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Landesman–Lazer type results for second order Hamilton–Jacobi–Bellman equations

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Abstract

We study the boundary-value problem

$$\begin{cases} F(D^2u, Du, u, x) + \lambda u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the second order differential operator F is of Hamilton–Jacobi–Bellman type, f is sub-linear in u at infinity and $\Omega \subset \mathbb{R}^N$ is a regular bounded domain. We extend the well-known Landesman–Lazer conditions to study various bifurcation phenomena taking place near the two principal eigenvalues associated to the differential operator. We provide conditions under which the solution branches extend globally along the eigenvalue gap. We also present examples illustrating the results and hypotheses.
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Keywords: Hamilton–Jacobi–Bellman equation; Landesman–Lazer condition; Bifurcation from infinity; Principal eigenvalues

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1. Introduction

We study the boundary-value problem

$$\begin{cases} F(D^2u, Du, u, x) + \lambda u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the second order differential operator F is of Hamilton–Jacobi–Bellman (HJB) type, that is, F is a supremum of linear elliptic operators, f is sub-linear in u at infinity, and $\Omega \subset \mathbb{R}^N$ is a regular bounded domain.

HJB operators have been the object of intensive study during the last thirty years – for a general review of their theory and applications we refer to [25,31,40,14]. Well-known examples include the Fucik operator $\Delta u + bu^+ + au^-$ [26], the Barenblatt operator $\max\{a\Delta u, b\Delta u\}$ [10,30], and the Pucci operator $\mathcal{M}^+_{\lambda,\Lambda}(D^2u)$ [35,15].

To introduce the problem we are interested in, let us first recall some classical results in the case when F is the Laplacian and $\lambda \in (-\infty, \lambda_2)$ (we shall denote with λ_i the i-th eigenvalue of the Laplacian). If f is independent of u, the solvability of (1.1) is a consequence of the Fredholm alternative, namely, if $\lambda \neq \lambda_1$, problem (1.1) has a solution for each f, while if $\lambda = \lambda_1$ (resonance) it has solutions if and only if f is orthogonal to φ_1, the first eigenfunction of the Laplacian. The existence result in the non-resonant case extends to nonlinearities $f(x, u)$ which grow sub-linearly in u at infinity, thanks to Krasnoselski–Leray–Schauder degree and fixed point theory, see [1].

A fundamental result, obtained by Landesman and Lazer [32] (see also [29]), states that in the resonance case $\lambda = \lambda_1$ the problem

$$\Delta u + \lambda_1 u = f(x, u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

is solvable provided f is bounded and, setting

$$f^{\pm}(x) := \limsup_{s\to\pm\infty} f(x, s), \qquad f_{\pm}(x) := \liminf_{s\to\pm\infty} f(x, s) \tag{1.2}$$

(this notation will be kept from now on), one of the following conditions is satisfied:

$$\int_\Omega f^-\varphi_1 < 0 < \int_\Omega f_+\varphi_1, \qquad \int_\Omega f_-\varphi_1 > 0 > \int_\Omega f_+\varphi_1. \tag{1.3}$$

This result initiated a huge amount of work on solvability of boundary-value problems in which the elliptic operator is at, or more generally close to, resonance. Various extensions of the results in [32] for resonant problems were obtained in [2,4,12]. Further, Mawhin and Schmitt [34] – see also [19,18] – considered (1.1) with $F = \Delta$ for λ close to λ_1, and showed that the first (resp. the second) condition in (1.3) implies that for some $\delta > 0$ problem (1.1) has at least one solution for $\lambda \in (\lambda_1 - \delta, \lambda_1]$ and at least three solutions for $\lambda \in (\lambda_1, \lambda_1 + \delta)$ (resp. at least one solution for $\lambda \in [\lambda_1, \lambda_1 + \delta)$ and at least three solutions for $\lambda \in (\lambda_1, \lambda_1 + \delta)$). These results rely on degree theory and, more specifically, on the notion of bifurcation from infinity, studied by Rabinowitz in [37].

The same results naturally hold if the Laplacian is replaced by any uniformly elliptic operator in divergence form. Further, they do remain true if a general linear operator in non-divergence form

$$L = a_{ij}(x)\partial^2_{x_i x_j} + b_i(x)\partial_{x_i} + c(x) \tag{1.4}$$

is considered, but we have to change φ_1 in (1.3) by the first eigenfunction of the adjoint operator of L. This fact is probably known to the experts, though we are not aware of a reference. Its proof – which will also easily follow from our arguments below – uses the Donsker–Varadhan [21] characterization of the first eigenvalue of L and the results in [11] which link the positivity of this eigenvalue to the validity of the maximum principle and to the Alexandrov–Bakelman–Pucci inequality (the degree theory argument remains the same as in the divergence case).

The interest in this type of problems has remained high in the PDE community over the years. Recently a large number of works have considered the extensions of the above results to quasi-linear equations (for instance, replacing the Laplacian by the p-Laplacian), where somewhat different phenomena take place, see [6,20,22,23]. There has also been a considerable interest in refining the Landesman–Lazer hypotheses (1.3) and finding general hypotheses on the nonlinearity which permit to determine on which side of the first eigenvalue the bifurcation from infinity takes place, see [3,5,27].

It is our goal here to study the boundary-value problem (1.1) under Landesman–Lazer conditions on f, when F is a Hamilton–Jacobi–Bellman (HJB) operator, that is, the supremum of linear operators as in (1.4):

$$F[u] := F\big(D^2u, Du, u, x\big) = \sup_{\alpha\in\mathcal{A}}\big\{\mathrm{tr}\big(A^\alpha(x)D^2u\big) + b^\alpha(x).Du + c^\alpha(x)u\big\}, \tag{1.5}$$

where $\mathcal{A}$ is an arbitrary index set. The following hypotheses on F will be kept throughout the paper: $A^\alpha \in C(\overline{\Omega})$, $b^\alpha, c^\alpha \in L^\infty(\Omega)$ for all $\alpha \in \mathcal{A}$ and, for some constants $0 < \lambda \leqslant \Lambda$, we have $\lambda I \leqslant A^\alpha(x) \leqslant \Lambda I$, for all $x \in \Omega$ and all $\alpha \in \mathcal{A}$. We stress however that all our results are new even for operators with smooth coefficients.

Let us now describe the most distinctive features of HJB operators – with respect to the operators considered in the previous papers on Landesman–Lazer type problems – which make our work and results different. The HJB operator $F[u]$ defined in (1.5) is nonlinear, yet positively homogeneous (that is, $F[tu] = tF[u]$ for $t \geqslant 0$), thus one may expect it has eigenvalues and eigenfunctions on the cones of positive and negative functions, but they may be different to each other. This fact was established by Lions in 1981, in the case of operators with regular coefficients, see [33]. In that paper he proved $F[u]$ has two real "demi"- or "half"-principal eigenvalues $\lambda_1^+, \lambda_1^- \in \mathbb{R}$ ($\lambda_1^+ \leqslant \lambda_1^-$), which correspond to a positive and a negative eigenfunction, respectively, and showed that the positivity of these numbers is a sufficient condition for the solvability of the related Dirichlet problem. Recently in [36] the second and the third author extended these results to arbitrary operators and studied the properties of the eigenvalues and the eigenfunctions, in particular the relation between the positivity of the eigenvalues and the validity of the comparison principle and the Alexandrov–Bakelman–Pucci estimate, thus obtaining extensions to nonlinear operators of the results of Berestycki, Nirenberg, and Varadhan in [11]. In what follows we always assume that F is indeed nonlinear in the sense that $\lambda_1^+ < \lambda_1^-$ – note the results in [36] easily imply that $\lambda_1^+ = \lambda_1^-$ can occur only if all linear operators which appear in (1.5) have the same principal eigenvalues *and* eigenfunctions.

In the subsequent works [39,24] we considered the Dirichlet problem (1.1) with f independent of u, and we obtained a number of results on the structure of its solution set, depending on the position of the parameter λ with respect to the eigenvalues λ_1^+ and λ_1^-. In particular, we proved that for each λ in the closed interval $[\lambda_1^+, \lambda_1^-]$ and each $h \in L^p$, $p > N$, which is not a multiple of the first eigenfunction φ_1^+, there exists a critical number $t^*_{\lambda,F}(h)$ such that the equation

$$F[u] + \lambda u = t\varphi_1^+ + h \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{1.6}$$

has solutions for $t > t^*_{\lambda,F}(h)$ and has no solutions for $t < t^*_{\lambda,F}(h)$. We remark this is in sharp contrast with the case of linear F, say $F = \Delta$, when (1.6) has a solution if and only if $t = t^*_{\lambda,\Delta}(h) = -\int_\Omega (h\varphi_1)$ (we shall assume all eigenfunctions are normalized so that their L^2-norm is one). Much more information on the solutions of (1.6) can be found in [39] and [24]. The value of $t^*_{\lambda_1^+,F}(h)$ in terms of F and h was computed by Armstrong [7], where he obtained an extension to HJB operators of the Donsker–Varadhan minimax formula.

We now turn to the statements of our main results. A standing assumption on the function f will be the following

(F0) $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and sub-linear in u at infinity:

$$\lim_{|s|\to\infty} \frac{f(x,s)}{s} = 0 \quad \text{uniformly in } x \in \overline{\Omega}.$$

Remark 1.1. For continuous f it is known [16,41,42] that all viscosity solutions of (1.1) are actually strong, that is, in $W^{2,p}(\Omega)$, for all $p < \infty$. Without serious additional complications we could assume that the dependence of f in x is only in L^p, for some $p > N$.

Remark 1.2. Some of the statements below can be divided into subcases by supposing that f is sub-linear in u only as $u \to \infty$ or as $u \to -\infty$ (such results for the Laplacian can be found in [19,18]). We have chosen to keep our theorems as simple as possible.

Now we introduce the hypotheses which extend the Landesman–Lazer assumptions (1.3) for the Laplacian to the case of general HJB operators. From now on we write the critical t-values at resonance as $t_+^* = t_+^*(h) = t^*_{\lambda_1^+,F}(h)$ and $t_-^* = t_-^*(h) = t^*_{\lambda_1^-,F}(h)$, and $p > N$ is a fixed number. We assume there are

(F_+^ℓ) a function $c_+ \in L^p(\Omega)$, such that $c_+(x) \leqslant f_+(x)$ in Ω and $t_+^*(c_+) < 0$,
(F_-^ℓ) a function $c^- \in L^p(\Omega)$, such that $c^-(x) \geqslant f^-(x)$ in Ω and $t_-^*(c^-) > 0$,
(F_+^r) a function $c^+ \in L^p(\Omega)$, such that $c^+(x) \geqslant f^+(x)$ in Ω and $t_+^*(c^+) > 0$,
(F_-^r) a function $c_- \in L^p(\Omega)$, such that $c_-(x) \leqslant f_-(x)$ in Ω and $t_-^*(c_-) < 0$.

Remark 1.3. Note that, decomposing $h(x) = (\int_\Omega h\varphi_1^+)\varphi_1^+(x) + h^\perp(x)$, where φ_1^+ is the eigenfunction associated to λ_1^+, we clearly have

$$t_\lambda^*(h) = t_\lambda^*(h^\perp) - \int_\Omega (h\varphi_1^+) \quad \text{for each } \lambda \in [\lambda_1^+, \lambda_1^-]. \tag{1.7}$$

So when $F = \Delta$ hypotheses (F_+^ℓ)–(F_-^ℓ) and (F_+^r)–(F_-^r) reduce to the classical Landesman–Lazer conditions (1.3), since for the Laplacian the critical t-value of a function orthogonal to φ_1 is always zero, by the Fredholm alternative.

We further observe that whenever one of the limits $f_\pm$, $f^\pm$ is infinite, the functions $c_\pm$, $c^\pm$ with the required in (F_+^ℓ)–(F_-^ℓ), (F_+^r)–(F_-^r) property always exist, while if any of $f_\pm$, $f^\pm$ is in $L^p(\Omega)$, we take the corresponding c to be equal to this limit. Note also that the strict inequalities in (F_+^ℓ)–(F_-^ℓ) and (F_+^r)–(F_-^r) are important, see Section 7.

Throughout the paper we denote by $\mathcal{S}$ the set of all pairs $(u, \lambda) \in C(\overline{\Omega}) \times \mathbb{R}$ which satisfy Eq. (1.1). For any fixed λ we set $\mathcal{S}(\lambda) = \{u \mid (u, \lambda) \in \mathcal{S}\}$ and if $\mathcal{C} \subset \mathcal{S}$ we denote $\mathcal{C}(\lambda) = \mathcal{C} \cap \mathcal{S}(\lambda)$.

Our first result gives a statement of existence of solutions for λ around λ_1^+ and λ_1^-, under the above Landesman–Lazer type hypotheses. We recall that for some constant $\delta_0 > 0$ (all constants in the paper will be allowed to depend on N, λ, Λ, γ, $\mathrm{diam}(\Omega)$), λ_1^+, λ_1^- are the only eigenvalues of F in the interval $(-\infty, \lambda_1^- + \delta_0)$ – see Theorem 1.3 in [36].

Theorem 1.1. *Assume* (F0) *and* (F_+^ℓ) *hold. Then there exist* $\delta > 0$ *and two disjoint closed connected sets of solutions of* (1.1), $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{S}$ *such that*

(1) $\mathcal{C}_1(\lambda) \neq \emptyset$ *for all* $\lambda \in (-\infty, \lambda_1^+]$,
(2) $\mathcal{C}_1(\lambda) \neq \emptyset$ *and* $\mathcal{C}_2(\lambda) \neq \emptyset$ *for all* $\lambda \in (\lambda_1^+, \lambda_1^+ + \delta)$.

