+^*, t_+^* + \epsilon)$, then $u_t > 0$ or*
   (ii) *there exists a function u^* such that $S_{t_+^*} = \{u^* + s\varphi_1^+ \mid s \ge 0\}$.*

In case the two eigenvalues have opposite signs, a multiplicity phenomenon occurs. This situation was studied in [27] and we recall it here.

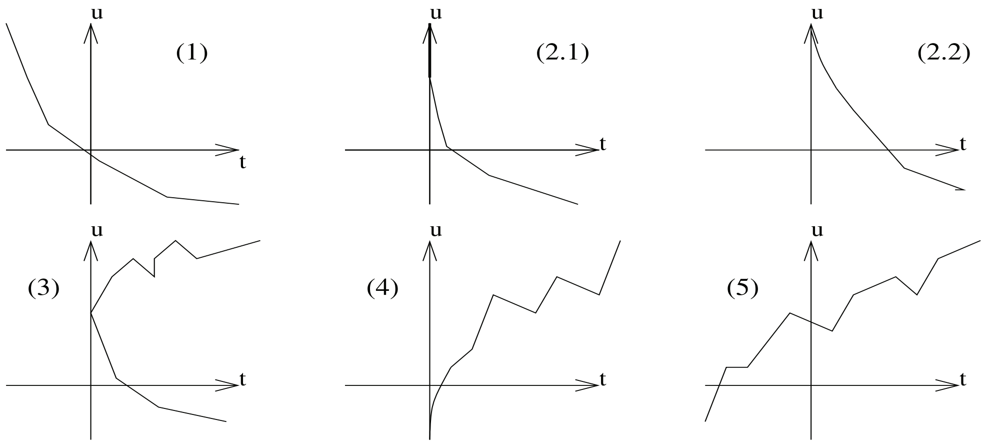

FIG. 1. *The number at each graph corresponds to the number of the theorem where the shown situation is described. When λ_1^+ crosses 0 the set S curves so that one region of nonexistence and one region of multiplicity of solutions appears for t. Similarly when λ_1^- crosses 0 the set S "uncurves" back. In this process, the set S evolves from being "decreasing," when both eigenvalues are positive, to being "increasing," at least for large $|t|$, when both eigenvalues are negative. Note* (1) *and* (2.1)–(2.2) *are exact, while in* (3)–(5) *there may be other solutions, except if Theorem* 1.6 *below holds.*

THEOREM 1.3 (see [27]). *Assume $\lambda_1^+(F) < 0 < \lambda_1^-(F)$. Then there exists a number $t^* = t^*(h)$ such that the following apply.*

1. *If $t < t^*$, then there is no solution of* (1.2).
2. *If $t > t^*$, then there are at least two solutions of* (1.2)*; more precisely, for $t \in (t^*, \infty)$ there is a continuous curve of minimal solutions u_t of* (1.2) *such that $t \to u_t(x)$ is convex and strictly decreasing for $x \in \Omega$ and a connected set of solutions different from the minimal ones.*
3. *If $t = t^*$, then there is at least one solution of* (1.2).

Note that in [27] the properties of the two branches were not described; however, by using the results there, our Lemma 2.1 and some topological and degree arguments, like in sections 3 through 5, they can be obtained easily.

Now we state our second main theorem, which describes properties of the set S when the second eigenvalue is at resonance, that is, when $\lambda_1^-(F) = 0$. Here the analysis is more difficult than in Theorem 1.2, but still the picture is quite clear.

THEOREM 1.4. *Assume $\lambda_1^-(F) = 0$. Then there exists $t_-^* = t_-^*(h)$ such that the following apply.*

1. *If $t < t_-^*$, then there is no solution of* (1.2).
2. *There is a closed connected set $\mathcal{C} \subset S$ such that $\mathcal{C}_t \neq \emptyset$ for all $t > t_-^*$.*
3. *The set S_I is bounded in $W^{2,p}(\Omega)$ for each compact $I \subset (t_-^*, \infty)$.*
4. *If we denote $\alpha_t = \inf\{\sup_\Omega u \,|\, u \in S_t\}$, we have $\lim_{t\to+\infty} \alpha_t = +\infty$.*
5. *The set $\mathcal{C}_{[t_-^*, t_-^*+\varepsilon)}$ is unbounded in $L^\infty(\Omega)$ for all $\varepsilon > 0$; there exists $C = C(h) > 0$ such that if $u \in S_{[t_-^*, t_-^*+\varepsilon)}$ and $\|u\|_{L^\infty(\Omega)} \geq C$, then $u < 0$ in Ω; if $u_n \in S_{[t_-^*, t_-^*+\varepsilon)}$ and $\|u_n\|_{L^\infty(\Omega)} \to \infty$, then $\max_K u_n \to -\infty$ for each compact $K \subset \Omega$.*
6. *If $S_{t_-^*}$ is unbounded in $L^\infty(\Omega)$, then there exists a function u_* such that $S_{t_-^*} = \{u_* + s\varphi_1^- \mid s \geq 0\}$.*

Both Theorems 1.2 and 1.4 are proved by a careful analysis of the behavior of the sets of solutions to equations with positive (resp., negative) eigenvalues when $\lambda_1^+(F) \searrow 0$ (resp., $\lambda_1^-(F) \nearrow 0$).

We note that not much is known on solutions of (1.2) when both eigenvalues are negative. Thus, before proving Theorem 1.4, we need to analyze solutions of problems in which $\lambda_1^-(F)$ is small and negative. This is the content of the next theorem, which is of clear independent interest.

THEOREM 1.5. *There exists $0 < L \leq \infty$ such that if $\lambda_1^-(F) \in (-L, 0)$, then*

1. *there exists a closed connected set $\mathcal{C} \subset S$ such that $\mathcal{C}_t \neq \emptyset$ for each $t \in \mathbb{R}$ (further, S_I is bounded in $W^{2,p}(\Omega)$ for each bounded $I \subset \mathbb{R}$);*
2. *setting $\alpha_t = \inf\{\sup_\Omega u \,|\, u \in S_t\}$ and $\overline{u}_t(x) = \sup\{u(x) \,|\, u \in S_t\}$, we have*

$$\lim_{t\to+\infty} \alpha_t = +\infty \qquad \text{and} \qquad \lim_{t\to-\infty} \sup_K \overline{u}_t(x) = -\infty$$

for each $K \subset\subset \Omega$, and $\overline{u}_t < 0$ in Ω for all t below some number $t^-(h)$.

The mere existence of solutions to (1.2) when $\lambda_1^-(F) \in (-L, 0)$ was recently proved in [1]. Here we describe qualitative properties of the set of solutions.

To summarize, the five theorems above give a global picture of the solutions of (1.2), depending on the values of the eigenvalues with respect to zero. This is shown on Figure 1.

A number of remarks on questions that are still open are now in order. First, it is clearly very important to give some characterization of the critical numbers t^* in terms of F, h, and λ. On submitting this paper we learned of a very recent work by Armstrong [2], where he studies this question in the case $\lambda = \lambda_1^+$ and proves part 1 of Theorem 1.2 by a different method. More specifically, he proves an interesting minimax formula for $\lambda_1^+(F)$, which generalizes the Donsker–Varadhan formula for linear operators to the nonlinear case. In particular, it is proved in [2] that

$$\lambda_1^+ = \min_{\mu\in\mathcal{M}(\Omega)} \sup_{u\in C_+^2(\overline{\Omega})} \int_\Omega \left(-\frac{F(D^2u(x), Du(x), u(x), x)}{u(x)} \right) d\mu(x).$$

Further, if $\mathcal{M}^*$ is the subset of the set of probability measures $\mathcal{M}$ on which this minimum is attained, then for each $\mu \in \mathcal{M}^*$, there exists a positive function $\varphi_\mu \in L^{N/(N-1)}(\Omega)$ such that $d\mu = \varphi_\mu \varphi_1^+ dx$, and the number t_+^* from Theorem 1.2 can be written as

$$t_+^* = -\min_{\mu\in\mathcal{M}^*} \int_\Omega h\varphi_\mu \, dx.$$

The results in [2] and our Theorem 1.2 are complementary to each other, as we describe the set of solutions, while the main theorems in [2] characterize the critical value $t_+^*(h)$.

Next, it is not clear how to distinguish between the two alternatives in statement 4 of Theorem 1.2 (that is, (2.1) and (2.2) on Figure 1) for any given operator F. A simple and important example where we have alternative (ii) is the Fučik operator $F(u) = \Delta u + \lambda_1(\Delta)u^+ + bu^-$; indeed, if we had (i), the fact that the solutions become positive for t close to t^* eliminates the term in u^-, giving a contradiction. A rather simple example of an operator for which both (i) and (ii) can happen (depending on f) is given in [2].

Naturally, the description of the set S when $\lambda_1^- = 0$, in contrast with $\lambda_1^+ = 0$, is less precise due to the fact that in this situation we only have degree theory at our disposal to get existence of solutions and that uniqueness of solutions above λ_1^- is not available in general (see, however, Theorem 1.6 below).

Further, a number of basic questions can be asked about exact multiplicity of solutions of (1.2) when $\lambda_1^+(F) < 0$. When $\lambda_1^-(F) > 0$ this question is a generalization of the famous Lazer–McKenna problem, which concerns the Fučik equation

$$\Delta u + bu^+ = \varphi_1 \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega. \tag{1.4}$$

Here $F(D^2u, Du, u) = \Delta u + bu^+$, $\lambda_1^+(F) = \lambda_1 - b, \lambda_1^-(F) = \lambda_1$, $b = \lambda_1^- - \lambda_1^+$ and λ_i are the eigenvalues of the Laplacian. It is known that equation (1.4) has exactly one solution if $b < \lambda_1$, exactly two solutions if $b \in (\lambda_1, \lambda_2)$, exactly four solutions if $b \in (\lambda_2, \lambda_3)$ and exactly six solutions if $b \in (\lambda_3, \lambda_3 + \delta)$, see [28] and the references in that paper. This example suggests that multiplicity of solutions when the two eigenvalues have opposite signs depends on the distance $\lambda_1^- - \lambda_1^+$. We conjecture that there exists a number C_0 such that if $\lambda_1^+(F) < 0 < \lambda_1^-(F) \le \lambda_1^+(F) + C_0$, then problem (1.2) has exactly two solutions, one solution, or no solution, depending on f.

In the same way it should be asked if uniqueness of solutions holds when $\lambda_1^-(F) \in (-L, 0)$ for some $L > 0$. In view of the discussion above one might expect that the answer is affirmative if the two eigenvalues are sufficiently close to each other. This fact constitutes our last main theorem.

THEOREM 1.6. *There exists a number $d_0 > 0$ such that if*

$$-d_0 \le \lambda_1^+(F) \le \lambda_1^-(F) < 0,$$

then problem (1.2) *has at most one solution.*

A consequence of this result is that if both Theorems 1.5 and 1.6 hold, then the sets $\mathcal{C}$ of solutions obtained in Theorems 1.4 and 1.5 are continuous curves, like in Theorems 1.1 and 1.2. We remark that d_0 is the difference between $\lambda_1^+(F, \Omega')$ and $\lambda_1^+(F, \Omega)$, where Ω' is some subset of Ω, whose Lebesgue measure is smaller than half the measure of Ω; see Proposition 6.1 and the proof of Theorem 1.6 in section 6.

The article is organized as follows. In section 2 we recall some known results which we use repeatedly in our analysis. We also complete the proof of Theorem 1.1. Section 3 is devoted to resonance phenomena at $\lambda_1^+ = 0$. In section 4 we analyze the existence and the properties of the set of solutions of (1.2) when $\lambda_1^- < 0$. This set serves to obtain the set of solutions at resonance when $\lambda_1^- = 0$, in section 5. Finally, in section 6 we prove Theorem 1.6.

Some notational conventions will be helpful in the following. When no confusion arises, we write $F[u] := F(D^2u, Du, u, x)$. We reserve the notation $\|\cdot\| = \|\cdot\|_{L^\infty(\Omega)}$; while for all other norms, we make precise mention to the corresponding space.

2. Preliminaries.

In this section we give, for the reader's convenience, some of the results of the general theory of viscosity solutions of HJB equations, which we use in the following. We start by restating the basic assumptions on the operator $F : S_N \times \mathbb{R}^N \times \mathbb{R} \times \Omega \to \mathbb{R}$.

