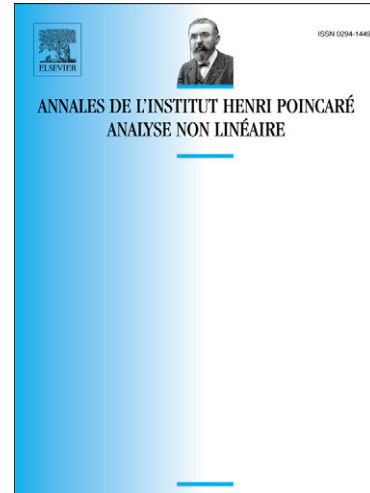


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Shengbing Deng, Monica Musso

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Uncited references

[27]

CRITICAL POINTS OF THE TRUDINGER-MOSER TRACE FUNCTIONAL WITH HIGH ENERGY LEVELS

SHENGBING DENG AND MONICA MUSSO

Abstract: Let Ω be a bounded domain in \mathbb{R}^2 with smooth boundary. In this paper we are concerned with the existence of critical points for the super critical Trudinger-Moser trace functional

$$(0.1) \quad \int_{\partial\Omega} e^{k\pi(1+\mu)u^2}$$

in the set $\{u \in H^1(\Omega) : \int_{\Omega} (|\nabla u|^2 + u^2) dx = 1\}$, where $k \geq 1$ is an integer and $\mu > 0$ is a small parameter. For any integer $k \geq 1$ and for any $\mu > 0$ sufficiently small, we prove the existence of a pair of k -peaks constrained critical points of the above problem.

Keywords: Trudinger-Moser trace functional; Reduction methods.

1. INTRODUCTION

Let Ω be a bounded domain in \mathbb{R}^2 with smooth boundary, and let $H^1(\Omega)$ be the Sobolev space, equipped with the norm

$$\|u\| = \left(\int_{\Omega} (|\nabla u|^2 + u^2) dx \right)^{\frac{1}{2}}.$$

Let α be a positive number, the Trudinger-Moser trace inequality states that

$$(1.1) \quad C_{\alpha}(\Omega) = \sup_{u \in H^1(\Omega), \|u\| \leq 1} \int_{\partial\Omega} e^{\alpha|u|^2} \begin{cases} \leq C < +\infty, & \text{if } \alpha \leq \pi \\ = +\infty, & \text{if } \alpha > \pi \end{cases}$$

[1, 2, 6, 7, 18, 22, 23]. Let us mention that the early works [6, 7] do not include the case when the constant in (1.1) is exactly π . For (1.1) there is a loss of compactness at the limiting exponent $\alpha = \pi$. Despite of that, it has been proven in [29] that the supremum $C_{\pi}(\Omega)$ is attained by a function $u \in H^1(\Omega)$ with $\int_{\Omega} [|\nabla u|^2 + u^2] = 1$, for any bounded domain Ω in \mathbb{R}^2 , with smooth boundary. Also, for any $\alpha \in (0, \pi)$, the supremum $C_{\alpha}(\Omega)$ is finite and it is attained. But the exponent $\alpha = \pi$ is critical in the sense that for any $\alpha > \pi$, $C_{\alpha}(\Omega) = \infty$. See also [8, 16, 17] for generalizations.

The aim of this paper is to study the existence of critical points of the Trudinger-Moser trace functional

$$(1.2) \quad E_{\alpha}(u) = \int_{\partial\Omega} e^{\alpha u^2},$$

constrained to functions

$$(1.3) \quad u \in M = \{u \in H^1(\Omega) : \|u\|^2 = 1\}$$

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in the super critical regime

$$\alpha > \pi.$$

In view of the results described above, we will be interested in critical points other than global supremum. As far as we know, no results are known in the literature concerning existence of critical points for the Trudinger-Moser trace constrained problem in the *super critical regime*. Nevertheless, much more is known for the corresponding Trudinger-Moser functional.

Let us recall that the Trudinger-Moser inequality in dimension 2 states that

$$(1.4) \quad \sup_{u \in H_0^1(\Omega), \|\nabla u\|_2 \leq 1} \int_{\Omega} e^{\mu |u|^2} dx \quad \begin{cases} \leq C < +\infty, & \text{if } \mu \leq 4\pi \\ = +\infty, & \text{if } \mu > 4\pi. \end{cases}$$

Here again Ω is a bounded domain of \mathbb{R}^2 , with smooth boundary. We refer the reader to [25, 23, 28, 30] for the first works on Problem (1.4), and to [3, 4] for some more recent contributions. For problem (1.4) there is a loss of compactness at the limiting exponent $\mu = 4\pi$ [21]. Despite of this loss of compactness, the supremum

$$\sup_{u \in H_0^1(\Omega), \|\nabla u\|_2 \leq 1} \int_{\Omega} e^{4\pi |u|^2} dx$$

is attained for any bounded domain $\Omega \subset \mathbb{R}^2$. This was proven first in the seminal work [5] for the ball $\Omega = B_1(0)$ (see also an alternative proof in [10]). In [26] the result was proven for domains Ω which are small perturbation of the ball. The general result in dimension 2 was proven by Flucher in [14], and Lin [20] extended it for the corresponding Trudinger-Moser inequality for general domain of \mathbb{R}^N , with $N > 2$.

Concerning the super critical regime for the Trudinger-Moser functional, namely

$$(1.5) \quad I_{\mu}(u) = \int_{\Omega} e^{\mu |u|^2} dx, \quad u \in H_0^1(\Omega), \|\nabla u\|_2^2 = 1, \quad \text{with } \mu > 4\pi,$$

some results are known. In the works [26] and [15] it has been proven that a local maxima and saddle point solutions in the supercritical regime $\mu \in (4\pi, \mu_0)$ for the functional (1.5) do exist, for some $\mu_0 > 4\pi$.

Our first result is an extension of the existence of a local maxima for the Trudinger-Moser trace functional in the super critical regime $\alpha \in (\pi, \alpha_0)$. Namely, a local maximizer for Problem (1.2)-(1.3) exists when the value of α is slightly to the right of π .

Theorem 1.1. *Let Ω be a bounded domain in \mathbb{R}^2 . Then there exists $\alpha_0 > \pi$, such that for any $\alpha \in (0, \alpha_0)$, there exists a function $u_{\alpha} \in M$ which locally maximizes of E_{α} on M .*

This result is proved in Section 2.

Much more is known for Problem (1.5) and $\mu > 4\pi$. Recently in [12] (see also [11]), the authors obtained several results concerning critical points for Problem (1.5) also in a *very* super critical regime. They found general conditions on the domain Ω under which there is a critical point for $I_{\mu}(u)$ with $\int_{\Omega} |\nabla u|^2 dx = 1$ when $\mu \in (4\pi k, \mu_k)$, for any integer $k \geq 1$ and for some μ_k slightly bigger than $4\pi k$. In particular, for any bounded domain Ω , they found a critical point for $I_{\mu}(u)$ with $\int_{\Omega} |\nabla u|^2 dx = 1$ when $\mu \in (4\pi, \mu_1)$, for some $\mu_1 > 4\pi$. The L^{∞} -norm of this solution converges to ∞ as $\mu \rightarrow 4\pi$ and its mass is concentrated, in some proper sense, as $\mu \rightarrow 4\pi$, around a point in the interior of Ω . On the other hand, if Ω has a hole, namely it is not simply connected, they proved the existence of a critical point for $I_{\mu}(u)$ with $\int_{\Omega} |\nabla u|^2 dx = 1$ also in the super critical range $\mu \in (8\pi, \mu_2)$, for some $\mu_2 > 8\pi$.

Again in this case, the L^∞ -norm of these solutions converges to ∞ as $\mu \rightarrow 8\pi$, but now its mass concentrates, as $\mu \rightarrow 8\pi$, around two distinct points inside Ω . Furthermore, if Ω is an annulus, taking advantage of the symmetry, a critical point for $I_\mu(u)$ with $\int_\Omega |\nabla u|^2 dx = 1$ and $\mu \in (4\pi k, \mu_k)$ does exist. In this latter case, the L^∞ -norm of the solution converges to ∞ as $\mu \rightarrow 4\pi k$ and its mass concentrates, as $\mu \rightarrow 4\pi k$, around k points distributed along the vertices of a proper regular polygon with k sides lying inside Ω .

The second result of this paper establishes the counterpart of the above situation for the Trudinger-Moser trace functional in the super critical regime: we will show the existence of critical points for E_α constrained to M , for $\alpha \in (k\pi, \alpha_k)$, for any $k \geq 1$ integer and for some α_k slightly to the right of $k\pi$. We next describe our result.

Let $G(x, y)$ be the Green's function of the problem

$$(1.6) \quad \begin{cases} -\Delta_x G(x, y) + G(x, y) = 0 & x \in \Omega; \\ \frac{\partial G(x, y)}{\partial \nu_x} = 2\pi \delta_y(x) & x \in \partial\Omega, \end{cases}$$

and H its regular part defined as

$$(1.7) \quad H(x, y) = G(x, y) - 2 \log \frac{1}{|x - y|}.$$

Our second result reads as follows.

Theorem 1.2. *Let Ω be any bounded domain in \mathbb{R}^2 with smooth boundary. Fix a positive integer $k \geq 1$. Then there exists $\alpha_k > k\pi$ such that for $\alpha \in (k\pi, \alpha_k)$, the functional $E_\alpha(u)$ restricted to M has at least two critical points u_α^1 and u_α^2 . Furthermore, for any $i = 1, 2$ there exist numbers $m_{j,\alpha}^i > 0$ and points $\xi_{j,\alpha}^i \in \partial\Omega$, for $j = 1, \dots, k$ such that*

$$(1.8) \quad \lim_{\alpha \rightarrow k\pi} m_{j,\alpha}^i = m_j^i \in (0, \infty),$$

$$(1.9) \quad \xi_{j,\alpha}^i \rightarrow \xi_j^i \in \partial\Omega, \quad \text{with } \xi_j^i \neq \xi_l^i \text{ for } j \neq l, \text{ as } \alpha \rightarrow k\pi$$

and

$$(1.10) \quad u_\alpha^i(x) = \sqrt{\frac{\alpha - k\pi}{\alpha}} \sum_{j=1}^k \left[m_{j,\alpha}^i G(x, \xi_{j,\alpha}^i) + o(1) \right], \quad i = 1, 2,$$

where $o(1) \rightarrow 0$ uniformly on compact sets of $\bar{\Omega} \setminus \{\xi_1^i, \dots, \xi_k^i\}$, as $\alpha \rightarrow k\pi$. In particular, $(\xi^i, m^i) = (\xi_1^i, \dots, \xi_k^i, m_1^i, \dots, m_k^i)$ in $(\partial\Omega)^k \times (0, \infty)^k$, for $i = 1, 2$, are two distinct critical points for the function

$$f_k(\xi, m) = \frac{2}{k} \left[2 \sum_{j=1}^k m_j^2 \log(2m_j^2) - \sum_{j=1}^k m_j^2 H(\xi_j, \xi_j) - \sum_{i \neq j} m_i m_j G(\xi_i, \xi_j) \right].$$

Moreover, for any $i = 1, 2$, for any $\delta > 0$ small, for any $j = 1, \dots, k$,

$$(1.11) \quad \sup_{x \in B(\xi_j^i, \delta)} u_\alpha^i(x) \rightarrow +\infty, \quad \text{as } \alpha \rightarrow k\pi.$$

There are two important differences between the result stated in Theorem 1.2 and the corresponding result obtained in [12] for the Trudinger-Moser functional (1.5). A first difference is that for Problem (1.2)-(1.3) existence of critical points in the range $\alpha \in (k\pi, \alpha_k)$ is guaranteed in *any* bounded domain Ω with smooth boundary, at any integer level k . No further hypothesis on Ω is needed, unlike the Trudinger-Moser case (1.5). The second

difference is that, we do find *two* families of critical points for Problem (1.2)-(1.3) when $\alpha \in (k\pi, \alpha_k)$, and not only one as in the Trudinger-Moser case (1.5).

In recent years a very successful method has been developed for studying elliptic equations in critical or supercritical regimes. The main idea is to try to guess the form of the solution (using the shape of the "standard bubble"), then linearize the equation at this approximate solution and use a Lyapunov-Schmidt reduction to arrive at a reduced finite dimensional variational problem, whose critical points yield actual solutions of the equation. In this paper we use this method to study problem (1.2)-(1.3) in the supercritical regime. We explain this in Section 3, where we also provide the proof of Theorem 1.2. Some technical results are postponed to Section 4 and Section 5.

Let us just mention that through out the paper, C will always denote an arbitrary positive constant, independent of λ , whose value changes from line to line.

2. THE LOCAL MAXIMIZER: PROOF OF THEOREM 1.1

We set

$$(2.1) \quad E(u) = \int_{\partial\Omega} e^{u^2},$$

and

$$(2.2) \quad M_\alpha = \{u \in H^1(\Omega) : \|u\|^2 = \alpha\}.$$

We note that by the obvious scaling property, finding critical points of E_α on M (see (1.2) and (1.3)) is equivalent to finding critical points of E on M_α (see (2.1) and (2.2)). In this section, we study the local maximizer for the functional E constrained on the set M_α with α in the right neighborhood of π .

We start with the following Lion's type Lemma. The proof is quite standard, but we reproduce it here for completeness.

Lemma 2.1. *Let u_m be a sequence of functions in $H^1(\Omega)$ with $\|u_m\| = 1$. Suppose that $u_m \rightharpoonup u_0$ weakly in $H^1(\Omega)$. Then either*

(i) $u_0 = 0$,

or

(ii) *there exists $\alpha > \pi$ such that the family $e^{u_m^2}$ is uniformly bounded in $L^\alpha(\partial\Omega)$.*

In particular, in case (ii), we have that

$$\int_{\partial\Omega} e^{\pi u_m^2} \rightarrow \int_{\partial\Omega} e^{\pi u_0^2} \quad \text{as } m \rightarrow \infty.$$

Proof. Since $\|u_m\| = 1$ and $u_m \rightharpoonup u_0$ weakly in $H^1(\Omega)$, we have

$$\int_{\Omega} (\nabla u_m \nabla u_0 + u_m u_0) \rightarrow \int_{\Omega} (|\nabla u_0|^2 + u_0^2) \quad \text{as } m \rightarrow \infty.$$

Thus we find that

$$\begin{aligned} \lim_{m \rightarrow \infty} \|u_m - u_0\|^2 &= \lim_{m \rightarrow \infty} \left\{ \int_{\Omega} [|\nabla(u_m - u_0)|^2 + (u_m - u_0)^2] \right\} \\ &= \lim_{m \rightarrow \infty} \left\{ \|u_m\|^2 - 2 \int_{\Omega} (\nabla u_m \nabla u_0 + u_m u_0) + \|u_0\|^2 \right\} \\ &= 1 - \|u_0\|^2. \end{aligned}$$

Assume $u_0 \neq 0$. Take $p \in (1, \frac{1}{1-\|u_0\|^2})$, and choose q_1 and q_2 such that $1 < pq_1 < \frac{1}{\|u_m - u_0\|^2}$ and $\frac{1}{q_1} + \frac{1}{q_2} = 1$. By Hölder inequality we have

$$\begin{aligned} \int_{\partial\Omega} e^{\pi p u_m^2} &= \int_{\partial\Omega} e^{\pi p (u_m - u_0 + u_0)^2} = \int_{\partial\Omega} e^{\pi p [(u_m - u_0)^2 + 2(u_m - u_0)u_0 + u_0^2]} \\ &= \int_{\partial\Omega} e^{\pi p [(u_m - u_0)^2 + 2u_m u_0 - u_0^2]} \leq \int_{\partial\Omega} e^{\pi p [(u_m - u_0)^2 + 2u_m u_0]} \\ &= \int_{\partial\Omega} e^{\pi p (u_m - u_0)^2} e^{2\pi p u_m u_0} \leq \left(\int_{\partial\Omega} e^{\pi p q_1 (u_m - u_0)^2} \right)^{\frac{1}{q_1}} \left(\int_{\partial\Omega} e^{2\pi p q_2 u_m u_0} \right)^{\frac{1}{q_2}}. \end{aligned}$$

We now recall that

$$(2.3) \quad \pi = \sup \left\{ \theta : \sup_{u \in H^1(\Omega), \|u\| \leq 1} \int_{\partial\Omega} e^{\theta u^2} d\sigma < \infty \right\}.$$

see for instance [2, 6, 7, 18]. Hence, given the choice of p and q_1 , we get that there exists a constant C , independent of m , such that

$$\int_{\partial\Omega} e^{\pi p q_1 (u_m - u_0)^2} < C.$$

On the other hand, Young's inequality implies that $2|u_m u_0| \leq \varepsilon^2 u_m^2 + \frac{1}{\varepsilon^2} u_0^2$, with $\varepsilon > 0$ small. Then from (2.3), we have

$$\int_{\partial\Omega} e^{2\pi p q_2 u_m u_0} < \int_{\partial\Omega} e^{\pi p q_2 [\varepsilon^2 u_m^2 + \frac{1}{\varepsilon^2} u_0^2]} = \int_{\partial\Omega} e^{\pi p q_2 \varepsilon^2 u_m^2} e^{\pi p q_2 \frac{1}{\varepsilon^2} u_0^2} < C$$

by choosing ε so that $p q_2 \varepsilon^2 < 1$. Here again C is a constant, independent of m . Thus, we have that there exists $\alpha = p\pi > \pi$ such that the family $e^{u_m^2}$ is uniformly bounded in $L^\alpha(\partial\Omega)$.

We shall now show that

$$(2.4) \quad \int_{\partial\Omega} e^{\pi u_m^2} \rightarrow \int_{\partial\Omega} e^{\pi u_0^2} \quad \text{as } m \rightarrow \infty.$$

Indeed, let l be a positive number and $p > 1$. We have

$$\begin{aligned} \left| \int_{\partial\Omega} e^{\pi u_m^2} - \int_{\partial\Omega \cap \{|u_m| \leq l\}} e^{\pi u_m^2} \right| &= \left| \int_{\partial\Omega \cap \{|u_m| > l\}} e^{\pi u_m^2} \right| \leq \frac{1}{l^{\frac{2(p-1)}{p}}} \int_{\partial\Omega} e^{\pi u_m^2} u_m^{\frac{2(p-1)}{p}} \\ &\leq \frac{1}{l^{\frac{2(p-1)}{p}}} \left(\int_{\partial\Omega} e^{\pi p u_m^2} \right)^{\frac{1}{p}} \left(\int_{\partial\Omega} u_m^2 \right)^{\frac{p-1}{p}} \leq \frac{C}{l^{\frac{2(p-1)}{p}}}. \end{aligned}$$

From the above relation, we conclude that

$$\int_{\partial\Omega} e^{\pi u_m^2} \leq |\partial\Omega| e^{\pi l^2} + \frac{C}{l^{\frac{2(p-1)}{p}}}.$$

Hence dominated convergence Theorem implies (2.4).

Suppose now that $e^{u_m^2}$ is not bounded in $L^\alpha(\partial\Omega)$ for any $\alpha > \pi$. Using Stokes theorem, for $\alpha > \pi$ we have

$$\begin{aligned} \int_{\partial\Omega} e^{\alpha u_m^2} d\sigma &= \int_{\Omega} \operatorname{div}(e^{\alpha u_m^2}) dx \leq C \int_{\Omega} |\nabla u_m| |u_m| e^{\alpha u_m^2} dx \\ &\leq C \left(\int_{\Omega} |\nabla u_m|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |u_m|^q dx \right)^{\frac{1}{q}} \left(\int_{\Omega} e^{\beta u_m^2} dx \right)^{\frac{\alpha}{\beta}} \end{aligned}$$

where $q > 1$ satisfies $\frac{1}{2} + \frac{1}{q} + \frac{\alpha}{\beta} = 1$ with $\beta > 2\pi$. Then we get that $\int_{\Omega} e^{\beta u_m^2} dx$ is unbounded for all $\beta > 2\pi$.

