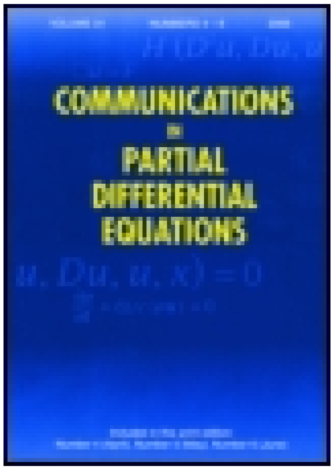

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Eigenvalues for Radially Symmetric Fully Nonlinear Operators

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In this paper we present an elementary theory about the existence of eigenvalues for fully nonlinear radially symmetric 1-homogeneous operators. A general theory for first eigenvalues and eigenfunctions of 1-homogeneous fully nonlinear operators exists in the framework of viscosity solutions. Here we want to show that for the radially symmetric operators or in the one dimensional case a much simpler theory, based on ode and degree theory arguments, can be established. We obtain the complete set of eigenvalues and eigenfunctions characterized by the number of zeroes.

Keywords Fully nonlinear equation; Fully nonlinear operator; Multiple eigenvalues; Principal eigenvalue; Radially symmetric solutions.

Mathematics Subject Classification 35P30; 34B15; 35J15.

1. Introduction

A fundamental step in the analysis of nonlinear equations is the understanding of the associated eigenvalue problem. In the case of our interest the question is the existence of nontrivial solutions (λ, u) of the boundary value problem

$$F(D^2u, Du, u, x) = -\lambda u \quad \text{in } \Omega \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

for a positively homogeneous elliptic operator F and a bounded smooth domain Ω in $\mathbb{R}^N$, $N \geq 1$.

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There is a well established theory for the first eigenvalue and eigenfunction for this problem in the framework of viscosity solutions. The first result in this direction is due to P.L. Lions who proved existence of a first eigenvalue and eigenfunction for the Bellman equation in [12] and for the Monge–Ampère equation in [11] by means of probabilistic arguments. More recently Quaas and Sirakov addressed, by purely partial differential equations arguments, the general case in [14] for the existence and qualitative theory and for the existence of solutions to the associated non-homogeneous Dirichlet problem when λ stays below the first eigenvalue. Results in this direction were also obtained by Armstrong [2] and Ishii and Yoshimura [9]. While in [14] convexity of F is required, in [2, 9] this hypothesis is not necessary. Earlier partial results were obtained by Felmer and Quaas [7] and Quaas [13], see also the detailed bibliography contained in [14]. Based on the eigenvalue theory just discussed, it is possible to build on the existence of positive (or negative) solution of the equation

$$F(D^2u, Du, u, x) = -\lambda u + f(x, u) \quad \text{in } \Omega \tag{1.3}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

by means of bifurcation theory, using the ideas of Rabinowitz [16–18]. See also [8].

A better understanding of the structure of the set of solutions of equations (1.3)–(1.4) can be obtained if further eigenvalues and eigenfunctions are known for (1.1)–(1.2), however this has been elusive in this general fully nonlinear setting, except in some particular cases in presence of radial symmetry as in the work by Berestycki [4], Arias and Campos [3] for the Fucik operator, by Busca et al. [5] for the Pucci operator and more recently for a more general class of extremal operators by Allendes and Quaas [1]. More precisely, in [1, 3, 5] a sequence of eigenvalues and eigenfunctions characterized by their number of zeroes is constructed and a global bifurcation theory is obtained upon them. In [1, 5] the problem is autonomous and has a variational structure, so that scale invariance and integration by parts techniques can be used. In order to find eigenvalues and eigenfunctions with a prescribed number of nodal points in an interval, the authors prove that the solution of the initial value problem is oscillatory. Then, a simple scaling argument allows to adjust the Dirichlet boundary condition. However, in the general case considered here, the problem is not necessarily autonomous and is not variational, so that the methods used in [1, 5] cannot be used, and we have to introduce new arguments to prove the existence of the eigenvalues and eigenfunctions.

The aim of this article is to prove the existence of a sequence of eigenvalues and eigenfunctions for a general fully nonlinear operator in the radially symmetric case, based on elementary arguments and in a self contained fashion. This construction is based on the existence of two semi-eigenvalues associated to positive and negative eigenfunctions in the ball and in concentric annuli, put together via degree theory through a Nehari type approach [15]. While the spectral theory for a ball and annuli can be obtained as particular cases of the general results in [2, 9, 14], the arguments needed to obtain the existence of semi-eigenvalues and positive (negative) eigenfunctions are quite sophisticated, based on the whole viscosity solutions theory. When dealing with the radially symmetric problem in the ball or an annulus, much simpler arguments can be given, based on a combination of purely ordinary differential equations and degree theory. It is our purpose to provide a simple, self contained spectral theory in the one dimensional and radially symmetric cases.

Now we present in precise terms our main theorem. On the operator F we assume the same general hypotheses as in [14], except for the convexity assumption. Namely, we assume that $F: S_N \times \mathbb{R}^N \times \mathbb{R} \times B_R \to \mathbb{R}$, is a continuous function, where B_R is the ball of radius R, centered at the origin and S_N is the set of all symmetric $N \times N$ matrices and

(F1) F is positively homogeneous of degree 1, that is, for all $s \geq 0$ and for all $(M, p, u, x) \in S_N \times \mathbb{R}^N \times \mathbb{R} \times \Omega$,

$$F(sM, sp, su, x) = sF(M, p, u, x).$$

(F2) There exist numbers $\Lambda \geq \lambda > 0$ and $\gamma, \delta > 0$ such that for all $M, N \in S_N$, $p, q \in \mathbb{R}^N$, $u, v \in \mathbb{R}$, $x \in \Omega$

$$\begin{aligned} \mathcal{M}^-_{\lambda,\Lambda}(M - N) - \gamma|p - q| - \delta|u - v| &\leq F(M, p, u, x) \\ -F(N, q, v, x) &\leq \mathcal{M}^+_{\lambda,\Lambda}(M - N) + \gamma|p - q| + \delta|u - v|. \end{aligned}$$

Here $\mathcal{M}^+_{\lambda,\Lambda}$ and $\mathcal{M}^-_{\lambda,\Lambda}$ are the maximal and minimal Pucci operators with parameters λ and Λ, respectively.
In this article we consider the extra assumption that the operator is radially invariant. For stating this, consider a smooth radially symmetric function $u = u(r)$, then we have

$$Du(x) = \frac{x}{r}\,u'(r) \quad \text{and} \quad D^2u(x) = \frac{u'(r)}{r}I + \left(u''(r) - \frac{u'(r)}{r}\right)\frac{x \otimes x}{r^2}.$$

Writing $m = u''(r)$ and $p = u'(r)$, we assume
(F3) The operator F is radially invariant, that is,

$$F\left(\frac{p}{r}I + \left(m - \frac{p}{r}\right)\frac{x \otimes x}{r^2}, \frac{p}{r}x, u, x\right)$$

depends on x only through r.

Now we can state our main theorem.

Theorem 1.1. *Under assumptions* (F1)–(F3), *the eigenvalue problem* (1.1)–(1.2) *in the ball* B_R *possesses sequences of classical radially symmetric solutions* $\{(\lambda_n^\pm, u_n^\pm)\}$, *both* u_n^+ *and* u_n^- *with n interior zeros* $0 < r_1 < \cdots < r_n < R$ *and* u_n^+ (*respectively* u_n^-) *is positive* (*respectively negative*) *in the interval* $(0, r_1)$. *Moreover the sequences* $\{\lambda_n^\pm\}$ *are increasing and the sequences* $\{(\lambda_n^\pm, u_n^\pm)\}$ *are complete in the sense that there are no other radially symmetric eigenpairs of* (1.1)–(1.2).

