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# ON THE EXISTENCE AND PROFILE OF NODAL SOLUTIONS FOR A TWO-DIMENSIONAL ELLIPTIC PROBLEM WITH LARGE EXPONENT IN NONLINEARITY

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Abstract

We study the existence of nodal solutions to the boundary value problem $-\Delta u = |u|^{p-1}u$ in a bounded, smooth domain $\Omega$ in $\mathbb{R}^2$, with homogeneous Dirichlet boundary condition, when $p$ is a large exponent. We prove that, for $p$ large enough, there exist at least two pairs of solutions which change sign exactly once and whose nodal lines intersect the boundary of $\Omega$.

## 1. *Introduction*

We consider the problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded, smooth domain in $\mathbb{R}^2$ and $p > 0$ is a large exponent.

In order to state old and new results, we need to recall some well-known definitions. The Green function of the Dirichlet Laplacian can be decomposed into a singular part and a regular part, that is,

$$G(x, y) = H(x, y) + \frac{1}{2\pi} \log \frac{1}{|x - y|}.$$

The regular part $H$ is a harmonic function with boundary values of opposite sign with respect to the singular part. The leading term $H(x, x)$ of the regular part of the Green function is called the Robin function of $\Omega$ at $x$.

Problem (1.1) always has a positive solution $u_p$, obtained by minimizing the function

$$I_p(u) := \int_\Omega |\nabla u|^2, \qquad \text{for } u \in \mathrm{H}_0^1(\Omega),$$

on the sphere

$$\{u \in \mathrm{H}_0^1(\Omega) : \|u\|_{p+1} = 1\}.$$

Here, $\|u\|_r$ denotes the $\mathrm{L}^r$-norm of $u$. In [**15**] and [**16**], it is proved that, as $p$ goes to infinity, the solution $u_p$ develops one interior peak, namely $u_p$ approaches zero except at one interior point where it stays bounded and bounded away from zero. More precisely, the authors proved that, up to a subsequence, the renormalized energy $pu_p^{p+1}$ concentrates as a Dirac mass around a critical point of the Robin function. Successively, in [**1**] and [**11**] the authors give a further description of the asymptotic behaviour of $u_p$ as $p$ goes to infinity, by identifying a limit profile

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problem of Liouville-type $\Delta u + e^u = 0$ in $\mathbb{R}^2$, $\int_{\mathbb{R}^2} e^u < +\infty$, and showing that $\|u_p\|_\infty \to \sqrt{e}$ as $p \to +\infty$.

In [**13**] it is proved that problem (1.1) can have many other positive solutions which concentrate, as $p$ goes to infinity, at some different points $\xi_1, \dots, \xi_k$ of $\Omega$, whose location depends on the geometry of $\Omega$. More precisely, if $k$ is a fixed integer, the authors introduce the function $\Psi_k : \mathcal{M} \to \mathbb{R}$ defined by

$$\Psi_k(\xi_1, \dots, \xi_k) := \sum_{i=1,\dots,k} H(\xi_i, \xi_i) + \sum_{\substack{i,j=1,\dots,k \\ i \neq j}} G(\xi_i, \xi_j),$$

where $\mathcal{M} = \Omega^k \setminus \Delta$ and $\Delta$ denotes the diagonal in $\Omega^k$, that is,

$$\Delta = \{(\xi_1, \dots, \xi_k) \in \Omega^k : \xi_i = \xi_j \text{ for some } i \neq j\}.$$

They prove that if $\Psi_k$ has a stable critical value $c$ (according to a definition of stable under $C^1$-perturbations), then, for $p$ large enough, there exists a positive solution $u_p$ to problem (1.1) with $k$ peaks, namely there is a $k$-tuple $\xi_p = (\xi_{1p}, \dots, \xi_{kp})$ converging (up to a subsequence) to a critical point $\xi^* = (\xi_1{}^*, \dots, \xi_k{}^*) \in \mathcal{M}$ of $\Psi_k$ at level $c$ such that

$$u_p \to 0 \;\text{ uniformly in } \Omega \setminus \bigcup_{i=1}^k B_\delta(\xi_{ip}), \qquad \sup_{x \in B_\delta(\xi_{ip})} u_p(x) \to \sqrt{e}\,, \quad \text{for } i = 1, \dots, k,$$

for any $\delta > 0$ and

$$p u_p^{p+1} \rightharpoonup 8\pi e \left( \sum_{i=1}^k \delta_{\xi_i^*} \right) \text{ weakly in the sense of measure in } \overline{\Omega},$$

as $p$ goes to $+\infty$.

After a detailed study of the existence and properties of positive solutions to problem (1.1), one is interested in studying the existence and properties of solutions which change sign.

Problem (1.1) is a particular case of the problems treated in [**4**] and [**7**], where the authors study the existence of sign-changing solutions and their properties for a larger class of nonlinearities. In particular, it is proved that problem (1.1) for any $p > 1$ has a sequence of distinct pairs of solutions $\pm u_p^n$ with $\|u_p^n\|_\infty \to +\infty$ as $n \to +\infty$, $u_p^n$ changes sign if $n \geqslant 2$ and it has at most $n$ nodal domains. Moreover, there exists a least energy nodal solution $\bar{u}_p$ to (1.1) which has precisely two nodal domains. More precisely, if $J_p : \mathrm{H}_0^1(\Omega) \to \mathbb{R}$ is defined by

$$J_p(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{1}{p+1} \int_\Omega |u|^{p+1} \, dx$$

and

$$\mathcal{N}_p = \{u \in \mathrm{H}_0^1(\Omega) : u^+ \neq 0,\, u^- \neq 0,\, J_p'(u)(u^+) = J_p'(u)(u^-) = 0\},$$

then

$$J_p(\bar{u}_p) = \min_{\mathcal{N}_p} J_p. \tag{1.2}$$

In this paper, we are interested in the existence and the profile of sign-changing solutions which concentrate positively at some different points $\xi_1, \dots, \xi_h$ of $\Omega$ and concentrate negatively at some other different points $\xi_{h+1}, \dots, \xi_k$ of $\Omega$.

First of all, let us state our general result. Let $k$ be a fixed positive integer, let $a_i \in \{\pm 1\}$, for $i = 1, \dots, k$, with $a_i a_j = -1$ for some $i \neq j$, and let $\Phi_k : \mathcal{M} \to \mathbb{R}$ be defined by

$$\Phi_k(\xi_1, \dots, \xi_k) = \sum_{i=1}^k H(\xi_i, \xi_i) + \sum_{\substack{i,j=1,\dots,k \\ i \neq j}} a_i a_j G(\xi_i, \xi_j). \tag{1.3}$$

Theorem 1.1. *Assume that* $\Phi_k$ *has a stable critical level* $c$ *in* $\mathcal{M}$, *according to Definition* 2.6. *Then there exists* $p_0 > 0$ *such that, for any* $p \geqslant p_0$, *problem* (1.1) *has one sign-changing solution* $u_p$ *such that*

$$p|u_p|^{p-1}u_p \rightharpoonup 8\pi e\left(\sum_{i=1}^{k} a_i \delta_{\xi_i{}^*}\right) \text{ weakly in the sense of measure in } \overline{\Omega} \tag{1.4}$$

*as* $p \to +\infty$, *for some* $\xi^* = (\xi_1^*, \dots, \xi_k^*) \in \mathcal{M}$ *such that* $\varphi_m(\xi_1^*, \dots, \xi_k^*) = c$. *More precisely, there is a* $k$*-tuple* $\xi_p = (\xi_{1p}, \dots, \xi_{kp})$ *converging (up to a subsequence) to* $\xi^*$ *such that, for any* $\delta > 0$, *as* $p$ *goes to* $+\infty$,

$$u_p \to 0 \text{ uniformly in } \Omega \setminus \bigcup_{i=1}^{k} B_\delta(\xi_{ip}) \tag{1.5}$$

*and*

$$\sup_{x \in B_\delta(\xi_{ip})} u_p(x) \to a_i\sqrt{e}. \tag{1.6}$$

Remark 1.2. In Theorem 1.1 we deal only with critical points of $\Phi_k$ stable with respect to $C^0$-perturbation according to Definition 2.6. As in [**13**], we could prove a stronger result concerning $C^1$-stable critical points by showing that some finite-dimensional functional is $C^1$-close to $\Phi_k$. However, all the applications in which we are interested do not need this generality and, to give a clear idea of our arguments, we will avoid it.

We can specialize the result as far as it concerns the existence and profile of sign-changing solutions which concentrate positively and negatively at two different points $\xi_1$ and $\xi_2$ of $\Omega$ respectively.

First of all, in this case the function $\Phi_2 : \mathcal{M} \to \mathbb{R}$ introduced in (1.3) reduces to

$$\Phi(\xi_1, \xi_2) = H(\xi_1, \xi_1) + H(\xi_2, \xi_2) - 2G(\xi_1, \xi_2). \tag{1.7}$$

The first result concerns the existence of a 'least energy' nodal solution.

Theorem 1.2. *There exists* $p_0 > 0$ *such that, for any* $p \geqslant p_0$,

(i) *problem* (1.1) *has a pair of sign-changing solutions* $\pm u_p$ *such that* (1.5) *and* (1.6) *hold and*

$$p|u_p|^{p-1}u_p \rightharpoonup 8\pi e\left(\delta_{\xi_1^*} - \delta_{\xi_2^*}\right) \text{ weakly in the sense of measure in } \overline{\Omega},$$

*as* $p \to +\infty$, *for some* $\xi_1^*, \xi_2^* \in \Omega$, *with* $\xi_1^* \neq \xi_2^*$, *such that* $\Phi(\xi_1^*, \xi_2^*) = \max_{\Omega\times\Omega} \Phi$;

(ii) *the set* $\Omega \setminus \{x \in \Omega : u_p(x) = 0\}$ *has exactly two connected components;*

(iii) *the set* $\overline{\{x \in \Omega : u_p(x) = 0\}}$ *intersects the boundary of* $\Omega$.

We conjecture that the solution found in Theorem 1.2 coincides with the least energy solution $\bar{u}_p$ found in (1.2).

The second result is a multiplicity result. In order to state it, let

$$C_2(\Omega) = \{A \subset \Omega : \#A = 2\} = \{(x, y) \in \Omega \times \Omega : x \neq y\}/(x, y) \sim (y, x)$$

be the configuration space of unordered pairs of elements of $\Omega$.

Theorem 1.3. *There exists* $p_0 > 0$ *such that, for any* $p \geqslant p_0$,

(i) *problem* (1.1) *has at least* $\operatorname{cat}(C_2(\Omega))$ *pairs of sign-changing solutions* $\pm u_p^i$, *with* $i = 1, \dots, \operatorname{cat}(C_2(\Omega))$, *such that* (1.5) *and* (1.6) *hold and*

$$p|u_p^i|^{p-1}u_p^i \rightharpoonup 8\pi e(\delta_{\xi_1^i} - \delta_{\xi_2^i}) \text{ weakly in the sense of measure in } \overline{\Omega},$$

*as* $p \to +\infty$, *for some* $\xi_1^i, \xi_2^i \in \Omega$, *with* $\xi_1^i \neq \xi_2^i$;

(ii) *the set* $\Omega \setminus \{x \in \Omega : u_p^i(x) = 0\}$ *has exactly two connected components;*

(iii) *the set* $\overline{\{x \in \Omega : u_p^i(x) = 0\}}$ *intersects the boundary of* $\Omega$.

