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A note on the strong maximum principle and the compact support principle

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Abstract

In this note we are concerned with the strong maximum principle (SMP) and the compact support principle (CSP) for non-negative solutions to quasilinear elliptic inequalities of the form

$$\operatorname{div}\big(A\big(|\nabla u|\big)\nabla u\big) + G\big(|\nabla u|\big) - f(u) \leqslant 0 \quad \text{in } \Omega,$$

and

$$\operatorname{div}\big(A\big(|\nabla u|\big)\nabla u\big) + G\big(|\nabla u|\big) - f(u) \geqslant 0 \quad \text{in } \mathbb{R}^N \setminus B_r(0),$$

respectively. We give new conditions on the data (A, G, f) to obtain (SMP) and (CSP). When these conditions are particularized to the m-Laplacian and pure power nonlinearities we completely classify the data according to the validity of the (CSP) or the (SMP). In doing so we clarify the general situation and we consider a case not covered in the literature.

Keywords: Maximum principle; Compact support; Uniqueness

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1. Introduction

In this note we are concerned with the strong maximum principle (SMP) and the compact support principle (CSP) for non-negative solutions to quasilinear elliptic equations in the presence of lower order terms, including the gradient. These two principles, which are somehow dual to each other, have been investigated in the last decades by many authors. A very complete account of the advances in this area since the seminal work of Eberhard Hopf can be found in the recent paper by Pucci and Serrin [8], where a thorough discussion and a complete bibliography is presented.

We start by recalling the precise meaning of these principles. We say that (SMP) holds for the inequality

$$\operatorname{div}\big(A\big(|\nabla u|\big)\nabla u\big) + G\big(|\nabla u|\big) - f(u) \leqslant 0 \quad \text{in } \Omega, \tag{1.1}$$

if any non-negative solution u of (1.1) in the domain Ω which vanishes at some point in Ω, vanishes everywhere in Ω. We say that (CSP) holds for the inequality

$$\operatorname{div}\big(A\big(|\nabla u|\big)\nabla u\big) + G\big(|\nabla u|\big) - f(u) \geqslant 0 \quad \text{in } \mathbb{R}^N \setminus B_r(0), \tag{1.2}$$

if any non-negative solution u of (1.2) in $\mathbb{R}^N \setminus B_r(0)$, for $r > 0$, satisfying $u(x) \to 0$ as $|x| \to \infty$, vanishes in $\mathbb{R}^N \setminus B_R(0)$, for some $R \geqslant r$. In this note a solution for these inequalities is a function in C^1 satisfying the inequalities in the weak sense. Our general assumptions on the function A are the following:

(A1) $A \in C(0, \infty)$;
(A2) the function $t \to tA(t)$ is strictly increasing and $tA(t) \to 0$ as $t \to 0^+$.

The case of the m-Laplacian is of particular interest for us, it is obtained with $A(t) = |t|^{m-2}$, with $m > 1$. In what follows we write $\Delta_m u = \operatorname{div}(|\nabla u|^{m-2}\nabla u)$.

Let us start by reviewing the results when $G \equiv 0$, case where the theory in completely understood. We consider the basic assumption on f:

(F1) $f : [0, +\infty) \to \mathbb{R}$ is an increasing, continuous function with $f(0) = 0$.

Regarding the (SMP), Vázquez proved in [12] that the (SMP) holds for (1.1), with $G \equiv 0$, if there exists $\delta > 0$ such that

$$\int_0^\delta \frac{1}{F(s)^{1/m}}\, ds = \infty, \tag{1.3}$$

where $F(s) = \int_0^s f(\tau)\, d\tau$. This condition is also necessary as shown by Benilan, Brezis and Crandall in [2] for $m = 2$ and by Diaz [5] for all $m > 1$. It was then showed by Pucci, Serrin and Zou in [10] that when condition (1.3) fails, that is, for some $\delta > 0$

$$\int_0^\delta \frac{1}{F(s)^{1/m}}\, ds < \infty, \tag{1.4}$$

then the (CSP) holds and that (1.4) is also a necessary condition for the (CSP) to hold. Thus, the possible data (m, f) for inequalities (1.1) and (1.2) with $G \equiv 0$ are completely classified in terms of the validity of the (SMP) or the (CSP). These results have been extended by Pucci, Serrin and Zou [10] and by Pucci and Serrin in [9] to the general class of operators characterized by the function A. In order to describe these results we need to introduce some notation. Let $\Omega(t) = tA(t)$ for $t > 0$, $\Omega(0) = 0$, and