As we already mentioned, we prove this theorem relying on ordinary differential equations arguments in combination of degree theory. As a first step we study the eigenvalues in an annulus which becomes a regular ordinary differential equations problem. In doing so we prove a one dimensional version of our main theorem whose precise statement is given in Theorem 4.1. The proof of the theorem uses the classical existence theory for the initial value problem together with maximum and comparison principles obtained by means of the Alexandrov–Bakelman–Pucci

(ABP) inequality. This allows us to prove an existence and uniqueness theorem for a Dirichlet boundary value problem upon which we set up a parameterized fixed point problem. We solve this fixed point problem via degree theory, through a version of the Krein Rutman theorem due to Rabinowitz [18]. Thus we obtain a spectral theory for the first positive and negative eigenvalues in an interval, which applies also to the annulus in the radially symmetric N-dimensional case.

In order to obtain the whole set of eigenvalues and eigenfunctions we consider a Nehari type argument. We emphasize that a crucial qualitative property needed in this approach is the monotonicity of the positive and negative semi-eigenvalues with respect to the domain (interval). When there is an underlying variational structure, this property is an easy consequence of the min–max characterization of the eigenvalues. But here the eigenvalues are obtained through nonlinear bifurcation theory and a new argument has to be used to prove their monotonicity. See Corollary 3.1.

As a second step in the proof of Theorem 1.1 we study the eigenvalue problem in a ball, following an approach similar to the one dimensional case, but studying in detail the singularity at the origin. Regularity and compactness properties are proved for solutions of this ordinary differential equation using elementary arguments.

The paper is organized as follows. After we prove some auxiliary results in Section 2, we treat the case of the principal eigenvalue for 1-dimensional problems in Section 3 and we prove some qualitative properties of the eigenvalues. In Section 4 we prove the existence of a complete sequence of eigenvalues and eigenfunctions in the one dimensional case. Finally, in Section 5 we extend the results to the radially symmetric multidimensional case.

2. The One-Dimensional Case: Preliminaries

In this section we assume that the operator F satisfies hypotheses (F1) and (F2) with $N = 1$ and we prove a preliminary result that essentially says that we can isolate the second derivative from the equations, allowing us to use ordinary differential equation arguments. We end the section with the maximum and comparison principles in this one dimensional setting.

Before continuing let us observe that, in particular, we are assuming that $F: \mathbb{R}^3 \times [a, b] \to \mathbb{R}$ is a continuous function that satisfies

(F2) There are constants $\Lambda \geq \lambda > 0$, $\gamma > 0$ and $\delta > 0$ so that for all $(m, p, u, t), (m', p', u', t) \in \mathbb{R}^3 \times [a, b]$,

$$\begin{aligned} &-\delta|u - u'| - \gamma|p - p'| + \lambda(m - m')^+ - \Lambda(m - m')^- \\ &\quad \leq F(m, p, u, t) - F(m', p', u', t) \\ &\quad \leq \Lambda(m - m')^+ - \lambda(m - m')^- + \gamma|p - p'| + \delta|u - u'|. \end{aligned}$$

Here and in what follows we write $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$ so that $x = x^+ - x^-$.

In the one dimensional setting the main goal of this paper is to study the eigenvalue problem

$$F(u'', u', u, t) = -\mu u, \quad \text{in } [a, b], \quad u(a) = u(b) = 0 \tag{2.1}$$

and the auxiliary Dirichlet problem

$$F(u'', u', u, t) = f(t), \quad \text{in } [a, b], \quad u(a) = u(b) = 0. \tag{2.2}$$

In what follows we denote by $C_2(a, b)$ the space $C^2(a, b) \cap C^1([a, b])$ and we say that u is a solution of problems (2.1) and (2.2) if $u \in C_2(a, b)$ and if it satisfies the corresponding equation in (a, b), together with the boundary conditions. We notice that with our definition, a solution always has well defined derivatives at the extremes of the interval (a, b).

Our first result allows us to isolate u'' in equations (2.1) and (2.2), a very convenient fact for existence and regularity analysis.

Lemma 2.1. *If* (F2) *holds, there is a continuous function* $G : \mathbb{R}^3 \times [a, b] \to \mathbb{R}$ *so that*

$$F(m, p, u, t) = q \quad \textit{if and only if } m = G(p, u, q, t),$$

G having the Lipschitz property in (p, u, q) and being monotone increasing in q.

Proof. Using (F2), we see that

$$\lambda m^+ - \Lambda m^- \le F(m, p, u, t) - F(0, p, u, t) \le \Lambda m^+ - \lambda m^-, \tag{2.3}$$

from which it follows that, for every (p, u, t) fixed, $F(\cdot, p, u, t)$ is onto $\mathbb{R}$. Indeed, (2.3) implies that F is not bounded. This, together with the continuity property, proves our claim. On the other hand, if there are m, m' so that

$$F(m, p, u, t) = F(m', p, u, t),$$

then, from (F2) again,

$$\lambda(m - m')^+ - \Lambda(m - m')^- \le 0 \le \Lambda(m - m')^+ - \lambda(m - m')^-$$

from where it follows that $m = m'$. Thus, given (p, u, q, t), there is a unique m so that $F(m, p, u, t) = q$, we denote by $G(p, u, q, t)$ such m. This function G is continuous. We also prove that it has the Lipschitz property in the first three variables. Assume that

$$q = F(m, p, u, t) \quad \text{and} \quad q' = F(m', p', u', t)$$

then from (F2) we have, in case $m \ge m'$,

$$q - q' \ge \lambda(m - m') - \gamma|p - p'| - \delta|u - u'|,$$

so that

$$0 \le G(p, u, q, t) - G(p', u', q', t) \le \frac{1}{\lambda}|q - q'| + \frac{\gamma}{\lambda}|p - p'| + \frac{\delta}{\lambda}|u - u'|,$$

and if $m < m'$, then

$$q - q' \le -\lambda(m - m')^- + \gamma|p - p'| + \delta|u - u'|,$$

so that

$$0 \le G(p', u', q', t) - G(p, u, q, t) \le \frac{1}{\lambda}|q - q'| + \frac{\gamma}{\lambda}|p - p'| + \frac{\delta}{\lambda}|u - u'|.$$

Thus, G has the Lipschitz property in (p, u, q).

Finally, let $q \le q'$ and m, m' such that $m = G(p, u, q, t)$ and $m' = G(p, u, q', t)$. Then, $F(m, p, u, t) = q \le q' = F(m', p, u, t)$, so that from (F2), we have

$$-\Lambda(m - m')^- + \lambda(m - m')^+ \le F(m, p, u, t) - F(m', p, u, t) = q - q' \le 0,$$

which implies that $m \le m'$, proving that $G(p, u, q, t) \le G(p, u, q', t)$. □

The following is a direct consequence of Lemma 2.1 and the standard uniqueness theorem for Cauchy problem for ordinary differential equations.

Corollary 2.1. *Assume that F satisfies* (F1) *and* (F2) *and that $u \in C_2(a, b)$ is a nontrivial solution of*

$$F(u'', u', u, t) = -\mu u, \quad in\ (a, b), \quad u(a) = 0$$

then $u'(a) \neq 0$.

An important ingredient in the study of fully nonlinear problems is the maximum and comparison principles as expressed by the ABP inequalities. Here we present a one dimensional version:

Proposition 2.1 (ABP). *Assume that $u \in C_2(a, b)$ is a solution of*

$$\Lambda(u'')^+ - \lambda(u'')^- + \gamma|u'| \ge -f^- \quad in\ \{u > 0\},$$

with $u(a), u(b) \le 0$, then

$$\sup_{(a,b)} u^+ \le B\,\|f^-\|_{L^1(a,b)}. \tag{2.4}$$

On the other hand, if u is a solution of

$$\lambda(u'')^+ - \Lambda(u'')^- - \gamma|u'| \le f^+ \quad in\ \{u < 0\}$$

with $u(a), u(b) \ge 0$, then

$$\sup_{(a,b)} u^- \le B\,\|f^+\|_{L^1(a,b)}. \tag{2.5}$$

The constant B depends on λ, γ and is linear in $b - a$. Moreover, by positivity, B is increasing in $b - a$.