It is easy to see that $\mathrm{cat}(C_2(\Omega)) \geqslant 2$ for any open domain $\Omega \subset \mathbb{R}^2$ (see, for example, [**6**]), so Theorem 1.3 provides at least two pairs of solutions. However, as has been pointed out in [**6**], under certain assumptions on the topology of $\Omega$, $\mathrm{cat}(C_2(\Omega))$ may be larger. We refer the reader to [**5**, Section 6] for some recent computations of $\mathrm{cat}(C_2(\Omega))$.

Section 4 deals with the existence of solutions to (1.1) with more than two nodal zones when $\Omega$ is a ball.

Let us make some comments and remarks. Results similar to those in Theorems 1.2 and 1.3 have been obtained in [**6**] for the slightly subcritical problem

$$\begin{cases} -\Delta u = |u|^{q-1-\epsilon} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where $\Omega$ is a bounded, smooth domain in $\mathbb{R}^N$, $q = (N+2)/(N-2)$, $N \geqslant 3$, and $\epsilon$ is a small positive parameter. In [**6**] the authors study the existence and the profile of sign-changing solutions to problem (1.8), which blow up positively at a point $\xi_1$ of $\Omega$ and blow up negatively at a point $\xi_2$ of $\Omega$, with $\xi_1 \neq \xi_2$, as the parameter $\epsilon$ goes to zero. They introduce the function $\varphi : \mathcal{M} \to \mathbb{R}$ defined by

$$\varphi(\xi_1, \xi_2) = H^{1/2}(\xi_1, \xi_1) H^{1/2}(\xi_2, \xi_2) - G(\xi_1, \xi_2),$$

where $G$ is the Green function of the Dirichlet Laplacian and $H$ is its regular part, that is,

$$G(x, y) = \alpha_N |x - y|^{2-N} - H(x, y), \qquad \text{for } x, y \in \Omega,$$

where $\alpha_N = 1/(N-2)\omega_N$ and $\omega_N$ denotes the surface area of the unit sphere in $\mathbb{R}^N$ (note that in this case $H$ is positive).

They prove that, if $\epsilon$ is small enough then problem (1.8) has at least $\mathrm{cat}(C_2(\Omega))$ pairs of sign-changing solutions $\pm u_\epsilon^i$, with $i = 1, \ldots, \mathrm{cat}(C_2(\Omega))$, such that, as $\epsilon$ goes to zero, $u_\epsilon^i$ blows up positively at a point $\xi_1^i$ and blows up negatively at a point $\xi_2^i$, with $\xi_1^i, \xi_2^i \in \Omega$, $\xi_1^i \neq \xi_2^i$ and $(\xi_1^i, \xi_2^i)$ a critical point of $\varphi$. Moreover, the set $\Omega \setminus \{x \in \Omega : u_\epsilon^i(x) = 0\}$ has exactly two connected components. Finally, if

$$H(\xi_1^i, \xi_1^i) = H(\xi_2^i, \xi_2^i), \tag{1.9}$$

then the nodal set $\overline{\{x \in \Omega : u_\epsilon^i(x) = 0\}}$ intersects the boundary of $\Omega$. We remark that condition (1.9) is satisfied when $\Omega$ is a ball.

We quote the fact that, as far as problem (1.1) is concerned, we do not need any assumption such as (1.9) to ensure that nodal lines of solutions found in Theorems 1.2 and 1.3 intersect the boundary of $\Omega$. We compare this result with the one found in [**2**], where the authors prove that the nodal line of a least energy solution to (1.1) intersects the boundary of $\Omega$, provided $\Omega$ is a ball or an annulus.

Finally, we would like to point out that the analogy between the almost critical problem (1.1) in $\mathbb{R}^2$ and the almost critical problem (1.8) in $\mathbb{R}^N$, with $N \geqslant 3$, is not complete. In fact, only the 'translation invariance' plays a role in the study of (1.1), while both the 'translation invariance' and the 'dilation invariance' are concerned in the study of (1.8). The same phenomenon occurs in the study of the mean field equation, as has already been observed in [**3**], [**10**] and [**12**].

The paper is organized as follows. In Section 2 we reduce the problem to a finite-dimensional one and we prove Theorem 1.1. In this section we use some technical computations developed in [**13**], which are contained in Appendices A and B. In Section 3 we study the profile of sign-changing solutions and we prove Theorems 1.2 and 1.3. In Section 4 we consider the case when $\Omega$ is a ball.

## 2. *Existence of nodal solutions*

Let us consider the problem

$$\begin{cases} -\Delta u = g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $g(u) := |u|^{p-1}u$. Let us introduce some functions which will be the basic elements for building sign-changing solutions to (2.1).

Firstly, let us consider the limit profile problem:

$$-\Delta u = e^u \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^u < +\infty. \tag{2.2}$$

All the solutions of (2.2) are given by

$$U_{\delta,\xi}(y) = \log \frac{8\delta^2}{(\delta^2 + |y-\xi|^2)^2}, \qquad \text{where } y \in \mathbb{R}^2, \tag{2.3}$$

with $\delta > 0$ and $\xi \in \mathbb{R}^2$ (see [**9**]). Let $PU_{\delta,\xi}$ denote the projection of $U_{\delta,\xi}$ onto $\mathrm{H}_0^1(\Omega)$, namely $\Delta PU_{\delta,\xi} = \Delta U_{\delta,\xi}$ in $\Omega$, $PU_{\delta,\xi} = 0$ on $\partial\Omega$. Linear combinations of the functions $PU_{\delta,\xi}$ will be used to build up a first approximation for a solution to (2.1). Unfortunately, such a first approximation happens to be not good enough to solve the problem, because of (2.26). So we need to improve this first approximation. In order to do so, we introduce the functions $w^0$ and $w^1$ below.

Let $U(y) := U_{1,0}(y)$. Define $w^0$ and $w^1$ to be radial solutions of

$$\Delta w^0 + e^U w^0 = f^0 \quad \text{in } \mathbb{R}^2, \qquad f^0(y) := \tfrac{1}{2} e^{U(y)} U^2(y), \tag{2.4}$$

and

$$\Delta w^1 + e^U w^1 = f^1 \quad \text{in } \mathbb{R}^2, \tag{2.5}$$

$$f^1(y) := e^{U(y)} \left( w^0 U - \tfrac{1}{2}(w^0)^2 - \tfrac{1}{3}U^3 - \tfrac{1}{8}U^4 + \tfrac{1}{2} w^0 U^2 \right)(y) \tag{2.6}$$

with the property that, for all $i = 1, 2$,

$$w^i(y) = C_i \log|y| + O\left(\frac{1}{|y|}\right) \qquad \text{as } |y| \to +\infty, \tag{2.7}$$

where

$$C_i = \int_0^{+\infty} t \frac{t^2 - 1}{t^2 + 1} f^i(t)\, dt$$

(see, for example, [**8**]). For any $\delta > 0$ and $\xi \in \mathbb{R}^2$, we define

$$w^0_{\delta,\xi}(x) := w^0\left(\frac{x-\xi}{\delta}\right), \qquad w^1_{\delta,\xi}(x) := w^1\left(\frac{x-\xi}{\delta}\right), \qquad \text{for } x \in \Omega.$$

Let $Pw^0_{\delta,\xi}$ and $Pw^1_{\delta,\xi}$ denote the projections onto $\mathrm{H}_0^1(\Omega)$ of $w^0_{\delta,\xi}$ and $w^1_{\delta,\xi}$, respectively. Define now

$$U_\xi(x) := \sum_{i=1}^{k} \frac{a_i}{\gamma \mu_i^{2/(p-1)}} \left( PU_{\delta_i,\xi_i}(x) + \frac{1}{p} Pw^0_{\delta_i,\xi_i}(x) + \frac{1}{p^2} Pw^1_{\delta_i,\xi_i}(x) \right) \tag{2.8}$$

where $a_i \in \{\pm 1\}$ for any $i = 1, \dots, k$ and

$$\gamma := p^{p/(p-1)} e^{-p/(2(p-1))}. \tag{2.9}$$

Furthermore, we assume that $\xi = (\xi_1, \dots, \xi_k) \in \mathcal{O}_\varepsilon$ for some $\varepsilon > 0$, where

$$\mathcal{O}_\varepsilon := \left\{ \xi \in \Omega^k : (\operatorname{dist} \xi_i, \partial\Omega) \geqslant 2\epsilon,\ |\xi_i - \xi_j| \geqslant 2\epsilon,\ i, j = 1, \dots, k,\ i \neq j \right\},$$

and that the parameters $\delta_i$ satisfy the following relation:

$$\delta_i := \delta_i(p, \xi) = \mu_i e^{-p/4}, \tag{2.10}$$

with $\mu_i := \mu_i(p, \xi)$, for $i = 1, \dots, k$, given by

$$\begin{aligned} \log(8\mu_i^4) = 8\pi H(\xi_i, \xi_i) \left(1 - \frac{C_0}{4p} - \frac{C_1}{4p^2}\right) + \frac{\log \delta_i}{p} \left(C_0 + \frac{C_1}{p}\right) \\ + 8\pi \sum_{j \neq i} (a_i a_j) \frac{\mu_i^{2/(p-1)}}{\mu_j^{2/(p-1)}} G(\xi_i, \xi_j) \left(1 - \frac{C_0}{4p} - \frac{C_1}{4p^2}\right). \end{aligned} \tag{2.11}$$

A direct computation shows that, for $p$ large, $\mu_i$ satisfies

$$\mu_i = e^{-3/4} \exp\left\{ 2\pi H(\xi_i, \xi_i) + 2\pi \sum_{j \neq i} a_i a_j G(\xi_j, \xi_i) \right\} \left(1 + O\left(\frac{1}{p}\right)\right). \tag{2.12}$$

By Lemmata A.1 and A.2 and by this choice of the parameters $\mu_i$, we deduce that, if $y = (x - \xi_i)/\delta_i$, then

$$U_\xi(x) = \frac{a_i}{\gamma \mu_i^{2/(p-1)}} \left( p + U(y) + \frac{1}{p} w_0(y) + \frac{1}{p^2} w_1(y) + O(e^{-p/4}|y| + e^{-p/4}) \right) \tag{2.13}$$

for $|y| \leqslant \epsilon/\delta_i$.

We will look for solutions to (2.1) of the form $u = U_\xi + \phi$, where $\phi$ is a higher order term in the expansion of $u$. It is useful to rewrite problem (2.1) in terms of $\phi$, namely

$$\begin{cases} L(\phi) = -\left[R + N(\phi)\right] & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.14}$$

where

$$L(\phi) := L(p, \xi, \phi) = \Delta\phi + g'(U_\xi), \tag{2.15}$$

$$R := R(p, \xi) = \Delta U_\xi + g(U_\xi), \tag{2.16}$$

$$N(\phi) := N(p, \xi, \phi) = \left[g(U_\xi + \phi) - g(U_\xi) - g'(U_\xi)\phi\right]. \tag{2.17}$$

A first step towards solving (2.14), or equivalently (2.1), consists in studying the invertibility properties of the linear operator $L$. In order to do so, we introduce a weighted $L^\infty$-norm defined as

$$\|h\|_* := \sup_{x \in \Omega} \left| \left( \sum_{i=1}^k \frac{\delta_i}{(\delta_i^2 + |x - \xi_i|^2)^{3/2}} \right)^{-1} h(x) \right| \tag{2.18}$$

for any $h \in L^\infty(\Omega)$. With respect to this norm, the error term $R(p, \xi)$ given in (2.16) can be estimated in the following way.

