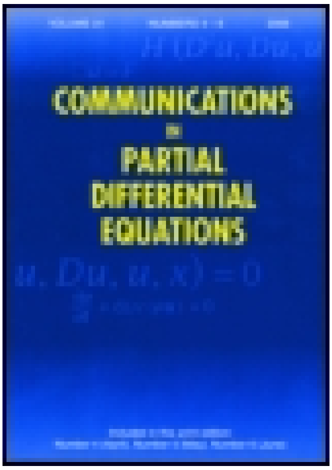

Communications in Partial Differential Equations
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/lpde20

Large Critical Exponents for Some Second Order Uniformly Elliptic Operators

Maria J. Esteban [a] , Patricio L. Felmer [b] & Alexander Quaas [c]

[a] Ceremade UMR CNRS 7534 , Université Paris-Dauphine , Paris, France

[b] Departamento de Ingeniería Matemática, and Centro de Modelamiento Matemático, UMR2071 CNRS-U Chile , Universidad de Chile , Santiago, Chile

[c] Departamento de Matemática , Universidad Santa María , Valparaíso, Chile

Published online: 12 Apr 2007.

Communications in Partial Differential Equations, 32: 543–556, 2007

ISSN 0360-5302 print/1532-4133 online
DOI: 10.1080/03605300500394397

Large Critical Exponents for Some Second Order Uniformly Elliptic Operators

MARIA J. ESTEBAN[1], PATRICIO L. FELMER[2], AND ALEXANDER QUAAS[3]

[1]Ceremade UMR CNRS 7534, Université Paris-Dauphine, Paris, France
[2]Departamento de Ingeniería Matemática, and Centro de Modelamiento Matemático, UMR2071 CNRS-U Chile, Universidad de Chile, Santiago, Chile
[3]Departamento de Matemática, Universidad Santa María, Valparaíso, Chile

In this paper we investigate the critical exponents of two families of Pucci's extremal operators. The notion of critical exponent that we have chosen for these fully nonlinear operators which are not variational is that of threshold between existence and nonexistence of the solutions for semilinear equations with pure power nonlinearities. Interesting new exponents appear in this context.

Keywords Critical exponent; Fully nonlinear; Pucci's operator; Second order elliptic operator.

Mathematics Subject Classification 35J60; 35J70.

1. Introduction

Associated to the Laplacian we have the Sobolev critical exponent, which is the largest number p_N^* having the property that the semi-linear equation

$$\Delta u + u^p = 0, \quad \text{in } \Omega \tag{1.1}$$

$$u = 0, \quad \text{on } \partial\Omega, \tag{1.2}$$

possesses a positive solution whenever $1 < p < p_N^*$ and for any given bounded domain Ω. This number, which depends on the dimension N, is given by $p_N^* = (N+2)/(N-2)$ and its name derives from the fact that the Sobolev space $H_0^1(\Omega)$ embeds itself continuously into $L^q(\Omega)$, for all domain Ω of $\mathbb{R}^N$, if and only if $1 < q \leq p_N^* + 1$, and the embedding is compact if and only if $1 < q < p_N^* + 1$ (and if the domain

Received August 1, 2005; Accepted September 1, 2005
Address correspondence to Maria J. Esteban, Ceremade UMR CNRS 7534, Université Paris-Dauphine 75–775, Paris Cedex 16, France; E-mail: esteban@ceremade.dauphine.fr

is bounded or at least very small at infinity). Moreover, if $p \geq p_N^*$ and the domain is star-shaped, Pohozaev's identity implies that the above equation does not have a positive solution (see Pohozaev, 1965). The Sobolev exponent has a dual property if the domain is $\mathbb{R}^N$, actually the equation has a positive solution whenever $p \geq p_N^*$ and it does not have a solution if $1 < p < p_N^*$.

If we replace the Laplacian by any linear second order uniformly elliptic operator with C^1 coefficients, say $Lu = \sum_i \sum_j a_{ij} \frac{\partial u^2}{\partial x_i \partial x_j}$ with $a_{ij} \in C^1$, then the semi-linear problem

$$Lu + u^p = 0, \quad \text{in } \Omega \tag{1.3}$$

$$u = 0, \quad \text{on } \partial\Omega, \tag{1.4}$$

has a positive solution for the same range of values of p, namely $1 < p < p_N^*$. That is, the existence property of the Sobolev exponent remains valid for all operators in this class.

In this note we consider two classes of uniformly second order elliptic operators for which the critical exponents are drastically changed with respect to p_N^*, in the case of radially symmetric solutions. Our aim is to prove that the corresponding existence property for these critical exponents persists when the domain is perturbed.

Our first class corresponds to the so-called Pucci's extremal operators (Cabré and Caffarelli, 1995; Pucci, 1966a,b). Given positive numbers $0 < \lambda \leq \Lambda$ we consider the operator $\mathcal{M}^+_{\lambda,\Lambda}(D^2u)$ where for any $N \times N$ symmetric matrix M,

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i,$$

$e_i = e_i(M)$ being M's eigenvalues. The case of the operator $\mathcal{M}^-_{\lambda,\Lambda}(D^2u)$ is also considered and it is defined by exchanging the roles between λ and Λ above.

