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Nonlinear Analysis ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Multiple blow-up solutions for an exponential nonlinearity with potential in $\mathbb{R}^2$

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ARTICLE INFO

Communicated by Enzo Mitidieri

Keywords:
Elliptic equation
Liouville problem
Singularly perturbed problem
Lyapunov–Schmidt reduction

ABSTRACT

We study the following boundary value problem

$$\begin{cases} \Delta u + \lambda a(x) u^{p-1} e^{u^p} = 0, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, $\lambda > 0$ is a small parameter, the function $a(x) \geq 0$ is a smooth potential, and the exponent p satisfies $0 < p < 2$. We construct a family of solutions to problem (0.1) which blows up, as $\lambda \to 0$, at some points of Ω which stay outside the zero set of $a(x)$. We relate the number of possible blow-up points with the zero set of $a(x)$.

1. Introduction

We consider the following boundary value problem

$$\begin{cases} \Delta u + \lambda a(x) u^{p-1} e^{u^p} = 0, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, $\lambda > 0$ is a small parameter and $0 < p < 2$. The function $a(x) \geq 0$ is smooth in Ω. This problem is the Euler–Lagrange equation for the functional

$$J^p_{a,\lambda}(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx - \frac{\lambda}{p}\int_\Omega a(x) e^{u^p} dx, \quad u \in H^1_0(\Omega). \tag{1.2}$$

If $a(x) \equiv 1$, problem (1.1) becomes

$$\begin{cases} \Delta u + \lambda u^{p-1} e^{u^p} = 0, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

This problem has been studied widely in the literature when $p = 1$. The asymptotic behavior of blowing up families of solutions can be referred to [1,4,13–16]: in these works it has been established that if u_λ is an unbounded family of solutions

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http://dx.doi.org/10.1016/j.na.2014.10.034
0362-546X/

to (1.3) for which $\lambda \int_\Omega e^{u_\lambda}$ remains uniformly bounded as $\lambda \to 0$, then there exists an integer K such that

$$\lambda \int_\Omega e^{u_\lambda}\, dx \to 8\pi K, \quad \text{as } \lambda \to 0.$$

Moreover there are K points $\xi_1, \dots, \xi_K$ in Ω, which are far away from the boundary of Ω and far away from each other, so that

$$\lambda e^{u_\lambda} \to \sum_{j=1}^K \delta_{\xi_j}$$

in the sense of measure. Furthermore, the location of the point $\xi = (\xi_1, \dots, \xi_K)$ is known to be related to the critical points of the function

$$\Phi_K(\xi) = \sum_{j=1}^K H(\xi_j, \xi_j) + \sum_{i\neq j}^K G(\xi_i, \xi_j).$$

Here $G(x, y)$ denotes Green's function for the negative Laplacian with Dirichlet boundary condition in Ω, namely

$$\begin{cases} -\Delta_x G(x, y) = \delta_y(x) & x \in \Omega; \\ G(x, y) = 0 & x \in \partial\Omega, \end{cases} \tag{1.4}$$

and $H(x, y)$ its regular part, given by

$$H(x, y) = G(x, y) - \frac{1}{2\pi} \log \frac{1}{|x - y|}. \tag{1.5}$$

Concerning the reciprocal issue, several results are already known in the literature, we refer to [1,10,7]. In particular, in [7] del Pino–Kowalczyk–Musso constructed bubbling solutions to problem (1.3) when $p = 1$. They showed that: *If the domain Ω is not simply connected*, and given any integer $K \geq 1$, there exist K points $\xi_1, \dots, \xi_K$ in Ω and a family of solutions u_λ, for any λ sufficiently small, which blows up at these K points in the sense that, as $\lambda \to 0$

$$\sup_{x\in\Omega\setminus\cup_{j=1}^K B(\xi_j,\delta)} u_\lambda(x) \to 0, \quad \text{and for any } j = 1, \dots, K, \quad \sup_{x\in B(\xi_j,\delta)} u_\lambda(x) \to \infty$$

for any positive fixed number δ. Furthermore,

$$\int_\Omega \lambda e^{u_\lambda} dx \to 8K\pi \quad \text{as } \lambda \to 0.$$

The location of these blow-up points $\xi_1, \dots, \xi_K$ is not arbitrary: indeed they correspond to critical points of the function Φ_K defined above.

The results have been extended in [9] for the whole range of values of exponents p with $0 < p < 2$. This result was surprising, since the scenario changes completely when $p = 2$: this situation was previously treated in [8].

In this paper, we construct bubbling solutions to Problem (1.1), with a non negative nontrivial potential. When $p = 1$, this situation was already treated in [7], under the condition that the concentration points $(\xi_1, \dots, \xi_K)$ belong to a region where the potential a is strictly positive. Our first result shows that this construction can be done for the whole range of exponents $0 < p < 2$.

Before stating our result, it is useful to introduce some notations. For an integer $K \geq 1$ and K distinct points ξ_j, $j = 1, \dots, K$, in Ω, separated uniformly from each other and from the boundary $\partial\Omega$, write $\xi = (\xi_1, \dots, \xi_K)$, let us define the following functional

$$\Phi^p_{a,K}(\xi) = \sum_{j=1}^K H(\xi_j, \xi_j) + \sum_{i\neq j}^K G(\xi_i, \xi_j) + \frac{2-p}{4p\pi} \sum_{j=1}^K \log a(\xi_j). \tag{1.6}$$

Definition 1.1. We say that ξ is a C^0-stable critical point of $\varphi : \mathcal{M} \to \mathbb{R}$ if for any sequence of functions $\varphi_n : \mathcal{M} \to \mathbb{R}$ such that $\varphi_n \to \varphi$ uniformly on compact sets of $\mathcal{M}$, φ_n has a critical point ξ^n such that $\varphi_n(\xi^n) \to \varphi(\xi)$.

In particular, if ξ is a strict local minimum or maximum point of φ, then ξ is C^0-stable critical point.

Let ε be a parameter, which depends on λ, defined as

$$p\lambda \left(-\frac{4}{p} \log \varepsilon\right)^{\frac{2(p-1)}{p}} \varepsilon^{\frac{2(p-2)}{p}} = 1. \tag{1.7}$$

Observe that, as $\lambda \to 0$, then $\varepsilon \to 0$, and $\lambda = \varepsilon^2$ if $p = 1$.

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The result we have is the following.

Theorem 1.2. *Let Ω be a bounded smooth domain in $\mathbb{R}^2$, $0 < p < 2$ and K an integer with $K \geq 1$, assume that $a(x) \geq 0$ is smooth in Ω, and $\xi^* = (\xi_1^*, \ldots, \xi_K^*)$ is a $\mathcal{C}^0$-stable critical point of $\Phi_{a,K}^p$. Then there exists $\lambda_0 > 0$ so that, for any $0 < \lambda < \lambda_0$, Problem (1.1) has a solution u_λ, which satisfies*

$$\lim_{\lambda\to 0} \varepsilon^{\frac{2(2-p)}{p}} \int_\Omega a(x) e^{u_\lambda^p} dx = 8K\pi, \tag{1.8}$$

where ε satisfies (1.7). Moreover, there exists a K-tuple $\xi^\lambda = (\xi_1^\lambda, \ldots, \xi_K^\lambda) \in \Omega^K$ such that $a(\xi_j^\lambda) > 0$, and as $\lambda \to 0$,

$$\Phi_{a,K}^p(\xi_1^\lambda, \ldots, \xi_K^\lambda) \to \Phi_{a,K}^p(\xi_1^*, \ldots, \xi_K^*),$$

and

$$u_\lambda(x) = \left(-\frac{4}{p}\log\epsilon\right)^{\frac{1-p}{p}} \left(8\pi \sum_{j=1}^K G(x, \xi_j^\lambda) + o(1)\right) \tag{1.9}$$

where $o(1) \to 0$, as $\lambda \to 0$, on each compact subset of $\bar{\Omega} \setminus \{\xi_1^\lambda, \ldots, \xi_K^\lambda\}$. Furthermore

$$J_{a,\lambda}^p(u_\lambda) = \frac{1}{p}\left(-\frac{4}{p}\log\epsilon\right)^{\frac{2(1-p)}{p}} \left[\frac{8K\pi}{(2-p)p}[-2 + p\log 8] - \frac{16K\pi}{p}\log\varepsilon - \frac{32\pi^2}{2-p}\Phi_{a,K}^p(\xi^\lambda) + O(|\log\varepsilon|^{-1})\right] \tag{1.10}$$

where $O(1)$ is uniformly bounded as $\lambda \to 0$.

In [7], the authors consider also the case in which the potential $a(x)$ has a zero of type $|x-q|^\alpha$ for some point $q \in \Omega$. When $p = 1$ and $K < 1+\alpha$, they show the existence of a family of solutions u_λ to Problem (1.1) blowing up at K points of Ω, which remain far from q. This result was generalized by [6] in the case in which the potential a has several zeros $q_1, \ldots, q_m$, of type $|x-q_j|^{\alpha_j}$ respectively. She studies how the concentration phenomena is affected by the presence of several zeros for the potential. Our next result concerns a generalization of these results when the exponent p belongs to the whole range $0 < p < 2$.

Define the set $Z \subset \Omega$ as

$$Z := \{q \in \Omega : a(q) = 0\}.$$

We make the following assumptions on $a(x)$.

(A_1) *For any $q \in Z$, there exists $\alpha_q > 0$ such that*

$$a_q(x) = a(x)|x-q|^{-2\alpha_q}$$

is a strictly positive continuous function in a neighborhood of q.

(A_2) *Assume $Z \subset \Omega$ is finite, and $K \geq 2$ is an integer such that there exist distinct points $q_1, \ldots, q_m \in Z$ and integers $K_1, \ldots, K_m$ with the following properties*:

$$\frac{2-p}{p}\alpha_{q_s} \neq 1, \ldots, K-1, \quad \text{for each } s = 1, \ldots, m,$$

$$1 \leq K_s < 1 + \frac{2-p}{p}\alpha_{q_s}, \quad \text{for each } s = 1, \ldots, m, \tag{1.11}$$

and $K = K_1 + \cdots + K_m$.

We have the following result.