The proof of this proposition can be obtained from the general N-dimensional case, see [6] for example, however in Section 5 we present a simplified proof adapted to this situation, including also the radial case. Some direct corollaries that follow from Proposition 2.1 are:

Corollary 2.2. *Assume that F satisfies* (F2), *$F(m, p, u, t)$ is nonincreasing in u and $u \in C_2$. If u satisfies $F(u'', u', u, t) \geq -f^-$ and $u(a), u(b) \leq 0$, then* (2.4) *holds, and if u satisfies $F(u'', u', u, t) \leq f^+$ and $u(a), u(b) \geq 0$, then* (2.5) *holds.*

And the comparison principle:

Corollary 2.3. *Assume that F satisfies* (F2) *and F is nonincreasing in u. If $u, v \in C_2$ satisfy*

$$F(u'', u', u, t) \geq F(v'', v', v, t) \quad in\ (a, b),$$

and $u(a) \leq v(a)$, $u(b) \leq v(b)$, then, $u \leq v$ in $[a, b]$.

Proof. Suppose by contradiction that $Z := \{x \in (a, b) \mid u > v\}$ is not empty. Since F is nonincreasing we have

$$F(u'', u', u, t) \geq F(v'', v', u, t) \quad \text{in } Z,$$

Let now Z^* be a connected component of Z. Then, using (F2) and the (ABP) estimate in Z^* for $z = u - v$, we get

$$\sup_{Z^*} z \leq 0,$$

which is a contradiction. Thus $Z = \emptyset$. □

Next corollary is a strong maximum principle type of result.

Corollary 2.4. *Assume that F satisfies* (F2). *If $u, v \in C_2$ satisfy*

$$F(u'', u', u, t) \geq F(v'', v', v, t) \quad in\ (a, b),$$

and $u \leq v$ in (a, b) and for some $t_0 \in [a, b]$, $u(t_0) = v(t_0)$, then, either $u = v$ in $[a, b]$ or $u'(t_0) \neq v'(t_0)$.

Proof. Define $z = u - v \leq 0$ then by (F2), z satisfies

$$H(z) := \Lambda(z'')^+ - \lambda(z'')^- + \gamma|z'| - \delta z \geq 0 \quad \text{in } (a, b).$$

Now we assume that $z \not\equiv 0$. then there exists a point $t_1 \in [a, b]$ such that $z(t_1) = 0$ and $z < 0$ in an interval to the left or to the right of t_1. Let us assume that the interval is to the left of t_1 and denote it by $[t_2, t_1)$. Define now $w(t) = -e^{-A(t-t_1)} + 1$. For large A, the function w satisfies $H(w) \leq 0$ in (t_2, t_1). Take now $\varepsilon > 0$ such that $\varepsilon w(t_2) = z(t_2)$. Then, by comparison (Corollary 2.3), we get $\varepsilon w \geq z$ in (t_2, t_1). But $w'(t_1) > 0$, so that $z'(t_1) > 0$, thus proving the result. The argument when the interval is to the right is similar. □

Remark 2.1. The above result can be made more precise : let $\epsilon > 0$. If $u < v$ in an interval $(t_0 - \varepsilon, t_0)$, then $u'(t_0) > v'(t_0)$, while if $u < v$ in an interval $(t_0, t_0 + \varepsilon)$, then $u'(t_0) < v'(t_0)$.

Corollary 2.5. *Assume that F satisfies* (F2) *and F is nonincreasing in u. If $u, v \in C_2$ satisfy*

$$F(u'', u', u, t) \geq F(v'', v', v, t) \quad in\ (a, b),$$

and either $u'(a) \geq v'(a)$, $u(b) \leq v(b)$, or $u(a) \leq v(a)$, $u'(b) \leq v'(b)$, then, $u \leq v$ in $[a, b]$.

Proof. We start with the case $u'(a) \geq v'(a)$, $u(b) \leq v(b)$. If $u(a) \leq v(a)$ we conclude by comparison (Corollary 2.3). If $u(a) > v(a)$ we define $\tilde{v}(t) = v(t) + u(a) - v(a)$, so that

$$F(u'', u', u, t) \geq F(\tilde{v}'', \tilde{v}', \tilde{v}, t) \quad \text{in } (a, b),$$

and then again by comparison for u and $\tilde{v}$ we conclude that $u \leq \tilde{v}$, which is a contradiction with the strong maximum principle (Corollary 2.4 and Remark 2.1). The other case is similar. □

3. A Theory for the First Eigenvalue and Eigenfunction

The purpose of this section is to present a simplified version of the first eigenvalue theory in the one-dimensional case. We start with an existence theorem for the Dirichlet problem in a finite interval.

Theorem 3.1. *Assume that F satisfies* (F1) *and* (F2). *Then, for any $\kappa > \delta$, for any $f \in C^0[a, b]$, the equation*

$$F(u'', u', u, t) - \kappa u = f(t), \quad in\ (a, b), \quad u(a) = u(b) = 0 \tag{3.1}$$

has a unique solution $u \in C_2(a, b)$.

Proof. First, for a given $d \in \mathbb{R}$, we consider the initial value problem

$$F(u'', u', u, t) - \kappa u = f, \quad \text{for } t \in (a, b),$$
$$u'(a) = d, \quad u(a) = 0,$$

which has a unique solution since, by Lemma 2.1 this equation is equivalent to

$$u'' = G(u', u, f(t) + \kappa u, t), \quad \text{for } t \in (a, b), \tag{3.2}$$
$$u'(a) = d, \quad u(a) = 0, \tag{3.3}$$

with G having the Lipschitz property. We observe that the solution can be extended for all $t \in (a, b)$ because of the Lipschitz property of G. If we denote by $u(d, t)$ the corresponding solution, we see that the map $d \mapsto u(d, b)$ is continuous.

Let $A > 0$ be a large constant and set $v(t) = e^{A(t-a)} - 1$, then by (F2) we have

$$F(v'', v', v, t) - \kappa v \geq (\lambda A^2 - \gamma A - \delta - \kappa)e^{A(t-a)} \geq f(t) \quad \text{for } t \in (a, b).$$

Next we observe that $u(d_1, t)$, with $d_1 > v'(a) = A$, satisfies $u(d_1, b) > v(b) > 0$ by the assumption $\kappa > \delta$ and Corollary 2.5. In a similar way, one finds $d_2 < 0$ such that the solution $u(d_2, t)$ satisfies $u(d_2, b) < 0$. Finally, using the continuity of $d \to u(d, b)$ we conclude to the existence of a solution of (3.1). The uniqueness is a consequence of Corollary 2.5. □

Now we state a compactness lemma.

Lemma 3.1. *If* (F1)–(F2) *hold true, let* u_n *be the solution of equation* (3.1) *with right hand side* f_n, *where* $\{f_n\}$ *is a uniformly bounded sequence of continuous functions in the interval* $[a, b]$. *Then, there is a constant* C, *independent of* n, *such that*

$$|u_n(t)| \leq C, \quad |u'_n(t)| \leq C \quad \textit{and} \quad |u''_n(t)| < C, \quad \textit{for all } t \in [a, b].$$

Proof. Suppose first $\beta_n := \|u_n\|_\infty + \|u'_n\|_\infty$ is unbounded. Define $v_n(r) = u_n(r)/\beta_n$. Then $\{v_n\}$ and $\{v'_n\}$ are bounded and v_n satisfies

$$F(v''_n, v'_n, v'_n, v_n, t) - \kappa v_n = \frac{f_n}{\beta_n} \quad \text{in } (a, b), \tag{3.4}$$

$$v_n(a) = v_n(b) = 0. \tag{3.5}$$

Thus we conclude from (3.4) that for a positive constant C

$$|v''_n(t)| < C, \quad \text{for all } t \in [a, b].$$

Now we use the Arzela–Ascoli theorem, and find a sequence $v_{n_k} \to v$ uniformly in $C^1([a, b])$ and $v \in C_2(0, R)$ is a solution of (3.1) with right-hand side equal to 0. At this point we may use the (ABP) inequality, to obtain $v \equiv 0$. But this is impossible since $\|v_n\|_\infty + \|v'_n\|_\infty = 1$ for all n. Thus there exists $C > 0$ such that $|u_n(t)| \leq C, \quad |u'_n(t)| \leq C$ for all $t \in [a, b]$ and so from (3.4)

$$|u''_n(t)| < C, \quad \text{for all } t \in [a, b].$$

Next we present an existence result that will be used in an approximation procedure in the multidimensional radial case in Section 5.