Observe now that we can assume that $\int_{\Omega} u_m dx = 0$, since otherwise we set $\bar{u}_m = u_m - \frac{1}{|\Omega|} \int_{\Omega} u_m dx$ and obtain $\int_{\Omega} u_m dx = 0$. We can also assume that $\int_{\Omega} |\nabla u_m|^2 = 1$. Furthermore, by Poincaré inequality, (u_m) is bounded in $H^1(\Omega)$, and also $(|u_m|)$ is bounded in $H^1(\Omega)$. Hence there exists $u \in H^1(\Omega)$ such that $|u_m| \rightharpoonup u_0$ weakly in $H^1(\Omega)$. We claim that

$$(2.5) \quad \lim_{m \rightarrow \infty} \int_{\Omega} |\nabla(u_m - \eta)^+|^2 dx = 1 \quad \forall \eta > 0.$$

By contradiction, assume there exists $\eta > 0$ such that $\lim_{m \rightarrow \infty} \int_{\Omega} |\nabla(u_m - \eta)^+|^2 dx \neq 1$. Define $\gamma = \inf_m \int_{\Omega} |\nabla(u_m - \eta)^+|^2 dx < 1$ and choose a sufficiently small $\varepsilon > 0$ such that $\alpha' := \frac{2\pi}{\gamma + \varepsilon} > 2\pi$. Let us recall that

$$(2.6) \quad 2\pi = \sup \left\{ \theta : \sup_{u \in H^1(\Omega), \int_{\Omega} |\nabla u|^2 \leq 1, \int_{\Omega} u = 0} \int_{\Omega} e^{\theta u^2} dx < \infty \right\},$$

(see [2, 6, 7, 29]). From (2.6), there exists a positive constant C such that

$$\int_{\Omega} e^{\alpha' [(|u_m| - \eta)^+ - \frac{1}{|\Omega|} \int_{\Omega} (|u_m| - \eta)^+]} dx = \int_{\Omega} e^{2\pi \left[\frac{(|u_m| - \eta)^+ - \frac{1}{|\Omega|} \int_{\Omega} (|u_m| - \eta)^+}{\sqrt{\gamma + \varepsilon}} \right]^2} dx < C,$$

where we use the fact that $\int_{\Omega} |\nabla \frac{(|u_m| - \eta)^+}{\sqrt{\gamma + \varepsilon}}|^2 dx < 1$.

Define $d_m = \frac{1}{|\Omega|} \int_{\Omega} (|u_m| - \eta)^+$. Choosing $\varepsilon' > 0$ small such that $\tilde{\alpha}' := \frac{\alpha'}{1 + \varepsilon'} > 2\pi$, and by the Young's inequality,

$$\begin{aligned} u_m^2 &\leq (\eta + d_m)^2 + 2(\eta + d_m)[(|u_m| - \eta)^+ - d_m] + [(|u_m| - \eta)^+ - d_m]^2 \\ &\leq (1 + \varepsilon')[(|u_m| - \eta)^+ - d_m]^2 + \left(\frac{1}{\varepsilon'} + 1\right)(\eta + d_m)^2. \end{aligned}$$

Thus, since there $d_m = O(1)$ as $m \rightarrow \infty$,

$$\int_{\Omega} e^{\tilde{\alpha}' u_m^2} dx = \int_{\Omega} e^{\frac{\alpha'}{1 + \varepsilon'} u_m^2} dx \leq C_1 \int_{\Omega} e^{\alpha' [(|u_m| - \eta)^+ - \frac{1}{|\Omega|} \int_{\Omega} (|u_m| - \eta)^+]} dx \leq C_2,$$

for some positive constants C_1 and C_2 . This is a contradiction, thus (2.5) holds.

Set $v_m = \min\{|u_m|, \eta\}$, then v_m is bounded in $H^1(\Omega)$ and, up to subsequence, we have that $v_m \rightharpoonup v$. Observe now that $|u_m| = v_m + (|u_m| - \eta)^+$, and

$$1 = \int_{\Omega} |\nabla u_m|^2 \geq \int_{\Omega} |\nabla |u_m||^2 dx = \int_{\Omega} |\nabla v_m|^2 dx + \int_{\Omega} |\nabla (|u_m| - \eta)^+|^2 dx.$$

Therefore (2.5) implies that that $\int_{\Omega} |\nabla v_m|^2 dx \rightarrow 0$ as $m \rightarrow \infty$, so v is constant. On the other hand,

$$\lim_{m \rightarrow \infty} \int_{\Omega} |\nabla v_m|^2 dx = \lim_{m \rightarrow \infty} \int_{\Omega \cap \{|u_m| \leq \eta\}} |\nabla |u_m||^2 dx = 0.$$

This implies that $|\{x : |u_m| \geq \eta\}| \rightarrow 0$ as $m \rightarrow \infty$. By Fatou Lemma,

$$|\{x : u_0 \geq \eta\}| \leq \liminf_{m \rightarrow \infty} |\{x : |u_m| \geq \eta\}| = 0,$$

then $|\{x : u_0 \geq \eta\}| = 0$ for any $\eta > 0$. Hence we get $u_0 = 0$. \square

We denote $\beta := \sup_{u \in M_{\pi}} E(u) = \sup_{u \in M} E_{\pi}(u)$. A direct consequence of the previous Lemma is the following

Proposition 2.1. *Let u_m be a bounded sequence in $H^1(\Omega)$ with $\|u_m\| = 1$. Suppose that $u_m \rightharpoonup u_0$ weakly in $H^1(\Omega)$. Suppose $E_\pi(u_m) \rightarrow \beta$ with $\beta > |\partial\Omega|$. Then there exists $\alpha > \pi$ such that the family $e^{u_m^2}$ is uniformly bounded in $L^\alpha(\partial\Omega)$. In particular $E_\pi(u_m) \rightarrow E_\pi(u_0)$ and $u_0 \neq 0$.*

Proof. Suppose $e^{u_m^2}$ is unbounded in $L^\alpha(\partial\Omega)$ for all $\alpha > \pi$, and assume the supremum of E_π on M is not attained. Then by Lemma 2.1, we have that $u_0 = 0$, which is impossible because $E_\pi(u_m) \rightarrow \beta > |\partial\Omega|$. \square

Let K_π be the set defined by

$$K_\pi = \{u \in M : E_\pi(u) = \beta\}.$$

Lemma 2.2. *The set K_π is compact.*

Proof. Let $\{u_m\} \subset K_\pi$ be such that $u_m \rightharpoonup u_0$ weakly in $H^1(\Omega)$, then by Proposition 2.1,

$$E_\pi(u_m) \rightarrow E_\pi(u_0).$$

Moreover, $\|u_0\| \leq \|u_m\| = 1$, then

$$E_\pi(u_0) \leq E_\pi\left(\frac{u_0}{\|u_0\|}\right) \leq \sup_{v \in M} E_\pi(v) = \beta.$$

Then we get $E_\pi(u_0) = \beta$, and $\|u_0\| = 1$, hence $u_m \rightarrow u_0$ strongly in $H^1(\Omega)$, hence K_π is compact. \square

The property of K_π of being compact implies that the family of norm-neighborhoods

$$N_\varepsilon = \{u \in M \mid \exists v \in K_\pi : \|u - v\| < \varepsilon\}$$

constitutes a basic neighborhood for K_π in M .

Lemma 2.3. *For sufficiently small $\varepsilon > 0$, one has*

$$(2.7) \quad \sup_{N_{2\varepsilon} \setminus N_\varepsilon} E_\pi < \beta = \sup_{N_\varepsilon} E_\pi.$$

Proof. We argue by contradiction. We suppose that there is a sequence $u_m \in N_{2\varepsilon} \setminus N_\varepsilon$ such that $E_\pi(u_m) \rightarrow \beta$. Then we have $u_m \in H^1(\Omega)$ with $\|u_m\|^2 = 1$. Up to subsequence, we can assume that $u_m \rightharpoonup u_0$ weakly in $H^1(\Omega)$. By the definition of $N_{2\varepsilon}$, there is $z_m \in K_\pi$ such that $\|z_m - u_m\| < 2\varepsilon$. By the compactness of K_π , we have that $z_m \rightarrow z$ strongly, with $z \in K_\pi$, and z satisfies

$$-\Delta z + z = 0 \quad \text{in } \Omega, \quad \frac{\partial z}{\partial \nu} = \frac{\pi z e^{z^2}}{\int_{\partial\Omega} z^2 e^{z^2}} \quad \text{on } \partial\Omega.$$

By the maximum principle, we have $z \in L^\infty(\Omega)$.

By the lower-semi continuity, we have $\|z - u_0\| \leq 2\varepsilon$. Then

$$\left\|z - \frac{u_0}{\|u_0\|}\right\| \leq \|z - u_0\| + \left\|u_0 - \frac{u_0}{\|u_0\|}\right\| = \|z - u_0\| + 1 - \|u_0\| \leq 4\varepsilon.$$

Thus $\frac{u_0}{\|u_0\|} \in N_{4\varepsilon}$, and so $E_\pi(u_0) \leq E_\pi\left(\frac{u_0}{\|u_0\|}\right) \leq \beta$. If $E_\pi(u_0) = \beta$ then $\|u_0\| = 1$, and $u_m \rightarrow u_0$. On the other hand, our assumption implies that $u_0 \notin N_\varepsilon$, thus u_0 does not belong to K_π and u_0 can not be relatively maximal. Thus we necessarily get $E_\pi(u_0) < \beta$.

Set $w_m = u_m - z_m + z$, so we have $w_m \rightharpoonup u_0$ weakly in $H^1(\Omega)$. Since

$$e^{\pi|w_m|^2} = e^{\pi|u_m - z_m + z|^2} \leq e^{2\pi|u_m - z_m|^2} e^{2\pi|z|^2} = e^{2\pi\|u_m - z_m\|^2} e^{2\pi|z|^2} \leq e^{8\pi\varepsilon^2 \left(\frac{\|u_m - z_m\|}{\|u_m - z_m\|}\right)^2} e^{2\pi|z|^2} \leq e^{8\pi\varepsilon^2} e^{2\pi|z|^2}.$$

Choosing ε small such that $16\varepsilon^2 \leq 1$, then from (2.3) we have that $e^{\pi|w_m|^2}$ is uniformly bounded in $L^2(\partial\Omega)$, as $m \rightarrow \infty$. Thus $\lim_{m \rightarrow \infty} E_\pi(w_m) = E_\pi(u_0)$. On the other hand, we have

$w_m - u_m \rightarrow 0$ strongly in $H^1(\Omega)$. By uniform local continuity of E_π , and compactness of K_π , we obtain that $E_\pi(w_m) - E_\pi(u_m) \rightarrow 0$, and $E_\pi(u_0) = \beta$. This is a contradiction. \square

Lemma 2.4. *There exists $\alpha^* > \pi$, $\varepsilon > 0$ such that for all $\alpha \in [\pi, \alpha^*]$, then we have*

(i)

$$(2.8) \quad \sup_{N_{2\varepsilon} \setminus N_\varepsilon} E_\alpha < \sup_{N_\varepsilon} E_\alpha.$$

(ii) $\beta_\alpha := \sup_{N_\varepsilon} E_\alpha$ is achieved in N_ε .

(iii) $K_\alpha = \{u \in N_\varepsilon \mid E_\alpha(u) = \beta_\alpha\}$ is compact.

Proof. (i) Since K_π is compact, there is a neighborhood N of K_π such that, for any $\varsigma > 0$ there exists $\delta' > 0$ such that for all $|\alpha - \pi| < \delta$ then $E_\alpha(u) - E_\pi(u) \leq \varsigma$, for all $u \in N$. Choose $\varepsilon > 0$ such that (2.7) holds and $N_\varepsilon \subset N$, then (2.8) will be valid for all α in a small neighborhood of π .

(ii) For such α , and let $u_m \in N_\varepsilon$ be a maximizing sequence of E_α , that is, $E_\alpha(u_m) \rightarrow \beta_\alpha$ and let $v_m \in K_\pi$ satisfy $\|u_m - v_m\| \leq \varepsilon$. We may assume that $v_m \rightarrow v$ strongly in $H^1(\Omega)$ with $v \in L^\infty$, and $u_m \rightarrow u$ weakly in $H^1(\Omega)$. Set $w_m = u_m - v_m + v$, as the proof of Lemma 2.3, we obtain that for $\varepsilon > 0$ small, α in a neighborhood of π we have that

$$E_\alpha(w_m) \rightarrow E_\alpha(u), \quad E_\alpha(u_m) - E_\alpha(w_m) \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Then $E_\alpha(u) = \beta_\alpha$. Moreover, by the lower-semi continuity, we have $\|v - u\| \leq \varepsilon$. Then

$$\|v - \frac{u}{\|u\|}\| \leq \|v - u\| + \|u - \frac{u}{\|u\|}\| = \|v - u\| + 1 - \|u\| \leq 2\varepsilon.$$

We get that $\frac{u}{\|u\|} \in N_{2\varepsilon}$ and $E_\alpha(\frac{u}{\|u\|}) \leq \beta_\alpha$. Furthermore, since $\|u\| \leq 1$, we can get $E_\alpha(\frac{u}{\|u\|}) \leq E_\alpha(u)$ and $\|u\| = 1$. It implies that $u \in M$, that is $u \in N_\varepsilon$ and β_α is attained. Moreover, $u_m \rightarrow u$ strongly in $H^1(\Omega)$.

(iii) As the proof of (ii), if $u_m \in K_\alpha$, we may assume that $u_m \rightharpoonup u$ weakly in $H^1(\Omega)$, we then get $u \in K_\alpha$, that is K_α is compact. \square

Proof of Theorem 1.1: From (2.3), we have that $\sup E$ is achieved for $\alpha < \pi$. Moreover, since $\sup_{u \in M_\pi} E(u) > |\partial\Omega|$, from Lemma 2.4 we have that for α sufficiently close to π , then E has relative maximizers on M_α .

3. THE PROOF OF THEOREM 1.2

In this section, we consider critical points of functional $E(u)$ constrained on the set M_α (which is equivalent to consider critical points of $E_\alpha(u)$ constrained on the set M with $\alpha = k\pi(1 + \mu)$, where $\mu > 0$ small). We define a critical point of E_α constrained on M to be a solution of the following problem

$$(3.1) \quad \begin{cases} -\Delta u + u = 0 & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = \lambda u e^{u^2} & \text{on } \partial\Omega, \end{cases}$$

where

$$(3.2) \quad \lambda = \frac{\alpha}{\int_{\partial\Omega} u^2 e^{u^2}} = \frac{k\pi(1 + \mu)}{\int_{\partial\Omega} u^2 e^{u^2}}.$$

In this section we shall prove the existence of solutions to Problem (3.1)-(3.2) with the properties described in Theorem 1.2. In fact, we will construct a solution to (3.1)-(3.2) of the form

$$(3.3) \quad u = U + \phi,$$

where U is the principal part while ϕ represents a lower order correction. In what follows we shall first describe explicitly the function $U(x)$. The definition of this function depends on several parameters: some points ξ on the boundary of Ω and some positive numbers m . Next we find the correction ϕ so that $U + \phi$ solves our Problem in a certain *projected sense* (see Proposition 3.1). Finally we select proper points ξ and numbers m in the definition of U to get an exact solution to Problem (3.1)-(3.2).

To define the function U , first we introduce the following limit problem

$$(3.4) \quad \begin{cases} \Delta w = 0 & \text{in } \mathbb{R}_+^2; \\ \frac{\partial w}{\partial \nu} = e^w & \text{on } \partial\mathbb{R}_+^2; \\ \int_{\partial\mathbb{R}_+^2} e^w < \infty. \end{cases}$$

A family solutions to (3.4) is given by

$$(3.5) \quad w_{t,\mu}(x) = w_{t,\mu}(x_1, x_2) = \log \frac{2\mu}{(x_1 - t)^2 + (x_2 + \mu)^2},$$

where $t \in \mathbb{R}$ and $\mu > 0$ are parameters. See [19, 24, 31]. Set

$$(3.6) \quad w_\mu(x) := w_{0,\mu}(x) = \log \frac{2\mu}{x_1^2 + (x_2 + \mu)^2}.$$

Let ξ_1, \dots, ξ_k be k distinct points on the boundary and m_1, \dots, m_k be k positive numbers. We assume there exists a sufficiently small but fixed number $\delta > 0$ such that

$$(3.7) \quad |\xi_i - \xi_j| > \delta \quad \text{for } i \neq j, \quad \delta < m_j < \frac{1}{\delta}.$$

For notational convenience through out the paper we will use the notation

$$(\xi, m) = (\xi_1, \dots, \xi_k, m_1, \dots, m_k).$$

For any $j = 1, \dots, k$, we define ε_j to be the positive numbers given by the relation

$$(3.8) \quad 2\lambda m_j^2 \left(\log \frac{1}{\varepsilon_j^2} + 2 \log(2m_j^2) \right) = 1.$$

Since the parameters m_j satisfy assumption (3.7), it follows that $\lim_{\lambda \rightarrow 0} \varepsilon_j = 0$. Define moreover μ_j to be the positive constants given by

$$(3.9) \quad \log(2\mu_j) = -2 \log(2m_j^2) + H(\xi_j, \xi_j) + \sum_{i \neq j} m_i m_j^{-1} G(\xi_i, \xi_j).$$

Using once more assumption (3.7), we get that there exists two positive constants c and C , such that $c \leq \mu_j \leq C$, as $\lambda \rightarrow 0$.

We define the function U in (3.3) to be given by

$$(3.10) \quad U(x) = \sqrt{\lambda} \sum_{j=1}^k m_j [u_j(x) + H_j(x)],$$

where

$$(3.11) \quad u_j(x) = \log \frac{1}{|x - \xi_j - \varepsilon_j \mu_j \nu(\xi_j)|^2},$$

$\nu(\xi_j)$ denoting the unit outer normal to $\partial\Omega$ at the point ξ_j , and where H_j is a correction term given as the solution of

$$(3.12) \quad \begin{cases} -\Delta H_j + H_j = -u_j & \text{in } \Omega; \\ \frac{\partial H_j}{\partial \nu} = 2\varepsilon_j \mu_j e^{u_j} - \frac{\partial u_j}{\partial \nu} & \text{on } \partial\Omega. \end{cases}$$

Arguing as in Lemma 3.1 in [9], one can show that the maximum principle allows a precise asymptotic description of the functions H_j , namely we have that

$$(3.13) \quad H_j(x) = H(x, \xi_j) + O(\varepsilon_j^\sigma) \quad \text{for } 0 < \sigma < 1$$

uniformly in Ω , as $\lambda \rightarrow 0$. Recall that H is the regular part of the Green's function, as defined in (1.6). Therefore, the function U can be described as follows

$$(3.14) \quad U(x) = \sqrt{\lambda} \sum_{j=1}^k m_j \left[G(x, \xi_j) + O(\varepsilon_j^\sigma) \right]$$

uniformly on compact sets of $\bar{\Omega} \setminus \{\xi_1, \dots, \xi_k\}$, as $\lambda \rightarrow 0$. On the other hand, if we consider a region close to ξ_j , for some j fixed, say for $|x - \xi_j| < \delta$, with sufficiently small but fixed δ , we can rewrite

$$(3.15) \quad U(x) = \sqrt{\lambda} m_j \left(w_j(x) + \log \varepsilon_j^{-2} + \beta_j + \theta(x) \right),$$

where

$$(3.16) \quad w_j(x) = w_{\mu_j} \left(\frac{x - \xi_j}{\varepsilon_j} \right) = \log \frac{2\mu_j}{|y - \xi'_j - \mu_j \nu(\xi'_j)|^2}, \quad y = \frac{x}{\varepsilon_j}, \quad \xi'_j = \frac{\xi_j}{\varepsilon_j},$$

and

$$\beta_j = -\log(2\mu_j) + H(\xi_j, \xi_j) + \sum_{i \neq j} m_i^{-1} m_i G(\xi_j, \xi_i), \quad \theta(x) = O(|x - \xi_j|) + \sum_{j=1}^k O(\varepsilon_j^\sigma).$$