(H0) F is positively homogeneous of degree one; that is, for all $t \ge 0$ and for all $(M, p, u, x) \in S_N \times \mathbb{R}^N \times \mathbb{R} \times \Omega$,

$$F(tM, tp, tu, x) = tF(M, p, u, x).$$

(H1) There exist $\gamma, \delta > 0$ such that for all $M, N \in S_N$, $p, q \in \mathbb{R}^N$, $u, v \in \mathbb{R}$, and a.e. $x \in \Omega$,

$$\begin{aligned}\mathcal{M}^-_{\lambda,\Lambda}(M - N) - \gamma|p - q| - \delta|u - v| &\le F(M, p, u, x) - F(N, q, v, x)\\ &\le \mathcal{M}^+_{\lambda,\Lambda}(M - N) + \gamma|p - q| + \delta|u - v|.\end{aligned}$$

(H2) $F(M, 0, 0, x)$ is continuous in $S_N \times \overline{\Omega}$.
(H3) If we denote $G(M, p, u, x) = -F(-M, -p, -u, x)$, then

$$G(M - N, p - q, u - v, x) \leq F(M, p, u, x) - F(N, q, v, x) \leq F(M - N, p - q, u - v, x).$$

Under (H0) the last assumption (H3) is equivalent to the convexity of F in (M, p, u). The simple proof of this fact can be found for instance in Lemma 1.1 in [24]. We recall that the Pucci extremal operators [10], [23] are defined by $\mathcal{M}^+(M) = \sup_{A\in\mathcal{A}} \operatorname{tr}(AM)$, $\mathcal{M}^-_{\lambda,\Lambda}(M) = \inf_{A\in\mathcal{A}} \operatorname{tr}(AM)$, where $\mathcal{A} \subset \mathcal{S}_N$ denotes the set of matrices whose eigenvalues lie in the interval $[\lambda, \Lambda]$.

We often use the following results from [24] (Theorems 1.2–1.4 of that paper), which state that the principle eigenvalues are simple and isolated.

Theorem 2.1 (see [24]). *Assume F satisfies* (H0)−(H3) *and there exists a viscosity solution $u \in C(\overline{\Omega})$ of*

$$F(D^2u, Du, u, x) = -\lambda_1^+ u \quad in \quad \Omega, \qquad u = 0 \quad on \quad \partial\Omega, \tag{2.1}$$

or of one of the problems

$$\begin{cases} F(D^2u, Du, u, x) \leq -\lambda_1^+ u & in \quad \Omega, \\ u > 0 & in \quad \Omega, \end{cases} \tag{2.2}$$

$$\begin{cases} F(D^2u, Du, u, x) \geq -\lambda_1^+ u & in \quad \Omega, \\ u(x_0) > 0, \qquad u \leq 0 & on \quad \partial\Omega \end{cases} \tag{2.3}$$

for some $x_0 \in \Omega$. Then $u \equiv t\varphi_1^+$ for some $t \in \mathbb{R}$. If a function $v \in C(\overline{\Omega})$ satisfies either (2.1) *or the inverse inequalities in* (2.2) *or* (2.3)*, with λ_1^+ replaced by λ_1^-, then $v \equiv t\varphi_1^-$ for some $t \in \mathbb{R}$.*

Theorem 2.2 (see [24]). *There exists $\varepsilon_0 > 0$ depending on $N, \lambda, \Lambda, \gamma, \delta, \Omega$ such that the problem*

$$F(D^2u, Du, u, x) = -\lambda u \quad in \quad \Omega, \qquad u = 0 \quad on \quad \partial\Omega, \tag{2.4}$$

has no solutions $u \not\equiv 0$ for $\lambda \in (-\infty, \lambda_1^- + \varepsilon_0) \setminus \{\lambda_1^+, \lambda_1^-\}$.

We shall need the following *one-sided* ABP estimate, which was obtained in [24] as well. The ABP inequality for proper operators can be found in [12] (an ABP inequality for the Pucci operator was first proved in [11]). We recall that λ_1^+, λ_1^- are bounded above and below by constants which depend only on $N, \lambda, \Lambda, \gamma, \delta, \Omega$, and that both principal eigenvalues of any proper operator are positive; see [24].

Theorem 2.3 (see [24]). *Suppose the operator F satisfies* (H0)−(H3).

I. *If $\lambda_1^-(F, \Omega) > 0$, then for any $u \in C(\overline{\Omega})$, $f \in L^N(\Omega)$, the inequality*

$$F(D^2u, Du, u, x) \leq f$$

implies

$$\sup_\Omega u^- \leq C \left(\sup_{\partial\Omega} u^- + \|f^+\|_{L^N(\Omega)} \right),$$

where C depends on $N, \lambda, \Lambda, \gamma, \delta$, Ω, and $1/\lambda_1^-$.

II. *In addition, if* $\lambda_1^+(F,\Omega) > 0$*, then* $F(D^2u, Du, u, x) \geq f$ *implies*

$$\sup_\Omega u \leq C\left(\sup_{\partial\Omega} u^+ + \|f^-\|_{L^N(\Omega)}\right).$$

Hence, if $\lambda_1^+(F,\Omega) > 0$*, then the comparison principle holds: if* $u, v \in C(\overline{\Omega})$ *are such that* $F[u] \leq F[v]$ *in* Ω*,* $u \geq v$ *on* $\partial\Omega$*, and either* u *or* v *is in* $W^{2,N}(\Omega)$*, then* $u \geq v$ *in* Ω*.*

Note that this result with $f = 0$ gives *one-sided maximum principles*. We also recall the following strong maximum principle or Hopf's lemma, which is a consequence of the results in [3] (a simple proof can be found in the appendix of [1]).

THEOREM 2.4 (see [3]). *Suppose* $w \in C(\overline{\Omega})$ *is a viscosity solution of*

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2w) - \gamma|Dw| - \delta w \leq 0 \quad in \quad \Omega,$$

and $w \geq 0$ *in* Ω*. Then either* $w \equiv 0$ *in* Ω *or* $w > 0$ *in* Ω*, and at any point* $x_0 \in \partial\Omega$ *at which* $w(x_0) = 0$*, we have* $\liminf_{t\searrow 0} \frac{w(x_0+t\nu)-w(x_0)}{t} > 0$*, where* ν *is the interior normal to* $\partial\Omega$ *at* x_0*.*

We are going to use the following regularity result. It was proved in this generality in [30] (interior estimate) and in [32] (global estimate).

THEOREM 2.5 (see [30] and [32]). *Suppose the operator* F *satisfies* (H0)–(H2) *and* u *is a viscosity solution of* $F(D^2u, Du, u, x) = f$ *in* Ω*,* $u = 0$ *on* $\partial\Omega$*. Then* $u \in W^{2,p}(\Omega)$*, and*

$$\|u\|_{W^{2,p}(\Omega)} \leq C\left(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)}\right),$$

where C *depends only on* $N, p, \lambda, \Lambda, \gamma, \delta, \Omega$*.*

Next we quote the existence result from [21] and [24].

THEOREM 2.6 (see [24]). *Suppose the operator* F *satisfies* (H0)–(H3)*.*

I. *If* $\lambda_1^-(F,\Omega) > 0$*, then for any* $f \in L^p(\Omega), p \geq N$*, such that* $f \geq 0$ *in* Ω*, there exists a solution* $u \in W^{2,p}(\Omega)$ *of* $F(D^2u, Du, u, x) = f$ *in* Ω*,* $u = 0$ *on* $\partial\Omega$*, such that* $u \leq 0$ *in* Ω*.*

II. *In addition, if* $\lambda_1^+(F,\Omega) > 0$*, then for any* $f \in L^p(\Omega)$*,* $p \geq N$*, there exists a unique viscosity solution* $u \in W^{2,p}(\Omega)$ *of* $F(D^2u, Du, u, x) = f$ *in* Ω*,* $u = 0$ *on* $\partial\Omega$*.*

The next theorem is a simple consequence of the compact embedding $W^{2,p}(\Omega) \hookrightarrow C^{1,\alpha}(\Omega)$, Theorem 2.5, and the convergence properties of viscosity solutions (see Theorem 3.8 in [12]).

THEOREM 2.7. *Let* $\lambda_n \to \lambda$ *in* $\mathbb{R}$ *and* $f_n \to f$ *in* $L^p(\Omega)$*,* $p > N$*. Suppose* F *satisfies* (H1) *and* u_n *is a solution of* $F(D^2u_n, Du_n, u_n, x) + \lambda_n u_n = f_n$ *in* Ω*,* $u_n = 0$ *on* $\partial\Omega$*. If* $\{u_n\}$ *is bounded in* $L^\infty(\Omega)$*, then a subsequence of* $\{u_n\}$ *converges in* $C^1(\overline{\Omega})$ *to a function* u*, which solves* $F(D^2u, Du, u, x) + \lambda u = f$ *in* Ω*,* $u = 0$ *on* $\partial\Omega$*.*

We now give the proof of Theorem 1.1.

Proof of Theorem 1.1. Part 1 is a consequence of Theorems 2.3 and 2.6.

Let us prove part 2. For $t \in \mathbb{R}$, let u_t be the solution of (1.2) with f as in (1.3); that is, $F[u_t] = t\varphi_1^+ + h$, where $\lambda_1^+(F) > 0$. Then $\|u_t\|/t$ is bounded as $t \to -\infty$. Indeed, if this is not the case, there exists a sequence $\{t_n\}$ such that we have $t_n \to -\infty$ and $\|u_{t_n}/t_n\| \to \infty$, in particular, $\|u_{t_n}\| \to \infty$. Defining $\hat{u}_n = u_{t_n}/\|u_{t_n}\|$, we get by (H0)

$$F(D^2\hat{u}_n, D\hat{u}_n, \hat{u}_n, x) = \frac{t_n}{\|u_{t_n}\|}\varphi_1^+ + \frac{h}{\|u_{t_n}\|} \quad \text{in} \quad \Omega, \qquad \hat{u}_n = 0 \quad \text{on} \quad \partial\Omega.$$

The right-hand side of this equation converges to zero in $L^p(\Omega)$, so $\hat{u}_n$ converges uniformly to zero by Theorem 2.7 (note the limit equation $F[\hat{u}] = 0$ has only the trivial solution, since $\lambda_1^+(F,\Omega) > 0$). This contradicts $\|\hat{u}_n\| = 1$.

Thus, by Theorem 2.7, for some sequence $t_n \to -\infty$, we have that

$$\lim_{n\to\infty} \frac{u_{t_n}}{-t_n} = v^* \quad \text{in} \quad C^1(\overline{\Omega}), \tag{2.5}$$

where v^* satisfies

$$F(D^2v^*, Dv^*, v^*, x) = -\varphi_1^+ \quad \text{in} \quad \Omega, \qquad v^* = 0 \quad \text{on} \quad \partial\Omega.$$

By Theorems 2.3 and 2.4, we have $v^* > 0$ in Ω and $\frac{\partial v}{\partial \nu} > 0$ on $\partial\Omega$. These facts, (2.5), and the monotonicity of u_t in t imply the last two statements of part 2 (the analysis for $t \to \infty$ is similar).

That S is Lipschitz follows from (H3) and Theorem 2.3, applied to

$$F[u_t - u_s] \geq (t-s)\varphi_1^+ \quad \text{and} \quad F[u_s - u_t] \geq (s-t)\varphi_1^+.$$

Finally, the convexity property of the curve is a consequence of the following simple lemma and the comparison principle, Theorem 2.3.

LEMMA 2.1. *Let $t_0, t_1 \in \mathbb{R}$ and $t_k = kt_1 + (1-k)t_0$ for $k \in [0,1]$. Suppose $u_{t_i} \in S_{t_i}$, $i = 0, 1$. Then the function $ku_{t_1} + (1-k)u_{t_0}$ is a supersolution of*

$$F(D^2u, Du, u, x) = t_k\varphi_1^+ + h \quad \textit{in} \quad \Omega, \qquad u = 0 \quad \textit{on} \quad \partial\Omega.$$

Proof. Use $F[ku_{t_1} + (1-k)u_{t_0}] \leq kF[u_{t_1}] + (1-k)F[u_{t_0}]$. ◻

Notation. In what follows it will be convenient for us to write problem (1.1) in the form

$$\begin{cases} F(D^2u, Du, u, x) + \lambda u = t\varphi_1(x) + h(x) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \Omega, \end{cases} \tag{2.6}$$

where F is supposed to be proper (if necessary, we replace F by $F - \delta$ and λ by $\lambda + \delta$), and study its solvability in terms of the value of the parameter $\lambda \in \mathbb{R}^+$. For instance, Theorem 1.2 corresponds to $\lambda = \lambda_1^+$, Theorem 1.4 corresponds to $\lambda = \lambda_1^-$, Theorem 1.1 corresponds to $\lambda < \lambda_1^+$, etc.

3. Resonance at $\boldsymbol{\lambda = \lambda_1^+}$. Proof of Theorem 1.2. We first set up some preliminaries. Let $\{\lambda_n\}$ be a sequence such that $\lambda_n < \lambda_1^+$ for all n, and $\lim_{n\to\infty} \lambda_n = \lambda_1^+$. We consider the problem

$$F(D^2u, Du, u, x) + \lambda_n u = t\varphi_1^+ + h \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega,$$

and its unique solution $u(n,t)$. In the following we shall write $u_n(t) = u(n,t)$ and also sometimes u_n or u_t instead of $u(n,t)$ when one of the parameters is kept fixed.

We define $\Gamma_n^+ = \{u_n(t) \,|\, t \in \mathbb{R}\}$. Recall that, by Theorem 1.1, if $s < t$, then $u_n(t) < u_n(s)$.