$$H(t) = t\Omega(t) - \int_0^t \Omega(s)\,ds, \quad t \geqslant 0. \tag{1.5}$$

It is easy to see that the function H is strictly increasing and that

$$H(t) = \int_0^{\Omega(t)} \Omega^{-1}(s)\,ds. \tag{1.6}$$

It is shown in [10] and [9] that the (SMP) and (CSP) hold if and only if

$$\int_0^\delta \frac{1}{H^{-1}(F(s))}\,ds = \infty \tag{1.7}$$

and

$$\int_0^\delta \frac{1}{H^{-1}(F(s))}\,ds < \infty, \tag{1.8}$$

respectively. These integral conditions replace (1.3) and (1.4). Thus, the data (A, f) of the problem are completely classified according to the validity of the (SMP) or (CSP) as above. See also [8].

An extension of these results to the case of a nontrivial functions G was given in [9] and [10]. Assuming that there exists $c > 0$ such that

$$G(s) \geqslant -c\Omega(s) \quad \text{for small } s > 0, \tag{1.9}$$

the authors proved that the (SMP) is still valid for (1.1) under (1.7). On the other hand, if (1.8) holds and there exists $c > 0$ such that

$$G(s) \leqslant c\Omega(s) \quad \text{for small } s > 0, \tag{1.10}$$

then (CSP) holds for (1.2). However, the following example was given in [10] for the case of the m-Laplacian and pure powers. For $q \in (0, m-1)$ it is possible to find $0 < q < p < m-1$ such that the inequality

$$\Delta_m u + |\nabla u|^q - u^p \geqslant 0 \tag{1.11}$$

has $u = \text{const.}|x|^{-\ell}$ as solution in $\{|x| > R\}$, for R and ℓ large, so that even if (1.4) holds the (CSP) may fail for (1.11). Moreover, in page 37 of [6] it was shown that for this situation it is actually the (SMP) that holds. In the case of inequality

$$\Delta_m u - |\nabla u|^q - u^p \leqslant 0 \tag{1.12}$$

it was shown in page 36 of [6] that for $q \in (0, m-1)$ the (CSP) is true even though (1.3) holds.

It is the purpose of this note to we give new conditions on the data (A, G, f) to obtain (SMP) and (CSP). We are not able to find conditions so that (A, G, f) gets completely classified by the validity of the (SMP) or the (CSP), as in the case of $G \equiv 0$. However in the case of the m-Laplacian and pure powers we completely classify (m, q, p) with this criterion. In doing so we will consider the case $0 < p \leqslant q < m-1$, a situation not covered before in the literature. It is worth mentioning that the ordinary differential equations associated to the inequalities with $G \not\equiv 0$ are not integrable, as in Vázquez situation [12], so an Osgood type condition is not directly available.

We describe next our results in a precise way. On the function G we will consider the following basic hypotheses:

(G1) $G : [0, +\infty) \to \mathbb{R}$ is a continuous function with $G(0) = 0$.

Our first two results have to do with the case of G positive. We assume:

(G2) G is a positive and increasing function.
(G3) For all $c > 0$ there exists $\delta > 0$ such that

$$G(s) \geqslant c\Omega(s) \quad \text{for every } s \in [0, \delta].$$

The main result for the case of G positive is the following.

Theorem 1.1. *Assume* (A1)*,* (A2)*,* (F1)*,* (G1) *and* (G2)*.*

(1) *If for some* $\delta > 0$ *we have*

$$\int_0^\delta \frac{1}{\Gamma^{-1}(\frac{1}{2}F(s))}\, ds < \infty, \tag{1.13}$$

then (CSP) *holds for* (1.2)*. Here* $\Gamma(t) := \int_0^{2t} G(s)\, ds + H(t)$*.*

(2) *In addition, if* (G3) *holds and for all* $\delta > 0$ *small we have*

$$\int_0^\delta \frac{1}{G^{-1}(2f(s))}\, ds = \infty, \tag{1.14}$$

then (SMP) *holds for* (1.1)*.*

We observe that under hypothesis (G3), G satisfies (1.9) so the results of [10] apply if (1.3) holds. The interesting case is when (1.4) holds.