Define on the boundary $\partial\Omega$ the error of approximation

$$(3.17) \quad R := f(U) - \frac{\partial U}{\partial \nu}.$$

Here and in what follows f denotes the nonlinearity

$$f(\tilde{u}) = \lambda \tilde{u} e^{\tilde{u}^2}.$$

The choice we made of μ_j in (3.9) and of ε_j in (3.8) gives that in the region $|x - \xi_j| < \delta$, the error of approximation can be described as follows

$$(3.18) \quad R = m_j \sqrt{\lambda} \left\{ \left(1 + 2\lambda m_j^2 (w_j + O(1)) \right) e^{\lambda m_j^2 w_j^2} (1 + O(\lambda w_j)) - 1 \right\} \varepsilon_j^{-1} e^{w_j},$$

where w_j is defined in (3.16). Indeed, for $x \in \partial\Omega$ with $|x - \xi_j| < \delta$, we have that

$$\begin{aligned}
\lambda^{-\frac{1}{2}} f(U) &= \lambda \left[m_j \left(w_j(x) + \log \varepsilon_j^{-2} + \beta_j + \theta(x) \right) \right] e^{\lambda \left[m_j (w_j(x) + \log \varepsilon_j^{-2} + \beta_j + \theta(x)) \right]} \\
&= \left(\lambda m_j \left(\log \frac{1}{\varepsilon_j^2} + \beta_j \right) + \lambda m_j (w_j + O(1)) \right) \\
&\quad \times e^{\lambda m_j^2 \left(\log \frac{1}{\varepsilon_j^2} + \beta_j \right)^2} e^{2\lambda m_j^2 \left(\log \frac{1}{\varepsilon_j^2} + \beta_j \right) w_j} e^{2\lambda m_j^2 \left(\log \frac{1}{\varepsilon_j^2} + \beta_j \right) \theta(x)} e^{\lambda m_j^2 (w_j + \theta(x))^2} \\
&= \lambda m_j \left(\log \frac{1}{\varepsilon_j^2} + \beta_j \right) \left(1 + \left(\log \frac{1}{\varepsilon_j^2} + \beta_j \right)^{-1} (w_j + O(1)) \right) \\
&\quad \times e^{\lambda m_j^2 \left(\log \frac{1}{\varepsilon_j^2} + \beta_j \right)^2} e^{2\lambda m_j^2 \left(\log \frac{1}{\varepsilon_j^2} + \beta_j \right) w_j} e^{2\lambda m_j^2 \left(\log \frac{1}{\varepsilon_j^2} + \beta_j \right) \theta(x)} e^{\lambda m_j^2 (w_j + \theta(x))^2} \\
&= \frac{1}{2m_j} \left(1 + 2\lambda m_j^2 (w_j + O(1)) \right) e^{\frac{1}{2} \left(\log \frac{1}{\varepsilon_j^2} + \beta_j \right)} e^{w_j} e^{\theta(x)} e^{\lambda m_j^2 (w_j + \theta(x))^2} \\
&= \frac{1}{2m_j} \varepsilon_j^{-1} e^{\beta_j/2} \left(1 + 2\lambda m_j^2 (w_j + O(1)) \right) e^{w_j} e^{\theta(x)} e^{\lambda m_j^2 w_j^2} (1 + O(\lambda w_j))
\end{aligned}$$

thanks to the definition of ε_j in (3.8). On the other hand, in the same region, we have

$$\lambda^{-\frac{1}{2}} \frac{\partial U}{\partial \nu} = \frac{\partial}{\partial \nu} \left[m_j \left(w_j(x) + \log \varepsilon_j^{-2} + \beta_j + \theta(x) \right) \right] = m_j \varepsilon_j^{-1} e^{w_j} + \sum_{j=1}^k O(\varepsilon_j^2), \quad \text{as } \lambda \rightarrow 0.$$

The definition of μ_j in (3.9) allows to match at main order the two terms $\frac{\partial \tilde{U}}{\partial \nu}$ and $f(\tilde{U})$ in the region under consideration, since we, we easily get that

$$\lambda^{-\frac{1}{2}} f(\tilde{U}) = m_j \left(1 + 2\lambda m_j^2 (w_j + O(1)) \right) \varepsilon_j^{-1} e^{w_j} e^{\lambda m_j^2 w_j^2} (1 + O(\lambda w_j)).$$

These facts imply the validity of expansion (3.18). Let us now observe that a direct computation shows that $R(x) \sim \lambda^{\frac{3}{2}} \varepsilon_j^{-1} e^{w_j(x)}$ in the region $|x - \xi_j| = O(\lambda)$; while, in the region $|x - \xi_j| > \delta$ for all j , we have that $|R(x)| \leq C\lambda^{\frac{3}{2}}$, for some positive constant C . We thus conclude that the error of approximation satisfies the global bound

$$|R| \leq C\lambda^{\frac{3}{2}} \rho(x),$$

where

$$\rho(x) := \sum_{j=1}^k \rho_j(x) \chi_{B_\delta(\xi_j)}(x) + 1.$$

Here $\chi_{B_\delta(\xi_j)}$ is the characteristic function on $B_\delta(\xi_j) \cap \partial\Omega$ and

$$\rho_j(x) := \frac{1}{2\lambda m_j^2} \left\{ \left(1 + 2\lambda m_j^2 (w_j + O(1)) \right) e^{\lambda m_j^2 w_j^2} (1 + O(\lambda w_j)) - 1 \right\} \varepsilon_j^{-1} e^{w_j}$$

From now on, let us write

$$(3.19) \quad \rho_j(x) = c\gamma_j \left\{ \left(1 + \frac{1}{\gamma_j} (w_j + 1) \right) \left(1 + \frac{1}{\gamma_j} (1 + |w_j|) \right) e^{\frac{w_j^2}{2\gamma_j}} - 1 \right\} \varepsilon_j^{-1} e^{w_j},$$

where $\gamma_j = \log \varepsilon_j^{-2}$. We define the L^∞ -weight norm

$$(3.20) \quad \|h\|_{*,\partial\Omega} = \sup_{x \in \partial\Omega} \rho(x)^{-1} |h(x)|.$$

We thus have the validity of the following key estimate for the error term R

$$(3.21) \quad \|R\|_{*,\partial\Omega} \leq C\lambda^{\frac{3}{2}}.$$

Up to this point, we have defined a function U , whose expression depends of ξ_1, \dots, ξ_k points on $\partial\Omega$, and depends of m_1, \dots, m_k positive numbers. These points and numbers satisfy the bounds (3.7). We next describe the problem that the function ϕ in (3.3) solves.

Define in $\mathbb{R}_+^2 = \{(x_1, x_2) : x_2 > 0\}$ the functions

$$z_{0j}(x_1, x_2) = \frac{1}{\mu_j} - 2 \frac{x_2 + \mu_j}{x_1^2 + (x_2 + \mu_j)^2}, \quad z_{1j}(x_1, x_2) = -2 \frac{x_1}{x_1^2 + (x_2 + \mu_j)^2}.$$

It has been shown in [9] that these functions are all the bounded solutions to the linearized equation around w_{μ_j} (3.6) associated to Problem (3.4), that is they are the only bounded solutions to

$$(3.22) \quad \Delta\psi = 0 \quad \text{in } \mathbb{R}_+^2, \quad -\frac{\partial\psi}{\partial x_2} = e^{w_{\mu_j}}\psi \quad \text{on } \partial\mathbb{R}_+^2.$$

For $\xi_j \in \partial\Omega$, we define $F_j : B_\delta(\xi_j) \rightarrow \mathcal{O}$ to be a diffeomorphism, where \mathcal{O} is an open neighborhood of the origin in \mathbb{R}_+^2 such that $F_j(\Omega \cap B_\delta(\xi_j)) = \mathbb{R}_+^2 \cap \mathcal{O}$, $F_j(\partial\Omega \cap B_\delta(\xi_j)) = \partial\mathbb{R}_+^2 \cap \mathcal{O}$. We can select F_j so that it preserves area. Define

$$(3.23) \quad Z_{ij}(x) = z_{ij}(\varepsilon_j^{-1}F_j(x)), \quad i = 0, 1, \quad j = 1, \dots, k.$$

Next, let us consider a large but fixed number $R_0 > 0$ and a nonnegative radial and smooth cut-off function χ with $\chi(r) = 1$ if $r < R_0$ and $\chi(r) = 0$ if $r > R_0 + 1$, $0 \leq \chi \leq 1$. Then set

$$(3.24) \quad \chi_j(x) = \varepsilon_j^{-1}\chi(\varepsilon_j^{-1}F_j(x)).$$

The problem we solve is the following: given ξ_1, \dots, ξ_k and m_1, \dots, m_k satisfying the bounds (3.7), find a function ϕ and numbers c_{ij} such that

$$(3.25) \quad \begin{cases} -\Delta(U + \phi) + (U + \phi) = 0 & \text{in } \Omega; \\ \frac{\partial(U + \phi)}{\partial\nu} = \lambda(U + \phi)e^{(U + \phi)^2} + \sqrt{\lambda} \sum_{i=0,1} \sum_{j=1}^k c_{ij}\chi_j Z_{ij} & \text{on } \partial\Omega; \\ \int_{\Omega} \chi_j Z_{ij} \phi = 0 & \text{for } i = 0, 1, \quad j = 1, \dots, k. \end{cases}$$

Consider the norm

$$\|\phi\|_\infty = \sup_{x \in \Omega} |\phi(x)|.$$

In [13], we have the following result.

Proposition 3.1. *Let $\delta > 0$ be a small but fixed number and assume points the $\xi_1, \dots, \xi_k \in \partial\Omega$ and the numbers m_1, \dots, m_k satisfy (3.7). Furthermore we assume that ε_j and μ_j are given by (3.8) and (3.9). Then there exist positive numbers λ_0 and C , such that for any $0 < \lambda < \lambda_0$, there is a unique solution $\phi = \phi(\lambda, \xi, m)$, $c_{ij} = c_{ij}(\lambda, \xi, m)$ to (3.25). Moreover,*

$$(3.26) \quad \|\phi\|_\infty \leq C\lambda^{\frac{3}{2}}, \quad |c_{ij}| \leq C\lambda.$$

Furthermore, function ϕ and constant c_{ij} are C^1 with respect to (ξ, m) , and we have

$$(3.27) \quad \|D_{\xi, m}\phi\|_\infty \leq C\lambda^{\frac{3}{2}}, \quad |D_{\xi, m}c_{ij}| \leq C\lambda.$$

We will sketch the proof in Section 4, leaving some technical details to the Appendix 6.

Assuming for the moment the validity of the statement in the above Proposition, we observe that $U + \phi$ is an exact solution to Problem (3.1), if there exists a proper choice of λ , of the points ξ_j and the parameters m_j , such that

$$(3.28) \quad \lambda = \frac{k\pi(1 + \mu)}{\int_{\partial\Omega} (U + \phi)^2 e^{(U + \phi)^2}} \quad \text{and} \quad c_{ij} = 0, \quad \text{for all } i, j,$$

or equivalently

$$(3.29) \quad \int_{\Omega} [|\nabla(U + \phi)|^2 + (U + \phi)^2] dx = k\pi(1 + \mu) \quad \text{and} \quad c_{ij} = 0, \quad \text{for all } i, j.$$

In order to solve (3.29), we are in the need of understanding the asymptotic expansion, as $\lambda \rightarrow 0$, of $\int_{\Omega} [|\nabla(U + \phi)|^2 + (U + \phi)^2] dx$ in terms of the localization of the points ξ and the values of the parameters m . Next Proposition contains this result, together with the asymptotic expansion of $\int_{\partial\Omega} e^{(U+\phi)^2}$, as $\lambda \rightarrow 0$, again in terms of in terms of ξ and m .

Proposition 3.2. *Under the conditions of Proposition 3.1, Assume that ε_j and μ_j are given by (3.8) and (3.9). Furthermore, we assume that λ is a free parameter. Then, as $\lambda \rightarrow 0$, we have*

$$(3.30) \quad \int_{\Omega} [|\nabla(U + \phi)|^2 + (U + \phi)^2] dx = k\pi \{1 + \lambda f_k(\xi, m) + \lambda^2 \Theta_{\lambda}(\xi, m)\}$$

where

$$(3.31) \quad f_k(\xi, m) = \frac{2}{k} \left[2 \sum_{j=1}^k m_j^2 \log(2m_j^2) - \sum_{j=1}^k m_j^2 H(\xi_j, \xi_j) - \sum_{i \neq j} m_i m_j G(\xi_i, \xi_j) \right].$$

Moreover, as $\lambda \rightarrow 0$,

$$(3.32) \quad \int_{\partial\Omega} e^{(U+\phi)^2} = |\partial\Omega| + 4\pi \sum_{j=1}^k m_j^2 + \lambda \sum_{j=1}^k m_j^2 \left[\tilde{c} + \int_{\partial\Omega} G^2(x, \xi_j) \right] + \lambda^2 \Theta_{\lambda}(\xi, m),$$

where \tilde{c} is a positive constant. In (3.31) and (3.32) the function $\Theta_{\lambda}(\xi, m)(x)$ denotes a generic smooth function, uniformly bounded together with its derivatives, as $\lambda \rightarrow 0$, for (ξ, m) satisfying (3.7). In (3.31) and (3.32), G is the Green function defined in (1.6) and H its regular part, as defined in (1.7).

Next Proposition will suggest how to solve Problem in (3.29).

Proposition 3.3. *Under the conditions of Proposition 3.1, let R be the set of points (ξ, m) satisfy (3.7). then there exist $\mu_0 > 0$ and a subregion R' of R such that for all $0 < \mu < \mu_0$ and for all $(\xi, m) \in R'$, there exists a function $\lambda = \lambda(\mu, \xi, m)$ such that*

$$(3.33) \quad \int_{\Omega} [|\nabla(U + \phi)|^2 + (U + \phi)^2] dx = k\pi(1 + \mu) \quad \text{for all } \mu > 0, \mu \rightarrow 0.$$

Moreover, λ is a smooth function of the free parameter μ , of the points ξ_1, \dots, ξ_k and of the parameters m_1, \dots, m_k . Furthermore, $\lambda \rightarrow 0$ as $\mu \rightarrow 0$ for points ξ_1, \dots, ξ_k and parameters m_1, \dots, m_k belonging to R' . With this definition of λ , we have that the function ϕ and the constants c_{ij} are C^1 with respect to (ξ, m) . We finally have that

$$(3.34) \quad D_{\xi, m} E(U + \phi) = 0 \implies c_{ij} = 0 \quad \text{for all } i, j.$$

See (2.1) for the definition of E .

The proofs of Proposition 3.2 and of Proposition 3.3 are postponed to Section 5.

Given the choice of λ defined through formula (3.33), for all $\mu > 0$ small, Proposition 3.3 gives that $U + \phi$ is a solution to problem (3.1)-(3.2) if we can find (ξ, m) to be a critical point of the function

$$(3.35) \quad \mathcal{I}(\xi, m) := E(U + \phi).$$

We have now all the elements to give the

Proof of Theorem 1.2: Let \mathcal{D} be the open set such that

$$\bar{\mathcal{D}} \subset \{(\xi, m) \in (\partial\Omega)^k \times \mathbb{R}_+^k : \xi_i \neq \xi_j, \forall i \neq j\}$$

Let $U(x)$ be defined as in (3.10), and $\phi(x)$ be the solution of problem (3.25), whose existence and properties are stated in Proposition 3.1. Proposition 3.3 gives that

$$u(x) = U(x) + \phi(x)$$

is a solution to problem (3.1)-(3.2) if we can find (ξ, m) to be a critical point of the function

$$\mathcal{I}(\xi, m) := E(U + \phi).$$

From (3.33) and (3.30), we have

$$(3.36) \quad \lambda f_k(\xi, m) + \lambda^2 \Theta_\lambda(\xi, m) = \mu$$

where

$$f_k(\xi, m) = \frac{2}{k} \left[2 \sum_{j=1}^k m_j^2 \log(2m_j^2) - \sum_{j=1}^k m_j^2 H(\xi_j, \xi_j) - \sum_{i \neq j} m_i m_j G(\xi_i, \xi_j) \right].$$

In (3.36), $\Theta_\lambda(\xi, m)(x)$ denotes a smooth function, uniformly bounded together with its derivatives, as $\lambda \rightarrow 0$, for (ξ, m) satisfying (3.7). Make the change of variables $s_j = m_j^2$. So we write, with abuse of notation,

$$f_k(\xi, s) = \frac{2}{k} \left[2 \sum_{j=1}^k s_j \log(2s_j) - \sum_{j=1}^k s_j H(\xi_j, \xi_j) - \sum_{i \neq j} \sqrt{s_i s_j} G(\xi_i, \xi_j) \right]$$

Fix ξ . Observe that the function $s \rightarrow f_k(\xi, s)$ has a unique zero, namely there exists a unique $\bar{s} = (\bar{s}_1(\xi), \dots, \bar{s}_k(\xi)) \in \mathbb{R}_+^k$ satisfying $f_k(\xi, \bar{s}) = 0$. We have the following properties:

- (i) \bar{s}_j is a C^1 function with respect to ξ defined in $(\partial\Omega)^k$;
- (ii) There is a positive constant c_0 , independent of the points ξ , such that $\bar{s}_j \geq c_0$ for each $j = 1, \dots, k$;
- (iii) $\bar{s}_j \rightarrow +\infty$ as $|\xi_i - \xi_j| \rightarrow 0$ for some $i \neq j$;
- (iv) Define

$$M^+ = \{(\xi, s) \in (\partial\Omega)^k \times \mathbb{R}_+^k : s_1 s_2 \dots s_k \neq 0, f_k(\xi, s) > 0\}.$$

Then $(\xi, (1+r)\bar{s}) \in M^+$ for $r > 0$ small.

Proof of (i). Since $f(\xi, \bar{s}) = 0$, and for j fixed,

$$\partial_{s_j} f_k(\xi, s) \Big|_{s=\bar{s}} = \frac{2}{k} \left\{ 2 \log(2\bar{s}_j) + 2 - \left[H(\xi_j, \xi_j) - \frac{1}{2} \sum_{i \neq j} \sqrt{\bar{s}_i \bar{s}_j} G(\xi_i, \xi_j) \right] \right\}.$$

Then

$$(3.37) \quad \nabla_s f_k(\xi, \bar{s}) \cdot \bar{s} = \partial_{s_1} f_k(\xi, \bar{s}) \bar{s}_1 + \dots + \partial_{s_k} f_k(\xi, \bar{s}) \bar{s}_k = \frac{4}{k} \sum_{j=1}^k \bar{s}_j > 0$$

Thus we get $\nabla_s f_k(\xi, s) \Big|_{s=\bar{s}} \neq 0$. The implicit function theorem implies the validity of (i).

Proof of (ii). According to the definition of \bar{s} , we know that

$$\frac{2}{k} \sum_{j=1}^k \bar{s}_j \left[2 \log(2\bar{s}_j) - H(\xi_j, \xi_j) - \sum_{i \neq j} \sqrt{\frac{\bar{s}_i}{\bar{s}_j}} G(\xi_i, \xi_j) \right] = 0.$$

It yields that

$$2 \log(2\bar{s}_j) - H(\xi_j, \xi_j) = \sum_{i \neq j} \sqrt{\frac{\bar{s}_i}{\bar{s}_j}} G_{\xi_i, \xi_j} > 0.$$

So

$$\bar{s}_j > \frac{1}{2} e^{\frac{H(\xi_j, \xi_j)}{2}}$$

Then we get (ii).

Proof of (iii). Since $G(\xi_i, \xi_j) \rightarrow +\infty$ if $|\xi_i - \xi_j| \rightarrow 0$, for some $i \neq j$, if we suppose that \bar{s}_l is bounded, for some l , then the relation $f_k(\xi, \bar{s}) = 0$ would provide a contradiction. This proves (iii).

Proof of (iv). For $r > 0$ small, by the Taylor expansion, from (3.37) we have

$$\begin{aligned} f_k(\xi, (1+r)\bar{s}) &= f_k(\xi, \bar{s}) + [\partial_{s_1} f_k(\xi, \bar{s})\bar{s}_1 + \dots + \partial_{s_k} f_k(\xi, \bar{s})\bar{s}_k] r + o(r) \\ (3.38) \quad &= \frac{4}{k} r \sum_{j=1}^k \bar{s}_j + o(r) > 0. \end{aligned}$$

Making the change of variable, define $s = (1+r)\bar{s}$ with $r > 0$ small, we have $(\xi, (1+r)\bar{s}) \in M^+$.

Thanks to the above properties, we conclude that relation (3.36) defines λ as a function of the free parameter μ and (ξ, s) . More precisely,

$$(3.39) \quad \lambda = \frac{\mu}{f_k(\xi, (1+r)\bar{s})} + \frac{\mu^2}{f_k(\xi, (1+r)\bar{s})^3} \Theta_\lambda(\xi, s)$$

where $\Theta_\lambda(\xi, s)$ is a smooth function, uniformly bounded together with its derivatives, as $\lambda \rightarrow 0$.

Taking (3.39) into (3.32), we get that

$$\begin{aligned} \mathcal{I}(\xi, (1+r)\bar{s}) &= |\partial\Omega| + 4(1+r)\pi \sum_{j=1}^k \bar{s}_j + \mu \frac{\sum_{j=1}^k \bar{s}_j [\bar{c} + \int_{\partial\Omega} G^2(x, \xi_j)]}{f_k(\xi, (1+r)\bar{s})} \\ &\quad + \left(\frac{\mu}{f_k(\xi, (1+r)\bar{s})} \right)^2 \Theta_\mu(\xi, s) \\ (3.40) \quad &= |\partial\Omega| + 4(1+r)\pi \sum_{j=1}^k \bar{s}_j + \mu \frac{\sum_{j=1}^k \bar{s}_j [\bar{c} + \int_{\partial\Omega} G^2(x, \xi_j)]}{\frac{4}{k} r \sum_{j=1}^k \bar{s}_j} + \mu \Theta_\mu(\xi, s), \end{aligned}$$

where $\Theta_\mu(\xi, s)$ is a smooth function, uniformly bounded together with its derivatives, as $\mu \rightarrow 0$.