We parameterize Γ_n^+ in the following way. We take a reference function $\tilde{u}_n = u_n(\tilde{t}_n) \in \Gamma_n^+$, which is arbitrary but fixed for each $n \in \mathbb{N}$ (later we choose an appropriate sequence $\{\tilde{u}_n\}$), and we define the function

$$\begin{cases} d_n \;:\; \Gamma_n^+ \to \mathbb{R} \\ d_n(u) = \operatorname{sign}(u - \tilde{u}_n)\|u - \tilde{u}_n\|. \end{cases} \tag{3.1}$$

LEMMA 3.1. *The function $d_n : \Gamma_n^+ \to \mathbb{R}$ is a bijection for each $n \in \mathbb{N}$. In addition, d_n is (Lipschitz) continuous.*

Proof. By (H3) for any $t_1, t_2 \in \mathbb{R}$ (say $t_1 > t_2$), we have

$$F[u_n(t_1) - u_n(t_2)] + \lambda_n(u_n(t_1) - u_n(t_2)) \geq (t_1 - t_2)\varphi_1^+. \tag{3.2}$$

The ABP inequality (Theorem 2.3) applies to this inequality—here we use $\lambda_n < \lambda_1^+$—so we have

$$\|u_n(t_1) - u_n(t_2)\| \leq C_n|t_1 - t_2|.$$

If $t_1 > t_2 > \tilde{t}_n$ (the argument is the same if $t_2 < t_1 < \tilde{t}_n$), we get

$$|d_n(u_1) - d_n(u_2)| = \|u_{t_1} - \tilde{u}_n\| - \|u_{t_2} - \tilde{u}_n\| \leq \|u_{t_1} - u_{t_2}\| \leq C_n|t_1 - t_2|.$$

If $t_1 > \tilde{t}_n > t_2$, we have

$$|d_n(u_1) - d_n(u_2)| \leq \|u_{t_1} - \tilde{u}_n\| + \|u_{t_2} - \tilde{u}_n\| \leq C_n(t_1 - \tilde{t}_n + \tilde{t}_n - t_2) = C_n|t_1 - t_2|,$$

which proves the Lipschitz continuity.

Assume that $d_n(u_n(t_1)) = d_n(u_n(t_2))$; then $\|u_n(t_1) - \tilde{u}_n\| = \|u_n(t_2) - \tilde{u}_n\|$ and $u_n(t_i) > \tilde{u}_n$ (or $u_n(t_i) < \tilde{u}_n$) for $i = 1, 2$. On the other hand, if $t_1 \neq t_2$, say $t_1 < t_2$, then $u_n(t_1) > u_n(t_2)$ and, consequently, $\|u_n(t_1) - \tilde{u}_n\| \neq \|u_n(t_2) - \tilde{u}_n\|$, which is impossible. Thus, d_n is one-to-one. By part 2 in Theorem 1.1, we see that d_n is onto. ∎

Now we start the analysis of the resonance at $\lambda = \lambda_1^+$ (recall we are working with (2.6)). Given $s \in \mathbb{R}$, we define the proposition $\mathcal{P}(s)$ as follows:

$\mathcal{P}(s)$: There exist sequences $\{\lambda_n\}$, $\{h_n\}$ and $\{u_n\}$ such that $\lambda_n < \lambda_1^+$ for all n, $\lim_{n\to\infty} \lambda_n = \lambda_1^+$, $h_n \to h$ in $L^p(\Omega)$ as $n \to \infty$,

$$F(D^2u_n, Du_n, u_n, x) + \lambda_n u_n = s\varphi_1^+ + h_n, \tag{3.3}$$

and $\|u_n\|$ is unbounded.

By dividing (3.3) by $\|u_n\|$—thanks to (H0), Theorem 2.1, and Theorem 2.7—we easily see that this definition is equivalent to the following:

$\mathcal{P}(s)$: There exist sequences $\{\lambda_n\}$ and $\{h_n\}$ such that $\lambda_n < \lambda_1^+$ for all n, $\lambda_n \to \lambda_1^+$, $h_n \to h$ in $L^p(\Omega)$, the solution of $F(D^2u_n, Du_n, u_n, x) + \lambda_n u_n = s\varphi_1^+ + h_n$ satisfies $\|u_n\| \to \infty$, and

$$\frac{u_n}{\|u_n\|} \to \varphi_1^+ > 0 \quad \text{in} \quad C^1(\overline{\Omega}).$$

We define

$$t_+^* = \sup\{t \in \mathbb{R} \,|\, \mathcal{P}(s) \text{ for all } s < t\}. \tag{3.4}$$

The next lemmas give meaning to this definition.

LEMMA 3.2. *Given $\bar{t} \in \mathbb{R}$, $\mathcal{P}(\bar{t})$ implies $\mathcal{P}(t)$ for all $t < \bar{t}$.*

Proof. Assuming the contrary, there is $t_0 < \bar{t}$ such that $\mathcal{P}(t_0)$ is false. This means that for some sequences $\{\lambda_n\}$, $\{h_n\}$ as above, the sequence of the solutions of

$$F(D^2v_n, Dv_n, v_n, x) + \lambda_n v_n = \bar{t}\varphi_1^+ + h_n \quad \text{in} \quad \Omega, \qquad v_n = 0 \quad \text{on} \quad \partial\Omega,$$

is unbounded; while the sequence of the solutions of

$$F(D^2u_n, Du_n, u_n, x) + \lambda_n u_n = t_0\varphi_1^+ + h_n \quad \text{in} \quad \Omega, \qquad u_n = 0 \quad \text{on} \quad \partial\Omega,$$

is bounded in $L^\infty(\Omega)$. By the comparison principle (Theorem 2.3), $v_n \le u_n$ for all n, since $\bar{t} > t_0$ and $\varphi_1^+ > 0$. On the other hand, by the one-sided ABP inequality, Theorem 2.3(I) (note λ_n is uniformly away from λ_1^-; that is, $\lambda_1^-(F+\lambda_n) \ge \lambda_1^-(F) - \lambda_1^+(F) > 0$), the sequence $\{v_n\}$ is bounded below. Thus, $\{v_n\}$ is bounded, which is a contradiction. ∎

LEMMA 3.3. *There exists a real number $\bar{t}_1 = \bar{t}_1(h)$ such that the problem*

$$F(D^2u, Du, u, x) + \lambda_1^+ u = t\varphi_1^+ + h \quad in \quad \Omega, \qquad u = 0 \quad on \quad \partial\Omega, \tag{3.5}$$

has no solutions for $t < \bar{t}_1$.

Proof. Let v be the solution of the Dirichlet problem

$$F(D^2v, Dv, v, x) = -h \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega$$

(this problem is uniquely solvable by the well-known results on proper equations or by Theorem 2.6). We are going to show that the statement of the lemma is true with

$$\bar{t}_1 = -1 - \lambda_1^+ \sup_{x\in\Omega} \frac{v(x)}{\varphi_1^+(x)}.$$

The last quantity is finite by Theorems 2.3–2.5.

Indeed, if (3.5) has a solution $u = u(t)$ for some $t < \bar{t}_1$, we get

$$\begin{aligned} F[u+v] + \lambda_1^+(u+v) &\le F[u] + F[v] + \lambda_1^+ u + \lambda_1^+ v \\ &\le t\varphi_1^+ + \lambda_1^+ v \le -\varphi_1^+ < 0, \end{aligned} \tag{3.6}$$

where we have used $F[u+v] \le F[u] + F[v]$, which follows from (H3). Since we have $\lambda_1^-(F+\lambda_1^+, \Omega) = \lambda_1^- - \lambda_1^+ > 0$, Theorem 2.3(I) again applies and yields $u + v > 0$ in Ω. We can now use Theorem 2.1 and conclude that $u + v$ is a multiple of φ_1^+, which contradicts the strict inequality in (3.6). ∎

LEMMA 3.4. *The set $T = \{t \in \mathbb{R} \,|\, \mathcal{P}(t)\}$ is not empty.*

Proof. Assuming the contrary, we find sequences $\{t_n\}$, $\{u_n^m\}$ such that $\mathcal{P}(t_n)$ is false, $t_n \to -\infty$ as $n \to \infty$, u_n^m satisfies

$$F(D^2u_n^m, Du_n^m, u_n^m, x) + (\lambda_1^+ - 1/m)u_n^m = t_n\varphi_1^+ + h \quad \text{in} \quad \Omega,\ u_n^m = 0 \quad \text{on} \quad \partial\Omega,$$

for each n, and $\{u_n^m\}$ is bounded in $L^\infty(\Omega)$ as $m \to \infty$. Hence, by Theorem 2.7, u_n^m converges as $m \to \infty$ (up to a subsequence), for each fixed n, to a function u_n which satisfies (3.5) with $t = t_n$. This and the previous lemma give a contradiction when t_n is sufficiently small. ∎

LEMMA 3.5. *The set T is bounded above; that is, t_+^* is a real number.*

Proof. Let $\lambda_n \nearrow \lambda_1^+$, $h_n \to h$ in $L^p(\Omega)$, and let $u_n = u_n(t)$ be such that

$$F(D^2u_n, Du_n, u_n, x) + \lambda_n u_n = t\varphi_1^+ + h_n \quad \text{in} \quad \Omega, \qquad u_n = 0 \quad \text{on} \quad \partial\Omega$$

(we recall that this problem has a unique solution, since $\lambda_n < \lambda_1^+$ and comparison holds). We need to show $\{u_n\}$ is bounded in $L^\infty(\Omega)$ if t is large enough.

First, Theorem 2.3(I) implies that u_n is bounded below independently of n (we recall once again that $\lambda_1^-(F+\lambda_n) \ge \lambda_1^- - \lambda_1^+ > 0$).

Next, let v_n be the solution of the Dirichlet problem

$$F(D^2 v_n, Dv_n, v_n, x) = \min\{h_n, 0\} \le 0 \quad \text{in} \quad \Omega, \qquad v = 0 \quad \text{on} \quad \partial\Omega.$$

Then $v_n \ge 0$ in Ω by the maximum principle, $\{v_n\}$ is bounded in $C^1(\overline{\Omega})$ by Theorems 2.3 and 2.5, and

$$F[v_n] + \lambda_n v_n \le \min\{h_n, 0\} + \lambda_1^+ v_n \le h_n + t\varphi_1^+ = F[u_n] + \lambda_n u_n,$$

provided

$$t > \lambda_1^+ \sup_{x\in\Omega, n\in\mathbb{N}} \frac{v_n(x)}{\varphi_1^+(x)}\,. \tag{3.7}$$

By the comparison principle, $u_n \le v_n$; hence, u_n is bounded above independently of n. So $\mathcal{P}(t)$ is false if (3.7) holds. ▯

The following two propositions give existence and uniqueness of solutions to our problem at resonance, provided $t > t_+^*$.

PROPOSITION 3.1. 1. *If $t > t_+^*$, then the equation*

$$F(D^2 u, Du, u, x) + \lambda_1^+ u = t\varphi_1^+ + h \quad \textit{in} \quad \Omega, \qquad u = 0 \quad \textit{on} \quad \partial\Omega, \tag{3.8}$$

possesses at least one solution.

2. *If $t < t_+^*$, then* (3.8) *has no solutions.*

Proof. 1. Given a sequence $\{\lambda_n\}$ such that $\lambda_n < \lambda_1^+$ for all $n \in \mathbb{N}$ and $\lambda_n \to \lambda_1^+$ as $n \to \infty$, there is a sequence $\{u_n\}$ such that

$$F(D^2 u_n, Du_n, u_n, x) + \lambda_n u_n = t\varphi_1^+ + h \quad \text{in} \quad \Omega, \qquad u_n = 0 \quad \text{on} \quad \partial\Omega. \tag{3.9}$$

Then $t > t^*$ implies that $\{u_n\}$ is bounded, so by Theorem 2.7 $\{u_n\}$ converges, up to a subsequence, to a function u satisfying (3.8).

2. Suppose, for contradiction, (3.8) has a solution u for some $t < t_+^*$. Fix $t_1 \in (t, t_+^*)$. Then $\mathcal{P}(t_1)$ holds, so for some sequences $\lambda_n \nearrow \lambda_1^+$, $h_n \to h$, the sequence of solutions u_n of

$$F(D^2 u_n, Du_n, u_n, x) + \lambda_n u_n = t_1\varphi_1^+ + h_n \quad \text{in} \quad \Omega, \qquad u_n = 0 \quad \text{on} \quad \partial\Omega,$$

is such that $u_n \ge k_n \varphi_1^+$ for some $k_n \to \infty$. Now let w_n be the solution of

$$F(D^2 w_n, Dw_n, w_n, x) = h_n - h \quad \text{in} \quad \Omega, \qquad u_n = 0 \quad \text{on} \quad \partial\Omega. \tag{3.10}$$

By Theorems 2.3 and 2.5, we know that (up to a subsequence) $w_n \to 0$ in $C^1(\overline{\Omega})$. Hence, by the boundary Lipschitz estimates (see Theorem 2.5 or Proposition 4.9 in [24]) applied to (3.8) and (3.10), we have

$$\|u\| + \|w_n\| \le C\text{dist}(x, \partial\Omega),$$

which implies

$$u_n - w_n - u > 0$$

for n sufficiently large. Since $w_n \to 0$ and $\lambda_n \to \lambda_1^+$, we also have

$$t_1\varphi_1^+ - 2\lambda_1^+|w_n| > t\varphi_1^+ \qquad \text{and} \qquad |u| \le \frac{t_1 - t}{2(\lambda_1^+ - \lambda_n)}\varphi_1^+ \quad \text{in} \quad \Omega.$$

However, (H3) implies $F[u_n - w_n - u] \geq F[u_n] - F[w_n] - F[u]$, so

$$F[u_n - w_n - u] + \lambda_n(u_n - w_n - u) \geq (t_1 - t)\varphi_1^+ - \lambda_1^+|w_n| + (\lambda_1^+ - \lambda_n)u \geq 0.$$

Then the maximum principle (Theorem 2.3) gives $u_n - w_n - u \leq 0$, which is a contradiction. ∎

Next we prove the uniqueness of solutions above t_+^*. In order to do this, we need the following simple result on convex functions.