Theorem 3.2. *Assume that* F *satisfies* (F1) *and* (F2). *Then, for every* $\kappa > \delta$, $c \in [a, b)$ *and* $f \in C^0[a, b]$, *the equation*

$$F(u'', u', u, t) - \kappa u = f(t), \quad \textit{in } (a, b), \quad u'(c) = u(b) = 0 \tag{3.6}$$

has a unique solution $u \in C_2(a, b)$.

Proof. For a given $d \in \mathbb{R}$, we consider the initial value problem

$$u'' = G(u', u, f(t) + \kappa u, t), \quad \text{for } t \in (a, b),$$
$$u'(c) = 0, \quad u(c) = d.$$

We denote by $u(d, t)$ the corresponding solution and we observe that the map $d \mapsto u(d, b)$ is continuous. Let $A > 0$ be a large constant and set $v(t) = -e^{A(t-b)} + 1$, then by (F2) we have

$$F(v'', v', v, t) - \kappa v \leq f(t) \quad \text{for } t \in (c, b).$$

Next we observe that $u(d_1, t)$ with $d_1 = v(c)$ satisfies $u(d_1, b) \geq v(b) = 0$, by the assumption $\kappa > \delta$ and Corollary 2.3. In a similar way, one finds $d_2 < 0$ such that the solution $u(d_2, t)$ satisfies $u(d_2, b) < 0$. Finally, using the continuity of $d \to u(d, b)$ we conclude to the existence of a solution of (3.6). The uniqueness is a consequence of Corollary 2.5. □

Now that we have completed the basic existence theory for (3.1) we address the existence of the first eigenvalue and eigenfunction as an application of global bifurcation theory. We follow the ideas of Rabinowitz [18], where global bifurcation theory is used to give a proof of a version of Krein–Rutman theorem in the linear case (Theorem VIII.3 in [18]). Here we adapt this result to our nonlinear setting. See also [7]. This approach will also allow us to obtain a monotonicity property of the first eigenvalues with respect to the domain (interval).

More precisely we will use Corollary 1 of Theorem VIII.1 in [18] that is:

Theorem 3.3. *Let E be a Banach space and K be a closed cone in E with a vertex at* 0. *Let $T : \mathbb{R}^+ \times K \to K$ be a compact operator such that $T(0, u) = 0$ for all $u \in E$, then there exists an unbounded connected component $\mathscr{C} \subset \mathbb{R}^+ \times K$ of solution of $u = T(\mu, u)$ containing* $(0, 0)$.

Theorem 3.4. *Under assumptions* (F1) *and* (F2), *the eigenvalue problem*

$$F(u'', u', u, t) = -\mu u, \quad \text{in } (a, b), \quad u(a) = u(b) = 0 \tag{3.7}$$

has a solution (u^+, λ^+), with $u^+ > 0$ in (a, b) and another solution (u^-, λ^-) with $u^- < 0$ in (a, b). Moreover, every positive (*resp. negative*) *solution of equation* (3.7) *is a multiple of u^+* (*resp. u^-*).

Proof. We define $K = \{u \in C[a, b]/u \geq 0, u(a) = u(b) = 0\}$ and fix a $\kappa > \delta$. Then, we use Theorem 3.1 to solve the Dirichlet problem

$$F(u'', u', u, t) - \kappa u = -g(t), \quad \text{in } (a, b), \quad u(a) = u(b) = 0 \tag{3.8}$$

for $g \in K$. We denote this solution by $\mathscr{L}(g)$ and define the operator $T : \mathbb{R}^+ \times K \to K$ as $T(\mu, f) = \mu\mathscr{L}(f)$. The operator T is well defined and, as a consequence of Corollaries 2.4 and 2.3, $T(\mu, f) > 0$ for every $f \in K\backslash\{0\}$, $\mu > 0$. Moreover T is compact by Lemma 3.1 and $T(0, g) = 0$ for every $g \in K$. Thus T satisfies the hypothesis of Theorem 3.3. Take $u_0 \in K\backslash\{0\}$, then there exists $M > 0$ such that $M\mathscr{L}(u_0) \geq u_0$ as a consequence of Corollary 2.4. Define now $\mathscr{L}_\varepsilon : \mathbb{R}^+ \times K \to K$ as $\mathscr{L}_\varepsilon(\mu, u) = \mu\mathscr{L}(u + \varepsilon u_0)$, for $\varepsilon > 0$. Then, from Theorem 3.3 there exists an unbounded connected component $\mathscr{C}_\varepsilon$ of solutions to $\mathscr{L}_\varepsilon(\mu, u) = u$, moreover $\mathscr{C}_\varepsilon \subset [0, M] \times K$. To see this fact, let $(\mu, u) \in \mathscr{C}_\varepsilon$, then

$$u = \mu\mathscr{L}(u + \varepsilon u_0)\,.$$

Hence by comparison $u \geq \mu\varepsilon\mathscr{L}(u_0) \geq \frac{\mu}{M}\varepsilon u_0$. If we apply $\mathscr{L}$ we get

$$\mathscr{L}(u) \geq \frac{\mu}{M}\varepsilon\mathscr{L}(u_0) \geq \frac{\mu}{M^2}\varepsilon u_0.$$

But $u \geq \mu\mathscr{L}(u)$, then $u \geq \left(\frac{\mu}{M}\right)^2\varepsilon u_0$. By recurrence we get

$$u \geq \left(\frac{\mu}{M}\right)^n \varepsilon u_0 \quad \text{for all } n \geq 2$$

and we conclude that $\mu \leq M$. This and the fact that $\mathscr{C}_\varepsilon$ is unbounded implies that there exists $(\mu_\varepsilon, u_\varepsilon) \in \mathscr{C}_\varepsilon$ such that $\|u_\varepsilon\|_\infty = 1$. This and Lemma 3.1 imply a uniform bound in $C_2(a, b)$, allowing us to pass to the limit as $\varepsilon \to 0$ to find $\mu^+ \in [0, M]$ and $u^+ > 0$ such that $u^+ = \mu^+\mathscr{L}(u^+)$. From here we also deduce that $\mu^+ > 0$ and then we define $\lambda^+ = -\kappa + \mu^+$. The isolation of the eigenvalue is a direct consequence of Lemma 2.1. For the simplicity assume by contradiction that the exists $h \in K\backslash\{0\}$ and μ a solution to

$$F(h'', h', h, t) = -\mu h, \quad \text{in } (a, b), \quad u(a) = u(b) = 0 \tag{3.9}$$

By Corollary 2.1 there exists a constant such that $h < su^+$. Define now

$$\tau = \inf\{s > 0 : h \leq su^+ \text{ in } (a, b)\}$$

Suppose that $\mu \leq \mu^+$. Then by Corollary 2.4 $h = \tau u^+$ and $\mu = \mu^+$ or $h < \tau u^+$, this last fact is a contradiction the definition of τ and Corollary 2.4. The case $\mu > \mu^+$ is similar. □

In what follows we denote by $\lambda^+(t_1, t_2)$ the first eigenvalue associated to a positive eigenfunction, and $\lambda^-(t_1, t_2)$ the first eigenvalue associated to a negative eigenfunction, given in Theorem 3.4 for the problem (3.7) in the interval $(t_1, t_2) \subset (a, b)$.