Theorem 1.3. *Let Ω be a bounded smooth domain in $\mathbb{R}^2$, $0 < p < 2$, and assume that $a(x)$ and K satisfy* (A_1) *and* (A_2). *Then there is $\lambda_0 > 0$ small such that for any $0 < \lambda < \lambda_0$, Problem (1.1) has a family of solutions u_λ with the property:*

$$\lim_{\lambda\to 0} \varepsilon^{\frac{2(2-p)}{p}} \int_\Omega a(x) e^{u_\lambda^p} dx = 8K\pi, \tag{1.12}$$

where ε is defined in (1.7). Moreover, there exists a K-tuple $\tilde{\xi}^\lambda = (\tilde{\xi}_1^\lambda, \ldots, \tilde{\xi}_K^\lambda) \in (\Omega \setminus Z)^K$ such that as $\lambda \to 0$

$$\nabla\Phi_{a,K}^p(\tilde{\xi}_1^\lambda, \ldots, \tilde{\xi}_K^\lambda) \to 0,$$

and

$$u_\lambda(x) = \left(-\frac{4}{p}\log\varepsilon\right)^{\frac{1-p}{p}} \left(8\pi \sum_{j=1}^K G(x, \tilde{\xi}_j^\lambda) + o(1)\right) \tag{1.13}$$

where $o(1) \to 0$, *as* $\lambda \to 0$, *on each compact subset of* $\overline{(\Omega \setminus Z)} \setminus \{\tilde{\xi}_1^\lambda, \ldots, \tilde{\xi}_K^\lambda\}$. *Furthermore*

$$J^p_{a,\lambda}(u_\lambda) = \frac{1}{p}\left(-\frac{4}{p}\log\varepsilon\right)^{\frac{2(1-p)}{p}} \left[\frac{8K\pi}{(2-p)p}[-2+p\log 8] - \frac{16K\pi}{p}\log\varepsilon - \frac{32\pi^2}{2-p}\Phi^p_{a,K}(\tilde{\xi}^\lambda) + O(|\log\varepsilon|^{-1})\right] \tag{1.14}$$

where $O(1)$ *is uniformly bounded as* $\lambda \to 0$.

For the special case that Ω is the unit ball B in $\mathbb{R}^2$ and $a(x) = |x|^{2\alpha}$ with $\alpha > 0$, that is, consider

$$\begin{cases} \Delta u + \lambda |x|^{2\alpha} u^{p-1} e^{u^p} = 0, \quad u > 0 & \text{in } B; \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.15}$$

where $\lambda > 0$ is a small parameter. A direct consequence of Theorem 1.3 is that there exists a bubbling solution to (1.15) concentrating at points, which are outside the origin; furthermore the number of bubbling points depends on α. Set

$$K_\alpha = \max\left\{k \in \mathbb{N} : k < \frac{2-p}{p}\alpha + 1\right\}.$$

The result we obtain for (1.15) can be stated as follows.

Theorem 1.4. *Let* $0 < p < 2$, *there exists* $\lambda_0 > 0$ *such that for any* $1 \le K \le K_\alpha$, *for any* $0 < \lambda < \lambda_0$, *the problem* (1.15) *has a solution* u_λ *which concentrates at* K *different points of* $B \setminus \{0\}$ *and*

$$\lim_{\lambda \to 0} \varepsilon^{\frac{2(2-p)}{p}} \int_B |x|^{2\alpha} e^{u_\lambda^p} dx = 8K\pi, \tag{1.16}$$

where ε *satisfies* (1.7). *Moreover,* (1.9) *and* (1.10) *hold.*

Remark 1.5. To prove Theorem 1.3 we follow the approach developed in [6]: we apply a max–min argument to establish a topologically nontrivial critical value of $\Phi^p_{a,K}$ under the assumptions (A_1) and (A_2) on $a(x)$ in any bounded smooth domain. Observe that we are not assuming the condition that domain is not simply connected. Observe that $Z = \emptyset$, the condition that Ω is not simply connected guarantees existence of a nontrivial critical value $\Phi^p_{a,K}$, see [7].

Remark 1.6. Theorem 1.4 is the special case of Theorem 1.3 for $a(x) = |x|^{2\alpha}$ and domain $\Omega = B$.

Remark 1.7. We construct bubbling solutions to (1.1), whose location of concentration occurs at points different from the zero set of the potential $a(x)$. The problem of finding solutions with additional concentration around at the zero points of $a(x)$ is of different type, indeed from the works [2,3,17] it follows that the contribution of each blow-up point in the limit (1.12) is of $8\pi(1+\alpha)$. The asymptotic analysis in this situation is completely different.

In order to cover the case $p = 2$ in (1.1), we believe that a different approach is needed, given the known result for $a(x) \equiv 1$ contained in [8].

The paper is organized as follows: Section 2 is devoted to describe a first approximation solution to problem (1.1) and to estimate its error. We consider the linear problem and the nonlinear problem in Sections 3 and 4. Furthermore, we reduced problem into the finite-dimensional problem and solve it, we sketch it in Section 5. In Section 6, we prove the main results.

2. The first approximation solution

In this section, we build a good approximation solution and we estimate its error. Let us introduce the radially symmetric solutions of the following limit equation

$$\Delta w + e^w = 0 \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^w < +\infty,$$

which are given by the one parameter family of functions

$$w_\mu(z) = \log\frac{8\mu^2}{(\mu^2 + |z|^2)^2}. \tag{2.1}$$

Let K be an integer, set $\xi = (\xi_1, \ldots, \xi_K)$, let $\delta > 0$ small but fixed, define

$$\mathcal{O} := \left\{\xi \in (\Omega \setminus Z)^K : \mathrm{dist}(\xi_j, \partial(\Omega \setminus Z)) \ge \delta, |\xi_i - \xi_j| \ge \delta \text{ for } i \ne j\right\}. \tag{2.2}$$

Moreover, consider K positive numbers μ_j such that

$$\delta < \mu_j < \delta^{-1}, \quad \text{for all } j = 1, \ldots, K. \tag{2.3}$$

The parameters μ_j will be chosen properly later on. Define the function

$$U_{\mu_j,\xi_j}(x) = \log \frac{8\mu_j^2}{(\mu_j^2\varepsilon^2 + |x-\xi_j|^2)^2 a(\xi_j)} = w_{\mu_j}\left(\frac{x-\xi_j}{\varepsilon}\right) + 4\log\frac{1}{\varepsilon} - \log a(\xi_j). \tag{2.4}$$

Let us denote $PU_{\mu_j,\xi_j}(x)$ the projection of U_{μ_j,ξ_j} into the space $H_0^1(\Omega)$, in other words, $PU_{\mu_j,\xi_j}(x)$ is the unique solution of

$$\begin{cases} \Delta PU_{\mu_j,\xi_j} = \Delta U_{\mu_j,\xi_j} & \text{in } \Omega; \\ PU_{\mu_j,\xi_j} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

By Maximum Principle, we have that $\xi \in \mathcal{O}$ and μ_j satisfies (2.3), then

$$PU_{\mu_j,\xi_j}(x) = U_{\mu_j,\xi_j}(x) + 8\pi H(x,\xi_j) - \log\frac{8\mu_j^2}{a(\xi_j)} + O(\mu_j^2\varepsilon^2) \tag{2.6}$$

in $C^1(\bar{\Omega})$ as $\varepsilon \to 0$, and

$$PU_{\mu_j,\xi_j}(x) = 8\pi G(x,\xi_j) + O(\mu_j^2\varepsilon^2) \tag{2.7}$$

in $C^1_{\mathrm{loc}}(\overline{(\Omega \setminus Z)} \setminus \{\xi_j\})$ as $\varepsilon \to 0$, where $G(\cdot,\cdot)$ and $H(\cdot,\cdot)$ are Green's function and its regular part as defined in (1.4) and (1.5).

We now define that the first ansatz is given by

$$U(x) = \frac{1}{p\gamma^{p-1}} \sum_{j=1}^{K} PU_{\mu_j,\xi_j}(x),$$

with some number γ, to be fixed later on. We want to show that $U(x)$ is a good approximation for a solution to (1.1), and so that the solution to problem (1.1) like the formula $U(x)$ plus a small term. In order to perform the fixed point argument to find the lower order term, we need to improve our ansatz, adding two other terms in the expansion of the solution. In order to do this, we set

$$w_j(y) = w_{\mu_j}(y-\xi_j') = \log\frac{8\mu_j^2}{(\mu_j^2 + |y-\xi_j'|^2)^2},$$

and

$$\widetilde{w}_j(y) = w_{\mu_j}(y) - \log a(\xi_j) = \log\frac{8\mu_j^2}{(\mu_j^2 + |y-\xi_j'|^2)^2 a(\xi_j)}. \tag{2.8}$$

Let w_j^i be the radial solution of

$$\Delta w_j^i + e^{w_j} w_j^i = e^{w_j} f^i \quad \text{in } \mathbb{R}^2, \text{ for } i = 0, 1, \tag{2.9}$$

where

$$f^0 = -\left[\widetilde{w}_j + \frac{1}{2}(\widetilde{w}_j)^2\right],$$

and

$$f^1 = -\left[w_j^0 + \frac{p-2}{2(p-1)}(\widetilde{w}_j)^2 + \frac{1}{2}(w_j^0)^2 + \frac{1}{8}(\widetilde{w}_j)^4 + 2\widetilde{w}_j w_j^0 + \frac{1}{2}(\widetilde{w}_j)^3 + \frac{1}{2}w_j^0(\widetilde{w}_j)^2\right].$$

In fact, as shown in [11] (see also [5,9]), there exists a radially symmetric solution with the properties that

$$w_j^i(y) = C_{ij}\log\frac{|y-\xi_j'|}{\mu_j} + O\left(\frac{1}{|y-\xi_j'|}\right) \quad \text{as } |y-\xi_j'| \to \infty, \tag{2.10}$$

for some explicit constants C_{ij}, which can be explicitly computed. In particular, when $i = 0$, the constant C_{0j} is given by

$$C_{0j} = -8\int_0^{+\infty} t\frac{t^2-1}{(t^2+1)^3}\left[\log\frac{8\mu_j^{-2}}{(1+t^2)^2 a(\xi_j)} + \frac{1}{2}\left(\log\frac{8\mu_j^{-2}}{(1+t^2)^2 a(\xi_j)}\right)^2\right]dt = 4\log 8 - 8 - 8\log\mu_j - 4\log a(\xi_j). \tag{2.11}$$

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Let us define

$$w^0_{\mu_j,\xi_j}(x) := w^0_j\left(\frac{x}{\varepsilon}\right), \qquad w^1_{\mu_j,\xi_j}(x) := w^1_j\left(\frac{x}{\varepsilon}\right) \quad \text{for } x \in \Omega.$$

Let $Pw^0_{\mu_j,\xi_j}$ and $Pw^1_{\mu_j,\xi_j}$ denote the projections into $H^1_0(\Omega)$ of $w^0_{\mu_j,\xi_j}$ and $w^1_{\mu_j,\xi_j}$, respectively. We write $y = \frac{x}{\varepsilon}$, $\xi'_j = \frac{\xi_j}{\varepsilon}$, by (2.10), we have that

$$Pw^i_{\mu_j,\xi_j}(x) = w^i_j\left(\frac{x}{\varepsilon}\right) - 2\pi C_{ij}H(x,\xi_j) + C_{ij}\log(\mu_j\varepsilon) + O(\mu_j\varepsilon) \tag{2.12}$$

in $C^1(\bar{\Omega})$ as $\varepsilon \to 0$, and

$$Pw^i_{\mu_j,\xi_j}(x) = P\left(w^i_j\left(\frac{x}{\varepsilon}\right)\right) = -2\pi C_{ij}G(x,\xi_j) + O(\mu_j\varepsilon) \tag{2.13}$$

in $C^1_{loc}(\overline{(\Omega\setminus Z)}\setminus\{\xi_j\})$ as $\varepsilon \to 0$.