The set $\mathcal{C}_2$ *is a branch of solutions "bifurcating from plus infinity to the right of* λ_1^+*", that is,* $\mathcal{C}_2 \subset C(\overline{\Omega}) \times (\lambda_1^+, \infty)$ *and there is a sequence* $\{(u_n, \lambda_n)\} \in \mathcal{C}_2$ *such that* $\lambda_n \to \lambda_1^+$ *and* $\|u_n\|_\infty \to \infty$*. Moreover, for every sequence* $\{(u_n, \lambda_n)\} \in \mathcal{C}_2$ *such that* $\lambda_n \to \lambda_1^+$ *and* $\|u_n\|_\infty \to \infty$*,* u_n *is positive in* Ω*, for n large enough.*

If we assume (F_-^ℓ) *holds, then there is a branch of solutions of* (1.1) *"bifurcating from minus infinity to the right of* λ_1^-*", that is, a connected set* $\mathcal{C}_3 \subset \mathcal{S}$ *such that* $\mathcal{C}_3 \subset C(\overline{\Omega}) \times (\lambda_1^-, \infty)$ *for which there is a sequence* $\{(u_n, \lambda_n)\} \in \mathcal{C}_3$ *such that* $\lambda_n \to \lambda_1^-$ *and* $\|u_n\|_\infty \to \infty$*. Moreover, for every sequence* $\{(u_n, \lambda_n)\} \in \mathcal{C}_3$ *such that* $\lambda_n \to \lambda_1^-$ *and* $\|u_n\|_\infty \to \infty$*,* u_n *is negative in* Ω *for n large.*

Under the sole hypothesis (F_+^ℓ) it cannot be guaranteed that the sets of solutions $\mathcal{C}_1$ and $\mathcal{C}_2$ extend much beyond λ_1^+. This important fact will be proved in Section 7, where we find $\delta_0 > 0$ such that for each $\delta \in (0, \delta_0)$ we can construct a nonlinearity $f(x, u)$ which satisfies (F0), (F_+^ℓ) and (F_-^ℓ), but for which $\mathcal{S}(\lambda_1^+ + \delta)$ is empty.

It is clearly important to give hypotheses on f under which we can get a global result, that is, existence of continua of solutions which extend over the gap between the two principal eigenvalues (this gap accounts for the nonlinear nature of the HJB operator!). The next theorems deal with that question, and use the following additional assumptions.

(F1) $f(x, 0) \geqslant 0$ and $f(x, 0) \not\equiv 0$ in Ω.
(F2) $f(x, \cdot)$ is locally Lipschitz, that is, for each $R \in \mathbb{R}$ there is C_R such that $|f(x, s_1) - f(x, s_2)| \leqslant C_R |s_1 - s_2|$ for all $s_1, s_2 \in (-R, R)$ and $x \in \overline{\Omega}$.

A discussion on these hypotheses, together with examples and counterexamples, will be given in Section 7.

Theorem 1.2. *Assume* (F0), (F1), (F2), (F_+^ℓ) *and* (F_-^ℓ) *hold. Then there exist a constant* $\delta > 0$ *and three disjoint closed connected sets of solutions* $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \subset \mathcal{S}$, *such that*

(1) $\mathcal{C}_1(\lambda) \neq \emptyset$ *for all* $\lambda \in (-\infty, \lambda_1^+]$,
(2) $\mathcal{C}_i(\lambda) \neq \emptyset$, $i = 1, 2$, *for all* $\lambda \in (\lambda_1^+, \lambda_1^-]$,
(3) $\mathcal{C}_i(\lambda) \neq \emptyset$, $i = 1, 2, 3$, *for all* $\lambda \in (\lambda_1^-, \lambda_1^- + \delta)$.

The sets $\mathcal{C}_2$ *end* $\mathcal{C}_3$ *have the same "bifurcation from infinity" properties as in the previous theorem.*

While Theorem 1.2 deals with bifurcation branches going to the right of the corresponding eigenvalues, the next theorem takes care of the case where the branches go to the left of the eigenvalues.

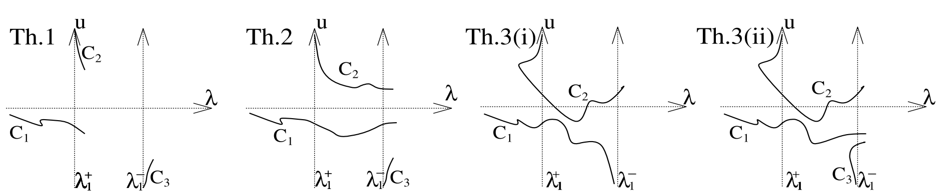
Theorem 1.3. *Assume* (F0), (F1), (F2), (F_+^r) *and* (F_-^r) *hold. Then there exist* $\delta > 0$ *and disjoint closed connected sets of solutions* $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{S}$ *such that*

(1) $\mathcal{C}_1(\lambda) \neq \emptyset$ *for all* $\lambda \in (-\infty, \lambda_1^+ - \delta]$,
(2) $\mathcal{C}_1(\lambda) \neq \emptyset$, $\mathcal{C}_2(\lambda)$ *contains at least two elements for all* $\lambda \in (\lambda_1^+ - \delta, \lambda_1^+)$, *and* $\mathcal{C}_2$ *is a branch "bifurcating from plus infinity to the left of* λ_1^+*"*,
(3) $\mathcal{C}_1(\lambda) \neq \emptyset$ *and* $\mathcal{C}_2(\lambda) \neq \emptyset$ *for all* $\lambda \in [\lambda_1^+, \lambda_1^-)$, *and either*:
  (i) $\mathcal{C}_1$ *is the branch "bifurcating from minus infinity to the left of* λ_1^-*"*,
  (ii) *there is a closed connected set of solutions* $\mathcal{C}_3 \subset \mathcal{S}$, *disjoint of* $\mathcal{C}_1$ *and* $\mathcal{C}_2$, *"bifurcating from minus infinity to the left of* λ_1^-*" such that* $\mathcal{C}_3(\lambda)$ *has at least two elements for all* $\lambda \in (\lambda_1^- - \delta, \lambda_1^-)$,
(4) $\mathcal{C}_2(\lambda) \neq \emptyset$ *for all* $\lambda \in [\lambda_1^-, \lambda_1^- + \delta]$. *In case* (ii) *in* (3), $\mathcal{C}_2(\lambda) \neq \emptyset$ *and* $\mathcal{C}_3(\lambda) \neq \emptyset$ *for all* $\lambda \in [\lambda_1^-, \lambda_1^- + \delta]$.

Note alternative (3)(ii) in this theorem is somewhat anomalous. While we are able to exclude it in a number of particular cases (in particular for the model nonlinearities which satisfy the hypotheses of the theorem), we do not believe it can be ruled out in general. See Proposition 6.1 in Section 6.

Going back to the case when F is linear, a well-known "rule of thumb" states that the number of expected solutions of (1.1) changes by two when the parameter λ crosses the first eigenvalue of F. A heuristic way of interpreting our theorems is that, when F is a supremum of linear operators, crossing a "half"-eigenvalue leads to a change of the number of solutions by one.

The following graphs illustrate our theorems.



The paper is organized as follows. The next section contains some definitions, known results, and continuity properties of the critical values t^*. In Section 3 we obtain a priori bounds for the solutions of (1.1), and construct super-solutions or sub-solutions in the different cases. In Section 4 bifurcation from infinity for HJB operators is established through the classical method of Rabinowitz, while in Section 5 we construct and study a bounded branch of solutions of (1.1). These results are put together in Section 6, where we prove our main theorems. Finally, a discussion on our hypotheses and some examples which highlight their role are given in Section 7.

2. Preliminaries and continuity of t^*

First, we list the properties shared by HJB operators of our type. The function $F : S_N \times \mathbb{R}^N \times \mathbb{R} \times \Omega \to \mathbb{R}$ satisfies (with $S, T \in S_N \times \mathbb{R}^N \times \mathbb{R}$):

(H0) F is positively homogeneous of order 1: $F(tS, x) = tF(S, x)$ for $t \geqslant 0$.
(H1) There exist $\lambda, \Lambda, \gamma > 0$ such that for $S = (M, p, u)$, $T = (N, q, v)$

$$\mathcal{M}^-_{\lambda,\Lambda}(M - N) - \gamma\big(|p - q| + |u - v|\big) \leqslant F(S, x) - F(T, x) \leqslant \mathcal{M}^+_{\lambda,\Lambda}(M - N) + \gamma\big(|p - q| + |u - v|\big).$$

(H2) The function $F(M, 0, 0, x)$ is continuous in $S_N \times \overline{\Omega}$.
(DF) We have $-F(T - S, x) \leqslant F(S, x) - F(T, x) \leqslant F(S - T, x)$ for all S, T.

In (H1) $\mathcal{M}^-_{\lambda,\Lambda}$ and $\mathcal{M}^+_{\lambda,\Lambda}$ denote the Pucci extremal operators, defined as $\mathcal{M}^+_{\lambda,\Lambda}(M) = \sup_{A\in\mathcal{A}} \operatorname{tr}(AM)$, $\mathcal{M}^-_{\lambda,\Lambda}(M) = \inf_{A\in\mathcal{A}} \operatorname{tr}(AM)$, where $\mathcal{A} \subset \mathcal{S}_N$ denotes the set of matrices whose eigenvalues lie in the interval $[\lambda, \Lambda]$, see for instance [15]. Note under (H0) assumption (DF) is equivalent to the convexity of F in S – see Lemma 1.1 in [36]. Hence for each $\phi, \psi \in W^{2,p}(\Omega)$ we have the inequalities $F[\phi + \psi] \leqslant F[\phi] + F[\psi]$ and $F[\phi - \psi] \geqslant F[\phi] - F[\psi]$.

We recall the definition of the principal eigenvalues of F from [36]

$$\lambda_1^+(F, \Omega) = \sup\big\{\lambda \;\big|\; \Psi^+(F, \Omega, \lambda) \neq \emptyset\big\}, \qquad \lambda_1^-(F, \Omega) = \sup\big\{\lambda \;\big|\; \Psi^-(F, \Omega, \lambda) \neq \emptyset\big\},$$

where $\Psi^\pm(F, \Omega, \lambda) = \{\psi \in C(\overline{\Omega}) \mid \pm(F[\psi] + \lambda\psi) \leqslant 0,\ \pm\psi > 0 \text{ in } \Omega\}$. Many properties of the eigenvalues (simplicity, isolation, monotonicity and continuity with respect to the domain, relation with the maximum principle) are established in Theorems 1.1–1.9 of [36]. We shall repeatedly use these results. We shall also often refer to the statements on the solvability of the Dirichlet problem, given in [36] and [24].

We recall the following Alexandrov–Bakelman–Pucci (ABP) and $C^{1,\alpha}$ estimates, see [28,17,42].

Theorem 2.1. *Suppose F satisfies* (H0), (H1), (H2), *and u is a solution of $F[u] + cu = f(x)$ in Ω, with $u = 0$ on $\partial\Omega$. Then there exist $\alpha \in (0, 1)$ and $C_0 > 0$ depending on N, λ, Λ, γ, c and Ω such that $u \in C^{1,\alpha}(\overline{\Omega})$, and*

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leqslant C_0\big(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)}\big).$$

Moreover, if one chooses $c = -\gamma$ (so that by (H1) *$F - \gamma$ is proper) then this equation has a unique solution which satisfies $\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leqslant C_0\|f\|_{L^\infty(\Omega)}$. More precisely, any solution of $F[u] - \gamma u \geqslant f(x)$ satisfies $\sup_\Omega u \leqslant \sup_{\partial\Omega} u + C\|f\|_{L^N}$.*

For readers' convenience we state a version of Hopf's lemma (for viscosity solutions it was proved in [9]).

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^N$ be a regular domain and let $\gamma > 0$, $\delta > 0$. Assume $w \in C(\overline{\Omega})$ is a viscosity solution of $\mathcal{M}^-_{\lambda,\Lambda}(D^2 w) - \gamma|Dw| - \delta w \leqslant 0$ in Ω, and $w \geqslant 0$ in Ω. Then either $w \equiv 0$ in Ω or $w > 0$ in Ω and at any point $x_0 \in \partial\Omega$ at which $w(x_0) = 0$ we have $\limsup_{t \searrow 0} \frac{w(x_0+t\nu)-w(x_0)}{t} < 0$, where ν is the interior normal to $\partial\Omega$ at x_0.*

The next theorem is a consequence of the compact embedding $C^{1,\alpha}(\Omega) \hookrightarrow C^1(\Omega)$, Theorem 2.1, and the convergence properties of viscosity solutions (see Theorem 3.8 in [17]).

Theorem 2.3. *Let $\lambda_n \to \lambda$ in $\mathbb{R}$ and $f_n \to f$ in $L^p(\Omega)$. Suppose F satisfies* (H1) *and u_n is a viscosity solution of $F[u_n] + \lambda_n u_n = f_n$ in Ω, $u_n = 0$ on $\partial\Omega$. If $\{u_n\}$ is bounded in $L^\infty(\Omega)$ then a subsequence of $\{u_n\}$ converges in $C^1(\overline{\Omega})$ to a function u, which solves $F[u] + \lambda u = f$ in Ω, $u = 0$ on $\partial\Omega$.*

As a simple consequence of this theorem, the homogeneity of F and the simplicity of the eigenvalues we obtain the following proposition.