LEMMA 2.1. *Let $\epsilon > 0$ be fixed. There exist $c > 0$ and $p_0 > 0$ such that, for any $\xi \in \mathcal{O}_\epsilon$ and $p \geqslant p_0$,*

$$\|\Delta U_\xi + g(U_\xi)\|_* \leqslant \frac{c}{p^4}.$$

*Proof.* We give a sketch of the proof. We refer the reader to [**13**] for all the details. A direct computation of $\Delta U_\xi$ and the estimates given by Lemmata A.1 and A.2 readily imply that, far away from the points $\xi_i$, namely for $|x-\xi_i|>\varepsilon$ for all $i=1,\dots,k$, the following estimate holds true:

$$\left|\left(\sum_{j=1}^{k}\frac{\delta_j}{(\delta_j^2+|x-\xi_j|^2)^{3/2}}\right)^{-1}(\Delta U_\xi+g(U_\xi))(x)\right|\leqslant Cpe^{-p/4}$$

for some positive constant $C$.

Let us now fix the index $i$ in $\{1,\dots,k\}$. Taking into account (2.13) and the fact that

$$\left(\frac{p}{\gamma\mu_i^{2/(p-1)}}\right)^p=\frac{1}{\gamma\delta_i^2\mu_i^{2/(p-1)}},$$

we get, for $|x-\xi_i|\leqslant\varepsilon\sqrt{\delta_i}$,

$$\left|\left(\sum_{j=1}^{k}\frac{\delta_j}{(\delta_j^2+|x-\xi_j|^2)^{3/2}}\right)^{-1}(\Delta U_\xi+g(U_\xi))(x)\right|\leqslant Cp^{-4}$$

by means of the Taylor expansion

$$\begin{aligned}&\left(1+\frac{a}{p}+\frac{b}{p^2}+\frac{c}{p^3}\right)^p\\&\quad=e^a\left[1+\frac{1}{p}\left(b-\frac{a^2}{2}\right)+\frac{1}{p^2}\left(c-ab+\frac{a^3}{3}+\frac{b^2}{2}+\frac{a^4}{8}-\frac{a^2b}{2}\right)+O\left(\frac{\log^6(|y|+2)}{p^3}\right)\right]\end{aligned}$$

provided that

$$-4\log(|y|+2)\leqslant a(y)\leqslant C\qquad\text{and}\qquad|b(y)|+|c(y)|\leqslant C\log(|y|+2).$$

While, for $\varepsilon\sqrt{\delta_i}\leqslant|x-\xi_i|\leqslant\varepsilon$, we get

$$\left|\left(\sum_{j=1}^{k}\frac{\delta_j}{(\delta_j^2+|x-\xi_j|^2)^{3/2}}\right)^{-1}(\Delta U_\xi+g(U_\xi))(x)\right|\leqslant Cpe^{-p/8}.$$

This concludes the proof. □

Next, we will solve the following projected linear problem: given $h\in\mathrm{C}(\bar\Omega)$, find a function $\phi$ and constants $c_{i,j}$, for $i=1,\dots,k$ and $j=1,2$, such that

$$L(\phi)=h+\sum_{i,j}c_{i,j}e^{U_{\delta_i,\xi_i}}Z_{i,j},\quad\text{in }\Omega,\tag{2.19}$$

$$\phi=0,\qquad\text{on }\partial\Omega,\tag{2.20}$$

$$\int_\Omega e^{U_{\delta_i,\xi_i}}Z_{i,j}\phi=0,\qquad\text{for all } j=1,2,\ i=1,\dots,k.\tag{2.21}$$

Here, for $i=1,\dots,k$ and $j=1,2$ we set

$$Z_{i,j}(x):=z_j\left(\frac{x-\xi_i}{\delta_i}\right),\qquad\text{with}\quad z_j(y):=-\frac{\partial U}{\partial y_j}(y)=\frac{4y_j}{1+|y|^2}.\tag{2.22}$$

This linear problem is uniquely solvable, for $p$ sufficiently large, with an $L^\infty$-estimate for $\phi$ in terms of $\|h\|_*$. This is the content of the next lemma; its proof is given in Appendix B.

Lemma 2.2. *Let $\epsilon>0$ be fixed. There exist $c>0$ and $p_0>0$ such that for any $p>p_0$ and $\xi\in\mathcal{O}_\epsilon$ there is a unique solution $\phi$ to problem* (2.19)–(2.21) *which satisfies*

$$\|\phi\|_\infty\leqslant cp\|h\|_*.\tag{2.23}$$

Let us now introduce the following non-linear auxiliary problem:

$$\begin{cases} \Delta(U_\xi+\phi)+g\left(U_\xi+\phi\right)=\sum_{i,j} c_{i,j}e^{U_{\delta_i,\xi_i}}Z_{i,j} & \text{in } \Omega, \\ \phi=0 & \text{on } \partial\Omega, \\ \int_\Omega e^{U_{\delta_i,\xi_i}}Z_{i,j}\phi=0 & \text{if } i=1,\dots,k,\ j=1,2, \end{cases} \tag{2.24}$$

for some coefficients $c_{i,j}$. The following result holds.

PROPOSITION 2.3. *Let $\epsilon>0$ be fixed. There exist $c>0$ and $p_0>0$ such that for any $p>p_0$ and $\xi\in\mathcal{O}_\varepsilon$ problem (2.24) has a unique solution $\phi_\xi$ depending on $p$ which satisfies $\|\phi_\xi\|_\infty\leqslant c/p^3$. Furthermore, the function $\xi\to\phi\xi$ is a $C^1$-function in $\mathrm{C}(\bar\Omega)$ and in $H_0^1(\Omega)$.*

*Proof.* Using (2.15)–(2.17) we can rewrite problem (2.24) in the following way:

$$L(\phi)=-\left(R+N(\phi)\right)+\sum_{i,j} c_{i,j}e^{U_{\delta_i,\xi_i}}Z_{i,j}.$$

Let us denote by $\mathrm{C}_*$ the function space $\mathrm{C}(\bar\Omega)$ endowed with the norm $\|\cdot\|_*$. Lemma 2.2 ensures that the unique solution $\phi=T(h)$ of (2.19)–(2.21) defines a continuous linear map from the Banach space $\mathrm{C}_*$ into $\mathrm{C}(\bar\Omega)$, with a norm bounded by a multiple of $p$. Then, problem (2.24) becomes

$$\phi=\mathcal{A}(\phi):=-T\big[R+N(\phi)\big]. \tag{2.25}$$

Let

$$\mathcal{B}_r:=\left\{\phi\in\mathrm{C}(\overline{\Omega}):\phi=0 \text{ on } \partial\Omega,\ \|\phi\|_\infty\leqslant r/p^3\right\},\qquad \text{for some } r>0.$$

We have the following estimates: there exists a positive constant $C$ such that

$$\|N(\phi)\|_*\leqslant Cp\|\phi\|_\infty^2,\qquad \|N(\phi_1)-N(\phi_2)\|_*\leqslant Cp\Big(\max_{i=1,2}\|\phi_i\|_\infty\Big)\|\phi_1-\phi_2\|_\infty, \tag{2.26}$$

for any $\phi,\phi_1,\phi_2\in\mathcal{F}_\gamma$. In fact, by the Lagrange theorem we have

$$|N(\phi)|\leqslant p(p-1)\left(U_\xi+O(1/p^3)\right)^{p-2}\phi^2,$$
$$|N(\phi_1)-N(\phi_2)|\leqslant p(p-1)\left(U_\xi+O(1/p^3)\right)^{p-2}\Big(\max_{i=1,2}|\phi_i|\Big)|\phi_1-\phi_2|$$

for any $x\in\Omega$, and hence, by the fact that

$$p\left|U_\xi+O\left(1/p^3\right)\right|^{p-2}\leqslant C\sum_{j=1}^m e^{U_j(x)}$$

we get (2.26) since $\|\sum_{j=1}^m e^{U_j}\|_*=O(1)$.

By (2.26) and Lemmata 2.1 and 2.2, we easily deduce that $\mathcal{A}$ is a contraction mapping on $\mathcal{B}_r$ for a suitable $r>0$. Finally, a unique fixed point $\phi_\xi$ of $\mathcal{A}$ exists in $\mathcal{B}_r$. As for the regularity of the map $\xi\to\phi_\xi$, we can proceed in a standard way by means of the Implicit Function Theorem. The proof is now complete. □

After problem (2.24) has been solved, we find a solution to problem (2.14) (and hence to the original problem (2.1)) if we find a point $\xi$ such that coefficients $c_{i,j}(\xi)$ in (2.24) satisfy

$$c_{i,j}(\xi)=0\qquad \text{for } i=1,\dots,k,\ j=1,2. \tag{2.27}$$

Let us introduce the energy functional $J_p:\mathrm{H}_0^1(\Omega)\to\mathbb{R}$ given by

$$J_p(u):=\tfrac12\int_\Omega|\nabla u|^2\,dx-\frac{1}{p+1}\int_\Omega|u|^{p+1}\,dx, \tag{2.28}$$

whose critical points are solutions to (2.1). We also introduce the finite-dimensional restriction $\widetilde{J}_p : \mathcal{O}_\epsilon \to \mathbb{R}$ given by

$$\widetilde{J}_p(\xi) := J_p\big(U_\xi + \phi_\xi\big). \tag{2.29}$$

The following result holds.

LEMMA 2.4. *If $\xi$ is a critical point of $\widetilde{J}_p$, then $U_\xi + \phi_\xi$ is a critical point of $J_p$, namely a solution to problem* (2.1).

*Proof.* The function $\widetilde{J}_p$ is of class $C^1$ since the map $\xi \to \phi_\xi$ is a $C^1$-function in $H_0^1(\Omega)$. Then, $D_\xi F(\xi) = 0$ is equivalent to

$$\sum_{i,j} c_{i,j}(\xi) \int_\Omega e^{U_{\delta_i,\xi_i}} Z_{i,j} D_\xi U_\xi - \sum_{i,j} c_{i,j}(\xi) \int_\Omega D_\xi (e^{U_{\delta_i,\xi_i}} Z_{i,j}) \phi_\xi = 0$$

taking into account (2.24). Writing $D_\xi U_\xi$ explicitly, we find that direct computations then show that, for $p$ large and uniformly in $\xi \in \mathcal{O}_\varepsilon$,

$$0 = \frac{D}{\gamma \delta_i \mu_i^{2/(p-1)}} c_{i,j}(\xi) + O\left( \frac{1}{p\gamma\delta_i} \sum_{l,h} |c_{l,h}(\xi)| \right)$$

where $D$ is a given positive constant. This fact implies that $c_{i,j}(\xi) = 0$ for all $i$ and $j$. ☐

Next, we need to write the expansion of $\widetilde{J}_p$ as $p$ goes to $+\infty$.