We define

$$U_\lambda(x) = \frac{1}{p\gamma^{p-1}}\sum_{j=1}^K\left[PU_{\mu_j,\xi_j}(x) + \frac{p-1}{p}\frac{1}{\gamma^p}Pw^0_{\mu_j,\xi_j}(x) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}Pw^1_{\mu_j,\xi_j}(x)\right]. \tag{2.14}$$

From (2.7) and (2.13), one has, away from the points ξ_j,

$$U_\lambda(x) = \frac{8\pi}{p\gamma^{p-1}}\sum_{j=1}^K G(x,\xi_j)\left[1 - \frac{p-1}{p}\frac{1}{\gamma^p}\frac{C_{0j}}{4} - \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\frac{C_{1j}}{4} + O(\varepsilon^2)\right]. \tag{2.15}$$

Consider now the change of variables

$$v(y) = p\gamma^{p-1}u(\varepsilon y) - p\gamma^p, \quad \text{with } \gamma^p = -\frac{4}{p}\log\varepsilon.$$

Then problem (1.1) reduces to

$$\begin{cases}\Delta v + g(v) = 0, \quad v > -p\gamma^p & \text{in } \Omega_\varepsilon;\\ v = -p\gamma^p & \text{on } \partial\Omega_\varepsilon,\end{cases} \tag{2.16}$$

where $\Omega_\varepsilon = \varepsilon^{-1}\Omega$, and

$$g(v) = a(\varepsilon y)\left(1 + \frac{v}{p\gamma^p}\right)^{p-1} e^{\gamma^p\left[\left(1+\frac{v}{p\gamma^p}\right)^p - 1\right]}. \tag{2.17}$$

Let us define the first approximation solution to (2.16) as

$$V_\lambda(y) = p\gamma^{p-1}U_\lambda(\varepsilon y) - p\gamma^p, \tag{2.18}$$

with the numbers μ_j, $j = 1,\dots,K$ defined by

$$\log\frac{8\mu_j^2}{a(\xi_j)} = \left[\frac{2(p-1)}{2-p}(1-\log 8) + \frac{8\pi}{2-p}\left(H(\xi_j,\xi_j) + \sum_{i\neq j}^K G(\xi_i,\xi_j)\right)\right]\left(1 + O\left(\frac{1}{\gamma^p}\right)\right). \tag{2.19}$$

Lemma 2.1. *We write $y = \varepsilon^{-1}x$, $\xi'_j = \varepsilon^{-1}\xi_j$. If μ_j, $j = 1,\dots,K$, are given by (2.19), then for $|y - \xi'_j| < \delta/\varepsilon$ with δ sufficiently small but fixed, we have*

$$V_\lambda(y) = \widetilde{w}_j(y) + \frac{p-1}{p}\frac{1}{\gamma^p}w^0_j(y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w^1_j(y) + \theta(y), \tag{2.20}$$

with $\widetilde{w}_j$ defined by (2.8) and

$$w^0_j(y) := w^0_j\left(\frac{y-\xi'_j}{\mu_j}\right), \qquad w^1_j(y) := w^1_j\left(\frac{y-\xi'_j}{\mu_j}\right)$$

and

$$\theta(y) = O(\varepsilon|y-\xi'_j|) + O(\varepsilon^2).$$

Proof. From (2.6), (2.7), (2.12), (2.13) and the fact that $U_{\mu_j,\xi_j}(\varepsilon y) - p\gamma^p = \widetilde{w}_j(y)$, we have

$$\begin{aligned}
V_\lambda(y) &= p\gamma^{p-1}U_\lambda(\varepsilon y) - p\gamma^p \\
&= \sum_{j=1}^{K}\left[PU_{\mu_j,\xi_j}(x) + \frac{p-1}{p}\frac{1}{\gamma^p}Pw^0_{\mu_j,\xi_j}(x) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}Pw^1_{\mu_j,\xi_j}(x)\right] - p\gamma^p \\
&= PU_{\mu_j,\xi_j}(x) + \frac{p-1}{p}\frac{1}{\gamma^p}Pw^0_{\mu_j,\xi_j}(x) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}Pw^1_{\mu_j,\xi_j}(x) - p\gamma^p \\
&\quad + \sum_{i\neq j}^{K}\left[PU_{\mu_i,\xi_i}(x) + \frac{p-1}{p}\frac{1}{\gamma^p}Pw^0_{\mu_i,\xi_i}(x) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}Pw^1_{\mu_i,\xi_i}(x)\right] \\
&= U_{\mu_j,\xi_j}(x) - p\gamma^p + 8\pi H(x,\xi_j) - \log\frac{8\mu_j^2}{a(\xi_j)} + O(\mu_j^2\varepsilon^2) \\
&\quad + \frac{p-1}{p}\frac{1}{\gamma^p}\left[w^0_j(y) - 2\pi C_{0j}H(x,\xi_j) + C_{0j}\log(\mu_j\varepsilon) + O(\mu_j\varepsilon)\right] \\
&\quad + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\left[w^1_j(y) - 2\pi C_{1j}H(x,\xi_j) + C_{1j}\log(\mu_j\varepsilon) + O(\mu_j\varepsilon)\right] \\
&\quad + 8\pi\sum_{i\neq j}^{K}G(\xi_i,\xi_j)\left[1 - \frac{C_{0j}}{4}\frac{p-1}{p}\frac{1}{\gamma^p} - \frac{C_{1j}}{4}\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\right] + O(\varepsilon^2) \\
&= \widetilde{w}_j(y) + \frac{p-1}{p}\frac{1}{\gamma^p}w^0_j(y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w^1_j(y) + O(\varepsilon|y-\xi'|) + O(\varepsilon^2) \\
&\quad - \log\frac{8\mu_j^2}{a(\xi_j)} + \left[C_{0j}\frac{p-1}{p}\frac{1}{\gamma^p} + C_{1j}\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\right]\left(\log(\mu_j) + \log\varepsilon\right) \\
&\quad + 8\pi\left[1 - \frac{C_{0j}}{4}\frac{p-1}{p}\frac{1}{\gamma^p} - \frac{C_{1j}}{4}\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\right]\left(H(\xi_j,\xi_j) + \sum_{i\neq j}^{K}G(\xi_i,\xi_j)\right).
\end{aligned}$$

Since numbers μ_j satisfy (2.19), we note that $p\gamma^p = -4\log\varepsilon$, then find

$$\begin{aligned}
&-\log\frac{8\mu_j^2}{a(\xi_j)} + \left[C_{0j}\frac{p-1}{p}\frac{1}{\gamma^p} + C_{1j}\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\right]\left(\log(\mu_j) + \log\varepsilon\right) \\
&\quad + 8\pi\left[1 - \frac{C_{0j}}{4}\frac{p-1}{p}\frac{1}{\gamma^p} - \frac{C_{1j}}{4}\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\right]\left(H(\xi_j,\xi_j) + \sum_{i\neq j}^{K}G(\xi_i,\xi_j)\right) = 0.
\end{aligned}$$

Thus (2.20) holds. □

We will look for solutions to (2.16) of the form

$$v = V_\lambda + \phi,$$

where V_λ is defined as in (2.18), and ϕ represents a lower order correction. We aim at finding a solution for ϕ small provided that the points ξ_j are suitably chosen. For small ϕ, we can rewrite problem (2.16) as a nonlinear perturbation of its linearization, namely,

$$\begin{cases} L(\phi) = -[E_\lambda + N(\phi)], & x \in \Omega_\varepsilon; \\ \phi = 0, & x \in \partial\Omega_\varepsilon, \end{cases} \tag{2.21}$$

where

$$L(\phi) := \Delta\phi + g'(V_\lambda)\phi, \tag{2.22}$$

$$E_\lambda := \Delta V_\lambda + g(V_\lambda), \tag{2.23}$$

$$N(\phi) := g(V_\lambda + \phi) - g(V_\lambda) - g'(V_\lambda)\phi. \tag{2.24}$$

We recall that $g(t) = a(\varepsilon y)(1 + \frac{t}{p\gamma^p})^{p-1}e^{\gamma^p[(1+\frac{t}{p\gamma^p})^p - 1]}$.

In order to solve the problem (2.21), first we have to study the invertibility properties of the linear operator L. In order to do this, we introduce a weighted L^∞-norm defined as

$$\|h\|_* := \sup_{y\in\Omega_\varepsilon} \left(\sum_{j=1}^{K}(1+|y-\xi_j'|)^{-3}+\varepsilon^2\right)^{-1} |h(y)| \tag{2.25}$$

for any $h \in L^\infty(\Omega_\varepsilon)$. With respect to this norm, the error term E_λ given in (2.23) can be estimated in the following way.

Lemma 2.2. *Let $\delta > 0$ be a small but fixed number and assume that the points $\xi \in \mathcal{O}$. There exists $C > 0$, such that we have*

$$\|E_\lambda\|_* \le \frac{C}{\gamma^{3p}} = \frac{C}{|\log\varepsilon|^3} \tag{2.26}$$

for all λ small enough.