Proposition 2.1. *Let $\lambda_n \to \lambda_1^\pm$ in $\mathbb{R}$ and f_n be bounded in $L^p(\Omega)$. Suppose F satisfies* (H1) *and u_n is a viscosity solution of $F[u_n] + \lambda_n u_n = f_n$ in Ω, $u_n = 0$ on $\partial\Omega$. If $\{u_n\}$ is unbounded in $L^\infty(\Omega)$ then a subsequence of $\frac{u_n}{\|u_n\|}$ converges in $C^1(\overline{\Omega})$ to $\varphi_1^\pm$. In particular, u_n is positive* (*negative*) *for large n, and for each $K > 0$ there is N such that $|u_n| \geqslant K\varphi_1^+$ for $n \geqslant N$.*

For shortness, from now on the zero boundary condition on $\partial\Omega$ will be understood in all differential (in)equalities we write, and $\|\cdot\|$ will refer to the $L^\infty(\Omega)$-norm.

We devote the remainder of this section to the definition and some basic continuity properties of the critical t-values for (1.6). These numbers are crucial in the study of existence of solutions at resonance and in the gap between the eigenvalues. For each $\lambda \in [\lambda_1^+, \lambda_1^-]$ and each $d \in L^p$, which is not a multiple of the first eigenfunction φ_1^+, the number

$$t^*_\lambda(d) = \inf\bigl\{t \in \mathbb{R} \bigm| F[u] + \lambda u = s\varphi_1^+ + d \text{ has solutions for } s \geqslant t\bigr\}$$

is well defined and finite. The non-resonant case $\lambda \in (\lambda_1^+, \lambda_1^-)$ was considered in [39], while the resonant cases $\lambda = \lambda_1^+$ and $\lambda = \lambda_1^-$ were studied in [24].

In what follows we prove the continuity of $t^*_\lambda : L^p(\Omega) \to \mathbb{R}$ for any fixed $\lambda \in [\lambda_1^+, \lambda_1^-]$. Actually, in [39] the continuity of this function is proved for all $\lambda \in (\lambda_1^+, \lambda_1^-)$, so we only need to take care of the resonant cases $\lambda = \lambda_1^+$ and $\lambda = \lambda_1^-$, that is, to study t^*_+ and t^*_-. In doing so, it is convenient to use the following equivalent definitions of t^*_+ and t^*_- (see [24])

$$\begin{aligned} t^*_+(d) = \inf\bigl\{t \in \mathbb{R} \bigm| &\text{ for each } s > t \text{ and } \lambda_n \nearrow \lambda_1^+ \text{ there exists } u_n \\ &\text{such that } F[u_n] + \lambda_n u_n = s\varphi_1^+ + d \text{ and } \|u_n\| \text{ is bounded}\bigr\} \end{aligned} \tag{2.1}$$

and

$$t_-^*(d) = \inf\bigl\{t \in \mathbb{R} \bigm| \text{for each } s > t \text{ and } \lambda_n \searrow \lambda_1^- \text{ there exists } u_n \text{ such that } F[u_n] + \lambda_n u_n = s\varphi_1^+ + d \text{ and } \|u_n\| \text{ is bounded}\bigr\}. \tag{2.2}$$

Proposition 2.2. *The functions $t_+^*, t_-^* : L^p(\Omega) \to \mathbb{R}$ are continuous.*

Proof. If we assume t_+^* is not continuous, then there is $d \in L^p(\Omega)$, $\varepsilon > 0$ and a sequence $d_n \to d$ in $L^p(\Omega)$ such that either $t_+^*(d_n) \geqslant t_+^*(d) + 3\varepsilon$ for all $n \in \mathbb{N}$ or $t_+^*(d_n) \leqslant t_+^*(d) - 3\varepsilon$ for all $n \in \mathbb{N}$.

First we suppose that $t_+^*(d_n) \geqslant t_+^*(d) + 3\varepsilon$ for all $n \in \mathbb{N}$. Then, for any sequence $\lambda_m \nearrow \lambda_1^+$ we find solutions u_n^m of the equation

$$F\bigl[u_n^m\bigr] + \lambda_m u_n^m = \bigl(t_+^*(d_n) - 2\varepsilon\bigr)\varphi_1^+ + d \quad \text{in } \Omega,$$

and the sequence $\{\|u_n^m\|\}$ is bounded as $m \to \infty$, for each fixed n – see (2.1). We also consider the solutions w_n of $F[w_n] - \gamma w_n = d_n - d$. By Theorem 2.1 we know that $w_n \to 0$ in $C^1(\overline{\Omega})$. Then by the structural hypotheses on F (recall $F[u+v] \leqslant F[u] + F[v]$) we have

$$\begin{aligned} F\bigl[u_n^m + w_n\bigr] + \lambda_m\bigl(u_n^m + w_n\bigr) &\leqslant \bigl(t_+^*(d_n) - 2\varepsilon\bigr)\varphi_1^+ + d_n + (\gamma + \lambda_m)w_n \\ &\leqslant \bigl(t_+^*(d_n) - \varepsilon\bigr)\varphi_1^+ + d_n, \end{aligned}$$

where the last inequality holds if n is large, independently of m. Fix one such n. On the other hand we can take solutions z_n^m of

$$F\bigl[z_n^m\bigr] + \lambda_m z_n^m = \bigl(t_+^*(d_n) - \varepsilon\bigr)\varphi_1^+ + d_n \quad \bigl(\geqslant F\bigl[u_n^m + w_n\bigr] + \lambda_m\bigl(u_n^m + w_n\bigr)\bigr).$$

By (2.1) for any n we have $\|z_n^m\| \to \infty$ as $m \to \infty$. By the comparison principle (valid by $\lambda_m < \lambda_1^+$ and Theorem 1.5 in [36]) we obtain $z_n^m \leqslant u_n^m + w_n$ in Ω, hence z_n^m is bounded above as $m \to \infty$. Since z_n^m is bounded below, by Theorem 1.7 in [36] and $\lambda_m \leqslant \lambda_1^+ < \lambda_1^-$, we obtain a contradiction.

Assume now that $t_+^*(d_n) \leqslant t_+^*(d) - 3\varepsilon$. Let u_n be a solution of

$$F[u_n] + \lambda_1^+ u_n = \bigl(t_+^*(d) - 2\varepsilon\bigr)\varphi_1^+ + d_n \quad \text{in } \Omega,$$

which exists since $t_+^*(d_n) < t_+^*(d) - 2\varepsilon$ – Theorem 1.2 in [24]. Let w_n be the solution of $F[w_n] + cw_n = d - d_n$ in Ω, with $w_n \to 0$ in $C^1(\overline{\Omega})$. Then there exists n_0 large enough so that $(\lambda_1^+ + \gamma)w_{n_0} < \varepsilon\varphi_1^+$, and consequently $u_{n_0} + w_{n_0}$ is a super-solution of

$$F[u] + \lambda_1^+ u = \bigl(t_+^*(d) - \varepsilon\bigr)\varphi_1^+ + d \quad \text{in } \Omega. \tag{2.3}$$

Now, if w is the solution of $F[w] - \gamma w = -d$ in Ω, by defining $v_k = k\varphi_1^- - w$ we obtain

$$F[v_k] + \lambda_1^+ v_k \geqslant k\bigl(\lambda_1^+ - \lambda_1^-\bigr)\varphi_1^- + d - \bigl(\lambda_1^+ + \gamma\bigr)w > \bigl(t_+^*(d) - \varepsilon\bigr)\varphi_1^+ + d,$$

for k large enough. By taking k large we also have $v_k < u_{n_0} - w_{n_0}$ in Ω, thus Eq. (2.3) possesses ordered super- and sub-solutions. Consequently it has a solution (by Perron's method – see for instance Lemma 4.3 in [36]), a contradiction with the definition of $t_+^*(d)$. This completes the proof of the continuity of the function t_+^*.

The rest of the proof is devoted to the analysis of continuity of t_-^*. Assuming t_-^* is not continuous, there is $\varepsilon > 0$ and a sequence $d_n \to d$ in $L^p(\Omega)$ such that either $t_-^*(d_n) \geqslant t_-^*(d) + 3\varepsilon$ or $t_-^*(d_n) \leqslant t_-^*(d) - 3\varepsilon$. In the first case, let us consider a sequence $\lambda_m \searrow \lambda_1^-$, and a solution v_m of the equation

$$F[v_m] + \lambda_m v_m = \big(t_-^*(d) + \varepsilon\big)\varphi_1^+ + d \quad \text{in } \Omega.$$

We recall v_m exists, by the results in [7] and [24]. We have shown in [24] that $t_-^*(d_n) \geqslant t_-^*(d) + 3\varepsilon > t_-^*(d) + \varepsilon$ implies that v_m can be chosen to be bounded as $m \to \infty$ (see (2.2)). Let w_n be the solution to $F[w_n] - \gamma w_n = d_n - d$ in Ω, as above. Then $z_{n_0}^m = v_m + w_{n_0}$ satisfies for some large n_0

$$F\big[z_{n_0}^m\big] + \lambda_m z_{n_0}^m \leqslant \big(t_-^*(d) + \varepsilon\big)\varphi_1^+ + (\lambda_m + \gamma)w_n + d_n \leqslant \big(t_-^*(d) + 2\varepsilon\big)\varphi_1^+ + d_n,$$

since again $w_n \to 0$ in $C^1(\overline{\Omega})$. On the other hand, we consider a solution of

$$F\big[u_n^m\big] + \lambda_m u_n^m = \big(t_-^*(d) + 2\varepsilon\big)\varphi_1^+ + d_n \quad \text{in } \Omega.$$

As $t_-^*(d_n) \geqslant t_-^*(d) + 3\varepsilon > t_-^*(d) + 2\varepsilon$ for all n, the sequence u_n^m is not bounded (again by (2.2) and [24]) and $u_n^m/\|u_n^m\|_\infty \to \varphi_1^-$ as $m \to \infty$, for each fixed n. Therefore for large m the function $\Psi = u_{n_0}^m - (v_m + w_{n_0}) < 0$ and $F[\Psi] + \lambda_m \Psi \geqslant 0$, which is a contradiction with the definition of λ_1^-, since $\lambda_m > \lambda_1^-$.

Let us assume now that for $\varepsilon > 0$ and the sequence $d_n \to d$ in $L^p(\Omega)$ we have $t_-^*(d_n) \leqslant t_-^*(d) - 3\varepsilon$, for all n. Let $\lambda_m \searrow \lambda_1^-$ and v_m be a solution of the equation $F[v_m] + \lambda_m v_m = (t_-^*(d) - \varepsilon)\varphi_1^+ + d$ in Ω (by (2.2) v_m is unbounded), and let w_n be the solution to $F[w_n] - \gamma w_n = d - d_n$ in Ω. Then, $v_m/\|v_m\|_\infty \to \varphi_1^-$ and $w_n \to 0$ in $C^1(\overline{\Omega})$. We take a solution u_n^m to

$$F\big[u_n^m\big] + \lambda_m u_n^m = \big(t_-^*(d) - 2\varepsilon\big)\varphi_1^+ + d_n \quad \text{in } \Omega,$$

and note that, since $t_-^*(d_n) \leqslant t_-^*(d) - 3\varepsilon < t_-^*(d) - 2\varepsilon$, for any given n there exists a constant c_n such that $\|u_n^m\|_\infty \leqslant c_n$, for all m. Now, as above, we define $\Psi = v_m - (u_n^m + w_n)$, and see that

$$\begin{aligned} F[\Psi] + \lambda_m \Psi &\geqslant \big(t_-^*(d) - \varepsilon\big)\varphi_1^+ + d - \big(t_-^*(d) - 2\varepsilon\big)\varphi_1^+ - d_n - F[w_n] - \lambda_m w_n \\ &\geqslant \varepsilon\varphi_1^+ - (\lambda_m + \gamma)w_n. \end{aligned}$$

We choose n large enough to have $(\lambda_m + \gamma)w_n < \varepsilon\varphi_1^+$ in Ω. Then, keeping n fixed, we can choose m large enough to have $\Psi < 0$ in Ω, a contradiction with the definition of λ_1^-, since $\lambda_m > \lambda_1^-$. □

Finally we prove that the function $t_\lambda^*(d)$ is also continuous in λ at the end points of the interval $[\lambda_1^+, \lambda_1^-]$, when d is kept fixed. This fact will be needed in Section 7.

Proposition 2.3. *For every $d \in L^p(\Omega)$*

$$\lim_{\lambda \searrow \lambda_1^+} t_\lambda^*(d) = t_+^*(d) \quad \textit{and} \quad \lim_{\lambda \nearrow \lambda_1^-} t_\lambda^*(d) = t_-^*(d).$$

Proof. Let us assume that there are $\varepsilon > 0$ and a sequence $\lambda_n \searrow \lambda_1^+$ such that $t_{\lambda_n}^* < t_+^* - \varepsilon$ (since d is fixed, we do not write it explicitly). Then by the definition of $t_{\lambda_n}^*$ there is a function u_n satisfying

$$F[u_n] + \lambda_n u_n = \left(t_+^* - \varepsilon\right)\varphi_1^+ + d \quad \text{in } \Omega.$$

Since $\lambda_n \searrow \lambda_1^+$, u_n cannot be bounded, for otherwise we get a contradiction with the definition of t_+^* by finding a solution with $\bar{t} < t_+^*$ – from Theorem 2.3. Then by Proposition 2.1 $u_n/\|u_n\|_\infty \to \varphi_1^+$, u_n is positive for large n, and

$$F[u_n] + \lambda_1^+ u_n = \left(t_+^* - \varepsilon\right)\varphi_1^+ + d + \left(\lambda_1^+ - \lambda_n\right)u_n < \left(t_+^* - \varepsilon\right)\varphi_1^+ + d,$$

that is, u_n is a super-solution. On the other hand, for $t > t_+^*$, let u be a solution of $F[u] + \lambda_1^+ u = t\varphi_1^+ + d$, in Ω, then u is a sub-solution for this equation with $(t_+^* - \varepsilon)\varphi_1^+ + d$ as a right-hand side. By taking n large enough, we have $u_n \geqslant u$, so that the equation

$$F[u] + \lambda_1^+ u = \left(t_+^* - \varepsilon\right)\varphi_1^+ + d \quad \text{in } \Omega$$

has a solution, a contradiction with the definition of t_+^*.

