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On uniqueness for nonlinear elliptic equation involving the Pucci's extremal operator

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Abstract

In this article we study uniqueness of positive solutions for the nonlinear uniformly elliptic equation $M^{+}_{\lambda,\Lambda}(D^2u) - u + u^p = 0$ in R^N, $\lim_{r\to\infty} u(r) = 0$, where $M^{+}_{\lambda,\Lambda}(D^2u)$ denotes the Pucci's extremal operator with parameters $0 < \lambda \leqslant \Lambda$ and $p > 1$. It is known that all positive solutions of this equation are radially symmetric with respect to a point in R^N, so the problem reduces to the study of a radial version of this equation. However, this is still a nontrivial question even in the case of the Laplacian ($\lambda = \Lambda$). The Pucci's operator is a prototype of a nonlinear operator in no-divergence form. This feature makes the uniqueness question specially challenging, since two standard tools like Pohozaev identity and global integration by parts are no longer available. The corresponding equation involving $M^{-}_{\lambda,\Lambda}$ is also considered.

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1. Introduction

Let $0 < \lambda \leqslant \Lambda$ be two given positive real numbers. For a C^2 scalar function u defined in R^N, the Pucci's extremal operators are given by

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$$M^+_{\lambda,\Lambda}(D^2u) = \lambda \sum_{e_i<0} e_i + \Lambda \sum_{e_i>0} e_i \quad \text{and} \quad M^-_{\lambda,\Lambda}(D^2u) = \Lambda \sum_{e_i<0} e_i + \lambda \sum_{e_i>0} e_i,$$

where $e_i = e_i(D^2u)$, $i = 1, \ldots, N$, are the eigenvalues of the Hessian matrix D^2u. For more details and equivalent definitions see the monograph of Caffarelli and Cabré [2]. Clearly, in the special case $\lambda = \Lambda$ the two operators become the same and

$$M^+_{\lambda,\lambda}(D^2u) = M^-_{\lambda,\lambda}(D^2u) = \lambda \Delta u,$$

where Δu is the usual Laplacian of u. The Pucci's extremal operators provide important prototypes of fully nonlinear uniformly elliptic operators and even though they retain positive homogeneity and some properties associated to the maximum principle, they are no longer in divergence form, thus deviating in a fundamental manner away from the Laplacian.

Recently in [8,9], Felmer and Quaas studied the nonlinear elliptic equation

$$M^{\pm}_{\lambda,\Lambda}(D^2u) + u^p = 0, \tag{1.1}$$

for positive radially symmetric solutions, $p > 1$. Here, for convenience, we write $M^{\pm}_{\lambda,\Lambda}$ in (1.1) to mean the two equations, one with the operator $M^+_{\lambda,\Lambda}$ and the other with the operator $M^-_{\lambda,\Lambda}$. In the special case of the Laplacian, that is $\lambda = \Lambda$, the range of existence and nonexistence for the ball or R^N is characterized by the Sobolev critical number $p_N = (N+2)/(N-2)$. For a ball of radius R, denoted by B_R, Eq. (1.1) has a solution with zero Dirichlet boundary condition in ∂B_R if and only if $1 < p < p_N$. When dealing with radially symmetric positive solutions in all R^N the situation is dual, that is, (1.1) has a solution in R^N if and only if $p \geqslant p_N$. These basic facts can be proved, for example, by doing a phase plane analysis after the Emden–Fowler transformation. When $0 < \lambda < \Lambda$ a similar situation occurs as proved in [9]. For the operator $M^+_{\lambda,\Lambda}$ it is shown in [9] that there exists a number p_+ playing the role of p_N regarding existence and nonexistence in the ball and in R^N, and for the operator $M^-_{\lambda,\Lambda}$ it is shown that there exists a corresponding number p_-. For these numbers the following inequality holds

$$p_- < p_N < p_+.$$

See [9], for a more detailed description of positive entire solutions in the range $p \geqslant p_+$ ($p \geqslant p_-$), where a new phenomenon appears. We also notice that no formula is known for this number p_+.

We observe that the nonexistence results in R^N just described may allow to find existence results for more general nonlinearities via blow-up analysis and degree theory. This is precisely the work done by Felmer and Quaas in [10], where the equation

$$M^{\pm}_{\lambda,\Lambda}(D^2u) + f(u) = 0, \quad u > 0 \text{ in } B_R \quad \text{and} \quad u = 0 \quad \text{on } \partial B_R, \tag{1.2}$$

was studied for various nonlinearities. For the canonical model case

$$f(u) = -u + u^p, \quad p > 1, \tag{1.3}$$

they found that whenever $1 < p < p_{\pm}$ Eq. (1.2) has at least one radially symmetric solution. Moreover they show that the equation possesses a *ground state*, that is an entire positive solution satisfying $\lim_{r\to\infty} u(r) = 0$. In view of these results it would be interesting to study other qualitative properties of the solutions of (1.2) and its associated initial value problem, for different values of p. For notational convenience, we allow in (1.2) that $R=\infty$ and we interpret the boundary condition as $\lim_{r\to\infty} u(r)=0$.

In the case of the Laplacian, where $p_N=p_+=p_-$, it is also known that the number p_N is optimal for existence in (1.2), that is, if $p\geqslant p_N$ then (1.2)–(1.3) does not have a solution. This nonexistence result for solutions for (1.2), and other qualitative properties of the associated initial value problem, have been historically proved using the well-known powerful Pohozaev identity. Naturally, one may ask if we can establish an analogue of the Pohozaev identity for the general Pucci's operator, and then use this identity to obtain nonexistence results. The answer to this question is *yes* to the first part and *no* to the second part. We can easily derive the homologue of the Pohozaev identity for the Pucci's operator, but this identity is nearly useless. To see this more clearly, we define the function

$$P(r)=\theta r^{\widetilde N}(u')^2(r)+2r^{\widetilde N}F\bigl(u(r)\bigr)+(\widetilde N-2)\theta r^{\widetilde N-1}u(r)u'(r), \tag{1.4}$$

where $u=u(r)$ is a radial solution of (1.2) and θ takes the value λ or Λ according to the sign of u' and u'', as described at the beginning of Section 2. We refer to (3.2) for the precise definition of the dimension-like parameter $\widetilde N$. It is straightforward to verify that

$$P'(r)=r^{\widetilde N-1}\bigl[2\widetilde N F(u)-(\widetilde N-2)uf(u)\bigr].$$

This is the homologue of the classical Pohozaev identity for the Pucci's operator. For our function $f(u)=-u+u^p$, a further calculation gives

$$P'(r)=2r^{\widetilde N-1}\bigl(\sigma u^{p+1}-u^2\bigr),\quad \sigma=\frac{2\widetilde N-(\widetilde N-2)(p+1)}{2(p+1)}.$$

Two fatal factors prevent any effective application of this generalized Pohozaev identity to the Pucci's operators. First, the function $P(r)$ has jumps at points where u' and u'' vanishes, that is at critical or inflection points of u. In particular, this function could possibly jump from a positive value to a negative one at those points. Second, for the range of p of interest here, the corresponding parameter σ is not always positive, and thus it is nearly impossible to understand how the function $P(r)$ behaves without any a priori knowledge of the concavity and critical points changes of u.