We claim that, given $\delta > 0$, for all $\mu > 0$ small enough, the function

$$\varphi_\mu(\xi, \bar{s}, r) := |\partial\Omega| + 4\pi \sum_{j=1}^k \bar{s}_j + 4r\pi \sum_{j=1}^k \bar{s}_j + \mu \frac{\sum_{j=1}^k \bar{s}_j [\bar{c} + \int_{\partial\Omega} G^2(x, \xi_j)]}{\frac{4}{k} r \sum_{j=1}^k \bar{s}_j}$$

has a critical point in the region $|\xi_i - \xi_j| > \delta$ for $i \neq j$, $\xi_j \in \partial\Omega$, and $\delta\sqrt{\mu} < r < \delta^{-1}\sqrt{\mu}$, with value $|\partial\Omega| + 4\pi \sum_{j=1}^k \bar{s}_j + O(\sqrt{\mu})$, as $\mu \rightarrow 0$, in the region considered. By construction,

the critical point situation is stable under proper small C^1 perturbation of φ_μ : to be more precise, any function ψ such that $\|\psi - \varphi_\mu\|_\infty + \|\nabla\psi - \nabla\varphi_\mu\|_\infty \leq C\mu$ in the region considered, also has a critical point. This fact will conclude the proof of Theorem 1.2.

Observe that the function

$$r \mapsto \varphi_\mu(\xi, \bar{s}, r) := |\partial\Omega| + 4\pi \sum_{j=1}^k \bar{s}_j + 4r\pi \sum_{j=1}^k \bar{s}_j + \mu \frac{\sum_{j=1}^k \bar{s}_j \left[\bar{c} + \int_{\partial\Omega} G^2(x, \xi_j) \right]}{\frac{4}{k} r \sum_{j=1}^k \bar{s}_j}$$

has a critical point \bar{r} given by

$$\bar{r} = \frac{\sqrt{\sum_{j=1}^k \bar{s}_j \left[\bar{c} + \int_{\partial\Omega} G^2(x, \xi_j) \right]}}{4 \frac{\sqrt{\pi}}{\sqrt{k}} \sum_{j=1}^k \bar{s}_j} \sqrt{\mu},$$

which is a non-degenerate minimum, since

$$\partial_{rr}^2 \varphi_\mu(\xi, \bar{s}, r) = \mu \frac{\sum_{j=1}^k \bar{s}_j \left[\bar{c} + \int_{\partial\Omega} G^2(x, \xi_j) \right]}{\frac{2}{k} \sum_{j=1}^k \bar{s}_j} \frac{1}{r^3} > 0.$$

Inserting the value of \bar{r} in φ_μ , in the new variables $\xi \in (\partial\Omega)^k$, we get

$$\begin{aligned} \Phi(\xi) &:= \mathcal{I}(\xi, (1 + \bar{r})\bar{s}) \\ &= |\partial\Omega| + 4\pi \sum_{j=1}^k \bar{s}_j + 2\sqrt{k\pi} \sqrt{\sum_{j=1}^k \bar{s}_j \left[\bar{c} + \int_{\partial\Omega} G^2(x, \xi_j) \right]} \sqrt{\mu} + \mu \Theta_\mu(\xi, s) \\ &= |\partial\Omega| + 4\pi \sum_{j=1}^k \bar{s}_j + O(\sqrt{\mu}) \quad \text{as } \mu \rightarrow 0 \end{aligned}$$

for $\xi \in \tilde{\Omega}_k = \{(\xi_1, \dots, \xi_k) \in (\partial\Omega)^k : \xi_i \neq \xi_j \text{ if } i \neq j\}$.

Next we show that functional $\Phi(\xi)$ has at least two critical points. Let C_0 be a component of $\partial\Omega$. Let $\Lambda : S^1 \rightarrow C_0$ be a continuous bijective function that parametrizes C_0 . Set

$$\tilde{\Omega}_k = \{(\xi_1, \dots, \xi_k) \in C_0^k : |\xi_i - \xi_j| > \delta \text{ for } i \neq j\}.$$

The function Φ is C^1 , bounded from below in $\tilde{\Omega}_k$, and from (iii) we have

$$\Phi(\xi) = \Phi(\xi_1, \dots, \xi_k) \rightarrow +\infty \text{ as } |\xi_i - \xi_j| \rightarrow 0 \text{ for some } i \neq j.$$

Hence, since δ is arbitrarily small, Φ has an absolute minimum c_m in $\tilde{\Omega}_k$.

On the other hand, using the Ljusternik-Schnirelmann theory, we get that Φ has at least two distinct points in $\tilde{\Omega}_k$. Let $\text{cat}(\tilde{\Omega}_k)$ be the Ljusternik-Schnirelmann category of $\tilde{\Omega}_k$ relative to $\tilde{\Omega}_k$, which is the minimum number of closed and contractible sets in $\tilde{\Omega}_k$ whose union covers $\tilde{\Omega}_k$. We will estimate the number of critical points for Φ by $\text{cat}(\tilde{\Omega}_k)$.

Claim: $\text{cat}(\tilde{\Omega}_k) > 1$.

Indeed, by contradiction, suppose that $\text{cat}(\tilde{\Omega}_k) = 1$. This means that $\tilde{\Omega}_k$ is contractible in itself, namely there exist a point $\xi^0 \in \tilde{\Omega}_k$ and a continuous function $\Gamma : [0, 1] \times \tilde{\Omega}_k \rightarrow \tilde{\Omega}_k$, such that, for all $\xi \in \tilde{\Omega}_k$,

$$\Gamma(0, \xi) = \xi, \quad \Gamma(1, \xi) = \xi^0.$$

Define $f : S^1 \rightarrow \tilde{\Omega}_k$ to be the continuous function given by

$$f(\bar{\xi}) = \left(\Lambda(\bar{\xi}), \Lambda(e^{2\pi i \frac{1}{k}} \bar{\xi}), \dots, \Lambda(e^{2\pi i \frac{k-1}{k}} \bar{\xi}) \right).$$

Let $\eta : [0, 1] \times S^1 \rightarrow S^1$ be the well defined continuous map given by

$$\eta(t, \bar{\xi}) = \Lambda^{-1} \circ \pi_1 \circ \Gamma(t, f(\bar{\xi})),$$

where π_1 is the projection on the first component. The function η is a contraction of S^1 to a point and this gives a contradiction, then claim follows.

Therefore we have that $\text{cat}(\tilde{\Omega}_k) \geq 2$ for any $k \geq 1$. Define

$$c = \sup_{C \in \Xi} \inf_{\xi \in C} \Phi(\xi)$$

where

$$\Xi = \{C \subset \tilde{\Omega}_k : C \text{ closed and } \text{cat}(C) \geq 2\}.$$

Then by Ljusternik-Schnirelmann theory we obtain that c is a critical level.

If $c \neq c_m$, we conclude that Φ has at least two distinct critical points in $\tilde{\Omega}_k$. If $c = c_m$, there is at least one set C such that $\text{cat}(C) \geq 2$, where the function Φ reaches its absolute minimum. In this case we conclude that there are infinitely many critical points for Φ in $\tilde{\Omega}_k$.

Thus we obtain that the function Φ has at least two distinct critical points in $\tilde{\Omega}_k$, denoted say by ξ^1, ξ^2 . Hence, for μ sufficiently small, the function $\mathcal{I}(\xi, s)$ has two distinct points (ξ_μ^1, s_μ^1) and (ξ_μ^2, s_μ^2) close respectively to $(\xi^1, (1 + \bar{r}(\xi^1))\bar{s}(\xi^1))$ and to $(\xi^2, (1 + \bar{r}(\xi^2))\bar{s}(\xi^2))$. This implies the existence of a solution to our Problem of the form $U + \phi$. Finally, let us remark that (1.10) holds as a direct consequence of the construction of U and of the fact that ϕ is a smaller perturbation. This ends the proof of the Theorem.

4. PROOF OF PROPOSITION 3.1

The proof of Proposition 3.1 is based on a fixed point argument and the invertibility property of the following linear Problem: Given $h \in L^\infty(\partial\Omega)$, find a function ϕ and constants c_{ij} such that

$$(4.1) \quad \begin{cases} -\Delta\phi + \phi = 0 & \text{in } \Omega; \\ L(\phi) = h + \sum_{i=0,1} \sum_{j=1}^k c_{ij} \chi_j Z_{ij} & \text{on } \partial\Omega; \\ \int_{\Omega} \chi_j Z_{ij} \phi = 0 & \text{for } i = 0, 1, \quad j = 1, \dots, k. \end{cases}$$

We shall prove the validity of the following

Proposition 4.1. *Let $\delta > 0$ be a small but fixed number and assume we have $\xi_1, \dots, \xi_k \in \partial\Omega$ and m_1, \dots, m_k with*

$$(4.2) \quad |\xi_i - \xi_j| \geq \delta, \quad \forall i \neq j, \quad \delta < m_j < \frac{1}{\delta}.$$

Then there exist positive numbers λ_0 and C such that, for any $0 < \lambda < \lambda_0$ and any $h \in L^\infty(\partial\Omega)$, there is a unique solution $\phi \equiv T_\lambda(h)$, and $c_{ij} \in \mathbb{R}$ to (4.1). Moreover,

$$(4.3) \quad \|\phi\|_\infty \leq C \|h\|_{*, \partial\Omega}.$$

The proof of this result is postponed to Appendix 6.

The result of Proposition 4.1 implies that the unique solution $\phi = T_\lambda(h)$ of (4.1) defines a continuous linear map from the Banach space C_* of all functions h in $L^\infty(\partial\Omega)$ for which $\|h\|_{*, \partial\Omega} < \infty$ into L^∞ , with norm bounded uniformly in λ .

Lemma 4.1. *The operator T_λ is differentiable with respect to the variable ξ_1, \dots, ξ_k on $\partial\Omega$ satisfying 4.2, and m_1, \dots, m_k , one has the estimate*

$$(4.4) \quad \|D_\xi T_\lambda(h)\|_\infty \leq C\|h\|_{*,\partial\Omega}, \quad \|D_m T_\lambda(h)\|_\infty \leq C\|h\|_{*,\partial\Omega}.$$

for a given positive C , independent of λ , and for all λ small enough.

Proof. Differentiating equation (4.1), formally $Z := \partial_{\xi_{sl}}\phi$, for all s, l , should satisfy in Ω the equation

$$-\Delta Z + Z = 0 \quad \text{in } \Omega,$$

and on the boundary $\partial\Omega$

$$L(Z) = -\partial_{\xi_{sl}} \left(\sum_{j=1}^k \varepsilon_j^{-1} e^{w_j} \right) \phi + \sum_{i=0,1} \sum_{j=1}^k c_{ij} \partial_{\xi_{sl}} (\chi_j Z_{ij}) + \sum_{i=0,1} \sum_{j=1}^k d_{ij} Z_{ij} \chi_j$$

with $d_{ij} = \partial_{\xi_{sl}} c_{ij}$, and the orthogonality conditions now become

$$\int_{\Omega} Z_{ij} \chi_j Z = 0 \quad \text{if } s \neq j.$$

$$\int_{\Omega} Z_{is} \chi_s Z = - \int_{\Omega} \partial_{\xi_{sl}} (Z_{is} \chi_s) \phi.$$

We consider the constants α_{ab} , $a = 0, 1$, $b = 1, \dots, k$, defined as

$$\alpha_{ab} \int_{\Omega} \chi_b^2 |Z_{ab}|^2 = \int_{\Omega} \partial_{\xi_{sl}} (Z_{ab} \chi_b) \phi, \quad \text{for } a = 0, 1, b = 1, \dots, k.$$

Define

$$\tilde{Z} = Z + \sum_{a=0,1} \sum_{b=1}^k \alpha_{ab} \chi_b Z_{ab}.$$

We then have

$$\begin{cases} -\Delta \tilde{Z} + \tilde{Z} = f_1 & \text{in } \Omega; \\ L(\tilde{Z}) = h_1 + \sum_{i=0,1} \sum_{j=1}^k d_{ij} Z_{ij} \chi_j & \text{on } \partial\Omega; \\ \int_{\Omega} \chi_j Z_{ij} \tilde{Z} = 0 & \text{for } i = 0, 1, j = 1, \dots, k, \end{cases}$$

where

$$f_1 = \sum_{a=0,1} \sum_{b=1}^k \alpha_{ab} (-\Delta(\chi_b Z_{ab}) + \chi_b Z_{ab}),$$

$$h_1 = -\partial_{\xi_{sl}} \left(\sum_{j=1}^k \varepsilon_j^{-1} e^{w_j} \right) \phi + \sum_{i=0,1} \sum_{j=1}^k c_{ij} \partial_{\xi_{sl}} (Z_{ij} \chi_j) + \sum_{a=0,1} \sum_{b=1}^k \alpha_{ab} L(\chi_b Z_{ab}).$$

Hence, using the result of Lemma 6.1 we have that

$$\|\tilde{Z}\|_\infty \leq C(\|h_1\|_{*,\partial\Omega} + \|f_1\|_{**,\Omega}).$$

By the definition of α_{ab} , we get $|\alpha_{ab}| \leq C\|\phi\|_\infty$. Since $\|\phi\|_\infty \leq C\|h\|_{*,\partial\Omega}$, $|c_{ij}| \leq C\|h\|_{*,\partial\Omega}$ we obtain that

$$\|\tilde{Z}\|_\infty \leq C\|h\|_{*,\partial\Omega}.$$

Hence we get

$$\|\partial_{\xi_{sl}} T_\lambda(h)\|_\infty \leq C\|h\|_{*,\partial\Omega} \quad \text{for all } s, l.$$

Analogous computation holds true if we differentiate with respect to m_j . \square

We are now in the position to prove Proposition 3.1.

Proof of Proposition 3.1. In terms of the operator T_λ defined in Proposition 4.1, problem (3.25) becomes

$$(4.5) \quad \phi = T_\lambda(R + N(\phi)) := A(\phi),$$

where R is defined in (3.17). For a given number $\gamma > 0$, let us consider the region

$$\mathcal{F}_\gamma := \left\{ \phi \in C(\bar{\Omega}) : \|\phi\|_\infty \leq \gamma \lambda^{\frac{3}{2}} \right\}.$$

From Proposition 4.1, we get

$$\|A(\phi)\|_\infty \leq C [\|R\|_{*,\partial\Omega} + \|N(\phi)\|_{*,\partial\Omega}].$$

An involved but direct computation shows that

$$(4.6) \quad \left\| f'(\tilde{U}) - \sum_{j=1}^k \varepsilon_j^{-1} e^{w_j} \right\|_{*,\partial\Omega} \leq C \lambda^{\frac{3}{2}},$$

and

$$(4.7) \quad \|f''(\tilde{U})\|_{*,\partial\Omega} \leq C.$$

From (3.21), (4.6) and (4.7), from the definition of $N(\phi)$ in (4.5), namely

$$(4.8) \quad N(\phi) := f(\tilde{U} + \phi) - f(\tilde{U}) - f'(\tilde{U})\phi + \left[f'(\tilde{U}) - \sum_{j=1}^k \varepsilon_j^{-1} e^{w_j} \right] \phi,$$

it follows that

$$\|A(\phi)\|_\infty \leq C \left(\lambda^{\frac{3}{2}} + \|\phi\|_\infty^2 + \lambda \|\phi\|_\infty \right).$$

We then get that $A(\mathcal{F}_\gamma) \subset \mathcal{F}_\gamma$ for a sufficiently large but fixed γ and all small λ . Moreover, for any $\phi_1, \phi_2 \in \mathcal{F}_\gamma$, one has

$$\|N(\phi_1) - N(\phi_2)\|_{*,\partial\Omega} \leq C \left[\left(\max_{i=1,2} \|\phi_i\|_\infty \right) + \lambda \right] \|\phi_1 - \phi_2\|_\infty.$$

In fact, using directly (4.8),

$$\begin{aligned} & N(\phi_1) - N(\phi_2) \\ &= f(\tilde{U} + \phi_1) - f(\tilde{U} + \phi_2) - f'(\tilde{U})(\phi_1 - \phi_2) + \left[f'(\tilde{U}) - \sum_{j=1}^k \varepsilon_j^{-1} e^{w_j} \right] (\phi_1 - \phi_2) \\ &= \int_0^1 \left(\frac{d}{dt} f(\tilde{U} + \phi_2 + t(\phi_1 - \phi_2)) \right) dt - f'(\tilde{U})(\phi_1 - \phi_2) + \left[f'(\tilde{U}) - \sum_{j=1}^k \varepsilon_j^{-1} e^{w_j} \right] (\phi_1 - \phi_2) \\ &= \int_0^1 \left(f'(\tilde{U} + \phi_2 + t(\phi_1 - \phi_2)) - f'(\tilde{U}) \right) dt (\phi_1 - \phi_2) + \left[f'(\tilde{U}) - \sum_{j=1}^k \varepsilon_j^{-1} e^{w_j} \right] (\phi_1 - \phi_2) \end{aligned}$$

Thus, for a certain $t^* \in (0, 1)$, and $s \in (0, 1)$

$$\begin{aligned} & |N(\phi_1) - N(\phi_2)| \\ & \leq C \left[|f'(\tilde{U} + \phi_2 + t^*(\phi_1 - \phi_2)) - f'(\tilde{U})| + \left(f'(\tilde{U}) - \sum_{j=1}^k \varepsilon_j^{-1} e^{w_j} \right) \right] \|\phi_1 - \phi_2\|_\infty \\ & \leq C \left[|f''(\tilde{U} + s\phi_2 + t^*(\phi_1 - \phi_2))| (\|\phi_1\|_{L^\infty(\Omega)} + \|\phi_2\|_\infty) \right. \\ & \quad \left. + |f'(\tilde{U}) - \sum_{j=1}^k \varepsilon_j^{-1} e^{w_j}| \right] \|\phi_1 - \phi_2\|_\infty. \end{aligned}$$

Thanks to (4.6), (4.7) and the fact that $\|\phi_1\|_\infty, \|\phi_2\|_\infty \rightarrow 0$ as $\lambda \rightarrow 0$, we conclude that

$$\|N(\phi_1) - N(\phi_2)\|_{*,\partial\Omega} \leq C [\|\phi_1\|_\infty + \|\phi_2\|_\infty + \lambda] \|\phi_1 - \phi_2\|_\infty.$$

Then we have

$$\|A(\phi_1) - A(\phi_2)\|_\infty \leq C \|N(\phi_1) - N(\phi_2)\|_{*,\partial\Omega} \leq C \left[\max_{i=1,2} \|\phi_i\|_\infty + \lambda \right] \|\phi_1 - \phi_2\|_\infty.$$

Thus the operator A has a small Lipschitz constant in \mathcal{F}_γ for all small λ , and therefore a unique fixed point of A exists in this region.