Remark 1.1. In the case of the m-Laplacian, and with $f(t) = t^p$ and $G(t) = t^q$, for $p > 0$ and $q > 0$, hypothesis (G3) implies $q < m - 1$. Part (1) of the theorem implies that (CSP) holds for $p < q < m - 1$ and also for $p < m - 1 \leqslant q$. While part (2) implies that (SMP) holds for $q \leqslant p < m - 1$. The case $q \geqslant m - 1$ was already covered in the work by Pucci, Serrin and Zou [10], they proved that $p < m - 1$ implies (CSP), while for $p \geqslant m - 1$ (SMP) holds. Thus, all possible cases of (m, q, p) are covered.

Now we consider the case of negative G. We assume

(G2)′ G is a negative and decreasing function.

The following is the main result for the case of G negative.

Theorem 1.2. *Assume* (A1), (A2), (F1), (G1) *and* (G2)′.

(1) *If for some* $\delta > 0$ *we have*

$$\text{either} \quad \int_0^\delta \frac{1}{-G(\Omega^{-1}(s))}\,ds < \infty \quad \text{or} \quad \int_0^\delta \frac{1}{H^{-1}(F(s))} < \infty, \tag{1.15}$$

then (CSP) *holds for* (1.2).

(2) *If for every small* $\delta > 0$ *and all* $k > 0$ *we have*

$$\int_0^\delta \frac{1}{ks - G(\Omega^{-1}(s)) + f(\Omega^{-1}(s))}\,ds = \infty, \tag{1.16}$$

then (SMP) *holds for* (1.1).

Notice that for negative G condition (1.10) holds trivially, so the interesting case in part (1) occurs when the first integral in (1.15) is finite and the second one is infinite, case not covered by Pucci, Serrin and Zou in [10].

Remark 1.2. In the case of the m-Laplacian and for $f(t) = t^p$ and $G(t) = -t^q$, with $p > 0$ and $q > 0$, we see that all cases are covered now. Part (1) of Theorem 1.2 implies that whenever $q < m - 1$ (CSP), regardless the value of $p > 0$.

If $p < m - 1$ then part (1) of Theorem 1.2 implies (CSP), regardless the value of $q > 0$. If $p \geqslant m - 1$ and $q \geqslant m - 1$ then (SMP) holds. This last two cases were already covered in [10].

Remark 1.3. Our Theorem 1.2 applies to the following two interesting examples in the case of the m-Laplacian: Part (1) applies to $G(t) = t^q(\log|t|)^\alpha$ and $q \leqslant m - 1$ and $\alpha > 1$. Part (2) applies to $f(t) = t^{m-1}(|\log t|)^\beta$, $G(t) = t^{m-1}(\log t)^\alpha$ and $\max\{\beta, \alpha\} \leqslant 1$.

Remark 1.4. We observe that in Theorem 1.1 the integral conditions implying the (SMP) is almost the converse to the integral condition implying (CSP). About Theorem 1.2 we could say the same. We do not know if one can find integral conditions such that one is the converse of the other.

The results in Theorems 1.1 and 1.2 could be contrasted with those in [6]. The advantage of those here is that conditions are given directly on the data of the problem, so they are easily checkable. We cannot say the same about the uniqueness conditions given in [6].

Let us briefly look at a simple but illuminating consequence of our results. Consider the following ordinary differential equation, for $p > 0$ and $q > 0$,

$$u'' = u^p \pm (u')^q, \quad u(0) = 0 \quad \text{and} \quad u'(0) = 0. \tag{1.17}$$

One may ask for which values of p and q this equation has a unique solution. Of course when $p \geqslant 1$ and $q \geqslant 1$ uniqueness holds, since in this case the involved functions are Lipschitz continuous. All other situations give non-uniqueness, except when $p \leqslant q < 1$ and $-(u')^q$ appears in the equation. This is a consequence of Theorem 1.1, part (2). This case is very special because both nonlinearities are non-Lipschitz, but its combination still gives uniqueness.