LEMMA 2.5. *Let $\epsilon > 0$. Then*

$$\widetilde{J}_p(\xi) = k\frac{A_1}{p} - 2A_1 k \frac{\log p}{p^2} + k\frac{A_2}{p^2} - \frac{A_3}{p^2}\Phi_k(\xi) + O\left(\frac{\log^2 p}{p^3}\right)$$

*uniformly with respect to $\xi \in \mathcal{O}_\epsilon$. Here*

$$A_1 := 4\pi e, \qquad A_2 := 8\pi e + \frac{e}{2}\int_{\mathbb{R}^2} (e^U U - \Delta w^0)(y)\, dy, \qquad A_3 := 32\pi^2 e \tag{2.30}$$

*and the function $\Phi_k : \mathcal{M} \to \mathbb{R}$ is defined by* (*see* (1.3))

$$\Phi_k(\xi_1, \ldots, \xi_k) = \sum_{i=1,\ldots,k} H(\xi_i, \xi_i) + \sum_{\substack{i,j=1,\ldots,k \\ i \neq j}} a_i a_j G(\xi_i, \xi_j).$$

*Proof.* Multiplying the equations in (2.24) by $U_\xi + \phi_\xi$ and integrating by parts, we get

$$\widetilde{J}_p(\xi) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_\Omega |\nabla (U_\xi + \phi_\xi)|^2 - \frac{1}{p+1} \sum_{i,j} c_{i,j}(\xi) \int_\Omega e^{U_{\delta_i,\xi_i}} Z_{i,j}\, (U_\xi + \phi_\xi)\,.$$

Now, from equation (B.12) contained in the proof of Lemma 2.2, we have

$$|c_{i,j}(\xi)| = O\left( \|N(\phi_p) + R\|_* + \frac{1}{p}\|\phi_\xi\|_\infty \right) = O\left(\frac{1}{p^4}\right).$$

Hence, we have

$$\widetilde{J}_p(\xi) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \left( \int_\Omega |\nabla U_\xi|^2 + 2\int_\Omega \nabla U_\xi \nabla \phi_\xi + \int_\Omega |\nabla \phi_\xi|^2 \right) + O\left(\frac{1}{p^5}\right)$$

since $U_\xi$ is a bounded function. We expand the term $\int_\Omega |\nabla U_\xi|^2$: in view of (2.13) we have

$$\begin{aligned}\int_\Omega |\nabla U_\xi|^2 &= \sum_{j=1}^k \frac{a_j}{\gamma \mu_j^{2/(p-1)}} \int_{B(\xi_j,\varepsilon)} \bigg( e^{U_{\delta_j,\xi_j}} - \frac{1}{p\delta_j^2}\Delta w^0\Big(\frac{x-\xi_j}{\delta_j}\Big) \\ &\quad - \frac{1}{p^2\delta_j^2}\Delta w^1\Big(\frac{x-\xi_j}{\delta_j}\Big) + O(pe^{-p/2})\bigg) U_\xi + O(pe^{-p/2}) \\ &= \sum_{j=1}^k \frac{a_j^2}{\gamma^2 \mu_j^{4/(p-1)}} \int_{B(0,\varepsilon/\delta_j)} \Big(\frac{8}{(1+|y|^2)^2} - \frac{1}{p}\Delta w^0 - \frac{1}{p^2}\Delta w^1 + O(pe^{-p})\Big) \\ &\quad \times \Big(p + U + \frac{1}{p}w^0 + \frac{1}{p^2}w^1 + O(e^{-p/4}|y| + e^{-p/4})\Big) + O(pe^{-p/2}) \\ &= \sum_{j=1}^k \frac{1}{\gamma^2\mu_j^{4/(p-1)}}\Big(8\pi p + \int_{\mathbb{R}^2}\Big(\frac{8}{(1+|y|^2)^2}U - \Delta w^0\Big) + O\Big(\frac{1}{p}\Big)\Big) \\ &= \frac{8\pi kp}{\gamma^2} - \frac{32\pi}{\gamma^2}\sum_{j=1}^k \log\mu_j + \frac{k}{\gamma^2}\int_{\mathbb{R}^2}\Big(\frac{8}{(1+|y|^2)^2}U - \Delta w^0\Big) + O\Big(\frac{1}{p^3}\Big)\end{aligned}$$

since $\mu_j^{-4/(p-1)} = 1 - (4/p)\log\mu_j + O(1/p^2)$. Recalling the expression of $\mu_j$ in (2.11) and (2.12), we get

$$\begin{aligned}\int_\Omega |\nabla U_\xi|^2 &= \frac{8\pi kp}{\gamma^2} - \frac{64\pi^2}{\gamma^2}\Phi_k(\xi_1,\dots,\xi_k) + \frac{24\pi k}{\gamma^2} \\ &\quad + \frac{k}{\gamma^2}\int_{\mathbb{R}^2}\Big(\frac{8}{(1+|y|^2)^2}U - \Delta w^0\Big) + O\Big(\frac{1}{p^3}\Big)\end{aligned} \tag{2.31}$$

uniformly for points in $\mathcal{O}_\varepsilon$. By Lemma 2.1 and Remark B.1 we get

$$\|\phi_\xi\|_{\mathrm{H}_0^1(\Omega)} \leqslant C(\|\phi_\xi\|_\infty + \|N(\phi_\xi)\|_* + \|R\|_*) = O(1/p^3)$$

since $\|N(\phi_\xi)\|_* = O(p\|\phi_\xi\|_\infty^2)$ and $\|R\|_* = O(1/p^4)$. Hence, by (2.31) we get

$$\int_\Omega \nabla U_\xi \nabla\phi_\xi + \frac{1}{2}\int_\Omega |\nabla\phi_\xi|^2 = O\Big(\frac{1}{p^3}\Big). \tag{2.32}$$

Finally, inserting (2.31) and (2.32) in the expression of $\widetilde{J}_p$, we get

$$\begin{aligned}\widetilde{J}_p(\xi) &= \frac{4\pi kp}{\gamma^2} - \frac{32\pi^2}{\gamma^2}\Phi_k(\xi_1,\dots,\xi_k) + \frac{4\pi k}{\gamma^2} \\ &\quad + \frac{k}{2\gamma^2}\int_{\mathbb{R}^2}\Big(\frac{8}{(1+|y|^2)^2}U - \Delta w^0\Big) + O\Big(\frac{1}{p^3}\Big)\end{aligned}$$

uniformly on points in $\mathcal{O}_\varepsilon$. Since $\gamma = p^{p/(p-1)}e^{-p/(2(p-1))}$ we obtain the desired expansion. □

Let us introduce now the following definition.

Definition 2.6. We say that $c$ is a *stable critical level* of $\Phi$ in $\mathcal{M}$ if there exist an open set $\mathcal{D}$ compactly contained in $\mathcal{M}$, and $B$ and $B_0$ that are closed subsets of $\bar{\mathcal{D}}$, with $B$ connected and $B_0 \subset B$, such that the following conditions hold:

$$c := \inf_{\gamma\in\Gamma}\sup_{\xi\in B}\Phi(\gamma(\xi)) > \sup_{\xi\in B_0}\Phi(\xi) \tag{2.33}$$

and

$$\{\xi \in \partial\mathcal{D} : \Phi(\xi) = c\} = \emptyset. \tag{2.34}$$

Here, $\Gamma$ denotes the class of all maps $\gamma \in C(B, \bar{\mathcal{D}})$ such that there exists a homotopy $\Psi \in C([0,1] \times B, \bar{\mathcal{D}})$ satisfying

$$\Psi(0, \cdot) = \mathrm{Id}_B, \qquad \Psi(1, \cdot) = \gamma, \qquad \Psi(t, \cdot)|_{B_0} = \mathrm{Id}_{B_0} \quad \text{for all } t \in [0,1].$$

Under these assumptions, a critical point $\bar{\xi} \in \mathcal{D}$ of $\Phi$ exists at level $c$, as a standard deformation argument shows. As an example, taking $B = B_0 = \partial\mathcal{D}$, one can easily check that (2.33) and (2.34) hold if

$$\sup_{\xi \in \partial\mathcal{D}} \Phi(\xi) < \sup_{\xi \in \mathcal{D}} \Phi(\xi) \quad \text{or} \quad \inf_{\xi \in \partial\mathcal{D}} \Phi(\xi) > \inf_{\xi \in \mathcal{D}} \Phi(\xi),$$

namely the case of (possibly degenerate) local maximum/minimum values of $\varphi_m$.

We are now in position to carry out the proof of our main result.

*Proof of Theorem* 1.1. In view of Lemma 2.4, the function $u_p = U_{\xi_p} + \phi_{\xi_p}$ is a solution to problem (2.1) if we show that $\xi_p$ is a critical point of the function $\widetilde{J}_p$. This is equivalent to showing that

$$F_p(\xi) = -p^2 A_3^{-1} \left( \widetilde{J}_p(\xi) - k\frac{A_1}{p} + 2kA_1 \frac{\log p}{p^2} - k\frac{A_2}{p^2} \right) \tag{2.35}$$

has a critical point $\xi_p$. Lemma 2.5 implies that $F_p$ is uniformly close to $\Phi$, as $p \to \infty$, on compact sets of $\mathcal{M}$. Moreover, assumptions (2.33) and (2.34) are stable with respect to $C^0$-perturbation and still hold for the function $F_p$ provided $p$ is large enough. Therefore, $F_p$ has a critical point $\xi_p \in \mathcal{D}$, whose critical value $c_p$ approaches $c$, as $p \to \infty$. By the definition of $U_\xi$ and since $\|\phi_\xi\|_\infty \leqslant c/p^3$, it is straightforward to show the validity of (1.4)–(1.6) for $u_p$. This proves our claim. □

## 3. *The case* $k = 2$

This section is devoted to the problem of finding nodal solutions to problem (1.1) in a general domain $\Omega \subset \mathbb{R}^2$ with exactly two nodal regions.

Assume that $k = 2$, $a_1 = 1$ and $a_2 = -1$. Then the function $\Phi$ defined in (1.3) reduces to (see (1.7))

$$\Phi(\xi) = H(\xi_1, \xi_1) + H(\xi_2, \xi_2) - 2G(\xi_1, \xi_2), \qquad \text{with } (\xi_1, \xi_2) \in \mathcal{M},$$

where $\mathcal{M} = \Omega \times \Omega \setminus \Delta$ and $\Delta$ is the diagonal in $\Omega \times \Omega$.

First of all, let us prove the existence part (i) in Theorems 1.2 and 1.3.

*Proof of Theorem* 1.2(i). We point out that $c := \max_{\mathcal{M}} \Phi$ is finite since $\Phi(\xi) \to -\infty$ as $\xi$ approaches $\partial\mathcal{M}$, and it is a stable critical value of the function $\Phi$ according to Definition 2.6. Therefore, the claim follows by Theorem 1.1. □

*Proof of Theorem* 1.3(i). By Lemma 2.4, we need to prove that, if $p$ is large enough, the function $\widetilde{J}_p$ has at least $\mathrm{cat}(C_2(\Omega))$ pairs of critical points. In order to prove that $\widetilde{J}_p$ has at least $\mathrm{cat}(C_2(\Omega))$ pairs of critical points, it is enough to show that the $F_p(\xi)$ in (2.35) has at least $\mathrm{cat}(C_2(\Omega))$ pairs of critical points. From Lemma 2.5 we see that $F_p$ is uniformly close to $\Phi$ on compact sets of $\mathcal{M}$ as $p \to \infty$. Moreover, we point out that

$$\Phi(\xi) \to -\infty \qquad \text{as } \xi \to \partial\mathcal{M}. \tag{3.1}$$