We shall next analyze the differentiability of the map $(\xi, m) = (\xi_1, \dots, \xi_k, m_1, \dots, m_k) \mapsto \phi$. Assume for instance that the partial derivative $\partial_{\xi_{sl}} \phi$ exists, for $s = 1, \dots, k, l = 1, 2$. Since $\phi = T_\lambda(N(\phi) + R)$, formally we have that

$$\partial_{\xi_{sl}} \phi = (\partial_{\xi_{sl}} T_\lambda)(N(\phi) + R) + T_\lambda(\partial_{\xi_{sl}} N(\phi) + \partial_{\xi_{sl}} R).$$

From (4.4), we have

$$\|\partial_{\xi_{sl}} T_\lambda(N(\phi) + R)\|_\infty \leq C \|N(\phi) + R\|_{*,\partial\Omega} \leq C \lambda^{\frac{3}{2}}.$$

On the other hand,

$$\begin{aligned} \partial_{\xi_{sl}} N(\phi) &= [f'(\tilde{U} + \phi) - f'(\tilde{U}) - f''(\tilde{U})\phi] \partial_{\xi_{sl}} \tilde{U} + \partial_{\xi_{sl}} \left(\frac{\partial Z_{ij}}{\partial v} - \left[\sum_{j=1}^k \varepsilon_j^{-1} e^{w_j} \right] \right) \phi \\ &\quad + [f'(\tilde{U} + \phi) - f'(\tilde{U})] \partial_{\xi_{sl}} \phi + \left(f'(\tilde{U}) - \left[\sum_{j=1}^k \varepsilon_j^{-1} e^{w_j} \right] \right) \partial_{\xi_{sl}} \phi. \end{aligned}$$

Then,

$$\|\partial_{\xi_{sl}} N(\phi)\|_{*,\partial\Omega} \leq C \left\{ \|\phi\|_\infty^2 + \lambda \|\phi\|_\infty + \|\phi\|_\infty \|\partial_{\xi_{sl}} \phi\|_\infty + \lambda \|\partial_{\xi_{sl}} \phi\|_\infty \right\}.$$

Since $\|\partial_{\xi_{sl}} R\|_{*,\partial\Omega} \leq \lambda^{\frac{3}{2}}$, Proposition 4.1 guarantees that

$$\|\partial_{\xi_{sl}} \phi\|_\infty \leq C \lambda^{\frac{3}{2}}$$

for all s, l . Analogous computation holds true if we differentiate with respect to m_j . Then, the regularity of the map $(\xi, m) \mapsto \phi$ can be proved by standard arguments involving the implicit function theorem and the fixed point representation (4.5). This concludes proof of the Proposition. \square

5. PROOFS OF PROPOSITION 3.2 AND OF PROPOSITION 3.3

5.1. Proof of Proposition 3.2.

Proof. Let us write

$$U(x) = \sum_{j=1}^k U_j(x), \quad \text{with } U_j(x) = \sqrt{\lambda} m_j [u_j(x) + H_j(x)]$$

where u_j and H_j are given by (3.11) and (3.12). We observe that U_j satisfies

$$(5.1) \quad \begin{cases} -\Delta U_j(x) + U_j(x) = 0 & \text{in } \Omega; \\ \frac{\partial U_j(x)}{\partial \nu} = 2\sqrt{\lambda} m_j \varepsilon_j \mu_j e^{u_j(x)} & \text{on } \partial\Omega. \end{cases}$$

We have

$$(5.2) \quad \begin{aligned} & \int_{\Omega} [|\nabla(U + \phi)|^2 + (U + \phi)^2] \\ &= \int_{\Omega} (|\nabla U|^2 + U^2) + \int_{\Omega} [2(\nabla U \nabla \phi + U\phi) + (|\nabla \phi|^2 + \phi^2)] := I_a + I_b. \end{aligned}$$

For I_a , we have

$$(5.3) \quad I_a = \sum_{j=1}^k \int_{\Omega} (|\nabla U_j|^2 + U_j^2) + \sum_{i \neq j} \int_{\Omega} (\nabla U_i \nabla U_j + U_i U_j) := I_{a,1} + I_{a,2}.$$

Multiplying (5.1) by U_j and integrating on Ω , by (3.13) we find

$$\begin{aligned} I_{a,1} &= \sum_{j=1}^k 2\sqrt{\lambda} m_j \varepsilon_j \mu_j \int_{\partial\Omega} e^{u_j(x)} U_j(x) = \sum_{j=1}^k 2\lambda m_j^2 \varepsilon_j \mu_j \int_{\partial\Omega} e^{u_j} (u_j + H_j) \\ &= \sum_{j=1}^k 2\lambda m_j^2 \int_{\partial\Omega} \frac{\varepsilon_j \mu_j}{|x - \xi_j - \varepsilon_j \mu_j \nu(\xi_j)|^2} \left(\log \frac{1}{|x - \xi_j - \varepsilon_j \mu_j \nu(\xi_j)|^2} + H(x, \xi_j) + O(\varepsilon_j^\sigma) \right) \\ &= \sum_{j=1}^k 2\lambda m_j^2 \int_{\partial\Omega_{\varepsilon_j \mu_j}} \frac{1}{|y - \nu(0)|^2} \left[\log \frac{1}{|y - \nu(0)|^2} + H(\xi_j, \xi_j) - 2 \log(\varepsilon_j \mu_j) + O(\varepsilon_j^\sigma) \right] \end{aligned}$$

where $\Omega_{\varepsilon_j \mu_j} = \frac{\Omega - \xi_j}{\varepsilon_j \mu_j}$. Using the following facts

$$\int_{\partial\Omega_{\varepsilon_j \mu_j}} \frac{1}{|y - \nu(0)|^2} = \pi + O(\varepsilon_j^\sigma), \quad \int_{\partial\Omega_{\varepsilon_j \mu_j}} \frac{1}{|y - \nu(0)|^2} \log \frac{1}{|y - \nu(0)|^2} = -2\pi \log 2 + O(\varepsilon_j^\sigma),$$

and the definition of ε_j given in (3.8), we obtain

$$(5.4) \quad \begin{aligned} I_{a,1} &= \sum_{j=1}^k 2\lambda m_j^2 \left[-2\pi \log 2 + \pi H(\xi_j, \xi_j) - 2\pi \log(\varepsilon_j \mu_j) + O(\varepsilon_j^\sigma) \right] \\ &= k\pi + 2\pi\lambda \sum_{j=1}^k m_j^2 \left[H(\xi_j, \xi_j) - 2 \log(2m_j^2) - 2 \log(2\mu_j) + O(\varepsilon_j^\sigma) \right]. \end{aligned}$$

Multiplying (5.1) by U_i and integrating on Ω , we find

$$\begin{aligned} I_{a,2} &= \sum_{i \neq j} \int_{\partial\Omega} 2\sqrt{\lambda} m_j \varepsilon_j \mu_j e^{\mu_j(x)} U_i(x) = 2 \sum_{i \neq j} \lambda m_i m_j \varepsilon_j \mu_j \int_{\partial\Omega} e^{\mu_j} (u_i + H_i) \\ &= 2 \sum_{i \neq j} \lambda m_i m_j \int_{\partial\Omega_{\varepsilon_j \mu_j}} \frac{1}{|y - \nu(0)|^2} \left[\log \frac{1}{|\xi_j - \xi_i + \varepsilon_j \mu_j y - \varepsilon_i \mu_i \nu(\xi_i)|^2} + H_i(\varepsilon_j \mu_j y + \xi_j) \right] \\ (5.5) \quad &= 2\pi\lambda \sum_{i \neq j} m_i m_j \left[G(\xi_i, \xi_j) + O\left(\varepsilon_i \log \frac{1}{\varepsilon_i} + \varepsilon_j \log \frac{1}{\varepsilon_j}\right) + O(\varepsilon_i^\sigma + \varepsilon_j^\sigma) \right]. \end{aligned}$$

Thus from (5.3), (5.4), (5.5) and the definition of μ_j given in (3.9) we get

$$(5.6) \quad \int_{\Omega} (|\nabla U|^2 + U^2) = k\pi \left\{ 1 + \lambda f_k(\xi, m) + \sum_{j=1}^k \varepsilon_j \log \frac{1}{\varepsilon_j} \Theta_\lambda(\xi, m) \right\}$$

where f_k is the function defined in (3.31) and $\Theta_\lambda(\xi, m)$ is a smooth function, uniformly bounded as $\lambda \rightarrow 0$, in the region for (ξ, m) satisfying (3.7). This is an estimate in the C^0 -sense. For C^1 -closeness, the derivatives in ξ and in m , by the same argument of C^0 -estimate, we have

$$(5.7) \quad D_\xi \left(\int_{\Omega} (|\nabla U|^2 + U^2) \right) = k\pi\lambda D_\xi (f_k(\xi, m)) + \sum_{j=1}^k \varepsilon_j \log \frac{1}{\varepsilon_j} \Theta_\lambda(\xi, m),$$

$$(5.8) \quad D_m \left(\int_{\Omega} (|\nabla U|^2 + U^2) \right) = k\pi\lambda D_m (f_k(\xi, m)) + \sum_{j=1}^k \varepsilon_j \log \frac{1}{\varepsilon_j} \Theta_\lambda(\xi, m),$$

where $\Theta(\xi, m)$ is uniformly bounded, as $\lambda \rightarrow 0$, in the region for (ξ, m) satisfying (3.7). From the choice of ε_j in (3.8), we note that $\varepsilon_j \log \frac{1}{\varepsilon_j} = o(\lambda^3)$.

On the other hand, for I_b given in (5.2). We have

$$I_b \leq 2 \left| \int_{\Omega} [\nabla(U + \phi)\nabla\phi + (U + \phi)\phi] \right|$$

Multiplying (3.25) by ϕ and integrating on Ω , we find

$$\int_{\Omega} [\nabla(U + \phi)\nabla\phi + (U + \phi)\phi] = \lambda \int_{\partial\Omega} (U + \phi) e^{(U+\phi)^2} \phi.$$

By (3.26) we have $\|\phi\|_\infty \leq C\lambda^{\frac{3}{2}}$ for some fixed constant C independent of λ , and using a Taylor expansion, we find

$$\lambda \int_{\partial\Omega} (U + \phi) e^{(U+\phi)^2} \phi \leq \lambda \|\phi\|_\infty \left| \int_{\partial\Omega} (U + \phi) e^{(U+\phi)^2} \right| \leq C\lambda^{\frac{5}{2}} \left| \int_{\partial\Omega} U e^{U^2} \right| + C\lambda^4.$$

Since, for some $\delta > 0$ small, we write

$$\int_{\partial\Omega} U e^{U^2} = \sum_{j=1}^k \int_{\partial\Omega \cap B(\xi_j, \delta \sqrt{\varepsilon_j})} U e^{U^2} + \int_{\partial\Omega \setminus \bigcup_{j=1}^k B(\xi_j, \delta \sqrt{\varepsilon_j})} U e^{U^2} := I_c + I_d,$$

where

$$\int_{\partial\Omega \cap B(\xi_j, \delta \sqrt{\varepsilon_j})} U e^{U^2} = \int_{\partial\Omega \cap B(\xi_j, \delta \varepsilon_j | \log \varepsilon_j |)} U e^{U^2} + \int_{\partial\Omega \cap (B(\xi_j, \delta \sqrt{\varepsilon_j}) \setminus B(\xi_j, \delta \varepsilon_j | \log \varepsilon_j |))} U e^{U^2} := I_{c,1} + I_{c,2}.$$

From (3.8) and (3.15), for x close to point ξ_j , we have $U = \sqrt{\lambda}m_j \left(w_j + \frac{1}{2\lambda m_j^2} + O(1) \right)$ and $e^{U^2} = 2m_j^2 \varepsilon_j^{-1} e^{w_j} (1 + O(\lambda))$, where w_j is defined in (3.16). Hence,

$$\begin{aligned} I_{c,1} &= 2\sqrt{\lambda}m_j^3 \varepsilon_j^{-1} \int_{\partial\Omega \cap B(\xi_j, \delta \varepsilon_j |\log \varepsilon_j|)} \left(w_j + \frac{1}{2\lambda m_j^2} + O(1) \right) e^{w_j} (1 + O(\lambda)) \\ &= 2\sqrt{\lambda}m_j^3 \int_{\frac{\partial\Omega - \xi_j}{\varepsilon_j \mu_j} \cap B(0, \frac{\delta |\log \varepsilon_j|}{\mu_j})} \left(\log \frac{2\mu_j^{-1}}{|y - \nu(0)|^2} + \frac{1}{2\lambda m_j^2} + O(1) \right) \frac{2}{|y - \nu(0)|^2} (1 + O(\lambda)). \end{aligned}$$

Moreover,

$$|I_{c,2}| \leq C\sqrt{\lambda} \int_{\partial \log \varepsilon_j}^{\delta \varepsilon_j^{-\frac{1}{2}}} \frac{1}{r^2} e^{\frac{\log^2 r}{\gamma_j^2}} r dr = C\sqrt{\lambda} \int_{R_1 + \log \gamma_j^2}^{R_2 + \frac{\gamma_j^2}{4}} e^{-2t + \frac{4t^2}{\gamma_j^2}} dt \leq C\sqrt{\lambda} \int_{R_1 + \log \gamma_j^2}^{R_2 + \frac{\gamma_j^2}{4}} e^{-t} dt = O(\lambda^{\frac{3}{2}}).$$

For I_d , since in the region $\partial\Omega \setminus \bigcup_{j=1}^k B(\xi_j, \delta \sqrt{\varepsilon_j})$, the function $U(x)$ satisfies $U(x) = \sqrt{\lambda} [\sum_{j=1}^k m_j G(x, \xi_j) + o(1)]$, with $o(1) \rightarrow 0$ as $\lambda \rightarrow 0$, we then have

$$\begin{aligned} I_d &= \int_{\partial\Omega \setminus \bigcup_{j=1}^k B(\xi_j, \delta \sqrt{\varepsilon_j})} U e^{U^2} = \sqrt{\lambda} \sum_{j=1}^k m_j \int_{\partial\Omega} G(x, \xi_j) \left[1 + \lambda \left(\sum_{j=1}^k m_j G(x, \xi_j) \right)^2 \right] (1 + o(1)) \\ &= \sqrt{\lambda} \sum_{j=1}^k m_j \int_{\partial\Omega} G(x, \xi_j) (1 + o(1)). \end{aligned}$$

Thanks to above facts, we obtain

$$(5.9) \quad I_b = \lambda^3 \Theta_\lambda(m, \xi)$$

with $\Theta_\lambda(m, \xi)$ is a function, uniformly bounded, in the region for (ξ, m) satisfying (3.7), as $\lambda \rightarrow 0$. Therefore, from (5.2), (5.6) and (5.9) we obtain that estimate (3.30) holds in the C^0 sense.

Next let us show the C^1 -closeness in estimate (3.30). From (3.25) and (3.27) we have

$$\begin{aligned} D_\xi \left(\int_{\Omega} (|\nabla(U + \phi)|^2 + (U + \phi)^2) \right) &= 2 \int_{\Omega} [\nabla(U + \phi) \nabla(\partial_\xi U + \partial_\xi \phi) + (U + \phi)(\partial_\xi U + \partial_\xi \phi)] \\ (5.10) \quad &= 2 \int_{\partial\Omega} \frac{\partial(U + \phi)}{\partial \nu} (\partial_\xi U + \partial_\xi \phi) = 2 \int_{\partial\Omega} \frac{\partial U}{\partial \nu} \partial_\xi U + \lambda^2 \Theta_\lambda(m, \xi) \end{aligned}$$

where $\Theta_\lambda(m, \xi)$ is a function, uniformly bounded, in the region for (ξ, m) satisfying (3.7), as $\lambda \rightarrow 0$, here we use the facts $\|\partial_\xi \phi\|_\infty \leq C\lambda^{\frac{3}{2}}$ and $\int_{\partial\Omega} \frac{\partial U}{\partial \nu} \leq C\sqrt{\lambda}$. On the other hand, we note that $-\Delta U + U = 0$ in Ω , hence

$$(5.11) \quad D_\xi \left(\int_{\Omega} (|\nabla U|^2 + U^2) \right) = 2 \int_{\Omega} [\nabla U \nabla \partial_\xi U + U \partial_\xi U] = 2 \int_{\partial\Omega} \frac{\partial U}{\partial \nu} \partial_\xi U.$$

From (5.7), (5.10) and (5.11), we obtain the C^1 -closeness in estimate (3.30)

$$(5.12) \quad D_\xi \left(\int_{\Omega} (|\nabla(U + \phi)|^2 + (U + \phi)^2) \right) = k\pi\lambda D_\xi(f_k(\xi, m)) + \lambda^2 \Theta_\lambda(\xi, m),$$

and by the same argument, we have

$$(5.13) \quad D_m \left(\int_{\Omega} (|\nabla(U + \phi)|^2 + (U + \phi)^2) \right) = k\pi\lambda D_m(f_k(\xi, m)) + \lambda^2 \Theta_{\lambda}(\xi, m),$$

where $\Theta_{\lambda}(m, \xi)$ is a function, uniformly bounded, in the region for (ξ, m) satisfying (3.7), as $\lambda \rightarrow 0$.

Finally, let us evaluate $\int_{\partial\Omega} e^{(U+\phi)^2}$. By a Taylor expansion, we find

$$(5.14) \quad \int_{\partial\Omega} e^{(U+\phi)^2} = \int_{\partial\Omega} e^{U^2} + \lambda^2 \Theta_{\lambda}(m, \xi).$$

We write

$$(5.15) \quad \int_{\partial\Omega} e^{U^2} = \sum_{j=1}^k \int_{\partial\Omega \cap B(\xi_j, \delta \sqrt{\varepsilon_j})} e^{U^2(x)} + \int_{\partial\Omega \setminus \bigcup_{j=1}^k B(\xi_j, \delta \sqrt{\varepsilon_j})} e^{U^2(x)} := I_e + I_f.$$

Since

$$\int_{\partial\Omega \cap B(\xi_j, \delta \sqrt{\varepsilon_j})} e^{U^2(x)} = \int_{\partial\Omega \cap B(\xi_j, \delta \varepsilon_j |\log \varepsilon_j|)} e^{U^2(x)} + \int_{\partial\Omega \cap (B(\xi_j, \delta \sqrt{\varepsilon_j}) \setminus B(\xi_j, \delta \varepsilon_j |\log \varepsilon_j|))} e^{U^2(x)} := I_{e,1} + I_{e,2}.$$

From (3.8), (3.9), (3.15) and definition of β_j , we have

$$(5.16) \quad \begin{aligned} I_{e,1} &= \int_{\partial\Omega \cap B(\xi_j, \delta \varepsilon_j |\log \varepsilon_j|)} e^{U^2(x)} = \varepsilon_j^{-1} e^{\frac{\beta_j}{2}} \int_{\partial\Omega \cap B(\xi_j, \delta \varepsilon_j |\log \varepsilon_j|)} e^{w_j} e^{\theta(x)} e^{\lambda m_j^2 [w_j^2 + 2w_j \theta(x) + \theta^2(x)]} \\ &= 2m_j^2 \int_{\frac{\partial\Omega - \xi_j}{\varepsilon_j \mu_j} \cap B(0, \frac{\delta |\log \varepsilon_j|}{\mu_j})} \frac{2}{|y - \nu(0)|^2} (1 + O(\lambda)) = 4\pi m_j^2 (1 + O(\lambda)), \end{aligned}$$

with $\Theta_{\lambda}(m, \xi)$ a function, uniformly bounded, in the region for (ξ, m) satisfying (3.7), as $\lambda \rightarrow 0$. Moreover,

$$(5.17) \quad |I_{e,2}| \leq C \int_{\delta |\log \varepsilon_j|}^{\delta \varepsilon_j^{-\frac{1}{2}}} \frac{1}{r^2} e^{\frac{\log^2 r}{\gamma_j}} r dr = C \int_{R_1 + \log \gamma_j^2}^{R_2 + \frac{\gamma_j^2}{4}} e^{-2t + \frac{4t^2}{\gamma_j^2}} dt \leq C \int_{R_1 + \log \gamma_j^2}^{R_2 + \frac{\gamma_j^2}{4}} e^{-t} dt = O(\lambda).$$

Furthermore, we have

$$(5.18) \quad \begin{aligned} I_f &= \int_{\partial\Omega \setminus \bigcup_{j=1}^k B(\xi_j, \delta \sqrt{\varepsilon_j})} e^{U^2} = \int_{\partial\Omega \setminus \bigcup_{j=1}^k B(\xi_j, \delta \sqrt{\varepsilon_j})} \left[1 + \lambda \sum_{j=1}^k m_j^2 G^2(x, \xi_j) \right] (1 + o(1)) \\ &= |\partial\Omega| + \lambda \sum_{j=1}^k m_j^2 \int_{\partial\Omega} G^2(x, \xi_j) + \lambda^2 \Theta_{\lambda}(m, \xi) \end{aligned}$$

with $|\partial\Omega|$ denotes the measure of domain $\partial\Omega$, and $\Theta_{\lambda}(m, \xi)$ is a function, uniformly bounded, in the region for (ξ, m) satisfying (3.7), as $\lambda \rightarrow 0$. Then from (5.14)-(5.18) we get that estimate (3.32) hold true in C^0 -sense.

On the other hand, by a Taylor expansion and the facts $\|\phi\|_{\infty} \leq C\lambda^{\frac{3}{2}}$ and $\int_{\partial\Omega} U \leq C\sqrt{\lambda}$, we have

$$D_{\xi} \left(\int_{\partial\Omega} e^{(U+\phi)^2} \right) = 2 \int_{\partial\Omega} e^{U^2} U \partial_{\xi} U + \lambda^2 \Theta_{\lambda}(m, \xi) = D_{\xi} \left(\int_{\partial\Omega} e^{U^2} \right) + \lambda^2 \Theta_{\lambda}(m, \xi),$$

and

$$D_m \left(\int_{\partial\Omega} e^{(U+\phi)^2} \right) = D_m \left(\int_{\partial\Omega} e^{U^2} \right) + \lambda^2 \Theta_\lambda(m, \xi)$$

with $\Theta_\lambda(m, \xi)$ is a function, uniformly bounded, in the region for (ξ, m) satisfying (3.7), as $\lambda \rightarrow 0$. Then we obtain that the C^1 -closeness in (3.32) by the same way as in the proof of C^1 -closeness in (3.30). \square

5.2. Proof of Proposition 3.3.

Proof. Define the set

$$R' = \{(\xi, m) \in R : f_k(\xi, m) \neq 0\}.$$

From Proposition 3.2, replacing expansion (3.30) into (3.33), we see that (3.33) gives

$$(5.19) \quad \lambda f_k(\xi, m) + \lambda^2 \Theta_\lambda(\xi, m) = \mu.$$

In R' , (5.19) defines λ as a function of μ, ξ and m , which is smooth in (ξ, m) in the region R' . Furthermore, as $\mu \rightarrow 0$,

$$\lambda = \frac{\mu}{f_k(\xi, m)} + \frac{\mu^2}{f_k^3(\xi, m)} \Theta_\mu(\xi, m)$$

with $\Theta_\mu(m, \xi)$ is a function, uniformly bounded with its derivatives, as $\mu \rightarrow 0$.

