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Calculus of Variations

Weighted Hardy inequality with higher dimensional singularity on the boundary

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Abstract Let Ω be a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 3$ and let Σ_k be a closed smooth submanifold of $\partial\Omega$ of dimension $1 \leq k \leq N-2$. In this paper we study the weighted Hardy inequality with weight function singular on Σ_k. In particular we provide necessary and sufficient conditions for existence of minimizers.

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1 Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^N$, $N \geq 2$ and let Σ_k be a smooth closed submanifold of $\partial\Omega$ with dimension $0 \leq k \leq N-1$. Here Σ_0 is a single point and $\Sigma_{N-1} = \partial\Omega$. For $\lambda \in \mathbb{R}$, consider the problem of finding minimizers for the quotient:

$$\mu_\lambda(\Omega, \Sigma_k) := \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 p dx - \lambda \int_\Omega \delta^{-2}|u|^2 \eta dx}{\int_\Omega \delta^{-2}|u|^2 q dx}, \tag{1}$$

where $\delta(x) := \text{dist}(x, \Sigma_k)$ is the distance function to Σ_k and where the weights p, q and η satisfy

$$p, q \in C^2(\overline{\Omega}), \qquad p, q > 0 \quad \text{in } \overline{\Omega}, \qquad \eta > 0 \quad \text{in } \overline{\Omega} \setminus \Sigma_k, \qquad \eta \in Lip(\overline{\Omega}) \tag{2}$$

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and

$$\max_{\Sigma_k} \frac{q}{p} = 1, \qquad \eta = 0 \qquad \text{on } \Sigma_k. \tag{3}$$

We put

$$I_k = \int_{\Sigma_k} \frac{d\sigma}{\sqrt{1 - (q(\sigma)/p(\sigma))}}, \quad 1 \le k \le N-1 \quad \text{and} \quad I_0 = \infty. \tag{4}$$

It was shown by Brezis and Marcus [4] that there exists λ^* such that if $\lambda > \lambda^*$ then $\mu_\lambda(\Omega, \Sigma_{N-1}) < \frac{1}{4}$ and it is attained while for $\lambda \le \lambda^*$, $\mu_\lambda(\Omega, \Sigma_{N-1}) = \frac{1}{4}$ and it is not achieved for every $\lambda < \lambda^*$. The critical case $\lambda = \lambda^*$ was studied by Brezis, Marcus and Shafrir [5], where they proved that $\mu_{\lambda^*}(\Omega, \Sigma_{N-1})$ admits a minimizer if and only if $I_{N-1} < \infty$. The case where $k = 0$ (Σ_0 is reduced to a point on the boundary) was treated by the first author in [11] and the same conclusions hold true.
Here we obtain the following

Theorem 1.1 *Let Ω be a smooth bounded domain of $\mathbb{R}^N$, $N \ge 3$ and let $\Sigma_k \subset \partial\Omega$ be a closed submanifold of dimension $k \in [1, N-2]$. Assume that the weight functions p, q and η satisfy (2) and (3). Then, there exists $\lambda^* = \lambda^*(p, q, \eta, \Omega, \Sigma_k)$ such that*

$$\mu_\lambda(\Omega, \Sigma_k) = \frac{(N-k)^2}{4}, \quad \forall \lambda \le \lambda^*,$$
$$\mu_\lambda(\Omega, \Sigma_k) < \frac{(N-k)^2}{4}, \quad \forall \lambda > \lambda^*.$$

The infimum $\mu_\lambda(\Omega, \Sigma_k)$ is attained if $\lambda > \lambda^*$ and it is not attained when $\lambda < \lambda^*$.

Concerning the critical case we get

Theorem 1.2 *Let λ^* be given by Theorem 1.1 and consider I_k defined in (4). Then $\mu_{\lambda^*}(\Omega, \Sigma_k)$ is achieved if and only if $I_k < \infty$.*

By choosing $p = q \equiv 1$ and $\eta = \delta^2$, we obtain the following consequence of the above theorems.

Corollary 1.3 *Let Ω be a smooth bounded domain of $\mathbb{R}^N$, $N \ge 3$ and $\Sigma_k \subset \partial\Omega$ be a closed submanifold of dimension $k \in \{1, \dots, N-2\}$. For $\lambda \in \mathbb{R}$, put*

$$\nu_\lambda(\Omega, \Sigma_k) = \inf_{u \in H^1_0(\Omega)} \frac{\int_\Omega |\nabla u|^2 \, dx - \lambda \int_\Omega |u|^2 \, dx}{\int_\Omega \delta^{-2} |u|^2 \, dx}.$$

Then, there exists $\bar{\lambda} = \bar{\lambda}(\Omega, \Sigma_k)$ such that

$$\nu_\lambda(\Omega, \Sigma_k) = \frac{(N-k)^2}{4}, \quad \forall \lambda \le \bar{\lambda},$$
$$\nu_\lambda(\Omega, \Sigma_k) < \frac{(N-k)^2}{4}, \quad \forall \lambda > \bar{\lambda}.$$

Moreover $\nu_\lambda(\Omega, \Sigma_k)$ is attained if and only if $\lambda > \bar{\lambda}$.

The proof of the above theorems are mainly based on the construction of appropriate sharp H^1-subsolution and H^1-supersolutions for the corresponding operator

$$\mathcal{L}_\lambda := -\Delta - \frac{(N-k)^2}{4} q\delta^{-2} + \lambda\delta^{-2}\eta$$

(with $p \equiv 1$). These super and sub-solutions are perturbations of an approximate "virtual" ground-state for the Hardy constant $\frac{(N-k)^2}{4}$ near Σ_k. For that we will consider the *projection distance* function $\tilde{\delta}$ defined near Σ_k as

$$\tilde{\delta}(x) := \sqrt{|\mathrm{dist}^{\partial\Omega}(\overline{x}, \Sigma_k)|^2 + |x - \overline{x}|^2},$$

where $\overline{x}$ is the orthogonal projection of x on $\partial\Omega$ and $\mathrm{dist}^{\partial\Omega}(\cdot, \Sigma_k)$ is the geodesic distance to Σ_k on $\partial\Omega$ endowed with the induced metric. While the distances δ and $\tilde{\delta}$ are equivalent, $\Delta\delta$ and $\Delta\tilde{\delta}$ differ and δ does not, in general, provide the right approximate solution for $k \le N-2$. Letting $d_{\partial\Omega} = \mathrm{dist}(\cdot, \partial\Omega)$, we have

$$\tilde{\delta}(x) := \sqrt{|\mathrm{dist}^{\partial\Omega}(\overline{x}, \Sigma_k)|^2 + d_{\partial\Omega}(x)^2}.$$

Our approximate virtual ground-state near Σ_k reads then as

$$x \mapsto d_{\partial\Omega}(x)\,\tilde{\delta}^{\frac{k-N}{2}}(x). \tag{5}$$

In some appropriate Fermi coordinates $y = (y^1, y^2, \ldots, y^{N-k}, y^{N-k+1}, \ldots, y^N) = (\tilde{y}, \bar{y}) \in \mathbb{R}^N$ with $\tilde{y} = (y^1, y^2, \ldots, y^{N-k}) \in \mathbb{R}^{N-k}$ and $\bar{y} = (y^{N-k+1}, \ldots, y^N)$ (see next section for a precise definition), the function in (5) then becomes

$$y \mapsto y^1 |\tilde{y}|^{\frac{k-N}{2}}$$

which is the "virtual" ground-state for the Hardy constant $\frac{(N-k)^2}{4}$ in the flat case $\Sigma_k = \mathbb{R}^k$ and $\Omega = \mathbb{R}^N$. We refer to Sect. 2 for more details about the constructions of the super and sub-solutions.

The proof of the existence part in Theorem 1.2 is inspired from [5]. It amounts to obtain a uniform control of a specific minimizing sequence for $\mu_{\lambda^*}(\Omega, \Sigma_k)$ near Σ_k via the H^1-super-solution constructed.

We recall that the existence and non-existence of extremals for (1) and related problems were studied in [1,6–9,12–14,16,19–21] and some references therein. We would like to mention that some of the results in this paper can be useful in the study of semilinear equations with a Hardy potential singular at a submanifold of the boundary. We refer to [2,3,10], where existence and nonexistence for semilinear problems were studied via the method of super/sub-solutions.

2 Preliminaries and notations

In this section we collect some notations and conventions we are going to use throughout the paper.

Let $\mathcal{U}$ be an open subset of $\mathbb{R}^N$, $N \ge 3$, whose boundary $\mathcal{M} := \partial\mathcal{U}$ is a smooth closed hypersurface of $\mathbb{R}^N$. Assume that $\mathcal{M}$ contains a smooth closed submanifold Σ_k of dimension $1 \le k \le N-2$. In the following, for $x \in \mathbb{R}^N$, we let $d(x)$ be the distance function of $\mathcal{M}$ and $\delta(x)$ the distance function of Σ_k. We denote by $N_{\mathcal{M}}$ the unit normal vector field of $\mathcal{M}$ pointed into $\mathcal{U}$.