Pucci's extremal operators appear in the context of stochastic control when the diffusion coefficient is a control variable, see Bensoussan and Lions (1982) or the papers of (Lions, 1981/1982, 1983a,b) for the relation between a general Hamilton–Jacobi–Bellman and stochastic control. They also provide natural extremal equations in the sense that if F is any (from linear to fully nonlinear) uniformly elliptic operator, with ellipticity constants λ, Λ, and depends only on the Hessian D^2u, then

$$(\mathcal{M})^-_{\lambda,\Lambda} \leq F(M) \leq (\mathcal{M})^+_{\lambda,\Lambda}(M) \tag{1.5}$$

for any symmetric matrix M. Moreover, these operators are also extremal with respect to the first half eigenvalue of all second order elliptic operators with constant coefficients and ellipticity constants between λ and Λ (see for instance Busca et al., 2005).

It is obvious that when $\lambda = \Lambda$, then $\mathcal{M}^\pm_{\lambda,\Lambda}$ coincides with a multiple of the Laplace operator. We also notice that given any number $s \in [\lambda, \Lambda]$ the operator $s\Delta$ belongs to the class defined by (1.5).

The second family of operators that we consider are defined as

$$Q^+_{\lambda,\Lambda}u = \lambda\Delta u + (\Lambda - \lambda)Q^0 u, \tag{1.6}$$

where Q^0 is the second order operator

$$Q^0 u = \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{x_i x_j}{|x|^2} \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

These operators are also considered by Pucci (1966a), being extremal with respect to some spectral properties. We notice that these operators belong to the class defined by (1.5) and when $\lambda = \Lambda$ they also become a multiple of the Laplacian. If we interchange the role of λ by Λ in definition (1.6), then we obtain the operator $Q^-_{\lambda,\Lambda}$, which is also considered later.

The operators $\mathcal{M}^\pm_{\lambda,\Lambda}$ are autonomous, but not linear, even if they enjoy some properties of the Laplacian. The operators $Q^\pm_{\lambda,\Lambda}$ are still linear, but their coefficients are not continuous at the origin. In both cases, when one considers a ball and the set of radially symmetric functions in it, there are critical exponents for the operators $\mathcal{M}^+$ and Q^+ which are greater than the Sobolev exponent p^*_N. On the contrary, for the operators $Q^-_{\lambda,\Lambda}$ and $\mathcal{M}^-_{\lambda,\Lambda}$, the critical exponents for radially symmetric solutions in a ball are smaller than the Sobolev exponent p^*_N. These facts were proved in Felmer and Quaas (2002, 2003) for $\mathcal{M}^\pm_{\lambda,\Lambda}$ and for $Q^+_{\lambda,\Lambda}$ and $Q^-_{\lambda,\Lambda}$ the proof is given here, in Sec. 2.

More precisely, in the case of operators $Q^+_{\lambda,\Lambda}$, there exists a number $\widetilde{N}_+ = \frac{\lambda}{\Lambda}(N-1)+1$, such that if Ω is a ball of $\mathbb{R}^N$ and if $\widetilde{N}_+ > 2$, (1.3)–(1.4) has a unique positive radially symmetric solution for any $1 < p < (\widetilde{N}_+ + 2)/(\widetilde{N}_+ - 2)$ and no positive radially symmetric solution for $p \geq (\widetilde{N}_+ + 2)/(\widetilde{N}_+ - 2)$. Notice that for any $\lambda < \Lambda$, $\widetilde{N}_+ < N$ and so, the critical exponent here is strictly larger than the Sobolev critical exponent p^*_N.

In the case of the Pucci's extremal operators $\mathcal{M}^+_{\lambda,\Lambda}$, the critical exponent is a number p^*_+ such that

$$p^*_N < p^*_+ < \frac{\widetilde{N}_+ + 2}{\widetilde{N}_+ - 2}.$$

The number p^*_+ depends on λ, Λ and the dimension N, however an explicit formula for it is not known.

Similarly, for the operator $Q^-_{\lambda,\Lambda}$ we may also define a dimension like number $\widetilde{N}_- = \frac{\Lambda}{\lambda}(N-1)+1$, so that its critical exponent in the radially symmetric case is precisely $(\widetilde{N}_- + 2)/(\widetilde{N}_- - 2)$. In the case of the operator $\mathcal{M}^-_{\lambda,\Lambda}$ we recall that the critical exponent of the operator $\mathcal{M}^-_{\lambda,\Lambda}$ is a number p^*_- satisfying

$$\frac{\widetilde{N}_- + 2}{\widetilde{N}_- - 2} < p^*_- < p^*_N,$$

as it was shown in Felmer and Quaas (2003).

It is the purpose of this note to prove that this phenomenom of critical exponent increase (or decrease) does not appear only in the radially symmetric case. By a perturbation argument, based on a work by Dancer (1988), we show that these critical exponents, with respect to existence properties in bounded domains, persist when the ball is perturbed not necessarily in a radial manner. This result provides us with evidence that the critical exponents for these operators, obtained in radial versions, are also the critical exponents in the general case.

At this point we would like to stress some surprising properties of the critical exponents of operators in the class given by (1.5). For the first property we consider all linear elliptic operators with bounded coefficients and belonging to the class defined by (1.5). If we take the L^∞ topology for the coefficients of these operators, we see that the critical exponent is not a continuous function of the operator. In particular, as shown in Sec. 2, the operators $\mathcal{Q}^{\pm}_{\lambda,\Lambda}$ can be "approximated" in L^∞ (the coefficients) by a sequence of operators with C^∞ coefficients, for which the critical exponent in the radially symmetric case is p^*_N.