Proof. Far away from the points ξ_j, namely for $|x-\xi_j| > \delta$, i.e. $|y-\xi_j'| > \frac{\delta}{\varepsilon}$, for all $j = 1, \dots, K$, from (2.7) and (2.13) we have that

$$\Delta V_\lambda(y) = p\gamma^{p-1}\varepsilon^2\Delta U(\varepsilon y) = O(\varepsilon^4).$$

On the other hand, in this region we have

$$1+\frac{V_\lambda(y)}{p\gamma^p} = 1+\frac{4\log\varepsilon+O(1)}{p\gamma^p} = \frac{O(1)}{|\log\varepsilon|} \tag{2.27}$$

where $O(1)$ denotes a smooth function, uniformly bounded, as $\varepsilon \to 0$, in the considered region. Hence

$$g(V_\lambda) = a(\varepsilon y)\left(1+\frac{V_\lambda}{p\gamma^p}\right)^{p-1} e^{\gamma^p\left[\left(1+\frac{V_\lambda}{p\gamma^p}\right)^p-1\right]} = \frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{p-1}}O(1).$$

Thus if we are far away from the points ξ_j, or equivalently for $|y-\xi_j'| > \frac{\delta}{\varepsilon}$, the size of the error, measured with respect to the $\|\cdot\|_*$-norm, is relatively small. In other words, if we denote by 1_{outer} the characteristic function of the set $\{y : |y-\xi_j'| > \frac{\delta}{\varepsilon},\ j = 1,\dots,K\}$, then in this region we have

$$\|E_\lambda 1_{\text{outer}}\|_* \le C\frac{\varepsilon^{\frac{2(2-p)}{p}}}{|\log\varepsilon|^{p-1}}. \tag{2.28}$$

Let us now fix the index j in $\{1,\dots,K\}$, for $|y-\xi_j'| < \frac{\delta}{\varepsilon}$, we have

$$\Delta V_\lambda(y) = -e^{w_j(y)} + \frac{p-1}{p}\frac{1}{\gamma^p}\Delta w_j^0(y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\Delta w_j^1(y) + O(\varepsilon^2). \tag{2.29}$$

On the other hand, for any $R > 0$ large but fixed, in the ball $|y-\xi_j'| < R_\varepsilon := R|\log\varepsilon|^\alpha$, with $\alpha \ge 3$, we can use Taylor expansion to first get

$$\left(1+\frac{V_\lambda}{p\gamma^p}\right)^{p-1} = 1+\frac{p-1}{p}\frac{1}{\gamma^p}\widetilde{w}_j + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\left[w_j^0+\frac{p-2}{2(p-1)}(\widetilde{w}_j)^2\right] + \left(\frac{p-1}{p}\right)^3\frac{1}{\gamma^{3p}}(\log|y-\xi_j'|),$$

$$\gamma^p\left[\left(1+\frac{V_\lambda}{p\gamma^p}\right)^p-1\right] = \widetilde{w}_j + \left(\frac{p-1}{p}\right)\frac{1}{\gamma^p}\left[w_j^0+\frac{(\widetilde{w}_j)^2}{2}\right] + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}(w_j^1+\widetilde{w}_jw_j^0) + \frac{1}{\gamma^{3p}}(\log|y-\xi_j'|)$$

and

$$e^{\gamma^p\left[\left(1+\frac{V_\lambda}{p\gamma^p}\right)^p-1\right]} = e^{\widetilde{w}_j}\left[1+\left(\frac{p-1}{p}\right)\frac{1}{\gamma^p}\left[w_j^0+\frac{(\widetilde{w}_j)^2}{2}\right]\right.$$
$$\left. + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\left[w_j^1+\widetilde{w}_jw_j^0+\frac{1}{2}(w_j^0+(\widetilde{w}_j)^2)^2\right] + \frac{1}{\gamma^{3p}}(\log|y-\xi_j'|)\right].$$

Thus we obtain

$$g(V_\lambda) := a(\varepsilon y)\left(1+\frac{V_\lambda}{p\gamma^p}\right)^{p-1} e^{\gamma^p\left[\left(1+\frac{V_\lambda}{p\gamma^p}\right)^p-1\right]}$$
$$= \left[a(\xi_j)+O(\varepsilon)\right]e^{\widetilde{w}_j}\left[1+\left(\frac{p-1}{p}\right)\frac{1}{\gamma^p}\left[w_j^0+\frac{(\widetilde{w}_j)^2}{2}+\widetilde{w}_j\right]\right.$$

$$+\left(\frac{p-1^2}{p}\right)\frac{1}{\gamma^{2p}}\left[w_j^1+2\widetilde{w}_jw_j^0+\frac{1}{2}\left(w_j^0+\frac{(\widetilde{w}_j)^2}{2}\right)^2+w_j^0+\frac{p-2}{2(p-1)}(\widetilde{w}_j)^2+\frac{\widetilde{w}_j^3}{2}\right]$$
$$+\ O\left(\frac{\log|y-\xi_j'|}{\gamma^{3p}}\right)\Bigg].$$

Thus, thanks to the fact that we have improved our original approximation with the terms w_j^0 and w_j^1, and the definition of $*$-norm, and we get that

$$\|E_\lambda 1_{B(\xi_j',R_\varepsilon)}\|_* \le \frac{C}{\gamma^{3p}}=\frac{C}{|\log\varepsilon|^3},\quad \text{for any } j=1,\dots,K. \tag{2.30}$$

Here $1_{B(\xi_j',R_\varepsilon)}$ denotes the characteristic function of $B(\xi_j,R_\varepsilon)$. Finally, in the remaining region, namely where $R_\varepsilon<|y-\xi_j'|<\frac{\delta}{\varepsilon}$, for any $j=1,\dots,K$, we have from one hand that $|\Delta V_\lambda(y)|\le Ce^{w_j(y)}$, and also $|g(V_\lambda(y))|\le Ce^{w_j(y)}$ as a consequence of (2.20). This fact, together with (2.30) and (2.28) we obtain estimate (2.26). □

As the same proof of above lemma, we have the following result.

Lemma 2.3. *For x very close to the point ξ_j in Ω, we have*

$$\|g'(V_\lambda)-e^{w_j}\|_*\to 0\quad \text{as } \lambda\to 0, \tag{2.31}$$

and there exists some positive constant D_0 such that

$$g'(V_\lambda)\le D_0\sum_{j=1}^{K}e^{w_j}. \tag{2.32}$$

Moreover,

$$\|g''(V_\lambda)\|_*\le C. \tag{2.33}$$

3. The linear problem

The main objective of this section is to study the invertibility of the linearized operator L. Let us recall that for x close point ξ_j,

$$\|g'(V_\lambda)-e^{w_j}\|_*\to 0\quad \text{as } \lambda\to 0.$$

Thus we can see that the operator L can be approximated by the family

$$L_j(\phi)=\Delta\phi+e^{w_j}\phi=\Delta\phi+\frac{8\mu_j^2}{\left(\mu_j^2+|y-\xi_j'|^2\right)^2}\phi,$$

which is non-degenerate, in the sense that the bounded solutions of $L_j(\phi)=0$, excepting rescaling and translation are (see [1])

$$z_{0,j}=\partial_{\mu_j}w_{\mu_j},\qquad z_{i,j}(y)=\partial_{y_i}w_{\mu_j}(y),\quad i=1,2.$$

We consider $\xi=(\xi_1,\dots,\xi_K)\in\mathcal{O}$, let us consider a large but fixed number $R_0>0$ and a smooth cut-off function $\eta(\rho)$ with $\eta(\rho)=1$ if $\rho<R_0$ and $\eta(\rho)=0$ if $\rho>R_0+1$, and we denote

$$\eta_j(y)=\eta(|y-\xi_j'|).$$

Moreover, let $h\in C^{0,\alpha}(\Omega_\varepsilon)$, we consider the linear problem of finding a function ϕ and scalars $c_{ij},\ i=1,2,\ j=1,\dots,K$, such that

$$L(\phi)=h+\sum_{j=1}^{2}\sum_{j=1}^{K}c_{ij}\eta_jZ_{ij}\quad \text{in } \Omega_\varepsilon$$
$$\phi=0\quad \text{on } \partial\Omega_\varepsilon$$
$$\int_{\Omega_\varepsilon}\eta_jZ_{ij}\phi=0\quad \text{for all } i=1,2,\ j=1,\dots,K.$$

Lemma 3.1. *Let $\tilde{\Omega}_\varepsilon:=\Omega\setminus\cup_{j=1}^K B(\xi_j,R)$. There exists R large enough such that if $L(\phi)\le 0$ in $\tilde{\Omega}_\varepsilon$ and $\phi\ge 0$ on $\partial\tilde{\Omega}_\varepsilon$ then $\phi\ge 0$ in $\tilde{\Omega}_\varepsilon$.*

Proof. We define the function

$$Z(y)=\sum_{j=1}^{K} z_0(a|y-\xi_j|),\quad y\in\Omega_\varepsilon,$$

where $z_0(r)=\frac{r^2-1}{r^2+1}$.