Assume now (3.33), we shall prove (3.34). Let us denote ∂ by the partial derivative with respect to m_j for any $j = 1, \dots, k$, or the partial derivative with respect to ξ_{j1} for $j = 1, \dots, k$. By a direct computation we have

$$J'(U + \phi) [\partial(U + \phi)] = \frac{1}{2} \partial \left(\int_{\Omega} (|\nabla(U + \phi)|^2 + (U + \phi)^2) \right) - \frac{\lambda}{2} \partial \left(\int_{\partial\Omega} e^{(U+\phi)^2} \right).$$

From (3.33) we have that $\partial \left(\int_{\Omega} (|\nabla(U + \phi)|^2 + (U + \phi)^2) \right) = 0$. Thus $\partial \left(\int_{\partial\Omega} e^{(U+\phi)^2} \right) = 0$ if and only if $J'(U + \phi) [\partial(U + \phi)] = 0$. Let us now rewrite

$$\frac{1}{\sqrt{\lambda}} (U + \phi)(\xi, m)(x) = m_l v_l \left(\frac{x - \xi_l}{\varepsilon_l} \right) + \frac{1}{2\lambda m_l}$$

for some $l = 1, \dots, k$, with

$$v_l(y) := w_{\mu_l}(y) + \sum_{j=1}^k \left(O(|\varepsilon_l y + \xi_l - \xi_j|) + O(\varepsilon_j^2) \right) \quad \text{for } |y| \leq \frac{\delta}{\varepsilon_l}.$$

Since $U + \phi$ is the solution of (3.25), then v_l satisfies

$$\begin{cases} -\Delta v_l + \varepsilon_l^2 \left(v_l + \frac{1}{2\lambda m_l^2} \right) = 0 & \text{in } \Omega_l; \\ \frac{\partial v_l}{\partial \nu} - (1 + 2\lambda m_l^2 v_l) e^{v_l} e^{\lambda m_l^2 v_l^2} = m_l^{-1} \varepsilon_l \sum_{i=0,1} \sum_{j=1}^k c_{ij} \varepsilon_j^{-1} \chi \left(\frac{F_j(\varepsilon_l y + \xi_l - \xi_j)}{\varepsilon_j} \right) z_{ij} \left(\frac{F_j(\varepsilon_l y + \xi_l - \xi_j)}{\varepsilon_j} \right) & \text{on } \partial\Omega_l, \end{cases}$$

where $\Omega_l = \frac{\Omega - \xi_l}{\varepsilon_l}$. For any l , we define

$$I_l(v_l) = \frac{1}{2} \int_{\Omega_l} \left[|\nabla v_l|^2 + \varepsilon_l^2 \left(v_l + \frac{1}{2\lambda m_l^2} \right)^2 \right] - \int_{\partial\Omega_l} e^{v_l} e^{\lambda m_l^2 v_l^2}.$$

We observe that

$$J'(U + \phi) [\partial(U + \phi)] = \lambda m_l^2 I'_l(v_l) [\partial v_l].$$

and

$$\begin{aligned} & \lambda m_l^2 I_l'(v_l)[\partial v_l] \\ &= \lambda m_l \varepsilon_l \sum_{i=0,1} \sum_{j=1}^k \left(\int_{\partial\Omega_l} \varepsilon_j^{-1} \chi \left(\frac{F_j(\varepsilon_l y + \xi_l - \xi_j)}{\varepsilon_j} \right) z_{ij} \left(\frac{F_j(\varepsilon_l y + \xi_l - \xi_j)}{\varepsilon_j} \right) \partial v_l dy \right) c_{ij}. \end{aligned}$$

Now, fix i and j , we compute the coefficient in front of c_{ij} , we choose $l = j$, $\partial v_l = D_{m_s} v_l(y)$, and obtain

$$\begin{aligned} & \int_{\partial\Omega_l} \varepsilon_j^{-1} \chi \left(\frac{F_j(\varepsilon_l y + \xi_l - \xi_j)}{\varepsilon_j} \right) z_{ij} \left(\frac{F_j(\varepsilon_l y + \xi_l - \xi_j)}{\varepsilon_j} \right) D_{m_s} v_l(y) dy \\ &= \int_{\partial\Omega_l} \varepsilon_j^{-1} \chi(y) z_{ij}(y) D_{m_s} \left[w_{\mu_j}(y) + \sum_{j=1}^k (O(|\varepsilon_j y|) + O(\varepsilon_j^2)) \right] dy \\ &= \frac{\partial \mu_j}{\partial m_s} \int_{\partial\mathbb{R}_+^2} z_{0j}^2(y) dy (1 + o(1)). \end{aligned}$$

Thus we concludes that for any $s = 1, 2, \dots, k$, we have

$$J'(U + \phi) [\partial_{m_s}(U + \phi)] = \lambda m_l \varepsilon_l \sum_{j=1}^k \frac{\partial \mu_j}{\partial m_s} \int_{\partial\mathbb{R}_+^2} z_{0j}^2(y) dy c_{0j} (1 + o(1)).$$

Similarly, we get that for all s, l

$$J'(U + \phi) [\partial_{\xi_{s1}}(U + \phi)] = \lambda m_l \varepsilon_l \left[\sum_{j=1}^k \left(\frac{\partial \mu_j}{\partial \xi_{s1}} \int_{\partial\mathbb{R}_+^2} z_{0j}^2(y) dy \right) c_{0j} + \left(\int_{\partial\mathbb{R}_+^2} z_{1s}^2(y) dy \right) c_{1s} \right] (1 + o(1)).$$

Thus, we can conclude that $J'(U + \phi) [\partial(U + \phi)] = 0$, that is $D_{\xi, m} E(U + \phi) = 0$ then we have the following system

$$(5.20) \quad \left[\sum_{j=1}^k \frac{\partial \mu_j}{\partial m_s} c_{0j} \right] (1 + o(1)) = 0, \quad s = 1, 2, \dots, k,$$

$$(5.21) \quad \left[A \sum_{j=1}^k \frac{\partial \mu_j}{\partial \xi_{s1}} c_{0j} + c_{1s} \right] (1 + o(1)) = 0, \quad \text{for all } s,$$

for some fixed constant A , with $o(1)$ small in the sense of the L^∞ norm as $\lambda \rightarrow 0$. Then (3.34) follows if we show that the matrix $\frac{\partial \mu_j}{\partial m_s}$ of dimension $k \times k$ is invertible in the region for (ξ, m) satisfying (3.7). Indeed, this fact implies unique solvability of (5.20). Inserting this in (5.21) we get unique solvability of (5.21).

Consider the definition of the μ_j , in terms of m'_j s and points ξ_j given in (3.7). These relations correspond to the gradient $D_m F(m, \xi)$ of the function $F(m, \xi)$ defined as follows

$$F(m, \xi) = \frac{1}{2} \sum_{j=1}^k m_j^2 \left[-2 \log(2m_j^2) - \log(2\mu_j) + 2 + H(\xi_j, \xi_j) \right] + \sum_{i \neq j} m_i m_j G(\xi_i, \xi_j).$$

We set $s_j = m_j^2$, then the above function can be written as follows

$$F(s, \xi) = \frac{1}{2} \sum_{j=1}^k s_j \left[-2 \log(2s_j) - \log(2\mu_j) + 2 + H(\xi_j, \xi_j) \right] + \sum_{i \neq j} \sqrt{s_i s_j} G(\xi_i, \xi_j).$$

This function is strictly convex function of the parameters s_j , for parameters s_j uniformly bounded and uniformly bounded away from 0 and for points ξ_j in Ω uniformly far away

from each other and from the boundary. For this reason, the matrix $(\frac{\partial^2 F}{\partial s_i \partial s_j})$ is invertible in the range of parameters and points we are considering. Thus, by the implicit function theorem, relation (3.9) defines a diffeomorphism between μ_j and m_j . This fact gives the invertibility of $(\frac{\partial \mu_j}{\partial m_s})$. Thus we finish the proof of Proposition 3.3. \square

6. APPENDIX

This section is devoted to the proof of Proposition 4.1. The proof of this result is based on the a-priori estimate for solutions to the following problem

$$(6.1) \quad \begin{cases} -\Delta\phi + \phi = f & \text{in } \Omega; \\ L(\phi) = h + \sum_{i=0,1} \sum_{j=1}^k c_{ij} \chi_j Z_{ij} & \text{on } \partial\Omega; \\ \int_{\Omega} \chi_j Z_{ij} \phi = 0 & \text{for } i = 0, 1, j = 1, \dots, k. \end{cases}$$

Define

$$(6.2) \quad \|f\|_{**,\Omega} := \sup_{x \in \Omega} \left(\sum_{j=1}^k \frac{\varepsilon_j^\sigma}{(1 + |x - \xi_j - \varepsilon_j \mu_j \nu(\xi_j)|)^{2+\sigma}} + 1 \right)^{-1} |f(x)|$$

where $0 < \sigma < 1$.

Lemma 6.1. *Under the assumptions of Proposition 4.1, if ϕ is a solution of (6.1) for some $h \in L^\infty(\partial\Omega)$ and for some $f \in L^\infty(\Omega)$ with $\|h\|_{*,\partial\Omega}, \|f\|_{**,\Omega} < \infty$ and $c_{ij} \in \mathbb{R}$, then*

$$(6.3) \quad \|\phi\|_\infty \leq C [\|h\|_{*,\partial\Omega} + \|f\|_{**,\Omega}],$$

$$|c_{ij}| \leq C (\|h\|_{*,\partial\Omega} + \|f\|_{**,\Omega}), \quad \forall i = 0, 1, j = 1, \dots, k$$

hold for C independent of λ .

Proof. We will carry out the proof of the a priori estimate (6.3) by contradiction. We assume then the existence of sequences $\lambda_n \rightarrow 0$, points $\xi_j^n \in \partial\Omega$ and numbers m_j^n, μ_j^n which satisfy relations (4.2) and (3.9), functions h_n, f_n with $\|h_n\|_{*,\partial\Omega}, \|f_n\|_{**,\Omega} \rightarrow 0$, ϕ_n with $\|\phi_n\|_\infty = 1$, constants $c_{ij,n}$,

$$(6.4) \quad -\Delta\phi_n + \phi_n = f_n, \quad \text{in } \Omega,$$

$$(6.5) \quad L(\phi_n) = h_n + \sum_{i=0}^2 \sum_{j=1}^k c_{ij,n} Z_{ij} \chi_j, \quad \text{on } \partial\Omega,$$

$$(6.6) \quad \int_{\Omega} Z_{ij} \chi_j \phi_n = 0, \quad \text{for all } i, j.$$

We will prove that in reality under the above assumption we must have that $\phi_n \rightarrow 0$ uniformly in $\bar{\Omega}$, which is a contradiction that concludes the result of the Lemma.

Passing to a subsequence we may assume that the points ξ_j^n approach limiting, distinct points ξ_j^* in $\partial\Omega$. We claim that $\phi_n \rightarrow 0$ in C^1 local sense on compacts of $\bar{\Omega} \setminus \{\xi_1^*, \dots, \xi_k^*\}$. Indeed, let us observe that $f_n \rightarrow 0$ locally uniformly in $\bar{\Omega}$, away from the points ξ_j . Away from the ξ_j^* 's we have then $-\Delta\phi_n + \phi_n \rightarrow 0$ uniformly on compact subsets on $\bar{\Omega} \setminus \{\xi_1^*, \dots, \xi_k^*\}$. Since ϕ_n is bounded it follows also that passing to a further subsequence, ϕ_n approaches in C^1 local sense on compacts of $\bar{\Omega} \setminus \{\xi_1^*, \dots, \xi_k^*\}$ a limit ϕ^* which is bounded and satisfies $-\Delta\phi^* + \phi^* = 0$ in $\Omega \setminus \{\xi_1^*, \dots, \xi_k^*\}$. Furthermore, observe that far from $\{\xi_1^*, \dots, \xi_k^*\}$, $h_n \rightarrow 0$ locally uniformly on $\partial\Omega \setminus \{\xi_1^*, \dots, \xi_k^*\}$ and so we also have $\frac{\partial \phi_n}{\partial \nu} \rightarrow 0$ on $\partial\Omega \setminus \{\xi_1^*, \dots, \xi_k^*\}$.

Hence ϕ^* extends smoothly to a function which satisfies $-\Delta\phi^* + \phi^* = 0$ in Ω , and $\frac{\partial\phi^*}{\partial\nu} = 0$ on $\partial\Omega$. We conclude that $\phi^* = 0$, and the claim follows.

For notational convenience, we shall omit the explicit dependence on n in the rest of the proof. We shall next show that

$$(6.7) \quad |c_{ij}| \leq C(\|\phi\|_\infty + \|h\|_{**,\partial\Omega} + \|f\|_{**,\Omega}).$$

Multiplying the first equation of (6.1) by Z_{ij} and integrating over $B(\xi_j, \delta)$, we find

$$(6.8) \quad \sum_{l=0,1} c_{lj} \int_{\partial\Omega \cap B(\xi_j, \delta)} \chi_j Z_{lj} Z_{ij} = - \int_{\partial\Omega \cap B(\xi_j, \delta)} h Z_{ij} + \int_{\partial\Omega \cap B(\xi_j, \delta)} L(Z_{ij})\phi - \int_{\Omega \cap \partial B(\xi_j, \delta)} \frac{\partial\phi}{\partial\nu} Z_{ij} \\ + \int_{\Omega \cap B(\xi_j, \delta)} (-\Delta Z_{ij} + Z_{ij})\phi - \int_{\Omega \cap B(\xi_j, \delta)} f Z_{ij}$$

Having in mind that $\phi_n \rightarrow 0$ in C^1 sense in $\Omega \cap \partial B(\xi_j, \delta)$, we have that $\int_{\Omega \cap \partial B(\xi_j, \delta)} \frac{\partial\phi}{\partial\nu} Z_{ij} \rightarrow 0$ as $\lambda \rightarrow 0$. Furthermore, a direct computation shows that

$$(6.9) \quad \int_{\partial\Omega \cap B(\xi_j, \delta)} \chi_j Z_{lj} Z_{ij} = M_l \delta_{li} + o(1), \quad \text{as } \lambda \rightarrow 0$$

where M_l is some universal constant and $\delta_{li} = 1$ if $i = l$, and $= 0$ if $i \neq l$. On the other hand, we have that

$$(6.10) \quad \int_{\partial\Omega \cap B(\xi_j, \delta)} \left(\frac{\partial Z_{ij}}{\partial\nu} - \left[\sum_{j=1}^k \varepsilon_j^{-1} e^{w_j} \right] Z_{ij} \right) \phi + \int_{\Omega \cap B(\xi_j, \delta)} (-\Delta Z_{ij} + Z_{ij})\phi \leq C\|\phi\|_\infty$$

and

$$(6.11) \quad \left| \int_{\Omega} f Z_{ij} \right| \leq C\|f\|_{**,\Omega}.$$

In fact, estimate (6.11) is a direct consequence of the definition of the $\|\cdot\|_{**,\Omega}$ -norm. Let us prove the validity of (6.10). Recall that in $\Omega \cap B(\xi_j, \delta)$, we have that $Z_{ij}(x) = z_{ij}(\varepsilon_j^{-1} F_j(x))$, where F_j is chosen to preserve area (see (3.23)). Performing the change of variables $y = \varepsilon_j^{-1} F_j(x)$, we get that

$$(6.12) \quad \int_{\Omega \cap B(\xi_j, \delta)} (-\Delta Z_{ij} + Z_{ij})\phi = (1 + o(1)) \int_{\mathbb{R}_+^2 \cap B(0, \frac{\delta}{\varepsilon_j})} (\mathcal{L}z_{ij} + \varepsilon_j^2 z_{ij}) \tilde{\phi}$$

where $\tilde{\phi}(y) = \phi(F_j^{-1}(\varepsilon_j y))$ and \mathcal{L} is a second order differential operator defined as follows

$$(6.13) \quad \mathcal{L} = -\Delta + O(\varepsilon_j |y|) \nabla^2 + O(\varepsilon_j) \nabla, \quad \text{in } \mathbb{R}_+^2 \cap B(0, \frac{\delta}{\varepsilon_j}).$$

Hence

$$\left| \int_{\Omega \cap B(\xi_j, \delta)} (-\Delta Z_{ij} + Z_{ij})\phi \right| \leq C\|\phi\|_\infty.$$

On the other hand, we observe that, after a possible rotation, we can assume that $\nabla F_j(\xi_j) = I$. Hence, using again the change of variables $y = \varepsilon_j^{-1} F_j(x)$, we get

$$(6.14) \quad \int_{\partial\Omega \cap B(\xi_j, \delta)} L(Z_{ij})\phi = (1 + o(1)) \int_{\partial\mathbb{R}_+^2 \cap B(0, \frac{\delta}{\varepsilon_j})} (B(z_{ij}) - \tilde{W}z_{ij})b(y)\tilde{\phi}$$

where $\tilde{W}(y) = \varepsilon_j W(F_j^{-1}(\varepsilon_j y))$ with $W(x) = \sum_{j=1}^k \varepsilon_j^{-1} e^{w_j}$, and $b(y)$ is a positive function, coming from the change of variables, which is uniformly positive and bounded as $\lambda \rightarrow 0$. Furthermore B is a differential operator of order one on $\partial\mathbb{R}_+^2$. In fact, we have that

$$(6.15) \quad B = -\frac{\partial}{\partial y_2} + O(\varepsilon_j |y|) \nabla \quad \text{on} \quad \partial\mathbb{R}_+^2 \cap B(0, \frac{\delta}{\varepsilon_j})$$

On the other hand, since

$$W(x) = \varepsilon_j^{-1} \frac{2\mu_j \varepsilon_j^2}{|x - \xi_j - \varepsilon_j \mu_j \nu(\xi_j)|^2} \left(1 + \sum_{l \neq j} \varepsilon_l \varepsilon_j O(1) \right)$$

we get

$$(6.16) \quad \tilde{W}(y) = \frac{2\mu_j}{y_1^2 + \mu_j^2} + \sum_l \frac{\varepsilon_l^\alpha}{(1 + |y|)} \quad \text{on} \quad \partial\mathbb{R}_+^2 \cap B(0, \frac{\delta}{\varepsilon_j}),$$

for some $0 < \alpha < 1$. Thus we can conclude that

$$\left| \int_{\partial\Omega \cap B(\xi_j, \delta)} L(Z_{ij}) \phi \right| \leq C \|\phi\|_\infty.$$

This shows the validity of (6.10).

