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Some Recent Results on Equations Involving the Pucci's Extremal Operators

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Abstract. In this article we review some recent results on equations involving the Pucci's extremal operators. We discuss the existence of eigenvalues and applications to bifurcation analysis. Then we turn to the study of critical exponents for positive solutions, reviewing some results for general solutions and for radially symmetric solutions. Then, some consequences for the existence of solutions for some semilinear equations are obtained. We finally indicate some open problems.

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1. Introduction

In this article we review some recent results in the theory of existence of solutions for some nonlinear equations involving the Pucci's extremal operators. These operators are prototypes for fully nonlinear second order differential operators and they are obtained as perturbations of the Laplacian. While retaining many properties of the Laplacian, they lose some crucial ones, opening many interesting and challenging questions regarding the existence of solutions.

In this respect let us remark that the theory of viscosity solutions provides a very general and flexible theory for the study of a large class of partial differential equations. While originally developed to understand first order equations, it was successfully extended to cover fully nonlinear second order elliptic and parabolic equations. Very general existence results are combined with regularity theory to obtain a complete theory. We refer to Crandall, Ishii and Lions [11] and, Cabré and Caffarelli [7] for the basic elements of the theory. For this theory to be applicable the fully nonlinear operator has to satisfy some structural hypotheses, deeply linked to the Perron's method of super and sub-solutions: maximum and comparison principles.

On the other hand, when the second order differential operator has divergence form, again there are many methods to study existence of solutions. These methods will take for granted the possibility of testing functions by integration, providing so a rich tool for the analysis. In this direction, this structural hypothesis allows to construct an associated functional, whose critical points provide the solutions one is looking for.

The hypotheses on the equation we are about to describe do not include maximum and comparison principles nor varational structure. Then we realize that there are few techniques available and the attempt to solve some seemingly simple and standard problems leads to some difficult questions.

Let us first recall the definition of the Pucci's extremal operators. Given two parameters $0 < \lambda \leq \Lambda$, the matrix operators $\mathcal{M}^+_{\lambda,\Lambda}$ and $\mathcal{M}^-_{\lambda,\Lambda}$ are defined as follows: if M is a symmetric $N \times N$ matrix

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i$$

and

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i,$$

where $e_i = e_i(M), i = 1, ..., N$, are the eigenvalues of M. The Pucci's operators are obtained applying $\mathcal{M}^+_{\lambda,\Lambda}$ or $\mathcal{M}^-_{\lambda,\Lambda}$ to the Hessian D^2u of the scalar function u. We observe that when $\lambda = \Lambda$ then both Pucci's operators become equal to a multiple of the Laplacian. These two operators have many properties in common, but they are not equivalent. For more details and equivalent definitions see the monograph of Caffarelli and Cabré [7].

We start in Section §2 with the basic eigenvalues problems for the Pucci's operator $\mathcal{M}^+_{\lambda,\Lambda}$, namely

$$\begin{array}{rcll} -\mathcal{M}^+_{\lambda,\Lambda}(D^2u) & = & \mu u & \text{in} \quad \Omega, \\ u & = & 0 & \text{on} \quad \partial\Omega. \end{array} \tag{1.1}$$

One first question is the existence of a positive eigenfunction. It is addressed by Felmer and Quaas in [21] in the radial case and by Quaas in [44] for the case of a bounded domain using general Krein-Rutman's Theorem in positive cones as in [48]. These results are related with a general result for positive homogeneous fully nonlinear elliptic operators by Rouy [49]. The method used there is due to P.L. Lions who proved results for the Bellman operator in [34] and for the Monge-Ampère operator in [35].

When the analysis is restricted to radially symmetric functions, then the full spectrum for (1.1) can be obtained and nice properties of complementarity among the spectra are disclosed in [6].

Once the spectra of the Pucci's operator is understood, Busca, Esteban and Quaas in [6] made a bifurcation analysis as developed by Rabinowitz in [47] and [48]. In this direction, several existence results are obtained in [6] for the equation

$$\begin{aligned} -\mathcal{M}^{+}_{\lambda,\Lambda}(D^2u) &= \mu u + f(u,\mu) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.2}$$

where f is continuous, $f(s,\mu) = o(|s|)$ near $s = 0$, uniformly for $\mu \in I\!R$, and Ω is a general bounded domain.

The second main question we address in this review has to do with the so-called Liouville type theorems and is started in Section §4. In general terms the problem consists in determining the range for $p > 1$ for which the nonlinear elliptic equation

$$\mathcal{M}^{+}_{\lambda,\Lambda}(D^2u) + u^p = 0, \quad u \geq 0 \quad \text{in} \quad I\!R^N, \tag{1.3}$$

does not have a non-trivial solution. Here $N \geq 3$.

The non-existence of positive solutions for (1.3) is evidently complementary to the question of existence and is related to the problem of existence in a bounded domain, via degree theory. This approach requires a priori bounds for the solutions that can be obtained via blow-up technique once a Liouville type theorem is available. This is the crucial importance of these non-existence results.

The first result in this direction is due to Cutri and Leoni [12] who obtained a general non-existence result for (1.3) whenever $1 < p \leq p^s_+ := \tilde{N}_+/(\tilde{N}_+ - 2)$, where the dimension-like number $\tilde{N}_+$ is given by $\tilde{N}_+ = \frac{\lambda}{\Lambda}(N-1)+1$. This remarkable result is actually true for supersolutions of (1.3), even in the viscosity sense.