Given $P \in \Sigma_k$, the tangent space $T_P\mathcal{M}$ of $\mathcal{M}$ at P splits naturally as

$$T_P\mathcal{M} = T_P\Sigma_k \oplus N_P\Sigma_k,$$

where $T_P\Sigma_k$ is the tangent space of Σ_k and $N_P\Sigma_k$ stands for the normal space of $T_P\Sigma_k$ at P. We assume that these subspaces are spanned respectively by $\big(E_a\big)_{a=N-k+1,\ldots,N}$ and $\big(E_i\big)_{i=2,\ldots,N-k}$. We will assume that $N_\mathcal{M}(P) = E_1$.

A neighborhood of P in Σ_k can be parameterized via the map

$$\bar{y} \mapsto f^P(\bar{y}) = \mathrm{Exp}_P^{\Sigma_k}\left(\sum_{a=N-k+1}^{N} y^a E_a\right),$$

where, $\bar{y} = (y^{N-k+1}, \ldots, y^N)$ and where $\mathrm{Exp}_P^{\Sigma_k}$ is the exponential map at P in Σ_k endowed with the metric induced by $\mathcal{M}$. Next we extend $(E_i)_{i=2,\ldots,N-k}$ to an orthonormal frame $(X_i)_{i=2,\ldots,N-k}$ in a neighborhood of P. We can therefore define the parameterization of a neighborhood of P in $\mathcal{M}$ via the mapping

$$(\breve{y}, \bar{y}) \mapsto h^P_\mathcal{M}(\breve{y}, \bar{y}) := \mathrm{Exp}^\mathcal{M}_{f^P(\bar{y})}\left(\sum_{i=2}^{N-k} y^i X_i\right),$$

with $\breve{y} = (y^2, \ldots, y^{N-k})$ and $\mathrm{Exp}^\mathcal{M}_Q$ is the exponential map at Q in $\mathcal{M}$ endowed with the metric induced by $\mathbb{R}^N$. We now have a parameterization of a neighborhood of P in $\mathbb{R}^N$ defined via the above Fermi coordinates by the map

$$y = (y^1, \breve{y}, \bar{y}) \mapsto F^P_\mathcal{M}(y^1, \breve{y}, \bar{y}) = h^P_\mathcal{M}(\breve{y}, \bar{y}) + y^1 N_\mathcal{M}(h^P_\mathcal{M}(\breve{y}, \bar{y})).$$

Next we denote by g the metric induced by $F^P_\mathcal{M}$ whose components are defined by

$$g_{\alpha\beta}(y) = \langle \partial_\alpha F^P_\mathcal{M}(y), \partial_\beta F^P_\mathcal{M}(y)\rangle.$$

Then we have the following expansions (see for instance [15])

$$\begin{aligned} &g_{11}(y) = 1 \\ &g_{1\beta}(y) = 0, \qquad\qquad \text{for } \beta = 2, \ldots, N \\ &g_{\alpha\beta}(y) = \delta_{\alpha\beta} + \mathcal{O}(|\tilde{y}|), \qquad \text{for } \alpha, \beta = 2, \ldots, N, \end{aligned} \tag{6}$$

where $\tilde{y} = (y^1, \breve{y})$ and $\mathcal{O}(r^m)$ is a smooth function in the variable y which is uniformly bounded by a constant (depending only $\mathcal{M}$ and Σ_k) times r^m.

In concordance to the above coordinates, we will consider the "half"-geodesic neighborhood contained in $\mathcal{U}$ around Σ_k of radius ρ

$$\mathcal{U}_\rho(\Sigma_k) := \{x \in \mathcal{U} : \quad \tilde{\delta}(x) < \rho\}, \tag{7}$$

where $\tilde{\delta}$ is the projection distance function given by

$$\tilde{\delta}(x) := \sqrt{|\mathrm{dist}^\mathcal{M}(\overline{x}, \Sigma_k)|^2 + |x - \overline{x}|^2},$$

where $\overline{x}$ is the orthogonal projection of x on $\mathcal{M}$ and $\mathrm{dist}^\mathcal{M}(\cdot, \Sigma_k)$ is the geodesic distance to Σ_k on $\mathcal{M}$ with the induced metric. Observe that

$$\tilde{\delta}(F^P_\mathcal{M}(y)) = |\tilde{y}|, \tag{8}$$

where $\tilde{y} = (y^1, \breve{y})$. We also define $\sigma(\overline{x})$ to be the orthogonal projection of $\overline{x}$ on Σ_k within $\mathcal{M}$. Letting

$$\hat{\delta}(\overline{x}) := \mathrm{dist}^{\mathcal{M}}(\overline{x}, \Sigma_k),$$

one has

$$\overline{x} = \mathrm{Exp}^{\mathcal{M}}_{\sigma(\overline{x})}(\hat{\delta}\,\nabla\hat{\delta}) \quad \text{or equivalently} \quad \sigma(\overline{x}) = \mathrm{Exp}^{\mathcal{M}}_{\overline{x}}(-\hat{\delta}\,\nabla\hat{\delta}).$$

Next we observe that

$$\tilde{\delta}(x) = \sqrt{\hat{\delta}^2(\bar{x}) + d^2(x)}. \tag{9}$$

In addition it can be easily checked via the implicit function theorem that there exists a positive constant $\beta_0 = \beta_0(\Sigma_k, \Omega)$ such that $\tilde{\delta} \in C^\infty(\mathcal{U}_{\beta_0}(\Sigma_k))$.

It is clear that for ρ sufficiently small, there exists a finite number of Lipschitz open sets $(T_i)_{1 \le i \le N_0}$ such that

$$T_i \cap T_j = \emptyset \quad \text{for } i \neq j \quad \text{and} \quad \mathcal{U}_\rho(\Sigma_k) = \bigcup_{i=1}^{N_0} \overline{T_i}.$$

We may assume that each T_i is chosen, using the above coordinates, so that

$$T_i = F^{p_i}_{\mathcal{M}}(B^{N-k}_+(0, \rho) \times D_i) \quad \text{with} \quad p_i \in \Sigma_k,$$

where the D_i's are Lipschitz disjoint open sets of $\mathbb{R}^k$ such that

$$\bigcup_{i=1}^{N_0} \overline{f^{p_i}(D_i)} = \Sigma_k.$$

In the above setting we have

Lemma 2.1 *As $\tilde{\delta} \to 0$, the following expansions hold*

(1) $\delta^2 = \tilde{\delta}^2(1 + O(\tilde{\delta}))$,
(2) $\nabla\tilde{\delta} \cdot \nabla d = \frac{d}{\tilde{\delta}}$,
(3) $|\nabla\tilde{\delta}| = 1 + O(\tilde{\delta})$,
(4) $\Delta\tilde{\delta} = \frac{N-k-1}{\tilde{\delta}} + O(1)$,

where $O(r^m)$ is a function for which there exists a constant $C = C(\mathcal{M}, \Sigma_k)$ such that

$$|O(r^m)| \le C r^m.$$

Proof (1) Let $P \in \Sigma_k$. With an abuse of notation, we write $x(y) = F^P_{\mathcal{M}}(y)$ and we set

$$\vartheta(y) := \frac{1}{2}\delta^2(x(y)).$$

The function ϑ is smooth in a small neighborhood of the origin in $\mathbb{R}^N$ and a Taylor expansion yields

$$\begin{aligned} \vartheta(y) &= \vartheta(0, \bar{y})\tilde{y} + \nabla\vartheta(0, \bar{y})[\tilde{y}] + \frac{1}{2}\nabla^2\vartheta(0, \bar{y})[\tilde{y}, \tilde{y}] + \mathcal{O}(\|\tilde{y}\|^3) \\ &= \frac{1}{2}\nabla^2\vartheta(0, \bar{y})[\tilde{y}, \tilde{y}] + \mathcal{O}(\|\tilde{y}\|^3). \end{aligned} \tag{10}$$

Here we have used the fact that $x(0,\bar{y}) \in \Sigma_k$ so that $\delta(x(0,\bar{y})) = 0$. We write

$$\nabla^2 \vartheta(0,\bar{y})[\tilde{y},\tilde{y}] = \sum_{i,l=1}^{N-k} \Lambda_{il} y^i y^l,$$

with

$$\begin{aligned}\Lambda_{il} &:= \frac{\partial^2 \vartheta}{\partial y^i \partial y^l} /_{\tilde{y}=0} \\ &= \frac{\partial}{\partial y^l}\left(\frac{\partial}{\partial x^j}\left(\frac{1}{2}\delta^2(x)\frac{\partial x^j}{\partial y^i}\right)\right)/_{\tilde{y}=0} \\ &= \frac{\partial^2}{\partial x^i \partial x^s}\left(\frac{1}{2}\delta^2\right)(x)\frac{\partial x^j}{\partial y^i}\frac{\partial x^s}{\partial y^l}/_{\tilde{y}=0} + \frac{\partial}{\partial x^j}(\delta^2)(x)\frac{\partial^2 x^s}{\partial y^i \partial y^l}/_{\tilde{y}=0}.\end{aligned}$$

Now using the fact that

$$\frac{\partial x^s}{\partial y^l}/_{\tilde{y}=0} = g_{ls} = \delta_{ls} \quad \text{and} \quad \frac{\partial}{\partial x^j}(\delta^2)(x)/_{\tilde{y}=0} = 0,$$

we obtain

$$\begin{aligned}\Lambda_{il} y^i y^l &= y^i y^s \frac{\partial^2}{\partial x^i \partial x^s}\left(\frac{1}{2}\delta^2\right)(x)/_{\tilde{y}=0} \\ &= |\tilde{y}|^2,\end{aligned}$$

where we have used the fact that the matrix $\left(\frac{\partial^2}{\partial x^i \partial x^s}(\frac{1}{2}\delta^2)(x)/_{\tilde{y}=0}\right)_{1\le i,s\le N}$ is the matrix of the orthogonal projection onto the normal space of $T_{f^P(\bar{y})}\Sigma_k$. Hence using (10), we get

$$\delta^2(x(y)) = |\tilde{y}|^2 + \mathcal{O}(|\tilde{y}|^3).$$

This together with (8) prove the first expansion.