LEMMA 3.6. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be positively homogeneous of degree one and convex. If for some $u, v \in \mathbb{R}^n$ and for some $\tau > 0$ we have*

$$f(u + \tau v) = f(u) + \tau f(v), \tag{3.11}$$

then (3.11) *holds for all $\tau \geq 0$.*

Proof. Using (3.11) and the homogeneity of f, we find that

$$f(\lambda_0 u + (1 - \lambda_0)v) = \lambda_0 f(u) + (1 - \lambda_0)f(v),$$

with $\lambda_0 = 1/(1 + \tau)$. If there is $\lambda \in (\lambda_0, 1)$ such that

$$f(\lambda u + (1 - \lambda)v) < \lambda f(u) + (1 - \lambda)f(v), \tag{3.12}$$

we take $\theta = 1 - \lambda_0/\lambda \in (0, 1)$—that is, $(1 - \theta)\lambda = \lambda_0$—and note that

$$\begin{aligned}\lambda_0 f(u) + (1 - \lambda_0)f(v) &= f(\lambda_0 u + (1 - \lambda_0)v)\\ &= f(\theta v + (1 - \theta)(\lambda u + (1 - \lambda)v))\\ &\leq \theta f(v) + (1 - \theta)f(\lambda u + (1 - \lambda)v)\\ &< (1 - \lambda_0)f(v) + \lambda_0 f(u),\end{aligned}$$

which is a contradiction. If there is $\lambda \in (0, \lambda_0)$ such that (3.12) holds, we proceed similarly. Thus, $f(\lambda u + (1 - \lambda)v) = \lambda f(u) + (1 - \lambda)f(v)$ for all $\lambda \in [0, 1]$. From here we get the conclusion, taking $\lambda = 1/(1 + t)$. ∎

PROPOSITION 3.2. 1. *If $t > t_+^*$ and u_1, u_2 satisfy*

$$F(D^2u_i, Du_i, u_i, x) + \lambda_1^+ u_i = t\varphi_1^+ + h \quad \text{in} \quad \Omega, \quad u_i = 0 \quad \text{on} \quad \partial\Omega,$$

$i = 1, 2$, then $u_1 = u_2$.

2. *If $t = t_+^*$ and u_1, u_2 are as in part 1, then $u_1 = u_2 + s\varphi_1$ for some $s \in \mathbb{R}$.*

Proof. Suppose $u_1 \neq u_2$, then we may assume there exists $x_0 \in \Omega$ such that $u_1(x_0) > u_2(x_0)$. By (H3), we have $F[u_1 - u_2] + \lambda_1^+(u_1 - u_2) \geq 0$, so by Theorem 2.1, there exists $\tau > 0$ such that $u_1 - u_2 = \tau\varphi_1^+$. This implies

$$F[u_1 + \tau\varphi_1^+] = F[u_1] + \tau F[\varphi_1^+] \quad \text{a.e. in } \Omega \tag{3.13}$$

(note $u_1, \varphi_1^+ \in W^{2,N}(\Omega)$). Consider the function $f(X) = F(X, x)$, where $X = (M, p, u) \in S_N \times \mathbb{R}^N \times \mathbb{R} = \mathbb{R}^{N^2+N+1}$, and $x \in \Omega$ is fixed. By hypotheses (H0) and (H3), the function f is positively homogeneous of degree one and convex in X. Therefore, we can use Lemma 3.6 to conclude that (3.13) holds for every $\tau > 0$.

We obtain that for every $n \in \mathbb{N}$, the function $v_n = u_1 + n\varphi_1^+$ satisfies

$$F(D^2v_n, Dv_n, v_n, x) + \lambda_1^+ v_n = t\varphi_1^+ + h \quad \text{in} \quad \Omega, \qquad u_i = 0 \quad \text{on} \quad \partial\Omega.$$

It follows that

$$F[v_n] + \left(\lambda_1^+ - \frac{1}{n^2}\right) v_n = t\varphi_1^+ + h - \frac{1}{n^2}u_1 - \frac{1}{n}\varphi_1^+ =: t\varphi_1^+ + h_n$$

in Ω. Note $h_n \to h$ in $L^p(\Omega)$. However, this is impossible if $t > t_+^*$ by the definition of t_+^*, since $\|v_n\|$ is unbounded, which means $\mathcal{P}(t)$ holds. ∎

Now we study the behavior of the branch Γ_n^+ as $n \to \infty$. Let $\tilde{u}$ be the unique solution (given by Proposition 3.1) of

$$F(D^2\tilde{u}, D\tilde{u}, \tilde{u}, x) + \lambda_1^+\tilde{u} = (1 + t_+^*)\varphi_1^+ + h \quad \text{in} \quad \Omega, \quad \tilde{u} = 0 \quad \text{on} \quad \partial\Omega,$$

and set

$$d(u) = \operatorname{sign}(u - \tilde{u})\|u - \tilde{u}\|.$$

Lemma 3.7. *If u_i and t_i, $i = 1, 2$, are such that $d(u_1) = d(u_2)$ and*

$$F(D^2u_i, Du_i, u_i, x) + \lambda_1^+ u_i = t_i\varphi_1^+ + h \quad \textit{in} \quad \Omega, \quad u_i = 0 \quad \textit{on} \quad \partial\Omega,$$

for $i = 1, 2$, then $t_1 = t_2$ and $u_1 = u_2$.

Proof. By Proposition 3.2 then $u_1 \neq u_2$ implies $t_1 \neq t_2$. If $t_1 \neq t_2$ (say $t_1 > t_2$),

$$F[u_1 - u_2] + \lambda_1^+(u_1 - u_2) \geq (t_1 - t_2)\varphi_1^+ > 0 \quad \text{in} \quad \Omega, \quad u_1 - u_2 = 0 \quad \text{on} \quad \partial\Omega.$$

Either there exists $x_0 \in \Omega$ such that $u_1(x_0) > u_2(x_0)$ or $u_1 \leq u_2$ in Ω. In the first case, Theorem 2.1 implies the existence of $\tau > 0$ such that $u_1 - u_2 = \tau\varphi_1^+$ so that $u_1 > u_2$ in Ω. In the second case, by the strong maximum principle, we have that $u_1 = u_2$ (excluded by $t_1 \neq t_2$) or $u_1 < u_2$ in Ω.

Thus, if $u_1 \neq u_2$, then either $u_1 > u_2$ or $u_1 < u_2$ in Ω, and in both cases $d(u_1) \neq d(u_2)$, completing the proof of the lemma. ∎

We recall (Lemma 3.1) that the set Γ_n^+ can be reparameterized as a curve by using the function d_n. In the definition of d_n we used the arbitrary function $\tilde{u}_n$, which we choose now as the unique solution of

$$F(D^2\tilde{u}_n, D\tilde{u}_n, \tilde{u}_n, x) + \lambda_n\tilde{u}_n = (1 + t_+^*)\varphi_1^+ + h \quad \text{in} \quad \Omega, \quad \tilde{u}_n = 0 \quad \text{on} \quad \partial\Omega.$$

By the definition of t_+^*, $\{\|\tilde{u}_n\|\}$ is bounded, so by Theorem 2.7 and the uniqueness property proved in Proposition 3.2, we find that $\tilde{u}_n \to \tilde{u}$, where $\tilde{u}$ is as above, the unique solution of

$$F(D^2\tilde{u}, D\tilde{u}, \tilde{u}, x) + \lambda_1^+\tilde{u} = (1 + t_+^*)\varphi_1^+ + h \quad \text{in} \quad \Omega, \quad \tilde{u} = 0 \quad \text{on} \quad \partial\Omega.$$

By Lemma 3.7, for fixed $d \in \mathbb{R}$, the following system in (u, t)

$$\begin{cases} F(D^2u, Du, u, x) + \lambda_n u = t\varphi_1^+ + h \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \\ d_n(u) = d, \end{cases} \tag{3.14}$$

has a unique solution (u_n, t_n) in $C(\overline{\Omega}) \times \mathbb{R}$. The sequence $\{u_n\}$ is bounded, since $\|u_n - \tilde{u}_n\| = |d|$ and $\{\tilde{u}_n\}$ is bounded. Hence $\{t_n\}$ is also bounded (if not, $u_n/t_n \to 0$, so by passage to the limit, $F[0] = \varphi_1^+$).

Then subsequences of $\{u_n\}$ and $\{t_n\}$ converge to a function $u = u(d)$ and a number $t = t(d)$, which satisfy

$$F(D^2u, Du, u, x) + \lambda_1^+ u = t\varphi_1^+ + h \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega. \tag{3.15}$$

By Lemma 3.7, the whole sequences $\{u_n\}$ and $\{t_n\}$ converge to the same limit that we call $u(d)$ and $t(d)$.

LEMMA 3.8. *The map*

$$\begin{cases} U : \mathbb{R} \to C(\overline{\Omega}) \times \mathbb{R} \\ U(d) = (u(d), t(d)) \end{cases}$$

is continuous.

Proof. Take $d_k \to d$ as $k \to \infty$. Then the sequences $u_k = u(d_k)$, $t_k = t(d_k)$ are bounded, as above. Any convergent subsequence of $\{(u_k, t_k)\}$ tends to a solution of an equation to which Lemma 3.7 applies, so the whole sequences u_k, t_k converge to $u(d)$, $t(d)$. ☐

We define $\Gamma^+ = \{u(d) \mid d \in \mathbb{R}\}$. The last lemma allows us to say that Γ^+ is actually a continuous curve, the pointwise limit of the curves $\{\Gamma_n^+\}$.

LEMMA 3.9. *If $t_1 > t_2 \geq t_+^*$, then any two solutions u_1, u_2 of*

$$F(D^2u_i, Du_i, u_i, x) + \lambda_1^+ u_i = t_i\varphi_1^+ + h \quad \text{in} \quad \Omega, \quad u_i = 0 \quad \text{on} \quad \partial\Omega,$$

are such that $u_1 < u_2$ in Ω.

Proof. We already showed in the proof of Lemma 3.7 that either $u_1 > u_2$ or $u_1 < u_2$ in Ω. Since the curve Γ^+ is the limit of Γ_n^+, which is strictly decreasing in t, $u_1 > u_2$ is impossible. ☐

Proof of Theorem 1.2. The set of solutions is $\{(u(d), t(d)) \mid d \in \mathbb{R}\}$, as the above discussion shows. Part 1 of the theorem was proved in Proposition 3.1. The first two statements of part 2 follow from Proposition 3.2 and Lemma 3.9.

For $t > t^*$, let u_t be the solution of

$$F(D^2u_t, Du_t, u_t, x) + \lambda_1^+ u_t = t\varphi_1^+ + h \quad \text{in} \quad \Omega, \quad u_t = 0 \quad \text{on} \quad \partial\Omega. \tag{3.16}$$

By Lemma 3.9, u_t is strictly decreasing in t.

When $t \to t_+^*$, two cases may occur: either $\|u_t\|$ is bounded or $\|u_t\| \to \infty$. In the first case the monotonous sequence u_t converges in $C^1(\overline{\Omega})$ to a solution u^* of (3.16) with $t = t_+^*$. Then by Proposition 3.2, all solutions $u \in \Gamma^+$ with $d(u) \geq d(u^*)$ are solutions of (3.16) with $t = t_+^*$, which is the situation described in part 4(ii). In the second case $u_t/\|u_t\|$ converges in $C^1(\overline{\Omega})$ to $\varphi_1^+ > 0$, which implies part 4(i). Note in this case there cannot be solutions with $t = t_+^*$ because of Lemma 3.9.

Let us now consider the limit $t \to \infty$. First, if for some sequence $t_n \to \infty$ we have $\|u_{t_n}\|/t_n \to 0$, then we divide (3.16) by t_n, pass to the limit, and get a contradiction. So $\|u_t\| \to \infty$ as $t \to \infty$. By the monotonicity of u_t in t, $\min_\Omega u_t < -1$ for sufficiently large t.

Assume for some sequence $t_n \to \infty$, we have $\|u_{t_n}\|/t_n \to \infty$. Then we divide (3.16) by $\|u_{t_n}\|$ and see that $u_{t_n}/\|u_{t_n}\|$ converges uniformly to φ_1^+, which is impossible, since u_{t_n} takes negative values and $\varphi_1^+ > 0$.