One important open question is to obtain the full range of exponents for the general Liouville theorem. In the case of radial solutions, this problem was addressed by Felmer and Quaas in [19] and [20]. The existence of a critical number p^*_+ is proved by means of a phase plane analysis after an Emden-Fowler transformation. This existence result is complemented by a uniqueness analysis resembling the study of uniqueness of ground states. The result in [20] clarifies the whole range of exponents, however it only gives an estimate of the critical exponent, whose value is between $(N+2)/(N-2)$ and $(\tilde{N}_+ + 2)/(\tilde{N}_+ - 2)$, remaining open to find a formula, in terms of the values of N, λ and Λ. It is important to mention the existence of an intermediate range of supercritical exponents where positive solutions in $\mathbb{R}^N$ exist, but their behavior differs from those of the usual Laplacian.

Strongly related to Liouville type theorems in $\mathbb{R}^N$ and also crucial for existence theory in bounded domains, are the non-existence results in half space. Here we will review a recent result of Quaas and Sirakov [45], where a dimension reduction approach in combination with Cutri and Leoni result is taken. In this way, a Liouville theorem is proved for general functions in the half space if the exponent is smaller than $(\bar{N})/(\bar{N}-2)$, where $\bar{N} = \frac{\lambda}{\Lambda}(N-2)+1$.

The third theme of this review is the existence theory for nonlinear equations with general form

$$\begin{aligned} \mathcal{M}^+_{\lambda,\Lambda}(D^2u) + f(u) &= 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.4}$$

The results obtained so far are bounded by available Liouville type theorems. The idea used in all results is originated in a paper by de Figueiredo, Lions and Nussbaum [14], where a related problem for the Laplacian is considered. Through an ingenious homotopy it is possible to prove that the degree of a large set not including the origin is non-trivial, thus providing an existence theorem.

Using these techniques, Felmer and Quaas [21] proved the existence of a radially symmetric ground state for the equation

$$\begin{aligned} \mathcal{M}^+_{\lambda,\Lambda}(D^2u) - \gamma u + u^p &= 0 \quad \text{in} \quad R^N, \\ \lim_{r\to\infty} u &= 0, \end{aligned} \tag{1.5}$$

if the exponent p is subcritical for the operator $\mathcal{M}^+_{\lambda,\Lambda}$. In a recent paper Felmer, Quaas and Tang in [26] have proved that this equation has actually only one solution. However, a second look at the problem reveals another open question. While solutions for (1.5) exists for all $1 < p < p^*_+$ we do not know if this exponent is optimal.

We will see also some recent existence results for equation (1.4), when Ω is a bounded domain in $\mathbb{R}^N$. In [18] Esteban, Felmer and Quaas obtain existence of positive solutions for the equation (1.3) for domains which are perturbations of a ball. These results provide with evidence that the critical exponent p^+_*, whose validity so far is confined to radially symmetric functions, is also a critical exponent for general domains. In [18] other related operators are also considered.

Finally we want to point out that all results discussed above can also be obtained for the operator $\mathcal{M}^-_{\lambda,\Lambda}$, without substantial changes. For other results concerning singular solutions for the Pucci's operators, we refer the reader to the work of Labutin in [32] and [33].

2. Eigenvalues for the Pucci's operator

As already mentioned in the introduction, the solvability of fully nonlinear elliptic equations of the form

$$F(x, u, Du, D^2u) = 0 \tag{2.1}$$

is very well understood for *coercive* uniformly elliptic operators F. On the contrary, little is known when coercivity (that is, monotonicity in u) is dropped. The aim is to study the model problem (1.4) when Ω is a bounded regular domain. In relation to (1.4) it is convenient to consider an eigenvalue problem that could provide some information on the general case. Since $\mathcal{M}^+_{\lambda,\Lambda}$ is homogeneous of degree one, it is natural to consider the "eigenvalue problem" (1.1).

Before continuing with our analysis we want to mention that Pucci's extremal operators appear in the context of stochastic control when the diffusion coefficient is a control variable, see the book of Bensoussan and J.L. Lions [1] or the papers of P.L. Lions [37], [38], [39] for the relation beetween a general Hamilton-Jacobi-Bellman and stochastic control. They also provide natural extremal equations in the sense that if F in (2.1) is uniformly elliptic, with ellipticity constants λ, Λ, and depends only on the Hessian D^2u, then

$$\mathcal{M}^-_{\lambda,\Lambda}(M) \leq F(M) \leq \mathcal{M}^+_{\lambda,\Lambda}(M) \tag{2.2}$$

for any symmetric matrix M. When $\lambda = \Lambda = 1$, (1.1) simply reduces to

$$\begin{array}{rcll} -\Delta u &=& \mu u & \text{in} \quad \Omega, \\ u &=& 0 & \text{on} \quad \partial\Omega. \end{array} \tag{2.3}$$

It is a very well known fact that there exists a sequence of solutions

$$\{(\mu_n, \varphi_n)\}_{n\geq 1}$$

to (2.3) such that:

i) The eigenvalues $\{\mu_n\}_{n\geq 1}$ are real, with $\mu_n > 0$ and $\mu_n \to \infty$ as $n \to \infty$.
ii) The set of all eigenfunctions $\{\varphi_n\}_{n\geq 1}$ is a basis of $L^2(\Omega)$.

Building on these eigenvalues, the classical Rabinowitz bifurcation theory [47], [48] allows to give general answers on existence of solutions to semilinear problems for the Laplacian.

When $\lambda < \Lambda$, problems (1.1) and (1.4) are fully nonlinear and it is interesting to know to which extent the known results about the Laplace operator can be generalized to this context. A few partial results in this direction have been established in the recent years. In [6] the authors provide a bifurcation result for general nonlinearities from the first two "half-eigenvalues" in general bounded domains. And in the radial case a complete description of the spectrum and the bifurcation branches for a general nonlinearity from any point in the spectrum.