Corollary 3.1. *If* $(a_1, b_1) \subset (a, b)$ *and* $(a_1, b_1) \neq (a, b)$ *then*

$$\lambda^\pm(a_1, b_1) > \lambda^\pm(a, b).$$

Proof. We consider the eigenpair (λ_1^+, u_1^+), $\mu_1^+ = \kappa + \lambda_1^+$, given by Theorem 3.4 on the interval $(a_1, b_1) \subset (a, b)$, so that

$$F((u_1^+)'', (u_1^+)', u_1^+, t) = -\lambda_1^+ u_1^+ \quad \text{in } (a_1, b_1).$$

If $\underline{u}$ is the function obtained by extending u_1^+ by zero to the whole interval $[a, b]$, we define $\tilde{u}$ as the unique solution of

$$F(\tilde{u}'', \tilde{u}', \tilde{u}, t) - \kappa\tilde{u} = -\mu_1^+\underline{u} \quad \text{in } (a, b), \quad u(a) = u(b) = 0.$$

Then, using Comparison and Strong Maximum Principle in the interval $[a_1, b_1]$ we see that $\tilde{u} > u_1^+$ in $[a_1, b_1]$ and consequently $\tilde{u} > \underline{u}$ in (a, b). If we define $w = \mathscr{L}(\underline{u})$ and $v = \mathscr{L}(\tilde{u})$, then

$$F(w'', w', w, t) - \kappa w = -\underline{u} > -\tilde{u} = F(v'', v', v, t) - \kappa v,$$

so, again by Comparison and Strong Maximum Principle, $w < v$, which implies $\tilde{u} = \mu_1^+ \mathscr{L}(\underline{u}) < \mu_1^+ \mathscr{L}(\tilde{u})$. Here we may replace μ_1^+ by a slightly smaller value $M < \mu_1^+$, without changing the strict inequality. Now we repeat the arguments of the proof of Theorem 3.4, with $u_0 = \tilde{u}$ and $M < \mu_1^+$, to obtain that $\mu^+ := \lambda^+(a, b) + \kappa \leq M < \mu_1^+$, thus completing the proof for λ^+. The proof for λ^- is similar. □

Corollary 3.2. *The functions* $\lambda^+, \lambda^- : \{(t_1, t_2)/a \leq t_1 < t_2 \leq b\} \to \mathbb{R}$ *are continuous and*

$$\lim_{t_2 - t_1 \to 0^+} \lambda^+(t_1, t_2) = \lim_{t_2 - t_1 \to 0^+} \lambda^-(t_1, t_2) = \infty.$$

Proof. The continuity of these functions is a consequence of the uniqueness of the eigenvalues for positive (negative) eigenfunctions. While the limit is a consequence of Proposition 2.1, in fact denoting $\mu^+ = \lambda + \kappa$, with κ as in Theorem 3.1, from (2.4) we obtain that

$$\sup_{(t_1, t_2)} u^+ \leq B\mu^+ \|u^+\|_{L^1(t_1, t_2)} \leq B\mu^+(t_2 - t_1) \sup_{(t_1, t_2)} u^+,$$

which completes the proof. □

4. Multiple Eigenvalues and Eigenfunctions in the One-Dimensional Case

In this section we consider the existence of higher eigenvalues, associated to changing-sign eigenfunctions in the general setting already defined in Section 2. More precisely, we prove the following theorem

Theorem 4.1. *Under assumptions* (F1) *and* (F2) *the eigenvalue problem*

$$F(u'', u', u, t) = -\mu u, \quad in\ (a, b), \quad u(a) = u(b) = 0 \tag{4.10}$$

has two sequences of solutions $\{(\lambda_n^\pm, u_n^\pm)\}$ *such that* $u_n^\pm$ *have both n interior zeros* $t_1 < \cdots < t_n$ *and* u_n^+ *(resp.* u_n^-*) is positive (resp. negative) in the interval* (a, t_1), *negative (resp. positive) on* (t_1, t_2). *Moreover the sequence* $\{\lambda_n^\pm\}$ *is increasing and the sequence* $\{(\lambda_n^\pm, u_n^\pm)\}$ *is complete in the sense that there are no eigenpairs of* (4.10) *outside these sequences.*

We devote this section to the proof of this theorem using degree theory. We start with a given $n \in \mathbb{N}$, $n \geq 1$ and we define

$$\Delta_n = \{(t_1, \ldots, t_n)/a < t_1 < t_2 < \cdots < t_n < b\},$$

$t_0 = a$ and $t_{n+1} = b$ and the function $V : \Delta_n \to \mathbb{R}^n$ as

$$V_i(\vec{t}) = \lambda^{(-1)^i}(t_{i-1}, t_i) - \lambda^{(-1)^{i+1}}(t_i, t_{i+1}), \qquad i = 1, \ldots, n,$$

whereby $\lambda^{(\pm 1)}$ we mean $\lambda^{\pm}$. We observe that under our assumptions, Corollary 3.2 implies that the function V is continuous in Δ_n. We have the following

Theorem 4.2. *Under assumptions* (F1) *and* (F2), *for every* $n \in \mathbb{N}$ *there is* $\vec{t} \in \Delta_n$ *such that*

$$V(\vec{t}) = 0. \tag{4.11}$$

Proof. Let us consider a point $\vec{t} \in \partial\Delta_n$, then there are $0 \le k < \ell \le n+1$ such that $t_k = t_{k+1} = \cdots = t_\ell$, such that they additionally satisfy $k = 0$ or $t_{k-1} < t_k$ and $\ell = n+1$ or $t_\ell < t_{\ell+1}$. We further assume that k is the smallest integer for which the situation described occur. We observe that simultaneously we cannot have $k = 0$ and $\ell = n+1$.

In what follows we denote by $e_1, e_2, \ldots, e_n$ the canonical basis of $\mathbb{R}^n$. If $k = 0$ then we have $t_\ell < t_{\ell+1}$ and we define $T(\vec{t}) = -e_\ell$. If $0 < k < \ell < n+1$ then we define $T(\vec{t}) = e_k - e_\ell$. And if $0 < k$ and $\ell = n+1$ then we define $T(\vec{t}) = e_k$. In this way we have defined $T : \partial\Delta_n \to \mathbb{R}^n$ as a function. We observe that T defines a 'normal vector field', which is not continuous on the edges of $\partial\Delta_n$.

Assume now that we have a sequence $\{\vec{t}_m\} \subset \Delta_n$ such that $\vec{t}_m \to \vec{t} \in \partial\Delta_n$, as $m \to \infty$. Then we have

$$\lim_{m\to\infty} V(\vec{t}_m) \cdot T(\vec{t}) = -\infty. \tag{4.12}$$

In order to prove (4.12) we have three cases, according to the numbers $k < \ell$ associated to $\vec{t} \in \partial\Delta_n$. First, if $k = 0$ and $\ell < n+1$, then

$$(\vec{t}_m)_\ell - (\vec{t}_m)_{\ell-1} \to 0 \quad \text{and} \quad (\vec{t}_m)_{\ell+1} - (\vec{t}_m)_\ell > c > 0,$$

so that $V_\ell(\vec{t}_m) \to \infty$ as m goes to $+\infty$, proving (4.12). Second, if $k > 0$ and $\ell < n+1$ then we have

$$(\vec{t}_m)_\ell - (\vec{t}_m)_{\ell-1} \to 0 \quad \text{and} \quad (\vec{t}_m)_{\ell+1} - (\vec{t}_m)_\ell > c > 0$$

and also

$$(\vec{t}_m)_{k+1} - (\vec{t}_m)_k \to 0 \quad \text{and} \quad (\vec{t}_m)_k - (\vec{t}_m)_{k-1} > c > 0.$$

Thus, by definition of V we have $V_\ell(\vec{t}_m) \to \infty$ and $V_k(\vec{t}_m) \to -\infty$ as m goes to $+\infty$, proving (4.12). The third case, when $k > 0$ and $\ell = n+1$ is similar. This completes the proof of (4.12).

Now we consider the point $\vec{t}_0$ defined as

$$(\vec{t}_0)_i = \frac{(n-i+1)a + ib}{n+1} \quad \text{for } i = 1, \ldots, n.$$

Note that $\vec{t}_0 \in \Delta_n$ is the average of the vertices of Δ_n. Next we define the field $F: \Delta_n \to \mathbb{R}^n$ as $F(\vec{t}) = -\vec{t} + \vec{t}_0$ and we claim that

$$F(\vec{t}) \cdot T(\vec{t}) < 0 \quad \text{for all } \vec{t} \in \partial\Delta_n. \tag{4.13}$$

In fact, given the numbers $k < \ell$ associated to $\vec{t}$ we have three cases. First, if $k = 0$ and $\ell < n + 1$, then

$$F(\vec{t}) \cdot T(\vec{t}) = (\vec{t} - \vec{t}_0)_\ell = a - \frac{(n - \ell + 1)a + \ell b}{n + 1} = \frac{\ell(a - b)}{n + 1} < 0.$$

Second, if $k > 0$ and $\ell < n + 1$ then we have

$$F(\vec{t}) \cdot T(\vec{t}) = (\vec{t} - \vec{t}_0)_\ell - (\vec{t} - \vec{t}_0)_k = \frac{(\ell - k)(a - b)}{n + 1} < 0.$$

The third case, when $k > 0$ and $\ell = n + 1$ is again similar to the previous one. This completes the proof of (4.13).