So there is a sequence $t_n \to \infty$ such that $u_{t_n}/\|u_{t_n}\|$ converges in $C^1(\overline{\Omega})$ to a solution of

$$F(D^2v, Dv, v, x) + \lambda_1^+ v = k\varphi_1^+ \quad \text{in} \quad \Omega, \qquad v = 0 \quad \text{on} \quad \partial\Omega, \tag{3.17}$$

for some $k > 0$. This problem is the particular case of (2.6) when $h = 0$. It is clear that (3.17) has solutions for $k \geq 0$ (by Theorem 2.6) and does not have solutions for $k < 0$ (by the definition of λ_1^+ and Theorem 2.1). Further, this problem obviously has solutions for $k = 0$ (in other words, for $h = 0$ we always are in the case of part 4(ii)), and the minimal solution at $k = 0$ is $u^* = 0$. Then, by the properties of the curve of solutions we already proved, (3.17) has a unique solution which satisfies $v < u^* = 0$, since $k > 0$.

This means $u_{t_n}/\|u_{t_n}\|$ converges in $C^1(\overline{\Omega})$ to a negative function v such that $\frac{\partial v}{\partial \nu} < 0$ on $\partial\Omega$ (by (3.17) and Hopf's lemma). This implies statement 3 for the subsequence u_{t_n}. Since u_t is monotonous, we have statement 3 for all u_t.

Finally, let us show that $t \to u_t(x)$ is convex. With the notations from Lemma 2.1, we note that for each compact interval $[t_0, t_1] \subset [t_+^*, \infty)$, there exists a function $v \in W^{2,p}(\Omega)$ which is a subsolution of

$$F(D^2u, Du, u, x) + \lambda_1^+ u = t_k \varphi_1^+ + h \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \tag{3.18}$$

and $v < ku_{t_1} + (1-k)u_{t_0}$ for each $k \in (0,1)$ (we take $u_{t_0} = u^*$ if $t_0 = t_+^*$). For instance, we can take v to be the negative solution—given by Theorem 2.6(I)—of the problem

$$F[v] + \lambda_1^+ v = \max\{t_1, 1\}\varphi_1^+ + \max\{h, 0\} \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega,$$

and then a take a multiple of v, by a sufficiently large constant, to ensure that $v < u_{t_1} \le ku_{t_1} + (1-k)u_{t_0}$ for each $k \in (0,1)$. Then by Lemma 2.1 and the usual sub- and supersolution method, there exists a solution of (3.18) which is below $ku_{t_1} + (1-k)u_{t_0}$. By the uniqueness which we already proved, this solution has to be u_{t_k}.

Theorem 1.2 is proved. ∎

4. The case $\lambda > \lambda_1^-$. Proof of Theorem 1.5. In this section we prove Theorem 1.5 and some auxiliary results which will be helpful in our analysis of the resonance phenomena at $\lambda = \lambda_1^-$.

We start with some simple preliminary lemmas which will lead us to the proof of the first part in Theorem 1.5. Our arguments for Lemmas 4.1–4.2 below are similar to those in [7], [24], and [1], but we sketch them here for completeness. We define the operators

$$F_\tau(D^2u, Du, u, x) = \tau F(D^2u, Du, u, x) + (1-\tau)\Delta u,$$

and we write $\lambda_1^-(\tau) = \lambda_1^-(F_\tau)$ for $\tau \in [0,1]$. Note that F_τ satisfies (H0)−(H3), and, recalling that we work with (2.6), F_τ is proper, since F is proper.

LEMMA 4.1. *The function $\tau \to \lambda_1^-(\tau)$ is continuous in the interval $[0,1]$, and there exists $\bar{\varepsilon} > 0$ so that there is no eigenvalue of F_τ in the interval $(\lambda_1^-(\tau), \lambda_1^-(\tau) + \bar{\varepsilon}]$ for $\tau \in [0,1]$.*

Proof. Let $\{\tau_n\}$ be a sequence in $[0,1]$; then it follows by Proposition 4.1 in [24] that the sequence $\{\lambda_1^-(\tau_n)\}$ is bounded. Then, by a compactness argument and the simplicity of the eigenvalues, the continuity follows. The isolation property follows by the same argument as the one used in the proof of Theorem 1.3 in [24]. ∎

LEMMA 4.2. *There exists $\varepsilon > 0$ such that for each $\lambda \in (\lambda_1^-, \lambda_1^- + \varepsilon)$ and each $n \in \mathbb{N}$, there is a closed connected set $\mathcal{C}(\lambda, n) \subset C(\overline{\Omega}) \times [-n, n]$, with the property that for all $(u,t) \in \mathcal{C}(\lambda, n)$, we have*

$$F(D^2u, Du, u, x) + \lambda u = t\varphi_1^+ + h \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega.$$

Moreover, if we define the projection $\mathcal{P} : C(\overline{\Omega}) \times \mathbb{R} \to \mathbb{R}$ as $\mathcal{P}(u,t) = t$, we have $\mathcal{P}(\mathcal{C}(\lambda, n)) = [-n, n]$.

Proof. For $\tau \in [0,1]$, let us define

$$\lambda_2(\tau) = \inf\{\mu > \lambda_1^-(\tau) \,|\, \mu \text{ is an eigenvalue of } F_\tau \text{ in } \Omega\}.$$

Observe that $\lambda_2(\tau) = +\infty$ is possible. Then, given $\lambda \in (\lambda_1^-, \lambda_1^- + \bar{\varepsilon})$, by the previous lemma there exists a continuous function $\mu : [0,1] \to \mathbb{R}$ such that $\mu(1) = \lambda$, $\lambda_1^-(\tau) < \mu(\tau) < \lambda_2(\tau)$, and the equation

$$F_\tau(D^2u, Du, u, x) + \mu(\tau)u = 0 \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \tag{4.1}$$

has no nontrivial solution for all $\tau \in [0,1]$. Now we define the operator $G : \mathbb{R} \times [0,1] \times C(\overline{\Omega}) \to C(\overline{\Omega})$ for $(t, \tau, v) \in \mathbb{R} \times [0,1] \times C(\overline{\Omega})$ as $u = G(t, \tau, v)$, where u is the solution of the equation

$$F_\tau(D^2u, Du, u, x) = -\mu(\tau)v + t\varphi_1^+ + h \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega. \tag{4.2}$$

When we restrict the variable t to the interval $[-n, n]$, the operator G becomes compact. Moreover, there exists $R > 0$ such that the Leray–Schauder degree $d(I - G(t, \tau, \cdot), B_R, 0)$ is well defined. Indeed, a priori bounds follow directly from the nonexistence property of (4.1); in fact, if (4.2) has a sequence of solutions $u_n = v_n$ with $\|u_n\| \to \infty$, we divide (4.2) by $\|u_n\|$, pass to the limit, and get a contradiction. Then, by the homotopy invariance of the Leray–Schauder degree, we have

$$d(I - G(t, 1, \cdot), B_R, 0) = d(I - G(t, 0, \cdot), B_R, 0) = -1.$$

The last equality is a standard fact, since the operator F_0 is the Laplacian. Thus, by the well-known results of [26], see in particular Corollary 10 in chapter V of that work (alternatively, we refer to [13]), the lemma follows. ∎

We will need the following topological result, whose proof is a direct consequence of Lemma 3.5.2 in [13].

LEMMA 4.3. *Let $R \subset C(\overline{\Omega}) \times [-n, n]$ be a compact connected set such that $\mathcal{P}(R) = [-n, n]$. If $R_0 = \{(u, t) \in R \,|\, t \in [t_-, t_+]\}$, with $[t_-, t_+] \subset [-n, n]$, then there exists a connected component E_0 of R_0 such that $\mathcal{P}(E_0) = [t_-, t_+]$.*

Proof of Theorem 1.5, *statement* 1. The boundedness of S_I for each bounded interval I is trivial; indeed, if we have a sequence of solutions to the problem which is unbounded in $L^\infty(\Omega)$, we divide each equation by the norm of its solution, as we have already done a number of times, and we find a solution which contradicts Theorem 2.2. Recall the regularity result in Theorem 2.5.

For each $n \in \mathbb{N}$ we define $E_n = \mathcal{C}(\lambda, n)$ as the connected set given in Lemma 4.2. Then, by Lemma 4.3, there are closed connected subsets E_n^N of $\{(u, t) \in E_n \,|\, t \in [-N, N]\}$, for $1 \le N \le n$, such that $\mathcal{P}(E_n^N) = [-N, N]$ and $E_n^N \subset E_n^{N+1}$ for $N = 1, 2, \ldots, n-1$. In order to get the last property, we proceed step-by-step, defining E_n^N through Lemma 4.3 by decreasing N starting from n. Now we define the sets E^N for $N \in \mathbb{N}$ as follows:

$$\begin{aligned} E^N = \{(u, t) \in C(\overline{\Omega}) \times \mathbb{R} \,|\, &\text{there exist } (u_{\ell_k}, t_{\ell_k}) \in E_{\ell_k}^N, \\ &\ell_k \ge k \text{ for all } k \in \mathbb{N}, (u_{\ell_k}, t_{\ell_k}) \to (u, t) \text{ as } k \to \infty\}. \end{aligned}$$

We notice that E^N is closed and $\mathcal{P}(E^N) = [-N, N]$. Since the pairs $(u, t) \in E_n^N$ are solutions of

$$F(D^2u, Du, u, x) + \lambda u = t\varphi_1^+ + h \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \quad t \in [-N, N],$$

we see that the set E^N is composed of solutions of these equations, and consequently, it is compact in $C^1(\overline{\Omega})$. Then it is easy to see that for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $E_n^N \subset B(E^N, \varepsilon)$ for all $n \geq n_0$. Here we denote by $B(U, \varepsilon)$ the ε-neighborhood of the set U. Indeed, if there exist $\varepsilon > 0$ and a sequence $\ell_k \geq k$ such that $(u_{\ell_k}, t_{\ell_k}) \in E_{\ell_k}^N \setminus B(E^N, \varepsilon)$, then t_{ℓ_k} and u_{ℓ_k} are bounded, and a subsequence of (u_{ℓ_k}, t_{ℓ_k}) converges to some (u, t) in E^N, which is a contradiction.

By the convergence property just proved, we see that E^N is connected. In fact, if it is not connected, there exist nonempty closed subsets U, V of E^N such that $U \cap V = \emptyset$ and $U \cup V = E^N$. By compactness, there exists $\varepsilon > 0$ such that $\mathrm{dist}(U, V) > \varepsilon$, and then $B(U, \varepsilon/4) \cap B(V, \varepsilon/4) = \emptyset$ which is impossible, since the connected set E_n^N is contained in $B(U, \varepsilon/4) \cup B(V, \varepsilon/4)$ for n large enough, as stated in the claim above.

We observe that, according to our construction of the sets E_n^N and E^N, we have $E^N \subset E^{N+1}$ for all $N \in \mathbb{N}$. So to complete the proof of part 1, we just need to define $\mathcal{C} = \mathcal{C}(\lambda) = \cup_{N \in \mathbb{N}} E^N$, which is clearly a closed connected set of solutions, and $\mathcal{P}(\mathcal{C}) = \mathbb{R}$. ∎

Before proceeding to the proof of part 2 of Theorem 1.5, we give a generalized version of the antimaximum principle for fully nonlinear equations, recently proved in [1].

PROPOSITION 4.1. *Let $f \in L^p(\Omega)$, $p > N$, be such that $f \leq 0$, $f \not\equiv 0$ in Ω.*

1. *There is $\varepsilon_0 > 0$ (depending on f) such that any solution of the equation*

$$F(D^2u, Du, u, x) + \lambda u = kf \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \tag{4.3}$$

satisfies $u < 0$ in Ω, provided $\lambda \in (\lambda_1^-, \lambda_1^- + \varepsilon_0)$ and $k \in (0, \infty)$.

2. *Equation (4.3) has no solutions if $\lambda = \lambda_1^-$ and $k > 0$.*

Proof. We first prove statement 2. Suppose there is a solution u of (4.3) with $\lambda = \lambda_1^-$ and $k > 0$. If there exists $x_0 \in \Omega$ such that $u(x_0) < 0$, then by Theorem 2.1, there exists $k_0 > 0$ such that $u = k_0 \varphi_1^-$, which is a contradiction with $f \not\equiv 0$. Therefore, $u \geq 0$ in Ω, and then, by the strong maximum principle, $u > 0$ in Ω. The existence of such a function contradicts Theorem 2.1.

Let us now prove statement 1. Suppose there are sequences $k_n > 0$, $\lambda_n > \lambda_1^-$, $\lambda_n \to \lambda_1^-$, and $\tilde{u}_n$ of solutions of (4.3) such that $\tilde{u}_n$ is positive or zero somewhere in Ω. Then $u_n = \tilde{u}_n / k_n$ has the same property and solves (4.3) with $k = 1$. Suppose first that u_n is bounded; then a subsequence of u_n converges uniformly to a solution of (4.3) with $\lambda = \lambda_1^-$ and $k = 1$, which is a contradiction with the result we already proved in 2. If u_n is unbounded, then a subsequence of $u_n / \|u_n\|$ converges in $C^1(\overline{\Omega})$ to the negative function φ_1^-—a contradiction as well. ∎

We now prove that the solutions of our equation are negative for small t.