The second property is related to the non-monotonicity of the critical exponents. Notice the following operator's inequalities,

$$\lambda\Delta \leq \mathcal{M}^{+}_{\lambda,\Lambda} \quad \text{and} \quad \mathcal{Q}^{+}_{\lambda,\Lambda} \leq \mathcal{M}^{+}_{\lambda,\Lambda},$$

while for the corresponding critical exponents we have

$$p^N_* < p^*_+ \quad \text{and} \quad \frac{\widetilde{N}_+ + 2}{\widetilde{N}_+ - 2} > p^*_+.$$

We finally observe that all operators of the form $\mathcal{M}^{\pm}_{s,S}$ and $\mathcal{Q}^{\pm}_{s,S}$, with $s, S \in [\lambda, \Lambda]$, have critical exponents in the interval

$$\left[\frac{\widetilde{N}_- + 2}{\widetilde{N}_- - 2}, \frac{\widetilde{N}_+ + 2}{\widetilde{N}_+ - 2}\right].$$

We conjecture that in the class of operators defined by (1.5), the critical exponents are all in the same interval, that is, the operators $\mathcal{Q}^{\pm}_{\lambda,\Lambda}$ are extremal for critical exponents.

This article is organized in two sections. In Sec. 2 we discuss the case of the operator $\mathcal{Q}^{+}_{\lambda,\Lambda}$. We first consider the radial case analyzing, in analogy with the case of the Laplacian, the critical exponent and proving that in the subcritical case the positive solution is non-degenerate. Then, by linearization, we show that this solution is non-degenerate in the space of functions not necessarily symmetric and we apply a perturbation argument via degree theory. In Sec. 3 we consider the case of the operator $\mathcal{M}^{+}_{\lambda,\Lambda}$. The situation here is somehow simpler since we know that in the ball all positive solutions are radially symmetric by using a moving plane argument, and then, in the subcritical case, the radial solution is isolated. We conclude by a homotopy invariance with respect to the ellipticity constant, as in Busca et al. (2005) and then by using a perturbation argument.

2. The Extremal Operator $\mathcal{Q}^{+}_{\lambda,\Lambda}$

In this section we analyze the equations (1.3)–(1.4) in the case of the extremal operator $\mathcal{Q}^{+}_{\lambda,\Lambda}$, with $0 < \lambda < \Lambda$. This is a uniformly elliptic operator, whose coefficients have a discontinuity at the origin. This feature is what makes this operator interesting. We need to make precise the very notion of a solution for equations (1.3)–(1.4).

We observe that the operator Q^0 corresponds to the second derivative with respect to the radial coordinate $r = |x|$, that is

$$Q^0 u = \frac{\partial^2 u}{\partial r^2}.$$

Because of this observation, we see that the analysis of the radial case is very simple, it amounts to change the notion of the dimension taking into account λ and Λ, and to perform a phase plane analysis. In fact, we easily see that if u is a solution of

$$\Lambda u'' + \lambda \frac{N-1}{r} u' + u^p = 0, \quad u'(0) = u(1) = 0. \tag{2.1}$$

then $u(x) = u(|x|)$ is a solution of (1.3)–(1.4). Defining

$$\widetilde{N}_+ = \frac{\lambda}{\Lambda}(N-1) + 1 \quad \text{and} \quad v(r) = (\Lambda)^{1/(1-p)} u(r)$$

we see that v satisfies

$$v'' + \frac{\widetilde{N}_+ - 1}{r} v' + v^p = 0, \quad v'(0) = v(1) = 0. \tag{2.2}$$

The following theorem gives the critical exponent for equation (2.2) and the non-degeneracy property of its solutions in the subcritical case.

Theorem 2.1. *Equation* (2.2) *does not have a positive solution if*

$$p \geq (\widetilde{N}_+ + 2)/(\widetilde{N}_+ - 2),$$

and it possesses exactly one positive solution if

$$1 < p < (\widetilde{N}_+ + 2)/(\widetilde{N}_+ - 2) \quad \textit{and} \quad \widetilde{N}_+ > 2,$$

or

$$1 < p \quad \textit{and} \quad 1 < \widetilde{N}_+ \leq 2.$$

Moreover, if v is a solution of (2.2), *then the linearized equation*

$$h'' + \frac{\widetilde{N}_+ - 1}{r} h' + p v^{p-1} h = 0, \quad h'(0) = h(1) = 0, \tag{2.3}$$

has no non-trivial solution, that is, 0 *is not in the spectrum of the linearized operator.*

Proof. The criticality of the number $(\widetilde{N}_+ + 2)/(\widetilde{N}_+ - 2)$ can be proved in a way similar to the case of the Laplacian, by using the Emden–Fowler transformation. Let v be a positive solution of equation (2.2), then we have that

$$v_\gamma(r) = \gamma v(\gamma^{(p-1)/2} r),$$

also satisfies the equation in (2.2) together with boundary conditions $v_\gamma'(0) = 0$ and $v_\gamma(\gamma^{(1-p)/2}) = 0$, for all positive γ. From here we see that the function

$$h_1(r) = \frac{\partial v_\gamma}{\partial \gamma}(r)|_{\gamma=1},$$

satisfies (2.3) and $h_1'(0) = 0$, $h_1(0) > 0$.

Assume that h_2 is a second solution, linearly independent of h_1. Then necessarily we have that $h_2'(r)$ stays away from zero, for r near 0, since the contrary implies that h_1 and h_2 are linearly dependent. Now, given any solution h of (2.3), we have $h = c_1 h_1 + c_2 h_2$. But then $c_2 = 0$ since $h'(0) = 0$ and $c_1 = 0$ since $h_1(1) < 0$, proving that $h \equiv 0$. □

Continuing with our analysis, we observe that since the operator Q^0, and then also $Q^+_{\lambda,\Lambda}$, has discontinuous coefficients, we should start making precise the notion of solution for the equation

$$Q^+_{\lambda,\Lambda} u = f \quad \text{in } \Omega, \tag{2.4}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{2.5}$$

For notational simplicity, in the rest of this section we simply write Q for $Q^+_{\lambda,\Lambda}$, since no confusion will arise.