Now we define the (continuous) homotopy $H: \Delta_n \times [0, 1] \to \mathbb{R}^n$ as $H(\vec{t}, s) = sV(\vec{t}) + (1 - s)F(\vec{t})$. Then we claim that there is $\varepsilon > 0$ so that for all $s \in [0, 1]$ and all $\vec{t} \in \Delta_n$ satisfying $\operatorname{dist}(\vec{t}, \partial\Delta_n) < \varepsilon$, we have

$$H(\vec{t}, s) \neq 0.$$

Assuming that the above claim is true, we apply homotopy invariance of the degree, together with $\deg(F, \Delta_n, 0) = (-1)^n$, to get the existence of a zero for V.

In order to prove the claim we assume the contrary. Then there is a sequence $(\vec{t}_m, s_m)$ such that $\vec{t}_m \to \vec{t} \in \partial\Delta_n$ and $s_m \to s \in [0, 1]$ as $m \to \infty$ and such that $H(\vec{t}_m, s_m) = 0$ for all m. Thus we have

$$\lim_{m\to\infty} H(\vec{t}_m, s_m) \cdot T(\vec{t}) = 0,$$

contradicting (4.12) and (4.13).

If we observe the definition of V we see that the first component V_1 is associated to $\lambda^-(t_0, t_1)$ and $\lambda^+(t_1, t_2)$, so that the eigenfunction that we can construct out of solutions of equation (4.11) will start being negative. For eigenfunctions starting with positive values in the first interval (t_0, t_1) we need to define the above arguments to the slightly modified function

$$\widetilde{V}_i(\vec{t}) = \lambda^{(-1)^{i+1}}(t_{i-1}, t_i) - \lambda^{(-1)^i}(t_i, t_{i+1}), \quad i = 1, \ldots, n. \qquad \square$$

Proof of Theorem 4.1. Given a solution $\vec{t} \in \Delta_n$ of (4.11) we proceed to construct an eigenfunction as follows. On the interval (a, t_1) we define u_n^- as $u^-(a, t_1)$. Then, on (t_1, t_2) the function u_n^- will be equal to $\alpha_1 u^+(t_1, t_2)$, where α_1 is chosen so that $(u^-)'(a, t_1)(t_1) = \alpha_1 (u^+)'(t_1, t_2)(t_1)$. The existence of α_1 is a consequence of Corollary 2.1. Here we denote by $u^\pm(t, s)$ the corresponding positive or negative eigenfunction on the interval (t, s). Repeating this argument we will finally arrive to the function u_n^-, which is of class $C^1[a, b]$ and of class C^2 in the interior of every interval of the form (t_i, t_{i+1}). Then we use the equation satisfied by each partial eigenfunction and the continuity of F, rather than that of G, to find that u_n^- is of class $C^2(a, b)$. The associated eigenvalue is simply $\lambda_n^- = \lambda^-(a, t_1)$.

For proving uniqueness, we assume that we have a second eigenpair (λ, v) associated with n such that there exist values $a < s_1 < s_2 < \cdots < s_n < b$ and v changes sign at those points, starting with negative values in the interval (a, s_1). If $\lambda = \lambda_n^-$ then by Corollary 3.1, we necessarily have $s_i = t_i$ for all $i = 1, 2, \ldots, n$ and then the simplicity and isolation of the first eigenfunctions proved in Theorem 3.4 completes the argument.

Now we assume that $\lambda > \lambda_n^-$, then by Corollary 3.1 we have $s_1 < t_1$ and then

$$\lambda > \lambda^-(a, t_1). \tag{4.14}$$

We either have $1 \le i \le n-1$ such that $(t_i, t_{i+1}) \subset (s_i, s_{i+1})$ or $s_n \le t_n$. In the first case, if i is odd $\lambda^+(t_i, t_{i+1}) \ge \lambda$ and if i is even $\lambda^-(t_i, t_{i+1}) \ge \lambda$, contradicting (4.14) in both cases. In the second case, $\lambda \le \lambda^+(t_n, b)$, if n is odd, contradicting (4.14) again and similarly if i is even. □

5. The Eigenvalue and Eigenfunction Theory in the Radial Case

We devote this section to proving our main theorem. We assume that $N > 1$ and that the operator F satisfies (F1), (F2) and it is radially invariant, that is, it satisfies also (F3). Our purpose is to study the eigenvalue problem (1.1)–(1.2) where $\Omega = B_R$, is the ball of radius R centered at the origin.

We start with some notation. Given our operator F we define $\mathcal{F} : \mathbb{R}^4 \times [0, R] \to \mathbb{R}$ as

$$\mathcal{F}(m, \ell, p, u, r) = F(\ell I + (m - \ell)e_1 \otimes e_1, pe_1, u, re_1)$$

and consider the operators

$$P^+(a, b) = \Lambda(a^+ + (N-1)b^+) - \lambda(a^- + (N-1)b^-)$$

and

$$P^-(a, b) = \lambda(a^+ + (N-1)b^+) - \Lambda(a^- + (N-1)b^-).$$

Here m stands for $u''(r)$, p for $u'(r)$ and ℓ for $\frac{u'(r)}{r}$. Under assumption (F3), we may write the equivalent of Hypothesis (F2) in this radially symmetric setting as follows

(F2′) There exist $\gamma, \delta > 0$ such that for all $m, m', \ell, \ell', p, p', u, u' \in \mathbb{R}$, $r \in [0, R]$,

$$\begin{aligned} P^-(m - m', \ell - \ell') - \gamma|p - p'| - \delta|u - u'| &\le \mathcal{F}(m, \ell, p, u, r) \\ -\mathcal{F}(m', \ell', p', u', r) &\le P^+(m - m', \ell - \ell') + \gamma|p - p'| + \delta|u - u'| \end{aligned}$$

The proof of Theorem 1.1 follows the general lines of that of Theorem 4.1. The new difficulty here is the singularity at $r = 0$. We deal with it by using an approximation procedure: in the interval $[\varepsilon, R)$ we apply the results of the previous sections. Then we obtain uniform estimates on the approximated solutions and their derivatives in order to pass to the limit. In the rest of this section we do this and then we complete the proof of Theorem 1.1. We also prove the (ABP) inequality for the multidimensional radial case, and thus also that of Proposition 2.1.

The next lemma is the analogue of Lemma 2.1 and it can be proved following the same arguments.

Lemma 5.1. *If* (F1), (F2′) *and* (F3) *hold true, then,*

1. *There is a continuous function* $\mathcal{G} : \mathbb{R}^4 \times [0, R] \to \mathbb{R}$ *so that*

$$\mathcal{F}(m, \ell, p, u, r) = q \quad \text{if and only if } m = \mathcal{G}(\ell, p, u, q, r)$$

and $\mathcal{G}$ *has the Lipschitz property in* (ℓ, p, u, q).

2. *There is a continuous function* $\mathcal{G}_1 : \mathbb{R}^3 \times [0, R] \to \mathbb{R}$ *such that*

$$\mathcal{F}(\ell, \ell, p, u, r) = q \quad \text{if and only if } \ell = \mathcal{G}_1(p, u, q, r)$$

and $\mathcal{G}_1$ *has the Lipschitz property in* (p, u, q).

The following is a regularity result, extending the second derivative of a solution to the origin, the only point in the domain that makes a difference with the one dimensional case.