The discontinuities experienced by the Pohozaev function $P(r)$ is indeed an intrinsic property of the Pucci's operator, posing in this way many interesting questions about qualitative properties of positive radial solutions to (1.2). While the qualitative analysis for equations involving Laplacian may still be nontrivial, in the case of Pucci's operators the lack of continuity of proper functions or their derivatives creates major technical difficulties for any argument relying upon a variety of applications of the method of integration by parts.

In this paper, we shall examine deeply some qualitative properties of positive radially symmetric solutions to (1.2) and prove uniqueness of the radial solutions found in [10]. This result, which in particular includes the uniqueness of the ground state for (1.2), is important by itself and open the fundamental question of the behavior of positive solutions for (1.2) when $p\geqslant p_+$. For further discussion see concluding remarks in Section 4.

Now we state our main theorem.

Theorem 1.1. *Assume* $1 < p < p_+$ *in case of the operator* $M^+_{\lambda,\Lambda}(D^2u)$, *or* $1 < p < p_-$ *in case of* $M^-_{\lambda,\Lambda}(D^2u)$. *Then the problem* (1.2) *admits exactly one positive radial solution in each finite ball* B_R *and exactly one ground state* ($R = \infty$).

Let $u = u(r)$ *be this unique solution defined for* $r \in (0, R)$, $R \leqslant \infty$. *The we have*:

(i) *the maximum value of* u, *attained at the origin, is larger than one*;
(ii) $u(r)$ *and* ru'/u *are strictly decreasing in the radial direction for* $r \in (0, R)$, *and*
(iii) u *changes concavity exactly once, that is, there is a unique* $r_c \in (0, R)$ *with* $u(r_c) > 1$ *such that*

$$u'' < 0 \quad \text{for } 0 < r < r_c, \qquad u'' > 0 \quad \text{for } r_c < r < R.$$

Remark 1.1. We mention that part (iii) is new even for the Laplacian case. It is by no means trivial, as the example of pseudo-slow decaying solutions for

$$M^+_{\lambda,\Lambda}\big(D^2u\big) + u^p = 0$$

show in case $p_+ < p < (\widetilde{N} + 2)/(\widetilde{N} - 2)$. These solutions are decreasing, positive and change concavity *infinitely many times*. The definition of $\widetilde{N}$ is given in (3.2). See details in [9].

Remark 1.2. The Pucci's operators retain maximum principle and comparison properties of the Laplacian, so that the moving planes method is applicable to study radial symmetry of solutions of (1.2). Da Lio and Sirakov in [7] proved, among other things, that all solutions of (1.2) are radially symmetric, even in the case $R = \infty$.

The study of uniqueness questions for Eq. (1.2) in the case of the Laplacian has a long history. The main step we can distinguish are the contributions of Ni [17] and Ni and Nussbaum [18] who treated the case of a ball. In the case of R^N, the study of uniqueness is traced back to Coffman [4], Peletier and Serrin [19,20] and McLeod and Serrin [15]. Then the fundamental work by Kwong [11] treating the all range of exponents. Subsequent contributions have been given by many authors, see among them the following [3,5,12–14,16]. More recently we mention the work by Serrin and Tang [21] and Tang [22]. The difficult case of the annulus with Dirichlet boundary conditions is treated by Tang [23]. In this article we use many ideas from [23].

This article is organized in four sections. In Section 2 we shall derive the monotonicity of u. In the Laplacian case, this follows from a very simple argument of an energy function. However in our context this analysis is very delicate since we do not have an appropriate energy function for all r. We need to combine different energies in order to overcome the discontinuity each of them may have. In Section 3, we shall prove the monotonicity of ru'/u and part (iii) of Theorem 1.1 by showing that the useful function

$$Q(r) = r^{\widetilde{N}}\big[\theta u'^2 + uf(u)\big] + \theta(\widetilde{N} - 2)r^{\widetilde{N}-1}uu' \tag{1.5}$$

is positive on $(0, R)$. Indeed, the proof of the positivity of Q constitutes the major technical part of this paper and it is here where we introduce new ideas. In Section 4, we prove the uniqueness of positive radial solutions. For this purpose we study the variations of the solution with respect to the initial value following an idea of Coffman [4]. Then a major step is obtained by modifying ideas from the recent work [23]. However, this is not a simple trivial generalization as a variety of technical complexities arise due to the discontinuities of the functions involved.

2. Properties of the solutions

Let $u = u(x)$ be a radial C^2 function in R^N. As usual we abuse the notation to write $u(x) = u(r),\ r = |x|$, without causing any further confusion. As calculated in [8] we have

$$D^2u(x) = \frac{u'(r)}{r} Id + \left[\frac{u''(r)}{r^2} - \frac{u'(r)}{r^3}\right] X,$$

where Id is the $N \times N$ identity matrix, and X is the matrix whose entries are $x_i x_j$. Observing further that

$$D^2u(x)\frac{x}{r} = u''(r)\frac{x}{r} \quad \text{and} \quad D^2u(x)y = \frac{u'(r)}{r} y,$$

for every vector in the hyperplane $x \cdot y = 0$, we find that the eigenvalues of the Hessian matrix $D^2u(x)$ are $u''(r)$, which is simple, and $u'(r)/r$, which has multiplicity $N - 1$. Therefore, for a radial function $u(r)$ there holds

$$M^{\pm}_{\lambda,\Lambda}\big(D^2u\big) = \theta u''(r) + \frac{N-1}{r}\Theta u'(r),$$

where θ and Θ take the values of either λ or Λ, depending on the operators $M^{\pm}_{\lambda,\Lambda}$ and the signs of $u'(r)$ and $u''(r)$. Corresponding to the operator $M^{+}_{\lambda,\Lambda}$, we have

$$\theta = \Lambda \quad \text{when } u'' > 0 \quad \text{and} \quad \theta = \lambda \quad \text{when } u'' < 0; \tag{2.1}$$

$$\Theta = \Lambda \quad \text{when } u' > 0 \quad \text{and} \quad \Theta = \lambda \quad \text{when } u' < 0; \tag{2.2}$$

and corresponding to the operator $M^{-}_{\lambda,\Lambda}$, we have

$$\theta = \lambda \quad \text{when } u'' > 0 \quad \text{and} \quad \theta = \Lambda \quad \text{when } u'' < 0,$$

$$\Theta = \lambda \quad \text{when } u' > 0 \quad \text{and} \quad \Theta = \Lambda \quad \text{when } u' < 0.$$