(2) Thanks to (8) and (6), we infer that

$$\nabla\tilde{\delta}\cdot\nabla d(x(y)) = \frac{\partial\tilde{\delta}(x(y))}{\partial y^1} = \frac{y^1}{|\tilde{y}|} = \frac{d(x(y))}{\tilde{\delta}(x(y))}$$

as desired.

(3) We observe that

$$\frac{\partial\tilde{\delta}}{\partial x^\tau}\frac{\partial\tilde{\delta}}{\partial x^\tau}(x(y)) = g^{\tau\alpha}(y)g^{\tau\beta}(y)\frac{\partial\tilde{\delta}(x(y))}{\partial y^\alpha}\frac{\partial\tilde{\delta}(x(y))}{\partial y^\beta},$$

where $g^{\alpha\beta}$ are the entries of the inverse of the matrix $(g_{\alpha\beta})_{\alpha,\beta=1,\dots,N}$. Therefore using again (6) and (8), we get the desired result.

(4) Finally using the expansion of the Laplace-Beltrami operator Δ_g, see Lemma 3.3 in [18], applied to (8), we get the last estimate. □

In the following of – only – this section, the function $q : \overline{\mathcal{U}} \to \mathbb{R}$ will be such that

$$q \in C^2(\overline{\mathcal{U}}), \quad \text{and} \quad q \le 1 \quad \text{on } \Sigma_k. \tag{11}$$

Let $M, a \in \mathbb{R}$, we consider the function

$$W_{a,M,q}(x) = X_a(\tilde{\delta}(x))\, e^{Md(x)}\, d(x)\, \tilde{\delta}(x)^{\alpha(x)}, \tag{12}$$

where

$$X_a(t) = (-\log(t))^a \quad 0 < t < 1$$

and

$$\alpha(x) = \frac{k-N}{2} + \frac{N-k}{2}\sqrt{1 - q(\sigma(\bar{x})) + \tilde{\delta}(x)}.$$

In the above setting, the following useful result holds.

Lemma 2.2 *As the parameter $\delta \to 0$, the laplacian of the function $W_{a,M,q}$ defined in (12) can be expanded as*

$$\begin{aligned}\Delta W_{a,M,q} = &-\frac{(N-k)^2}{4}\, q\, \delta^{-2}\, W_{a,M,q} - 2\,a\,\sqrt{\tilde{\alpha}}\, X_{-1}(\delta)\,\delta^{-2}\, W_{a,M,q}\\ &+a(a-1)\, X_{-2}(\delta)\,\delta^{-2}\, W_{a,M,q} + \frac{h+2M}{d}\, W_{a,M,q} + O(|\log(\delta)|\,\delta^{-\frac{3}{2}})\, W_{a,M,q},\end{aligned}$$

where $\tilde{\alpha}(x) = \frac{(N-k)^2}{4}\left(1 - q(\sigma(\overline{x})) + \tilde{\delta}(x)\right)$ and $h = \Delta d$. Here the lower order term satisfies

$$|O(r)| \le C|r|,$$

where C is a positive constant only depending on a, M, $\Sigma_k, \mathcal{U}$ and $\|q\|_{C^2(\mathcal{U})}$.

Proof We put $s = \frac{(N-k)^2}{4}$. Let $w = \tilde{\delta}(x)^{\alpha(x)}$ then the following formula can be easily verified

$$\Delta w = w\left(\Delta \log(w) + |\nabla \log(w)|^2\right). \tag{13}$$

Since

$$\log(w) = \alpha \log(\tilde{\delta}),$$

we get

$$\Delta \log(w) = \Delta\alpha \log(\tilde{\delta}) + 2\nabla\alpha \cdot \nabla(\log(\tilde{\delta})) + \alpha\Delta \log(\tilde{\delta}). \tag{14}$$

We have

$$\Delta\alpha = \Delta\sqrt{\tilde{\alpha}} = \sqrt{\tilde{\alpha}}\left(\frac{1}{2}\Delta \log(\tilde{\alpha}) + \frac{1}{4}|\nabla \log(\tilde{\alpha})|^2\right), \tag{15}$$

$$\nabla \log(\tilde{\alpha}) = \frac{\nabla\tilde{\alpha}}{\tilde{\alpha}} = \frac{-s\nabla(q\circ\sigma) + s\nabla\tilde{\delta}}{\tilde{\alpha}}$$

and using the formula (13), we obtain

$$\begin{aligned}\Delta \log(\tilde{\alpha}) &= \frac{\Delta\tilde{\alpha}}{\tilde{\alpha}} - \frac{|\nabla\tilde{\alpha}|^2}{\tilde{\alpha}^2}\\ &= \frac{-s\Delta(q\circ\sigma) + s\Delta\tilde{\delta}}{\tilde{\alpha}} - \frac{s^2|\nabla(q\circ\sigma)|^2 + s^2|\nabla\tilde{\delta}|^2}{\tilde{\alpha}^2} + 2s^2\frac{\nabla(q\circ\sigma)\cdot\nabla\tilde{\delta}}{\tilde{\alpha}^2}.\end{aligned}$$

Putting the above in (15), we deduce that

$$\Delta\alpha = \frac{1}{2\sqrt{\tilde\alpha}}\left\{ -s\Delta(q\circ\sigma) + s\Delta\tilde\delta - \frac{1}{2}\frac{s^2|\nabla(q\circ\sigma)|^2 + s^2|\nabla\tilde\delta|^2 - 2s^2\nabla(q\circ\sigma)\cdot\nabla\tilde\delta}{\tilde\alpha}\right\}. \tag{16}$$

Using Lemma 2.1 and the fact that q is in $C^2(\overline{\mathcal{U}})$, together with (16) we get

$$\Delta\alpha = O(\tilde\delta^{-\frac{3}{2}}). \tag{17}$$

On the other hand

$$\nabla\alpha = \nabla\sqrt{\tilde\alpha} = \frac{1}{2}\frac{\nabla\tilde\alpha}{\sqrt{\tilde\alpha}} = -\frac{s}{2\sqrt{\tilde\alpha}}\nabla(q\circ\sigma) + \frac{s}{2}\frac{\nabla\tilde\delta}{\sqrt{\tilde\alpha}}$$

so that

$$\nabla\alpha\cdot\nabla\tilde\delta = -\frac{s}{2\sqrt{\tilde\alpha}}\nabla(q\circ\sigma)\cdot\nabla\tilde\delta + \frac{s}{2}\frac{|\nabla\tilde\delta|^2}{\sqrt{\tilde\alpha}} = O(\tilde\delta^{-\frac{1}{2}})$$

and from which we deduce that

$$\nabla\alpha\cdot\nabla\log(\tilde\delta) = \frac{1}{\tilde\delta}\nabla\alpha\cdot\nabla\tilde\delta = O(\tilde\delta^{-\frac{3}{2}}). \tag{18}$$

By Lemma 2.1 we have that

$$\alpha\Delta\log(\tilde\delta) = \alpha\frac{N-k-2}{\tilde\delta^2}(1+O(\tilde\delta)).$$

Taking back the above estimate together with (18) and (17) in (14), we get

$$\Delta\log(w) = \alpha\frac{N-k-2}{\tilde\delta^2}(1+O(\tilde\delta)) + O(|\log(\tilde\delta)|\tilde\delta^{-\frac{3}{2}}). \tag{19}$$

We also have

$$\nabla(\log(w)) = \nabla(\alpha\log(\tilde\delta)) = \alpha\frac{\nabla\tilde\delta}{\tilde\delta} + \log(\tilde\delta)\nabla\alpha$$

and thus

$$|\nabla(\log(w))|^2 = \frac{\alpha^2}{\tilde\delta^2} + \frac{2\alpha\log(\tilde\delta)}{\tilde\delta}\nabla\tilde\delta\cdot\nabla\alpha + |\log(\tilde\delta)|^2|\nabla\alpha|^2 = \frac{\alpha^2}{\tilde\delta^2} + O(|\log(\tilde\delta)|\tilde\delta^{-\frac{3}{2}}).$$

Putting this together with (19) in (13), we conclude that

$$\frac{\Delta w}{w} = \alpha\frac{N-k-2}{\tilde\delta^2} + \frac{\alpha^2}{\tilde\delta^2} + O(|\log(\tilde\delta)|\,\tilde\delta^{-\frac{3}{2}}). \tag{20}$$