If a is taken small and fixed, and R large enough such that $a|y-\xi_j'|>aR\gg 1$ then $Z(y)>0$. From (2.32) we have

$$\begin{aligned}L(Z)&=-\sum_{j=1}^{K}\frac{8a^2(a^2|y-\xi_j|^2-1)}{(1+a^2(|y-\xi_j'|^2))^3}+g'(V_\lambda)Z(y)\le-\sum_{j=1}^{K}\frac{c}{a^2|y-\xi_j|^4}+D_0\sum_{j=1}^{K}e^{w_j}Z(y)\\&\le-\sum_{j=1}^{K}\frac{c}{a^2|y-\xi_j|^4}+\sum_{j=1}^{K}\frac{C}{|y-\xi_j|^4}\le 0.\quad\square\end{aligned}$$

We consider for R as in Lemma 3.1, define the inner norm as follows:

$$\|\phi\|_i:=\sup_{y\in\cup_{j=1}^{K}B(\xi_j',R)}|\phi(y)|.$$

Lemma 3.2. *Let $h\in L^\infty(\Omega_\varepsilon)$ if we consider the equation*

$$L(\phi)=h\quad in\ \Omega_\varepsilon \tag{3.1}$$

$$\phi=0\quad on\ \partial\Omega_\varepsilon, \tag{3.2}$$

then there exists $C>0$ such that

$$\|\phi\|_\infty\le C[\|\phi\|_i+\|h\|_*]. \tag{3.3}$$

Proof. Let us take the following barrier

$$\tilde\phi(y)=2\|\phi\|_iZ(y)+\|h\|_*\sum_{j=1}^{K}\psi_j(y)$$

where ψ_j is a solution of the equation:

$$-\Delta\psi_j=\frac{2}{|y-\xi_j'|^3}+2\varepsilon^2,\quad R<|y-\xi_j'|<\frac{M}{\varepsilon}, \tag{3.4}$$

$$\psi_j=0\quad \text{if } |y-\xi_j'|=R,\ |y-\xi_j'|=\frac{M}{\varepsilon}, \tag{3.5}$$

where M is such that $\Omega_\varepsilon\subset B(\xi_j',\frac{M}{\varepsilon})$. A direct computation shows that

$$\psi(r)=-\frac{2}{r}-\frac{\varepsilon^2r^2}{2}+a\log(r)+b$$

where $a=\frac{\frac{2}{R}+\frac{\varepsilon^2R^2}{2}-\frac{\varepsilon}{M}-\frac{M^2}{2}}{\log(\frac{\varepsilon R}{M})}$ and $b=\frac{2}{R}+\frac{\varepsilon^2R^2}{2}-a\log R$. Hence $\psi(r)$ is a uniform bound function independent of ε as long as $1<R<\frac{1}{2\varepsilon}$. By the Maximum Principle one has $\psi_j\ge 0$. Therefore, by the definition of $Z(y)$ and for R large enough

$$\tilde\phi(y)\ge|\phi(y)|\quad \text{in } |y-\xi_j'|=R,$$

$$\tilde\phi\ge 0=\phi(y)\quad \text{on } \partial\Omega_\varepsilon.$$

Moreover

$$\begin{aligned}L(\tilde\phi)&=2\|\phi\|_iL(Z)+\|h\|_*L\left(\sum_{j=1}^{K}\psi_j\right)\le\|h\|_*\sum_{j=1}^{K}\left(\Delta\psi_j+g'(V_\lambda)\psi_j\right)\\&=\|h\|_*\sum_{j=1}^{K}\left(-\frac{2}{|y-\xi_j'|^3}-2\varepsilon^2+g'(V_\lambda)\psi_j\right)\end{aligned}$$

$$
\begin{aligned}
&\le \|h\|_* \sum_{j=1}^{K}\left(-\frac{2}{|y-\xi_j'|^3} - 2\varepsilon^2 + \frac{2KD_0}{R}e^{w_j}\right)\\
&\le -\|h\|_*\left(\sum_{j=1}^{K}(1+|y-\xi_j'|)^{-3} + \varepsilon^2\right)\\
&\le -|h(y)| \le |L(\phi)(y)|.
\end{aligned}
$$

From Lemma 3.1 one has

$$|\phi| \le |\tilde{\phi}(y)|; \quad y \in \Omega_\varepsilon.$$

Since ψ_j is uniformly bounded over ε, there exists $C > 0$ such that

$$\|\phi\|_\infty \le C[\|\phi\|_i + \|h\|_*]. \quad \square$$

Lemma 3.3. *We consider the equation*

$$
\begin{aligned}
&L(\phi) = h \quad \text{in } \Omega_\varepsilon,\\
&\phi = 0 \quad \text{in } \partial\Omega_\varepsilon,\\
&\int_{\Omega_\varepsilon} \eta_j Z_{ij}\phi = 0 \quad \text{for } i = 0, 1, 2,\ j = 1, 2 \dots, K.
\end{aligned}
$$

Then there exist positive numbers λ_0, C such that, for all $\xi \in \mathcal{O}$ we have

$$\|\phi\|_\infty \le C\|h\|_*$$

for all $\lambda < \lambda_0$.

Proof. By contradiction, we suppose that there exist $\lambda_n \to 0$, $(\xi_1^n, \xi_2^n, \dots, \xi_K^n) \in \mathcal{O}$, functions h_n and ϕ_n, satisfy the above equation, with $\|h_n\|_* \to 0$, $\|\phi_n\|_\infty = 1$. By Lemma 3.2, we have that $\|\phi_n\|_i > \kappa > 0$. Let $\hat{\phi}_n(z) = \phi_n((\xi_j^n)' + z)$, where the index j is such that $\sup_{|y-(\xi_j^n)'|<R} |\phi_n| \ge \kappa$. We assume j is the same for all n. By local elliptic estimates, we get that $\hat{\phi}_n$ converges uniformly over compact set to a bounded solution $\hat{\phi} \ne 0$ of the following equation

$$\Delta\phi + \frac{8\mu_j^2}{(\mu_j^2 + |z|^2)^2}\phi = 0 \quad \text{in } \mathbb{R}^2.$$

The non degeneracy of the equation and the orthogonality condition give us the contradiction. $\square$

Lemma 3.4. *Let $\delta > 0$ be fixed and small. There exist positive numbers λ_0 and C, such that for $\xi \in \mathcal{O}$, and any solution ϕ to the following problem*

$$
\begin{cases}
L(\phi) = h, \quad \text{in } \Omega_\varepsilon\\
\phi = 0, \quad \text{on } \partial\Omega_\varepsilon\\
\displaystyle\int_{\Omega_\varepsilon} \eta_j Z_{ij}\phi = 0, \quad \text{for } i = 1, 2,\ j = 1, \dots, K.
\end{cases}
\tag{3.6}
$$

Then

$$\|\phi\|_\infty \le C(-\log\varepsilon)\|h\|_*$$

for all $\lambda < \lambda_0$.

Proof. Let $R > R_0 + 1$ be a large and fixed number, and $\hat{z}_0$ be the solution of the problem

$$
\begin{cases}
\Delta\hat{z}_{0j} + \dfrac{8\mu_j^2}{\left(\mu_j^2 + |y-\xi_j'|^2\right)^2}\hat{z}_{0j} = 0, & \\
\hat{z}_{0j}(y) = z_{0j}(R) & \text{for } |y-\xi_j'| = R,\\
\hat{z}_{0j}(y) = 0 & \text{for } |y-\xi_j'| = \dfrac{\delta}{3\varepsilon}.
\end{cases}
$$

By computation, this function is explicitly given by

$$\hat{z}_{0j}(y) = z_{0j}(y)\left[1 - \frac{\int_R^r \frac{ds}{sz_{0j}^2(s)}}{\int_R^{\frac{\delta}{3\varepsilon}} \frac{ds}{sz_{0j}^2(s)}}\right], \quad r = |y - \xi_j'|.$$

Next we consider the radial smooth cut-off functions χ_1 and χ_2 with the following properties

$$0 \le \chi_1 \le 1,\ \chi_1 \equiv 1 \quad \text{in } B(0, R), \qquad \chi_1 \equiv 0 \quad \text{in } B(0, R+1)^c; \quad \text{and}$$
$$0 \le \chi_2 \le 1,\ \chi_2 \equiv 1 \quad \text{in } B\left(0, \frac{\delta}{4\varepsilon}\right), \qquad \chi_1 \equiv 0 \quad \text{in } B\left(0, \frac{\delta}{3\varepsilon}\right)^c,$$

and $|\chi_2'(r)| \le C\varepsilon$, $|\chi_2''(r)| \le C\varepsilon^2$. Then we set

$$\chi_{1j}(y) = \chi_1(|y - \xi_j'|), \qquad \chi_{2j}(y) = \chi_2(|y - \xi_j'|),$$

and define

$$\tilde{z}_{0j} = \chi_{1j} Z_{0j} + (1 - \chi_{1j})\chi_{2j}\hat{z}_{0j}.$$

Let ϕ be a solution to Eq. (3.6), we will modify ϕ so that the extra orthogonality condition with respect to Z_{0j} holds. We set

$$\tilde{\phi} = \phi + \sum_{j=1}^{K} d_j \tilde{z}_{0j}$$

with the number d_j defined as

$$d_j = -\frac{\int_{\Omega_\varepsilon} \eta_j Z_{0j}\phi}{\int_{\Omega_\varepsilon} \eta_j |Z_{0j}|^2}.$$

Then

$$L(\tilde{\phi}) = h + \sum_{j=1}^{K} d_j L(\tilde{z}_{0j}), \tag{3.7}$$

and

$$\int_{\Omega_\varepsilon} \eta_j Z_{0i}\tilde{\phi} = 0, \quad \text{for all } i = 0, 1, 2.$$

Then from the previous lemma we have the following estimate

$$\|\tilde{\phi}\|_\infty \le C\left[\|h\|_* + \sum_{j=1}^{K} |d_j|\, \|L(\tilde{z}_{0j})\|_*\right]. \tag{3.8}$$

Next, we show that

$$\|L(\tilde{z}_{0j})\|_* \le \frac{C}{\log\frac{1}{\varepsilon}}, \quad \text{and} \quad |d_j| \le C\left(\log\frac{1}{\varepsilon}\right)^2 \|h\|_*. \tag{3.9}$$

Indeed, we have

$$L(\tilde{z}_{0j}) = 2\nabla\chi_{1j}\nabla(Z_{0j} - \hat{z}_{0j}) + \Delta\chi_{1j}(Z_{0j} - \hat{z}_{0j}) + 2\nabla\chi_{2j}\nabla\hat{z}_{0j} + \Delta\chi_{2j}\hat{z}_{0j} + O(\varepsilon^4).$$

We consider the following four regions

$$\Omega_1 = \{y : |y - \xi_j'| \le R\}, \qquad \Omega_2 = \{y : R < |y - \xi_j'| < R+1\},$$
$$\Omega_3 = \left\{y : R+1 \le |y - \xi_j'| \le \frac{\delta}{4\varepsilon}\right\}, \qquad \Omega_2 = \left\{y : \frac{\delta}{4\varepsilon} < |y - \xi_j'| < \frac{\delta}{3\varepsilon}\right\}.$$

First, we note that $L(\tilde{z}_0) = O(\varepsilon^4)$ for $y \in \Omega_1 \cup \Omega_3$. For $y \in \Omega_2$, we have

$$\hat{z}_{0j} - Z_{0j} = -z_{0j}(r)\frac{\int_R^r \frac{ds}{sz_{0j}^2(s)}}{\int_R^{\frac{\delta}{3\varepsilon}} \frac{ds}{sz_{0j}^2(s)}}.$$

Then

$$|\hat{z}_{0j} - Z_{0j}| \leq \frac{C}{\log \frac{1}{\varepsilon}}.$$

Similarly, in this region, we have

$$|\hat{z}'_{0j} - Z'_{0j}| \leq \frac{C}{\log \frac{1}{\varepsilon}}.$$

On the other hand, for $y \in \Omega_4$, we have

$$\hat{z}_{0j}(r) \leq \frac{C}{\log \frac{1}{\varepsilon}}, \quad \text{and} \quad \hat{z}'_{0j}(r) \leq \frac{C\varepsilon}{\log \frac{1}{\varepsilon}}.$$

Therefore, by the definition of the $*$-norm, we get

$$\|L(\tilde{z}_{0j})\|_* \leq \frac{C}{\log \frac{1}{\varepsilon}}, \tag{3.10}$$

where the number C depends in principle of the chosen large constant R.