Consequently, if $u = u(r)$ is a positive C^2 radial solution of (1.2) and we write $u(0) = \alpha > 0$, then u is also a solution to the initial value problem of the ordinary differential equation

$$\theta u'' + \frac{N-1}{r}\Theta u' + f(u) = 0, \qquad u(0) = \alpha > 0, \quad u'(0) = 0, \tag{2.3}$$

satisfying additionally the conditions: $u(r) > 0$ in the interval $[0, R)$ and $u(R) = 0$. Thus, in order to prove the uniqueness property for (1.2) it is sufficient to prove that there is exactly one $\alpha > 0$ such that the solution of (2.3) satisfies the two additional conditions.

The existence and uniqueness of C^2 solution to the initial value problem (2.3) can be analyzed using the ideas of Ni and Nussbaum [18] and Felmer and Quaas [10].

2.1. Energy functions

Motivated by the case of the Laplacian, that is, $\lambda = \Lambda = 1$, we may define the energy function

$$E_\theta(r) = \frac{\theta}{2}(u')^2(r) + F\big(u(r)\big).$$

Using (2.3) we find that

$$E'_\theta(r) = -\frac{(N-1)\Theta}{r}(u')^2(r). \tag{2.4}$$

However, this is not sufficient to imply that $E_\theta(r)$ is a decreasing function over the whole range where u is defined and positive. In fact $E_\theta(r)$ can be discontinuous at points where u changes concavity. Indeed, let $r = r_I$ be a point of inflection which is not a critical point of u, that is, u'' changes sign near $r = r_I$, $u''(r_I) = 0$ and $u'(r_I) \neq 0$, then $E_\theta(r)$ must have a jump at r_I. Thus $E_\theta(r)$ is only a piecewise C^1 function and decreases over each subinterval where u has the same concavity.

Alternatively, we may use

$$E_\lambda(r) = \frac{\lambda}{2}(u')^2(r) + F\big(u(r)\big) \quad \text{or} \quad E_\Lambda(r) = \frac{\Lambda}{2}(u')^2(r) + F\big(u(r)\big) \tag{2.5}$$

as more appropriate energy-type functions, since they are obviously C^1 functions. In an interval where u'' does not vanish, either $E_\lambda(r)$ or $E_\Lambda(r)$ agrees with $E_\theta(r)$ and is therefore decreasing. However, over the whole interval of definition of u, we cannot easily claim the monotonicity of $E_\lambda(r)$ or $E_\Lambda(r)$, since there are no simple formulas like (2.4), available for the calculation of $E'_\lambda(r)$ or $E'_\Lambda(r)$. The following lemmas, which suffice for our purposes, give some partial results on the monotonicity of $E_\lambda(r)$ and $E_\Lambda(r)$.

Lemma 2.1. *Consider the operator $M^+_{\lambda,\Lambda}(D^2u)$ and let L be an interval in which u is positive.*

(i) *If $u' > 0$ in L, then both $E_\lambda(r)$ and $E_\Lambda(r)$ are decreasing in L.*
(ii) *If $u' < 0$ in L, then $E_\lambda(r)$ decreases when $f(u) > 0$, and $E_\Lambda(r)$ decreases when $f(u) < 0$.*

Proof. (i) Assume $u' > 0$ in L. Consider $E_\lambda(r)$ first. At points where $u'' < 0$, $\theta = \lambda$ and then by (2.4) we get that $E_\lambda(r)$ is decreasing; at points where $u'' > 0$, we have

$$E_\lambda(r) = -\frac{\Lambda - \lambda}{2}(u')^2(r) + E_\theta(r)$$

and by (2.4) we obtain

$$E'_\lambda(r) \leqslant -(\Lambda - \lambda)u'(r)u''(r) < 0.$$

A similar argument yields the same result for $E_\Lambda(r)$.

(ii) Assume $u' < 0$ in L. If $u'' < 0$, then $E_\lambda(r)$ is decreasing by (2.4) again and, if $u'' > 0$ and $f(u) > 0$, then

$$E'_\lambda(r) = \lambda u'u'' + f(u)u' < 0.$$

Finally, if $f(u) < 0$, then (2.3) implies $u'' > 0$, and so $E'_\Lambda(r) < 0$ by (2.4). □

A similar argument establishes the next lemma.

Lemma 2.2. *Consider the operator $M^-_{\lambda,\Lambda}(D^2u)$ and let L be an interval in which u is positive.*

(i) *If $u' < 0$ in L, then both $E_\lambda(r)$ and $E_\Lambda(r)$ are decreasing in L.*
(ii) *If $u' > 0$ in L, then $E_\lambda(r)$ decreases when $f(u) < 0$ and $E_\Lambda(r)$ decreases when $f(u) > 0$.*

Remark 2.1. Our discussion for the three energy-type functions above reveals a rather delicate feature appearing in our study on the Pucci's operators: functions involving both u and u' either have discontinuities somewhere, or their derivatives do not have a universal formula over the whole interval of definition of u. This fact creates major technical difficulties in our further discussion which relies upon the implications of integration by parts.

2.2. Monotonicity

The validity of the next result in the classical Laplacian case can be verified very easily using the universal decreasing property of the energy function $E(r)$. In the current case, the proof is nontrivial and uses Lemmas 2.1 and 2.2 in a delicate way.

Lemma 2.3. *Let $u = u(r)$ be a solution of* (2.3). *If it attains a positive minimum value at some $r_0 \geqslant 0$, then it is positive and bounded in (r_0, ∞), and $\liminf_{r\to\infty} u > 0$.*

Proof. If u is the constant solution, that is, $u \equiv 1$, then the lemma is trivially true. Suppose u is a nonconstant solution of (2.3). Then u' and u'' do not vanish simultaneously at any $r > 0$, as follows from the uniqueness of solutions to this initial value problem. Thus if u takes a local minimum value at $r_0 \geqslant 0$, then $u''(r_0) > 0$, and so by (2.3) it holds that $f(u(r_0)) < 0$ and $u(r_0) < 1$. We shall prove that u is bounded and

$$u(r) > u(r_0) \quad \text{for all } r > r_0. \tag{2.6}$$

We have two cases:

Case 1. $M^+_{\lambda,\Lambda}(D^2u)$. Let (r_0, r_1), $r_0 < r_1 \leqslant \infty$, be the maximal interval in which $u' > 0$. Then for any $r \in (r_0, r_1)$ it follows from Lemma 2.1 that

$$F\big(u(r)\big) < E_\lambda(r) \leqslant E_\lambda(r_0) = F\big(u(r_0)\big) < 0, \tag{2.7}$$

from where we see that $u(r)$ is bounded above by

$$\beta = \big((p+1)/2\big)^{1/(p-1},$$

the positive number making $F(\beta) = 0$.