Lemma 5.2. *Assume that* (F1), (F2′) *and* (F3) *hold true and assume also that* f *is a continuous function in* $[0, R]$ *and* $u : [0, R] \to \mathbb{R}$ *is a solution of*

$$\mathcal{F}\left(u'', \frac{u'}{r}, u', u, r\right) = f(r) \quad \text{in } (0, R) \tag{5.1}$$

with boundary conditions

$$u'(0) = 0, \quad u(R) = 0. \tag{5.2}$$

If the functions $|u''(r)|$ *and* $\left|\frac{u'(r)}{r}\right|$ *are bounded in* $(0, R)$ *then*:

1. *The limit*

$$\lim_{r\to 0} \frac{u'(r)}{r}$$

exists and consequently $u''(0)$ *is well defined.*

2. *The function* $u(x) = u(|x|)$ *is a* $C^2(B_R)$*-solution to the partial differential equation*

$$F(D^2u, Du, u, x) = f \quad \text{in } B_R, \tag{5.3}$$

with boundary condition

$$u = 0 \quad \text{on } \partial B_R. \tag{5.4}$$

Proof. We use Lemma 5.1 to write

$$u'' = \mathcal{G}\left(\frac{u'}{r}, u', u, f, r\right),$$

and then, using the boundary condition and writing $\ell = \frac{u'}{r}$, we find

$$r\ell = \int_0^r \mathcal{G}(\ell, u', u, f, s)ds.$$

Differentiating the above functional equality, we get

$$r\ell' + \ell = \mathcal{G}(\ell, u', u, f, r).$$

Assume, by contradiction, that ℓ does not converge as $r \to 0^+$. Then there are two numbers $a < b$ and two sequences $\{r_n^+\}$, $\{r_n^-\}$ such that

$$\lim_{n\to\infty} r_n^+ = \lim_{n\to\infty} r_n^- = 0,$$

and

$$\ell'(r_n^+) = \ell'(r_n^-) = 0, \quad \lim_{n\to\infty} \ell(r_n^+) = b, \quad \lim_{n\to\infty} \ell(r_n^-) = a.$$

Then we have

$$\ell(r_n^\pm) = \mathcal{G}(\ell(r_n \pm), u'(r_n^\pm), u(r_n^\pm), f(r_n^\pm), r_n^\pm)$$

and also,

$$\ell(r_n^\pm) = \mathcal{G}_1(u'(r_n^\pm), u(r_n^\pm), f(r_n^\pm), r_n^\pm).$$

Since f, u and u' are continuous at $r = 0$ as well as $\mathcal{G}_1$ we have that

$$\lim_{n\to\infty} \ell(r_n^+) = \lim_{n\to\infty} \ell(r_n^-),$$

which is a contradiction. Regarding assertion 2, we notice that by the above facts and Lemma 5.1, $u(x) = u(|x|)$ is a $C^2(B_R)$ function and by (F3), u is a solution to (5.3) and (5.4). □

Next we prove the (ABP) inequality in the multidimensional radial case.

Proposition 5.1. *Assume that $u \in C_2(0, R)$ is a solution of*

$$P^+\left(u'', \frac{u'}{r}\right) + \gamma|u'| \geq -f^- \quad in\ \{u > 0\},$$

with $u(R) \leq 0$ and $u'(0) = 0$, then

$$\sup_{(0,R)} u^+ \leq B\|f^-\|_{L^N(B_R)}. \tag{5.5}$$

On the other hand, if u is a solution of

$$P^-\left(u'', \frac{u'}{r}\right) - \gamma|u'| \leq f^+ \quad in\ \{u < 0\}$$

with $u(R) \geq 0$ *and* $u'(0) = 0$, *then*

$$\sup_{(0,R)} u^- \leq B\|f^+\|_{L^N(B_R)}. \tag{5.6}$$

The constant B depends on N, λ, γ *and* R. *Moreover, by positivity,* B *is increasing in* R.

Proof. Assume that $\sup_{(0,R)} u > 0$ and define $l_0 = \frac{\sup_{(0,R)} u}{R}$ and denote by r_0 a maximum point of u in $[0, R)$. There exists a point $r_- \in (0, R)$ such that $-u'(r_-) = l_0$ and $-u'(r) \leq l_0$ in the interval (r_0, r_-). Moreover, we can find a set I (union of intervals) in (r_0, r_-) so that $u'' \leq 0$ in I and $-u'(I) = (0, l_0)$. We observe that on I both u'' and u' are non-positive and then

$$P^+\left(u'', \frac{u'}{r}\right) = \lambda\left(u'' + (N-1)\frac{u'}{r}\right) \quad \text{for all } r \in I.$$

Then, for any $k > 0$, making a change of variables we find

$$\begin{aligned}\ln\left(1 + \frac{l_0^N}{k}\right) &= \int_0^{l_0^N} \frac{dz}{z+k} \\ &\leq \int_I \frac{-N(-u'(r))^{N-1}u''(r)\,dr}{(-u'(r))^N + k},\end{aligned}$$

where the inequality holds because $-u'$ is not necessarily injective in I. Following the inequality we have

$$\begin{aligned}\ln\left(1 + \frac{l_0^N}{k}\right) &\leq N\int_I \left(\frac{-u'(r)}{r}\right)^{N-1}(-u''(r))\frac{r^{N-1}dr}{(-u'(r))^N + k} \\ &\leq N^{1-N}\int_I \left(-u''(r) - (N-1)\frac{u'(r)}{r}\right)^N \frac{r^{N-1}dr}{(-u'(r))^N + k} \\ &\leq \frac{(N/2)^{1-N}}{\lambda^N}\int_I \left(\frac{|f^-|^N}{k} + \gamma^N\right) r^{N-1}dr \\ &\leq \frac{(N/2)^{1-N}}{\lambda^N}\left(\frac{1}{k\omega_N}\|f^-\|^N_{L^N(B_R)} + \frac{(\gamma R)^N}{N}\right).\end{aligned}$$

Here we have used comparison between the arithmetic and geometric mean and the inequality $(a+b)^N \leq 2^{N-1}(a^N + b^N)$. We denoted by ω_N the measure of the sphere S^{N-1}. We observe that the above inequality implies that $\|f^-\|_{L^N(B_R)} > 0$, since k is arbitrary, then we may choose $k = \|f^-\|^N_{L^N(B_R)}$, and find $l_0 \leq C\|f^-\|_{L^N(B_R)}$, for some constant $C > 0$ depending on N, λ, γ and R. □

Remark 5.1. When $N = 1$ the proof just presented reduces to a proof of Proposition 2.1 with the obvious change in the domain in order to consider a general interval (a, b).

Next corollaries follow from Proposition 5.1.

Corollary 5.1. *Assume that hypotheses* (F1)–(F2′)–(F3) *hold and additionally that* $\mathcal{F}(m, \ell, p, u, r)$ *is nonincreasing in* u. *If* $u \in C_2(0, R)$ *satisfies* $\mathcal{F}(u'', u'/r, u', u, r) \geq$

$-f^-$ and $u(R) \le 0$, $u'(0) = 0$, then (5.5) *holds, and if $u \in C_2(0, R)$ satisfies $\mathcal{F}(u'', u'/r, u', u, r) \le f^+$ and $u(R) \ge 0$, $u'(0) = 0$, then* (5.6) *holds.*

The following comparison principle also follows from Proposition 5.1:

Corollary 5.2. *Assume that hypotheses* (F1)–(F2′)–(F3) *hold and additionally that $\mathcal{F}(m, \ell, p, u, r)$ is decreasing in u. If $u, v \in C_2(0, R)$ satisfy*

$$\mathcal{F}(u'', u'/r, u', u, t) \ge \mathcal{F}(v'', v'/r, v', v, t) \quad \text{in } (0, R),$$

and $u(R) \le v(R)$, $u'(0) = v'(0) = 0$, then, $u \le v$ in $[0, R]$.

As in Section 3, before proving the existence of eigenvalues and eigenfunctions, we prove the existence of solutions for a related Dirichlet problem, as follows.