Next we show the other inequality of (3.9) holds. Testing Eq. (3.7) against $\tilde{z}_{0l}$ we have

$$\langle \tilde{\phi}, L(\tilde{z}_{0l})\rangle = \langle h, \tilde{z}_{0l}\rangle + d_l \langle L(\tilde{z}_{0l}), \tilde{z}_{0l}\rangle,$$

where $\langle f, g\rangle = \int_{\Omega_\varepsilon} fg$. This relation and (3.8) provide that

$$d_l \langle L(\tilde{z}_{0l}), \tilde{z}_{0l}\rangle \leq C\|h\|_*[1 + \|L(\tilde{z}_{0l})\|_*] + C\sum_{j=1}^{K} |d_j|\, \|L(\tilde{z}_{0l})\|_*^2. \tag{3.11}$$

We want to measure the size of $\langle L(\tilde{z}_{0l}), \tilde{z}_{0l}\rangle$. We decompose

$$\langle L(\tilde{z}_{0l}), \tilde{z}_{0l}\rangle = \int_{\Omega_2} L(\tilde{z}_{0l})\tilde{z}_{0l} + \int_{\Omega_4} L(\tilde{z}_{0l})\tilde{z}_{0l} + O(\varepsilon). \tag{3.12}$$

Since

$$\begin{aligned} \left|\int_{\Omega_4} L(\tilde{z}_{0l})\tilde{z}_{0l}\right| &\leq C\int |\nabla\chi_{2l}|\,|\nabla\hat{z}_{0l}|\,|\hat{z}_{0l}| + C\int |\Delta\chi_{2l}|\,|\hat{z}_{0l}|^2 + O(\varepsilon^2) \\ &\leq \frac{C}{\left(\log \frac{1}{\varepsilon}\right)^2}. \end{aligned} \tag{3.13}$$

Moreover, for $y \in \Omega_2$, we have

$$\begin{aligned} \int_{\Omega_2} L(\tilde{z}_{0l})\tilde{z}_{0l} &= 2\int \nabla\chi_{1l}\nabla(Z_{0l} - \hat{z}_{0l})\hat{z}_0 + \int \Delta\chi_{1l}(Z_{0l} - \hat{z}_{0l})\hat{z}_{0l} + O(\varepsilon) \\ &= \int \nabla\chi_{1l}\nabla(Z_{0l} - \hat{z}_{0l})\hat{z}_{0l} - \int \nabla\chi_{1l}(Z_{0l} - \hat{z}_{0l})\nabla\hat{z}_{0l} + O(\varepsilon). \end{aligned}$$

Now, we observe that in the considered region Ω_2, $|\hat{z}_{0l} - Z_{0l}| \leq \frac{C}{\log \frac{1}{\varepsilon}}$, while $|\hat{z}'_{0l}| \sim \frac{1}{R^3} + \frac{1}{R}\frac{1}{\log \frac{1}{\varepsilon}}$. Then, for R is large but independent of ε we have

$$\left|\int \nabla\chi_{1l}(Z_{0l} - \hat{z}_{0l})\nabla\hat{z}_{0l}\right| \leq \frac{C_1}{R^3}\frac{1}{\log \frac{1}{\varepsilon}},$$

where C_1 is a constant to be chosen independent of R. Moreover

$$\begin{aligned} \int \nabla\chi_{1l}\nabla(Z_{0l} - \hat{z}_{0l})\hat{z}_{0l} &= 2\pi\int_R^{R+1} \chi'_{1l}(z_{0l} - \hat{z}_{0l})'\hat{z}_{0l} r\, dr \\ &= \frac{2\pi}{\int_R^{\frac{\delta}{3\varepsilon}} \frac{ds}{s z_{0l}^2}} \int_R^{R+1} \chi'_{1l}\left[1 - \frac{4\mu_l^2 r^2 z_{0l}\int_R^r \frac{ds}{s z_{0l}^2}}{(\mu_l^2 + r^2)^2}\right] dr \\ &= -\frac{C_2}{\log \frac{1}{\varepsilon}}\left[1 + O\left(\frac{1}{\log \frac{1}{\varepsilon}}\right)\right], \end{aligned}$$

Please cite this article in press as: S. Deng, et al., Multiple blow-up solutions for an exponential nonlinearity with potential in $\mathbb{R}^2$, Nonlinear Analysis (2014), http://dx.doi.org/10.1016/j.na.2014.10.034

where C_2 is a positive constant independent on ε. Thus, choosing R large enough, we get

$$\int_{\Omega_2} L(\tilde{z}_{0l})\tilde{z}_{0l} \sim -\frac{C_2}{\log\frac{1}{\varepsilon}}.$$

Combining this and (3.12), (3.13) we get

$$\langle L(\tilde{z}_{0l}), \tilde{z}_{0l}\rangle \leq -\frac{C_2}{\log\frac{1}{\varepsilon}}\left[1 + O\left(\frac{1}{\log\frac{1}{\varepsilon}}\right)\right]. \tag{3.14}$$

From (3.10), (3.11) and (3.13) we have

$$|d_j| \leq C\left(\log\frac{1}{\varepsilon}\right)^2 \|h\|_*.$$

We thus have from estimate (3.8) that

$$\|\phi\|_\infty \leq C\left(\log\frac{1}{\varepsilon}\right)\|h\|_*. \quad \square$$

We are ready to obtain the principal result of this section.

Proposition 3.5. *There exist positive numbers λ_0 and C, such that for $\xi \in \mathcal{O}$, there is unique solution $\phi = T_\lambda(h)$ to:*

$$\begin{aligned} &L(\phi) = h + \sum_{j=1}^{2}\sum_{j=1}^{K} c_{ij}\eta_j Z_{ij} \quad \text{in } \Omega_\varepsilon \\ &\phi = 0 \quad \text{on } \partial\Omega_\varepsilon \\ &\int_{\Omega_\varepsilon} \eta_j Z_{ij}\phi = 0 \quad \text{for all } i = 1, 2,\ j = 1, \dots, K, \end{aligned} \tag{3.15}$$

for all $\lambda < \lambda_0$. Moreover

$$\|\phi\|_\infty \leq C\left(\log\frac{1}{\varepsilon}\right)\|h\|_*. \tag{3.16}$$

We just considered the orthogonality conditions with respect to the elements of the approximate kernel due to translation.

Proof. Let us consider the cut-off function χ_{2j} introduced before. Testing Eq. (3.15) against $Z_{ij}\chi_{2j}$ we get

$$\langle L(\phi), Z_{ij}\chi_{2j}\rangle = \langle h, Z_{ij}\chi_{2j}\rangle + c_{ij}\int_{\Omega_\varepsilon} \eta_j |Z_{ij}|^2. \tag{3.17}$$

Moreover

$$\langle L(\phi), Z_{ij}\chi_{2j}\rangle = \langle \phi, L(Z_{ij}\chi_{2j})\rangle.$$

We have

$$L(Z_{ij}\chi_{2j}) = \Delta\chi_{2j}Z_{ij} + 2\nabla Z_{ij}\nabla\chi_{2j} + \varepsilon O((1+r)^{-3}),$$

with $r = |y - \xi_j'|$. Since $\Delta\chi_{2j} = O(\varepsilon^2)$, $\nabla\chi_{2j} = O(\varepsilon)$, and $Z_{ij} = O(r^{-1})$, $\nabla Z_{ij} = O(r^{-2})$, we get

$$L(Z_{ij}\chi_{2j}) = O(\varepsilon^3)\varepsilon O((1+r)^{-3}).$$

Then we have

$$|\langle L(\phi), Z_{ij}\chi_{2j}\rangle| = |\langle \phi, L(Z_{ij}\chi_{2j})\rangle| \leq C\varepsilon\|\phi\|_\infty.$$

From the previous lemma we find

$$\|\phi\|_\infty \leq C\left(\log\frac{1}{\varepsilon}\right)\left[\|h\|_* + \sum_{i=1}^{2}\sum_{j=1}^{K} c_{ij}\right]. \tag{3.18}$$

Combining this with (3.17) and (3.18)

$$|c_{ij}| \le C\left[\|h\|_* + \varepsilon \log\frac{1}{\varepsilon}\sum_{l,m}|c_{lm}|\right]. \tag{3.19}$$

Then,

$$|c_{ij}| \le C\|h\|_*.$$

Combining this with (3.18) we obtain the estimate

$$\|\phi\|_\infty \le C\left(\log\frac{1}{\varepsilon}\right)\|h\|_*.$$

Next prove the solvability assertion. We consider the Hilbert space

$$\mathbb{H} = \left\{\phi \in H_0^1(\Omega_\varepsilon) : \int_{\Omega_\varepsilon}\phi Z_{ij}\eta_j = 0 \text{ for } i = 1, 2,\ j = 1, 2, \dots, K\right\},$$

endowed with the usual inner product $\langle\phi, \psi\rangle = \int_{\Omega_\varepsilon}\nabla\phi\nabla\psi$. Problem (3.15), expressed in a weak form, is equivalent to find $\phi \in \mathbb{H}$ such that

$$\langle\phi, \psi\rangle = \int_{\Omega_\varepsilon}(W\phi - h)\psi\, dx, \quad \text{for all } \psi \in \mathbb{H},$$

where $W = g'(V_\lambda)$. With the aid of Riesz's representation theorem, this equation gets rewritten in $\mathbb{H}$ in the operator form

$$(Id - R)\phi = \tilde{h}, \tag{3.20}$$

for certain $\tilde{h} \in \mathbb{H}$, where R is a compact operator in $\mathbb{H}$. The homogeneous equation $\phi = R\phi$ in $\mathbb{H}$, which is equivalent to (3.15) with $h \equiv 0$, has only the trivial solution in view of the a priori estimate (3.16). Now, Fredholm's alternative guarantees unique solvability of (3.20) for any $\tilde{h} \in \mathbb{H}$. This finishes the proof. □