If $r_1 = \infty$ we are clearly done. If $r_1 < \infty$, then u has a strict maximum at r_1 and by (2.3) we have $f(u(r_1)) > 0$ and $u(r_1) > 1$. Let (r_1, r_2), $r_1 < r_2 \leqslant \infty$, be the maximal interval over which $u' < 0$. We claim that

$$u(r) > u(r_0) \quad \text{for all } r \in (r_1, r_2). \tag{2.8}$$

In fact, if $u(r) \geqslant 1$ in (r_1, r_2), then (2.8) is obviously valid. Otherwise, let $\hat{r}_0 \in (r_0, r_1)$ and $\hat{r}_1 \in (r_1, r_2)$ be the unique numbers such that $u = 1$. By Lemma 2.1 $E_\lambda(r)$ is decreasing on $(\hat{r}_0, \hat{r}_1)$. Hence

$$\frac{(u')^2(\hat{r}_0)}{2} = \frac{E_\lambda(\hat{r}_0) - F(1)}{\lambda} > \frac{E_\lambda(\hat{r}_1) - F(1)}{\lambda} = \frac{(u')^2(\hat{r}_1)}{2}.$$

This, together with the decreasing property of $E_\Lambda(r)$ over $(r_0, \hat{r}_0)$ and $(\hat{r}_1, r)$ for any $r \in (\hat{r}_1, r_2)$ we find that

$$\begin{aligned} F\big(u(r)\big) &\leqslant E_\Lambda(r) < E_\Lambda(\hat{r}_1) = \frac{\Lambda}{2}(u')^2(\hat{r}_1) + F(1) \\ &< \frac{\Lambda}{2}(u')^2(\hat{r}_0) + F(1) = E_\Lambda(\hat{r}_0) < E_\Lambda(r_0) = F\big(u(r_0)\big), \end{aligned}$$

implying (2.8), our claim.

Now if $r_2 = \infty$, then (2.8) implies (2.6) and we are done. If not, then u assumes a local minimum value at r_2 and we can repeat the argument above successively to get $u(r_2) < u(r) < \beta$ for all $r > r_2$. This completes the proof for Case 1.

Case 2. $M^-_{\lambda,\Lambda}(D^2u)$. We shall use the same notation as in Case 1. Without loss of generality, we may only discuss the situation when u has critical points at r_1 and r_2.

Since $E_\lambda(r)$ decreases in $(r_0, \hat{r}_0)$, we have that $E_\lambda(\hat{r}_0) < E_\lambda(r_0) = F(u(r_0)) < 0$, and so $(u')^2(\hat{r}_0) < -2F(1)/\lambda$. Hence

$$F\big(u(r_1)\big) < E_\Lambda(\hat{r}_0) = \frac{\Lambda}{2}(u')^2(\hat{r}_0) + F(1) < \left(1 - \frac{\Lambda}{\lambda}\right)F(1), \tag{2.9}$$

which clearly provides a upper bound for u in $(\hat{r}_0, r_1)$.

It remains to show that (2.8) holds in this case too. It follows from Lemma 2.2 that $E_\Lambda(r)$ is decreasing on $(\hat{r}_0, \hat{r}_1)$, so we obtain $(u')^2(\hat{r}_0) > (u')^2(\hat{r}_1)$ again. The rest of the proof is the same as in Case 1, except one has to replace Λ with λ there. □

Remark 2.2. We observe that, as the ratio $\Lambda/\lambda \to \infty$, the last term in (2.9) and hence the upper bound for u provided by (2.9) tends to ∞ too. This distinguishes from Case 1 where we derive a simple estimate $u < \beta$. For $M^-_{\lambda,\Lambda}(D^2u)$, it remains unclear whether or not all the positive solutions have a universal upper bound independent of λ and Λ. Constructing a sharper estimate in this case is nontrivial as it is likely that the energy function $E_\Lambda(r)$ could become positive in a subinterval of (r_0, r_1) when Λ is sufficiently large.

Lemma 2.4. *If $u = u(r)$ is a positive radial solution of* (1.2), *then $u(0) > 1$ and $u'(r) < 0$ for $r \in (0, R)$.*

Proof. If $u(0) \leqslant 1$, then u attains a positive minimum value at zero, and by Lemma 2.3 we have $u > u(0)$ for all $r > 0$, which is impossible for a positive radial solution of (1.2). Furthermore, if $u(0) > 1$ then u has a strict maximum at $r = 0$, and by Lemma 2.3 again, u can only be decreasing on $(0, R)$. □

Remark 2.3. We may obtain a much stronger result than Lemma 2.3. In fact, it can be proved that if a solution u of (2.3) has a positive minimum then it is oscillatory, with infinitely many positive minima and maxima in $(0,\infty)$. The difficult situation to consider is when u is monotonically approaching 1. In this case the nonlinearity f approach a linear function, so that one can get a contradiction, by slightly modifying the arguments developed in [1, Lemma 3.1], for the eigenvalue problem.

Remark 2.4. Summarizing, we have the following classification of the solutions for the initial value problem (2.3): every solutions belongs to one of the following three classes:

(i) u is a *crossing solution*. Here by a crossing solution we mean that there exists a finite number R such that $u>0$ for $r\in(0,R)$, $u(R)=0$ and $u'(R)<0$.
(ii) u is a *ground state*.
(iii) u is a *positive, oscillatory solution* with infinitely many positive minima and maxima in $(0,\infty)$.

From here we observe that given $0<R\leqslant\infty$ only certain values of $\alpha>0$ give rise to a solution of (1.2). In fact, our purpose is to show that there exists a unique $\alpha(R)$ with this property. In the particular and important case of a ground state and in the range of p where uniqueness holds, it can further been proved that there is α^* such that for all $\alpha\in(0,\alpha^*)$, solutions of (2.3) are positive and oscillatory and if $\alpha\in(\alpha^*,\infty)$ the solutions of (2.3) are crossing.