Given i, j we consider a sequence of C^∞ functions $a^n_{i,j}$ so that $a^n_{i,j}(x) = x_i x_j/|x|^2$ for all $|x| \geq 1/n$ and $|a^n_{i,j}(x)| \leq 1$ for $|x| \leq 1/n$. For example, we may consider a cut-off function η so that $\eta(r) = 0$ if $r < 1/2$ and $\eta(r) = 1$ if $r \geq 1$, and then define

$$a^n_{i,j}(x) = \eta(nr)\frac{x_i x_j}{|x|^2}. \tag{2.6}$$

Then we define the operators

$$Q^n u = \lambda\Delta u + (\Lambda - \lambda)\sum_{i=1}^{n}\sum_{j=1}^{n} a^n_{i,j}\frac{\partial^2 u}{\partial x_i \partial x_j}.$$

We assume that the function f is continuous in $\overline{\Omega}$, then the problem

$$Q^n u = f \quad \text{in } \Omega, \tag{2.7}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{2.8}$$

possesses a unique smooth solution u_n. Moreover, since the coefficients are C^∞ functions it is well known the existence of a Green function $G_n : \Omega \times \Omega \to \mathbb{R}$ allowing to represent this solution as

$$u_n(x) = \int_\Omega G_n(x, y) f(y) dy.$$

It follows from Alexandrof–Bakelman–Pucci's estimate that the sequence $\{G_n(x, \cdot)\}$ is bounded in $L^{N/(N-1)}(\Omega)$ and hence, up to subsequence, it has a weak limit in this space. Moreover, since our operator Q is discontinuous just at one point, the origin, the weak limit is unique as shown by the arguments in Cerutti et al. (1991). Thus, our problem has a unique Green function $G : \Omega \times \Omega \to \mathbb{R}$, such that $G(x, \cdot)$ is in $L^{N/(N-1)}(\Omega)$, and we define

$$u(x) = \int_{\Omega} G(x, y) f(y) dy,$$

as the solution to equations (2.4) and (2.5).

On the other hand, the sequence of solutions $\{u_n\}$ is bounded in $C_0^{\alpha}(\overline{\Omega})$, for $\alpha > 0$ as follows from basic estimates, see Gilbarg and Trudinger (1983) or Cabré and Caffarelli (1995). Then, the solution u is actually of class $C^{\alpha}(\overline{\Omega})$ and we have

$$\|u\|_{C^{\alpha}(\overline{\Omega})} \leq C\|u\|_{L^{\infty}(\partial\Omega)} + C\|f\|_{L^{N}(\Omega)}. \tag{2.9}$$

Here the constant C depends only on the ellipticity constants, the L^{∞} bounds on the coefficients and on the domain Ω, which we assume to have a regular boundary.

Let u_0 be the unique solution of (2.1). In what follows we show that, when u_0 is considered as a function of the N variables, it satisfies the equation in the sense given above.

Lemma 2.1. *The function $u_0(x) = u_0(|x|)$ satisfies the equation*

$$Qu = -u^p \quad \textit{in } B, \tag{2.10}$$

$$u = 0 \quad \textit{on } \partial B, \tag{2.11}$$

in the sense just defined above.

Proof. By direct computation we see that, pointwise, we have

$$Q^n u = -u^p - c_n(x) \quad \text{in } B,$$

where c_n is a function with support in the ball $B(0, 1/n)$ and which is bounded, with a bound independent of n. Then we certainly have

$$u(x) = \int_B G_n(x, y)(-u^p(y) + c_n(y)) dy.$$

Taking limits here we conclude. □

Remark 2.1. The notion of solution defined above is known as *good solution* and it was introduced by Cerutti et al. (1991). In a recent paper by Jensen et al. (2001), this notion of solutions is shown to be equivalent to L^p viscosity solution.

Remark 2.2. We do not know whether equations (2.10)–(2.11) possesses a non-radial solution or not.

Our existence result is for domains which are close to the unit ball. More precisely we assume that we have a sequence of domains $\{\Omega_n\}$ such that for all $0 < r < 1 < R$ there exists $n_0 \in \mathbb{N}$ such that

$$B(0, r) \subset \Omega_n \subset B(0, R), \quad \text{for all } n \geq n_0.$$

We consider a reference bounded domain D such that $\Omega_n \subset D$, for all $n \in \mathbb{N}$. We may take, for example, D as the ball of radius 2.

Next we prove a continuity property for the Green functions associated to the domains Ω_n

Lemma 2.2. *Under the conditions given above we have, for every $f \in C(D)$ and for every $x \in B$,*

$$\lim_{n\to\infty} \int_{\Omega_n} G_{\Omega_n}(x, y)f(y)dy = \int_B G_B(x, y)f(y)dy,$$

where G_B and G_{Ω_n} are the Green functions of the unit ball B and of Ω_n, respectively.