Now we define the function

$$v(x) := d(x)\,w(x),$$

where we recall that d is the distance function to the boundary of $\mathcal{U}$. It is clear that

$$\Delta v = w\Delta d + d\Delta w + 2\nabla d\cdot\nabla w. \tag{21}$$

Notice that

$$\nabla w = w\,\nabla\log(w) = w\left(\log(\tilde\delta)\nabla\alpha + \alpha\frac{\nabla\tilde\delta}{\tilde\delta}\right)$$

and so

$$\nabla d \cdot \nabla w = w \left(\log(\tilde{\delta}) \nabla d \cdot \nabla \alpha + \frac{\alpha}{\tilde{\delta}} \nabla d \cdot \nabla \tilde{\delta} \right). \tag{22}$$

Recall the second assertion of Lemma 2.1 that we rewrite as

$$\nabla d \cdot \nabla \tilde{\delta} = \frac{d}{\tilde{\delta}}. \tag{23}$$

Therefore

$$\nabla d \cdot \nabla \alpha = \nabla d \cdot \left(-\frac{s}{2\sqrt{\tilde{\alpha}}} \nabla (q \circ \sigma) + \frac{s}{2} \frac{\nabla \tilde{\delta}}{\sqrt{\tilde{\alpha}}} \right) = \frac{s}{2\sqrt{\tilde{\alpha}}} \frac{d}{\tilde{\delta}} - \frac{s}{2\sqrt{\tilde{\alpha}}} \nabla d \cdot \nabla (q \circ \sigma). \tag{24}$$

Notice that if x is in a neighborhood of some point $P \in \Sigma_k$ one has

$$\nabla d \cdot \nabla (q \circ \sigma)(x) = \frac{\partial}{\partial y^1} q(\sigma(\overline{x})) = \frac{\partial}{\partial y^1} q(f^P(\overline{y})) = 0.$$

This with (24) and (23) in (22) give

$$\begin{aligned} \nabla d \cdot \nabla w &= w \left(O(\tilde{\delta}^{-\frac{3}{2}} |\log(\tilde{\delta})|)\, d + \frac{\alpha}{\tilde{\delta}^2} d \right) \\ &= v \left(O(\tilde{\delta}^{-\frac{3}{2}} |\log(\tilde{\delta})|) + \frac{\alpha}{\tilde{\delta}^2} \right). \end{aligned} \tag{25}$$

From (20), (21) and (25) (recalling the expression of α above), we get immediately

$$\begin{aligned} \Delta v &= \left(\alpha \frac{N-k}{\tilde{\delta}^2} + \frac{\alpha^2}{\tilde{\delta}^2} \right) v + O(|\log(\tilde{\delta})|\, \tilde{\delta}^{-\frac{3}{2}})\, v + \frac{h}{d} v \\ &= \left(-\frac{(N-k)^2}{4} \frac{q(x)}{\tilde{\delta}^2} + O(|\log(\tilde{\delta})|\, \tilde{\delta}^{-\frac{3}{2}}) \right) v + \frac{h}{d} v, \end{aligned} \tag{26}$$

where $h = \Delta d$. Here we have used the fact that $|q(x) - q(\sigma(\bar{x}))| \leq C\tilde{\delta}(x)$ for x in a neighborhood of Σ_k.

Recall the definition of $W_{a,M,q}$

$$W_{a,M,q}(x) = X_a(\tilde{\delta}(x))\, e^{Md(x)}\, v(x), \quad \text{with} \quad X_a(\tilde{\delta}(x)) := (-\log(\tilde{\delta}(x)))^a,$$

where M and a are two real numbers. We have

$$\Delta W_{a,M,q} = X_a(\tilde{\delta})\, \Delta(e^{Md}\, v) + 2\nabla X_a(\tilde{\delta}) \cdot \nabla(e^{Md}\, v) + e^{Md}\, v\, \Delta X_a(\tilde{\delta})$$

and thus

$$\begin{aligned} \Delta W_{a,M,q} &= X_a(\tilde{\delta}) e^{Md}\, \Delta v + X_a(\tilde{\delta}) \Delta(e^{Md})\, v + 2X_a(\tilde{\delta}) \nabla v \cdot \nabla(e^{Md}) \\ &\quad + 2\nabla X_a(\tilde{\delta}) \cdot \left(v\, \nabla(e^{Md}) + e^{Md} \nabla v \right) + e^{Md}\, v\, \Delta X_a(\tilde{\delta}). \end{aligned} \tag{27}$$

We shall estimate term by term the above expression.
First we have form (26)

$$X_a(\tilde{\delta}) e^{Md}\, \Delta v = -\frac{(N-k)^2}{4} \frac{q}{\tilde{\delta}^2} W_{a,M,q} + \frac{h}{d} W_{a,M,q} + O(|\log(\tilde{\delta})|\, \tilde{\delta}^{-\frac{3}{2}})\, W_{a,M,q}. \tag{28}$$

On the other hand it is plain that

$$X_a(\tilde{\delta})\, \Delta(e^{Md})\, v = O(1)\, W_{a,M,q}. \tag{29}$$

It is clear that

$$\nabla v = w\,\nabla d + d\,\nabla w = w\,\nabla d + d\left(\log(\tilde\delta)\,\nabla\alpha + \alpha\frac{\nabla\tilde\delta}{\tilde\delta}\right)w. \tag{30}$$

From which and (23) we get

$$\begin{aligned} X_a(\tilde\delta)\,\nabla v\cdot\nabla(e^{Md}) &= M\,X_a(\tilde\delta)\,e^{Md}\,w\left\{|\nabla d|^2 + d\left(\log(\tilde\delta)\,\nabla d\cdot\nabla\alpha + \frac{\alpha}{\tilde\delta}\nabla\tilde\delta\cdot\nabla d\right)\right\}\\ &= M\,X_a(\tilde\delta)\,e^{Md}\,w\left\{1 + O(|\log(\tilde\delta)|\,\tilde\delta^{-\frac12})\,d + O(\tilde\delta^{-1})\,d\right\}\\ &= W_{a,M,q}\left\{\frac{M}{d} + O(|\log(\tilde\delta)|\,\tilde\delta^{-1})\right\}. \end{aligned} \tag{31}$$

Observe that

$$\nabla(X_a(\tilde\delta)) = -a\,\frac{\nabla\tilde\delta}{\tilde\delta}X_{a-1}(\tilde\delta).$$

This with (30) and (23) imply that

$$\nabla X_a(\tilde\delta)\cdot\left(v\,\nabla(e^{Md}) + e^{Md}\nabla v\right) = -\frac{a(\alpha+1)}{\tilde\delta^2}\,X_{-1}\,W_{a,M,q} + O(|\log(\tilde\delta)|\tilde\delta^{-\frac32})\,W_{a,M,q}. \tag{32}$$

By Lemma 2.1, we have

$$\Delta(X_a(\tilde\delta)) = \frac{a}{\tilde\delta^2}X_{a-1}(\tilde\delta)\{2+k-N+O(\tilde\delta)\} + \frac{a(a-1)}{\tilde\delta^2}X_{a-2}(\tilde\delta).$$

Therefore we obtain

$$e^{Md}v\,\Delta(X_a(\tilde\delta)) = \frac{a}{\tilde\delta^2}\{2+k-N+O(\tilde\delta)\}\,X_{-1}\,W_{a,M,q} + \frac{a(a-1)}{\tilde\delta^2}X_{-2}\,W_{a,M,q}. \tag{33}$$

Collecting (28), (29), (31), (32) and (33) in the expression (27), we get as $\tilde\delta\to 0$

$$\begin{aligned} \Delta W_{a,M,q} &= -\frac{(N-k)^2}{4}\,q\,\tilde\delta^{-2}\,W_{a,M,q} - 2\,a\,\sqrt{\tilde\alpha}\,X_{-1}(\tilde\delta)\,\tilde\delta^{-2}\,W_{a,M,q}\\ &\quad + a(a-1)\,X_{-2}(\tilde\delta)\,\tilde\delta^{-2}\,W_{a,M,q} + \frac{h+2M}{d}\,W_{a,M,q} + O(|\log(\tilde\delta)|\,\tilde\delta^{-\frac32})\,W_{a,M,q}. \end{aligned}$$

The conclusion of the lemma follows then from the first assertion of Lemma 2.1. □

2.1 Construction of a subsolution

For $\lambda\in\mathbb{R}$ and $\eta\in Lip(\overline{\mathcal{U}})$ with $\eta = 0$ on Σ_k, we define the operator

$$\mathcal{L}_\lambda := -\Delta - \frac{(N-k)^2}{4}\,q\,\delta^{-2} + \lambda\,\eta\,\delta^{-2}, \tag{34}$$

where q is as in (11). We have the following lemma

Lemma 2.3 *There exist two positive constants M_0, β_0 such that for all $\beta\in(0,\beta_0)$ the function $V_\varepsilon := W_{-1,M_0,q} + W_{0,M_0,q-\varepsilon}$ (see (12)) satisfies*

$$\mathcal{L}_\lambda V_\varepsilon \le 0 \quad \text{in } \mathcal{U}_\beta, \quad \text{for all } \varepsilon\in[0,1). \tag{35}$$