Lemma 3.6. *The operator T_λ is differentiable with respect to the variable $(\xi_1, \dots, \xi_K) \in \mathcal{O}$. Moreover one has the estimate*

$$\|\partial_{(\xi'_m)_l}T_\lambda(h)\|_\infty \le C\left(\log\frac{1}{\varepsilon}\right)^2\|h\|_*.$$

Proof. Let $\phi = T_\lambda(h)$ where ϕ satisfies the equation

$$L(\phi) = h + \sum_{i,j}c_{ij}Z_{ij}\eta_j$$

with additional conditions, for some unique constants c_{ij}. Formally $Z = \partial_{(\xi'_m)_l}\phi$ should satisfy

$$L(Z) = -\partial_{(\xi'_m)_l}(g'(V_\lambda))\phi + \sum_{i=1}^{2}c_{im}\partial_{(\xi'_m)_l}(\eta_m Z_{im}) + \sum_{i=1}^{2}\sum_{j=1}^{K}d_{ij}Z_{ij}\eta_j \tag{3.21}$$

with $d_{ij} = \partial_{(\xi'_m)_l}c_{ij}$ and the orthogonality conditions become

$$\int_{\Omega_\varepsilon}Z_{im}\eta_m Z = -\int_{\Omega_\varepsilon}\partial_{(\xi'_m)_l}(Z_{im}\eta_m)\phi.$$

We consider the projected function

$$\tilde{Z} = Z + \sum b_{im}\eta_m Z_{im}$$

such that

$$\int_{\Omega_\varepsilon}\eta_j Z_{ij}\tilde{Z} = 0.$$

Then

$$b_{im}\int_{\Omega_\varepsilon}\eta_m Z_{im}^2 = \int_{\Omega_\varepsilon}\partial_{(\xi'_m)_l}(Z_{im}\eta_m)\phi.$$

We write Eq. (3.21) in the way that (3.15)

$$\begin{cases} L(\tilde{Z}) = f + \sum_{i=1}^{2}\sum_{j=1}^{K} b_{im}\eta_m Z_{im} & \text{in } \Omega_\varepsilon \\ \tilde{Z} = 0 \quad \text{on } \partial\Omega_\varepsilon \\ \int_{\Omega_\varepsilon} \eta_j Z_{im}\tilde{Z} = 0 \quad \text{for } i = 0, 1, 2, \end{cases} \tag{3.22}$$

and by Proposition (3.15), we get

$$\tilde{Z} = T_\lambda(f), \tag{3.23}$$

where f satisfies

$$\|f\|_* \le C\|\phi\|_\infty.$$

Using (3.16) we find

$$\|\partial_{(\xi'_m)_l} T_\lambda(h)\|_\infty \le C\left(\log\frac{1}{\varepsilon}\right)\|f\|_* \le C\left(\log\frac{1}{\varepsilon}\right)\|\phi\|_\infty \le C\left(\log\frac{1}{\varepsilon}\right)^2\|h\|_*. \quad \square$$

4. The nonlinear problem

Following the approach in [9] for $a(x) = 1$, we have the following result.

Lemma 4.1. *There exist $\lambda_0 > 0$ and a constant $C > 0$ such that for any $\lambda \in (0, \lambda_0)$ and each $\xi \in \mathcal{O}$, there exists a unique ϕ satisfying*

$$\begin{cases} \Delta(V_\lambda + \phi) + g\,(V_\lambda + \phi) = \sum_{i=1}^{2}\sum_{j=1}^{K} c_{ij}Z_{ij}\eta_j & \text{in } \Omega_\varepsilon; \\ \phi = 0 \quad \text{on } \partial\Omega_\varepsilon; \\ \int_{\Omega_\varepsilon} \phi Z_{ij}\eta_j = 0 \quad \text{for } i = 1, 2,\ j = 1, \ldots, K \end{cases} \tag{4.1}$$

for some $c_{ij} \in \mathbb{R}$. Moreover,

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \le \frac{C}{|\log\varepsilon|^2}.$$

Furthermore, the map $\xi' \mapsto \phi \in H_0^1(\Omega_\varepsilon)$ is $\mathcal{C}^1$, and

$$\|D_{\xi'}\phi\|_{L^\infty(\Omega_\varepsilon)} \le \frac{C}{|\log\varepsilon|}.$$

We included the proof just for completeness.

Proof. From Proposition 3.5 Eq. (4.1) is equivalent to find ϕ such that

$$\phi = T_\lambda(-(N(\phi) + E_\lambda)) := A(\phi) \tag{4.2}$$

where

$$\|A(\phi)\|_\infty \le C\left(\log\frac{1}{\varepsilon}\right)\left[\|N(\phi)\|_* + \|E_\lambda\|_*\right]. \tag{4.3}$$

To $N(\phi)$ we have that there exist $s \in (0, 1)$ such that

$$|N(\phi)| \le C|g''(V_\lambda + s\phi)|\,|\phi|^2 \le C|g''(V_\lambda + s\phi)|\,\|\phi\|_\infty^2.$$

From the previous step, we know that $\|\phi\|_\infty \to 0$ as $\lambda \to 0$ and from (2.33)

$$\|g''(V_\lambda)\|_* \le C;$$

then we get

$$\|N(\phi)\|_* \le C\|\phi\|_*$$

we combine this with (4.3) to get

$$\|A(\phi)\|_\infty \le C|\log\varepsilon|\left(C\|\phi\|_\infty^2 + \frac{1}{|\log\varepsilon|^3}\right).$$

For a given number $M > 0$, let us consider the region

$$\mathcal{F}_M := \left\{\phi \in C(\bar{\Omega}) : \|\phi\|_\infty \le \frac{M}{|\log\varepsilon|^2}\right\}.$$

We then get that $A(\mathcal{F}_M) \subset \mathcal{F}_M$ for a sufficiently large but fixed M and all small λ. Moreover, for any $\phi_1, \phi_2 \in \mathcal{F}_M$, using standard argument on mean value integral, one has

$$\|N(\phi_1) - N(\phi_2)\|_* \le C\left(\max_{i=1,2}\|\phi_i\|_\infty\right)\|\phi_1 - \phi_2\|_\infty.$$

Thanks to (2.33) and the fact that $\|\phi_1\|_\infty, \|\phi_2\|_\infty \to 0$ as $\lambda \to 0$, we conclude that

$$\|N(\phi_1) - N(\phi_2)\|_* \le C\|g''(V_\lambda)\|_*(\|\phi_1\|_\infty + \|\phi_2\|_\infty)\|\phi_1 - \phi_2\|_\infty \le C(\|\phi_1\|_\infty + \|\phi_2\|_\infty)\|\phi_1 - \phi_2\|_\infty.$$

Then we have

$$\|A(\phi_1) - A(\phi_2)\|_\infty \le C|\log\varepsilon|\,\|N(\phi_1) - N(\phi_2)\|_* \le C|\log\varepsilon|\left(\max_{i=1,2}\|\phi_i\|_\infty\right)\|\phi_1 - \phi_2\|_\infty.$$

Thus the operator A has a small Lipschitz constant in $\mathcal{F}_M$ for all small λ, and therefore a unique fixed point of A exists in this region.

We shall next analyze the differentiability of the map $\xi' = (\xi_1', \ldots, \xi_K') \mapsto \phi$. Assume for instance that the partial derivative $\partial_{(\xi_j')_i}\phi$ exists for $i = 1, 2$. Since $\phi = T_\lambda(-(N(\phi) + E_\lambda))$, formally that

$$\partial_{(\xi_j')_i}\phi = (\partial_{(\xi_j')_i}T_\lambda)(-(N(\phi) + E_\lambda)) + T_\lambda\left(-(\partial_{(\xi_j')_i}N(\phi) + \partial_{(\xi_j')_i}E_\lambda)\right).$$

From Lemma 3.6, we have

$$\|\partial_{(\xi_j')_i}T_\lambda(-(N(\phi) + E_\lambda))\|_\infty \le C|\log\varepsilon|^2\|N(\phi) + E_\lambda\|_* \le C\frac{1}{|\log\varepsilon|}.$$

On the other hand,

$$\begin{aligned}\partial_{(\xi_j')_i}N(\phi) &= [g'(V_\lambda + \phi) - g'(V_\lambda) - g''(V_\lambda)\phi]\partial_{(\xi_j')_i}V_\lambda + \partial_{(\xi_j')_i}[g'(V_\lambda) - e^{w_j}]\phi \\ &\quad + [g'(V_\lambda + \phi) - g'(V_\lambda)]\partial_{(\xi_j')_i}\phi + [g'(V_\lambda) - e^{w_j}]\partial_{(\xi_j')_i}\phi.\end{aligned}$$

Then,

$$\|\partial_{(\xi_j')_i}N(\phi)\|_* \le C\left\{\|\phi\|_\infty^2 + \frac{1}{|\log\varepsilon|}\|\phi\|_\infty + \|\partial_{(\xi_j')_i}\phi\|_\infty\|\phi\|_\infty + \frac{1}{|\log\varepsilon|}\|\partial_{(\xi_j')_i}\phi\|_\infty\right\}.$$

Since $\|\partial_{(\xi_j)_i}E_\lambda\|_* \le \frac{C}{|\log\varepsilon|^3}$, and by Proposition 3.5 we have

$$\|\partial_{(\xi_j')_i}\phi\|_\infty \le \frac{C}{|\log\varepsilon|},$$

for all $i = 1, 2, j = 1, \ldots, K$. Then, the regularity of the map $\xi' \mapsto \phi$ can be proved by standard arguments involving the implicit function theorem and the fixed point representation (4.2). This concludes proof of the lemma. □

5. The finite dimensional reduction

After problem (4.1) has been solved, we find a solution to problem (2.21), if we can find a point $\xi' = \frac{\xi}{\varepsilon} = (\xi_1', \ldots, \xi_K')$ such that coefficients $c_{ij}(\xi')$ in (4.1) satisfy

$$c_{ij}(\xi') = 0 \quad \text{for all } i = 1, 2,\ j = 1, \ldots, K. \tag{5.1}$$

We now introduce the finite dimensional restriction $\mathcal{J}_\lambda(\xi) : \mathcal{O} \to \mathbb{R}$, given by

$$\mathcal{J}_\lambda(\xi) = J_{a,\lambda}^p\left(\left(U_\lambda + \tilde{\phi}\right)(x, \xi)\right) \tag{5.2}$$

where

$$\left(U_\lambda + \tilde{\phi}\right)(x,\xi) = \gamma + \frac{1}{p\gamma^{p-1}}\left((V_\lambda + \phi)\left(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\right)\right) \tag{5.3}$$

with V_λ defined in (2.18), ϕ is the unique solution to problem (4.1) given by Lemma 4.1.