Proof. Let $f \in C(D)$ and let u_n be the solution of the equation

$$Qu = f, \quad \text{in } \Omega_n, \qquad u = 0 \quad \text{on } \partial\Omega_n.$$

Consider also the function u, the solution of the equation

$$Qu = f, \quad \text{in } B_R, \qquad u = 0 \quad \text{on } \partial B_R.$$

Here we assume $r < 1 < R$ are close to 1 and n is large enough so that $B_r \subset \Omega_n \subset B_R$. Using (2.9) we find

$$\|u\|_{C^\alpha(\overline{B}_R)} \leq C\|f\|_{L^N(B_R)},$$

which implies

$$|u(x)| \leq C(R - r)^\alpha \|f\|_{L^N(B_R)}, \quad x \in B_R \backslash B_r.$$

Since $u_n - u$ satisfies $Q(u_n - u) = 0$ in Ω_n, by standard estimates (Cabré and Caffarelli, 1995), we obtain

$$\|u_n - u\|_{L^\infty(\Omega_n)} \leq C\|u_n - u\|_{L^\infty(\partial\Omega_n)} \leq C(R - r)^\alpha \|f\|_{L^N(B_R)},$$

where the constant C here is uniform in n. Similarly if v is the solution of

$$Qu = f, \quad \text{in } B, \ \ u = 0 \quad \text{on } \partial B,$$

then we have

$$\|v - u\|_{L^\infty(B)} \leq C(R - r)^\alpha \|f\|_{L^N(B)}.$$

Since for all $x \in B_r$ we have

$$u_n(x) - v(x) = \int_{\Omega_n} G_{\Omega_n}(x, y)f(y) - \int_B G_B(x, y)f(y),$$

the result follows. □

Now we state and prove our main theorem of this section.

Theorem 2.2. *Assume* $\widetilde{N}_+ > 2$ *and that* $1 < p < (\widetilde{N}_+ + 2)/(\widetilde{N}_+ - 2)$. *Then there is* $n_0 \in \mathbb{N}$ *so that for all* $n \geq n_0$, *the equation*

$$Qu + u^p = 0 \quad \text{in } \Omega_n, \tag{2.12}$$

$$u = 0 \quad \text{on } \partial\Omega_n, \tag{2.13}$$

possesses at least one nontrivial solution.

In order to prove our theorem we will follow some ideas from Dancer (1988). We start setting up the functional analytic framework. We consider the inclusions $i : C_0(\overline{D}) \to C(\overline{B})$ and $j : C_0^\alpha(\overline{D}) \to C_0(\overline{D})$. Here C stands for continuous functions, C_0 for continuous functions vanishing on the boundary and C_0^α for Hölder continuous functions vanishing on the boundary. Given $f \in C(\overline{B})$, we let $\overline{\mathscr{L}}(f)$ be the unique solution to (2.4)–(2.5) with $\Omega = B$. Then we extend this solution to define $\mathscr{L}(f)$ as

$$\mathscr{L}(f)(x) = \begin{cases} \overline{\mathscr{L}}(f)(x) & \text{if } x \in \overline{B}, \\ 0 & \text{if } x \in \overline{D} \backslash B. \end{cases} \tag{2.14}$$

Thus, the operator $\mathscr{L} : C(\overline{B}) \to C_0^\alpha(\overline{D})$ is well defined as a linear bounded operator. Next we define the nonlinear operator $\mathscr{F} : C_0(\overline{D}) \to C_0(\overline{D})$ as

$$\mathscr{F}(u) = -j \circ \mathscr{L}(i(u^p)). \tag{2.15}$$

If we consider $N\colon C_0(\overline{D}) \to C_0(\overline{D})$, the Nemitsky operator defined as $N(u) = u^p$, we easily see that N is of class C^1 and $N'(u)(h) = pu^{p-1}h$ for $h \in C_0(\overline{D})$. Thus, the operator $\mathscr{F}$ is compact and of class C^1.

In the definition of the operators i and $\mathscr{L}$ we can replace B by Ω_n and we obtain i_n and $\mathscr{L}_n$. The operator $\mathscr{F}_n$ is then defined as $\mathscr{F}_n(u) = -j \circ \mathscr{L}_n(i_n(u^p))$, and naturally it is compact and of class C^1 as an operator in $C_0(\overline{D})$.

Before giving the proof of Theorem 2.2 we need still another preliminary result. It guarantees that u_0 is isolated not only in the space of radial functions, as shown in Theorem 2.1, but also in the space of all functions of class C^α in the ball $B = B(0, 1)$.

Proposition 2.1. *Under the hypotheses of Theorem* 2.2, *the linear equation*

$$Qh + pu_0^{p-1}h = 0 \quad \text{in } B, \tag{2.16}$$

$$h = 0 \quad \text{on } \partial B, \tag{2.17}$$

has only the trivial solution $h \equiv 0$ *in* $C_0^\alpha(B)$.

Proof. Our proof uses a standard argument by Smoller and Wasserman (1984). We let $\{\phi_k(\theta)\}_{k=0}^\infty$, $\theta \in S^{N-1}$, be the eigenfunctions of the Laplacian in S^{N-1}, whose eigenvalues are

$$\lambda_k = -k(k + N - 2).$$

Assume h is a solution of (2.16)–(2.17). Then we consider the approximation operator Q^n as given by 2.6 and we solve the equation

$$Q^n h_n = -pu_0^{p-1}h \quad \text{in } B, \qquad h_n = 0, \quad \text{on } \partial B. \tag{2.18}$$

We observe that if $u \in C^2$ and $u = u(r, \theta)$ with $\theta \in S^{N-1}$ then for the operator Q^n we can write

$$Q^n u = \{\lambda + (\Lambda - \lambda)\eta(nr)\}u'' + \lambda\frac{N-1}{r}u' + \frac{\lambda}{r^2}\Delta_\theta u,$$

where $'$ denotes derivative with respect to r and Δ_θ is the Laplacian on the sphere. Let us define

$$a_k^n(r) = \int_{S^{N-1}} h_n\phi_k \, d\theta \quad \text{and} \quad a_k(r) = \int_{S^{N-1}} h\phi_k \, d\theta.$$