Let us mention that besides the fact that (1.1)-(1.4) appears to be a favorable case from which one might hope to address general problems like (2.1), there are other reasons why one should be interested in Pucci's extremal operators or, more generally, in Hamilton-Jacobi-Bellman operators, which are envelopes of linear operators. The Pucci's operators are related to the Fučík operator as we describe next. Let u be a solution of nonlinear elliptic equation

$$-\Delta u = \mu u^+ - \alpha\mu u^-,$$

where α is a fixed positive number. One easily checks that if $\alpha \geq 1$, then u satisfies

$$\max\{-\Delta u\,,\, \frac{-1}{\alpha}\Delta u\} = \mu u,$$

whereas if $\alpha \leq 1$, u satisfies

$$\min\{-\Delta u\,,\, \frac{-1}{\alpha}\Delta u\} = \mu u.$$

These relations mean that the Fučík spectrum can be seen as the spectrum of the maximum or minimum of two linear operators, whereas (1.1)-(1.4) deal the maximum or minimum of an infinite family of operators.

We recall here that understanding the "spectrum" the Fučík operator, even in dimension $N = 2$, is still largely an open question. Only partial results are known and, in general, they refer to a region near the usual spectrum, (that is for α near 1). For a further discussion of this topic, we refer the interested reader to the works of de Figueiredo and Gossez [24], H. Berestycki [2], E.N. Dancer [13], S. Fučík [27], P. Drábek [17], T.Gallouet and O. Kavian [28], M. Schechter [50] and the references therein.

The first result in [6] deals with the existence and characterizations of the two first "half-eigenvalues" for $\mathcal{M}^+_{\lambda,\Lambda}$.

Theorem 2.1. *Let Ω be a regular domain, then there exist two positive constants μ_1^+, μ_1^-, that we call first half-eigenvalues such that:*

i) *There exist two functions $\varphi_1^+, \varphi_1^- \in C^2(\Omega) \cap C(\bar{\Omega})$ such that (μ_1^+, φ_1^+), (μ_1^-, φ_1^-) are solutions to* (1.1) *and $\varphi_1^+ > 0, \varphi_1^- < 0$ in Ω. Moreover, these two half-eigenvalues are simple, that is, all positive solutions to* (1.1) *are of the form $(\mu_1^+, \alpha\varphi_1^+)$, with $\alpha > 0$. The same holds for the negative solutions.*

ii) *The two first half-eigenvalues satisfy*

$$\mu_1^+ = \inf_{A\in\mathcal{A}} \mu_1(A), \quad \mu_1^- = \sup_{A\in\mathcal{A}} \mu_1(A),$$

where $\mathcal{A}$ is the set of all symmetric measurable matrices such that $0 < \lambda I \leq A(x) \leq \Lambda I$ and $\mu_1(A)$ is the principal eigenvalue of the nondivergent second order linear elliptic operator associated to A.

iii) *The two half-eigenvalues have the following characterization*

$$\mu_1^+ = \sup_{u>0} \operatorname*{essinf}_{\Omega}\Big(-\frac{\mathcal{M}^+_{\lambda,\Lambda}(D^2u)}{u}\Big), \quad \mu_1^- = \sup_{u<0} \operatorname*{essinf}_{\Omega}\Big(-\frac{\mathcal{M}^-_{\lambda,\Lambda}(D^2u)}{u}\Big).$$

The supremum is taken over all functions $u \in W^{2,N}_{loc}(\Omega) \cap C(\bar{\Omega})$.

iv) *The first half-eigenvalues can be also characterized by*

$$\mu_1^+ = \sup\{\mu \,|\; \textit{there exists } \phi > 0 \textit{ in } \Omega \textit{ satisfying } \mathcal{M}^+_{\lambda,\Lambda}(D^2\phi) + \mu\phi \leq 0\}$$

and

$$\mu_1^- = \sup\{\mu \,|\; \textit{there exists } \phi < 0 \textit{ in } \Omega \textit{ satisfying } \mathcal{M}^+_{\lambda,\Lambda}(D^2\phi) + \mu\phi \geq 0\}.$$

The above existence result, that is part i) of Theorem 2.1, is proved using a modified version, for convex (or concave) operators, of Krein-Rutman's Theorem in positive cones (see [21] in the radial symmetric case and see [44] in the case of a regular bounded domain).

This existence result has been also proved in the case of general positive homogeneous fully nonlinear elliptic operators in the paper by Rouy [49]. The method used there is due to P.L. Lions who proved the result i) of Theorem 2.1 for the Bellman operator in [34] and for the Monge-Ampère operator in [35].

Properties ii) of Theorem 2.1 can be generalized to any fully nonlinear elliptic operator F that is positively homogeneous of degree one, with ellipticity constants λ, Λ. This follows by the proof of ii) and (2.2). These properties were established by C. Pucci in [42], for related extremal operators.

The characterization of the form iii) and iv) for the first eigenvalue, was introduced by Berestycki, Nirenberg and Varadhan for second order linear elliptic operators in [5]. From iv) it follows that

$$\mu_1^+(\Omega) \le \mu_1^+(\Omega') \quad \text{and} \quad \mu_1^-(\Omega) \le \mu_1^-(\Omega') \quad \text{if} \quad \Omega' \subset \Omega.$$

In [6] many other properties for the two first half-eigenvalues are deduced from Theorem 2.1. For example, whenever $\lambda \neq \Lambda$, we have $\mu_1^+ < \mu_1^-$, since $\mu_1^+ \le \lambda\,\mu_1(-\Delta) \le \Lambda\,\mu_1(-\Delta) \le \mu_1^-$. Another interesting and useful property is the following maximum principle.