In this note we provide the proof of Theorems 1.1 and 1.2 only regarding the construction of the appropriate super and sub-solutions for the corresponding problems. All extra work needed to complete the proof of the theorems is to use a comparison principle to get the conclusions. There are various versions of comparison theorems that can be used in our context, see [6–9] and [10].

More precisely, under hypotheses (A1), (A2), (F1) and (G1), assumptions considered in this note, we can use the comparison result proved in [6] for the m-Laplacian, but extendable to our more general class of operator in a straightforward way. See Lemma 2.1 in [6].

On the other hand we could weaken assumption (F1) by assuming only that f is non-decreasing near the origin. However in this case we need to use the comparison result proved in [10] that requires differentiability of the function A, something we do not assumed here. See Lemma 4 in [10].

We finally want to mention some other recent works concerning (SMP) in the context of viscosity solution in [1] and [11] and for degenerate elliptic operators in [3].

This paper is organized as follows. In Section 2 we prove our theorems assuming the existence of solutions for a boundary value problem. In Section 3 we prove the existence result for these equations.

2. Proofs of the main results

As we mentioned above, we will only discuss the existence of the appropriate comparison super and sub-solutions, and leave the reader to complete the proofs using the comparison theorem given in [6–10].

The strategy for proving the (CSP) is to construct a super-solution of a one-dimensional problem upon which we construct a super-solution on $\mathbb{R}^N \setminus B(0, R)$ having small values in $\partial B(0, R)$ and vanishing outside a larger ball. For proving the (SMP) one constructs a positive sub-solution v of an appropriate ODE in an interval $(0, a)$ such that $u'(0) > 0$ and then we use it for comparison following the usual Hopf proof.

Proof of Theorem 1.1. (1) Let v be the function implicitly defined by

$$\int_0^{v(t)} \frac{1}{\Gamma^{-1}(\frac{1}{2}F(s))}\, ds = t$$

for $0 \leqslant t \leqslant T$, where T is the value of integral in (1.13) and $v(T) = \delta$. Differentiating we find

$$\Gamma\bigl(v'(t)\bigr) = \frac{1}{2} F\bigl(v(t)\bigr). \tag{2.1}$$

Then, using the fact that Γ^{-1} is positive and the monotonicity of G and f one obtain in the interval $(0, T)$

$$\begin{aligned} G\bigl(v'(t)\bigr)v'(t) &\leqslant \int_0^{2v'(t)} G(s)\,ds \\ &\leqslant \Gamma\bigl(v'(t)\bigr) = \frac{1}{2} F\bigl(v(t)\bigr) \leqslant \frac{1}{2} f\bigl(v(t)\bigr)v(t), \end{aligned} \tag{2.2}$$

from where it follows that $v'(0) = 0$. Moreover, from (2.1) we also see that $v'(t)$ is increasing. In fact, we first observe that v' is positive, then v is increasing, but then the left-hand side is increasing implying that v' is increasing. From here we get $v(t) \leqslant v'(t)$ for small $t > 0$ and from (2.2)

$$G\bigl(v'(t)\bigr) \leqslant \frac{1}{2} f\bigl(v(t)\bigr). \tag{2.3}$$

Next we observe that $H(v'(t))$ and $\int_0^{2v'(t)} G(s)\,ds$ are increasing functions, then they are differentiable a.e. and consequently

$$\bigl\{H\bigl(v'(t)\bigr)\bigr\}' \leqslant \frac{1}{2} f\bigl(v(t)\bigr)v'(t), \quad \text{a.e.}$$

But we see that $\Omega(v'(s))$ is also differentiable a.e., then from (1.6) we see find that

$$\bigl\{H\bigl(v'(t)\bigr)\bigr\}' = v'(t)\bigl\{\Omega\bigl(v'(t)\bigr)\bigr\}', \quad \text{a.e.}$$

so that

$$\bigl\{\Omega\bigl(v'(t)\bigr)\bigr\}' \leqslant \frac{1}{2} f\bigl(v(t)\bigr), \quad \text{a.e.} \tag{2.4}$$

Adding (2.3) and (2.4) we see that v satisfies

$$\bigl\{\Omega\bigl(v'(t)\bigr)\bigr\}' + G\bigl(v'(t)\bigr) \leqslant f\bigl(v(t)\bigr)$$

for small $t > 0$. From here we construct a super-solution and we deduce (CSP).