By the C^α convergence of h_n to h we see that a_k^n converges uniformly in $[0, 1]$ to a_k, for all $k \geq 0$. Next we multiply equation (2.18) by ϕ_k and integrate over S^{N-1} to obtain

$$\{\lambda + (\Lambda - \lambda)\eta(nr)\}(a_k^n)'' + \lambda\frac{N-1}{r}(a_k^n)' + \lambda\lambda_k\frac{a_k^n}{r^2} = -pu_0^{p-1}a_k.$$

From here we can prove that for $r \in (0, 1]$ the convergence of a_k^n to a_k is even C^2 and a_k satisfies

$$a_k'' + \frac{\widetilde{N}-1}{r}a_k' + \frac{\lambda\lambda_k}{\Lambda}\frac{a_k}{r^2} + \frac{p}{\Lambda}u_0^{p-1}a_k = 0. \tag{2.19}$$

Moreover, we can prove that $a_k(0) = 0$ and that $r^{\widetilde{N}_+-1}a_k'(r)$ is bounded as $r \to 0$. To prove the last statement we use that $\widetilde{N}_+ > 2$. On the other hand we have that $w = u_0'$ satisfies the equation

$$w'' + \frac{\widetilde{N}-1}{r}w' - \frac{\widetilde{N}-1}{r^2}w + \frac{p}{\Lambda}u_0^{p-1}w = 0. \tag{2.20}$$

Multiplying equation (2.19) by w and equation (2.20) by a_k, integrating between 0 and the first zero of a_k, and subtracting we can prove that $a_k \equiv 0$ for all $k \geq 1$ (Smoller and Wasserman, 1984). For $k = 0$ we use Theorem 2.1 to prove that also $a_0 \equiv 0$. □

We finally complete the proof of our main theorem in this section.

Proof of Theorem 2.2. Since $I - \mathcal{F}'(u_0)$ has trivial kernel, as we proved in Proposition 2.1, there is $\delta > 0$ such that $u - \mathcal{F}(u) \neq 0$ for all $u \in \partial\mathcal{B}$, where $\mathcal{B} = \{u \in C_0(\overline{D})/\|u - u_0\|_{C_0(\overline{D})} < \delta\}$. Moreover, the Leray–Schauder degree of $I - \mathcal{F}$ is well defined in $\mathcal{B}$ and $\deg(I - \mathcal{F}, \mathcal{B}, 0) = 1$ or -1. To finish the proof we just need to prove that there exists $n_0 \in \mathbb{N}$ so that

$$u \neq t\mathcal{F}(u) + (1-t)\mathcal{F}_n(u), \quad \text{for all} \quad t \in [0, 1], \; u \in \partial\mathcal{B}, \tag{2.21}$$

since this implies that

$$\deg(I - \mathcal{F}_n, \mathcal{B}, 0) = \deg(I - \mathcal{F}, \mathcal{B}, 0) \neq 0.$$

Let us assume that (2.21) is not true. Then there exist sequences $\{t_n\} \subset [0, 1]$ and $\{u_n\} \subset \partial\mathcal{B}$ such that

$$u_n = t_n \mathcal{F}(u_n) + (1 - t_n)\mathcal{F}_n(u_n).$$

We may assume that $u_n \to \bar{u}$ uniformly in $\overline{D}$, up to a subsequence, as a consequence of the compactness of the inclusion j. We may also assume $t_n \to \bar{t}$. Then we see that we will get a contradiction if we prove that

$$\lim_{n\to\infty} \mathcal{F}_n(u_n) = \mathcal{F}(\bar{u}). \tag{2.22}$$

From the definition of $\mathcal{F}_n$ we see that if $v_n = \mathcal{F}_n(u_n)$ then

$$v_n(x) = -\int_{\Omega_n} G_{\Omega_n}(x, y) u_n^p(y) dy,$$

where G_{Ω_n} is the Green function of Ω_n. Since u_n is uniformly convergent to $\bar{u}$ in $\overline{D}$, to complete the proof we just use Lemma 2.2. □

Remark 2.3. Our Theorem 2.2 is concerned with the operator $Q^+_{\lambda,\Lambda}$, for $0 < \lambda \leq \Lambda$. A completely analogous theorem can be proved for the operator $Q^-_{\lambda,\Lambda}$. Naturally, our hypotheses have to be changed to: $\widetilde{N}_- > 2$ and $1 < p < (\widetilde{N}_- + 2)/(\widetilde{N}N_- - 2)$.

3. The Extremal Operator $\mathcal{M}^+_{\lambda,\Lambda}$

In this section we analyze equations (1.3)–(1.4) in the case $L = M^+_{\lambda,\Lambda}$, the extremal Pucci's operator for $0 < \lambda \leq \Lambda$. This is an elliptic operator which is nonlinear, but homogenous of degree 1. In comparison with $Q^+_{\lambda,\Lambda}$, for the operator $\mathcal{M}^+_{\lambda,\Lambda}$ there is a good regularity theory that guarantees that the solutions to equations (1.3)–(1.4) are classical solutions (Cabré and Caffarelli, 1995).

We start this section recalling the existence of positive solutions for (1.3)–(1.4), when Ω is a ball as was proved by Felmer and Quaas (2002, 2003).