Now we assume that there are $\varepsilon > 0$ and a sequence $\lambda_n \searrow \lambda_1^+$ such that $t_n = t_{\lambda_n}^* > t_+^* + 2\varepsilon$. Let v be a solution to

$$F[v] + \lambda_1^+ v = \left(t_+^* + \varepsilon/2\right)\varphi_1^+ + d \quad \text{in } \Omega,$$

then $F[v] + \lambda_n v = (t_+^* + \varepsilon)\varphi_1^+ + d - \varepsilon/2\varphi_1^+ + (\lambda_n - \lambda_1^+)v$. Since $t_+^* + \varepsilon < t_{\lambda_n}^* - \varepsilon/2$, by choosing n large we find $F[v] + \lambda_n v < (t_{\lambda_n}^* - \varepsilon/2)\varphi_1^+ + d$, so that v is a super-solution of

$$F[u] + \lambda_n u = \left(t_{\lambda_n}^* - \varepsilon/2\right)\varphi_1^+ + d. \tag{2.4}$$

Next we consider a solution u_K of $F[u] + (\lambda_1^+ + \nu)u = K$ (where we have set $\nu = (\lambda_1^- - \lambda_1^+)/2 > 0$), for each $K > 0$. Such a solution exists by Theorem 1.9 in [36], and it further satisfies $u_K < 0$ in Ω and $\|u_K\|_\infty \to \infty$ as $K \to \infty$, so $|u_K| \geqslant C(K)\varphi_1^+$, where $C(K) \to \infty$ as $K \to \infty$. Let w be the (unique) solution of $F[w] - \gamma w = -d$ in Ω. Since $F[u_K - w] \geqslant F[u_K] - F[w]$, we easily see that the function $u_K - w$ is a sub-solution of (2.4) and $u_K - w < v$, for large K. Then Perron's method leads again to a contradiction with the definition of t_{λ_n}. This shows t_λ^* is right-continuous at λ_1^+.

Now we prove the second statement of Lemma 2.3. Assume there are $\varepsilon > 0$ and a sequence $\lambda_n \nearrow \lambda_1^-$ such that $t_{\lambda_n}^* < t_-^* - \varepsilon$. Let u_n be a solution to

$$F[u_n] + \lambda_n u_n = \left(t_-^* - \varepsilon\right)\varphi_1^+ + d \quad \text{in } \Omega.$$

Since $\lambda_n \to \lambda_1^-$, u_n cannot be bounded (as before) and then $u_n/\|u_n\|_\infty \to \varphi_1^-$. Thus, for large n we have $u_n < 0$ and

$$F[u_n] + \lambda_1^- u_n = (t_-^* - \varepsilon)\varphi_1^+ + d + (\lambda_1^- - \lambda_n)u_n < (t_-^* - \varepsilon)\varphi_1^+ + d,$$

so that u_k is a super-solution for some large (fixed) k. Consider now a sequence $\tilde{\lambda}_n \searrow \lambda_1^-$ and let v_n be the solution to

$$F[v_n] + \tilde{\lambda}_n v_n = (t_-^* - \varepsilon)\varphi_1^+ + d \quad \text{in } \Omega,$$

whose existence was proved in [7] and [24]. Then v_n cannot be bounded, so $v_n/\|v_n\|_\infty \to \varphi_1^-$, and for large n we have

$$F[v_n] + \lambda_1^- v_n = (t_-^* - \varepsilon)\varphi_1^+ + d + (\lambda_1^- - \tilde{\lambda}_n)v_n > (t_-^* - \varepsilon)\varphi_1^+ + d,$$

that is, v_n is a sub-solution. For the already fixed u_k, we can find n sufficiently large so that $u_k > v_n$, which implies that the equation

$$F[u] + \lambda_1^- u = (t_-^* - \varepsilon)\varphi_1^+ + d \quad \text{in } \Omega$$

has a solution, a contradiction with the definition of t_-^*.

Finally, assume that there are $\varepsilon > 0$ and a sequence $\lambda_n \nearrow \lambda_1^-$ such that $t_{\lambda_n}^* > t_{\lambda_n}^* - 2\varepsilon > t_-^* + \varepsilon$. By Theorem 1.4 in [24] we can find a function u which solves the equation $F[u] + \lambda_1^- u = (t_-^* + \varepsilon)\varphi_1^+ + d$ in Ω. Then

$$F[u] + \lambda_n u < (t_{\lambda_n}^* - \varepsilon)\varphi_1^+ + d - \varepsilon\varphi_1^+ + (\lambda_n - \lambda_1^-)\varphi_1^+ < (t_{\lambda_n}^* - \varepsilon)\varphi_1^+ + d,$$

so that u is a super-solution of $F[u] + \lambda_n u = (t_{\lambda_n}^* - \varepsilon)\varphi_1^+ + d$, for some large fixed n. As we explained above, since $\lambda_n < \lambda_1^-$, by Theorem 1.9 in [36] we can construct an arbitrarily negative sub-solution of this problem, hence a solution as well, contradicting the definition of $t_{\lambda_n}^*$. □

3. Resonance and a priori bounds

In this section we assume that the nonlinearity $f(x,s)$ satisfies the one-sided Landesman–Lazer conditions at resonance, that is, one of (F_+^ℓ), (F_-^ℓ), (F_+^r) and (F_-^r). Under each of these conditions we analyze the existence of super-solutions, sub-solutions and a priori bounds when λ is close to the eigenvalues λ_1^+ and λ_1^-. This information will allow us to obtain existence of solutions by using degree theory and bifurcation arguments. In particular we will get branches bifurcating from infinity which curve right or left depending on the a priori bounds obtained here.

We start with the existence of a super-solution and a priori bounds at λ_1^+, under hypothesis (F_+^ℓ).

Proposition 3.1. *Assume f satisfies* (F0) *and (F_+^ℓ). Then there exists a super-solution z such that $F[z] + \lambda z < f(x,z)$ in Ω, for all $\lambda \in (-\infty, \lambda_1^+]$. Moreover, for each $\lambda_0 < \lambda_1^+$ there exist $R > 0$ and a super-solution z_0 such that if u is a solution of* (1.1) *with $\lambda \in [\lambda_0, \lambda_1^+]$, then $\|u\| \leqslant R$ and $u \leqslant z_0$ in Ω.*

Proof. We first replace c_+ by a more appropriate function: we claim that for each $\varepsilon > 0$ there exist $R > 0$ and a function $d \in L^p(\Omega)$ such that

$$\|d - c_+\|_{L^p(\Omega)} \leqslant \varepsilon \quad \text{and} \quad u \geqslant R\varphi_1^+ \quad \text{implies} \quad f\big(x, u(x)\big) \geqslant d(x) \quad \text{in } \Omega.$$

In fact, setting $\sigma = \frac{\varepsilon}{2|\Omega|^{1/p}}$, we can find s_0 such that $f(x,s) \geqslant c_+(x) - \sigma$ in Ω, for all $s \geqslant s_0$. Let $\Omega^R = \{x \in \Omega \mid R\varphi_1^+(x) > s_0\}$ and define the function d_R as $d_R(x) = c_+(x) - \sigma$ if $x \in \Omega^R$, and $d_R(x) = -M$ for $x \in \Omega \setminus \Omega^R$, where M is such that $f(x,s) \geqslant -M$, for all $s \in [0, s_0]$. It is then trivial to check that the claim holds for $d = d_R$, if R is taken such that $|\Omega \setminus \Omega^R| < (\varepsilon/2M)^p$.

Now, by (F_+^ℓ) and the continuity of t_+^* (Proposition 2.2) we can fix ε so small that the function d chosen above satisfies

$$t_+^*(d) < 0. \tag{3.1}$$

Let z_n be a solution to

$$F[z_n] + \lambda_1^+ z_n = t_n\varphi_1^+ + d \quad \text{in } \Omega,$$

where $t_n \to t_+^*(d) < 0$, $t_n \geqslant t_+^*(d)$, is a sequence such that z_n can be chosen to be unbounded – such a choice of t_n and z_n is possible thanks to Theorem 1.2 in [24]. Then $z_n/\|z_n\| \to \varphi_1^+$, which implies that for large n

$$F[z_n] + \lambda_1^+ z_n < d \quad \text{and} \quad z_n \geqslant R\varphi_1^+,$$

by (3.1), where R is as in the claim above. Thus z_n is a strict super-solution and, since z_n is positive, $F[z_n] + \lambda z_n < f(x, z_n)$, for all $\lambda \in (-\infty, \lambda_1^+]$. From now on we fix one such n_0 and drop the index, calling the super-solution z.

Suppose there exists an unbounded sequence u_n of solutions to

$$F[u_n] + \lambda_n u_n = f(x, u_n) \quad \text{in } \Omega,$$

with $\lambda_n \in [\lambda_0, \lambda_1^+]$ and $\lambda_n \to \bar\lambda$. If $\bar\lambda < \lambda_1^+$ then a contradiction follows since λ_1^+ is the first eigenvalue (divide the equation by $\|u_n\|$ and let $n \to \infty$). If $\bar\lambda = \lambda_1^+$ then $u_n/\|u_n\| \to \varphi_1^+$, so that for n large we have $u_n > z$ and $u_n \geqslant R\varphi_1^+$, consequently $f(x, u_n) \geqslant d(x)$ in Ω. Thus, setting $w = u_n - z$ we get, by $\lambda_n \leqslant \lambda_1^+$, $w > 0$,

$$F[w] + \lambda_1^+ w \geqslant F[u_n] - F[z] + \lambda_1^+(u_n - z) > f(x, u_n) - d \geqslant 0.$$

Since $w > 0$, Theorem 1.2 in [36] implies the existence of a constant $k > 0$ such that $w = k\varphi_1^+$, a contradiction with the last strict inequality. Now that we have an a priori bound for the solutions, we may choose an appropriate n_0 for the definition of $z_0 = z_{n_0}$, which makes it larger than all solutions. □

Next we state an analogous proposition on the existence of a sub-solution to our problem at λ_1^- under hypothesis (F_-^ℓ).

Proposition 3.2. *Assuming that f satisfies* (F0) *and* (F_-^ℓ)*, there exists a strict sub-solution z such that $F[z]+\lambda z > f(x,z)$ in Ω for all $\lambda \in (-\infty, \lambda_1^-]$. Moreover, for each $\delta > 0$ there exist $R > 0$ and a sub-solution z such that if u solves* (1.1) *with $\lambda \in [\lambda_1^+ + \delta, \lambda_1^-]$ then $\|u\|_\infty \leqslant R$ and $u \geqslant z$ in Ω.*

Proof. By using essentially the same proof as in Proposition 3.1, we can find $R > 0$ and a function $d \in L^p(\Omega)$ such that $t_-^*(d) > 0$, and $u \leqslant -R\varphi_1^+$ implies $f(x,u(x)) \leqslant d(x)$ in Ω. Consider a sequence $t_n \searrow t_-^*(d)$ and solutions z_n to

$$F[z_n] + \lambda_1^- z_n = t_n\varphi_1^+ + d \quad \text{in } \Omega, \tag{3.2}$$

chosen so that z_n is unbounded and $z_n/\|z_n\|_\infty \to \varphi_1^-$ – see Theorem 1.4 in [24]. Hence for n large enough

$$F[z_n] + \lambda_1^- z_n > d \quad \text{and} \quad z_n \leqslant -R\varphi_1^+. \tag{3.3}$$

Thus z_n is a strict sub-solution and, since z_n is negative for sufficiently large n, $F[z_n] + \lambda z_n > f(x,z_n)$, for all $\lambda \in (-\infty, \lambda_1^-]$. Fix one such n_0 and set $z = z_{n_0}$.

If u_n is an unbounded sequence of solutions to $F[u_n] + \lambda_n u_n = f(x,u_n)$, in Ω, with $\lambda_n \in [\lambda_1^+ + \delta, \lambda_1^-]$ and $\lambda_n \to \bar\lambda$ we obtain a contradiction like in the previous proposition. Namely, if $\bar\lambda \in [\lambda_1^+ + \delta, \lambda_1^-)$ then the conclusion follows since there are no eigenvalues in this interval. If $\bar\lambda = \lambda_1^-$ then $u_n/\|u_n\| \to \varphi_1^-$, so that for n large $u_n < z$ and $u_n \leqslant -R\varphi_1^+$, hence $f(x,u_n) \leqslant d(x)$, which leads to the contradiction $F[z-u_n] + \lambda_1^-(z-u_n) \geqslant 0$ and $z - u_n > 0$. Then, given the a priori bound, we can choose n_0 such that z_{n_0} is smaller than all solutions. □

The next two propositions are devoted to proving a priori bounds under hypotheses (F_+^r) and (F_-^r).