Remark 2.5. For a ground state, we can further prove that it decays to zero exponentially. For our later purpose, we mention given any $\varepsilon>0$

$$\lim_{r\to\infty} u(r)e^{(1-\varepsilon)r} = \lim_{r\to\infty} u'(r)e^{(1-\varepsilon)r} = 0. \tag{2.10}$$

This conclusion can be reached by using comparison techniques associated to the Laplacian, since there exists r_0 such that $u'(r)<0$ and $u''(r)>0$ for all $r\geqslant r_0$.

3. A useful function

For clarity of our discussion, through the rest of this paper we will only consider the operator $M^+_{\lambda,\Lambda}(D^2u)$, as the main idea applies equally well to both $M^+_{\lambda,\Lambda}$ and $M^-_{\lambda,\Lambda}$. Thus the parameters θ and Θ in equation

$$\theta u'' + \frac{N-1}{r}\Theta u' + f(u) = 0 \tag{3.1}$$

are determined by (2.1) and (2.2). As in [6,9], we introduce the *dimension-like* parameter

$$\widetilde{N} = \frac{\Theta}{\theta}(N-1)+1. \tag{3.2}$$

Using this notation we can write (3.1) as

$$\theta\big(r^{\widetilde{N}-1}u'\big)' = -r^{\widetilde{N}-1}f(u). \tag{3.3}$$

Note carefully that $\widetilde{N}$ is not a fixed constant, but depends on the concavity and monotonicity of u. In the special case $u'u'' > 0$, one simply has $\widetilde{N} = N$. For the other cases, $\widetilde{N}$ can be either larger than or smaller than N.

Now we recall the definition of the functional Q in (1.5), giving precise meaning to all the "constants" appearing in the definition. We shall devote the rest of this section to discussing about this Q function. We will see how useful it is in detecting some fundamental qualitative properties of the solutions of (2.3), and what is the advantage of using this function instead of the homologous form of the well-known Pohozaev identity.

3.1. Sign-retaining property

As we observed in Section 2, functions involving with both u and u' either has discontinuities somewhere, or does not have a universal formula for its derivative with respect to r in the whole interval of definition of u. In particular, it is easy to see that $Q(r)$ has jumps at points where u changes concavity. However, it possesses a crucial property: it does not change signs at these points. We call such a property the *sign-retaining* property.

Lemma 3.1. *The function $Q(r)$ has the sign-retaining property.*

Proof. By (3.1) and (3.2) we first derive

$$-f(u)/\theta = u'' + \frac{\widetilde{N} - 1}{r} u'.$$

Inserting this into (1.5) we obtain

$$Q(r) = \theta r^{\widetilde{N}-1}\bigl(ru'^2 - uu' - ruu''\bigr). \tag{3.4}$$

This formula gives the sign-retaining property immediately. □

3.2. Concavity

Let $u = u(r)$ be a positive radial solution of (1.2). Recall from Lemma 2.4 that $u(0) > 1$ and $u'(r) < 0$ for $r \in (0, R)$. Hence u must be concave down ($u'' < 0$) for r close to zero. On the other hand, by (3.1) it is clear that $u'' > 0$ whenever $f(u) < 0$, that is, $u < 1$. Consequently, u must change concavity at least once in $(0, R)$. Naturally, one may ask how many times u will change its concavity. In fact, to our knowledge, this question was not addressed before, even for the classical Laplacian equation. For the Pucci's operators, this question becomes an important one since the equation parameters θ and Θ depend on the concavity of u.

In the next proposition we show that if $Q > 0$ for all $r \in (0, R)$, then u changes concavity exactly once. Later we will show that Q is indeed positive in some cases.

Proposition 3.1. *Let $u = u(r)$ be a positive radial solution of* (1.2). *If $Q > 0$ for all $r \in (0, R)$, then u changes concavity exactly once.*

Proof. We first verify the identity

$$\frac{d}{dr}\left(\frac{ru'}{u}\right) = -\frac{Q}{\theta r^{\widetilde{N}-1}u^2}. \tag{3.5}$$

Indeed, by (3.3) we have

$$\begin{aligned}
\frac{d}{dr}\left(\frac{ru'}{u}\right)\left(\theta r^{\widetilde{N}-2}u\right)^2 &= \frac{d}{dr}\left(\frac{\theta r^{\widetilde{N}-1}u'}{\theta r^{\widetilde{N}-2}u}\right)\left(\theta r^{\widetilde{N}-2}u\right)^2 \\
&= -\theta r^{2\widetilde{N}-3}uf(u) - (\widetilde{N}-2)\theta^2 r^{2\widetilde{N}-4}uu' - \theta^2 r^{2\widetilde{N}-3}(u')^2 \\
&= -\theta r^{\widetilde{N}-3}\left(r^{\widetilde{N}}uf(u) + (\widetilde{N}-2)\theta r^{\widetilde{N}-1}uu' + \theta r^{\widetilde{N}}(u')^2\right) \\
&= -\theta r^{\widetilde{N}-3}Q.
\end{aligned}$$

Now suppose for contradiction that u has more than one points of inflection, and let $0 < c_1 < c_2$ be the first two of them such that

$$u''(c_1) = u''(c_2) = 0, \qquad u'''(c_1) \geqslant 0, \quad \text{and} \quad u'''(c_2) \leqslant 0.$$

Since u is always concave up as long as $u < 1$, there must hold that

$$u(r) > 1 \quad \text{for } 0 < r < c_2.$$

At $r = c_1$, $u'' = 0$ implies

$$\frac{\widetilde{N}-1}{r}u' = -\frac{f(u)}{\theta}.$$

Hence we have

$$0 \leqslant u'''(c_1) = \frac{\widetilde{N}-1}{r^2}u' - \frac{f'(u)}{\theta}\,u' = -\frac{f(u)}{\theta r} - \frac{f'(u)}{\theta}\,u'$$

and then we get

$$\frac{f(u)}{uf'(u)} + \frac{ru'}{u} \leqslant 0.$$

Similarly, we can show that at $r = c_2$ there holds

$$\frac{f(u)}{uf'(u)} + \frac{ru'}{u} \geqslant 0.$$

On one hand, from our hypothesis we have $Q > 0$ on $(0, R)$, then by (3.5) we see that ru'/u is decreasing. On the other hand, as $f(u) = -u + u^p$, with $p > 1$, we have that $f(u)/(uf'(u))$ is an increasing function of u for $u > 1$ and thus it is a decreasing function of $r \in (c_1, c_2)$. This leads to a contradiction and proves the uniqueness of the points of inflection. □

3.3. The positivity of Q for $1 < p < p_+$

By the existence result of [10] problem (1.2) admits radial solutions. Given $u = u(r)$ a positive radial solution of (1.2) we shall prove that indeed $Q(r) > 0$ for $r \in (0, R)$. This result, together with Proposition 3.1 and (3.5), reveals some further properties of radial solutions. Moreover, as we see in the next section, the positivity of Q is a crucial condition in our study of uniqueness of radial solutions.