Moreover $V_\varepsilon \in H^1(\mathcal{U}_\beta)$ *for any* $\varepsilon \in (0, 1)$ *and in addition*

$$\int_{\mathcal{U}_\beta} \frac{V_0^2}{\delta^2}\, dx \geq C \int_{\Sigma_k} \frac{1}{\sqrt{1 - q(\sigma)}}\, d\sigma. \tag{36}$$

Proof Let β_1 be a positive small real number so that d is smooth in $\mathcal{U}_{\beta_1}$. We choose

$$M_0 = \max_{x \in \overline{\mathcal{U}}_{\beta_1}} |h(x)| + 1.$$

Using this and Lemma 2.2, for some $\beta \in (0, \beta_1)$, we have

$$\mathcal{L}_\lambda W_{-1,M_0,q} \leq \left(-2\delta^{-2} X_{-2} + C|\log(\delta)|\, \delta^{-\frac{3}{2}} + |\lambda|\eta\delta^{-2}\right) W_{-1,M_0,q} \quad \text{in } \mathcal{U}_\beta. \tag{37}$$

Using the fact that the function η vanishes on Σ_k (this implies in particular that $|\eta| \leq C\delta$ in $\mathcal{U}_\beta$), we have

$$\mathcal{L}_\lambda(W_{-1,M_0,q}) \leq -\delta^{-2} X_{-2}\, W_{-1,M_0,q} = -\delta^{-2} X_{-3}\, W_{0,M_0,q} \quad \text{in } \mathcal{U}_\beta,$$

for β sufficiently small. Again by Lemma 2.2, and similar arguments as above, we have

$$\mathcal{L}_\lambda W_{0,M_0,q-\varepsilon} \leq C|\log(\delta)|\, \delta^{-\frac{3}{2}}\, W_{0,M_0,q-\varepsilon} \leq C|\log(\delta)|\, \delta^{-\frac{3}{2}}\, W_{0,M_0,q} \quad \text{in } \mathcal{U}_\beta, \tag{38}$$

for any $\varepsilon \in [0, 1)$. Therefore we get

$$\mathcal{L}_\lambda \left(W_{-1,M_0,q} + W_{0,M_0,q-\varepsilon}\right) \leq 0 \quad \text{in } \mathcal{U}_\beta,$$

if β is small. This proves (35).
The proof of the fact that $W_{a,M_0,q} \in H^1(\mathcal{U}_\beta)$, for any $a < -\frac{1}{2}$ and $W_{0,M_0,q-\varepsilon} \in H^1(\mathcal{U}_\beta)$, for $\varepsilon > 0$ can be easily checked using polar coordinates (by assuming without any loss of generality that $M_0 = 0$ and $q \equiv 1$), we therefore skip it.
We now prove the last statement of the theorem. Using Lemma 2.1, we have

$$\begin{aligned}
\int_{\mathcal{U}_\beta} \frac{V_0^2}{\delta^2}\, dx &\geq \int_{\mathcal{U}_\beta} \frac{W_{0,M_0,q}^2}{\delta^2}\, dx \\
&\geq C \int_{\mathcal{U}_\beta(\Sigma_k)} d^2(x)\tilde{\delta}(x)^{2\alpha(x)-2}\, dx \\
&\geq C \sum_{i=1}^{N_0} \int_{T_i} d^2(x)\tilde{\delta}(x)^{2\alpha(x)-2}\, dx \\
&= C \sum_{i=1}^{N_0} \int_{B_+^{N-k}(0,\beta)\times D_i} (y^1)^2\, |\tilde{y}|^{2\alpha(F_{\mathcal{M}}^{p_i}(y))-2}\, |\mathrm{Jac}(F_{\mathcal{M}}^{p_i})|(y)\, dy \\
&\geq C \sum_{i=1}^{N_0} \int_{B_+^{N-k}(0,\beta)\times D_i} (y^1)^2\, |\tilde{y}|^{k-N-2+(N-k)\sqrt{1-q(f^{p_i}(\tilde{y}))}}\, |\tilde{y}|^{-\sqrt{|\tilde{y}|}}\, dy.
\end{aligned}$$

Here we used the fact that $|\mathrm{Jac}(F_{\mathcal{M}}^{p_i})|(y) \geq C$. Observe that

$$|\tilde{y}|^{-\sqrt{|\tilde{y}|}} \geq C > 0 \quad \text{as } |\tilde{y}| \to 0.$$

Using polar coordinates, the above integral becomes

$$\int_{\mathcal{U}_\beta} \frac{V_0^2}{\delta^2}\,dx \geq C \sum_{i=1}^{N_0} \int_{D_i} \int_{S_+^{N-k-1}} \left(\frac{y^1}{|\tilde{y}|}\right)^2 d\theta \int_0^\beta r^{-1+(N-k)\sqrt{1-q(f^{p_i}(\bar{y}))}}\,d\bar{y}$$

$$\geq C \sum_{i=1}^{N_0} \int_{D_i} \int_0^{r_{i_1}} r^{-1+(N-k)\sqrt{1-q(f^{p_i}(\bar{y}))}}\,|\mathrm{Jac}(f^{p_i})|(\bar{y})\,d\bar{y}.$$

We therefore obtain

$$\int_{\mathcal{U}_\beta} \frac{V_0^2}{\delta^2}\,dx \geq C \int_{\Sigma_k} \int_0^\beta r^{-1+(N-k)\sqrt{1-q(\sigma)}}\,dr\,d\sigma$$

$$\geq C \int_{\Sigma_k} \frac{1}{\sqrt{1-q(\sigma)}}\,d\sigma.$$

This concludes the proof of the lemma. □

2.2 Construction of a supersolution

In this subsection we provide a supersolution for the operator $\mathcal{L}_\lambda$ defined in (34). We prove

Lemma 2.4 *There exist constants $\beta_0 > 0$, $M_1 < 0$, $M_0 > 0$ (the constant M_0 is as in Lemma 2.3) such that for all $\beta \in (0, \beta_0)$ the function $U := W_{0,M_1,q} - W_{-1,M_0,q} > 0$ in $\mathcal{U}_\beta$ and satisfies*

$$\mathcal{L}_\lambda U_a \geq 0 \quad in\ \mathcal{U}_\beta. \tag{39}$$

Moreover $U \in H^1(\mathcal{U}_\beta)$ provided

$$\int_{\Sigma_k} \frac{1}{\sqrt{1-q(\sigma)}}\,d\sigma < +\infty. \tag{40}$$

Proof We consider β_1 as in the beginning of the proof of Lemma 2.3 and we define

$$M_1 = -\frac{1}{2} \max_{x\in\overline{\mathcal{U}}_{\beta_1}} |h(x)| - 1. \tag{41}$$

Since

$$U(x) = (e^{M_1 d(x)} - e^{M_0 d(x)} X_{-1}(\tilde{\delta}(x)))d(x)\tilde{\delta}(x)^{\alpha(x)},$$

it follows that $U > 0$ in $\mathcal{U}_\beta$ for $\beta > 0$ sufficiently small. By (41) and Lemma 2.2, we get

$$\mathcal{L}_\lambda W_{0,M_1,q} \geq \left(-C|\log(\delta)|\,\delta^{-\frac{3}{2}} - |\lambda|\eta\delta^{-2}\right) W_{0,M_1,q}.$$

Using (37) we have

$$\mathcal{L}_\lambda(-W_{-1,M_0,q}) \geq \left(2\delta^{-2}X_{-2} - C|\log(\delta)|\,\delta^{-\frac{3}{2}} - |\lambda|\eta\delta^{-2}\right) W_{-1,M_0,q}.$$

Taking the sum of the two above inequalities, we obtain

$$\mathcal{L}_\lambda U \geq 0 \quad \text{in } \mathcal{U}_\beta,$$

which holds true because $|\eta| \leq C\delta$ in $\mathcal{U}_\beta$. Hence we get readily (39).
Our next task is to prove that $U \in H^1(\mathcal{U}_\beta)$ provided (40) holds, to do so it is enough to show that $W_{0,M_1,q} \in H^1(\mathcal{U}_\beta)$ provided (40) holds.
We argue as in the proof of Lemma 2.3. We have

$$\begin{aligned}
\int_{\mathcal{U}_\beta} |\nabla W_{0,M_1,q}|^2 &\leq C \int_{\mathcal{U}_\beta} d^2(x)\tilde{\delta}(x)^{2\alpha(x)-2}\,dx \\
&\leq C \sum_{i=1}^{N_0} \int_{B_+^{N-k}(0,\beta)\times D_i} d^2(F_{\mathcal{M}}^{p_i}(y))\tilde{\delta}(F_{\mathcal{M}}^{p_i}(y))^{2\alpha(F_{\mathcal{M}}^{p_i}(y))-2}|\mathrm{Jac}(F_{\mathcal{M}}^{p_i})|(y)dy \\
&\leq C \sum_{i=1}^{N_0} \int_{B_+^{N-k}(0,\beta)\times D_i} (y^1)^2\,|\tilde{y}|^{2\alpha(F_{\mathcal{M}}^{p_i}(y))-2}\,|\mathrm{Jac}(F_{\mathcal{M}}^{p_i})|(y)\,dy \\
&\leq C \sum_{i=1}^{N_0} \int_{B_+^{N-k}(0,\beta)\times D_i} (y^1)^2\,|\tilde{y}|^{k-N-2+(N-k)\sqrt{1-q(f^{p_i}(\bar{y}))}}\;|\tilde{y}|^{-\sqrt{|\tilde{y}|}}\,dy.
\end{aligned}$$