(2) Given $k > 0$ we let $\delta > 0$ so that (G3) holds with $c = 2k$. Next we use Lemma 3.1 to find a nontrivial solution v to the two-point boundary value problem

$$\begin{gathered} \bigl\{\Omega\bigl(v'(t)\bigr)\bigr\}' + G\bigl(|v'|\bigr) = k\Omega\bigl(|v'|\bigr) + f(v) \quad \text{in } [0, \delta], \\ v(0) = 0, \quad v(\delta) = b, \end{gathered}$$

with $b > 0$. Then by (G3) v satisfies

$$\big\{\Omega\big(v'(t)\big)\big\}' + \frac{1}{2}G\big(|v'|\big) \leqslant f(v) \quad \text{in } [0, \delta].$$

Now by Lemma 3.1 again let u be a solution of

$$\big\{\Omega\big(u'(t)\big)\big\}' + \frac{1}{2}G\big(|u'|\big) = f(u) \quad \text{in } [0, \delta],$$
$$u(0) = 0, \quad u(\delta) = v(\delta).$$

We claim that $u'(0) > 0$. To prove this we assume by contradiction that $u'(0) = 0$. As a first step we prove that $\{\Omega(u'(t))\}' \geqslant 0$ in $(0, \delta)$. Assume there exists $\varepsilon > 0$ and $t_1 \in (0, \delta)$ such that $\{\Omega(u'(t))\}' < 0$ in $(t_1, t_1 + \varepsilon)$ and $\{\Omega(u'(t_1))\}' = 0$. Then u' is decreasing in $(t_1, t_1 + \varepsilon)$ and since $u' > 0$ u is increasing. Thus, from the equation $\{\Omega(u'(t))\}'$ is increasing in $(t_1, t_1 + \varepsilon)$ since G and f are increasing, which is a contradiction. Therefore we have

$$G\big(|u'|\big) \leqslant 2f(u).$$

Inverting G and integrating, one obtain a contradiction to the hypothesis (1.14), proving our claim.

Next we use comparison principle for u and v to conclude that $u \leqslant v$, therefore $v'(0) > 0$. This function v is thus appropriate to obtain the (SMP) following the Hopf argument. □

Proof of Theorem 1.2. (1) Let h be the function defined by

$$h(t) = \int_0^t \frac{1}{-G(\Omega^{-1}(s))}\, ds,$$

which is an increasing and invertible function, as it is $h \circ \Omega$. Let $v(t) = \int_0^t (h \circ \Omega)^{-1}(s)\, ds$. Notice that this v satisfies

$$v'(t) = (h \circ \Omega)^{-1}(t) \quad \text{and} \quad v'(0) = 0.$$

Since v is strictly increasing and $h(\Omega(v'(t)) = t$, the derivative of $\Omega(v'(t)$ exists and satisfies

$$\big\{\Omega\big(v'(t)\big)\big\}' = -G\big(v'(t)\big).$$

Thus v is nontrivial and it satisfies

$$\big\{\Omega\big(v'(t)\big)\big\}' + G\big(|v'|\big) \leqslant f(v) \quad \text{in } [0, \delta]. \tag{2.5}$$

The function v can be used as a super-solution to prove (CSP).

In case the other integral in finite, we define v implicitly as

$$t = \int_0^{v(t)} \frac{1}{H^{-1}(F(s))}\, ds$$

and proceed as before differentiating to get

$$\{\Omega(v'(t))\}' = f(v(t)),$$

from where we see that v can be used as a super-solution to prove (CSP).

(2) Let v be a nontrivial solution of the two-point boundary value problem with $k > 0$:

$$\{\Omega(v'(t))\}' + G(|v'|) = k\Omega(|v'|) + f(v) \quad \text{in } [0,\delta], \quad v(0) = 0, \quad v(\delta) = b,$$

as given by Lemma 3.1. We will prove that $v'(0) > 0$. Since G is negative, from the equation we directly see that v' is non-decreasing. Then, assuming that $v'(0) = 0$, we find that $v \leqslant v'$ for t small. So v satisfies

$$\{\Omega(v'(t))\}' \leqslant -G(|v'|) + k\Omega(|v'|) + f(v') \quad \text{in } [0,\delta].$$

Integrating we get a contradiction. So, (SMP) by using this v as a sub-solution. □

3. Existence result for a boundary value problem

In this section we prove an existence lemma that is used to construct comparison functions. We need to assume $k \geqslant 0$, because of the comparison Lemma 2.1 of [6].