LEMMA 4.4. *Given $R > 0$, there are numbers $\varepsilon > 0$ and $\bar{t}$ such that for all $\lambda \in [\lambda_1^-, \lambda_1^- + \varepsilon)$, $t \leq \bar{t}$, and h with $\|h\|_{L^p(\Omega)} \leq R$, if u solves the equation*

$$F(D^2u, Du, u, x) + \lambda u = t\varphi_1^+ + h \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \tag{4.4}$$

then $u < 0$ in Ω.

Proof. Assuming the result is not true, there are sequences $\{t_n\}$, $\{u_n\}$, $\{\lambda_n\}$, and $\{h_n\}$ such that $\lambda_n \geq \lambda_1^-$, $\lambda_n \to \lambda_1^-$, $t_n \to -\infty$, $\|h_n\|_{L^p} \leq R$, u_n is positive or zero at a point in Ω, and

$$F(D^2u_n, Du_n, u_n, x) + \lambda_n u_n = t_n \varphi_1^+ + h_n \quad \text{in} \quad \Omega, \qquad u_n = 0 \quad \text{on} \quad \partial\Omega,$$

for all $n \in \mathbb{N}$. Defining $v_n = -u_n/t_n$, we can easily check that if $\{v_n\}$ is bounded, then a subsequence of it converges to a solution of $F(v) + \lambda_1^- v = -\varphi_1^+$ in Ω, which is a contradiction with part 2 of Proposition 4.1; while if $\{v_n\}$ is unbounded, then a subsequence of $v_n/\|v_n\|$ converges in $C^1(\overline{\Omega})$ to $\varphi_1^- < 0$, which is a contradiction, since these functions are positive or zero somewhere. ∎

Proof of Theorem 1.5, *statement* 2. It remains to analyze the asymptotic behavior of the set S. Take any $u_t \in S_t$, $t \in \mathbb{R}$. It is clear that there exist constants $C_0, T > 0$, depending only on F, Ω, and h such that $\|u_t\| \geq C_0|t|$ if $|t| \geq T$. Indeed, assuming that $\{t/\|u_t\|\}$ is not bounded, one easily gets the contradiction $0 = \pm\varphi_1^+$, after dividing the equation by t and passing to the limit.

First, suppose for contradiction that there exist a compact set $K \subset \Omega$ and sequences $t_n \to -\infty$, $u_n \in S_{t_n}$ such that $u_{t_n}(x_n) \geq -c$ for some constant c and some $x_n \in K$. Note that by the previous lemma, we already know that $u_{t_n} < 0$ in Ω for large n. Thus, setting $v_n = u_{t_n}/\|u_{t_n}\|$, we have $\|v_n\| = 1$, $v_n < 0$ in Ω, $v_n(x_n) \to 0$ as $n \to \infty$, and

$$F[v_n] + \lambda v_n = (t_n/\|u_{t_n}\|)\varphi_1^+ + h/\|u_{t_n}\| \quad \text{in} \quad \Omega, \qquad v_n = 0 \quad \text{on} \quad \partial\Omega.$$

Now, if $t_n/\|u_{t_n}\| \to 0$, a subsequence of v_n converges to a nontrivial solution of $F[v] + \lambda v = 0$, which is a contradiction with $\lambda \in (\lambda_1^-, \lambda_1^- + \varepsilon)$. On the contrary, if $t_n/\|u_{t_n}\| \not\to 0$, then a subsequence of v_n converges uniformly to a solution of $F(v) + \lambda v = -k\varphi_1^+$ for some $k > 0$. In addition $v(x_0) = 0$ for some $x_0 \in K$, which is a contradiction with the antimaximum principle, Proposition 4.1, provided $\varepsilon < \varepsilon_0(-\varphi_1^+)$, with ε_0 defined in that proposition.

Second, suppose there is a sequence $t_n \to +\infty$ such that $u_{t_n} \leq C$ for some constant C. Then, as above, either $v_n = u_{t_n}/\|u_{t_n}\|$ converges to a nontrivial solution of $F(v) + \lambda v = 0$, which is a contradiction with Theorem 2.2, or v_n converges to a nonpositive solution of $F(v) + \lambda v = k\varphi_1 > 0$, which is then negative by Hopf's lemma. This is a contradiction again, here with the definition of λ_1^- and $\lambda > \lambda_1^-$. Theorem 1.5 is proved. ∎

5. Resonance at $\boldsymbol{\lambda = \lambda_1^-}$. Proof of Theorem 1.4. In this section we study the behavior of the set of solutions of our problem in the second resonant case, that is, when $\lambda = \lambda_1^-$. For this purpose we consider a sequence $\{\lambda_n\}$, with $\lambda_n \in (\lambda_1^-, \lambda_1^- + \varepsilon)$ (everywhere in this section $\varepsilon = L$ will be the number which appears in Theorem 1.5, found in the previous section), which converge to λ_1^-, and we study the asymptotic behavior of the connected sets $\mathcal{C} = \mathcal{C}(\lambda_n) \subset S(\lambda_n)$ obtained in Theorem 1.5.

We modify the definition of condition $\mathcal{P}(s)$ as follows:

$\mathcal{P}(s)$: There exist sequences $\{\lambda_n\}$, $\{h_n\}$, and $\{u_n\}$ such that $\lambda_n > \lambda_1^-$ for all n, $\lim_{n\to\infty} \lambda_n = \lambda_1^-$, $h_n \to h$ in $L^p(\Omega)$,

$$F(D^2u_n, Du_n, u_n, x) + \lambda_n u_n = s\varphi_1^+ + h_n \quad \text{in} \quad \Omega, \qquad u_n = 0 \quad \text{on} \quad \partial\Omega,$$

and $\|u_n\|$ is unbounded.

Since no confusion arises with the definition given in section 3, we keep the same notation. As before, $\mathcal{P}(s)$ is equivalent to the following:

$\mathcal{P}(s)$: There exist sequences $\{\lambda_n\}$, $\{h_n\}$, and $\{u_n\}$ such that $\lambda_n > \lambda_1^-$ for all n, $\lim_{n\to\infty} \lambda_n = \lambda_1^-$, $h_n \to h$ in $L^p(\Omega)$, $\{u_n\}$ is a sequence of solutions of $F(D^2u_n, Du_n, u_n, x) + \lambda_n u_n = s\varphi_1^+ + h_n$ such that $\|u_n\| \to \infty$, and

$$\frac{u_n}{\|u_n\|} \to \varphi_1^- < 0 \quad \text{in} \quad C^1(\overline{\Omega}).$$

Then we define, as before,

$$t_-^* = \sup\{t \in \mathbb{R} \,|\, \mathcal{P}(s) \text{ for all } s < t\}. \tag{5.1}$$

The following lemmas are necessary to give sense to this definition.

LEMMA 5.1. *$\mathcal{P}(\bar{t})$ implies $\mathcal{P}(t)$ for all $t < \bar{t}$.*

Proof. Assume that there exists $t_0 < \bar{t}$ such that $\mathcal{P}(t_0)$ is false. Since $\mathcal{P}(\bar{t})$ holds, there exist sequences $\{\lambda_n\}$, $\{h_n\}$, and $\{v_n\}$ such that $\lambda_n > \lambda_1^-$ for all n, $\lim_{n\to\infty} \lambda_n = \lambda_1^-$, $h_n \to h$ in $L^p(\Omega)$, the solutions of

$$F(D^2 v_n, Dv_n, v_n, x) + \lambda_n v_n = \bar{t}\varphi_1^+ + h_n \quad \text{in} \quad \Omega, \quad v_n = 0 \quad \text{on} \quad \partial\Omega,$$

satisfy $\lim_{n\to\infty} \|v_n\| = \infty$, and $v_n/\|v_n\|$ converges to $\varphi_1^- < 0$ in $C^1(\overline{\Omega})$; in other words, $v_n \le k_n \varphi_1^-$ for some sequence $k_n \to \infty$. On the other hand, let $\{u_n\}$ be any sequence such that

$$F(D^2 u_n, Du_n, u_n, x) + \lambda_n u_n = t_0\varphi_1^+ + h_n \quad \text{in} \quad \Omega, \quad u_n = 0 \quad \text{on} \quad \partial\Omega.$$

Such a sequence exists thanks to Theorem 1.5. Since we are assuming that $\mathcal{P}(t_0)$ is false, $\{\|u_n\|\}$ is bounded, so a subsequence of $\{u_n\}$ converges in $C^1(\overline{\Omega})$.

Then $|u_n| \le C|\varphi_1^-|$ in Ω, by the boundary Lipschitz estimates (recall φ_1^- has nonzero normal derivative on the boundary, by Hopf's lemma), so the above convergence properties of v_n imply that for n large $\psi_n = v_n - u_n < 0$ in Ω. However, by (H3) we have $F[\psi_n] \ge F[v_n] - F[u_n]$, so

$$F(D^2\psi_n, D\psi_n, \psi_n, x) + \lambda_n \psi_n \ge (\bar{t} - t_0)\varphi_1^+ > 0 \quad \text{in} \quad \Omega, \quad \psi_n = 0 \quad \text{on} \quad \partial\Omega,$$

for large n, contradicting the definition of λ_1^-, since $\lambda_n > \lambda_1^-$. ◻

Now we prove that t_-^* is a real number. We set $T = \{t \in \mathbb{R} \,|\, \mathcal{P}(t)\}$.

LEMMA 5.2. *The set T is not empty.*

Proof. Assuming the contrary, we find a sequence $\{t_n\}$ such that $\mathcal{P}(t_n)$ is false and $t_n \to -\infty$, which implies the existence of a sequence u_n satisfying

$$F(D^2 u_n, Du_n, u_n, x) + \lambda_1^- u_n = t_n\varphi_1^+ + h \quad \text{in} \quad \Omega, \qquad u_n = 0 \quad \text{on} \quad \partial\Omega.$$

This statement follows from Theorem 2.7 through exactly the same argument as the one used in the proof of Lemma 3.4. Next, we see that $v_n = -u_n/t_n$ is unbounded, since the contrary implies the existence of a solution to $F(D^2v, Dv, v, x) + \lambda_1^- v = -\varphi_1^+$ in Ω, $v = 0$ on $\partial\Omega$, which was shown to be impossible in Proposition 4.1. Then a subsequence of $u_n/\|u_n\|$ converges in $C^1(\overline{\Omega})$ to a solution of the equation

$$F(D^2 w, Dw, w, x) + \lambda_1^- w = 0 \quad \text{in} \quad \Omega, \qquad w = 0 \quad \text{on} \quad \partial\Omega,$$

which implies that $w = \varphi_1^-$. We conclude that $\max_K u_n \to -\infty$ for each compact $K \subset \Omega$. To complete the proof, let v be the solution of

$$F(D^2 v, Dv, v, x) = -h \quad \text{in} \quad \Omega, \qquad v = 0 \quad \text{on} \quad \partial\Omega.$$

Then, for n large, the function $\psi = u_n + v$ is negative at some point and satisfies

$$F(D^2\psi, D\psi, \psi, x) + \lambda_1^- \psi \le t_n\varphi_1^+ + \lambda_1^- v \quad \text{in} \quad \Omega, \qquad \psi = 0 \quad \text{on} \quad \partial\Omega$$

(we use $F[\psi] \le F[u_n] + F[v]$ which is a consequence of (H0) and (H3)). The quantity $t_n\varphi_1^+ + \lambda_1^- v$ is strictly negative for large n, so by Theorem 2.1, we have $\psi = k\varphi_1^-$ for some $k > 0$, which is a contradiction with the strict inequality $F[\psi] + \lambda_1^- \psi < 0$. Hence $T \ne \emptyset$. ◻

LEMMA 5.3. *There exists* $\bar{t} = \bar{t}(h) \in \mathbb{R}$ *such that for any* $t \geq \bar{t}$, *we can find* $C, \delta > 0$ *such that if* $\|\tilde{h} - h\|_{L^p(\Omega)} < \delta$, *then all solutions to*

$$F(D^2u, Du, u, x) + \lambda v = t\varphi_1^+ + \tilde{h} \quad in \quad \Omega, \qquad u = 0 \quad on \quad \partial\Omega,$$

with $\lambda \in [\lambda_1^-, \lambda_1^- + \varepsilon)$, *satisfy* $\|u\| \leq C$. *In particular, the set* T *is bounded above by* $\bar{t}$; *that is,* t_-^* *is finite.*

Proof. Assuming the contrary, we may find sequences $t_n \to \infty$ as $n \to \infty$, $\lambda_n^{(m)} \in [\lambda_1^-, \lambda_1^- + \varepsilon)$, $h_n^{(m)} \to h$ in $L^p(\Omega)$ as $m \to \infty$ for each fixed n, and $\{u_n^{(m)}\}$ such that

$$F(D^2u_n^{(m)}, Du_n^{(m)}, u_n^{(m)}, x) + \lambda_n^{(m)} u_n^{(m)} = t_n\varphi_1^+ + h_n^{(m)} \text{ in } \Omega, \ u_n^{(m)} = 0 \text{ on } \partial\Omega,$$

and $\{u_n^{(m)}\}$ is unbounded as $m \to \infty$ for each n. Then, as we have done a number of times already, we can divide the last equation by $\|u_n^{(m)}\|$ and use Theorem 2.7, which implies that, up to a subsequence, $u_n^{(m)}/\|u_n^{(m)}\|$ converges in $C^1(\overline{\Omega})$ as $m \to \infty$ to a function $\hat{u}_n \not\equiv 0$, which solves

$$F(D^2\hat{u}_n, D\hat{u}_n, \hat{u}_n, x) + \lambda_n \hat{u}_n = 0 \quad \text{in} \quad \Omega, \quad \hat{u}_n = 0 \quad \text{on} \quad \partial\Omega.$$

This implies that $\lambda_n = \lambda_1^-$ and $\hat{u}_n = \varphi_1^- < 0$. Hence $u_n^{(m)} \leq k_n^{(m)}\varphi_1^-$ for some sequence $\{k_n^{(m)}\}$ such that $k_n^{(m)} \to \infty$ as $m \to \infty$.