Here we used the fact that $|\mathrm{Jac}(F_{\mathcal{M}}^{p_i})|(y) \leq C$. Note that

$$|\tilde{y}|^{-\sqrt{|\tilde{y}|}} \leq C \quad \text{as } |\tilde{y}| \to 0.$$

Using polar coordinates, it follows that

$$\begin{aligned}
\int_{\mathcal{U}_\beta} |\nabla W_{0,M_1,q}|^2 &\leq C \sum_{i=1}^{N_0} \int_{D_i} \int_{S_+^{N-k-1}} \left(\frac{y^1}{|\tilde{y}|}\right)^2 d\theta \int_0^\beta r^{-1+(N-k)\sqrt{1-q(f^{p_i}(\bar{y}))}}\,dr\,d\bar{y} \\
&\leq C \sum_{i=1}^{N_0} \int_{D_i} \frac{1}{\sqrt{1-q(f^{p_i}(\bar{y}))}}\,d\bar{y}.
\end{aligned}$$

Recalling that $|\mathrm{Jac}(f^{p_i})|(\bar{y}) = 1 + O(|\bar{y}|)$, we deduce that

$$\begin{aligned}
\sum_{i=1}^{N_0} \int_{D_i} \frac{1}{\sqrt{1-q(f^{p_i}(\bar{y}))}}\,d\bar{y} &\leq C \sum_{i=1}^{N_0} \int_{D_i} \frac{1}{\sqrt{1-q(f^{p_i}(\bar{y}))}}\,|\mathrm{Jac}(f)|(\bar{y})\,d\bar{y} \\
&= C \int_{\Sigma_k} \frac{1}{\sqrt{1-q(\sigma)}}\,d\sigma.
\end{aligned}$$

Therefore

$$\int_{\mathcal{U}_\beta} |\nabla W_{0,M_1,q}|^2\,dx \leq C \int_{\Sigma_k} \frac{1}{\sqrt{1-q(\sigma)}}\,d\sigma$$

and the lemma follows at once. □

3 Existence of λ*

We start with the following local improved Hardy inequality.

Lemma 3.1 *Let Ω be a smooth domain and assume that $\partial\Omega$ contains a smooth closed submanifold Σ_k of dimension $1 \le k \le N-2$. Assume that p, q and η satisfy (2) and (3). Then there exist constants $\beta_0 > 0$ and $c > 0$ depending only on Ω, Σ_k, q, η and p such that for all $\beta \in (0, \beta_0)$ the inequality*

$$\int_{\Omega_\beta} p|\nabla u|^2\, dx - \frac{(N-k)^2}{4}\int_{\Omega_\beta} q\frac{|u|^2}{\delta^2}\, dx \ge c\int_{\Omega_\beta} \frac{|u|^2}{\delta^2|\log(\delta)|^2}\, dx$$

holds for all $u \in H^1_0(\Omega_\beta)$.

Proof We use the notations in Sect. 2 with $\mathcal{U} = \Omega$ and $\mathcal{M} = \partial\Omega$.
Fix $\beta_1 > 0$ small and

$$M_2 = -\frac{1}{2}\max_{x\in\overline{\Omega}_{\beta_1}} (|h(x)| + |\nabla p \cdot \nabla d|) - 1. \tag{42}$$

Since $\frac{p}{q} \in C^1(\overline{\Omega})$, there exists $C > 0$ such that

$$\left|\frac{p(x)}{q(x)} - \frac{p(\sigma(\bar{x}))}{q(\sigma(\bar{x}))}\right| \le C\delta(x) \quad \forall x \in \Omega_\beta, \tag{43}$$

for small $\beta > 0$. Hence by (3) there exits a constant $C' > 0$ such that

$$p(x) \ge q(x) - C'\delta(x) \quad \forall x \in \Omega_\beta. \tag{44}$$

Consider $W_{\frac{1}{2},M_2,1}$ (in Lemma 2.2 with $q \equiv 1$). For all $\beta > 0$ small, we set

$$\tilde{w}(x) = W_{\frac{1}{2},M_2,1}(x), \quad \forall x \in \Omega_\beta. \tag{45}$$

Notice that $\operatorname{div}(p\nabla\tilde{w}) = p\Delta\tilde{w} + \nabla p \cdot \nabla\tilde{w}$. By Lemma 2.2, we have

$$-\frac{\operatorname{div}(p\nabla\tilde{w})}{\tilde{w}} \ge \frac{(N-k)^2}{4}p\delta^{-2} + \frac{p}{4}\delta^{-2}X_{-2}(\delta) + O(|\log(\delta)|\delta^{-\frac{3}{2}}) \quad \text{in } \Omega_\beta.$$

This together with (44) yields

$$-\frac{\operatorname{div}(p\nabla\tilde{w})}{\tilde{w}} \ge \frac{(N-k)^2}{4}q\delta^{-2} + \frac{c_0}{4}\delta^{-2}X_{-2}(\delta) + O(|\log(\delta)|\delta^{-\frac{3}{2}}) \quad \text{in } \Omega_\beta,$$

with $c_0 = \min_{\overline{\Omega_{\beta_1}}} p > 0$. Therefore

$$-\frac{\operatorname{div}(p\nabla\tilde{w})}{\tilde{w}} \ge \frac{(N-k)^2}{4}q\delta^{-2} + c\,\delta^{-2}X_{-2}(\delta) \text{ in } \Omega_\beta, \tag{46}$$

for some positive constant c depending only on Ω, Σ_k, q, η and p.

Let $u \in C^\infty_c(\Omega_\beta)$ and put $\psi = \frac{u}{\tilde{w}}$. Then one has $|\nabla u|^2 = |\tilde{w}\nabla\psi|^2 + |\psi\nabla\tilde{w}|^2 + \nabla(\psi^2)\cdot\tilde{w}\nabla\tilde{w}$. Therefore $|\nabla u|^2 p = |\tilde{w}\nabla\psi|^2 p + p\nabla\tilde{w}\cdot\nabla(\tilde{w}\psi^2)$. Integrating by parts, we get

$$\int_{\Omega_\beta} |\nabla u|^2 p\, dx = \int_{\Omega_\beta} |\tilde{w}\nabla\psi|^2 p\, dx + \int_{\Omega_\beta}\left(-\frac{\operatorname{div}(p\nabla\tilde{w})}{\tilde{w}}\right)u^2\, dx.$$

Putting (46) in the above equality, we get the desired result. □

We next prove the following result

Lemma 3.2 *Let* Ω *be a smooth bounded domain and assume that* $\partial\Omega$ *contains a smooth closed submanifold* Σ_k *of dimension* $1 \le k \le N-2$. *Assume that* (2) *and* (3) *hold. Then there exists* $\lambda^* = \lambda^*(\Omega, \Sigma_k, p, q, \eta) \in \mathbb{R}$ *such that*

$$\mu_\lambda(\Omega, \Sigma_k) = \frac{(N-k)^2}{4}, \quad \forall \lambda \le \lambda^*,$$
$$\mu_\lambda(\Omega, \Sigma_k) < \frac{(N-k)^2}{4}, \quad \forall \lambda > \lambda^*.$$

Proof We device the proof in two steps

Step 1: We claim that:

$$\sup_{\lambda\in\mathbb{R}} \mu_\lambda(\Omega, \Sigma_k) \le \frac{(N-k)^2}{4}. \tag{47}$$

Indeed, we know that $\nu_0(\mathbb{R}^N_+, \mathbb{R}^k) = \frac{(N-k)^2}{4}$, see [17] for instance. Given $\tau > 0$, we let $u_\tau \in C^\infty_c(\mathbb{R}^N_+)$ be such that

$$\int_{\mathbb{R}^N_+} |\nabla u_\tau|^2\, dy \le \left(\frac{(N-k)^2}{4} + \tau\right) \int_{\mathbb{R}^N_+} |\tilde{y}|^{-2} u_\tau^2\, dy. \tag{48}$$

By (3), we can let $\sigma_0 \in \Sigma_k$ be such that

$$q(\sigma_0) = p(\sigma_0).$$

Now, given $r > 0$, we let $\rho_r > 0$ such that for all $x \in B(\sigma_0, \rho_r) \cap \Omega$

$$p(x) \le (1+r) q(\sigma_0), \quad q(x) \ge (1-r) q(\sigma_0) \quad \text{and} \quad \eta(x) \le r. \tag{49}$$

We choose Fermi coordinates near $\sigma_0 \in \Sigma_k$ given by the map $F^{\sigma_0}_{\partial\Omega}$ (as in Sect. 2) and we choose $\varepsilon_0 > 0$ small such that, for all $\varepsilon \in (0, \varepsilon_0)$,

$$\Lambda_{\varepsilon,\rho,r,\tau} := F^{\sigma_0}_{\partial\Omega}(\varepsilon\, \mathrm{Supp}(u_\tau)) \subset B(\sigma_0, \rho_r) \cap \Omega$$

and we define the following test function

$$v(x) = \varepsilon^{\frac{2-N}{2}} u_\tau \left(\varepsilon^{-1} (F^{\sigma_0}_{\partial\Omega})^{-1}(x)\right), \quad x \in \Lambda_{\varepsilon,\rho,r,\tau}.$$