We shall now estimate the term $\int_{\partial\Omega} h Z_{ij}$. Using the definition of the $\|\cdot\|_{*,\partial\Omega}$ -norm, we observe that

$$(6.17) \quad \begin{aligned} \left| \int_{\partial\Omega} h Z_{ij} \right| &= \int_{\partial\Omega} \rho(x)^{-1} |h| \rho(x) Z_{ij} \leq \|h\|_{*,\partial\Omega} \int_{\partial\Omega} \rho(x) Z_{ij} \\ &= \|h\|_{*,\partial\Omega} \int_{\partial\Omega} \left(\sum_{l=1}^k \rho_l \chi_{B_\delta(\xi_l)}(x) + 1 \right) Z_{ij} \\ &\leq C \|h\|_{*,\partial\Omega} \sum_{l=1}^k \int_{\partial\Omega \cap B_\delta(\xi_l)} \gamma_l \left\{ \left(1 + \frac{w_l + 1}{\gamma_l} \right) \left(1 + \frac{1 + |w_l|}{\gamma_l} \right) e^{\frac{w_l^2}{2\gamma_l}} - 1 \right\} \varepsilon_l^{-1} e^{w_l} \\ &\quad + C \|h\|_{*,\partial\Omega} \int_{\partial\Omega \setminus \bigcup_{l=1}^k B_\delta(\xi_l)} Z_{ij}. \end{aligned}$$

Since Z_{ij} are uniformly bounded, as $\lambda \rightarrow 0$, in $\partial\Omega \setminus \bigcup_{l=1}^k B_\delta(\xi_l)$, we just need to estimate

$\int_{\partial\Omega \cap B_\delta(\xi_j)} \gamma_j \left\{ \left(1 + \frac{w_j + 1}{\gamma_j} \right) \left(1 + \frac{1 + |w_j|}{\gamma_j} \right) e^{\frac{w_j^2}{2\gamma_j}} - 1 \right\} \varepsilon_j^{-1} e^{w_j}$. Recall that the functions w_j are defined as

$$w_j(x) = \log \frac{2\mu_j}{|y - \xi_j - \mu_j \nu(\xi_j')|^2},$$

with $y = \frac{x}{\varepsilon_j}$, $\xi'_j = \frac{\xi_j}{\varepsilon_j}$, and $\gamma_j = -2 \log \varepsilon_j$. We decompose $\partial\Omega \cap B_\delta(\xi_j)$ into the union of $\partial\Omega \cap B_{\frac{\delta}{\gamma_j}}(\xi_j)$ and $\partial\Omega \cap (B_\delta(\xi_j) \setminus B_{\frac{\delta}{\gamma_j}}(\xi_j))$. We write

$$\begin{aligned}
& \int_{\partial\Omega \cap B_\delta(\xi_j)} \gamma_j \left\{ \left(1 + \frac{w_j + 1}{\gamma_j}\right) \left(1 + \frac{1 + |w_j|}{\gamma_j}\right) e^{\frac{w_j^2}{2\gamma_j}} - 1 \right\} \varepsilon_j^{-1} e^{w_j} \\
&= \int_{\partial\Omega \cap B_{\frac{\delta}{\gamma_j}}(\xi_j)} \gamma_j \left\{ \left(1 + \frac{w_j + 1}{\gamma_j}\right) \left(1 + \frac{1 + |w_j|}{\gamma_j}\right) e^{\frac{w_j^2}{2\gamma_j}} - 1 \right\} \varepsilon_j^{-1} e^{w_j} \\
&\quad + \int_{\partial\Omega \cap (B_\delta(\xi_j) \setminus B_{\frac{\delta}{\gamma_j}}(\xi_j))} \gamma_j \left\{ \left(1 + \frac{w_j + 1}{\gamma_j}\right) \left(1 + \frac{1 + |w_j|}{\gamma_j}\right) e^{\frac{w_j^2}{2\gamma_j}} - 1 \right\} \varepsilon_j^{-1} e^{w_j} \\
(6.18) \quad &= L_1 + L_2.
\end{aligned}$$

Using the change of variables $\varepsilon_j y = x - \xi_j$, we have

$$L_1 = \int_{\partial\Omega_{\varepsilon_j} \cap B_{\frac{\delta}{\gamma_j \varepsilon_j}}(0)} \gamma_j \left\{ \left(1 + \frac{\bar{w}_j + 1}{\gamma_j}\right) \left(1 + \frac{1 + |\bar{w}_j|}{\gamma_j}\right) e^{\frac{\bar{w}_j^2}{2\gamma_j}} - 1 \right\} e^{\bar{w}_j}$$

and

$$L_2 = \int_{\partial\Omega_{\varepsilon_j} \cap (B_{\frac{\delta}{\varepsilon_j}}(0) \setminus B_{\frac{\delta}{\gamma_j \varepsilon_j}}(0))} \gamma_j \left\{ \left(1 + \frac{\bar{w}_j + 1}{\gamma_j}\right) \left(1 + \frac{1 + |\bar{w}_j|}{\gamma_j}\right) e^{\frac{\bar{w}_j^2}{2\gamma_j}} - 1 \right\} e^{\bar{w}_j}$$

where $\Omega_{\varepsilon_j} = \frac{\Omega - \xi_j}{\varepsilon_j}$ and

$$\bar{w}_j = \log \frac{2\mu_j}{|y - \mu_j \nu(0)|^2}.$$

First we estimate L_1 :

$$\begin{aligned}
L_1 &= \int_{\partial\Omega_{\varepsilon_j} \cap B_{\frac{\delta}{\gamma_j \varepsilon_j}}(0)} \gamma_j \left\{ \left(1 + \frac{\bar{w}_j + 1}{\gamma_j}\right) \left(1 + \frac{1 + |\bar{w}_j|}{\gamma_j}\right) e^{\frac{\bar{w}_j^2}{2\gamma_j}} - 1 \right\} e^{\bar{w}_j} \\
&\leq C \int_{\partial\Omega_{\varepsilon_j} \cap B_{\frac{\delta}{\gamma_j \varepsilon_j}}(0)} e^{\bar{w}_j} = C \int_{\partial\Omega_{\varepsilon_j} \cap B_{\frac{\delta}{\gamma_j \varepsilon_j}}(0)} \frac{1}{|y - \mu_j \nu(0)|^2} \\
&\leq C \int_{\mu_j - \frac{\delta}{\gamma_j \varepsilon_j}}^{\mu_j} \frac{1}{r^2} dr \leq C.
\end{aligned}$$

On the other hand, using the fact that $\bar{w}_j = -2 \log r + O(1)$ with $r = |y - \mu_j \nu(0)|$, the term L_2 can be estimated as follows

$$\begin{aligned} L_2 &= \int_{\partial\Omega_{\varepsilon_j} \cap \left(B_{\frac{\delta}{\varepsilon_j}}(0) \setminus B_{\frac{\delta}{\gamma_j \varepsilon_j}}(0) \right)} \gamma_j \left\{ \left(1 + \frac{\bar{w}_j + 1}{\gamma_j} \right) \left(1 + \frac{1 + |\bar{w}_j|}{\gamma_j} \right) e^{\frac{\bar{w}_j^2}{2\gamma_j}} - 1 \right\} e^{\bar{w}_j} \\ &\leq C \int_{\partial\Omega_{\varepsilon_j} \cap \left(B_{\frac{\delta}{\varepsilon_j}}(0) \setminus B_{\frac{\delta}{\gamma_j \varepsilon_j}}(0) \right)} \gamma_j e^{\frac{\bar{w}_j^2}{2\gamma_j}} \frac{\gamma_j + \bar{w}_j}{\gamma_j} e^{\bar{w}_j} \leq C \int_{\frac{\delta}{\gamma_j \varepsilon_j}}^{\frac{\delta}{\varepsilon_j}} \frac{1}{r^2} e^{\frac{(\log r)^2}{|\log \varepsilon_j|}} (\gamma_j - 2 \log r) dr \\ &\leq C \int_{\log \frac{\delta}{\gamma_j \varepsilon_j}}^{\log \frac{\delta}{\varepsilon_j}} e^{-t} e^{\frac{t^2}{|\log \varepsilon_j|}} (\gamma_j - t) dt \leq C \int_{\log \frac{\delta}{\gamma_j \varepsilon_j}}^{\log \frac{\delta}{\varepsilon_j}} e^{-\sigma t} (\gamma_j - t) dt \leq C \end{aligned}$$

for some positive σ . Therefore we get

$$(6.19) \quad \left| \int_{\partial\Omega} h Z_{ij} \right| \leq C \|h\|_{*, \partial\Omega}.$$

Thus, from (6.8)-(6.19) we find the validity of (6.7).

We now conclude our argument by contradiction to prove (6.3). From (6.7), we have that $c_{ij,n}$ is bounded, thus we may assume that $c_{ij,n} \rightarrow c_{ij}$ as $n \rightarrow \infty$.

Let us fix $R > 0$ large sufficiently but fixed. By the maximum principle and the Hopf Lemma we find that,

$$\max_{\Omega \setminus \bigcup_{j=1}^k B_{R\varepsilon_j}(\xi_{j,n})} |\phi_n| = \max_{\Omega \setminus \bigcup_{j=1}^k \partial B_{R\varepsilon_j}(\xi_{j,n})} |\phi_n|.$$

Thus, from $\|\phi_n\|_\infty = 1$, we can find that there is some fixed $j_0 \in \{1, 2, \dots, k\}$ such that

$$(6.20) \quad \max_{\bar{\Omega} \cap \partial B_{R\varepsilon_{j_0}}(\xi_{j_0,n})} |\phi_n| = 1.$$

Set $\Omega_{\varepsilon_{j_0}} = \frac{\Omega - \xi_{j_0,n}}{\varepsilon_{j_0,n}}$, and consider

$$\begin{aligned} \hat{\phi}_n(z) &= \phi_n(\xi_{j_0,n} + \varepsilon_{j_0,n} z), & \hat{h}_n(z) &= h_n(\xi_{j_0,n} + \varepsilon_{j_0,n} z), \\ \hat{f}_n(z) &= f_n(\xi_{j_0,n} + \varepsilon_{j_0,n} z), & \hat{Z}_{ij}(z) &= Z_{ij}(\xi_{j_0,n} + \varepsilon_{j_0,n} z) \end{aligned}$$

Then

$$\begin{aligned} -\Delta \hat{\phi}_n(z) + \varepsilon_{j_0}^2 \hat{\phi}_n(z) &= \varepsilon_{j_0}^2 f_n(z) \quad \text{in } \Omega_{\varepsilon_{j_0}}, \\ \frac{\partial \hat{\phi}_n}{\partial \nu} - \varepsilon_{j_0} \left[\sum_{j=1}^k \varepsilon_j^{-1} e^{w_j} \right] \hat{\phi}_n &= \varepsilon_{j_0} \hat{h}_n + \sum_{i=0,1} \sum_{j=1}^k \varepsilon_{j_0} c_{ij,n} \chi_j \hat{Z}_{ij} \quad \text{on } \partial\Omega_{\varepsilon_{j_0}}. \end{aligned}$$

Then by elliptic estimate $\hat{\phi}_n$ (up to subsequence) converges uniformly on compact sets to a nontrivial solution $\hat{\phi} \neq 0$ of the problem

$$\begin{cases} \Delta \phi = 0, & \text{in } \mathbb{R}_+^2; \\ \frac{\partial \phi}{\partial \nu} - \frac{2\mu_j}{x_1^2 + \mu_j^2} \phi = 0 & \text{on } \partial\mathbb{R}_+^2. \end{cases}$$

By the nondegeneracy result ([9]), we conclude that $\hat{\phi}$ is a linear combination of z_{0j} and z_{1j} . On the other hand, we can take the limit in the orthogonality relation and we find that $\int_{\partial\mathbb{R}_+^2} \chi \hat{\phi} z_{ij} = 0$ for $i = 0, 1$. This contradicts the fact that $\hat{\phi} \neq 0$. This ends the proof of the Lemma. \square

Proof of Proposition 4.1 In proving the solvability of (4.1), we may first solve the following problem: for given $h \in L^\infty(\partial\Omega)$, with $\|h\|_{*,\partial\Omega}$ bounded, find $\phi \in L^\infty(\Omega)$ and $d_{ij} \in \mathbb{R}$, $i = 0, 1$, $j = 1, \dots, k$ such that

$$(6.21) \quad \begin{cases} -\Delta\phi + \phi = \sum_{i=0,1} \sum_{j=1}^k d_{ij} \chi_j Z_{ij} & \text{in } \Omega; \\ \frac{\partial\phi}{\partial\nu} - [\sum_{j=1}^k \varepsilon_j^{-1} e^{w_j}] \phi = h & \text{on } \partial\Omega; \\ \int_{\Omega} \chi_j Z_{ij} \phi = 0 & \text{for } i = 0, 1, j = 1, \dots, k. \end{cases}$$

First we prove that for any ϕ , d_{ij} solution to (6.21) the bound

$$(6.22) \quad \|\phi\|_\infty \leq C \|h\|_{*,\partial\Omega}$$

holds. In fact, by Lemma 6.1, we have

$$(6.23) \quad \|\phi\|_\infty \leq C \left(\|h\|_{*,\partial\Omega} + \sum_{i=0,1} \sum_{j=1}^k \varepsilon_j |d_{ij}| \right)$$

and therefore it is enough to prove that $\varepsilon_j |d_{ij}| \leq C \|h\|_{*,\partial\Omega}$.

Fix an integer j . To show that $\varepsilon_j |d_{ij}| \leq C \|h\|_{*,\partial\Omega}$, we shall multiply equation (6.21) against a test function, properly chosen. Let us observe that, the proper test function depends whether we are considering the case $i = 0$ or $i = 1$. We start with $i = 0$. We define $\hat{z}_{0j}(y) = h(y) z_{0j}(y)$, where $h(y) = \frac{\log(\frac{\delta}{\varepsilon_j}) - \log|y|}{\log\frac{\delta}{\varepsilon_j} - \log R}$. In fact, we recognize that $\Delta h = 0$ in $B(0, \frac{\delta}{\varepsilon_j}) \setminus B(0, R)$, $h = 1$ on $\partial B(0, R)$ and $h = 0$ on $\partial B(0, \frac{\delta}{\varepsilon_j})$.

Let η_1 and η_2 be two smooth cut-off functions defined in \mathbb{R}^2 as

$$\eta_1 \equiv 1 \quad \text{in } B(0, R), \quad \equiv 0 \quad \text{in } \mathbb{R}^2 \setminus B(0, R+1)$$

so that

$$0 \leq \eta_1 \leq 1, \quad |\nabla \eta_1| \leq C$$

and

$$\eta_2 \equiv 1 \quad \text{in } B(0, \frac{\delta}{4\varepsilon_j}), \quad \equiv 0 \quad \text{in } \mathbb{R}^2 \setminus B(0, \frac{\delta}{3\varepsilon_j})$$

so that

$$0 \leq \eta_2 \leq 1, \quad |\nabla \eta_2| \leq C \frac{\varepsilon_j}{\delta}, \quad |\nabla^2 \eta_2| \leq C \left(\frac{\varepsilon_j}{\delta}\right)^2.$$

We assume that $R > R_0$ (see (3.24)) and we define

$$(6.24) \quad \tilde{Z}_{0j}(x) = \eta_1(\varepsilon_j^{-1} F_j(x)) Z_{0j}(x) + (1 - \eta_1(\varepsilon_j^{-1} F_j(x))) \eta_2(\varepsilon_j^{-1} F_j(x)) \hat{z}_{0j}(\varepsilon_j^{-1} F_j(x)),$$

for $x \in B(\xi_j, \delta) \cap \Omega$.

We multiply equation (6.21) against \tilde{Z}_{0j} and we integrate by parts. We get

$$\sum_{a=0,1} d_{aj} \int_{\Omega} \chi_j Z_{aj} \tilde{Z}_{0j} = \int_{\Omega} (-\Delta \tilde{Z}_{0j} + \tilde{Z}_{0j}) \phi + \int_{\partial\Omega} h \tilde{Z}_{0j} + \int_{\partial\Omega} L(\tilde{Z}_{0j}) \phi$$

Observe first that, assuming $R > R_0$, we have

$$(6.25) \quad d_{aj} \int_{\Omega} \chi_j Z_{aj} \tilde{Z}_{0j} = d_{aj} \int_{\Omega} \chi_j Z_{aj} Z_{0j} = \varepsilon_j M_0 \delta_{a0} d_{aj} (1 + o(1)), \quad \text{as } \lambda \rightarrow 0.$$

Furthermore we have that

$$(6.26) \quad \left| \int_{\partial\Omega} h \tilde{Z}_{0j} \right| \leq C \|h\|_{*,\partial\Omega}.$$

We claim that

$$(6.27) \quad \| -\Delta \tilde{Z}_{0j} + \tilde{Z}_{0j} \|_{**,\Omega} \leq \frac{C}{|\log \varepsilon_j|},$$

$$(6.28) \quad \|L(\tilde{Z}_{0j})\|_{*,\partial\Omega} \leq \frac{C}{|\log \varepsilon_j|}.$$

The proof of estimates (6.27) and (6.28) is postponed to the end of the Appendix. Assuming for the moment the validity of (6.27) and (6.28), from estimates (6.25)–(6.28) we conclude that

$$(6.29) \quad |\varepsilon_j d_{0j}| \leq C \left(\|h\|_{*,\partial\Omega} + |\log \varepsilon_j|^{-1} \|\phi\|_\infty \right).$$

We shall now obtain an estimate similar to (6.29) for $\varepsilon_j d_{1j}$. To do so, we use another test function. Indeed we multiply equation (6.21) against $\eta_2 Z_{1j}$ and we integrate by parts. We get

$$\begin{aligned} \sum_{a=0,1} d_{aj} \int_{\Omega} \chi_j Z_{aj} \eta_2 Z_{1j} &= \int_{\Omega} (-\Delta(\eta_2 Z_{1j}) + \eta_2 Z_{1j}) \phi - \int_{\partial\Omega} h \eta_2 Z_{1,j} \\ &\quad + \int_{\partial\Omega} L(Z_{1j}) \eta_2 \phi + \int_{\partial\Omega} Z_{1j} \frac{\partial \eta_2}{\partial \nu} \phi \end{aligned}$$

Observe first that, assuming $R > R_0$, we have

$$d_{aj} \int_{\Omega} \chi_j Z_{aj} \eta_2 Z_{1j} = d_{aj} \int_{\Omega} \chi_j Z_{aj} Z_{1j} = M_1 \delta_{a1} \varepsilon_j d_{1j} (1 + o(1)), \quad \text{as } \lambda \rightarrow 0,$$

and $|\int_{\partial\Omega} h \eta_2 Z_{1j}| \leq C \|h\|_{*,\partial\Omega}$. Using the change of variables $y = \varepsilon_j^{-1} F_j(x)$, we get that

$$\int_{\partial\Omega} Z_{1j} \frac{\partial \eta_2}{\partial \nu} \phi = \int_{\partial\Omega_{\varepsilon_j}} z_{1j} \frac{\partial \eta_2}{\partial \nu} \tilde{\phi}$$

where $\Omega_{\varepsilon_j} = \frac{\Omega}{\varepsilon_j}$ and $\tilde{\phi}(y) = \phi(F_j^{-1}(\varepsilon_j^{-1}y))$. But $z_{1j} = O(\frac{1}{1+r})$ and $\nabla \eta_2 = O(\varepsilon_j)$ so $|\int_{\partial\Omega} Z_{1j} \frac{\partial \eta_2}{\partial \nu} \phi| \leq C \varepsilon_j |\log \varepsilon_j|$. Using again the change of variables $y = \varepsilon_j^{-1} F_j(x)$, and proceeding similarly to (6.14), (6.15) and (6.16), one gets

$$\int_{\partial\Omega} L(Z_{ij}) \eta_2 \phi = (1 + o(1)) \int_{\partial\Omega_{\varepsilon_j}} \left[\frac{\partial z_{ij}}{\partial \nu} - \tilde{W} z_{ij} \right] \eta_2 \tilde{\phi}$$

where $\tilde{\phi}(y) = \phi(F_j^{-1}(\varepsilon_j y))$ and $b(y)$ is a positive function, coming from the change of variables, which is uniformly positive and bounded as $\lambda \rightarrow 0$. Observe that $\frac{\partial z_{ij}}{\partial \nu} - \tilde{W} z_{ij} = O(\frac{\varepsilon_j}{1+r}) + O(\frac{\varepsilon_j^\alpha}{1+r^2})$ for $y \in \Omega_{\varepsilon_j}$ and $|y| < \delta \varepsilon_j^{-1}$, and this implies that

$$\int_{\partial\Omega_{\varepsilon_j}} \left| \frac{\partial z_{ij}}{\partial \nu} - \tilde{W} z_{ij} \right| \leq C \varepsilon_j^\alpha$$

for some $0 < \alpha < 1$. Thus we can conclude that

$$\left| \int_{\partial\Omega} L(Z_{ij}) \eta_2 \phi \right| \leq C \varepsilon_j^\alpha \|\phi\|_\infty.$$

Consider once again the change of variables $y = \varepsilon_j^{-1} F_j(x)$. Arguing as in (6.12) and (6.13) we get that

$$\int_{\Omega} (-\Delta(\eta_2 Z_{ij}) + \eta_2 Z_{ij}) \phi = (1 + o(1)) \int_{\Omega_{\varepsilon_j}} (-\Delta(\eta_2 z_{ij}) + \varepsilon_j^2 \eta_2 z_{ij}) \tilde{\phi}$$

where $\tilde{\phi}(y) = \phi(F_j^{-1}(\varepsilon_j y))$. We thus compute in $y \in \Omega_{\varepsilon_j}$, with $|y| < \delta \varepsilon_j^{-1}$,

$$\Delta(\eta_2 z_{1j}) = \Delta \eta_2 z_{1j} + 2\nabla \eta_2 \nabla z_{1j} + \eta_2 \Delta z_{1j} = O\left(\frac{\varepsilon_j^2}{1+r}\right) + O\left(\frac{\varepsilon_j}{1+r}\right) + \eta_2 \Delta z_{1j}.$$

On the other hand, in this region we have $-\Delta z_{1j} + \varepsilon_j^2 z_{1j} = O\left(\frac{\varepsilon_j}{1+r^2}\right) + O\left(\frac{\varepsilon_j^2}{1+r}\right)$. Thus

$$\int_{\Omega_{\varepsilon_j}} |-\Delta(\eta_2 z_{1j}) + \varepsilon_j^2 \eta_2 z_{1j}| \leq C \varepsilon_j |\log \varepsilon_j|$$

Summarizing all the above information, we get

$$(6.30) \quad |\varepsilon_j d_{1j}| \leq C (\|h\|_{*,\partial\Omega} + \varepsilon_j \|\phi\|_{\infty})$$

Estimates (6.29), (6.30) combined with (6.23) yields

$$|\varepsilon_j d_{ij}| \leq C \|h\|_{*,\partial\Omega}.$$

which gives the validity of (6.22). Now consider the Hilbert space

$$\mathbb{H} = \left\{ \phi \in H^1(\Omega) : \int_{\Omega} \chi_j Z_{ij} \phi = 0 \quad \forall i = 0, 1, \quad j = 1, \dots, k \right\},$$

endowed the norm $\|\phi\|_{\mathbb{H}^1}^2 = \int_{\Omega} (|\nabla \phi|^2 + \phi^2)$. Problem (6.21), expressed in a weak form, is equivalent to find $\phi \in \mathbb{H}$ such that

$$\int_{\Omega} (\nabla \phi \nabla \psi + \phi \psi) - \int_{\partial\Omega} \left[\sum_{j=1}^k \varepsilon_j^{-1} e^{w_j} \right] \psi = \int_{\partial\Omega} h \psi, \quad \text{for all } \psi \in \mathbb{H},$$

With the aid of Fredholm's alternative guarantees unique solvability of (6.21), which is guaranteed by (6.22).