Proposition 3.2. *Suppose $1 < p < p_+$. Let $u = u(r)$ be a positive radial solution of* (1.2). *Then for all $r \in (0, R)$, we have $Q > 0$, and consequently,*

(i) *the negative function ru'/u is decreasing, and*
(ii) *u changes concavity exactly once: there is a unique $r_c \in (0, R)$ with $u(r_c) > 1$ such that*

$$u'' < 0 \quad \text{for } 0 < r < r_c, \qquad u'' > 0 \quad \text{for } r_c < r < R.$$

The proof of the positivity of Q in the Laplacian case ($\lambda = \Lambda$) follows from a simple application of the well-known Pohozaev identity, see Tang [23, Lemma 2.2]. Unfortunately, the same approach does not work for the general case ($\lambda < \Lambda$) as the homologous form of the Pohozaev identity is no longer useful as observed in the introduction.

The proof of the positivity of Q constitutes the major technical part of the current work. We will begin with showing that Q must be positive when r is either close to zero or close to R, so we get the positivity at the two ends of the interval $(0, R)$. Next, we show that for some particular choice of p and $\widetilde{N}$, Q does not admit any local minimum value in the "middle" of $(0, R)$. This is of course sufficient to obtain the positivity of Q for the chosen values of p and $\widetilde{N}$. Finally, we show that whenever $1 < p < p_+$, the function Q cannot have a nonnegative minimum value. Using a homotopy type argument we can therefore establish the positivity of Q as claimed in Proposition 3.2.

Lemma 3.2. *Let $u = u(r)$ be a positive radial solution of* (1.2). *Then $Q(r) > 0$ if either r is close to zero, or $u(r) \leqslant 1$.*

Proof. Since $u' < 0$ for all $r \in (0, R)$ and $u'' < 0$ for r close to zero, it readily follows from (3.4) that $Q > 0$ as long as $r > 0$ is small.

Writing

$$Q(r) = r^{\widetilde{N}} u f(u) + \theta r^{\widetilde{N}-1} u' \big(r u' + (\widetilde{N} - 2) u \big),$$

we find

$$\begin{aligned} Q'(r) = {} & \widetilde{N} r^{\widetilde{N}-1} u f(u) + r^{\widetilde{N}} u f'(u) u' + r^{\widetilde{N}} f(u) u' \\ & - r^{\widetilde{N}-1} f(u) \big(r u' + (\widetilde{N} - 2) u \big) + \theta r^{\widetilde{N}-1} u' \big(-r f(u)/\theta \big) \end{aligned}$$

and so

$$Q'(r) = r^{\widetilde{N}} u' \big(u f'(u) - f(u) \big) + 2 r^{\widetilde{N}-1} u f(u). \tag{3.6}$$

Let $r_1 \in (0, R)$ be the unique number where $u = 1$, then for $r_1 < r < R$ we have $0 < u(r) < 1$, and $Q'(r) < 0$ by (3.6). Since $Q(R) = 0$, it follows readily that $Q(r) > 0$ as long as $r_1 < r < R$. □

Lemma 3.3. *Suppose* $p \leqslant \widetilde{N}/(\widetilde{N}-2)$. *Let* $u = u(r)$ *be a positive radial solution of* (1.2)*, then* $Q(r) > 0$ *for* $0 < r < R$.

Proof. In view of Lemma 3.2, it suffices to show that Q does not admit any nonpositive minimum value in $(0, r_1)$, where r_1 was defined in the proof above. We do this by showing that whenever $Q' = 0$ it holds that $Q'' < 0$. Starting with (3.6) we have

$$\begin{aligned} Q''(r) &= \frac{d}{dr}\big((p-1)r^{\widetilde{N}}u^p u' + 2r^{\widetilde{N}-1}uf(u)\big) \\ &= (p-1)r^{\widetilde{N}}u^p u'' + (p-1)\widetilde{N}r^{\widetilde{N}-1}u^p u' + p(p-1)r^{\widetilde{N}}u^{p-1}(u')^2 \\ &\quad + 2r^{\widetilde{N}-1}f(u)u' + 2r^{\widetilde{N}-1}uf'(u)u' + 2(\widetilde{N}-1)r^{\widetilde{N}-2}uf(u). \end{aligned}$$

At a critical point of Q, we use $Q' = 0$ to reduce Q'' to

$$\begin{aligned} Q''(r) &= (p-1)r^{\widetilde{N}}u^p u'' + r^{\widetilde{N}-1}u'\big[(p-1)\widetilde{N}u^p - 2pf(u) + 2f(u) \\ &\quad + 2uf'(u) - (\widetilde{N}-1)(p-1)u^p\big] \\ &= (p-1)r^{\widetilde{N}}u^p u'' + r^{\widetilde{N}-1}u'\big[(p+1)u^p + 2(p-2)u\big]. \end{aligned}$$

Now if $p \leqslant \widetilde{N}/(\widetilde{N}-2)$, then $p+1 \geqslant (p-1)(\widetilde{N}-1)$ and

$$(p+1)u^p + 2(p-2)u > (p-1)(\widetilde{N}-1)u^p,$$

yielding

$$Q'' < (p-1)r^{\widetilde{N}}u^p\left[u'' + \frac{\widetilde{N}-1}{r}u'\right] = -(p-1)r^{\widetilde{N}}u^p f(u)/\theta < 0$$

for $r \in (0, r_1)$. The proof is completed. □

Lemma 3.4. *Let* $u = u(r)$ *be a positive radial solution of* (1.2). *Then* $Q(r)$ *does not have any nonnegative minimum value in* $(0, R)$.

Proof. Defining

$$Q_1(r) = \frac{ru'(r)}{u(r)} + \frac{2f(u)}{uf'(u) - f(u)} \tag{3.7}$$

we can rewrite $Q'(r)$ as

$$Q'(r) = r^{\widetilde{N}-1}\big(uf'(u) - f(u)\big)uQ_1(r).$$

Differentiating $Q_1(r)$ we obtain

$$Q_1'(r) = -\frac{Q}{\theta r^{\widetilde{N}-1} u^2} + 2u^{-p}u'. \tag{3.8}$$

Suppose, for contradiction, that Q has a nonnegative minimum value at $r_0 \in (0, R)$. Then there exists a set $I \subset R$ having r_0 as accumulation point and $Q_1(r) \leqslant 0$ for $r \in I$ and $r < r_0$, and $Q_1(r) \geqslant 0$ for $r \in I$ and $r > r_0$. It follows then that $Q_1'(r_0) \geqslant 0$. On the other hand, from (3.8) and (3.5) we have $Q_1'(r_0) < 0$ as $Q(r_0) \geqslant 0$ and $u'(r_0) < 0$ yielding a contradiction. □

Proof of Proposition 3.2. For a given $p \in (1, p_+)$, if $p \leqslant \widetilde{N}/(\widetilde{N}-2)$, then Lemma 3.3 implies $Q(r) > 0$ for $0 < r < R$ as needed.