Next, we remark that we can find (thanks to Theorems 2.3 and 2.5) a constant $C_0 = C_0(h)$ such that for any $g \in L^p(\Omega)$ with $\|g\|_{L^p(\Omega)} \leq \|h\|_{L^p(\Omega)} + 1$, if w is a solution of

$$F(D^2w, Dw, w, x) = g \quad \text{in} \quad \Omega, \quad g = 0 \quad \text{on} \quad \partial\Omega, \tag{5.2}$$

then $\|w\|_{W^{2,p}(\Omega)} \leq C_0$. This, of course, implies $w \geq \tilde{C}_0\varphi_1^-$ for some $\tilde{C}_0 > 0$.

Now, for each n, we fix $m(n)$ such that $\lambda_n^{(m(n))} < \lambda_1^- + 1/n$, $h_n := h_n^{(m(n))}$ satisfies $\|h_n - h\|_{L^p(\Omega)} \leq 1/n$, and $u_n := u_n^{(m(n))} < w$ for each solution w of (5.2). So, in particular, $u_n < v_n$, where v_n is the solution of

$$F(D^2v_n, Dv_n, v_n, x) = h_n \quad \text{in} \quad \Omega, \quad v_n = 0 \quad \text{on} \quad \partial\Omega.$$

Then we choose n large enough so that $t_n\varphi_1^+ > \lambda_1^- v_n$, and we see that the function $\psi_n = u_n - v_n < 0$ satisfies

$$F(D^2\psi_n, D\psi_n, \psi_n, x) + \lambda_1^-\psi_n \geq t_n\varphi_1^+ - \lambda_1^- v_n > 0 \quad \text{in} \quad \Omega, \quad \psi_n = 0 \quad \text{on} \quad \partial\Omega.$$

By Theorem 2.1 we find that $\psi_n = \varphi_1^-$, which contradicts the last strict inequality. ☐

The next result contains part 1 of Theorem 1.4.

PROPOSITION 5.1. *The equation*

$$F(D^2u, Du, u, x) + \lambda_1^- u = t\varphi_1^+ + h \quad in \quad \Omega, \quad u = 0 \quad on \quad \partial\Omega,$$

(i) *has at least one solution if* $t > t_-^*$;

(ii) *does not have a solution if* $t < t_-^*$.

Proof. (i) is proved in exactly the same way as Proposition 3.1(1), using Theorems 1.5 and 2.7. In order to prove (ii), let $t_1 \in (t, t_-^*)$. By Lemma 5.1, $\mathcal{P}(t_1)$ holds; then there exist sequences $\{v_n\}$, $\{h_n\}$, and $\{\lambda_n\}$ such that $\{v_n\}$ is unbounded, $\lambda_n > \lambda_1^-$, $\lambda_n \to \lambda_1^-$, $h_n \to h$, $v_n/\|v_n\| \to \varphi_1^-$ in $C^1(\overline{\Omega})$, and

$$F(D^2 v_n, Dv_n, v_n, x) + \lambda_n v_n = t_1\varphi_1^+ + h_n \quad \text{in} \quad \Omega, \quad v_n = 0 \quad \text{on} \quad \partial\Omega.$$

Now, supposing (ii) is false, let u and w_n be solutions of

$$F(D^2 u, Du, u, x) + \lambda_1^- u = t\varphi_1^+ + h \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega,$$
$$F(D^2 w_n, Dw_n, w, x) = h_n - h \quad \text{in} \quad \Omega, \quad w_n = 0 \quad \text{on} \quad \partial\Omega.$$

Notice that $w_n \to 0$ in $C^1(\overline{\Omega})$. Then, for large n, we have $v_n < 0$, $\lambda_1^- v_n > \lambda_n v_n$, $v_n - w_n - u < 0$ in Ω, and

$$F[v_n - w_n - u] + \lambda_1^-(v_n - w_n - u) \geq (t_1 - t)\varphi_1^+/2 > 0 \quad \text{in } \Omega, \tag{5.3}$$

where we used (H3) which implies $F[v_n - w_n - u] \geq F[v_n] - F[w_n] - F[u]$. Then, by Theorem 2.1 once more, we have $v_n - w_n - u = k_n\varphi_1^-$ for some number $k_n \geq 0$, in contradiction with the strict inequality in (5.3). ∎

The next lemma contains statement 3 in Theorem 1.4.

Lemma 5.4. *For each compact interval $I \subset (t_-^*, \infty)$, there exists a constant C such that for all $\lambda \in [\lambda_1^-, \lambda_1^- + \varepsilon)$ and all $t \in I$, if u is a solution to*

$$F(D^2 u, Du, u, x) + \lambda v = t\varphi_1^+ + h \quad \textit{in} \quad \Omega, \qquad u = 0 \quad \textit{on} \quad \partial\Omega,$$

then $\|u\|_{W^{2,p}(\Omega)} \leq C$.

Proof. Recall we already proved in the previous section that the set of solutions is bounded for t in a bounded interval, provided λ is away from the eigenvalue λ_1^-. Hence, if the statement of Lemma 5.4 is false, then we can find sequences $t_n \to t_0$ with $t_0 > t_-^*$, $\lambda_n \to \lambda_1^-$ ($\lambda_n = \lambda_1^-$ is allowed), $\{u_n\}$ with $\|u_n\| \to \infty$, and $u_n/\|u_n\| \to \varphi_1^-$ such that

$$F[u_n] + \left(\lambda_n + \frac{1}{\|u_n\|^2}\right) u_n = t_0\varphi_1^+ + h + (t_n - t_0)\varphi_1^+ + \frac{1}{\|u_n\|}\left(\frac{u_n}{\|u_n\|}\right) = t_0\varphi_1^+ + h_n.$$

Clearly, $h_n \to h$ in $L^p(\Omega)$, so the existence of such a sequence contradicts the definition of the number t_-^* and $t_0 > t_-^*$. ∎

Before continuing, we set up some notation. The set of solutions $\mathcal{C}$ found in Theorem 1.5 will be denoted by $\mathcal{C}(\lambda)$, remembering we work with the equivalent equation (2.6). We define the function $\mathcal{Q} : C(\overline{\Omega}) \times \mathbb{R} \to \mathbb{R}$ as $\mathcal{Q}(u,t) = \|u\|$ for $(u,t) \in C(\overline{\Omega}) \times \mathbb{R}$, and we recall that $\mathcal{P}$ is the projection $\mathcal{P}(u,t) = t$. In the proof of Theorem 1.4 the function $\mathcal{Q}$ plays a role similar to that of $\mathcal{P}$ in the proof of Theorem 1.5. The following lemma will be needed later.

Lemma 5.5. *Given $t_1 > t_-^*$, there exists $N_0 \in \mathbb{N}$ such that for every $\lambda \in (\lambda_1^-, \lambda_1^- + \varepsilon)$ and $N > N_0$,*

$$N \in \mathcal{Q}(\mathcal{C}(\lambda)_{[t_1,\infty)}) \cap \mathcal{Q}(\mathcal{C}(\lambda)_{(-\infty,t_1]});$$

that is, for all λ larger than and sufficiently close to λ_1^- and all N large, we can find u' and u'' such that

$$\|u'\| = \|u''\| = N, \qquad \textit{and}$$

$$F[u'] + \lambda u' = t'\varphi_1^+ + h, \qquad F[u''] + \lambda u'' = t''\varphi_1^+ + h \quad \textit{in} \quad \Omega,$$

where $t' \geq t_1$ *and* $t'' \leq t_1$.

Proof. Given $t_1 > t_-^*$, we let $N_0 \in \mathbb{N}$ be an upper bound of the set $\mathcal{C}(\lambda)_{t_1}$, uniformly in the interval $\lambda \in (\lambda_1^-, \lambda_1^- + \varepsilon)$—such a bound exists by the previous lemma. The conclusion follows from Theorem 1.5, since the set $\mathcal{C}(\lambda)$ is connected and the sets $\mathcal{C}(\lambda)_t$ contain elements whose norms grow arbitrarily as $t \to \infty$ and as $t \to -\infty$. ∎

Proof of Theorem 1.4. The proof follows an idea similar to the one used in the proof of Theorem 1.5, but here we take as a parameter the norm of the solution, instead of t.

Fix $t_1 > t_-^*$. We start with a sequence $\{\lambda_n\}$ with $\lambda_n > \lambda_1^-$ and $\lambda_n \to \lambda_1^-$ as $n \to \infty$. Then we look at the connected set of solutions $\mathcal{C}(\lambda_n)$ given by Theorem 1.5, and we take $N \in \mathbb{N}, N > N_0$, where N_0 is the number from Lemma 5.5.

By an argument similar to the one given in the previous section (using Lemmas 4.3 and 5.5), we find that for each $N = n, n-1, \ldots, N_0+1, N_0$, there is a closed connected subset $E_n^N \subset \{(u,t) \in \mathcal{C}(\lambda_n) \,|\, \|u\| \leq N\}$ such that

$$\mathcal{Q}((E_n^N)_{[t_1,\infty)}) = [N_0, N] \quad \text{and} \quad \mathcal{Q}((E_n^N)_{(-\infty,t_1]}) = [N_0, N]$$

for $N = n, n-1, \ldots, N_0$. For each $n \in \mathbb{N}$ we construct the sets E_n^N, starting with $N = n$ and successively going down to $N = N_0$. Thus, $E_n^N \subset E_n^{N+1}$, $N = n - 1, \ldots, N_0+1, N_0$. Then we define

$$E^N = \{(u,t) \in C(\overline{\Omega}) \times \mathbb{R} \,|\, \text{there exists } (u_{\ell_k}, t_{\ell_k}) \in E_{\ell_k}^N, \\ \ell_k \geq k \text{ for all } k \in \mathbb{N}, (u_{\ell_k}, t_{\ell_k}) \to (u,t) \text{ as } k \to \infty\}.$$

We notice that E^N is closed,

$$\mathcal{Q}((E^N)_{[t_1,\infty)}) = [N_0, N], \quad \text{and} \quad \mathcal{Q}((E^N)_{(-\infty,t_1]}) = [N_0, N].$$

Since the pairs $(u,t) \in E_n^N$ are solutions of

$$F(D^2u, Du, u, x) + \lambda_n u = t\varphi_1^+ + h \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega,$$

the bounded in $L^\infty(\Omega)$ set E^N is made of solutions of such an equation, but with λ_1^- instead of λ_n, and, consequently, E^N is compact. By a similar argument as the one in the proof of Theorem 1.5, we can prove that E^N is connected. Since, according to our construction, we have that $E_n^N \subset E_n^{N+1}$ for all n, we see that $E^N \subset E^{N+1}$ for all $N \in \mathbb{N}$. Thus, the set $\mathcal{C} = \cup_{N\in\mathbb{N}} E^N$ is a closed connected set of solutions and

$$\mathcal{Q}(\mathcal{C}_{[t_1,\infty)}) = [N_0, \infty) \quad \text{and} \quad \mathcal{Q}(\mathcal{C}_{[-\infty,t_1]}) = [N_0, \infty). \tag{5.4}$$

Next we observe that by the definition of t_-^* and (5.4), we have $\mathcal{P}(\mathcal{C}_{[t_1,\infty)}) = [t_1, \infty)$. On the other hand, by Proposition 5.1, we know that

$$(t_-^*, t_1] \subset \mathcal{P}(\mathcal{C}_{[-\infty,t_1]}) \subset [t_-^*, t_1]$$

so that we also have $\mathcal{Q}(\mathcal{C}_{[t_-^*,t_1]}) = [N_0, \infty)$. This completes the proof of statement 2 and the first statement in 5 of Theorem 1.4.