Now, let $\widetilde{\mathcal{M}}$ denote the quotient manifold with respect to the equivalence $(\xi_1, \xi_2) \sim (\xi_2, \xi_1)$. Since through the map $(\xi_1, \xi_2) \to (\xi_2, \xi_1)$ we have $U_\xi \to -U_\xi$ and $\phi_\xi \to -\phi_\xi$, the induced functions $\widetilde{F}_p, \widetilde{\Phi} : \widetilde{\mathcal{M}} \to \mathbb{R}$ are well defined. Setting $m := \mathrm{cat}(\widetilde{\mathcal{M}})$, we see that there exists a compact set $C \subset \widetilde{\mathcal{M}}$ such that $\mathrm{cat}(C) = m$. By (3.1) we deduce that there exists an open bounded set $\mathcal{U} \subset \widetilde{\mathcal{M}}$ such that $C \subset \mathcal{U}$ and $\sup_{\partial\mathcal{U}} \widetilde{\Phi} < \min_C \widetilde{\Phi}$. Therefore, if $p$ is large enough, it follows that $\sup_{\partial\mathcal{U}} \widetilde{F}_p < \min_C \widetilde{F}_p$. Now, for $j = 1, \dots, m$ set

$$\begin{aligned} c_p^j &:= \sup\left\{c : \mathrm{cat}_{\widetilde{\mathcal{M}}}\left(\widetilde{F}_p^c \cap \mathcal{U}\right) \geqslant j\right\} \\ &= \sup\left\{\min_A \widetilde{F}_p : A \subset \mathcal{U} \text{ compact, } \mathrm{cat}_{\widetilde{\mathcal{M}}} A \geqslant j\right\}, \end{aligned}$$

where $\widetilde{F}_p^c = \{\xi \in \widetilde{\mathcal{M}} : \widetilde{F}_p(\xi) \geqslant c\}$. Standard arguments based on the Deformation Lemma show that $c_p^j$, for $j = 1, \dots, m$, are critical levels for $\widetilde{F}_p$. Finally, since $\widetilde{\mathcal{M}}$ is homotopy equivalent to the configuration space $C_2(\Omega)$, we have $m = \mathrm{cat}\widetilde{\mathcal{M}} = \mathrm{cat}(C_2(\Omega))$. This proves our claim. □

Let $u$ be a solution to problem (2.1) found in Theorems 1.2(i) and 1.3(i). We know that

$$\begin{aligned} u(x) = \frac{1}{\gamma}\Bigg[&\frac{1}{\mu_{1p}^{2/(p-1)}}\left(PU_{\delta_{1p},\xi_{1p}}(x) + \frac{1}{p} Pw^0_{\delta_{1p},\xi_{1p}}\right) \\ &-\frac{1}{\mu_{2p}^{2/(p-1)}}\left(PU_{\delta_{2p},\xi_{2p}}(x) + \frac{1}{p} Pw^0_{\delta_{2p},\xi_{2p}}\right) + \hat{\phi}_\xi(x)\Bigg], \end{aligned} \tag{3.2}$$

where

$$\gamma := p^{p/(p-1)} e^{-p/(2(p-1))}, \qquad \frac{\gamma}{p} \to \frac{1}{\sqrt{e}} \quad \text{as } p \to +\infty, \tag{3.3}$$

$$\xi_{ip} \to \xi_i^* \quad \text{as } p \to +\infty, \qquad \xi_1^*, \xi_2^* \in \Omega,\ \xi_1^* \neq \xi_2^*, \tag{3.4}$$

$$\delta_{ip} := \delta_i(p, \xi_p) = \mu_{ip} e^{-p/4}, \tag{3.5}$$

$$\mu_{ip} := \mu_i(p, \xi_p) \to e^{-3/4} e^{2\pi H(\xi_i^*, \xi_i^*) - 2\pi G(\xi_1^*, \xi_2^*)} \quad \text{as } p \to +\infty, \tag{3.6}$$

$$\|\hat{\phi}_\xi\|_\infty \leqslant C/p. \tag{3.7}$$

Here,

$$\hat{\phi}_\xi := \frac{1}{p^2}\left(\mu_{1p}^{-2/(p-1)} Pw^1_{\delta_{1p},\xi_{1p}} - \mu_{2p}^{-2/(p-1)} Pw^1_{\delta_{2p},\xi_{2p}}\right) + \gamma\phi_\xi$$

and the last estimate (3.7) follows by Proposition 2.3 and (A.5) of Lemma A.2. Let us prove that $u$ changes sign exactly once.

THEOREM 3.1. *Let $u$ be a solution to (2.1) as in (3.2)–(3.7). Then, if $p$ is large enough, the set $\Omega \setminus \{x \in \Omega : u(x) = 0\}$ has exactly two connected components.*

*Proof.* First of all, by (2.13) we deduce that there exist $r > 0$ small and $p_0 > 0$ such that, for any $p > p_0$,

$$u(x) > 0 \qquad \text{for any } x \in B\left(\xi_{1p}, r\right) \tag{3.8}$$

and

$$u(x) < 0 \qquad \text{for any } x \in B\left(\xi_{2p}, r\right) \tag{3.9}$$

since

$$U(y) \geqslant -p + \log\frac{8\mu_i^2}{4r^4} \qquad \text{and} \qquad \frac{1}{p}w_0(y) + \frac{1}{p^2}w_1(y) \geqslant -c \qquad \text{for } |y| \leqslant \frac{r}{\delta_i}.$$

Now, let $\Omega_p := \Omega \setminus \left[B\left(\xi_{1p}, r\right) \cup B\left(\xi_{2p}, r\right)\right]$. Formulas (3.4)–(3.7) and Lemma A.2 imply that

$$\begin{aligned} \|u\|_{\mathrm{L}^\infty(\Omega_p)} &\leqslant \frac{1}{\gamma}\left(\left\|\frac{1}{\mu_{1p}^{2/(p-1)}} PU_{\delta_{1p},\xi_{1p}}\right\|_{\mathrm{L}^\infty(\Omega_p)} + \left\|\frac{1}{\mu_{2p}^{2/(p-1)}} PU_{\delta_{2p},\xi_{2p}}\right\|_{\mathrm{L}^\infty(\Omega_p)}\right) \\ &\quad + \frac{1}{p\gamma}\left(\left\|Pw^0_{\delta_{1p},\xi_{1p}}\right\|_{\mathrm{L}^\infty(\Omega_p)} + \left\|Pw^0_{\delta_{2p},\xi_{2p}}\right\|_{\mathrm{L}^\infty(\Omega_p)}\right) + \frac{1}{\gamma}\left\|\hat{\phi}_\xi\right\|_{\mathrm{L}^\infty(\Omega_p)} \\ &\leqslant \frac{c}{\gamma} + \frac{c}{p\gamma}\left(|\log\delta_{1p}| + |\log\delta_{2p}|\right) + \frac{c}{p\gamma} \leqslant \frac{c}{p} \end{aligned} \tag{3.10}$$

since by Lemma A.1 we easily deduce that $\left\|PU_{\delta_{ip},\xi_{ip}}\right\|_{\mathrm{L}^\infty(\Omega_p)} \leqslant c$, for $i = 1, 2$. Therefore, by (3.10) we deduce that

$$\lim_{p\to+\infty} (p+1)\left\||u|^{p-1}\right\|_{\mathrm{L}^{p/(p-2)}(\Omega_p)} = 0. \tag{3.11}$$

Finally, it is clear that $\Omega \setminus \{x \in \Omega : u(x) = 0\}$ has at least two connected components

$$\Omega_p^+ \subset \{x \in \Omega : u(x) > 0\} \qquad \text{and} \qquad \Omega_p^- \subset \{x \in \Omega : u(x) < 0\}.$$

By (3.8) and (3.9) it follows that $B\left(\xi_{1p}, r\right) \subset \Omega_p^+$ and $B\left(\xi_{2p}, r\right) \subset \Omega_p^-$. By contradiction, we assume that there exists a third connected component $\omega \subset \Omega_p$. Therefore, $u$ solves $-\Delta u = |u|^{p-1}u$ in $\omega$, $u = 0$ on $\partial\omega$, and the weight $a := |u|^{p-1}$ satisfies (3.12) by means of (3.11). By Lemma 3.2 below, it follows that $u \equiv 0$ in $\omega$ and a contradiction arises. □

Lemma 3.2. *Let $\omega$ be a bounded domain in $\mathbb{R}^2$ and assume that $a : \omega \to \mathbb{R}$ satisfies*

$$\limsup_{p\to+\infty} (p+1)\|a\|_{\mathrm{L}^{p/(p-2)}(\omega)} < 8\pi e. \tag{3.12}$$

*Then the problem $-\Delta u = au$ in $\omega$, for $u \in H_0^1(\omega)$, has only the trivial solution.*

*Proof.* First of all, we point out that $u \in \mathrm{L}^p(\omega)$ for any $p > 1$ and

$$\|u\|^2_{\mathrm{H}_0^1(\omega)} = \int_\omega au^2 \leqslant \|a\|_{\mathrm{L}^{p/(p-2)}(\omega)}\|u\|^2_{\mathrm{L}^p(\omega)} \leqslant S_p^2\|a\|_{\mathrm{L}^{p/(p-2)}(\omega)}\|u\|^2_{\mathrm{H}_0^1(\omega)},$$

where $S_p$ denotes the best Sobolev constant of the embedding $\mathrm{H}_0^1(\omega) \hookrightarrow \mathrm{L}^p(\omega)$. Therefore, if $u$ is a non-trivial solution, the following condition has to be satisfied:

$$S_p^2\|a\|_{\mathrm{L}^{p/(p-2)}(\omega)} \geqslant 1.$$

On the other hand, in [**15**] it was proved that

$$\lim_{p\to\infty} \frac{p-1}{S_p^2} = 8\pi e.$$

Then, by (3.12) a contradiction arises. □

In order to prove that the nodal line of $u$ touches the boundary of $\Omega$, it is useful to describe the asymptotic behaviour of $u$ in a neighbourhood of the boundary, as $p$ goes to $+\infty$.

Proposition 3.3. *Let $u$ be a solution to* (2.1) *as in* (3.2)–(3.7). *Then*

$$pu(x) \to 8\pi\sqrt{e}\,[G(x,\xi_1^*) - G(x,\xi_2^*)] \quad \text{in } \mathrm{C}^1_{\mathrm{loc}}\left(\overline{\Omega} \setminus \{\xi_1^*, \xi_2^*\}\right) \tag{3.13}$$

*as $p \to +\infty$.*

*Proof.* By Lemmata A.1 and A.2, using (3.2)–(3.7), we deduce that estimate (3.13) holds in $C^0_{\rm loc}\left(\overline{\Omega}\setminus\{\xi_1^*,\xi_2^*\}\right)$. We are going to prove that

$$\|g(u)\|_{L^1(\Omega)} = \|u\|^p_{L^p(\Omega)} \leqslant c/p \tag{3.14}$$

for some positive constant $c$ and

$$\|g(u)\|_{L^\infty(\omega)} = \|u\|^p_{L^\infty(\omega)} \leqslant c_\omega/p, \tag{3.15}$$

for some positive constant $c_\omega$, for any $\omega$ neighbourhood of $\partial\Omega$. By Lemma 3.4, we deduce that $\|\nabla u\|_{C^{0,\alpha}(\omega')} \leqslant c/p$ and the claim follows by the Ascoli–Arzelá Theorem. Finally, (3.14) follows since

$$\int_\Omega |u|^p\,dx \leqslant |\Omega|^{1/(p+1)}\left(\int_\Omega |u|^{p+1}\,dx\right)^{p/(p+1)} \leqslant \frac{c}{p},$$

because $\lim_{p\to+\infty} p\int_\Omega |u|^{p+1} = 16\pi e$, and (3.15) follows exactly as (3.10). □

We recall the following lemma (see [**14**, Lemma 2]).