Assume $\widetilde{N}/(\widetilde{N}-2) \leqslant p < p_+$. If $Q(r)$ is zero or negative somewhere in the interval $(0, R)$, then Lemmas 3.2 and 3.4 imply that Q must assume a *negative* minimum value in the interval $(0, R)$.

Recall from the existence result of [10] that for each $p \in (1, p_+)$ there exists a positive radial solution of (1.2). Moreover, in the proof of existence in [10], we may think of p as a parameter and use homotopy properties of the degree to prove that given $1 < p_1 < p_2 < p_+$, the set of solutions of (1.2) contains a connected subset having inside a solution for p_1 and a solution for p_2. Thus we can use a continuity argument to find a number $\bar{p} \in [\widetilde{N}/(\widetilde{N}-2), p)$ such that the corresponding Q function possesses a nonnegative minimum value in $(0, R)$. This gives a contradiction to Lemma 3.4. Hence $Q > 0$ for all $r \in (0, R)$. We just recall that $Q > 0$ if $u(r) \leqslant 1$, so that the case $R = \infty$ is also well covered. □

4. Uniqueness

The uniqueness in Theorem 1.1 follows if we can show that there is at most one $\alpha > 0$ such that the solution of (2.3) satisfies, for $R < \infty$,

$$u(r) > 0 \quad \text{for } r \in (0, R) \quad \text{and} \quad u(R) = 0, \tag{4.1}$$

and for $R = \infty$,

$$u(r) > 0 \quad \text{for } r > 0 \quad \text{and} \quad \lim_{r \to \infty} u(r) = 0. \tag{4.2}$$

In either case we have $\alpha > 1$ and $u'(r) < 0$ for $r \in (0, R)$ as a consequence of Lemma 2.3, and in the first case $u'(R) < 0$ by the uniqueness theorem to the initial value problem (2.3). Therefore, whenever the case (4.1) occurs, the crossing number R is a C^1 function of α.

Denote the solution u by $u(r, \alpha)$ to emphasize its dependence on α. Differentiating the identity $u(R(\alpha), \alpha) = 0$ with respect to α, and denoting by

$$v(r, \alpha) = \frac{\partial u(r, \alpha)}{\partial \alpha},$$

the variation of u, we obtain

$$R'(\alpha) = -v(R, \alpha)/u'(R, \alpha).$$

(Here the prime in u' is indeed the *partial* derivative $\partial u/\partial r$.) Hence the sign of $v(R,\alpha)$ determines the monotonicity of R. As in the Laplacian case, to complete the proof of our uniqueness result it is sufficient to prove the following lemma.

Lemma 4.1. *There is a* $\tau \in (0, R)$ *such that*

$$v > 0 \quad \text{in } (0,\tau), \qquad v(\tau) = 0 \quad \text{and} \quad v < 0 \quad \text{in } (\tau, R). \tag{4.3}$$

Moreover, $v(R) < 0$ *if* $R < \infty$ *and* $\lim_{r\to\infty} v(r) = -\infty$ *if* $R = \infty$.

In fact, if this lemma is established, then $R'(\alpha) < 0$ whenever $R(\alpha) < \infty$. This implies that for all $\tilde{\alpha} > \alpha$, $R(\tilde{\alpha})$ is defined and $R(\tilde{\alpha}) < R(\alpha)$, yielding the uniqueness in the finite ball immediately. That this lemma implies the uniqueness of ground states is not so obvious. We notice that for all r large we have $u' < 0$ and $u'' > 0$, so that there is no more change in θ and Θ. Thus we may apply the classical arguments used for the Laplacian to conclude. See, for example, the paper by Peletier and Serrin [19] or the paper by Kwong [11].

Remark 4.1. By using the arguments in [11] one can prove that the unique solution u is nondegenerate in the sense that the "linearized" equation

$$\theta h'' + \frac{N-1}{r}\Theta h' + f'(u)h = 0, \qquad h(R) = 0, \quad h'(0) = 0. \tag{4.4}$$

has only the trivial solution.

The rest of this section is therefore devoted to the proof of Lemma 4.1. The basic strategy here is to modify the approaches recently developed by Tang in [23] for the Laplacian case, which largely simplified the technicalities of previous works in the study of the uniqueness problem for the semi-linear elliptic equations involving the Laplace operators. We notice that the approach here is not simply a trivial generalization of the work of [23] as new technical complexities arise due to the discontinuities of various functions. In the first part that follows, we shall generalize some key functions and identities from [23], which will be used in the second part to complete the proof of Lemma 4.1.

4.1. Several functionals

We start with a differentiation of (2.3) with respect to α to get

$$\theta v'' + \frac{N-1}{r}\Theta v' + f'(u)v = 0, \qquad v(0) = 1, \quad v'(0) = 0. \tag{4.5}$$

Similar to (3.3) we can write (4.5) as

$$\theta\big(r^{\widetilde{N}-1}v'\big)' = -r^{\widetilde{N}-1}f'(u)v. \tag{4.6}$$

Let

$$\xi(r) = \theta r^{\widetilde{N}-1}(u'v - uv')$$

denote the Wronskian of u and v. Using (3.3) and (4.6) we obtain

$$\xi'(r) = r^{\widetilde{N}-1}\big[uf'(u) - f(u)\big]v. \tag{4.7}$$

Then we introduce the following function

$$\delta(r) = r^{\widetilde{N}}\big[\theta u'v' + f(u)v\big] + (\widetilde{N}-2)\theta r^{\widetilde{N}-1}u'v. \tag{4.8}$$

To find the derivative of $\delta(r)$, we first compute

$$\theta\big[rv' + (\widetilde{N}-2)v\big]' = -rf'(u)v.$$

Using this and (3.3) we obtain

$$\begin{aligned}\delta'(r) &= \big[\theta r^{\widetilde{N}-1}u'\big(rv' + (\widetilde{N}-2)v\big) + r^{\widetilde{N}}f(u)v\big]'\\ &= -r^{\widetilde{N}-1}f(u)\big(rv' + (\widetilde{N}-2)v\big) - r^{\widetilde{N}}f'(u)u'v\\ &\quad + \widetilde{N}r^{\widetilde{N}-1}f(u)v + r^{\widetilde{N}}f'(u)u'v + r^{\widetilde{N}}f(u)v',\end{aligned}$$

and simplifying

$$\delta'(r) = 2r^{\widetilde{N}-1}f(u)v. \tag{4.9}$$

Finally, we introduce the functions T and g

$$T(r) = g(u)\xi(r) - \delta(r), \quad g(u) = \frac{2f(u)}{uf'(u) - f(u)}. \tag{4.10}$$