Lemma 3.1. *Assume* (A1), (A2), (F1) *and* (G1) *and let* λ *and* $k \geqslant 0$ *be real constants. If* $a > 0$ *is small enough, then equation*

$$\begin{cases} -(\Omega(u'(r)))' + \lambda G(|u'(r)|) + k\Omega(|u'(r)|) + f(u(r)) = 0 & \text{in } (0,T), \\ u(0) = 0, \quad u(T) = a, \end{cases} \tag{3.6}$$

has a solution $u \in C^1[0,T]$*, in the weak sense, such that* $u(r) \geqslant 0$*,* $u'(r) \geqslant 0$ *for all* $r \in (0,T)$ *and* $u(r) > 0$*,* $u'(r) > 0$ *for all* $r \in (T_0,T)$ *for some* $T_0 \in [0,T)$*.*

Proof. For every u in $C^1[0,T]$ and $\sigma \in [0,1]$ we define the integral operator

$$\mathcal{H}_\sigma(u)(r) = \int_0^r \Omega^{-1}\left(K_\sigma + \sigma \int_0^s \lambda G(|u'(t)|) + k\Omega(|u'(t)|) + f(u(t))\,dt\right) ds.$$

Here the constant K_σ is uniquely determined so $\mathcal{H}_\sigma(u)(T) = \sigma a$ is satisfied. Each pair $(u,\sigma) \in C^1[0,T] \times [0,1]$ is mapped to $\mathcal{H}_\sigma(u) = v$. Moreover, if $\mathcal{H}_\sigma(u) = u$ then u a solution of

$$\begin{cases} -(\Omega(u'(r)))' + \sigma\lambda G(|u'(r)|) + \sigma k\Omega(|u'(r)|) + \sigma f(u(r)) = 0 & \text{in } (0,T), \\ u(0) = 0, \quad u(T) = \sigma a. \end{cases} \tag{3.7}$$

If we endow the space $C^1[0,T]$ with the norm $\|u\|_{C^1[0,T]} = \|u\|_{L^\infty(0,T)} + \|u'\|_{L^\infty(0,T)}$, then by means of converging sequences in $C^1[0,T] \times [0,1]$ we easily infer the continuity of $\mathcal{H}_\sigma$. Note that if $u_n \to u$ in C^1 and $\sigma_n \to \sigma$, then the corresponding sequence of constants $K^n_{\sigma_n}$ converges to K_σ.

For $\sigma = 0$, the unique fixed point of $\mathcal{H}_\sigma$ is $u \equiv 0$, because the unique solution of $-(\Omega(u'(r)))' = 0$ in $(0,T)$ with $u(0) = u(T) = 0$ is $u \equiv 0$.

The compactness of $\mathcal{H}_\sigma$ for each $\sigma \in [0,1]$ follows by the C^1 a priori estimate below, since the difference between $\mathcal{H}_\sigma u(r)$ and $\mathcal{H}_\sigma u(s)$ is controlled by $|r-s|$. And the same is true for the derivatives of $\mathcal{H}_\sigma(u)(r)$, and this allows the use of Arzelá–Ascoli Theorem.

By definition of the operator $\mathcal{H}_\sigma$ we see that a fixed point u of $\mathcal{H}_\sigma$ is a weak solution of (3.7) and the functions $u(r)$ and $\Omega(u'(r))$ are differentiable. Moreover, if u is a solution of (3.7), then $u \geqslant 0$ in $[0,T]$, by comparison Lemma 2.1 in [6] and the conditions (F1) and (G1). By the sliding method, introduced in [4], as it was applied in the proof of Theorem 2.2 of [6], we see that u is non-decreasing and then $u' \geqslant 0$ in $[0,T]$. Now, in order to see the existence of T_0 as in the statement of the lemma, we assume that for some $\bar{r} \in (0,T)$ we have $u'(\bar{r}) > 0$ and at $r_0 > \bar{r}$ we have $u'(r_0) = 0$. Then r_0 is a minimum point of u' and $u(r_0) > 0$, so that from Eq. (3.7)

$$\big(\Omega\big(u'(r_0)\big)\big)' = \sigma\big\{\lambda G\big(\big|u'(r_0)\big|\big) + k\Omega\big(\big|u'(r_0)\big|\big) + f\big(u(r_0)\big)\big\} = \sigma f\big(u(r_0)\big) > 0.$$