Theorem 2.2. *The next two maximum principles hold:*

a) *Let $u \in W^{2,N}_{loc}(\Omega) \cap C(\bar{\Omega})$ satisfy*

$$\begin{aligned} \mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \mu u &\ge 0 \quad in \quad \Omega, \\ u &\le 0 \quad on \quad \partial\Omega. \end{aligned} \tag{2.4}$$

If $\mu < \mu_1^+$, then $u \le 0$ in Ω.

b) *Let $u \in W^{2,N}_{loc}(\Omega) \cap C(\bar{\Omega})$ satisfy*

$$\begin{aligned} \mathcal{M}^-_{\lambda,\Lambda}(D^2u) + \mu u &\le 0 \quad in \quad \Omega, \\ u &\ge 0 \quad on \quad \partial\Omega. \end{aligned} \tag{2.5}$$

If $\mu < \mu_1^-$, then $u \ge 0$ in Ω.

The study of higher eigenvalues for the Pucci's operator in a general domain is wide open, as for general second order linear operators. However, in the radial case a complete description of the whole "spectrum" is given in [6]. This result may shed some light on the general case. More precisely, we have the following theorem.

Theorem 2.3. *Let $\Omega = B_1$. The set of all the scalars μ such that* (1.1) *admits a nontrivial radial solution, consists of two unbounded increasing sequences*

$$0 < \mu_1^+ < \mu_2^+ < \cdots < \mu_k^+ < \cdots,$$

$$0 < \mu_1^- < \mu_2^- < \cdots < \mu_k^- < \cdots.$$

Moreover, the set of radial solutions of (1.1) *for $\mu = \mu_k^+$ is positively spanned by a function φ_k^+, which is positive at the origin and has exactly k-1 zeros in* $(0,1)$*, all these zeros being simple. The same holds for $\mu = \mu_k^-$, but considering φ_k^- negative at the origin.*

This theorem is proved solving an appropriate initial value problem and a corresponding scaling. All nodal eigenfunction are generated in this way. There are many interesting questions left open in [6] regarding the distribution of these eigenvalues. For example: is it true that $\mu_k^+ \leq \mu_k^-$?

3. Bifurcation Analysis for the Pucci's operator

In this section we describe the results obtained in [6] regarding bifurcation of solutions that can be obtained now that we know spectral properties of the extremal Pucci's operator. In precise terms we consider (1.2) when f is continuous, $f(s,\mu) = o(|s|)$ near $s = 0$, uniformly for $\mu \in I\!R$, and Ω is a general bounded domain. Concerning this problem we have the following theorem

Theorem 3.1. *The pair $(\mu_1^+, 0)$ (resp. $(\mu_1^-, 0)$) is a bifurcation point of positive (resp. negative) solutions to* (1.2)*. Moreover, the set of nontrivial solutions of* (1.2) *whose closure contains $(\mu_1^+, 0)$ (resp. $(\mu_1^-, 0)$), is either unbounded or contains a pair $(\bar{\mu}, 0)$ for some $\bar{\mu}$, eigenvalue of* (1.1) *with $\bar{\mu} \neq \mu_1^+$ (resp. $\bar{\mu} \neq \mu_1^-$).*

For the Laplacian this result is well known, see [46], [47] and [48]. In this case the "half-branches" become connected. Therefore, we observe a symmetry breaking phenomena when $\lambda < \Lambda$.

For the p-Laplacian the result is also known, in the general case, see the paper of del Pino and Manásevich [16]. See also the paper of del Pino, Elgueta and Manásevich [15], for the case $N = 1$. In this case the branches are also connected. The proof of these results uses an invariance under homotopy with respect to p for the Leray-Schauder degree. In the proof of Theorem 3.1 homotopy invariance with respect to the ellipticity constant λ is used instead, having to deal with a delicate region in which the degree is equal to zero.

A bifurcation result in the particular case $f(u,\mu) = -\mu|u|^{p-1}u$ can be found in the paper by P.L. Lions for the Bellman equation [34]. For the problem

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = \mu g(x,u) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega$$

with the following assumption on g:

(g0) $u \to g(x,u)$ is nondecreasing and $g(x,0) = 0$,

(g1) $u \to \frac{g(x,u)}{u}$ decreasing, and

(g2) $\displaystyle\lim_{u\to 0} \frac{g(x,u)}{u} = 1, \quad \lim_{u\to\infty} \frac{g(x,u)}{u} = 0,$

a similar result was proved by E. Rouy [49]. In [34] and [49] the assumptions on g play a crucial role to construct sub and super-solutions. By contrast, in [6] the use of a Leray-Schauder degree argument allows to treat more general nonlinearities.

In the radially symmetric case the authors obtain a more complete result. Their proof again is based on the invariance of the Leray-Schauder degree under homotopy.

Theorem 3.2. *Let $\Omega = B_1$. For each $k \in \mathbb{N}$, $k \geq 1$ there are two connected components $S_k^\pm$ of nontrivial solutions to* (1.2), *whose closures contains* $(\mu_k^\pm, 0)$. *Moreover, $S_k^\pm$ are unbounded and $(\mu, u) \in S_k^\pm$ implies that u possesses exactly $k-1$ zeros in* $(0,1)$.

Remark 3.1. *S_k^+ (resp. S_k^-) denotes the set of solutions that are positive (resp. negative) at the origin.*

For the Laplacian this result is well known. In this case, for all $k \geq 1$, $\mu_k^+ = \mu_k^-$ and the "half-branches" connect each other at the bifurcation point.