Lemma 3.4. *Let $u$ be a solution to $-\Delta u = f$ in $\Omega$, $u = 0$ on $\partial\Omega$. If $\omega$ is a neighbourhood of $\partial\Omega$, then*

$$\|\nabla u\|_{C^{0,\alpha}(\omega')} \leqslant c\left(\|f\|_{L^1(\Omega)} + \|f\|_{L^\infty(\omega)}\right),$$

*where $\alpha \in (0,1)$ and $\omega' \subset\subset \omega$ is a neighbourhood of $\partial\Omega$.*

Theorem 3.5. *Let $u$ be a solution to* (2.1) *as in* (3.2)–(3.7)*. Then if $p$ is large enough,*

$$\overline{\{x\in\Omega : u(x)=0\}} \cap \partial\Omega \neq \emptyset.$$

*Proof.* First of all, we remark that if $\overline{\{x\in\Omega : u(x)=0\}} \cap \partial\Omega = \emptyset$, then $\frac{\partial u}{\partial\nu}$ does not change sign on $\partial\Omega$. On the other hand, the normal derivative

$$\frac{\partial}{\partial\nu}\left[G(\cdot,\xi_1^*) - G(\cdot,\xi_2^*)\right]$$

changes sign on $\partial\Omega$, since

$$\int_{\partial\Omega} \frac{\partial}{\partial\nu}\left[G(x,\xi_1^*) - G(x,\xi_2^*)\right]dx = 0.$$

By Proposition 3.3 we deduce that

$$pV\partial u\partial\nu(x) \to \frac{\partial}{\partial\nu}\left[G(x,\xi_1^*) - G(x,\xi_2^*)\right]$$

uniformly on $\partial\Omega$ as $p\to\infty$. Therefore, if $p$ is large enough, $\frac{\partial u}{\partial\nu}$ also changes sign on $\partial\Omega$ and a contradiction arises. □

*Proof of* (ii) *and* (iii) *of Theorems* 1.2 *and* 1.3. Part (ii) follows by Theorem 3.1 and (iii) follows by Theorem 3.5. □

## 4. *The symmetric case*

In this section we describe three possible symmetric configurations for points of positive and negative concentration for nodal solutions to problem (1.1) when the domain $\Omega$ is the ball $\{x\in\mathbb{R}^2 : |x| < R\}$. In all cases, we strongly use the symmetry of the problem.

In the first example, we build a solution to (1.1) with $h$ points of positive concentration and $h$ points of negative concentration located on the vertices of a regular polygon, distributed with alternating sign.

Let $h \geqslant 1$ be a fixed integer and let $k = 2h$. Set

$$\xi_i^* = \left(\cos \frac{2\pi}{k}(i-1), \sin \frac{2\pi}{k}(i-1)\right) \qquad \text{for any } i = 1, \dots, k. \tag{4.1}$$

THEOREM 4.1. *For any even integer $k$ there exist $p_0 > 0$ and $\rho^* \in (0, R)$ such that for any $p \geqslant p_0$ problem* (1.1) *has a sign-changing solution $u$ such that* (1.5) *and* (1.6) *hold and*

$$p|u|^{p-1}u \rightharpoonup 8\pi e \sum_{i=1}^{k} (-1)^{i+1} \delta_{\rho^* \xi_i^*} \quad \textit{weakly in the sense of measure in } \overline{\Omega}$$

*as $p \to +\infty$.*

*Proof.* We will look for a solution to problem (1.1) as $u(x) = U_\rho(x) + \phi(x)$, where

$$U_\rho := \sum_{i=1}^{k} \frac{(-1)^{i+1}}{\gamma \mu_i^{2/(p-1)}} \left( PU_{\delta_i, \xi_i} + \frac{1}{p} Pw^0_{\delta_i, \xi_i} + \frac{1}{p^2} Pw^1_{\delta_i, \xi_i} \right), \tag{4.2}$$

where the $\delta_i$ are given in (2.10) and (2.11) and the concentration points are, for $i = 1, \dots, k$,

$$\xi_i := \xi_i(\rho) = \rho \xi_i^* = \left(\rho \cos \frac{2\pi}{k}(i-1), \rho \sin \frac{2\pi}{k}(i-1)\right) \qquad \text{with } \rho \in (0, R), \tag{4.3}$$

and the rest of the term $\phi$ is symmetric with respect to the variable $x_2$ and is symmetric with respect to each line $\{t\xi_i^* : t \in \mathbb{R}\}$ for $i = 1, \dots, k$.

Using the results obtained in previous sections and taking into account the symmetry of the domain, we reduce the problem of finding solutions to (1.1) to that of finding critical points of the function $\widetilde{J}_p : (0, R) \to \mathbb{R}$ defined as in (2.29) by $\widetilde{J}_p(\rho) := J_p(U_\rho + \phi(\rho))$. It is not difficult to check that

$$\widetilde{J}_p(\rho) = k\frac{A_1}{p} - 2kA_1 \frac{\log p}{p^2} + k\frac{A_2}{p^2} - k\frac{A_3}{p^2}\Phi(\rho) + O\left(\frac{\log^2 p}{p^3}\right),$$

where $A_1$, $A_2$ and $A_3$ are given in (2.30) and

$$\Phi(\rho) := H(\rho\xi_1^*, \rho\xi_1^*) - \sum_{i=2}^{k} (-1)^i G(\rho\xi_1^*, \rho\xi_i^*), \qquad \text{for } \rho \in (0, R). \tag{4.4}$$

Since, on a ball $\Omega$ centred at 0 of radius $R$,

$$G(x, y) = \frac{1}{2\pi}\left(\log \frac{1}{|y - x|} - \log \frac{R}{\sqrt{|x|^2|y|^2 + R^4 - 2R^2 x \cdot y}}\right),$$
$$H(x, x) = -\frac{1}{2\pi} \log \frac{R}{R^2 - |x|^2},$$

the function $\Phi$ reduces to (since $k$ is even)

$$\begin{aligned}\Phi(\rho) &= \frac{1}{2\pi} \log \frac{R^2 - \rho^2}{R} \\ &\quad + \frac{1}{2\pi} \sum_{i=2}^{k} (-1)^i \left[\log \rho|\xi_1^* - \xi_i^*| - \log \frac{\sqrt{\rho^4 + R^4 - 2\rho^2 R^2 \xi_1^* \cdot \xi_i^*}}{R}\right] \\ &= \frac{1}{2\pi}\left[\log(R^2 - \rho^2) + \log \rho - \sum_{i=2}^{k} (-1)^i \log \sqrt{\rho^4 + R^4 - 2\rho^2 R^2 \cos \frac{2\pi}{k}(i-1)}\right] \\ &\quad + \frac{1}{2\pi} \sum_{i=2}^{k} (-1)^i \log |\xi_1^* - \xi_i^*|.\end{aligned}$$

It is easy to check that $\lim_{\rho\to 0^+}\Phi(\rho)=\lim_{\rho\to R^-}\Phi(\rho)=-\infty$. Then there exists $\rho^*\in(0,R)$ such that $\Phi(\rho^*)=\max_{\rho\in(0,R)}\Phi(\rho)$, which is a critical value which persists under a small $C^0$-perturbation. This proves our claim. □

Our second example is a nodal solution to problem (1.1) with a negative (or positive) point of concentration at the origin of the ball and again $h$ points of positive concentration and $h$ points of negative concentration located at the vertices of a regular polygon with alternating signs.

THEOREM 4.2. *For any even integer $k$ there exist $p_0>0$ and $\rho^*\in(0,R)$ such that for any $p\geqslant p_0$ problem* (1.1) *has a sign-changing solution $u$ such that* (1.5) *and* (1.6) *hold and*

$$p|u|^{p-1}u\rightharpoonup 8\pi e\left(-\delta_0+\sum_{i=1}^{k}(-1)^{i+1}\delta_{\rho^*\xi_i^*}\right)\quad \textit{weakly in the sense of measure in } \overline{\Omega}$$

*as $p\to+\infty$.*

*Proof.* Here, we will look for a solution to problem (1.1) as $u(x)=U_\rho(x)+\phi(x)$, where

$$\begin{aligned} U_\rho := &\sum_{i=1}^{k}\frac{(-1)^{i+1}}{\gamma\mu_i^{2/(p-1)}}\left(PU_{\delta_i,\xi_i}+\frac{1}{p}Pw^0_{\delta_i,\xi_i}+\frac{1}{p^2}Pw^1_{\delta_i,\xi_i}\right)\\ &-\frac{1}{\gamma\mu_{k+1}^{2/(p-1)}}\left(PU_{\delta_{k+1},0}+\frac{1}{p}Pw^0_{\delta_{k+1},0}+\frac{1}{p^2}Pw^1_{\delta_{k+1},0}\right), \end{aligned}\tag{4.5}$$

where the $\delta_i$ are given in (2.10) and (2.11), the concentration points $\xi_i$ are given in (4.3) for any $i=1,\dots,k$ and $\xi_{k+1}=0$, and the rest of the term $\phi$ is symmetric with respect to the variable $x_2$ and is symmetric with respect to each line $\{t\xi_i^*: t\in\mathbb{R}\}$ for $i=1,\dots,k$.

Using the results obtained in previous sections and taking into account the symmetry of the domain, we reduce the problem of finding solutions to (1.1) to that of finding critical points of the function $\widetilde{J}_p:(0,R)\to\mathbb{R}$ defined as in (2.29) by $\widetilde{J}_p(\rho):=J_p\left(U_\rho+\phi(\rho)\right)$. It is not difficult to check that

$$\widetilde{J}_p(\rho)=k\frac{A_1}{p}-2kA_1\frac{\log p}{p^2}+k\frac{A_2}{p^2}-\frac{A_3}{p^2}H(0,0)-k\frac{A_3}{p^2}\Phi(\rho)+O\left(\frac{\log^2 p}{p^3}\right),$$

where $A_1$, $A_2$ and $A_3$ are given in (2.30) and $\Phi$ is defined in (4.4). By the proof of Theorem 4.1, it follows that $\Phi$ has a maximum value which persists under a small $C^0$-perturbation. This proves our claim. □

In the last example, we build a solution to problem (1.1) with a positive point of concentration at the origin of the ball and two antipodal points of negative concentration.

THEOREM 4.3. *For any even integer $k$ there exist $p_0>0$ and $\rho^*\in(0,R)$ such that for any $p\geqslant p_0$ problem* (1.1) *has a sign-changing solution $u$ such that* (1.5) *and* (1.6) *hold and*

$$p|u|^{p-1}u\rightharpoonup 8\pi e\left(\delta_0-\delta_{\xi(\rho)}-\delta_{-\xi(\rho)}\right)\quad \textit{weakly in the sense of measure in } \overline{\Omega}$$

*as $p\to+\infty$. Here $\xi(\rho):=(\rho,0)$.*

*Proof.* We will look for a solution to problem (1.1) as $u(x) = U_\rho(x) + \phi(x)$, where

$$U_\rho := \frac{1}{\gamma\mu_1^{2/(p-1)}}\left(PU_{\delta_1,\xi_1} + \frac{1}{p}Pw^0_{\delta_1,\xi_1} + \frac{1}{p^2}Pw^1_{\delta_1,\xi_1}\right) - \sum_{i=2}^{3}\frac{1}{\gamma\mu_i^{2/(p-1)}}\left(PU_{\delta_i,\xi_i} + \frac{1}{p}Pw^0_{\delta_i,\xi_i} + \frac{1}{p^2}Pw^1_{\delta_i,\xi_i}\right), \tag{4.6}$$

where the $\delta_i$ are given in (2.10) and (2.11) and the concentration points are, for $i = 1, 2, 3$,

$$\xi_1 := (0,0), \qquad \xi_2 := \xi(\rho) = (\rho, 0), \qquad \xi_3 = \xi(\rho) = (-\rho, 0) \qquad \text{with } \rho \in (0, R), \tag{4.7}$$

and the rest of the term $\phi$ is even with respect to both the variables $x_2$ and $x_3$. Using the results obtained in previous sections and taking into account the symmetry of the domain, we reduce the problem of finding solutions to (1.1) to that of finding critical points of the function $\widetilde{J}_p : (0,R) \to \mathbb{R}$ defined as in (2.29) by $\widetilde{J}_p(\rho) := J_p\left(U_\rho + \phi(\rho)\right)$. It is not difficult to check that

$$\widetilde{J}_p(\rho) = 3\frac{A_1}{p} - 6A_1\frac{\log p}{p^2} + 3\frac{A_2}{p^2} - \frac{A_3}{p^2}\Phi(\rho) + O\left(\frac{\log^2 p}{p^3}\right),$$

where $A_1$, $A_2$ and $A_3$ are given in (2.30) and

$$\Phi(\rho) := H(0,0) + 2H(\xi(\rho), \xi(\rho)) - 4G(\xi(\rho), 0) + 2G(\xi(\rho), -\xi(\rho)), \qquad \text{with } \rho \in (0, R). \tag{4.8}$$