Let us look at the asymptotic behavior of the set of solutions S as $t \to \infty$. First, it is easily proved that if $(u_t, t) \in S$, then $\lim_{t\to\infty} \|u\| = \infty$ (if not, we divide the equation by t and pass to the limit $t \to \infty$, as before). Suppose now that there is a sequence $t_n \to +\infty$ such that for some $u_{t_n} \in S_{t_n}$, we have $u_{t_n} \leq C$ for some constant C. Then, as in the proof of Theorem 1.5(2), either $v_n := u_{t_n}/\|u_{t_n}\|$ converges to a nonpositive solution v of $F[v] + \lambda_1^- v = k\varphi_1^+ > 0$, with $v = 0$ on $\partial\Omega$, which is negative by Hopf's lemma, providing a contradiction with Theorem 2.1, or v_n converges to a nontrivial solution of $F[v] + \lambda_1^- v = 0$, with $v = 0$ on $\partial\Omega$. In this case v_n converges to $\varphi_1^- < 0$ in $C^1(\overline{\Omega})$, which implies that for some sequence $k_n \to \infty$, we have $u_n \le -k_n\varphi_1^+$ in Ω. Now let w be the solution of $F[w] = h$ in Ω, with $w = 0$ on $\partial\Omega$. Then by $\|w\| \le C$ and the Lipschitz estimates, we have $u_n - w < 0$ in Ω if n is sufficiently large, so

$$\begin{aligned} F[u_n - w] + \lambda_1^-(u_n - w) &\ge F[u_n] + \lambda_1^- u_n - (F[w] + \lambda_1^- w) \\ &\ge t_n\varphi_1^+ - \lambda_1^- w. \end{aligned}$$

However, the last quantity is positive if n is sufficiently large, yielding a contradiction with Theorem 2.1. This gives statement 4 in Theorem 1.4.

Next, we see that there is $R > 0$ so that if $(u, t) \in S$, with $t \in [t_-^*, t_1]$ and $\|u\|_\infty \ge R$, then $u < 0$. In fact, if the contrary is true, then there is a sequence $(u_n, t_n) \in S$, with $t_n \in [t_-^*, t_1]$, $\|u_n\|_\infty \to \infty$, and such that u_n is positive or zero somewhere in Ω. But this is impossible since a subsequence of $u_n/\|u_n\|$ converges in $C^1(\overline{\Omega})$ to φ_1^-, which is negative. By the same argument, we have $\max_K u_n \to -\infty$ for each $(u_n, t_n) \in S$ such that $\|u_n\| \to \infty$ and $t_n \in [t_-^*, t_1]$. This completes the proof of statement 5 in Theorem 1.4.

We now turn to the proof of statement 6. Assume the equation

$$F(D^2u, Du, u, x) + \lambda_1^- u = t_-^*\varphi_1^+ + h \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega,$$

has an unbounded set of solutions; that is, $S_{t_-^*}$ is unbounded. Let $u_1, u_2 \in S_{t_-^*}$; then there exists $R_1 > 0$ so that whenever $u \in S_{t_-^*}$ and $\|u\| \ge R_1$, we have $u = u_1 + k_1\varphi_1^-$ for some $k_1 > 0$. In fact, we already know that if $\|u\|$ is large enough, then $u/\|u\|$ is close in $C^1(\overline{\Omega})$ to φ_1^- and then $\psi = u - u_1 < 0$ in Ω. Since ψ satisfies

$$F(D^2\psi, D\psi, \psi, x) + \lambda_1^-\psi \ge 0 \quad \text{in} \quad \Omega, \quad \psi = 0 \quad \text{on} \quad \partial\Omega,$$

Theorem 2.1 implies $\psi = k_1\varphi_1^-$. In the same way we get $u = u_2 + k_2\varphi_1^-$ if $\|u\| \ge \max\{R_1, R_2\}$ for some $R_2 > 0$, so $u_1 - u_2 = (k_2 - k_1)\varphi_1^-$.

Finally, we prove that if $u + k_1\varphi_1^-$ and $u + k_2\varphi_1^-$ are in S_t for some $k_2 > k_1 > 0$, then $u + k\varphi_1^- \in S_t$ for each $k \in (k_1, k_2)$. This is a simple consequence of the convexity and the homogeneity of F. Indeed, setting $\tilde{F} = F + \lambda_1^-$,

$$\begin{aligned} t\varphi_1^+ + h &= \tilde{F}[u_* + k_1\varphi_1^-] + (k - k_1)\tilde{F}[\varphi_1^-] \ge \tilde{F}[u_* + k_1\varphi_1^- + (k - k_1)\varphi_1^-] \\ &= \tilde{F}[u + k\varphi_1^-] \\ &= \tilde{F}[u_* + k_2\varphi_1^- - (k_2 - k)\varphi_1^-] \ge \tilde{F}[u_* + k_2\varphi_1^-] - (k_2 - k)\tilde{F}[\varphi_1^-] \\ &= t\varphi_1^+ + h. \end{aligned}$$

Theorem 1.4 is proved. ∎

6. Proof of Theorem 1.6. The proof of Theorem 1.6 relies on an estimate on the difference between the first eigenvalue of an operator on a domain and a proper subset of the domain, which was proved in [4, Theorem 2.4] in the context of general linear operators. We give here a nonlinear version of this result.

Given a smooth bounded domain $A \subset \Omega$, we write $\lambda_1^+(A)$ for the first eigenvalue of the operator F on A.

PROPOSITION 6.1. *Assume* (H0)–(H3). *Let* Γ *be a closed set in* Ω *such that* $|\Gamma| \geq \alpha_0 > 0$. *Then there exists a constant* $\alpha > 0$ *depending only on* $\lambda, \Lambda, N, \gamma, \delta, \Omega, \alpha_0$ *such that for any smooth subdomain* A *of* $\Omega \setminus \Gamma$, *we have*

$$\lambda_1^+(A) \geq \lambda_1^+(\Omega) + \alpha.$$

The proof of Proposition 6.1 is very similar to the proof of Theorem 2.4 in [4]. Below we will mention the points where some small changes have to be made, but before doing that we show how we get the proof of Theorem 1.6, assuming Proposition 6.1.

Proof of Theorem 1.6. We take $d_0 = \alpha/2$, where α is the number from Proposition 6.1, with $\alpha_0 = |\Omega|/2$. Suppose for contradiction that we have two different solutions u_1 and u_2 of (1.1), with F satisfying the hypothesis of Theorem 1.6. We distinguish two cases.

First, suppose the function $w = u_1 - u_2$ has a constant sign in Ω, say, $w \leq 0$ (otherwise we take $w = u_2 - u_1$). Then (H3) implies $F(w) \geq 0$ in Ω and then $w < 0$ in Ω, by Hopf's lemma. The existence of such a function contradicts the definition of $\lambda_1^-(\Omega)$ and the assumption $\lambda_1^-(\Omega) < 0$; see Theorem 2.1.

Second, if $w = u_1 - u_2$ changes sign in Ω, then the sets $\Omega_1 = \{x \in \Omega \mid u_1(x) > u_2(x)\}$ and $\Omega_2 = \{x \in \Omega \mid u_2(x) > u_1(x)\}$ are not empty. One of these sets, say Ω_1, satisfies $|\Omega_1| \leq |\Omega|/2$. Take $\tilde{\Omega}_1$ to be any connected component of Ω_1 and A to be any smooth subdomain of $\tilde{\Omega}_1$. Then the choice of d_0, Proposition 6.1, and $\lambda_1^+(\Omega) \geq -d_0$ imply

$$\lambda_1^+(A) \geq \alpha/2 > 0.$$

Take a sequence of smooth domains $A_n \subset \tilde{\Omega}_1$ which converges to $\tilde{\Omega}_1$. Then $\lambda_1^+(A_n) \geq \alpha/2 > 0$, so by applying the ABP inequality (Theorem 2.3) to $F(w) \geq 0$ in A_n, we get

$$\sup_{A_n} w \leq C \sup_{\partial A_n} w.$$

Letting $n \to \infty$ implies $w \leq 0$ in $\tilde{\Omega}_1$, since $w = 0$ on $\partial\tilde{\Omega}_1$. This is a contradiction with the definition of $\Omega_1 \neq \emptyset$ and proves Theorem 1.6. □

Proof of Proposition 6.1. We follow the proof of Theorem 2.4 in [4, section 9]. We write

$$F(M, p, u, x) = F(M, p, u, x) - \delta u + \delta u =: F_0(M, p, u, x) + \delta u$$

so that F_0 is a proper operator. The operator F plays the role of L in [4], F_0 plays the role of M, δ replaces c, and we let $q = 1 + \delta$ as in [4]. As shown in [12], the ABP inequality holds for F_0, with a constant which depends only on $\lambda, \Lambda, \gamma, \delta$, and $\operatorname{diam}(\Omega)$.

In what follows we list the results in [4] which lead to Proposition 6.1, and we only note the changes needed in order to cover the nonlinear case.

Theorem 9.1 in [4] is proved in the same way here, but we have to choose $\sigma > 0$ so that $G(D^2 e^{\sigma x_1}, De^{\sigma x_1}, e^{\sigma x_1}, x) \geq 1$—recall G is defined in (H3)—which is easily seen to be possible by (H1), and then we use the inequality $F(M - N, p - q, u - v, x) \leq F(M, p, u, x) - G(N, q, v, x)$, which follows from hypothesis (H3).

The proof of Lemma 9.1 in [4] is identical in our situation, as is the proof of Lemma 9.2, provided we have the concavity of $\lambda_1^+(F_0 + \delta, \Omega)$ in δ for any proper operator F_0 satisfying our hypotheses; see below.

Theorems 9.2 and 9.3 from [4] are well known to hold for strong solutions, which is actually the only case in which we use them, if the operators in their statements are replaced by the operator

$$\mathcal{L}[u] = \mathcal{M}^-_{\lambda,\Lambda}(D^2u) - \gamma|u| - \delta|u|,$$

which appears in the left-hand side of (H1)—simply because $\mathcal{L}[u]$ is equal to a linear operator acting on u, whose coefficients depend on u but their bounds do not. Extensions of these theorems to viscosity solutions can be found in [31], [8], and in the appendix of [25].

Corollary 9.1 from [4] is proved identically here. Further, we need to modify the proof of Proposition 9.3 in [4] in the following way: we take ν to be the solution of

$$G(D^2\nu, D\nu, \nu, x) - q\nu = -\chi_\Gamma \quad \text{in } \Omega, \qquad \nu = 0 \quad \text{on } \partial\Omega,$$

where Γ is as defined in Proposition 9.3 in [4]. We easily check that $G[\cdot] - q\cdot$ is proper, $G[u] - qu \leq G[u] \leq F[u] \leq 0$ in $\Omega \setminus \Gamma$,

$$F[u - t\nu] \leq F[u] - tG[\nu] \leq -tG[\nu] = -tq\nu \leq -t\nu$$

in $\Omega \setminus \Gamma$, and the rest of the proof of Proposition 9.3 is the same.

Finally, Proposition 6.1 follows from the above in exactly the same way as Theorem 2.4 in [4] follows from Proposition 9.3 in [4]. ∎

For completeness we shall briefly sketch the elementary proof of fact that $\lambda_1^+(F_0 + \delta, \Omega)$ is concave in δ. Note that we can repeat exactly the same reasonings as the ones given on pages 50 and 68 of [4], with the only difference being that here we need to have the convexity in z of the operator

$$\mathcal{F}(z)(x) = F_0(D^2z + Dz \otimes Dz, Dz, 1, x).$$

This is the content of the following lemma.

Lemma 6.1. *Suppose* $F = F(M, p, u)$ *satisfies* (H0), (H1), *and* (H3), *and let* $l : \mathbb{R}^N \to \mathcal{M}_N(\mathbb{R})$ *be a linear map. Then the function*

$$h(p) := F(l(p) + p \otimes p, p, 1) : \mathbb{R}^N \to \mathbb{R}$$

is convex.

Proof. Suppose F depends only on M. Then (H3) implies $F(M) - F(N_1) - F(N_2) \leq F(M - N_1 - N_2)$, so for any $t \in [0, 1]$ and any $p_1, p_2 \in \mathbb{R}^N$,

$$\begin{aligned} &h(tp_1 + (1-t)p_2) - th(p_1) - (1-t)h(p_2) \\ &\leq F\left((tp_1 + (1-t)p_2) \otimes (tp_1 + (1-t)p_2) - tp_1 \otimes p_1 - (1-t)p_2 \otimes p_2\right). \end{aligned} \tag{6.1}$$

By the ellipticity of F, it is enough to show that the argument of F in the last inequality is a seminegative definite matrix. Since $p \otimes q$ is linear in both p and q, this is trivially seen to be equivalent to the semipositive definiteness of

$$(t - t^2)(p_1 \otimes p_1 + p_2 \otimes p_2 - p_1 \otimes p_2 - p_2 \otimes p_1),$$

that is, of $(t - t^2)((p_1 - p_2) \otimes (p_1 - p_2))$, which is of course true, since $t \in [0, 1]$ and the eigenvalues of $q \otimes q$ are $0, \ldots, 0, |q|^2$ for each $q \in \mathbb{R}^N$.

If $F = F(M, p, u)$, we have exactly the same reasoning, since in (6.1) we get $F(\cdot, 0, 0)$. ∎

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