4. Critical Exponents for the Pucci's Operators

In this section we consider the study of solutions to the nonlinear elliptic equation (1.3) where $N \geq 3$, $p > 1$. When $\lambda = \Lambda = 1$ (1.3) becomes

$$\Delta u + u^p = 0, \quad u \geq 0 \quad \text{in} \quad \mathbb{R}^N. \tag{4.1}$$

This very well known equation has a solution set whose structure depends on the exponent p. When $1 < p < p^* := (N+2)/(N-2)$ then equation (4.1) has no nontrivial solution vanishing at infinity, as can be proved using the celebrated Pohozaev identity [43]. If $p = p^*$ then it is shown by Caffarelli, Gidas and Spruck in [9] that, up to scaling, equation (4.1) possesses exactly one solution. This solution behaves like $C|x|^{2-N}$ near infinity. When $p > p^*$ then equation (4.1) admits radial solutions behaving like $C|x|^{-\alpha}$ near infinity, where $\alpha = 2/(p-1)$. The critical character of p^* is enhanced by the fact that it intervenes in compactness properties of Sobolev spaces, a reason for being known as critical Sobolev exponent.

It is interesting to mention that the nonexistence of solutions to (4.1) when $1 < p < p^*$ holds even if we do not assume a given behavior at infinity. This Liouville type theorem was proved by Gidas and Spruck in [23]. When $1 < p \leq N/(N-2) := p^s$, then a Liouville type theorem is known for supersolutions of (4.1).

This number p^s is called sometimes the second critical exponent or Serrin exponent for (4.1). In a recent paper [12], Cutri and Leoni extend this result for the Pucci's extremal operators. They consider the inequality

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + u^p \leq 0, \quad u \geq 0 \quad \text{in} \quad \mathbb{R}^N, \tag{4.2}$$

and define the dimension-like number $\tilde{N}_+ = \frac{\lambda}{\Lambda}(N-1) + 1$. Then they prove that for $1 < p \leq p^s_+ := \tilde{N}_+/(\tilde{N}_+ - 2)$ equation (4.2) has only the trivial solution.

In view of the results for the semilinear equation (4.1) that we have discussed above and the new results for inequality (4.2) just mentioned, it is natural to ask about the existence of critical exponents of the Sobolev type for (1.3). In particular it would be interesting to understand the structure of solutions for equation (1.3) in terms for different values of $p > 1$. It would also be interesting to prove Liouville type theorems for positive solutions in $\mathbb{R}^N$ and to understand the mechanisms for existence of positive solutions in general bounded domains.

In [19] Felmer and Quaas obtained some results in the case of radially symmetric solutions. Before we state the results we give a definition to classify the possible radial solutions of equation (4.1).

Definition 4.1. *Assume u is a radial solution of* (1.3) *then we say that:*

i) *u is a* pseudo-slow decaying solution *if there exist constants $C_2 > C_1 > 0$ such that $C_1 = \liminf_{r\to\infty} r^\alpha u(r) < \limsup_{r\to\infty} r^\alpha u(r) = C_2$.*

ii) *u is a* slow decaying solution *if there exists $c^* > 0$ such that $\lim_{r\to\infty} r^\alpha u(r) = c^*$.*

iii) *u is a* fast decaying solution *if there exists $C > 0$ such that $\lim_{r\to\infty} r^{\tilde{N}-2} u(r) = C$.*

The main results in [19] are summarized in the following theorem.

Theorem 4.1. *Suppose that $\tilde{N}_+ > 2$. Then there are critical exponents $1 < p_+^s < p_+^* < p_+^p$, with $p_+^s = \tilde{N}_+/(\tilde{N}_+ - 2)$, $p_+^p = (\tilde{N}_+ + 2)/(\tilde{N}_+ - 2)$ and $\max\{p_+^s, p^*\} < p_+^* < p_+^p$, that satisfy:*

i) *If $1 < p < p_+^*$ then there is no nontrivial radial solution of* (1.3).

ii) *If $p = p_+^*$ then there is a unique fast decaying radial solution of* (1.3).

iii) *If $p^* < p \le p_+^p$ then there is a unique pseudo-slow decaying radial solution to* (1.3).

iv) *If $p_+^p < p$ then there is a unique slow decaying radial solution to* (1.3).

Here uniqueness is understood up to scaling. The approach in [19] consists in a combination of the Emden-Fowler phase plane analysis with the Coffman-Kolodner technique. We start considering the classical Emden-Fowler transformation that allows to view the problem in the phase plane. With the aid of suitable energy functions much of the behavior of the solutions is understood. Their asymptotic behavior is obtained in some cases using the Poincaré-Bendixon theorem. This phase plane analysis has been used in related problems by Clemons and Jones [10], Kajikiya [29] and Erbe and Tang [26] among many others.

On the other hand we use the Coffman-Kolodner technique which consists in differentiating the solution with respect to a parameter. The function so obtained possesses valuable information on the problem. This idea has been used by several authors in dealing with uniqueness questions differentiating with respect to the initial value. In particular, see the work by Kwong [30], Kwong and Zhang [31] and Erbe and Tang [26]. However in [20] the authors do not differentiate with respect to the initial value, which is kept fixed, but with respect to the power p. Thus the variation function satisfies a non-homogeneous equation, in contrast with the situations treated earlier.

When the Pucci's extremal operators is considered on radially symmetric functions, it takes a very simple form so we can consider the following initial value problem

$$u'' = M\left(\frac{-\lambda(N-1)}{r}u' - u^p\right),\ r > 0,\ u(0) = \gamma, u'(0) = 0, \tag{4.3}$$

where $\gamma > 0$ and $M(s) = s/\Lambda$ if $s \geq 0$ and $M(s) = s/\lambda$ if $s < 0$. We notice that this equation possesses a unique solution that we denote by $u(r, p, \gamma)$ and that non-negative solutions of (4.3) correspond to radially symmetric solutions of (1.3). It can be proved that the solutions of (4.3) are decreasing, while they remain positive and that they have the following scaling property: $\gamma u(\gamma^{1/\alpha} r, p, \gamma_0) = u(r, p, \gamma_0 \gamma)$, for all $\gamma_0, \gamma > 0$.