Clearly, for every $\varepsilon \in (0, \varepsilon_0)$, we have that $v \in C^\infty_c(\Omega)$ and thus by a change of variable, (49) and Lemma 2.1, we have

$$\begin{aligned}
\mu_\lambda(\Omega, \Sigma_k) &\le \frac{\int_\Omega p|\nabla v|^2\, dx + \lambda \int_\Omega \delta^{-2}\eta v^2\, dx}{\int_\Omega q(x)\, \delta^{-2}\, v^2\, dx} \\
&\le \frac{(1+r) \int_{\Lambda_{\varepsilon,\rho,r,\tau}} |\nabla v|^2\, dx}{(1-r) \int_{\Lambda_{\varepsilon,\rho,r,\tau}} \delta^{-2}\, v^2\, dx} + \frac{r|\lambda|}{(1-r) q(\sigma_0)} \\
&\le \frac{(1+r) \int_{\Lambda_{\varepsilon,\rho,r,\tau}} |\nabla v|^2\, dx}{(1-cr) \int_{\Lambda_{\varepsilon,\rho,r,\tau}} \tilde{\delta}^{-2}\, v^2\, dx} + \frac{r|\lambda|}{(1-r) q(\sigma_0)} \\
&\le \frac{(1+r)\varepsilon^{2-N} \int_{\mathbb{R}^N_+} \varepsilon^{-2} (g^\varepsilon)^{ij}\, \partial_i u_\tau \partial_j u_\tau \sqrt{|g^\varepsilon|}(y)\, dy}{(1-cr) \int_{\mathbb{R}^N_+} \varepsilon^{2-N}\, |\varepsilon\tilde{y}|^{-2}\, u_\tau^2 \sqrt{|g^\varepsilon|}(\tilde{y})\, dy} + \frac{cr}{1-r},
\end{aligned}$$

where g^ε is the scaled metric with components

$$g^\varepsilon_{\alpha\beta}(y) = \varepsilon^{-2}\langle \partial_\alpha F^{\sigma_0}_{\partial\Omega}(\varepsilon y), \partial_\beta F^{\sigma_0}_{\partial\Omega}(\varepsilon y)\rangle$$

for $\alpha, \beta = 1, \ldots, N$ and where we have used the fact that $\tilde\delta(F^{\sigma_0}_{\partial\Omega}(\varepsilon y)) = |\varepsilon\tilde y|^2$ for every $\tilde y$ in the support of u_τ. Since the scaled metric g^ε expands a $g^\varepsilon = I + O(\varepsilon)$ on the support of u_τ, we deduce that

$$\mu_\lambda(\Omega, \Sigma_k) \le \frac{1+r}{1-cr}\,\frac{1+c\varepsilon}{1-c\varepsilon}\,\frac{\int_{\mathbb{R}^N_+} |\nabla u_\tau|^2\, dy}{\int_{\mathbb{R}^N_+} |\tilde y|^{-2} u_\tau^2\, dy} + \frac{cr}{1-r},$$

where c is a positive constant depending only on Ω, p, q, η and Σ_k. Hence by (48) we conclude

$$\mu_\lambda(\Omega, \Sigma_k) \le \frac{1+r}{1-cr}\,\frac{1+c\varepsilon}{1-c\varepsilon}\left(\frac{(N-k)^2}{4} + \tau\right) + \frac{cr}{1-r}.$$

Taking the limit in ε, then in r and then in τ, the claim follows.

Step 2: We claim that there exists $\tilde\lambda \in \mathbb{R}$ such that $\mu_{\tilde\lambda}(\Omega, \Sigma_k) \ge \frac{(N-k)^2}{4}$.

Thanks to Lemma 3.1, the proof uses a standard argument of cut-off function and integration by parts (see [4]) and we can obtain

$$\int_\Omega \delta^{-2}u^2 q\, dx \le \int_\Omega |\nabla u|^2 p\, dx + C\int_\Omega \delta^{-2}u^2\eta\, dx \quad \forall u \in C^\infty_c(\Omega),$$

for some constant $C > 0$. We skip the details. The claim now follows by choosing $\tilde\lambda = -C$

Finally, noticing that $\mu_\lambda(\Omega, \Sigma_k)$ is decreasing in λ, we can set

$$\lambda^* := \sup\left\{\lambda \in \mathbb{R} \,:\, \mu_\lambda(\Omega, \Sigma_k) = \frac{(N-k)^2}{4}\right\} \tag{50}$$

so that $\mu_\lambda(\Omega, \Sigma_k) < \frac{(N-k)^2}{4}$ for all $\lambda > \lambda^*$. □

4 Non-existence result

Lemma 4.1 *Let Ω be a smooth bounded domain of $\mathbb{R}^N$, $N \ge 3$, and let Σ_k be a smooth closed submanifold of $\partial\Omega$ of dimension k with $1 \le k \le N-2$. Then, there exist bounded smooth domains $\Omega^\pm$ such that $\Omega^+ \subset \Omega \subset \Omega^-$ and*

$$\partial\Omega^+ \cap \partial\Omega = \partial\Omega^- \cap \partial\Omega = \Sigma_k.$$

Proof For $\beta > 0$ small, let Γ_β be a neighborhood of Σ_k in $\mathbb{R}^N$. Define $\Omega^\pm_\beta$ by $\Omega^+_\beta := \Gamma_\beta \cap \Omega$ and $\Omega^-_\beta := \Gamma_\beta \cap (\mathbb{R}^N \setminus \Omega)$. Consider the maps defined in $\Omega^\pm_\beta$ by

$$x \mapsto g^\pm(x) := \bar d_{\partial\Omega}(x) \mp \frac{1}{2}\,\hat\delta^2(\bar x),$$

where $\bar d_{\partial\Omega}$ is the signed distance function to $\partial\Omega$ and we recall the notations in Sect. 2. We observe that for a point $P \in \Sigma_k$, recalling once again the local coordinates defined in Sect. 2, we can see that

$$g^+(F^P_{\partial\Omega}(y^1, \check y, \bar y)) = y^1 - \frac{1}{2}|\check y|^2,$$

for $y^1 > 0$ and also

$$g^-(F^P_{\partial\Omega}(y^1, \breve{y}, \bar{y}))) = y^1 + \frac{1}{2}|\breve{y}|^2,$$

for $y^1 < 0$. It is clear that for small β, we have $|\nabla g^\pm| \ge C > 0$ in $\Omega^\pm_\beta$. Therefore the sets

$$\left\{x \in \Omega^\pm_\beta \; : \; g^\pm = 0\right\},$$

containing Σ_k, are smooth $(N-1)$-dimensional submanifolds of $\mathbb{R}^N$.In addition, by construction, they can be taken to be part of the boundaries of smooth bounded domains $\Omega^\pm$ with $\Omega^+ \subset \Omega \subset \Omega^-$ and such that

$$\partial\Omega^+ \cap \partial\Omega = \partial\Omega^- \cap \partial\Omega = \Sigma_k.$$

The proof then follows at once. □

Now, we prove the following non-existence result.

Theorem 4.2 *Let Ω be a smooth bounded domain of $\mathbb{R}^N$ and let Σ_k be a smooth closed submanifold of $\partial\Omega$ of dimension k with $1 \le k \le N-2$ and let $\lambda \ge 0$. Assume that p, q and η satisfy (2) and (3). Suppose that $u \in H^1_0(\Omega) \cap C(\Omega)$ is a non-negative function satisfying*

$$-\operatorname{div}(p\nabla u) - \frac{(N-k)^2}{4} q\delta^{-2}u \ge -\lambda\eta\delta^{-2}u \quad \text{in } \Omega. \tag{51}$$

If $\int_{\Sigma_k} \frac{1}{\sqrt{1-p(\sigma)/q(\sigma)}} d\sigma = +\infty$ then $u \equiv 0$.