Proposition 3.3. *Under assumptions* (F0) *and* (F_+^r) *for each $\delta > 0$ the solutions to* (1.1) *with $\lambda \in [\lambda_1^+, \lambda_1^- - \delta]$ are a priori bounded.*

Proof. As in the proof of Proposition 3.1, we may choose $R > 0$ and a function d so that $t_+^*(d) > 0$, that is, $\int_\Omega d\varphi_1^+ < t_+^*(d^\perp)$ (recall (1.7)), and whenever $u \geqslant R\varphi_1^+$ then $f(x,u) \leqslant d$. Let $\tilde t$ be fixed such that $\int_\Omega d\varphi_1^+ < \tilde t < t_+^*(d^\perp)$. If the proposition were not true, then there would be sequences $\lambda_n \searrow \lambda_1^+$ and u_n of solutions to $F[u_n] + \lambda_n u_n = f(x,u_n)$, such that u_n is unbounded. Then $u_n/\|u_n\| \to \varphi_1^+$, in particular, u_n is positive for large n. Then

$$F[u_n] + \lambda_1^+ u_n \leqslant f(x,u_n) \leqslant d < \tilde t\varphi_1^+ + d^\perp,$$

that is, u_n is a super-solution of $F[u_n] + \lambda_1^+ u_n = \tilde t\varphi_1^+ + d^\perp$. Next, take the solution w of $F[w] - \gamma w = -d^\perp$ in Ω, where, as before, γ is the constant from (H1), so that $F - \gamma$ is proper. For $\alpha > 0$ we define $v = \alpha\varphi_1^- - w$, then

$$F[v] + \lambda_1^+ v \geqslant \alpha\big(\lambda_1^+ - \lambda_1^-\big)\varphi_1^- - \big(\lambda_1^+ + \gamma\big)w + d^\perp,$$

exactly like in the proof of Proposition 2.2. If we choose α large enough, we see that v is a sub-solution for $F[u_n] + \lambda_1^+ u_n = \tilde{t}\varphi_1^+ + d^\perp$, and v is smaller than the super-solution we constructed before. The existence of a solution to this equation contradicts the definition of $t_+^*(d^\perp)$ and $\tilde{t} < t_+^*(d^\perp)$. $\square$

Proposition 3.4. *Under assumptions* (F0) *and* (F_-^r) *there exists* $\delta > 0$ *such that the solutions to* (1.1) *with* $\lambda \in [\lambda_1^-, \lambda_1^- + \delta]$ *are a priori bounded.*

Proof. We proceed like in the proof of the previous proposition. Now $\int_\Omega d\varphi_1^+ > t_-^*(d^\perp)$, and whenever $u \leqslant -R\varphi_1^+$ then $f(x,u) \geqslant d$. If $\tilde{t}$ is such that $\int_\Omega d\varphi_1^+ > \tilde{t} > t_+^*(d^\perp)$, and we assume there are sequences $\lambda_n \searrow \lambda_1^-$ and u_n of solutions to $F[u_n] + \lambda_n u_n = f(x, u_n)$ in Ω, such that u_n is unbounded, we get $u_n/\|u_n\| \to \varphi_1^-$, consequently

$$F[u_n] + \lambda_1^- u_n > \tilde{t}\varphi_1^+ + d^\perp.$$

On the other hand if z solves $F[z] + \lambda_1^- z = \tilde{t}\varphi_1^+ + d^\perp$ in Ω (such z exists by Theorem 1.4 in [24]), then $F[u_n - z] + \lambda_1^-(u_n - z) > 0$, and $u_n - z < 0$ in Ω, for large n. Thus, we may apply Theorem 1.4 in [36] to obtain $k > 0$ so that $u_n - z = k\varphi_1^-$, a contradiction with the strict inequality. $\square$

4. Bifurcation from infinity at λ_1^+ and λ_1^-

In this section we prove the existence of unbounded branches of solutions of (1.1), bifurcating from infinity at the eigenvalues λ_1^+ and λ_1^-. Then, thanks to the a priori bounds obtained in Section 3, for the two types of Landesman–Lazer conditions (see Propositions 3.1–3.4), we may determine to which side of the eigenvalues these branches curve.

We recall that $F(M, q, u, x) + cu$ is decreasing in u for any $c \leqslant -\gamma$, in other words, $F + c$ is a proper operator. Given $v \in C^1(\overline{\Omega})$ we consider the problem

$$F[u] + cu = (c - \lambda)v + f(x, v) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{4.1}$$

see Theorem 2.1. We define the operator $K : \mathbb{R} \times C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$ as follows: $K(\lambda, v)$ is the unique solution $u \in C^{1,\alpha}(\overline{\Omega})$ of (4.1). The operator K is compact in view of Theorem 2.1 and the compact embedding $C^{1,\alpha}(\overline{\Omega}) \to C^1(\overline{\Omega})$. With these definitions, our Eq. (1.1) is transformed into the fixed point problem $u = K(\lambda, u)$, $u \in C^1(\overline{\Omega})$, with $\lambda \in \mathbb{R}$ as a parameter. We are going to show that the sub-linearity of the function $f(x, \cdot)$, given by assumption (F0), implies bifurcation at infinity at the eigenvalues of F. The proof follows the standard procedure for the linear case, see for example [38] or [8], so we shall be sketchy, discussing only the main differences. We define

$$G(\lambda, v) = \|v\|_{C^1}^2 K\left(\lambda, \frac{v}{\|v\|_{C^1}^2}\right),$$

for $v \neq 0$, and $G(\lambda, 0) = 0$. Finding $u \neq 0$ such that $u = K(\lambda, u)$ is equivalent to solving the fixed point problem $v = G(\lambda, v)$, $v \in C^1(\overline{\Omega})$, for $v = u/\|u\|_{C^1}^2$. The important observation is that bifurcation from zero in v is equivalent to bifurcation from infinity for u.

Let $u = G_0(\lambda, v)$ be the solution of the problem

$$F[u] + cu = (c - \lambda)v \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{4.2}$$

and set $G_1 = G - G_0$, so that $G(\lambda, v) = G_0(\lambda, v) + G_1(\lambda, v)$.

Lemma 4.1. *Under the hypothesis* (F0) *we have* $\lim_{\|v\|_{C^1} \to 0} \frac{G_1(\lambda, v)}{\|v\|_{C^1}} = 0$.

Proof. Let $g = G(\lambda, v)$ and $g_0 = G_0(\lambda, v)$. Then we have

$$\frac{1}{\|v\|_{C^1}}\big(F[g] - F[g_0] + c(g - g_0)\big) = \|v\|_{C^1} f\left(x, \frac{v}{\|v\|_{C^1}^2}\right).$$

The right-hand side here goes to zero as $\|v\|_{C^1} \to 0$, by (F0). Then by (DF)

$$\frac{1}{\|v\|_{C^1}}\big(F\big[|g - g_0| + c|g - g_0|\big]\big) \geqslant -\|v\|_{C^1} \left| f\left(x, \frac{v}{\|v\|_{C^1}^2}\right)\right|,$$

so the ABP inequality (Theorem 2.1) implies

$$\sup_{\Omega}\left\{\frac{1}{\|v\|_{C^1}}|g - g_0|\right\} \leqslant C\|v\|_{C^1}\left\| f\left(x, \frac{v}{\|v\|_{C^1}^2}\right)\right\|_{L^p},$$

and the result follows. □

The next proposition deals with the equation $v = G_0(\lambda, v)$, $v \in C^1(\overline{\Omega})$ (recall we want to solve $v = G_0(\lambda, v) + G_1(\lambda, v)$), which is equivalent to

$$F\big(D^2 v, Dv, v, x\big) = -\lambda v \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega. \tag{4.3}$$

Proposition 4.1. *There exists* $\delta > 0$ *such that for all* $r > 0$ *and all* $\lambda \in (-\infty, \lambda_1^- + \delta) \setminus \{\lambda_1^+, \lambda_1^-\}$, *the Leray–Schauder degree* $\deg(I - G_0(\lambda, \cdot), B_r, 0)$ *is well defined. Moreover*

$$\deg\big(I - G_0(\lambda, \cdot), B_r, 0\big) = \begin{cases} 1 & \text{if } \lambda < \lambda_1^+, \\ 0 & \text{if } \lambda_1^+ < \lambda < \lambda_1^-, \\ -1 & \text{if } \lambda_1^- < \lambda < \lambda_1^- + \delta. \end{cases}$$

Proof. We recall it was proved in [36] that problem (4.3) has only the zero solution in $(-\infty, \lambda_1^- + \delta) \setminus \{\lambda_1^+, \lambda_1^-\}$, for certain $\delta > 0$. The compactness of G_0 follows from Theorem 2.1, so the degree is well defined in the given ranges for λ.

Suppose $\lambda < \lambda_1^+$ and consider the operator $I - tG_0(\lambda, \cdot)$ for $t \in [0, 1]$. Since $t\lambda$ is not an eigenvalue of (4.3), we have for $t \in [0, 1]$

$$\deg\big(I - G_0(\lambda, \cdot), B_r, 0\big) = \deg\big(I - tG_0(\lambda, \cdot), B_r, 0\big) = \deg(I, B_r, 0) = 1.$$

The case $\lambda_1^+ < \lambda < \lambda_1^-$ was studied in [39]. Consider the problem

$$F[u] + cu = (c - \lambda)v - t\varphi_1^+ \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{4.4}$$

for $t \in [0, \infty)$, whose unique solution is denoted by $\widetilde{G}_0(\lambda, v, t)$. It follows from the results in [36,39] that for $t > 0$ the equation

$$F[u] + cu = (c - \lambda)u - t\varphi_1^+ \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{4.5}$$

does not have a solution. On the other hand, since λ is not an eigenvalue, there is $R > 0$ such that the solutions of (4.5), for $t \in [0, \bar{t}]$, are a priori bounded, consequently

$$\begin{aligned} \deg\big(I - G_0(\lambda, \cdot), B_r, 0\big) &= \deg\big(I - \widetilde{G}_0(\lambda, \cdot, 0), B_R, 0\big) \\ &= \deg\big(I - \widetilde{G}_0(\lambda, \cdot, \bar{t}), B_R, 0\big) = 0. \end{aligned}$$

If $\lambda_1^- < \lambda < \lambda_1^- + \delta$ we proceed as in [24], where the computation of the degree was done by making a homotopy with the Laplacian (see the proof of Lemma 4.2 in that paper). □

Now we are in position to apply the general theory of bifurcation to $v = G(\lambda, v)$, see for instance the surveys [38] and [8], and obtain bifurcation branches emanating from $(\lambda_1^+, 0)$ and $(\lambda_1^-, 0)$, exactly like in [13]. In short, from $(\lambda_1^+, 0)$ bifurcates a continuum of solutions of $v = G(\lambda, v)$, which is either unbounded in λ, or unbounded in u, or connects to $(\bar{\lambda}, 0)$, where $\bar{\lambda} \neq \lambda_1^+$ is an eigenvalue (recall λ_1^+ and λ_1^- are the only eigenvalues in $(-\infty, \lambda_1^- + \delta)$, for some $\delta > 0$). A similar situation occurs at $(\lambda_1^-, 0)$. Inverting the variables we obtain bifurcation at infinity for our problem (1.1):

Theorem 4.1. *Under the hypotheses of Theorem* 1.1 *there are two connected sets $\mathcal{C}_2$, $\mathcal{C}_3 \subset \mathcal{S}$ such that*:

1) *There is a sequence (λ_n, u_n) with $u_n \in \mathcal{C}_2(\lambda_n)$ $(u_n \in \mathcal{C}_3(\lambda_n))$, and $\|u_n\|_\infty \to \infty$, $\lambda_n \to \lambda_1^+$ (λ_1^-).*
2) *If (λ_n, u_n) is a sequence such that $u_n \in \mathcal{C}_2(\lambda_n)$ $(\mathcal{C}_3(\lambda_n))$, $\|u_n\|_\infty \to \infty$ and $\lambda_n \to \lambda_1^+$ (λ_1^-), then u_n is positive* (*negative*) *for large n.*
3) *The branch $\mathcal{C}_2$ satisfies one of the following alternatives, for some $\delta > 0$*: (i) *$\mathcal{C}_2(\lambda) \neq \emptyset$ for all $\lambda \in (\lambda_1^+, \lambda_1^- + \delta)$*; (ii) *there is $\lambda \in (-\infty, \lambda_1^- + \delta]$ such that $0 \in \mathcal{C}_2(\lambda)$*; (iii) *$\mathcal{C}_2(\lambda) \neq \emptyset$ for all $\lambda \in (-\infty, \lambda_1^+)$*; (iv) *there is a sequence (λ_n, u_n) such that $u_n \in \mathcal{C}_2(\lambda_n)$, $\|u_n\|_\infty \to \infty$, $\lambda_n \to \lambda_1^-$, and $\lambda_n \leqslant \lambda_1^-$.*
4) *The branch $\mathcal{C}_3$ satisfies one of the following alternatives, for some $\delta > 0$*: (i) *$\mathcal{C}_3(\lambda) \neq \emptyset$ for all $\lambda \in (\lambda_1^-, \lambda_1^- + \delta)$*; (ii) *there is $\lambda \in (-\infty, \lambda_1^- + \delta]$ such that $0 \in \mathcal{C}_3(\lambda)$*; (iii) *$\mathcal{C}_3(\lambda) \neq \emptyset$ for all $\lambda \in (-\infty, \lambda_1^-)$*; (iv) *there is a sequence (λ_n, u_n) such that $u_n \in \mathcal{C}_3(\lambda_n)$, $\|u_n\|_\infty \to \infty$, and $\lambda_n \to \lambda_1^+$.*

We remark that (F1) excludes alternatives 3)(ii) and 4)(ii) in this theorem.

5. A bounded branch of solutions

In this section we prepare for the proof of our main theorems by establishing the existence of a continuum of solutions of (1.1) which is not empty for all $\lambda \in (-\infty, \lambda_1^+ + \delta)$, for some $\delta > 0$. Our first proposition concerns the behavior of solutions of (1.1) when $\lambda \to -\infty$.

Proposition 5.1.