In the next definition we classify the exponent p according to the behavior of the solution of the initial value problem (4.3) according to Definition 4.1. We define:

$$\begin{array}{lcllll} \mathcal{C} & = & \{p & \mid & p > 1, & u(r,p,\gamma) \;\; \text{has a finite zero}\}. \\ \mathcal{P} & = & \{p & \mid & p > 1, & u(r,p,\gamma) > 0 \quad \text{and is pseudo-slow decaying}\} \\ \mathcal{S} & = & \{p & \mid & p > 1, & u(r,p,\gamma) > 0 \quad \text{and is slow decaying}\} \\ \mathcal{F} & = & \{p & \mid & p > 1, & u(r,p,\gamma) > 0 \quad \text{and is fast decaying}\}. \end{array}$$

In view of the scaling property, we notice that these sets do not depend on the particular value of $\gamma > 0$.

An important step in the proof is to perform the classical Emden-Fowler change of variables $x(t) = r^\alpha u(r)$, $r = e^t$. This allows to use phase plane analysis. We have that the initial value problem (4.3) reduces to the autonomous differential equation

$$x'' = -\alpha(\alpha+1)x + (1+2\alpha)x' + M(\lambda(N-1)(\alpha x - x') - x^p)), \tag{4.4}$$

with boundary condition $x(-\infty) = 0$, $x'(-\infty) = 0$. Studying this dynamical system one can obtain the following basic properties:

a) If $p > \frac{\tilde N+2}{\tilde N-2}$ then $p \in \mathcal{S}$.

b) If $p \leq \max\{\frac{\tilde N}{\tilde N-2}, \frac{N+2}{N-2}\}$ then $p \in \mathcal{C}$.

c) $\frac{\tilde N+2}{\tilde N-2} \in \mathcal{P}$ and if $p \leq \frac{\tilde N+2}{\tilde N-2}$, then $p \notin \mathcal{S}$.

d) $\mathcal{P} \setminus \{\frac{\tilde N+2}{\tilde N-2}\}$ is open.

In the proof of these propositions we use two energy like functions

$$e(t) = \frac{(x')^2}{2} + \frac{\alpha x^{p+1}}{2\lambda(N-1)} - \frac{(\alpha x)^2}{2}, \quad E(t) = \frac{(x')^2}{2} + \frac{x^{p+1}}{\Lambda(p+1)} - \frac{\tilde b x^2}{2},$$

in order to understand the behavior of the trajectories. The Poincaré-Bendixon theorem is also used. It is interesting to note that in the range of p where the solution is pseudo-slow decaying, the periodic orbit of the dynamical system corresponds to a singular solution to $\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + u^p = 0$, which change infinitely many times its concavity. These solutions are not present in the case of the Laplacian and appear in trying to compensate the fact that $\lambda < \Lambda$.

The second main step in the proof of the main theorem in [20] is to understand the nature of the solutions obtained near a fast decaying solution. The goal is to prove that $\mathcal{F}$ is a Singleton. As we mentioned, the idea is to differentiate the solution of (4.3) with respect to p. The resulting function φ has valuable information on the solutions near the fast decaying one. By analyzing φ one can prove the following crucial proposition.

Proposition 4.1. a) *If $q \in \mathcal{F}$, then for $p < q$ close to q we have $p \in \mathcal{C}$.*
b) *If $q \in \mathcal{F}$, then for $p > q$ close to q we have $p \in \mathcal{S} \cup \mathcal{P}$.*

In order to understand the asymptotic behavior of φ it is convenient to study the function $w = w_\theta(r) = r^\theta u(r,q)$, for $\theta > 0$ chosen so that $\theta = (\tilde{N}-1)/2$ if $\tilde{N} > 3$ and $\theta = (\tilde{N}-2)/2$ if $2 < \tilde{N} \le 3$. This function was introduced by Erbe and Tang in [26], for a related problem. Defining $y(r) = \frac{\partial w(r)}{\partial p} = r^\theta \varphi$, when $\tilde{N} > 3$, y satisfies the equation

$$y'' + (\frac{(\tilde{N}-1)(3-\tilde{N})}{4r^2} + \frac{qu^{q-1}}{\Lambda})y + r^\theta \frac{u^q}{\Lambda} \log u = 0 \quad \text{if} \quad r > r_0. \tag{4.5}$$

Using the fact that u is a fast decaying solution we find that the coefficient in the second term of (4.5) is negative for r large. A similar situation occurs when $2 < \tilde{N} \le 3$. The following lemma on the asymptotic behavior of y is crucial in proving Proposition 4.1.

Lemma 4.1. *The function y defined above satisfies $y(r) > 0$ and $y'(r) > 0$ for r large.*

Finally, the proof of Theorem 4.1 is a direct consequence of previous Propositions, the openness of $\mathcal{C}$ and $\mathcal{P} \setminus \{\frac{\tilde{N}_+ + 2}{\tilde{N}_+ - 2}\}$.

5. Semi-linear equations and Liouville type Theorems

Having proved the existence of the critical exponents for (1.3) one can look for solutions for similar equations but in a bounded domain. Consider (1.4) when $\Omega = B_R$ is the ball of radius R in $I\!R^N$ and f is an appropriate nonlinearity. When $\lambda = \Lambda = 1$, (Laplace operator case) (1.4) has been studied by many authors, not only in a ball, but on general domains. We refer the reader to the review paper by P.L. Lions [36] and the references therein.

Continuing with the description of the results, let us introduce the precise assumptions on the nonlinearity f:

(f0) $f(u) = -\gamma u + g(u)$, $g \in C([0, +\infty))$ and is locally Lipschitz.