In order to solve (4.1), let $Y_{ls} \in L^\infty(\Omega_\varepsilon)$, $d_{ij}^{ls} \in \mathbb{R}$ be the solution of (6.21) with $h = \chi_s Z_{ls}$, that is

$$(6.31) \quad \begin{cases} -\Delta Y_{ls} + Y_{ls} = \sum_{i=0,1} \sum_{j=1}^k d_{ij}^{ls} \chi_j Z_{ij} & \text{in } \Omega; \\ \frac{\partial Y_{ls}}{\partial \nu} - \left[\sum_{j=1}^k \varepsilon_j^{-1} e^{w_j} \right] Y_{ls} = \chi_s Z_{ls} & \text{on } \partial\Omega; \\ \int_{\Omega} \chi_j Z_{ij} Y_{ls} = 0 & \text{for } l = 0, 1, \quad s = 1, \dots, k, \end{cases}$$

Then there is a unique solution $Y_{ls} \in L^\infty(\Omega)$ of (6.31), and

$$(6.32) \quad \|Y_{ls}\|_{\infty} \leq C, \quad \varepsilon_j |d_{ij}^{ls}| \leq C$$

for some constant C independent on λ .

Multiplying (6.31) by Z_{ij} , and integrates by parts, we have

$$\begin{aligned} \sum_{i=0,1} \sum_{j=1}^k \int_{B(\xi_j, \delta)} d_{ij,ls} \chi_j (Z_{ij})^2 &= \int_{\partial B(\xi_j, \delta)} \chi_s Z_{ls} Z_{ij} + \int_{B(\xi_j, \delta)} (-\Delta Z_{ij} + Z_{ij}) Y_{ls} \\ &\quad + \int_{\partial B(\xi_j, \delta)} \left(\frac{\partial Z_{ij}}{\partial \nu} - \left[\sum_{j=1}^k \varepsilon_j^{-1} e^{w_j} \right] Z_{ij} \right) Y_{ls} \\ &= \delta_{il} \delta_{js} \int_{\partial B(\xi_j, \delta)} \chi_j (Z_{ij})^2 + o(1) \end{aligned}$$

where δ_{il}, δ_{js} are Kronecker's delta. Then we get

$$(6.33) \quad d_{0j,0s} = a \delta_{js} + o(1), \quad d_{1j,1s} = b \delta_{js} + o(1)$$

with $a, b > 0$ are independent of ε_j . Hence the matrix D_1 (or D_2) with entries $d_{0,j,0s}$ (or $d_{1,j,1s}$) is invertible for small ε_j and $\|D_i^{-1}\| \leq C$ ($i = 1, 2$) uniformly in ε_j .

Now, given $h \in L^\infty(\partial\Omega)$ we find ϕ_1, d_{ij} , solution to (6.21). Define constants c_{ls} as

$$\sum_{l=0,1} \sum_{s=1}^k c_{ls} d_{ij}^{ls} = -d_{ij}, \quad \forall i = 0, 1, \quad j = 1, \dots, k.$$

The above linear system is almost diagonal, since arguing as before one can show that $d_{ij}^{ls} = \varepsilon_j^{-1} M_i \delta_{js} \delta_{il} (1 + o(1))$, as $\lambda \rightarrow 0$, where M_i is a positive universal constant. Then define

$$\phi = \phi_1 + \sum_{l=0,1} \sum_{s=1}^k c_{ls} Y_{ls},$$

A direct computation shows that ϕ satisfies (4.1) and furthermore

$$\|\phi\|_\infty \leq \|\phi_1\|_\infty + \sum_{l=0,1} \sum_{s=1}^k |c_{ls}| \leq C \|h\|_{*,\partial\Omega} + \sum_{i=0,1} \sum_{j=1}^k \varepsilon_j |d_{ij}| \leq C \|h\|_{*,\partial\Omega}$$

by (6.22). This finishes the proof of Proposition 4.1.

Proof of (6.27). We shall prove

$$\|-\Delta \tilde{Z}_{0j} + \tilde{Z}_{0j}\|_{*,\Omega} \leq \frac{C}{|\log \varepsilon_j|}$$

where \tilde{Z}_{0j} is defined in (6.24). Perform the change of variables $y = \varepsilon_j^{-1} F_j(x)$ and denote $\tilde{z}_{0j}(y) = \tilde{Z}_{0j}(F_j^{-1}(\varepsilon_j y))$. Then $-\Delta \tilde{Z}_{0j} + \tilde{Z}_{0j} = (\mathcal{L} \tilde{z}_{0j} + \varepsilon_j^2 \tilde{z}_{0j})$, where \mathcal{L} is defined in (6.13). We shall show that

$$\left| (\mathcal{L} \tilde{z}_{0j} + \varepsilon_j^2 \tilde{z}_{0j}) \right| \leq \frac{C}{|\log \varepsilon_j|} \left[\varepsilon_j^2 + \sum_{j=1}^m (1 + |y - \xi_j|)^{-2-\sigma} \right], \quad y \in \frac{\Omega}{\varepsilon_j}.$$

This fact implies (6.27).

Let us first consider the region where $|y| < R$. In this region, $\tilde{z}_{0j} = z_{0j}$. Since $\Delta z_{0j} = 0$ and since (6.13) holds, we have that

$$(6.34) \quad (\mathcal{L} \tilde{z}_{0j} + \varepsilon_j^2 \tilde{z}_{0j}) = O(\varepsilon_j) \quad \text{for } |y| < R.$$

In the region $R + 1 < |y| < \frac{\delta}{4\varepsilon_j}$, we have $\tilde{z}_{0j} = h z_{0j}$. Therefore, in this region,

$$|\Delta \tilde{z}_{0j}| = 2|\nabla h \nabla z_{0j}| \leq \frac{C}{r^3 \log \frac{\delta}{\varepsilon_j}} \quad R + 1 < r < \frac{\delta}{4\varepsilon_j}, \quad r = |y|.$$

For the other terms we find

$$\begin{aligned} |\nabla^2 \tilde{z}_{0j}| &\leq |\nabla^2 h| z_{0j} + 2|\nabla h \nabla z_{0j}| + h |\nabla^2 z_{0j}| \\ &= O\left(\frac{1}{r^2 \log \frac{\delta}{\varepsilon_j}}\right) + O\left(\frac{1}{r^3 \log \frac{\delta}{\varepsilon_j}}\right) + O\left(\frac{1}{r^3}\right) \quad R + 1 < r < \frac{\delta}{4\varepsilon_j} \end{aligned}$$

so

$$O(\varepsilon_j |y|) |\nabla^2 \tilde{z}_{0j}| = O\left(\frac{\varepsilon_j}{r \log \frac{\delta}{\varepsilon_j}}\right) + O\left(\frac{\varepsilon_j}{r^2}\right) \quad R + 1 < r < \frac{\delta}{4\varepsilon_j}.$$

Also

$$|\nabla \tilde{z}_{0j}| \leq |\nabla h|z_{0j} + h|\nabla z_{0j}| = O\left(\frac{1}{r \log \frac{\delta}{\varepsilon_j}}\right) + O\left(\frac{1}{r^2}\right) \quad R+1 < r < \frac{\delta}{4\varepsilon_j}.$$

Hence

$$(6.35) \quad (\mathcal{L}\tilde{z}_{0j} + \varepsilon_j^2 \tilde{z}_{0j}) = O\left(\frac{1}{r^3 \log \frac{\delta}{\varepsilon_j}}\right) + O\left(\frac{\varepsilon_j}{r \log \frac{\delta}{\varepsilon_j}}\right) + O\left(\frac{\varepsilon_j}{r^2}\right) + \varepsilon_j^2 \tilde{z}_{0j} \quad R+1 < r < \frac{\delta}{4\varepsilon_j}.$$

In the region $\frac{\delta}{4\varepsilon_j} < r < \frac{\delta}{3\varepsilon_j}$ the definition of \tilde{z}_{0j} is $\tilde{z}_{0j} = \eta_2 h z_{0j}$. We will estimate each term of (6.13) using the facts that $\nabla \eta_2 = O\left(\frac{\varepsilon_j}{\delta}\right)$, $|\nabla^2 \eta_2| = O\left(\frac{\varepsilon_j^2}{\delta^2}\right)$ and that in the considered region $h = O\left(\frac{1}{\log \frac{\delta}{\varepsilon_j}}\right)$ which implies also $\tilde{z}_{0j} = O\left(\frac{1}{\log \frac{\delta}{\varepsilon_j}}\right)$. We obtain

$$\begin{aligned} \Delta \tilde{z}_{0j} &= \Delta \eta_2 h z_{0j} + 2\nabla \eta_2 \nabla(h z_{0j}) + \eta_2 \Delta(h z_{0j}) \\ &= \Delta \eta_2 h z_{0j} + 2\nabla \eta_2 \nabla h z_{0j} + 2\nabla \eta_2 \nabla z_{0j} h + 2\eta_2 \nabla h \nabla z_{0j} \\ &= O\left(\frac{\varepsilon_j^2}{\delta^2 \log \frac{\delta}{\varepsilon_j}}\right) + O\left(\frac{\varepsilon_j}{r \delta \log \frac{\delta}{\varepsilon_j}}\right) + O\left(\frac{\varepsilon_j}{r^2 \delta \log \frac{\delta}{\varepsilon_j}}\right) + O\left(\frac{1}{r^3 \log \frac{\delta}{\varepsilon_j}}\right) \\ &= O\left(\frac{\varepsilon_j^2}{\delta^2 \log \frac{\delta}{\varepsilon_j}}\right) \quad \frac{\delta}{4\varepsilon_j} < r < \frac{\delta}{3\varepsilon_j}. \end{aligned}$$

Next

$$\nabla^2 \tilde{z}_{0j} = \nabla^2 \eta_2 h z_{0j} + 2\nabla \eta_2 \nabla(h z_{0j}) + \eta_2 \nabla^2(h z_{0j}) \quad \frac{\delta}{4\varepsilon_j} < r < \frac{\delta}{3\varepsilon_j}.$$

and by the above computations, for $\frac{\delta}{4\varepsilon_j} < r < \frac{\delta}{3\varepsilon_j}$,

$$\nabla^2 \tilde{z}_{0j} = O\left(\frac{\varepsilon_j^2}{\delta^2 \log \frac{\delta}{\varepsilon_j}}\right) + \eta_2 (\nabla^2 h z_{0j} + 2\nabla h \nabla z_{0j} + h \nabla^2 z_{0j}) = O\left(\frac{\varepsilon_j^2}{\delta^2 \log \frac{\delta}{\varepsilon_j}}\right).$$

Similarly, for $\frac{\delta}{4\varepsilon_j} < r < \frac{\delta}{3\varepsilon_j}$

$$\nabla \tilde{z}_{0j} = \nabla \eta_2 h z_{0j} + \eta_2 \nabla h z_{0j} + \eta_2 h \nabla z_{0j} = O\left(\frac{\varepsilon_j}{\delta \log \frac{\delta}{\varepsilon_j}}\right)$$

This shows that

$$(6.36) \quad (\mathcal{L}\tilde{z}_{0j} + \varepsilon_j^2 \tilde{z}_{0j}) = O\left(\frac{\varepsilon_j^2}{\delta^2 \log \frac{\delta}{\varepsilon_j}}\right) \quad \frac{\delta}{4\varepsilon_j} < r < \frac{\delta}{3\varepsilon_j}.$$

Thus we only need to estimate the size of $\mathcal{L}\tilde{z}_{0j} + \varepsilon_j^2 \tilde{z}_{0j}$ in the region $R < r < R+1$. In this region we have $\tilde{z}_{0j} = \eta_1 z_{0j} + (1 - \eta_1) h z_{0j}$ and hence

$$\begin{aligned} \Delta \tilde{z}_{0j} &= \Delta \eta_1 (1 - h) z_{0j} - 2\nabla \eta_1 \nabla h z_{0j} + 2\nabla \eta_1 \nabla z_{0j} (1 - h) + \eta_1 \Delta z_{0j} \\ &\quad + (1 - \eta_1) \Delta(h z_{0j}) \\ &= O\left(\frac{1}{\log \frac{\delta}{\varepsilon_j}}\right) + \eta_1 \Delta z_{0j} + (1 - \eta_1) \Delta(h z_{0j}) \quad R < r < R+1. \end{aligned}$$

First we recall that $\Delta z_{0j} = 0$ and, for $R < r < R + 1$,

$$\Delta(hZ_{0j}) = 2\nabla h \nabla z_{0j} + O(\varepsilon_j) = O\left(\frac{1}{\log \frac{\delta}{\varepsilon_j}}\right) + O(\varepsilon_j).$$

Thus

$$(6.37) \quad \mathcal{L}\tilde{z}_{0j} + \varepsilon_j^2 \tilde{z}_{0j} = O\left(\frac{1}{\log \frac{\delta}{\varepsilon_j}}\right) \quad R < r < R + 1.$$

This bound and (6.34), (6.35) and (6.36) imply (6.27).

Proof of (6.28). We shall prove

$$\|L(\tilde{Z}_{0j})\|_{*,\partial\Omega} \leq \frac{C}{|\log \varepsilon_j|}$$

We perform the change of variables $y = \varepsilon_j^{-1} F_j(x)$. We already observed that we can assume that $\nabla F_j(\xi_j) = I$. Hence,

$$L(\tilde{Z}_{0j}) = (1 + o(1)) [B(\tilde{z}_{0j}) - \tilde{W}\tilde{z}_{0j}]$$

where $\tilde{z}_{0j} = \tilde{Z}_{0j}(F_j^{-1}(\varepsilon_j y))$ and $\tilde{W}(y) = W(F_j^{-1}(\varepsilon_j y))$. B is the differential operator of order one on $\partial\mathbb{R}_+^2$, defined in (6.15) and \tilde{W} is described in (6.16). Thus in the region $y \in \partial\left(\frac{\Omega}{\varepsilon_j}\right)$, with $|y| < R$, we get

$$(6.38) \quad B(\tilde{z}_{0j}) - \tilde{W}\tilde{z}_{0j} = O(\varepsilon_j)$$

Next, in the region $R < |x| < R + 1$ we have

$$\begin{aligned} \nabla \tilde{z}_{0j} &= \nabla(\eta_1(1-h)z_{0j} + hz_{0j}) \\ &= \nabla\eta_1(1-h)z_{0j} - \eta_1\nabla h z_{0j} + \eta_1(1-h)\nabla z_{0j} + \nabla h z_{0j} + h\nabla z_{0j} \\ &= O\left(\frac{1}{\log \frac{\delta}{\varepsilon_j}}\right) + \eta_1(1-h)\nabla z_{0j} + h\nabla z_{0j}. \end{aligned}$$

Since h is radial this implies

$$B(\tilde{z}_{0j}) = -h \frac{\partial z_{0j}}{\partial x_2} + O\left(\frac{1}{R^2 \log \frac{\delta}{\varepsilon_j}}\right) + O\left(\frac{R\varepsilon_j}{\log \frac{\delta}{\varepsilon_j}}\right) \quad R < |y| < R + 1, y \in \partial\mathbb{R}_+^2.$$

Using (6.16) we see that

$$(6.39) \quad B(\tilde{z}_{0j}) - \tilde{W}\tilde{z}_{0j} = O\left(\frac{1}{R^2 \log \frac{\delta}{\varepsilon_j}}\right) + O\left(\frac{R\varepsilon_j}{\log \frac{\delta}{\varepsilon_j}}\right) \quad R < |y| < R + 1, y \in \partial\mathbb{R}_+^2.$$

Using the fact that h has zero normal derivative on $\partial\mathbb{R}_+^2$ we deduce

$$(6.40) \quad \begin{aligned} B(\tilde{h}z_{0j}) &= -h \frac{\partial z_{0j}}{\partial x_2} + O(\varepsilon_j r)(\nabla h z_{0j} + h\nabla z_{0j}) \\ &= -h \frac{\partial z_{0j}}{\partial x_2} + O\left(\frac{\varepsilon_j}{\log \frac{\delta}{\varepsilon_j}}\right) + O\left(\frac{\varepsilon_j}{r}\right) \quad R + 1 < r < \frac{\delta}{\varepsilon_j}. \end{aligned}$$

On the other hand, using (6.16) we have in $R + 1 < r < \frac{\delta}{\varepsilon_j}$

$$(6.41) \quad B(\tilde{z}_{0j}) - \tilde{W}\tilde{z}_{0j} = O\left(\frac{\varepsilon_j}{\log \frac{\delta}{\varepsilon_j}}\right) + O\left(\frac{\varepsilon_j^\alpha}{r}\right)$$

for some $0 < \alpha < 1$. Finally we consider $\frac{\delta}{4\varepsilon_j} < r < \frac{\delta}{3\varepsilon_j}$. Here we have $\tilde{z}_{0j} = \eta_2 h z_{0j}$ and $h, z_{0j} = O(\frac{1}{\log \frac{\delta}{\varepsilon_j}})$, $\nabla \bar{\eta}_2 = O(\frac{\varepsilon_j}{\delta})$. Using these facts, estimate (6.40) and that η_2 has zero normal derivative we find

$$\begin{aligned} B(\tilde{z}_{0j}) &= B(\eta_2) h z_{0j} + \eta_2 B(h z_{0j}) \\ &= O\left(\frac{\varepsilon_j^2 r}{\delta \log \frac{\delta}{\varepsilon_j}}\right) + O\left(\frac{1}{r^2}\right) + O\left(\frac{\varepsilon_j}{\log \frac{\delta}{\varepsilon_j}}\right) + O\left(\frac{\varepsilon_j}{r}\right) \quad \frac{\delta}{4\varepsilon_j} < r < \frac{\delta}{3\varepsilon_j}. \end{aligned}$$

From (6.16) we have

$$\tilde{W} = O\left(\frac{\varepsilon_j^\alpha}{r}\right) \quad \frac{\delta}{4\varepsilon_j} < r < \frac{\delta}{\varepsilon_j}.$$

Thus we conclude that for $y \in \partial\Omega_{\varepsilon_j}$, $\frac{\delta}{4\varepsilon_j} < r < \frac{\delta}{3\varepsilon_j}$

$$(6.42) \quad B(\tilde{z}_{0j}) - \tilde{W}\tilde{z}_{0j} = O\left(\frac{\varepsilon_j^2 r}{\delta \log \frac{\delta}{\varepsilon_j}}\right) + O\left(\frac{1}{r^2}\right) + O\left(\frac{\varepsilon_j}{\log \frac{\delta}{\varepsilon_j}}\right) + O\left(\frac{\varepsilon_j}{r}\right).$$

Estimates (6.38), (6.39), (6.41) and (6.42) give the validity of (6.28).

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DEPARTAMENTO DE INGENIERÍA MATEMÁTICA AND CMM, UNIVERSIDAD DE CHILE, CASILLA 170 CORREO 3, SANTIAGO, CHILE.

E-mail address: shbdeng65@gmail.com

M. MUSSO - DEPARTAMENTO DE MATEMÁTICA, PONTIFICIA UNIVERSIDAD CATOLICA DE CHILE, AVDA. VICUÑA MACKENNA 4860, MACUL, CHILE

E-mail address: mmusso@mat.puc.cl