(1) *Assume f satisfies* (F0). *Then there exists a constant $C_0 > 0$, depending only on F, f, and Ω, such that any solution of* (1.1) *satisfies $\|u\|_\infty \leqslant C_0\lambda^{-1}$ as $\lambda \to -\infty$.*
(2) *If in addition f is Lipschitz at zero, that is, for some $\varepsilon > 0$ and some $\overline{C} > 0$ we have $|f(x, s_1) - f(x, s_2)| \leqslant \overline{C}|s_1 - s_2|$ for $s_1, s_2 \in (-\varepsilon, \varepsilon)$, then* (1.1) *has at most one solution when λ is sufficiently large and negative.*

Proof. (1) Let u_λ be a sequence of solutions of (1.1), with $\lambda \to -\infty$. We first claim that $\|u_\lambda\|_\infty$ is bounded. Suppose this is not so, and say $\|u_\lambda^+\|_\infty \to \infty$ (with the usual notation for the positive part of u). Then, setting $v_\lambda = u_\lambda/\|u_\lambda^+\|_\infty$, on the set $\Omega_\lambda^+ = \{u_\lambda > 0\}$ we have the inequality

$$F[v_\lambda] - \gamma v_\lambda \geqslant \frac{f(x, u_\lambda)}{\|u_\lambda^+\|_\infty} \to 0, \quad \text{as } \lambda \to -\infty.$$

The ABP estimate (see Theorem 2.1) then implies $\sup_{\Omega_\lambda^+} v_\lambda \to 0$, which is a contradiction with $\sup_{\Omega_\lambda^+} v_\lambda = 1$. In an analogous way we conclude that $\|u_\lambda^-\|_\infty$ is bounded.

Hence there exists a constant C such that $|f(x, u_\lambda(x))| \leqslant C$ in $\overline{\Omega}$, so

$$F[u_\lambda] - \gamma u_\lambda \geqslant -(\lambda + \gamma)u_\lambda - C \geqslant 0 \quad \text{on the set } \widetilde{\Omega}_\lambda,$$

where $\widetilde{\Omega}_\lambda = \{u_\lambda > C/(|\lambda| + \gamma)\}$. Applying the maximum principle or the ABP inequality in this set implies it is empty, which means $u_\lambda \leqslant C/(|\lambda| + \gamma)$ in Ω. By the same argument we show u_λ is bounded below, and (1) follows.

(2) From statement (1) we conclude that for λ small, all solutions of (1.1) are in $(-\varepsilon, \varepsilon)$. If u_1, u_2 are two solutions of (1.1) then for $|\lambda| > \gamma + \overline{C}$ we have $F[u_1 - u_2] - \gamma(u_1 - u_2) \geqslant 0$ on $\{u_1 > u_2\}$ which means this set is empty. □

The next result is stated in the framework of Theorem 1.1 and gives a bounded family of solutions (u_λ, λ), for $\lambda \in (-\infty, \lambda_1^+ + \delta)$. No assumption of Lipschitz continuity on f is needed.

Proposition 5.2. *Assume f satisfies* (F0) *and* (F_+^ℓ). *Then there is a connected subset $\mathcal{C}_1$ of $\mathcal{S}$ such that $\mathcal{C}_1(\lambda) \neq \emptyset$, for all $\lambda \in (-\infty, \lambda_1^+ + \delta)$.*

Proof. According to Proposition 3.1, given $\lambda_0 < \lambda_1^+$, there is $R > 0$ so that all solutions of (1.1) with $\lambda \in [\lambda_0, \lambda_1^+]$ belong to the ball B_R. In particular, the equation does not have a solution $(\lambda, u) \in [\lambda_0, \lambda_1^+] \times \partial B_R$. Moreover, there is $\delta > 0$ such that (1.1) does not have a solution in $[\lambda_1^+, \lambda_1^+ + \delta] \times \partial B_R$ – otherwise we obtain a contradiction by a simple passage to the limit. Consequently the degree $\deg(I - K(\lambda, \cdot), B_R, 0)$ is well defined for all $\lambda \in [\lambda_0, \lambda_1^+ + \delta]$ ($K(\lambda, \cdot)$ is defined in the previous section). We claim that its value is 1.

To compute this degree, we fix $\lambda < \lambda_1^+$ and analyze the equation

$$F[u] + \lambda u = sf(x, u) \quad \text{in } \Omega,$$

for $s \in [0, 1]$. Since λ is not an eigenvalue of F in Ω, the solutions of this equation are a priori bounded, uniformly in $s \in [0, 1]$, that is, there is $R_1 \geqslant R$, such that no solution of the equation exists outside of the open ball B_{R_1}. Given $v \in C^1(\overline{\Omega})$ we denote by $K_s(\lambda, v)$ the unique solution of the equation $F[u] + \lambda u = sf(x, v)$ in Ω. Then we have

$$\begin{aligned}\deg\big(I - K(\lambda, \cdot), B_R, 0\big) &= \deg\big(I - K_1(\lambda, \cdot), B_{R_1}, 0\big)\\ &= \deg\big(I - K_0(\lambda, \cdot), B_{R_1}, 0\big) = 1,\end{aligned}$$

where the last equality is given by Proposition 4.1. Hence, again by the homotopy invariance of the degree, we have $\deg(I - K(\lambda, \cdot), B_R, 0) = 1$, for all $\lambda \in (\lambda_0, \lambda_1^+ + \delta)$.

The last fact together with standard degree theory implies that for every $\lambda \in [\lambda_0, \lambda_1^+ + \delta]$ there is at least one (λ, u), solution of (1.1), and, moreover, there is a connected subset $\mathcal{C}_1$ of $\mathcal{S}$ such that $\mathcal{C}_1(\lambda) \neq \emptyset$ for all λ in the interval $[\lambda_0, \lambda_1^+ + \delta]$. Since λ_0 is arbitrary, we can use the same argument for each element of a sequence $\{\lambda_0^n\}$, with $\lambda_0^n \to -\infty$. Then, by a limit argument (like the one in the proof of Theorem 1.5.1 in [24]), we find a connected set $\mathcal{C}_1$ with the desired properties. □

Next we study a branch of solutions driven by a family of super- and sub-solutions, assuming that f is locally Lipschitz continuous. In this case the statement of the previous proposition can be made more precise. Specifically, we assume that f satisfies (F2), and there exist $\underline{u}, \overline{u} \in C^1(\overline{\Omega})$, such that $\overline{u}$ is a super-solution and $\underline{u}$ is a sub-solution of (1.1), for all $\lambda \leqslant \overline{\lambda}$, where $\overline{\lambda}$ is fixed. We further assume that $\underline{u}$ and $\overline{u}$ are not solutions of (1.1), and

$$\underline{u} < \overline{u} \quad \text{in } \Omega, \qquad \underline{u} = \overline{u} = 0 \quad \text{and} \quad \frac{\partial \underline{u}}{\partial \nu} < \frac{\partial \overline{u}}{\partial \nu} \quad \text{on } \partial\Omega. \tag{5.1}$$

We define the set

$$\mathcal{O} = \left\{ v \in C^1(\overline{\Omega}) \;\middle|\; \underline{u} < v < \overline{u} \text{ in } \Omega \text{ and } \frac{\partial \underline{u}}{\partial \nu} < \frac{\partial v}{\partial \nu} < \frac{\partial \overline{u}}{\partial \nu} \text{ on } \partial\Omega \right\}, \tag{5.2}$$

which is open in $C^1(\overline{\Omega})$. Since $\mathcal{O}$ is bounded in $C(\overline{\Omega})$, we see that for every $\lambda_0 < \overline{\lambda}$ the set of solutions of (1.1) in $[\lambda_0, \overline{\lambda}] \times \mathcal{O}$ is bounded in $C^1(\overline{\Omega})$, that is, all solutions of (1.1) in $[\lambda_0, \overline{\lambda}] \times \mathcal{O}$ are inside the ball B_R, for some $R > 0$.

Lemma 5.1. *With the definitions given above, we have*

$$\deg\big(I - K(\lambda, \cdot), \mathcal{O} \cap B_R, 0\big) = 1, \quad \textit{for all } \lambda \in [\lambda_0, \overline{\lambda}].$$

Proof. First we have to prove that the degree is well defined. We just need to show that there are no fixed points of $K(\lambda, \cdot)$ on the boundary of $\mathcal{O} \cap B_R$. For this purpose it is enough to prove that, given $v \in C^1(\overline{\Omega})$ such that $\underline{u} \leqslant v \leqslant \overline{u}$ in Ω, we have $\underline{u} < K(\lambda, v) < \overline{u}$ in Ω. In what follows we write $u = K(\lambda, v)$.

By (F2) we can assume that the negative number c, chosen in Section 4, is such that the function $s \to f(x,s) + (c-\lambda)s$ is decreasing, for $s \in (-\tau, \tau)$, where $\tau = \max\{\|\underline{u}\|_\infty, \|\bar{u}\|_\infty\}$. Then

$$\begin{aligned} F[u] &= F[u] - f(x,\bar{u}) - (c-\lambda)\bar{u} + f(x,\bar{u}) + (c-\lambda)\bar{u} \\ &\geqslant F[u] - f(x,v) - (c-\lambda)v + f(x,\bar{u}) + (c-\lambda)\bar{u} \\ &= -cu + f(x,\bar{u}) + (c-\lambda)\bar{u} \\ &= c(\bar{u}-u) + f(x,\bar{u}) - \lambda\bar{u} \geqslant F[\bar{u}] + c(\bar{u}-u). \end{aligned}$$

By (H1) this implies $\mathcal{M}^+(D^2(u-\bar{u})) + \gamma|Du - D\bar{u}| + (\gamma - c)(u-\bar{u}) > 0$ in Ω. It follows from Theorem 2.2 that $u < \bar{u}$ in Ω and $\frac{\partial u}{\partial \nu} < \frac{\partial \bar{u}}{\partial \nu}$ on $\partial\Omega$. The other inequality is obtained similarly.

By using its homotopy invariance, the degree we want to compute is equal to the degree at λ_0. But the latter was shown to be one in the proof of Proposition 5.2, which completes the proof of the lemma. □

Now we can state a proposition on the existence of a branch of solutions for $\lambda \in (-\infty, \bar{\lambda}]$, whose proof is a direct consequence of Lemma 5.1 and general degree arguments.

Proposition 5.3. *Assume f satisfies* (F0) *and* (F2). *Suppose there are functions $\underline{u}, \bar{u} \in C^1(\overline{\Omega})$ such that $\bar{u}$ is a super-solution and $\underline{u}$ is a sub-solution of* (1.1) *for all $\lambda \leqslant \bar{\lambda}$, these functions are not solutions of* (1.1) *and satisfy* (5.1). *Then there is a connected subset $\mathcal{C}_1$ of $\mathcal{S}$ such that $\mathcal{C}_1(\lambda) \neq \emptyset$ for all $\lambda \in (-\infty, \bar{\lambda})$ and each $u \in \mathcal{C}_1(\lambda)$ is such that $\underline{u} \leqslant u \leqslant \bar{u}$.*

Remark 5.1. In the next section we use this proposition with appropriately chosen sub-solutions and super-solutions.

Remark 5.2. The branch $\mathcal{C}_1$ is isolated of other branches of solutions by the open set $\mathcal{O}$, since we know there are no solutions on $\partial\mathcal{O}$.

6. Proofs of the main theorems

In this section we put together the bifurcation branches emanating from infinity obtained in Theorem 4.1 with the bounded branches constructed in Section 5, and study their properties.

Proof of Theorem 1.1. This theorem is a consequence of Proposition 5.2, for the definition of $\mathcal{C}_1$, and of Theorem 4.1, 1)–2), for the definition of $\mathcal{C}_2$ and $\mathcal{C}_3$. Both $\mathcal{C}_2$ and $\mathcal{C}_3$ curve to the right of λ_1^+ and λ_1^-, respectively – as a consequence of the a priori bounds obtained in Propositions 3.1 and 3.2. □

Proof of Theorem 1.2. We first construct the branch $\mathcal{C}_1$, through Proposition 5.3. In view of (F1) we may take as a super-solution the function $\bar{u} \equiv 0$. In order to define the corresponding sub-solution we use Proposition 3.2, where a sub-solution is constructed for all $\lambda \in (-\infty, \lambda_1^-]$. We can rewrite inequality (3.2) in the following way

$$F[z_n] + (\lambda_1^- + \delta)z_n = \tilde{t}_n \varphi_1^+ + d, \quad \text{with } \tilde{t}_n(x) = \frac{\delta z_n(x)}{\varphi_1^+(x)} + t_n.$$

Since $z_n/\|z_n\|_\infty \to \varphi_1^- < 0$ in $C^1(\overline{\Omega})$ we find that for some $c > 0$

$$\frac{|z_n(x)|}{\|z_n\|_\infty \varphi_1^+(x)} \leqslant c, \quad \forall x \in \Omega.$$

Consequently, once n is chosen so that (3.3) holds, we can fix $\delta > 0$ such that $\tilde{t}_n(x) \geqslant -\delta c + t_-^*(d) > 0$, for all $x \in \Omega$, which means that z_n is a sub-solution also for $F[u] + (\lambda_1^- + \delta)u = f(x,u)$, as in the proof of Proposition 3.2.

Now we define $\underline{u} = z_n$, chosen as above, and take $\overline{\lambda} = \lambda_1^- + \delta$ in Proposition 5.3. Clearly $\underline{u}$ and $\overline{u}$ satisfy also (5.1), so the existence of the branch $\mathcal{C}_1$ (with the properties stated in Theorem 1.2) follows from Proposition 5.3.

Further, the branches $\mathcal{C}_2$ and $\mathcal{C}_3$ are given by Theorem 4.1 and both of them curve to the right of λ_1^+ and λ_1^-, respectively. Neither $\mathcal{C}_2$ or $\mathcal{C}_3$ connects to $\mathcal{C}_1$, since $\mathcal{C}_1$ is isolated from the exterior of the open set $\mathcal{O}$, see Remark 5.2. Observe that the elements of $\mathcal{C}_2$ (resp. $\mathcal{C}_3$) are outside $\mathcal{O}$ for λ close to λ_1^+ (resp. λ_1^-).