Theorem 5.1. *Assume that* (F1)–(F2′)–(F3) *hold true. There is $\kappa > 0$ so that the equation*

$$\mathcal{F}\left(u'', \frac{u'}{r}, u', u, r\right) - \kappa u = f \quad \text{in } (0, R), \tag{5.7}$$

$$u'(0) = 0, \quad u(R) = 0, \tag{5.8}$$

possesses a unique solution for any given continuous function f.

The proof of this theorem can be done through an approximation procedure and using only elementary ODE arguments. In this direction we have the following two results.

Lemma 5.3. *Assume assumptions* (F1)–(F2′)–(F3). *There is $\kappa > 0$ (independent of ε) so that for any given $f \in C^0[0, R]$ and $\varepsilon > 0$, there exists a unique solution u_ε of*

$$\mathcal{F}\left(u'', \frac{u'}{r}, u', u, r\right) - \kappa u = f \quad \text{in } (\varepsilon, R), \tag{5.9}$$

$$u'(\varepsilon) = 0, \quad u(R) = 0. \tag{5.10}$$

The proof of this proposition is completely similar to that of Theorem 3.2 so we omit it. The following lemma provides estimates for the solution u_ε, independent of ε and its proof is inspired of that of Lemma 2.2 in [7].

Lemma 5.4. *Assume that* (F1)–(F2′)–(F3) *hold true and let u_ε be the solution to* (5.9)–(5.10) *given by Lemma* 5.3. *Then there is a constant C, independent of ε, such that*

$$\left|\frac{u'_\varepsilon(r)}{r}\right| \le C \quad \text{and} \quad |u''_\varepsilon(r)| < C, \quad \text{for all } \varepsilon > 0, \quad r \in [\varepsilon, R].$$

Proof. We first claim that if $u_\varepsilon(r)$ and $u'_\varepsilon(r)$ are uniformly bounded in $[\varepsilon, R]$, then $u'_\varepsilon(r)/r$ and $u''_\varepsilon(r)$ are uniformly bounded in $[\varepsilon, R]$. By contradiction, suppose the existence of two sequences $\varepsilon_n \to 0$ and $r_n \in (\varepsilon_n, R]$ such that

$$\lim_{n\to+\infty} \frac{u'_n(r_n)}{r_n} = -\infty,$$

where we write $u_n = u_{\varepsilon_n}$. From (5.9), (F2′) and our assumption on $u_\varepsilon(r)$ and $u'_\varepsilon(r)$, we have that $u''_n(r_n) \to +\infty$ as $n \to +\infty$.

If $u''_n(r) > 0$ for all $r \in (\varepsilon_n, r_n]$, then $u'_n(r_n) > 0$, which is impossible. Thus, for all n there exists $\bar{r}_n \in (\varepsilon_n, r_n)$ such that $u''(\bar{r}_n) = 0$ and $u''_n(r) > 0$ for all $r \in (\bar{r}_n, r_n)$. Hence $u'(\bar{r}_n) < u'(r_n)$, which implies that

$$\lim_{n\to+\infty} \frac{u'(\bar{r}_n)}{\bar{r}_n} = -\infty \quad \text{and} \quad u''(\bar{r}_n) = 0,$$

which is again impossible by (5.9), (F2′) and our assumption on $u_\varepsilon(r)$ and $u'_\varepsilon(r)$. Suppose next that for a sequence of points $r_n \in (\varepsilon_n, R)$ we have

$$\lim_{n\to+\infty} \frac{u'_n(r_n)}{r_n} = +\infty,$$

then with a similar argument we also get a contradiction. Thus, we have that $\{u'_\varepsilon(r)/r\}$ is bounded and as before we conclude that $\{u''_\varepsilon(r)\}$ is bounded, proving the claim.

Suppose now that $\{\beta_\varepsilon\}$ is unbounded with $\beta_\varepsilon = \|u_\varepsilon\|_\infty + \|u'_\varepsilon\|_\infty$. Define $v_\varepsilon(r) = u_\varepsilon(r)/\beta_\varepsilon$. Then $\{v_\varepsilon\}$ and $\{v'_\varepsilon\}$ are bounded and v_ε satisfies

$$\mathscr{F}\left(v''_\varepsilon, \frac{v'_\varepsilon}{r}, v'_\varepsilon, v_\varepsilon, r\right) - \kappa v_\varepsilon = \frac{f}{\beta_\varepsilon} \quad \text{in } (\varepsilon, R), \tag{5.11}$$

$$v'_\varepsilon(\varepsilon) = 0, \quad v_\varepsilon(R) = 0. \tag{5.12}$$

Using the claim again, we conclude that for a positive constant C

$$\left|\frac{v'_\varepsilon(r)}{r}\right| < C, \quad |v''_\varepsilon(r)| < C, \quad \text{for all } r \in [\varepsilon, R].$$

To proceed with the proof now we use the Arzela–Ascoli theorem, and find a sequence $v_{\varepsilon_n} \to v$ uniformly in $C^1([0, R])$ to a solution $v \in C_2(0, R)$ of (5.7)–(5.8) with right-hand side equal to 0. At this point we may use (ABP) inequality as given in Proposition 5.1, to obtain that this equation has a unique solution by the comparison principle given in Corollary 5.2, so $v \equiv 0$. But this is impossible since $\|v_\varepsilon\|_\infty + \|v'_\varepsilon\|_\infty = 1$ for all ε. □

Remark 5.2. If we use the claim given in the first part of the proof of Lemma 5.4 we see that for the function v defined as the limit of v_ε, there is a constant C so that

$$\left|\frac{v'(r)}{r}\right| \le C \quad \text{and} \quad |v''(r)| < C, \quad \text{for all } r \in (0, R].$$

Then we may apply Lemma 5.2 to find that $v''(0)$ is well defined and thus v is a solution to the corresponding partial differential equation in the ball with zero right hand side.

Proof of Theorem 5.1. Using Proposition 5.1 we obtain a sequence of approximating solutions for (5.7)–(5.8). Then we use Lemma 5.4 to obtain estimates that allows us to use the Arzela–Ascoli theorem as at the end of the proof of Lemma 5.4 to obtain a solution of the problem. □

Finally we state a compactness lemma, whose proof is similar to those of Lemmas 5.2 and 5.4 which are necessary to use Krein–Rutman theory to find the first eigenvalues.

Lemma 5.5. *If* (F1)–(F2′)–(F3) *hold true, let* u_n *be the solution of equation* (5.7)–(5.8) *with right hand side* f_n, *where* $\{f_n\}$ *is a uniformly bounded sequence of continuous functions in the interval* $[0, R]$. *Then, there is a constant* C, *independent of* n, *such that*

$$|u_n(r)| \le C, \quad \left|\frac{u_n'(r)}{r}\right| \le C \quad \text{and} \quad |u_n''(r)| < C, \quad \text{for all } r \in (0, R].$$

Next we have the existence of the first eigenvalues in the ball. This theorem is a particular case of the general eigenvalue theory for fully nonlinear equations. Here we have provided a proof which relies on elementary arguments.

Theorem 5.2. *Under assumptions* (F1), (F2′) *and* (F3), *the radially symmetric eigenvalue problem* (1.1)–(1.2) *in* $\Omega = B_R$ *has a solution* (λ^+, u^+), *with* $u^+ > 0$ *and radially symmetric in* B_R *and another solution* (λ^-, u^-) *with* $u^- < 0$ *and radially symmetric in* B_R. *Moreover*

i) $\lambda^+ \le \lambda^-$.
ii) *Every positive* (*resp. negative*) *solution of equation* (3.7) *is a multiple of* u^+ (*resp.* u^-).
iii) *If* $\lambda^\pm(R)$ *denotes the eigenvalue in* B_R *then* $\lambda^\pm(R) < \lambda^\pm(R')$ *if* $R > R'$.
iv) $\lambda^\pm(R) \to \infty$ *if* $R \to 0$.

Proof. With the aid of Theorem 5.1 and Lemma 5.5 we can follow step by step the arguments given in the proof of Theorem 3.4 to obtain the existence of the eigenvalues and eigenfunctions. The qualitative properties are proved similarly as in the one dimensional case shown in Section 3. □

Proof of Theorem 1.1. The arguments are the same as those given in the proof of Theorem 4.1. □

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