Proof We first assume that $p \equiv 1$. Let Ω^+ be the set given by Lemma 4.1. We will use the notations in Sect. 2 with $\mathcal{U} = \Omega^+$ and $\mathcal{M} = \partial\Omega^+$. For $\beta > 0$ small we define

$$\Omega^+_\beta := \{x \in \Omega^+ : \quad \delta(x) < \beta\}.$$

We suppose by contradiction that u does not vanish identically near Σ_k and satisfies (51) so that $u > 0$ in Ω_β by the maximum principle, for some $\beta > 0$ small.
Consider the subsolution V_ε defined in Lemma 2.3 which satisfies

$$\mathcal{L}_\lambda V_\varepsilon \le 0 \quad \text{in } \Omega^+_\beta, \quad \forall \varepsilon \in (0,1). \tag{52}$$

Notice that $\overline{\partial\Omega^+_\beta \cap \Omega^+} \subset \Omega$ thus, for $\beta > 0$ small, we can choose $R > 0$ (independent on ε) so that

$$R\, V_\varepsilon \le R\, V_0 \le u \quad \text{on } \overline{\partial\Omega^+_\beta \cap \Omega^+} \quad \forall \varepsilon \in (0,1).$$

Again by Lemma 2.3, setting $v_\varepsilon = R\, V_\varepsilon - u$, it turns out that $v^+_\varepsilon = \max(v_\varepsilon, 0) \in H^1_0(\Omega^+_\beta)$ because $V_\varepsilon = 0$ on $\partial\Omega^+_\beta \setminus \overline{\partial\Omega^+_\beta \cap \Omega^+}$. Moreover by (51) and (52),

$$\mathcal{L}_\lambda v_\varepsilon \le 0 \quad \text{in } \Omega^+_\beta, \quad \forall \varepsilon \in (0,1).$$

Multiplying the above inequality by v^+_ε and integrating by parts yields

$$\int_{\Omega^+_\beta} |\nabla v^+_\varepsilon|^2\, dx - \frac{(N-k)^2}{4} \int_{\Omega^+_\beta} \delta^{-2} q |v^+_\varepsilon|^2\, dx + \lambda \int_{\Omega^+_\beta} \eta\delta^{-2}|v^+_\varepsilon|^2\, dx \le 0.$$

But then Lemma 3.1 implies that $v_\varepsilon^+ = 0$ in Ω_β^+ provided β small enough because $|\eta| \le C\delta$ near Σ_k. Therefore $u \ge R\, V_\varepsilon$ for every $\varepsilon \in (0, 1)$. In particular $u \ge R\, V_0$. Hence we obtain from Lemma 2.3 that

$$\infty > \int_{\Omega_\beta^+} \frac{u^2}{\delta^2} \ge R^2 \int_{\Omega_\beta^+} \frac{V_0^2}{\delta^2} \ge \int_{\Sigma_k} \frac{1}{\sqrt{1-q(\sigma)}} d\sigma$$

which leads to a contradiction. We deduce that $u \equiv 0$ in Ω_β^+. Thus by the maximum principle $u \equiv 0$ in Ω.
For the general case $p \ne 1$, we argue as in [5] by setting

$$\tilde{u} = \sqrt{p} u. \tag{53}$$

This function satisfies

$$-\Delta\tilde{u} - \frac{(N-k)^2}{4}\frac{q}{p}\delta^{-2}\tilde{u} \ge -\lambda\frac{\eta}{p}\delta^{-2}\tilde{u} + \left(-\frac{\Delta p}{2p} + \frac{|\nabla p|^2}{4p^2}\right)\tilde{u} \quad \text{in } \Omega.$$

Hence since $p \in C^2(\overline{\Omega})$ and $p > 0$ in $\overline{\Omega}$, we get the same conclusions as in the case $p \equiv 1$ and q replaced by q/p. □

5 Existence of minimizers for $\mu_\lambda(\Omega, \Sigma_k)$

Theorem 5.1 *Let Ω be a smooth bounded domain of $\mathbb{R}^N$ and let Σ_k be a smooth closed submanifold of $\partial\Omega$ of dimension k with $1 \le k \le N-2$. Assume that p, q and η satisfy (2) and (3). Then $\mu_\lambda(\Omega, \Sigma_k)$ is achieved for every $\lambda < \lambda^*$.*

Proof The proof follows the same argument of [4] by taking into account the fact that $\eta = 0$ on Σ_k so we skip it. □

Next, we prove the existence of minimizers in the critical case $\lambda = \lambda_*$.

Theorem 5.2 *Let Ω be a smooth bounded domain of $\mathbb{R}^N$ and let Σ_k be a smooth closed submanifold of $\partial\Omega$ of dimension k with $1 \le k \le N-2$. Assume that p, q and η satisfy (2) and (3). If $\int_{\Sigma_k} \frac{1}{\sqrt{1-p(\sigma)/q(\sigma)}} d\sigma < \infty$ then $\mu_{\lambda^*} = \mu_{\lambda^*}(\Omega, \Sigma_k)$ is achieved.*

Proof We first consider the case $p \equiv 1$.
Let λ_n be a sequence of real numbers decreasing to λ^*. By Theorem 5.1, there exits u_n minimizers for $\mu_{\lambda_n} = \mu_{\lambda_n}(\Omega, \Sigma_k)$ so that

$$-\Delta u_n - \mu_{\lambda_n}\delta^{-2} q u_n = -\lambda_n \delta^{-2}\eta u_n \quad \text{in } \Omega. \tag{54}$$

We may assume that $u_n \ge 0$ in Ω. We may also assume that $\|\nabla u_n\|_{L^2(\Omega)} = 1$. Hence $u_n \rightharpoonup u$ in $H_0^1(\Omega)$ and $u_n \to u$ in $L^2(\Omega)$ and pointwise. Let $\Omega^- \supset \Omega$ be the set given by Lemma 4.1. We will use the notations in Sect. 2 with $\mathcal{U} = \Omega^-$ and $\mathcal{M} = \partial\Omega^-$. It will be understood that q is extended to a function in $C^2(\overline{\Omega^-})$. For $\beta > 0$ small we define

$$\Omega_\beta^- := \{x \in \Omega^- : \quad \delta(x) < \beta\}.$$

We have that

$$\Delta u_n + b_n(x)\, u_n = 0 \quad \text{in } \Omega,$$

with $|b_n| \leq C$ in $\overline{\Omega \setminus \overline{\Omega^-_{\frac{\beta}{2}}}}$ for all integer n. Thus by standard elliptic regularity theory,

$$u_n \leq C \qquad \text{in } \overline{\Omega \setminus \overline{\Omega^-_{\frac{\beta}{2}}}}. \tag{55}$$

We consider the supersolution U in Lemma 2.4. We shall show that there exits a constant $C > 0$ such that for all $n \in \mathbb{N}$

$$u_n \leq CU \quad \text{in } \overline{\Omega^-_\beta}. \tag{56}$$

Notice that $\overline{\Omega \cap \partial\Omega^-_\beta} \subset \Omega^-$ thus by (55), we can choose $C > 0$ so that for any n

$$u_n \leq C\,U \quad \text{on } \overline{\Omega \cap \partial\Omega^-_\beta}.$$

Again by Lemma 2.4, setting $v_n = u_n - C\,U$, it turns out that $v_n^+ = \max(v_n, 0) \in H^1_0(\Omega^-_\beta)$ because $u_n = 0$ on $\partial\Omega \cap \Omega^-_\beta$. Hence we have

$$\mathcal{L}_{\lambda_n} v_n \leq -C(\mu_{\lambda^*} - \mu_n)qU - C(\lambda^* - \lambda_n)\eta U \leq 0 \quad \text{in } \Omega^-_\beta \cap \Omega.$$

Multiplying the above inequality by v_n^+ and integrating by parts yields

$$\int\limits_{\Omega^-_\beta} |\nabla v_n^+|^2\,dx - \mu_{\lambda_n} \int\limits_{\Omega^-_\beta} \delta^{-2} q |v_n^+|^2\,dx + \lambda_n \int\limits_{\Omega^-_\beta} \eta\delta^{-2}|v_n^+|^2\,dx \leq 0.$$

Hence Lemma 3.1 implies that

$$C \int\limits_{\Omega^-_\beta} \delta^{-2} X_{-2} |v_n^+|^2\,dx + \lambda_n \int\limits_{\Omega^-_\beta} \eta\delta^{-2}|v_n^+|^2\,dx \leq 0.$$

Since λ_n is bounded, we can choose $\beta > 0$ small (independent of n) such that $v_n^+ \equiv 0$ on Ω^-_β (recall that $|\eta| \leq C\delta$). Thus we obtain (56).
Now since $u_n \to u$ in $L^2(\Omega)$, we get by the dominated convergence theorem and (56), that

$$\delta^{-1} u_n \to \delta^{-1} u \quad \text{in } L^2(\Omega).$$

Since u_n satisfies

$$1 = \int\limits_\Omega |\nabla u_n|^2 = \mu_{\lambda_n} \int\limits_\Omega \delta^{-2} q u_n^2 + \lambda_n \int\limits_\Omega \delta^{-2} \eta u_n^2,$$

taking the limit, we have $1 = \mu_{\lambda^*} \int_\Omega \delta^{-2} q u^2 + \lambda^* \int_\Omega \delta^{-2}\eta u^2$. Hence $u \neq 0$ and it is a minimizer for $\mu_{\lambda^*} = \frac{(N-k)^2}{4}$.
For the general case $p \neq 1$, we can use the same transformation as in (53). So (56) holds and the same argument as a above carries over. □

6 Proof of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1 Combining Lemma 3.2 and Theorem 5.1, it remains only to check the case $\lambda < \lambda^*$. But this is an easy consequence of the definition of λ^* and of $\mu_\lambda(\Omega, \Sigma_k)$, see [4, Section 3].

Proof of Theorem 1.2 Existence is proved in Theorem 5.2 for $I_k < \infty$. Since the absolute value of any minimizer for $\mu_\lambda(\Omega, \Sigma_k)$ is also a minimizer, we can apply Theorem 4.2 to infer that $\mu_{\lambda^*}(\Omega, \Sigma_k)$ is never achieved as soon as $I_k = \infty$.

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