Theorem 3.1. *Suppose $\widetilde{N}_+ > 2$. Then there exist a number p^*_+ such that*

$$\frac{N+2}{N-2} < p^*_+ < \frac{\widetilde{N}_+ + 2}{\widetilde{N}_+ - 2}.$$

with the property that if $1 < p < p^+_*$ *then* (1.3)–(1.4) *has a nontrivial radially symmetric* C^2 *solution. Moreover, if* $p \geq p^+_*$ *then* (1.3)–(1.4) *does not have a solution.*

The number p^+_* is called critical exponents for the operator $M^+_{\lambda,\Lambda}$. We notice that $\lambda = \Lambda$ implies $p^+_* = p^N_*$.

Let Ω_n be a sequence of domains satisfying the conditions given in the previous section. Now we present the main theorem of this section.

Theorem 3.2. *Assume $\widetilde{N}_+ > 2$ and that $1 < p < p_+^*$. Then there is $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$, the equation*

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + u^p = 0 \quad in\ \Omega_n, \tag{3.1}$$

$$u = 0 \quad on\ \partial\Omega_n, \tag{3.2}$$

possesses at least one positive solution.

Remark 3.1. It can be seen that equations (1.3)–(1.4), when Ω is a ball, has only one positive radially symmetric solution. In fact, by using a classical moving planes technique (Berestycki and Nirenberg, 1991), all positive solutions of (1.3)–(1.4) are radially symmetric (for detail see Da Lio and Sirakov, 2007). This uniqueness property is crucial in our analysis, since it allows us to use a degree theory approach as in the previous section, avoiding the study of the linearized equation in order to obtain non-degeneracy of the radially symmetric solution.

Remark 3.2. a) Our Theorem 3.2 is concerned with the operator $M^+_{\lambda,\Lambda}$, for $0 < \lambda \leq \Lambda$. A completely analogous theorem can be proved for the operator $M^-_{\lambda,\Lambda}$. With the natural change in the hypothesis to $1 < p < p_-^*$.

b) For other existence result concerning Pucci operator we refer the reader to Felmer and Quaas (2004) and Quaas (2004).

Proof. Let $q : [\lambda, \Lambda] \to \mathbf{R}$ be a continuous function such that $q(s) < p_*^+(s)$ for all $s \in [\lambda, \Lambda]$ and $q(\lambda) = p$. Here $p_*^+(s)$ is the critical exponent for the operator $\mathcal{M}^+_{s,\Lambda}$, which is a continuous function of s. Next we consider $s \in [\lambda, \Lambda]$ and we define $\mathcal{L}^s(f)$ as the unique solution to

$$\mathcal{M}^+_{s,\Lambda}(D^2u) \ = \ f \quad \text{in } B, \tag{3.3}$$

$$u = 0 \quad \text{on } \partial B, \tag{3.4}$$

for $f \in C(\overline{B})$. This operator is well defined in $C(\overline{B})$, with values in $C_0^\alpha(\overline{B})$, and it is positive, that is if $f(x) \leq 0$ in B then $u \geq 0$ in B. These follow from existence and regularity theory for fully nonlinear operators and the maximum principle for $\mathcal{M}^+_{s,\Lambda}$. See the monography by Cabré and Caffarelli (1995) and the work by Bardi and Da Lio (2001). Now we extend $\overline{\mathcal{L}}^s$ as in (2.14) and define the operator $\mathcal{F}(s, \cdot)$ using (2.15), with $p = q(s)$.

Let u_0^s be the unique solution of (1.3)–(1.4) in B for $p = q(s)$ given by Theorem 3.1. We consider

$$M = \sup_{s\in[\lambda,\Lambda]} \|u_0^s\|_{L^\infty(B)} \quad \text{and} \quad m = \inf_{s\in[\lambda,\Lambda]} \|u_0^s\|_{L^\infty(B)} > 0,$$

and let $\delta > 0$ be such that $m - \delta > 0$. We define the set

$$\mathcal{B} = \{u \in C_0(\overline{D})/u(x) \geq 0 \text{ in } D, m - \delta < \|u\|_{C_0(\overline{D})} < M + 1\},$$

and we observe, as follows by the uniqueness of solutions in the ball and the strong maximum principle for $\mathcal{M}^+_{s,\Lambda}$, that $u - F(s, u) \neq 0$ for all $u \in \partial\mathcal{B}$ and $s \in [\lambda, \Lambda]$.

We notice that $\deg(I - \mathscr{F}(\Lambda, \cdot), \mathscr{B}, 0) \neq 0$, since in this case we are dealing with the Laplacian. Then by invariance under homotopy of the degree we obtain

$$\deg(I - \mathscr{F}(\lambda, \cdot), \mathscr{B}, 0) = \deg(I - \mathscr{F}(\Lambda, \cdot), \mathscr{B}, 0) \neq 0.$$

Next we define $\mathscr{F}_n$ perturbing the domain as in the previous section, keeping $s = \lambda$ and $p = q(\lambda)$. To finish the proof we just need to prove that there exists $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$

$$u \neq t\mathscr{F}(\lambda, u) + (1-t)\mathscr{F}_n(\lambda, u), \quad \text{for all } t \in [0,1], \ \ u \in \partial\mathscr{B}, \tag{3.5}$$

since this implies that

$$\deg(I - \mathscr{F}_n(\lambda, \cdot), \mathscr{B}, 0) = \deg(I - \mathscr{F}(\lambda, \cdot), \mathscr{B}, 0) \neq 0.$$