(f1) $g(s) \ge 0$ and there is $1 < p < p_+^*$ and a constant $C^* > 0$ such that

$$\lim_{s \to +\infty} \frac{g(s)}{s^p} = C^*.$$

(f2) There is a constant $c^* \ge 0$ such that $c^* - \gamma < \mu_1^+$ and

$$\lim_{s \to 0} \frac{f(s)}{s} = c^*,$$

where μ_1^+ is the first half eigenvalue for $\mathcal{M}^+_{\lambda,\Lambda}$ in B_R.

The first model problem is $f(s) = -u + s^p$, $1 < p < p_+^*$. The second model problem is $f(s) = \alpha s + s^p$, $1 < p < p_+^*$ and $0 \le \alpha < \mu_1^+$.

Now we are in a position to state the main theorem by Felmer and Quaas in [21]

Theorem 5.1. *Assume $N \ge 3$ and f satisfies the hypotheses (f0), (f1) and (f2). Then there exist a positive radially symmetric C^2 solution of* (1.4).

In case of the first model problem, Theorem 5.1 can be extended for positive solutions in $I\!R^N$. Precisely we have

Theorem 5.2. *Assume $N \ge 3$ and $1 < p < p_+^*$. Then there is a positive radially symmetric C^2 solution of the equation*

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) - u + u^p \quad = \quad 0 \quad in \quad I\!R^N. \tag{5.1}$$

In order to prove Theorem 5.1 the author use degree theory on positive cones as presented in the work by de Fugueiredo, Lions and Nussbaum in [14]. A priori bounds for solutions are obtained by blow up method introduced by Gidas and Spruck [23] in combination with the Liouville type Theorem 4.1.

Following in this direction and in view of Cutri and Leoni theorem in [12], it is interesting to ask if the theory of viscosity solutiuons allows to use a degree argument. Consider the existence of positive solutions for the equation (1.4) when Ω is a convex domain in $I\!R^N$ with boundary $\partial\Omega$ of clase $C^{2,\alpha}$ and f is an appropriate nonlinearity.

On the nonlinearity f we consider the hypotheses (f0), (f1) and (f2). With the difference that in (f1) we assume $1 < p < p_+^s$ and in (f2) μ_1^+ is the first half eigenvalue for $\mathcal{M}^+_{\lambda,\Lambda}$ in Ω, as given in Section §2.

Now we are in a position to state the main theorem in [44]

Theorem 5.3. *Assume $N \ge 3$, Ω is convex and f satisfies the hypotheses (f0), (f1) and (f2). Then there exist a positive $C^2(\Omega)$ solution of* (1.4).

Remark 5.1. *The missing piece to cover all $1 < p < p_+^*$ in (f1) is the Louville type theorem in the general case, which remains open.*

In order to prove our main theorem the author uses the Liouville type Theorem of Cutri and Leoni. At this point the convexity of the domain plays a crucial role. In fact, the convexity Ω allows to prove, via moving planes, that the blow-up point always converges to the interior of Ω. As we see in what follows the convexity of Ω can be lifted as proved by Quaas and Sirakov in [45].

Theorem 5.4. *Suppose $N \ge 3$ and set*

$$\tilde{p}_+ = \frac{\Lambda(N-2) + \lambda}{\Lambda(N-2) - \lambda}.$$

Then the problem

$$\begin{aligned} \mathcal{M}^+_{\lambda,\Lambda}(D^2u) + u^p \quad &= \quad 0 \quad in \quad \mathbb{R}+, \\ u \quad &= \quad 0 \quad on \quad \partial\mathbb{R}+ \end{aligned}$$

does not have a positive nontrivial bounded solution, provided $1 < p \le \tilde{p}_+$. *Observe that* $\tilde{p}_+ > p^s_+$.

A Theorem of this type for the equation $-\Delta u + f(u) = 0$ was first obtained by Dancer in [13]. Theorem 5.4 is proved by using a (simplified) version of the proof of Berestycki, Caffarelli and Nirenberg [3], who showed that solutions of $\Delta u + f(u) = 0$ in a half space which are at most exponential at infinity are necessarily monotone in x_N. Once this is proved it is possible to pass to the limit as $x_N \to \infty$, and this leads to a solution of the same problem in $\mathbb{R}^{N-1}$, which permits the use of Liouville Theorems of Cutri and Leoni in the whole space.

The following existence result is a consequence.

Theorem 5.5. *Assume* $N \ge 3$ *and* f *satisfies the same condition as Theorem* 5.3. *Then for any bounded regular domain* Ω *there exists a positive* $C^2(\Omega)$ *solution of* (1.4).

Remark 5.2. *The missing part to cover the range* $p^s_+ < p < p^*_+$ *in (f1) is the general Liouville type theorem in* $\mathbb{R}^N$ *which is still open. In fact, a Liouville type Theorem in all space would imply a Liouville type Theorem in the half space for a larger range of p.*

6. Further questions and open problems

For any linear second order uniformly elliptic operator with C^1 coefficients, say $Lu = \sum_i \sum_j a_{ij} \frac{\partial u^2}{\partial x_i \partial x_j}$ with $a_{ij} \in C^1$, the semilinear problem

$$\begin{aligned} Lu + u^p &= 0, \quad \text{in} \quad \Omega && (6.1)\\ u &= 0, \quad \text{on} \quad \partial\Omega, && (6.2) \end{aligned}$$

has a positive solution for the same range of values of p as for the Laplacian. That is, the existence property of the Sobolev exponent remains valid for all operators in this class.