As in [23], (4.7) and (4.9) we obtain the useful identity

$$T'(r) = g'(u)u'(r)\xi(r), \tag{4.11}$$

which can be further simplified, using that $f(u) = -u + u^p$, to obtain

$$T'(r) = 2u^{-p}u'(r)\xi(r). \tag{4.12}$$

4.2. Proof of Lemma 4.1

We first prove that v must vanish somewhere in $(0, R)$. Suppose this is not true, then v remains positive in $(0, R)$. Thus ξ increases over all subintervals where $u'' \neq 0$ (notice that as $u' < 0$ in $(0, R)$, we have $\Theta = \lambda$). Denote by r_1 the number such that $u(r_1) = 1$. Then $u'' > 0$ in (r_1, R), in which ξ is C^1 and increasing. It follows by the sign-retaining property of ξ that $\xi(R) > 0$, or $\lim_{r\to\infty}\xi(r) > 0$ when $R = \infty$. But the evaluation of $\xi(R)$ by the definition of ξ gives $\xi(R) \leqslant 0$. In case $R = \infty$ we have $\lim_{r\to\infty}\xi(r) = 0$ when $R = \infty$, where we use the exponential decay of u. In both cases we reach a contradiction.

Denote the first zero of v by τ. We next prove that v must stay negative in the remaining interval (τ, R). We prove this again by contradiction. Suppose this is not true then there is a number $\tilde{\tau} \in (\tau, R)$ such that $v(\tilde{\tau}) = 0$ and $v < 0$ in $(\tau, \tilde{\tau})$. Then, using the definition of ξ (4.7) we find that $\xi(\tau) < 0$ and $\xi(\tilde{\tau}) < 0$. Then, there must be a number $t \in (\tau, \tilde{\tau})$ such that

$$\xi(t) = 0 \quad \text{and} \quad \xi(r) > 0 \quad \text{for } r \in (0, t). \tag{4.13}$$

By (4.12) we see that $T(r)$ is decreasing on the subintervals of $(0, t)$ in which $u'' \neq 0$. To continue, using (4.7), (4.8) and (4.10), we rewrite $T(r)$ as

$$\begin{aligned} T(r) &= \theta g(u) r^{\widetilde{N}-1}(u'v - uv') - r^{\widetilde{N}}\bigl[\theta u'v' + f(u)v\bigr] - (\widetilde{N} - 2)\theta r^{\widetilde{N}-1} u'v \\ &= \theta g(u) r^{\widetilde{N}-1}(u'v - uv') - \theta r^{\widetilde{N}} u'v' + \theta r^{\widetilde{N}} u''v + \theta r^{\widetilde{N}-1} u'v \\ &= \theta r^{\widetilde{N}-1}\bigl[g(u)(u'v - uv') - ru'v' + u'v\bigr] + \theta r^{\widetilde{N}} u''v, \end{aligned}$$

where we used the substitution

$$rf(u) = -\theta\bigl(ru'' + (\widetilde{N} - 1)u'\bigr),$$

which follows from (2.3) and (3.2). It is therefore clear that T also has the sign-retaining property, since $\theta u''$ is continuous where u'' vanishes. This, together with (4.10) and (4.13), allows us to conclude

$$\delta(t) = -T(t) > 0.$$

A further calculation using (4.8) and (4.13) yields

$$\begin{aligned} \delta(t) &= t^{\widetilde{N}}\bigl[\theta u'v' + f(u)v\bigr] + (\widetilde{N} - 2)\theta t^{\widetilde{N}-1} u'v \\ &= \bigl[t^{\widetilde{N}}\bigl(\theta u'v'u/v + uf(u)\bigr) + (\widetilde{N} - 2)\theta t^{\widetilde{N}-1} uu'\bigr]v/u \\ &= \bigl[t^{\widetilde{N}}\bigl(\theta u'^2 + uf(u)\bigr) + (\widetilde{N} - 2)\theta t^{\widetilde{N}-1} uu'\bigr]v/u = Q(t)v(t)/u(t). \end{aligned}$$

Since $v(t)/u(t) < 0$, we obtain $Q(t) < 0$, which contradicts Proposition 3.2 proving (4.3).

To reach the remaining conclusion of Lemma 4.1, we first establish

$$\lim_{r \to R} \delta(r) > 0. \tag{4.14}$$

If $\tau > r_1$, then $v < 0$ and $f(u) < 0$ in (τ, R), recalling that $u(r_1) = 1$. Hence $u'' > 0$ and $\delta(r)$ is C^1 and increasing in (τ, R), yielding

$$\lim_{r \to R} \delta(r) > \delta(\tau) = \theta \tau^{\widetilde{N}} u'(\tau)v'(\tau) > 0$$

as desired. If $\tau \leqslant r_1$, then the same argument shows that $\delta(r)$ is C^1 and increasing over (r_1, R). Moreover, observing that the argument in last paragraph shows that $\xi(r) > 0$ for all $r \in (0, R)$, by (4.12) and the sign-retaining property of T we conclude that $T < 0$ in $(0, R)$, giving in particular

$$\delta(r_1) = -T(r_1) > 0.$$

This verifies (4.14) in the second case.

Now, if $R < \infty$ then $\delta(R) > 0$, which is incompatible with $v(R) = 0$, implying that $v(R) < 0$. For the case $R = \infty$, we first notice that as $r \to \infty$, either $v(r) \to -\infty$ or $v(r) \to 0$; see McLeod [16, Lemma 2(b)], recalling that the asymptotic behavior here is the same as in the Laplacian case, since eventually $u'' > 0$. If $v(r) \to -\infty$ occurs, then $\lim_{r\to\infty} \delta(r) = 0$, as follows from (4.8) and (2.10), providing an obvious contradiction to (4.14). Thus $\lim_{r\to\infty} v(r) = -\infty$, and the proof is completed.

Remark 4.2. In the case of grounds states, our theorem assures that for every $1 < p < p_+$, there is a number α_* separating the range of α in crossing and positive oscillating solutions for (2.3). The main open question left in this paper is the analysis of the solutions beyond p_+. We believe that the function Q defined here contains crucial information about this question.

On the other hand, we recall that there is not formula known for p_+. In the search for one we may also obtain valuable information about the question open here.

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