Therefore the uniqueness statement of Proposition 5.1 excludes the alternatives in Theorem 4.1, 3)(iii) and 4)(iii), since we already know that $\mathcal{C}_1$ contains solutions for arbitrary small λ. We already noted cases 3)(ii) and 4)(ii) are excluded by (F1). Finally, case 3)(iv) in Theorem 4.1 is excluded by the a priori bound in Proposition 3.2, so only case 3)(i) remains. □

Proof of Theorem 1.3. We fix a small number $\varepsilon > 0$ and for each $K > 0$ consider a solution u_K of $F[u_K] + (\lambda_1^- - \varepsilon)u_K = K$ in Ω, $u_K = 0$ on $\partial\Omega$, $u_K < 0$ in Ω. We know such a function u_K exists, by Theorem 1.9 in [36]. By (F0) we can fix K_0 such that $K_0 > f(x, K_0)$ in Ω, hence $\underline{u} = u_{K_0}$ is a sub-solution of (1.1), for all $\lambda \in (-\infty, \lambda_1^- - \varepsilon)$. The super-solution to consider is $\overline{u} \equiv 0$, as given by hypothesis (F1). Then Proposition 5.3 yields the existence of a branch $\mathcal{C}_1^\varepsilon$ such that $\mathcal{C}_1^\varepsilon(\lambda) \neq \emptyset$ for all $\lambda \in (-\infty, \lambda_1^- - \varepsilon)$.

Next we pass to the limit as $\varepsilon \to 0$, like in the proofs of Proposition 5.2 and Theorem 1.5.1 in [24], and obtain either a connected component of $\mathcal{S}$ which bifurcates from infinity to the left of λ_1^-, or a *bounded* branch of solutions which "survives" up to λ_1^-, and hence "continues" in some small right neighborhood of λ_1^-, again like in the proof of Proposition 5.2. The first of these alternatives is (3)(i). In case the second alternative is realized there is a connected set of solutions $\mathcal{C}_3$ bifurcating from minus infinity towards the left of λ_1^-, as predicted in Theorem 4.1. We claim this branch contains only negative solutions. To prove this, we set

$$A = \left\{(\lambda, u) \in \mathcal{C}_3 \;\middle|\; \lambda \in \left(-\infty, \lambda_1^-\right),\ \max_{\Omega} u > 0\right\}.$$

The set A is clearly open in $\mathcal{C}_3$, and $A \neq \mathcal{C}_3$. Hence if A is not empty, then A is not closed in $\mathcal{C}_3$, by the connectedness of $\mathcal{C}_3$. This means there is a sequence $(\lambda_n, u_n) \in A$ such that $\lambda_n \to \lambda$, $u_n \to u$, and the limit function u satisfies $u \leqslant 0$ in Ω, u vanishes somewhere in Ω, and solves the equation $F[u] + (\lambda - c)u = f(x,u) - cu \geqslant 0$ in Ω, for some large c. Hence by Hopf's lemma $u \equiv 0$, a contradiction with (F1).

Therefore $\mathcal{C}_3$ cannot connect with the branch bifurcating from plus infinity at λ_1^+. It is not connected to $\mathcal{C}_1$ either – by the isolation property of $\mathcal{C}_1(\lambda)$, see Remark 5.2. Further, $\mathcal{C}_3$ cannot contain solutions for arbitrarily small λ, since $\mathcal{C}_1$ does, and we know solutions are unique for sufficiently small λ. Hence $\mathcal{C}_3$ must eventually curve to the right, so extra solutions appear, proving (3)(ii) and (4).

Finally, a branch $\mathcal{C}_2$ bifurcating from plus infinity towards the left of λ_1^+ exists thanks to Theorem 4.1. This branch is kept away from $\mathcal{C}_1$ and $\mathcal{C}_3$, as we already saw, and, again by the uniqueness of solutions for sufficiently small λ, $\mathcal{C}_2$ has to curve to the right. This completes the proof. □

The occurrence of alternative (3)(ii) in Theorem 1.3 can be avoided if f satisfies some further hypotheses.

Proposition 6.1. *Under the hypotheses of Theorem* 1.3, *if in addition we make one of the following assumptions*

(1) *$f(x,s)$ is concave in s for $s<0$,*
(2) *for each $a_0>0$ there exists $k_0>0$ such that*

$$\frac{f(x,-k\varphi_1^+)}{k} < f\big(x,-a\varphi_1^+\big), \quad \text{for all } a\in(0,a_0),\ k>k_0, \tag{6.1}$$

then alternative (3)(ii) *in Theorem* 1.3 *does not occur.*

Remark. Note the model example of a sub-linear nonlinearity which satisfies the hypotheses of Theorem 1.3

$$f(x,s) = -s|s|^{\alpha-1} + h(x), \quad \alpha\in(0,1),\ h \gneqq 0,$$

satisfies both hypotheses in the above proposition.

Proof of Proposition 6.1. We are going to prove the following stronger claim: under the hypotheses of the proposition, there cannot exist sequences λ_n, u_n, v_n, such that $\lambda_n<\lambda_{n+1}$, $\lambda_n\to\lambda_1^-$, $u_n, v_n<0$ in Ω, $\|u_n\|$ is bounded, $\|v_n\|\to\infty$ and u_n and v_n are solutions of (1.1) with $\lambda=\lambda_n$.

Assume this is false and (1) holds. Then (passing to subsequences if necessary) u_n is convergent in $C^1(\overline{\Omega})$, and $v_n/\|v_n\|\to\varphi_1^-$ in $C^1(\overline{\Omega})$, so there is n_0 such that for all $n\geqslant n_0$ we have $v_n<u_{n+1}$ in Ω. The negative function u_{n+1} is clearly a strict sub-solution of $F[u]+\lambda_n u = f(x,u)$, and, since the zero function is a strict super-solution of this equation, it has a negative solution which is above u_{n+1}. We define

$$\overline{v}_n = \inf\big\{v \bigm| u_{n+1}<v<0,\ v \text{ is a super-solution of } F[u]+\lambda_n u = f(x,u)\big\}.$$

Then $\overline{v}_n$ is a solution of $F[u]+\lambda_n u = f(x,u)$ such that between u_{n+1} and $\overline{v}_n$ no other solution of this problem exists. Indeed, $\overline{v}_n$ is a super-solution (as an infimum of super-solutions), so between u_{n+1} and $\overline{v}_n$ there is a minimal solution, with which $\overline{v}_n$ has to coincide, by its definition. Note Hopf's lemma trivially implies that for some $\varepsilon>0$ we have $v_n<u_{n+1}-\varepsilon\varphi_1^+<\overline{v}_n-2\varepsilon\varphi_1^+$.

Next, by the convexity of F and the concavity of f we easily check that the function $u_\alpha = \alpha v_n+(1-\alpha)\overline{v}_n$ is a super-solution of $F[u]+\lambda_n u = f(x,u)$, for each $\alpha\in[0,1]$. This gives a contradiction with the definition of $\overline{v}_n$, for α small enough but positive.

Assume now our claim is false and (2) holds. We again have $-C_0\varphi_1^+ \leqslant u_n \leqslant -c_0\varphi_1^+ < 0$ and $v_n/\|v_n\| \to \varphi_1^-$, so the numbers

$$\varepsilon_n := \sup\{\varepsilon > 0 \mid u_n \leqslant \varepsilon v_n \text{ in } \Omega\}$$

clearly satisfy $\varepsilon_n > 0$ and $\varepsilon_n \to 0$. Hypothesis (6.1) implies that for sufficiently large n we have $\varepsilon_n f(x, v_n) < f(x, u_n)$, that is, $F[\varepsilon_n v_n] + \lambda_n \varepsilon_n v_n < F[u_n] + \lambda_n u_n$, and Hopf's lemma yields a contradiction with the definition of ε_n. $\square$

7. Discussion and examples

The main point of this section is to provide some examples showing that when (F1) or (F2) fails, then the bifurcation diagram for (1.1) may look very differently from what is described in Theorems 1.2–1.3. However, we begin with some general comments on our hypotheses and their use.

Hypothesis (F0) is classical sub-linearity for f, which guarantees bifurcation from infinity and also ensures the solutions of (1.1) tend to zero as $\lambda \to -\infty$. Condition (F1) guarantees the existence of a strict super-solution of (1.1) for all λ, while (F2) is used in some comparison statements and to prove uniqueness of solutions of (1.1) for sufficiently negative λ.

Further, conditions (F_+^ℓ)–(F_-^ℓ) and (F_+^r)–(F_-^r) are the Landesman–Lazer type hypotheses which give a priori bounds when λ stays on one side of the eigenvalues, and thus provide a solution at resonance and determine on which side of each eigenvalue the bifurcation from infinity takes place. The strict inequalities in (F_+^ℓ)–(F_-^r) are important and cannot be relaxed in general – for instance the problem $F[u] + \lambda_1^+ u = -\sqrt{\max\{1-u, 0\}}$ has no solutions (and hence Theorem 1.1 fails), as Theorems 1.6 and 1.4 in [36] show, even though the nonlinearity satisfies the hypotheses of Theorem 1.1, except for the strict inequality in (F_+^ℓ). On the other hand, for $F = \Delta$ it is known that in the case of equalities in (F_+^ℓ)–(F_-^r) one can give supplementary assumptions on f and the rate of convergence of f to its limits $f_\pm$, $f^\pm$, so that results like Theorem 1.1 still hold, see for instance Remark 21 in [5]. Extensions of these ideas to HJB operators are out of the scope of this work and could be the basis of future research.

Now we discuss examples where (F1) or (F2) fails.

Example 1. Our first example shows that for all sufficiently small $\delta > 0$ we can construct a nonlinearity f which does not satisfy (F1) and for which the set $\mathcal{S}(\lambda_1^+ + \delta)$ is empty. This means that, in the framework of Theorems 1.1–1.2, the branch bifurcating from infinity to the right of λ_1^+ "turns back" before it reaches $\lambda_1^+ + \delta$. A similar situation can be described for the branch bifurcating from minus infinity to the left of λ_1^- that "turns right", before reaching $\lambda_1^- - \delta$. In particular there cannot be a continuum of solutions along the gap between λ_1^+ and λ_1^-.

Consider the Dirichlet problem

$$F[u] + \lambda u = t\varphi_1^+ + h \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{7.1}$$

at resonance, that is, for $\lambda = \lambda_1^+$. When $t = t_+^*(h)$ Eq. (7.1) may or may not have a solution, depending on F and h. An example of such a situation was given in [7] and we recall it here. Take $F[u] = \max\{\Delta u, 2\Delta u\}$, and $h \in C(\overline{\Omega})$ such that $\int_\Omega h\varphi_1 = 0$ and h changes sign on $\partial\Omega$. Here $\lambda_1^+ = \lambda_1$, $\lambda_1^- = 2\lambda_1$, $\varphi_1^+ = -\varphi_1^- = \varphi_1$, where λ_1 and φ_1 are the first eigenvalue and eigenfunction of the Laplacian. Then (see Example 4.3 in [7]) under the above hypotheses on h we have $t_+^* = 0$ and problem (7.1) has no solutions if $\lambda = \lambda_1^+$ and $t = t_+^*$. By exactly the same reasoning it is possible to show that problem (7.1) has no solutions if $\lambda = \lambda_1^-$ and $t = t_-^*$.

Lemma 7.1. *If Eq.* (7.1) *with* $\lambda = \lambda_1^+$ *and* $t = t_+^*$ *does not have a solution then there exists* δ_0 *such that* $t_\lambda^* > t_+^*$ *provided* $\lambda \in (\lambda_1^+, \lambda_1^+ + \delta_0)$. *Similarly, if* (7.1) *with* $\lambda = \lambda_1^-$ *and* $t = t_-^*$ *does not have a solution then there exists* δ_0 *such that* $t_\lambda^* > t_-^*$ *whenever* $\lambda \in (\lambda_1^- - \delta_0, \lambda_1^-)$.

Before proving the lemma, we use it to construct a nonlinearity with the desired properties. For λ sufficiently close to λ_1^+ we have $t_+^* < t_\lambda^*$ so that we can choose $\bar{t} \in (t_+^*, t_\lambda^*)$. We then define

$$f(x,u) = \begin{cases} \bar{t}\varphi_1^+ + h & \text{if } u \geqslant -M, \\ (\frac{\bar{t} - t_+^* + \varepsilon}{M}(u+M) + \bar{t})\varphi_1^+ + h & \text{if } -2M \leqslant u \leqslant -M, \\ (t_+^* - \varepsilon)\varphi_1^+ + h & \text{if } u \leqslant -2M, \end{cases} \tag{7.2}$$

where ε and M are some positive constants. We readily see that f satisfies (F0) and (F_+^ℓ), the hypotheses of Theorem 1.1, but $\mathcal{S}(\lambda)$ is empty. Indeed, if $\bar{u} \in \mathcal{S}(\lambda)$, then $\bar{u}$ is a super-solution for (7.1). On the other hand, by Theorem 1.9 in [36], the equation $F[u] + \lambda u = K\|\bar{t}\varphi_1^+ + h\|_{L^\infty(\Omega)}$ with $\lambda < \lambda_1^-$ has a solution u_K, for each $K > 0$. Moreover, for large K, $\underline{u}_K$ is a sub-solution of (7.1) and $\underline{u}_K < \bar{u}$. Then by Perron's method (7.1) has a solution, a contradiction.