Using the explicit expression for the Green function in a ball, we see that the function $\Phi$ reduces to

$$\begin{aligned}\Phi(\rho) &= \frac{1}{2\pi}\log R - \frac{1}{\pi}\log\frac{R}{R^2-\rho^2} - \frac{2}{\pi}\left(\log\frac{1}{\rho} - \log\frac{1}{R}\right) + \frac{1}{\pi}\left(\log\frac{1}{2\rho} - \log\frac{R}{R^2+\rho^2}\right)\\ &= \frac{1}{\pi}\left(\log(R^2-\rho^2) + \log\rho\right) - \frac{7}{2\pi}\log R + \frac{1}{\pi}\log 2 + \frac{1}{\pi}\log(R^2+\rho^2).\end{aligned}$$

It is easy to check that $\lim_{\rho\to 0^+}\Phi(\rho) = \lim_{\rho\to R^-}\Phi(\rho) = -\infty$. Then there exists $\rho^* \in (0, R)$ such that $\Phi(\rho^*) = \max_{\rho\in(0,R)}\Phi(\rho)$, which is a critical value which persists under a small $C^0$-perturbation. This proves our claim. □

## Appendix A

Let $U_{\delta,\xi}$ be the function defined in (2.3). The following result holds.

LEMMA A.1. *We have*

$$PU_{\delta,\xi}(x) = U_{\delta,\xi}(x) - \log 8\delta^2 + 8\pi H(x,\xi) + O\left(\delta^2\right) \qquad \text{as } \delta \to 0 \tag{A.1}$$

*in* $C(\bar{\Omega})$ *and*

$$PU_{\delta,\xi}(x) = 8\pi G(x,\xi) + O\left(\delta^2\right) \qquad \text{as } \delta \to 0 \tag{A.2}$$

*in* $C_{\mathrm{loc}}(\bar{\Omega}\setminus\{\xi\})$*, uniformly for* $\xi$ *away from* $\partial\Omega$.

*Proof.* Since

$$PU_{\delta,\xi}(x) - U_{\delta,\xi}(x) + \log 8\delta^2 = -4\log\frac{1}{|x-\xi|} + O(\delta^2) \qquad \text{as } \delta \to 0$$

uniformly for $x \in \partial\Omega$ and $\xi$ away from $\partial\Omega$, by harmonicity and the maximum principle (A.1) readily follows.

On the other hand, away from $\xi$, we have

$$U_{\delta,\xi}(x) - \log 8\delta^2 = 4\log\frac{1}{|x-\xi|} + O(\delta^2).$$

This fact, together with (A.1) gives (A.2). □

Let $w^0$ be a radial solution of (2.4) and $w^1$ be one of (2.5) and (2.6). They are the unique radial solutions satisfying, respectively,

$$w^0(y) = C_0 \log|y| + O\left(\frac{1}{|y|}\right) \qquad \text{as } |y| \to +\infty, \tag{A.3}$$

and

$$w^1(y) = C_1 \log|y| + O\left(\frac{1}{|y|}\right) \qquad \text{as } |y| \to +\infty, \tag{A.4}$$

where

$$C_0 := \int_0^{+\infty} t\frac{t^2-1}{t^2+1} f^0(t) = 12 - 4\log 8$$

and $C_1$ is a suitable constant (see [**13**]). By (A.3) and (A.4) we deduce the following expansions.

Lemma A.2. *For $i = 1, 2$ we have*

$$Pw^i_{\delta,\xi}(x) = w^i_{\delta,\xi}(x) - 2\pi C_i H(x,\xi) + C_i \log\delta + O(\delta) \qquad \text{as } \delta \to 0$$

*in $C(\bar\Omega)$ and*

$$Pw^i_{\delta,\xi}(x) = -2\pi C_i G(x,\xi) + O(\delta) \qquad \text{as } \delta \to 0$$

*in $C_{\rm loc}(\bar\Omega \setminus \{\xi\})$, uniformly for $\xi$ away from $\partial\Omega$. In particular, the following global estimate holds: for any $\epsilon > 0$ there exists $c > 0$ such that for any $\delta$ small and $\xi \in \Omega$ with $\operatorname{dist}(\xi, \partial\Omega) \geqslant \epsilon$ we have*

$$\left\| Pw^i_{\delta,\xi} \right\|_\infty \leqslant c|\log\delta| \tag{A.5}$$

*for $i = 1, 2$.*

*Proof.* The proof follows from the same arguments as those used to prove Lemma A.1 and from estimates (A.3) and (A.4). □

## Appendix B

This appendix is devoted to the proof of Lemma 2.2.

First of all, in order to treat the invertibility properties of the linear operator $L$, we need to estimate $g'(U_\xi)(x)$. If $|x - \xi_i| \leqslant \varepsilon$ then, for some $i = 1, \ldots, k$,

$$\begin{aligned} g'(U_\xi)(x) &= a_i p \left(\frac{p}{\gamma\mu_i^{2/(p-1)}}\right)^{p-1} \left(1 + \frac{1}{p}U(y) + \frac{1}{p^2}w^0(y) + \frac{1}{p^3}w^1(y) \right.\\ &\qquad\qquad \left. + O\left(\frac{e^{-p/4}}{p}|y| + \frac{e^{-p/4}}{p}\right)\right)^{p-1} \\ &= a_i\delta_i^{-2}\left(1 + \frac{1}{p}U(y) + \frac{1}{p^2}w^0(y) + \frac{1}{p^3}w^1(y) + O\left(\frac{e^{-p/4}}{p}|y| + \frac{e^{-p/4}}{p}\right)\right)^{p-1}, \end{aligned} \tag{B.1}$$

where we use the notation $y = (x - \xi_i)/\delta_i$. In this region, then we have

$$|g'(U_\xi)(x)| \leqslant Ce^{p/2}e^{((p-1)/p)U(y)} = O\big(e^{U_{\delta_i,\xi_i}(x)}\big),$$

since $U(y) \geqslant -2p$. On the other hand, if $|x - \xi_i| \geqslant \varepsilon$ then, for any $i = 1, \ldots, k$,

$$|g'(U_\xi)(x)| = O(p(C/p)^{p-1}).$$

Summing up, we see that there exist $D > 0$ and $p_0 > 0$ such that

$$|g'(U_\xi)(x)| \leqslant D \sum_{j=1}^{k} e^{U_{\delta_j,\xi_j}(x)} \tag{B.2}$$

for any $\xi \in \mathcal{O}_\varepsilon$ and $p \geqslant p_0$.

*Proof of Lemma* 2.2. The proof consists of six steps.

*Step* 1. The operator $L$ satisfies the maximum principle in

$$\tilde{\Omega} := \Omega \setminus \bigcup_{j=1}^{m} B(\xi_j, R\delta_j)$$

for $R$ large (independent of $p$), namely

$$\text{if } L(\psi) \leqslant 0 \text{ in } \tilde{\Omega} \text{ and } \psi \geqslant 0 \text{ on } \partial\tilde{\Omega}, \text{ then } \psi \geqslant 0 \text{ in } \tilde{\Omega}.$$

Indeed, let

$$Z(x) = \sum_{j=1}^{m} z_0\left(\frac{a(x-\xi_j)}{\delta_j}\right),$$

where

$$z_0(y) = \frac{|y|^2 - 1}{1 + |y|^2}. \tag{B.3}$$

If $a$ is chosen positive and small and $R$ is chosen large, depending on $a$ but independent of $p$, it follows that $Z$ is a positive function in $\tilde{\Omega}$ and, taking into account (B.2), we deduce that it satisfies $LZ(x) \leqslant 0$ for all $x \in \tilde{\Omega}$, for $p$ sufficiently large. The existence of such a function $Z$ guarantees that $L$ satisfies the maximum principle in $\tilde{\Omega}$.

*Step* 2. Let $R$ be as before. Define

$$\|\phi\|_i = \sup_{x \in \bigcup_{j=1}^{m} B(\xi_j, R\delta_j)} |\phi(x)|.$$

Then, there is a constant $C > 0$ such that, if $L(\phi) = h$ in $\Omega$ and $h \in C^{0,\alpha}(\bar{\Omega})$, then

$$\|\phi\|_\infty \leqslant C[\|\phi\|_i + \|h\|_*]. \tag{B.4}$$

Indeed, consider the solution $\psi_j(x)$ of the problem

$$\begin{cases} -\Delta\psi_j = \dfrac{2\delta_j}{|x-\xi_j|^3} & \text{in } R\delta_j < |x-\xi_j| < M, \\ \psi_j(x) = 0 & \text{on } |x-\xi_j| = R\delta_j \text{ and } |x-\xi_j| = M. \end{cases}$$

Here, $M = 2 \operatorname{diam} \Omega$. The function $\psi_j(x)$ is a positive function, which is uniformly bounded from above for $p$ sufficiently large.

Define now the function

$$\tilde{\phi}(x) = 2\|\phi\|_i Z(x) + \|h\|_* \sum_{j=1}^{k} \psi_j(x),$$

where $Z$ was defined in the previous step. From the definition of $Z$, choosing $R$ larger if necessary, we see that

$$\tilde{\phi}(x) \geqslant |\phi(x)| \qquad \text{for } |x-\xi_j| = R\delta_j,\ j = 1, \dots, k,$$

and, by the positivity of $Z(x)$ and $\psi_j(x)$,

$$\tilde{\phi}(x) \geqslant 0 = |\phi(x)| \qquad \text{for } x \in \partial\Omega.$$

Furthermore, direct computation yields

$$L\tilde{\phi}(x) \leqslant |L\phi(x)|,$$

provided $p$ is large enough. Hence, by the maximum principle established in Step 1 we obtain

$$|\phi(x)| \leqslant \tilde{\phi}(x) \qquad \text{for } x \in \tilde{\Omega},$$

and therefore

$$\|\phi\|_\infty \leqslant C[\|\phi\|_i + \|h\|_*].$$

*Step* 3. Given $h \in C^{0,\alpha}(\bar{\Omega})$, assume that $\phi$ is a solution of problem $L\phi = h$ in $\Omega$ and $\phi = 0$ on $\partial\Omega$. When $\phi$ satisfies (2.21) and, in addition, the orthogonality conditions, we have

$$\int_\Omega e^{U_{\delta_j,\xi_j}} Z_{j,0}\phi = 0, \qquad \text{for } j = 1, \ldots, k, \tag{B.5}$$

where

$$Z_{j,0}(x) = z_0\left(\frac{x - \xi_j}{\delta_j}\right)$$

(see (B.3)). We prove that there exists a positive constant $C$ such that, for any $\xi \in \mathcal{O}_\varepsilon$,

$$\|\phi\|_\infty \leqslant C\|h\|_*, \tag{B.6}$$

for $p$ sufficiently large.