Hence $(\Omega(u'(r)))' > 0$ in a neighborhood $(r_0-\epsilon, r_0+\epsilon)$ for $\epsilon > 0$ small. Since $u'(r_0) = 0$ we must have $\Omega(u'(r)) < 0$ in $(r_0-\epsilon, r_0)$ and applying the increasing function Ω^{-1} we conclude that $u'(r) < 0$ in $(r_0-\epsilon, r_0)$, The existence of T_0 with the desired properties then follows, when $\sigma \in (0,1]$. Thus, a fixed point u of $\mathcal{H}_1$ turns out to be the solution we are looking for.

To find a fixed point of $\mathcal{H}_1$ we apply the Leray–Schauder Theorem and for that we just need to check that the solutions of (3.7) are a priori bounded in the C^1 norm.

Since $u' \geqslant 0$, it is easy to see that u attains its maximum on the boundary of the interval $[0,T]$ and then $\|u\|_{L^\infty(0,T)} \leqslant \sigma a \leqslant a$. We now find an a priori estimate for u'. Since u satisfies (3.7) we have

$$\big(H(u')\big)' = \big(\Omega(u')\big)'u' = \sigma\big\{\lambda G(u')u' + k\Omega(u')u' + f(u)u'\big\}$$

and then

$$\frac{(H(u'))'}{1+|G(u')|+\Omega(u')} \leqslant Cu',$$

for $C = \max(|\lambda|, k, f(a))$ and H as defined in (1.5) and (1.6). If c, d are two points in $[0,T]$ such that $u'(c) < u'(d)$, after integrating in r in the interval (c,d) we obtain

$$\begin{aligned}
&\int\limits_{H(u'(c))}^{H(u'(r))} \frac{1}{1+|G(H^{-1}(s))|+\Omega(H^{-1}(s))}\,ds \\
&\quad = \int\limits_c^r \frac{(H(u'(s)))'}{1+|G(u'(s))|+\Omega(u'(s))}\,ds \leqslant Cu(d) \leqslant Ca.
\end{aligned} \tag{3.8}$$

Now we claim that there exists a constant $C_1 > 0$ such that for any solution of (3.7) there is a point $s \in [0,T]$ such that $u'(s) \leqslant C_1$. In fact, taking C_1 such that $C_1 T > a$ and assuming that $u'(r) \geqslant C_1$ for all $r \in [0,T]$, we find

$$a \geqslant u(T) = \sigma a = \int\limits_0^T u'(s)\,ds \geqslant C_1 T,$$

that is a contradiction, from where the claim follows.

Now we choose a constant $M > 0$ so that

$$M < \int_{C_1}^{\infty} \frac{1}{1 + |G(H^{-1}(s))| + \Omega(H^{-1}(s))}\, ds, \tag{3.9}$$

and then make $a > 0$ smaller if necessary so that $Ca < M$.

Let us assume now that there exists a sequence of solutions u_n and a sequence of points $s_n \in [0, T]$, such that $|u_n'(s_n)| \to \infty$ as $n \to \infty$. As proved above there is a sequence $r_n \in [0, T]$, such that $u_n'(r_n) \leqslant C_1$. And we may apply (3.8) to u_n, with $c = r_n$ and $d = s_n$ to get

$$\int_{C_1}^{H(u_n'(s_n))} \frac{1}{1 + |G(H^{-1}(s))| + \Omega(H^{-1}(s))}\, ds \leqslant Ca.$$

But we have taken the constant a in such a way that $Ca < M$ where $M > 0$ satisfy (3.9). Since $u_n'(s_n) \to \infty$ as $n \to \infty$, we get a contradiction. □

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