Similarly, for $\lambda < \lambda_1^-$ sufficiently close to λ_1^- we choose $\bar{t} \in (t_-^*, t_\lambda^*)$ and define $f(x,u)$ being equal to $\bar{t}\varphi_1^+ + h$ if $u \leqslant M$ and to $(t_-^* - \varepsilon)\varphi_1^+ + h$ if $u \geqslant 2M$. By the same reasoning we find that $\mathcal{S}(\lambda)$ is empty.

We summarize: with these choices of λ and f there is a region of non-existence in the gap between λ_1^+ and λ_1^-. In other words, the connected sets of solutions of (1.1) $\mathcal{C}_2$ (resp. $\mathcal{C}_3$), predicted in Theorem 1.1, do not extend to the right (resp. to the left) of λ. The first graph at the end of this section is an illustration of this situation.

We observe that if we take M sufficiently large then all solutions of (7.1) and (1.1), with f as given in (7.2), coincide. In fact, we can take $-M$ to be a lower bound for all solutions of the inequality $F[u] + \lambda u \leqslant c + h$, where c is such that $f(x,u) \leqslant c + h$ in Ω. Such an M exists by the one-sided ABP inequality given in Theorem 1.7 in [36]. Now we see that (1.1) with this f has a unique solution for $\lambda < \lambda_1^+$, as an application of Theorem 1.8 in [36], and then the branch of solutions bifurcating from plus infinity must turn left and go towards infinity near the λ-axis, as drawn on the picture.

Proof of Lemma 7.1. Given $\lambda \in (\lambda_1^+, \lambda_1^-)$, let v_λ^* be a solution of

$$F[u] + \lambda u = t_\lambda^* \varphi_1^+ + h \quad \text{in } \Omega, \tag{7.3}$$

whose existence is guaranteed by the results in [39]. We notice that $\|v_\lambda^*\|$ is unbounded as $\lambda \searrow \lambda_1^+$, as otherwise v_λ^* a subsequence of v_λ^* would converge to a solution of (7.1) with $\lambda = \lambda_1^+$ and $t = t_+^*$, which is excluded by assumption. That $t_\lambda^* \to t_+^*$ as $\lambda \to \lambda_1^+$ was proved in Proposition 2.3. Then, by the simplicity of λ_1^+, we find that $v_\lambda^*/\|v_\lambda^*\|_\infty \to \varphi_1^+$ as $\lambda \to \lambda_1^+$, in particular, v_λ^* becomes positive in Ω, for λ larger than and close enough to λ_1^+. Suppose for contradiction that $t_+^* \geqslant t_\lambda^*$, then $v_\lambda^* \geqslant 0$ satisfies

$$F(v_\lambda^*) + \lambda_1^+ v_\lambda^* \leqslant F(v_\lambda^*) + \lambda v_\lambda^* = t_\lambda^* \varphi_1^+ + h \leqslant t_+^* \varphi_1^+ + h,$$

so v_λ^* is a super-solution for (7.1) with $\lambda = \lambda_1^+$ and $t = t_+^*$. As we already showed above, (7.1) has a sub-solution below v_λ^*, providing a contradiction.

In the same way, we see that $v_\lambda^*/\|v_\lambda^*\|_\infty \to \varphi_1^-$ as $\lambda \nearrow \lambda_1^-$ and then v_λ^* becomes negative in Ω, for $\lambda < \lambda_1^-$ and close enough to λ_1^-. Then $t_-^* \geqslant t_\lambda^*$ would imply that $v_\lambda^* \leqslant 0$ satisfies

$$F(v_\lambda^*) + \lambda_1^- v_\lambda^* \leqslant F(v_\lambda^*) + \lambda v_\lambda^* = t_\lambda^* \varphi_1^+ + h \leqslant t_-^* \varphi_1^+ + h,$$

so v_λ^* is a super-solution for (7.1) with $\lambda = \lambda_1^-$ and $t = t_-^*$. To construct a sub-solution we consider v_ε a solutions of $F[v_\varepsilon] + \lambda_1^- v_\varepsilon = (t_-^* + \varepsilon)\varphi_1^+ + h$, with $\varepsilon > 0$. By our assumption, $v_\varepsilon/\|v_\varepsilon\|_\infty \to \varphi_1^-$ as $\varepsilon \to 0$ (see also Theorem 1.4 in [24]). Hence there exists $\varepsilon = \varepsilon(\lambda)$ such that $v_\varepsilon < v_\lambda^*$ and $F[v_\varepsilon] + \lambda_1^- v_\varepsilon \geqslant t_-^* \varphi_1^+ + h$ and then Perron's method gives a contradiction again. □

Remark 7.1. The claim gives an idea of the behavior of t_λ^*, with respect to λ, near the extremes of the interval $[\lambda_1^+, \lambda_1^-]$. However we do not have any idea about the global behavior of t_λ^*, actually we do not even know how t_+^* and t_-^* compare.

For completeness we give a direct proof of the fact that in the above examples condition (F1) is not satisfied by nonlinearities like in (7.2). In this direction we have the following lemma, which is of independent interest.

Lemma 7.2. *For any $h \in L^p(\Omega)$, $p > N$, which is not a multiple of φ_1^+,*

(a) *if $h \geqslant 0$ and $h \not\equiv 0$ then $t_+^*(h) < 0$ and $t_-^*(h) < 0$;*
(b) *if $h \leqslant 0$ and $h \not\equiv 0$ then $t_+^*(h) > 0$ and $t_-^*(h) > 0$;*
(c) *the functions $t_+^*(h)\varphi_1^+ + h$ and $t_-^*(h)\varphi_1^+ + h$ change sign in Ω.*

Proof. (a) If $t_+^*(h) \geqslant 0$ then, as $h \geqslant 0$, by Theorem 1.9 in [36] the problem $F[u] + \lambda_1^+ u = t_+^*(h)\varphi_1^+ + h$ has a solution. Then by Theorem 1.2 in [24] $u + k\varphi_1^+$ is a solution of the same problem, for all $k > 0$. Since $u + k\varphi_1^+$ is positive for sufficiently large k, by Theorem 1.2 in [36] we get that u is a multiple of φ_1^+, a contradiction, since $h \neq 0$.

If $t_-^*(h) \geqslant 0$, by Theorem 1.5 in [24] either there exist sequences $\varepsilon_n \to 0$ and u_n of solutions of the problem $F[u_n] + \lambda_1^- u_n = (t_-^*(h) + \varepsilon_n)\varphi_1^+ + h$ such that u_n is unbounded and u_n is negative for large n, or $F[u + k\varphi_1^-] + \lambda_1^-(u + k\varphi_1^-) = t_-^*(h)\varphi_1^+ + h$ for some u and all $k > 0$. In both cases we get a negative solution of $F[u] + \lambda_1^- u \gneqq 0$, which by Theorem 1.4 in [36] is then a multiple of φ_1^-, a contradiction.

(b) If $t_+^*(h) \leqslant 0$ then $F[u] + \lambda_1^+ u = t_+^*(h)\varphi_1^+ + h$ has no solution by Theorems 1.6 and 1.4 in [36], since $t_+^*(h)\varphi_1^+ + h \lneqq 0$. If $t_-^*(h) \leqslant 0$ we again have $t_-^*(h)\varphi_1^+ + h \lneqq 0$, then $F[u] + \lambda_1^- u = t_-^*(h)\varphi_1^+ + h$ has no solutions by the anti-maximum principle, see for instance Proposition 4.1 in [24]. Hence by Theorems 1.2 and 1.4 in [24] there exist sequences $\varepsilon_n \to 0$, u_n^+ and u_n^- of solutions of $F[u_n^\pm] + \lambda_1^\pm u_n^\pm = (t_\pm^*(h) + \varepsilon_n)\varphi_1^+ + h$ such that $u_n^\pm/\|u_n^\pm\|_\infty \to \varphi^\pm$. Fix w to be the solution of the Dirichlet problem $F(w) - \gamma w = -h$ in Ω. This problem is uniquely solvable, with $w < 0$ in Ω, since by (H1) the operator $F - \gamma$ is decreasing in u (see for instance [17] and [36]). Then by the maximum principle and Hopf's lemma $\varepsilon_n \varphi_1^+ + (\lambda_1^+ + \gamma)w < 0$ in Ω, if n is sufficiently large. Hence $u_n^+ + w$ is positive and $F[u_n^+ + w] + \lambda_1^+(u_n^+ + w) < 0$ in Ω, which is a contradiction with Theorem 1.4 in [36]. Similarly, $u_n^- + w$ is negative and satisfies $F[u_n^- + w] + \lambda_1^-(u_n^- + w) < 0$ in Ω, which is a contradiction with Theorem 1.2 in [36].

(c) This is an immediate consequence of (a) and (b). Indeed, if (c) is false we just replace h by $t_\pm^*(h)\varphi_1^+ + h$ in (a) or (b). □

Remark 7.2. The statements on t_+^* in the preceding lemma also follow from Theorem 1.1 and formula (1.12) in [7].

The following example illustrate the role of hypothesis (F2), which allows the use of the method of sub- and super-solutions, and prevents the branches which bifurcate from infinity to survive for arbitrarily negative λ.

Example 2. Consider the function $\omega(u) = \frac{u}{\sqrt{|u|}}$, $\omega(0) = 0$ and the problem

$$\Delta u + \lambda u = -\omega(u) \quad \text{in } \Omega. \tag{7.4}$$

This problem is variational and its associated functional is

$$J(u) = \int_\Omega \left(|\nabla u|^2 - \lambda u^2 - |u|^{3/2}\right) dx,$$

which is even, bounded below, takes negative values and attains its minimum on $H_0^1(\Omega)$, for each $\lambda < \lambda_1$. The same is valid for $J_+(u) = J(u^+)$ and $J_-(u) = J(u^-)$, whose minima are then a positive and a negative solutions of (7.4).

In the context of nonlinear HJB operators we may consider

$$\max\{\Delta u, 2\Delta u\} + \lambda u = -\omega(u), \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{7.5}$$

For this problem we have bifurcation from plus infinity to the left of λ_1 and from minus infinity to the left of $2\lambda_1$. These branches cannot reach the trivial solution set $\mathbb{R} \times \{0\}$, since bifurcation of positive or negative solutions from the trivial solution does not occur for (7.4). Exactly as in the proof of Theorem 1.3 (see the definition of the set A in the previous section) we can show that they contain only positive or negative solutions. Actually these branches are curves which can never turn, since positive and negative solutions of (7.5) are unique – this can be proved in the same way as Proposition 7.1 below.

Example 3. Finally, let us look at an example of a sub-linear nonlinearity f which satisfies (F2) but $f(x, 0) \equiv 0$. For any HJB operator F satisfying our hypotheses consider

$$F[u] + \lambda u = \tilde{f}(u) := \begin{cases} -u & \text{if } |u| \leqslant 1, \\ -\omega(u) & \text{if } |u| \geqslant 1. \end{cases} \tag{7.6}$$

In this situation we have positive (resp. negative) bifurcation from zero at $\lambda = \lambda_1^+ - 1$ (resp. $\lambda = \lambda_1^- - 1$), more precisely, $(\lambda_1^+ - 1, k\varphi_1^+)$ and $(\lambda_1^- - 1, k\varphi_1^-)$ are solutions for $k \in [0, 1]$ (for more general results on bifurcation from zero see [13]). Further, note that there are only positive (resp. negative) solutions on these branches, as well on the branches which bifurcate from plus (resp. minus) infinity, given by Theorem 1.3. This is a simple consequence of the strong maximum principle and the fact that the right-hand side of (7.6) is positive (resp. negative) if u is negative (resp. positive), so if $u \leqslant (\geqslant)\ 0$ and u vanishes at one point then u is identically zero. The bifurcation branches connect, as shown by the following uniqueness result.

Proposition 7.1. *If u and v are two solutions of (7.6) having the same sign and $\|u\| > 1$ or $\|v\| > 1$ then $u \equiv v$. If $\|u\| \leqslant 1$ and $\|v\| \leqslant 1$ then (by the simplicity of the eigenvalues) $\lambda = \lambda_1^{\pm} - 1$ and $u = v + k\varphi_1^{\pm}$ for some $k \in [0, 1]$.*

Proof. Say $u > 0$, $v > 0$, $\|v\| > 1$. Set

$$\tau := \sup\{\mu > 0 \mid u \geqslant \mu v \text{ in } \Omega\}.$$

By Hopf's lemma $\tau > 0$ and we have $u \geqslant \tau v$.

First, suppose $\tau < 1$. By the definition of $\tilde{f}$ in (7.6) and $\|v\| > 1$ we easily see that

$$\tilde{f}(u) \leqslant \tilde{f}(\tau v) \lneqq \tau \tilde{f}(v) \quad \text{in } \Omega.$$

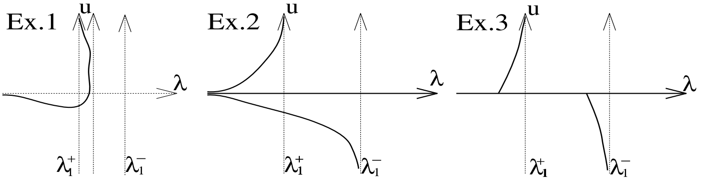
Hence (7.6) and the hypotheses on F imply

$$\mathcal{M}^-_{\lambda,\Lambda}\big(D^2(u - \tau v)\big) - \gamma\big|D(u - \tau v)\big| - (\gamma + \lambda)(u - \tau v) \lneqq 0$$

and $u - \tau v \geqslant 0$ in Ω, so Hopf's lemma implies $u \geqslant (\tau + \varepsilon)v$ for some $\varepsilon > 0$, a contradiction with the definition of τ.

Second, if $\tau \geqslant 1$ we repeat the above argument with u and v interchanged. This leaves $u \geqslant v$ and $v \geqslant u$ as the only case not excluded. □

The following picture summarizes the above examples.



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