In [18] Esteban, Felmer and Quaas consider two classes of uniformly second order elliptic operators for which the critical exponents in the radially symmetric case are drastically changed with respect to the Sobolev exponent p^*. The main point in [18] is to prove that the corresponding existence property for these critical exponents persists when the domain is perturbed, away from the ball.

The first class of operators corresponds to the Pucci's extremal ones, that is, $\mathcal{M}^+_{\lambda,\Lambda}(D^2u)$ and $\mathcal{M}^-_{\lambda,\Lambda}(D^2u)$, already discussed in this paper. We recall that these are extremal operators in the class defined by (2.2) and we notice that given any number $s \in [\lambda, \Lambda]$ the operator $s\Delta$ belongs to the class defined by (2.2).

The second family of operators that are considered in [18] are defined as

$$Q^+_{\lambda,\Lambda} u = \lambda \Delta u + (\Lambda - \lambda) Q^0 u, \tag{6.3}$$

where Q^0 is the second order linear operator

$$Q^0 u = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{x_i x_j}{|x|^2} \frac{\partial^2 u}{\partial x_i \partial x_j} .$$

These operators are also considered by Pucci [42], being extremal with respect to some spectral properties. We notice that these operators belong to the class defined by (2.2) and when $\lambda = \Lambda$ they also become a multiple of the Laplacian. If we interchange the role of λ by Λ in definition (6.3) then we obtain the operator $Q^-_{\lambda,\Lambda}$, which is also considered later.

The operators $\mathcal{M}^{\pm}_{\lambda,\Lambda}$ are autonomous, but not linear, even if they enjoy some properties of the Laplacian. The operators $Q^{\pm}_{\lambda,\Lambda}$ are still linear, but their coefficients are not continuous at the origin. When one considers a ball and the set of radially symmetric functions on it, there are critical exponents for the operators $\mathcal{M}^+$ and Q^+ which are greater than the Sobolev exponet p^*. On the contrary, for the operators $Q^-_{\lambda,\Lambda}$ and $\mathcal{M}^-_{\lambda,\Lambda}$, the critical exponents for radially symmetric solutions in a ball are smaller than the Sobolev exponet p^*. We recall

$$\frac{\widetilde{N}_- + 2}{\widetilde{N}_- - 2} < p^*_- < p^* < p^*_+ < \frac{\widetilde{N}_+ + 2}{\widetilde{N}_+ - 2} ,$$

where the numbers in the extreme are the critical exponents of Q^- and Q^+, respectively. The numbers p^*_+ and p^*_- depend on λ, Λ and the dimension N.

Open problem 1. Determine an explicit formula for the numbers p^*_+ and p^*_-, or at least describe in more precise terms the dependence with respect to the parameters.

The existense results in [18] are for domains which are close to the unit ball. More precisely it is assumed that there is a sequence of domains $\{\Omega_n\}$ such that for all $0 < r < 1 < R$ there exists $n_0 \in \mathbb{N}$ such that

$$B(0,r) \subset \Omega_n \subset B(0,R), \quad \text{for all } n \geq n_0.$$

Then the main theorem is

Theorem 6.1. *Assume $\tilde{N}_+ > 2$ and that $1 < p < (\tilde{N}_+ + 2)/(\tilde{N}_+ - 2)$. Then there is $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$, the equation*

$$Qu + u^p = 0 \quad in \quad \Omega_n, \tag{6.4}$$

$$u = 0 \quad on \quad \partial\Omega_n, \tag{6.5}$$

possesses at least one nontrivial solution.

A second theorem states a similar result replacing Q^+ by $\mathcal{M}^+_{\lambda,\Lambda}$. Corresponding theorems for Q^- and $\mathcal{M}^-_{\lambda,\Lambda}$ are also considered.

Thus, in this work it is proved that the phenomenon of critical exponent increase (or decrease) does not appear only in the radially symmetric case, but persists when the ball is perturbed not necessarily in a radial manner. This result is proved by a perturbation argument, based on a work by Dancer [13]. It provides evidence that the critical exponents for these operators, obtained in radial versions, are also the critical exponents in the general case.

At this point we would like to stress some surprising properties of the critical exponents of operators in the class given by (2.2). For the first property we consider all linear elliptic operators with bounded coefficients and belonging to the class defined by (2.2). If we take the L^∞ topology for the coefficients of these operators, we see that the critical exponent is not a continuous function of the operator. In particular, as shown in Section §2, the operators $Q^-_{\lambda,\Lambda}$ can be "approximated" in L^∞ (the coefficients) by a sequence of operators with C^∞ coefficients, for which the critical exponent in the radially symmetric case is p^*.

The second property is related to the non-monotonicity of the critical exponents. Notice the following inequality for operators holds,

$$\lambda\Delta \leq \mathcal{M}^+_{\lambda,\Lambda} \quad \text{and} \quad Q^+_{\lambda,\Lambda} \leq \mathcal{M}^+_{\lambda,\Lambda},$$

while for the corresponding critical exponents we have

$$p^* < p^*_+ \quad \text{and} \quad \frac{\widetilde{N}_+ + 2}{\widetilde{N}_+ - 2} > p^*_+.$$

Open problem 2. Is there a natural order in the operators that is compatible with the order of the critical exponents?

We finally observe that all operators of the form $\mathcal{M}^\pm_{s,S}$ and $Q^\pm_{s,S}$, with $s, S \in [\lambda, \Lambda]$, have critical exponents in the interval

$$\left[\frac{\widetilde{N}_- + 2}{\widetilde{N}_- - 2}, \quad \frac{\widetilde{N}_+ + 2}{\widetilde{N}_+ - 2} \right].$$

Open problem 3. Prove that in the class of operators defined by (2.2), all the critical exponents are in the same interval, that is, the operators $Q^\pm_{\lambda,\Lambda}$ are extremal for critical exponents.

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