Let us assume that 3.5 is not true. Then there exist sequences $\{t_n\} \subset [0,1]$ and $\{u_n\} \subset \partial\mathscr{B}$ such that

$$u_n = t_n\mathscr{F}(\lambda, u_n) + (1-t_n)\mathscr{F}_n(\lambda, u_n).$$

We may assume that $t_n \to \bar{t}$ and that $u_n \to \bar{u}$ uniformly in $\overline{D}$, up to a subsequence, as a consequence of the compactness of the inclusion j. We will get a contradiction if we prove that

$$\lim_{n\to\infty} \mathscr{F}_n(\lambda, u_n)(x) = \mathscr{F}(\lambda, \bar{u})(x) \quad \text{for all } x \in D. \tag{3.6}$$

We first notice that $\mathscr{F}_n(\lambda, u_n) := v_n \to v$ in $C_0(D)$. If $x \in D\backslash\overline{B}$, then for n large $\mathscr{F}_n(\lambda, u_n)(x) = 0$, and so $v(x) = 0$, therefore by continuity of v we also have $v = 0$ in $D\backslash B$. On the other hand

$$\mathscr{M}^+_{\lambda,\Lambda}(D^2 v_n) = -u_n^p \quad \text{in } B(0,r)$$

for n large. Passing to the limit in the viscosity sense we get $\mathscr{M}^+_{\lambda,\Lambda}(D^2 v) = -\bar{u}^p$ in B. Since $v = 0$ on ∂B then, by the definition of $\mathscr{F}$, we obtain $\mathscr{F}(\lambda, \bar{u}) = v$, concluding the proof. □

Acknowledgments

The second author was partially supported by Fondecyt Grant # 1030929 and FONDAP de Matemáticas Aplicadas. The third author was partially supported by Fondecyt Grant # 1040794. This work was partially supported by ECOS Grant # C02E08.

References

Bardi, M., Da Lio, F. (2001). Propagation of maxima and strong maximum principle for viscosity solutions of degenerate elliptic equation I: convex operators. *Nonlinear Anal.* 44:991–1006.

Bensoussan, A., Lions, J. L. (1982). *Applications of Variational Inequalities in Stochastic Control*. Translated from the French. Studies in Mathematics and Its Applications, 12. Amsterdam-New York: North-Holland Publishing Co.

Berestycki, H., Nirenberg, L. (1991). On the method of moving planes and the sliding method. *Boll. Soc. Brasil Mat. Nova Ser.* 22:237–275.

Busca, J., Esteban, M., Quaas, A. (2005). Nonlinear eigenvalues and bifurcation problems for Pucci's operator. *Ann. Inst. H. Poincaré, Analyse Non-Linéaire* 22:187–206.

Cabré, X., Caffarelli, L. (1995). *Fully Nonlinear Elliptic Equation*. American Mathematical Society. Vol. 43. Colloquium Publication.

Cerutti, C., Escauriaza, L., Fabes, E. (1991). Uniqueness for some diffusions with discontinuous coefficients. *The Annals of Probability* 19(2):525–537.

Da Lio, F., Sirakov, B. (2007). Symmetry results for viscosity solution of fully nonlinear elliptic equation. *J. E. M. S.* 9(2):317–330.

Dancer, N. (1988). The effect of domain shape on the number of positive solutions of certain nonlinear equations. *Journal of Differential Equations* 74:120–156.

Felmer, P., Quaas, A. (2002). Critical exponents for the Pucci's extremal operators. *C.R. Acad. Sci. Paris (I)* 335:909–914.

Felmer, P., Quaas, A. (2003). On critical exponents for the Pucci's extremal operators. *Ann Inst. Henri Poicaré, Analyse Non Linéaire* 20(5):843–865.

Felmer, P., Quaas, A. (2004). Positive solutions to 'semi-linear' equation involving the Pucci's operator. *Journal of Differential Equations* 199(2):376–393.

Gilbarg, D., Trudinger, N. S. (1983). *Elliptic Partial Differential Equation of Second Order*. 2nd ed. Springer-Verlag.

Jensen, R., Kocan, M., Swiech, A. (2001). Good and viscosity solutions of fully nonlinear elliptic equations. *Proceedings of the American Mathematical Society* 130(2):533–542.

Lions, P. L. (1981/1982). *Optimal Control of Diffusion Processes and Hamilton–Jacobi–Bellman Equations. III. Regularity of the Optimal Cost Function.* Nonlinear Partial Differential Equations and Their Applications. Collège de France Seminar, Vol. V (Paris, 1981/1982). Pitman, Boston, MA, pp. 95–205.

Lions, P. L. (1983a). Optimal control of diffusion processes and Hamilton–Jacobi–Bellman equations. I. The dynamic programming principle and applications. *Comm. Partial Differential Equations* 8(10):1101–1174.

Lions, P. L. (1983b). Optimal control of diffusion processes and Hamilton–Jacobi–Bellman equations. II. Viscosity solutions and uniqueness. *Comm. Partial Differential Equations* 8(11):1229–1276.

Pucci, C. (1966a). Maximum and minimum first eigenvalues for a class of elliptic operators. *Proc. Amer. Math. Soc.* 17:788–795.

Pucci, C. (1966b). Operatori ellittici estremanti. *Ann. Mat. Pure Appl.* 72:141–170.

Pohozaev, S. I. (1965). Eigenfunctions of the equation $\Delta u + f(u) = 0$. *Soviet Math.* 5:1408–1411.

Quaas, A. (2004). Existence of positive solutions to a 'semi-linear' equation involving the Pucci's operator in a convex domain. *Diff. Integral Equation* 17:481–494.

Smoller, J., Wasserman, G. (1984). Existence, uniqueness, and non-degeneracy of positive solutions of semi-linear elliptic equations. *Commun. Math. Phys.* 95:129–159.