By contradiction, assume the existence of sequences $p_n \to \infty$, points $\xi^n \in \mathcal{O}_\varepsilon$, functions $h_n$ and associated solutions $\phi_n$ such that $\|h_n\|_* \to 0$ and $\|\phi_n\|_\infty = 1$. Since $\|\phi_n\|_\infty = 1$, Step 2 shows that $\liminf_{n\to+\infty} \|\phi_n\|_i > 0$. Let us set

$$\hat{\phi}_j^n(y) = \phi_n(\delta_j^n y + \xi_j^n).$$

Elliptic estimates and the fact that $\liminf_{n\to+\infty} \|\phi_n\|_i > 0$ readily imply that, for some $j \in \{1, \ldots, n\}$, $\hat{\phi}_j^n$ converges uniformly over compact sets to a non-trivial bounded solution $\hat{\phi}_j^\infty$ of the following equation in $\mathbb{R}^2$:

$$\Delta\phi + \frac{8}{(1+|y|^2)^2}\phi = 0.$$

This implies that $\hat{\phi}_j^\infty$ is a linear combination of the functions $z_i$, for $i = 0, 1, 2$ (see (2.22) and (B.3)). Since $\|\hat{\phi}_j^n\|_\infty \leqslant 1$, by the Lebesgue theorem the orthogonality conditions (2.21) and (B.5) on $\phi_n$ pass to the limit and yield

$$\int_{\mathbb{R}^2} \frac{8}{(1+|y|^2)^2} z_i(y)\hat{\phi}_j^\infty = 0 \qquad \text{for any } i = 0, 1, 2.$$

Hence, $\hat{\phi}_j^\infty \equiv 0$, a contradiction.

*Step* 4. We prove that there exists a positive constant $C > 0$ such that any solution $\phi$ of the equation $L\phi = h$ in $\Omega$, with $\phi = 0$ on $\partial\Omega$, satisfies (2.23) when $h \in C^{0,\alpha}(\bar{\Omega})$ and we assume on $\phi$ only the orthogonality conditions (2.21).

Proceeding by contradiction as in Step 3, we can suppose further that

$$p_n\|h_n\|_* \to 0 \qquad \text{as } n \to +\infty \tag{B.7}$$

but we lose in the limit the condition

$$\int_{\mathbb{R}^2} \frac{8}{(1+|y|^2)^2} z_0(y)\hat{\phi}_j^\infty = 0.$$

Hence, we have

$$\hat{\phi}_j^n \to C_j\frac{|y|^2 - 1}{|y|^2 + 1} \qquad \text{in } C^0_{\text{loc}}(\mathbb{R}^2) \tag{B.8}$$

for some constants $C_j$. Testing (2.19) against properly chosen test functions and using the stronger convergence (B.7), one can show that $C_j = 0$ for any $j = 1, \dots, k$ (the construction of suitable test functions is the more delicate part in the whole proof of Lemma 2.2; see [**13**] for the details).

*Step* 5. We establish the validity of the *a priori* estimate (2.23) for $h \in C^{0,\alpha}(\bar{\Omega})$. The previous step yields

$$\|\phi\|_\infty \leqslant Cp\left(\|h\|_* + \sum_{i,j} |c_{i,j}|\right) \tag{B.9}$$

since

$$\|e^{U_{\delta_j,\xi_j}} Z_{i,j}\|_* \leqslant 2\|e^{U_{\delta_j,\xi_j}}\|_* \leqslant C.$$

Hence, proceeding by contradiction as in Step 3, we can suppose further that

$$p_n\|h_n\|_* \to 0, \qquad p_n \sum_{i,j} |c^n_{i,j}| \geqslant \delta > 0 \qquad \text{as } n \to +\infty. \tag{B.10}$$

We omit the dependence on $n$. It suffices to estimate the values of the constants $c_{ij}$. Let $PZ_{i,j}$ be the projection on $H^1_0(\Omega)$ of the functions $Z_{i,j}$. Testing equation (2.19) against $PZ_{i,j}$ and integrating by parts one gets

$$\begin{aligned} Dc_{i,j} + O\left(e^{-p/2}\sum_{l,h} |c_{l,h}| + \|h\|_*\right) \\ = \frac{1}{p}\int_{B(0,\varepsilon/\sqrt{\delta_j})} \frac{32y_i}{(1+|y|^2)^3}\left(w^0 - U - \tfrac{1}{2}U^2\right)\hat{\phi}_j + O\left(\frac{1}{p^2}\|\phi\|_\infty\right) \end{aligned} \tag{B.11}$$

where

$$D = 64\int_{\mathbb{R}^2} \frac{|y|^2}{(1+|y|^2)^4}, \qquad \hat{\phi}_j(y) = \phi(\delta_j y + \xi_j)$$

and $w^0$ is given by (2.4). Hence, we obtain

$$\sum_{l,h} |c_{l,h}| = O\left(\|h\|_* + \frac{1}{p}\|\phi\|_\infty\right). \tag{B.12}$$

Since $\sum_{l,h} |c_{l,h}| = o(1)$, as in Step 4 we have

$$\hat{\phi}_j \to C_j \frac{|y|^2 - 1}{|y|^2 + 1} \qquad \text{in } C^0_{\text{loc}}(\mathbb{R}^2) \tag{B.13}$$

for some constants $C_j$. Hence, in (B.11) we have a better estimate since by the Lebesgue theorem the term

$$\int_{B(0,\varepsilon/\sqrt{\delta_j})} \frac{32y_i}{(1+|y|^2)^3}\left(w^0 - U - \tfrac{1}{2}U^2\right)(y)\hat{\phi}_j(y)$$

converges to

$$C_j\int_{\mathbb{R}^2} \frac{32y_i(|y|^2-1)}{(1+|y|^2)^4}\left(w^0 - U - \tfrac{1}{2}U^2\right)(y) = 0.$$

Therefore, we get

$$\sum_{l,h} |c_{l,h}| = O(\|h\|_*) + o(1/p).$$

This contradicts the assumption that

$$p\sum_{i,j} |c_{i,j}| \geqslant \delta > 0,$$

and the claim is established.

*Step* 6. We prove the solvability for (2.19)–(2.21). For this purpose, we consider the space

$$K_\xi = \mathrm{span}\{PZ_{i,j} : i = 1, \dots, k,\ j = 1, 2\}$$

and its orthogonal space

$$K_\xi^\perp = \left\{ \phi \in H_0^1(\Omega) : \int_\Omega e^{U_{\delta_i,\xi_i}} Z_{i,j}\, \phi = 0 \text{ for } i = 1, \dots, k,\ j = 1, 2 \right\},$$

endowed with the usual inner product. Let $\Pi_\xi$ and $\Pi_\xi^\perp$ be the associated orthogonal projections in $H_0^1(\Omega)$.

Problem (2.19)–(2.21), expressed in weak form, is equivalent to that of finding $\phi \in K_\xi^\perp$ such that

$$(\phi, \psi)_{\mathrm{H}_0^1(\Omega)} = \int_\Omega (W\phi - h)\, \psi\, dx \qquad \text{for all } \psi \in K_\xi^\perp.$$

With the aid of Riesz's representation theorem, this equation can be rewritten in $K_\xi^\perp$ in the operator form

$$(\mathrm{Id} - K)\phi = \tilde{h}, \tag{B.14}$$

where $\tilde{h} = \Pi_\xi^\perp \Delta^{-1} h$ and $K(\phi) = -\Pi_\xi^\perp \Delta^{-1}(W\phi)$ is a linear compact operator in $K_\xi^\perp$. The homogeneous equation $\phi = K(\phi)$ in $K_\xi^\perp$, which is equivalent to (2.19)–(2.21) with $h \equiv 0$, has only the trivial solution in view of the *a priori* estimate (2.23). Now, Fredholm's alternative guarantees unique solvability of (B.14) for any $\tilde{h} \in K_\xi^\perp$. Finally, by density we obtain the validity of (2.23) also for $h \in C(\bar{\Omega})$ (not only for $h \in C^{0,\alpha}(\bar{\Omega})$). □

Remark B.1. Given $h \in C(\bar{\Omega})$, let $\phi$ be the solution of (2.19)–(2.21) given by Lemma 2.2. Multiplying (2.19) by $\phi$ and integrating by parts, we get

$$\|\phi\|^2_{\mathrm{H}_0^1(\Omega)} = \int_\Omega W\phi^2 - \int_\Omega h\phi.$$

Since (B.2) holds true, we get

$$\|\phi\|_{\mathrm{H}_0^1(\Omega)} \leqslant C\left(\|\phi\|_\infty + \|h\|_*\right).$$

## *References*


1. Adimurthi and M. Grossi, 'Asymptotic estimates for a two-dimensional problem with polynomial nonlinearity', *Proc. Amer. Math. Soc.* 132 (2004) 1013–1019.
2. A. Aftalion and F. Pacella, 'Qualitative properties of nodal solutions of semilinear elliptic equations in radially symmetric domains', *C. R. Acad. Sci. Paris* 339 (2004) 339–344.
3. S. Baraket and F. Pacard, 'Construction of singular limits for a semilinear elliptic equation in dimension 2', *Calc. Var. Partial Differential Equations* 6 (1998) 1–38.
4. T. Bartsch, 'Critical point theory on partially ordered Hilbert spaces', *J. Funct. Anal.* 186 (2001) 117–152.
5. T. Bartsch, M. Clapp and T. Weth, 'Configuration spaces, transfer, and 2-nodal solutions of a semiclassical nonlinear Schrödinger equation', *Math. Ann.* to appear.
6. T. Bartsch, A. M. Micheletti and A. Pistoia, 'On the existence and the profile of nodal solutions of elliptic equations involving critical growth', *Calc. Var. Partial Differential Equations* 26 (2006) 265–282.
7. T. Bartsch and T. Weth, 'A note on additional properties of sign changing solutions to superlinear elliptic equations', *Topol. Meth. Nonlinear Anal.* 22 (2003) 1–14.
8. D. Chae and O. Imanuvilov, 'The existence of non-topological multivortex solutions in the relativistic self-dual Chern–Simons theory', *Comm. Math. Phys.* 215 (2000) 119–142.
9. W. Chen and C. Li, 'Classification of solutions of some nonlinear elliptic equations', *Duke Math. J.* 63 (1991) 615–623.
10. M. del Pino, M. Kowalczyk and M. Musso, 'Singular limits in Liouville-type equations', *Calc. Var. Partial Differential Equations* 24 (2005) 47–81.
11. K. El Mehdi and M. Grossi, 'Asymptotic estimates and qualitative properties of an elliptic problem in dimension two', *Adv. Nonlinear Stud.* 4 (2004) 15–36.

**12.** P. Esposito, M. Grossi and A. Pistoia, 'On the existence of blowing-up solutions for a mean field equation', *Ann. Inst. H. Poincaré Anal. Non Linéaire* 22 (2005) 227–257.

**13.** P. Esposito, M. Musso and A. Pistoia, 'Concentrating solutions for a planar elliptic problem involving nonlinearities with large exponent', *J. Differential Equations* 227 (2006) 1–364.

**14.** Z. C. Han, 'Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent', *Ann. Inst. H. Poincaré Anal. Non Linéaire* 8 (1991) 159–174.

**15.** X. Ren and J. Wei, 'On a two-dimensional elliptic problem with large exponent in nonlinearity', *Trans. Amer. Math. Soc.* 343 (1994) 749–763.

**16.** X. Ren and J. Wei, 'Single point condensation and least energy solutions', *Proc. Amer. Math. Soc.* 124 (1996) 111–120.

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