The following result can be proved by using standard arguments, see Lemma 5.1 in [9].

Lemma 5.1. *For all $\lambda > 0$ sufficiently small, the functional $\mathcal{J}_\lambda(\xi)$ is of class C^1. Moreover, if $\xi \in \mathcal{O}$ is a critical point of $\mathcal{J}$, then $U_\lambda + \tilde{\phi}$ is a critical point of $J^p_{a,\lambda}$, namely a solution to the problem (1.1).*

Next we need to write the expansion of $\mathcal{J}_\lambda(\xi)$ as λ goes to 0.

Lemma 5.2. *Let $\delta > 0$ be fixed. There exists positive number λ_0, such that μ_j are given by (2.19), for any $0 < \lambda < \lambda_0$, the following expansion holds*

$$p\left(-\frac{4}{p}\log\varepsilon\right)^{\frac{2(p-1)}{p}} \mathcal{J}_\lambda(\xi) = \frac{8K\pi}{(2-p)p}[-2 + p\log 8] - \frac{16K\pi}{p}\log\varepsilon - \frac{32\pi^2}{2-p}\Phi^p_{a,K}(\xi) + |\log\varepsilon|^{-1}\theta_\lambda(\xi) \tag{5.4}$$

uniformly for any points $(\xi_1, \ldots, \xi_K) \in \mathcal{O}$, where

$$\Phi^p_{a,K}(\xi) = \sum_{j=1}^{K} H(\xi_j,\xi_j) + \sum_{i\neq j}^{K} G(\xi_i,\xi_j) + \frac{2-p}{4p\pi}\sum_{j=1}^{K}\log a(\xi_j).$$

Furthermore

$$p\left(-\frac{4}{p}\log\varepsilon\right)^{\frac{2(p-1)}{p}} \nabla_{(\xi_m)_l}\mathcal{J}_\lambda(\xi) = \frac{32\pi^2}{2-p}\nabla_{(\xi_m)_l}\Phi^p_{a,K}(\xi) + |\log\varepsilon|^{-1}\theta_\lambda(\xi). \tag{5.5}$$

In (5.4) and (5.5), the function θ_λ denotes a smooth function of the points ξ, which is uniformly bounded, as $\lambda \to 0$, for points $\xi \in \mathcal{O}$.

Proof. Define

$$I^p_{a,\lambda}(v) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla v|^2\,dy - \int_{\Omega_\varepsilon} a(\varepsilon y) e^{\gamma^p\left[\left(1+\frac{v}{p\gamma^p}\right)^p - 1\right]}\,dy. \tag{5.6}$$

By direct calculation,

$$J^p_{a,\lambda}\left(\left(U_\lambda + \tilde{\phi}\right)(x,\xi)\right) = \frac{1}{p^2\gamma^{2(p-1)}} I^p_{a,\lambda}\left((V_\lambda + \phi)\left(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\right)\right). \tag{5.7}$$

Using the fact $\left(U_\lambda + \tilde{\phi}\right)(x,\xi) = \gamma + \frac{1}{p\gamma^{p-1}}\left((V_\lambda + \phi)\left(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\right)\right)$, we have

$$\mathcal{J}_\lambda(\xi) - J^p_{a,\lambda}(U_\lambda(\xi)) = \frac{1}{p^2\gamma^{2(p-1)}}\left[I^p_{a,\lambda}(V_\lambda + \phi) - I^p_{a,\lambda}(V_\lambda)\right].$$

Since by construction $DI^p_{a,\lambda}(V_\lambda + \phi)[\phi] = 0$, we get

$$\begin{aligned}\mathcal{J}_\lambda(\xi) - J^p_{a,\lambda}(U_\lambda(\xi)) &= \frac{1}{p^2\gamma^{2(p-1)}}\int_0^1 D^2 I^p_{a,\lambda}(V_\lambda + t\phi)\phi^2(1-t)\,dt \\ &= \frac{1}{p^2\gamma^{2(p-1)}}\int_0^1\left[\int_{\Omega_\varepsilon}(E_\lambda + N(\phi))\phi + \int_{\Omega_\varepsilon}[g'(V_\lambda) - g'(V_\lambda + t\phi)]\phi^2\right](1-t)\,dt.\end{aligned}$$

Since $\|E_\lambda\|_* \le \frac{c}{|\log\varepsilon|^3}$, $\|\phi\|_{L^\infty(\Omega_\varepsilon)} \le \frac{c}{|\log\varepsilon|^2}$, $\|N(\phi)\|_* \le \frac{c}{|\log\varepsilon|^4}$ and (2.33), we get that

$$\left|\mathcal{J}_\lambda(\xi) - J^p_{a,\lambda}(U_\lambda(\xi))\right| \le \frac{C}{\gamma^{2(p-1)}|\log\varepsilon|^3}. \tag{5.8}$$

Next we expand $J^p_{a,\lambda}(U_\lambda(\xi))$. We first have

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla(U_\lambda(\xi))|^2 &= \frac{1}{2}\frac{1}{p^2\gamma^{2(p-1)}}\Bigg\{\sum_{j=1}^K\int_\Omega |\nabla PU_{\mu_j,\xi_j}|^2 + \sum_{l\neq j}\int_\Omega \nabla PU_{\mu_l,\xi_l}\nabla PU_{\mu_j,\xi_j} \\ &+ \frac{p-1}{p}\frac{1}{\gamma^p}\sum_{j=1}^K\int_\Omega \nabla PU_{\mu_j,\xi_j}(x)\nabla Pw^0_{\mu_j,\xi_j}(x) \\ &+ \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\sum_{j=1}^K\int_\Omega \nabla PU_{\mu_j,\xi_j}\nabla Pw^1_{\mu_j,\xi_j} \\ &+ \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\left[\sum_{j=1}^K\int_\Omega |\nabla Pw^0_{\mu_j,\xi_j}|^2 + \sum_{l\neq j}\int_\Omega \nabla Pw^0_{\mu_l,\xi_l}\nabla Pw^0_{\mu_j,\xi_j}\right] \\ &+ \left(\frac{p-1}{p}\right)^3\frac{1}{\gamma^{3p}}\sum_{j=1}^K\int_\Omega \nabla Pw^0_{\mu_j,\xi_j}\nabla Pw^1_{\mu_j,\xi_j} \\ &+ \left(\frac{p-1}{p}\right)^4\frac{1}{\gamma^{4p}}\left[\sum_{j=1}^K\int_\Omega |\nabla w^1_{\mu_j,\xi_j}|^2 + \sum_{l\neq j}\int_\Omega \nabla Pw^1_{\mu_l,\xi_l}\nabla Pw^1_{\mu_j,\xi_j}\right]\Bigg\}. \end{aligned} \tag{5.9}$$

Let us estimate the first two terms. We observe that the remaining terms are $O(\frac{1}{\gamma^{2(p-1)}\gamma^p})$. We note that PU_{μ_j,ξ_j} satisfies

$$-\Delta PU_{\mu_j,\xi_j} = \varepsilon^2 a(\xi_j)e^{U_{\mu_j,\xi_j}}, \quad \text{in } \Omega, \qquad PU_{\mu_j,\xi_j} = 0 \quad \text{on } \partial\Omega.$$

Then we have

$$\begin{aligned}\int_\Omega |\nabla PU_{\mu_j,\xi_j}(x)|^2\,dx &= \varepsilon^2\int_\Omega a(\xi_j)e^{U_{\mu_j,\xi_j}}PU_{\mu_j,\xi_j}(x) \\ &= \varepsilon^2\int_\Omega a(\xi_j)e^{U_{\mu_j,\xi_j}}\left(U_{\mu_j,\xi_j}(x) + 8\pi H(x,\xi_j) - \log\frac{8\mu_j^2}{a(\xi_j)} + O(\mu_j^2\varepsilon^2)\right) \\ &= \int_\Omega \frac{8\varepsilon^2\mu_j^2}{(\varepsilon^2\mu_j^2+|x-\xi_j|^2)^2}\left(\log\frac{1}{(\varepsilon^2\mu_j^2+|x-\xi_j|^2)^2} + 8\pi H(x,\xi_j) + O(\mu_j^2\varepsilon^2)\right) \\ &= \int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|z|^2)^2}\left(\log\frac{1}{(1+|z|^2)^2} + 8\pi H(\xi_j+\varepsilon\mu_j z,\xi_j) - 4\log(\varepsilon\mu_j)\right) + O(\mu_j^2\varepsilon^2) \\ &= \int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|z|^2)^2}\log\frac{1}{(1+|z|^2)^2} + 8\pi\int_{\Omega_{\varepsilon\mu_j}}\frac{8}{(1+|z|^2)^2}\big(H(\xi_j+\varepsilon\mu_j z,\xi_j) - H(\xi_j,\xi_j)\big) \\ &+ 8\pi\int_{\Omega_{\varepsilon\mu_j}}\frac{8}{(1+|z|^2)^2}H(\xi_j,\xi_j) - 4\log(\varepsilon\mu_j)\int_{\Omega_{\varepsilon\mu_j}}\frac{8}{(1+|y|^2)^2} + O(\mu_j^2\varepsilon^2). \end{aligned} \tag{5.10}$$

But

$$\int_{\Omega_{\varepsilon\mu_j}}\frac{8}{(1+|y|^2)^2} = 8\pi + O(\varepsilon), \tag{5.11}$$

and

$$\int_{\Omega_{\varepsilon\mu_j}}\frac{8}{(1+|y|^2)^2}\log\frac{1}{(1+|y|^2)^2} = -16\pi + O(\varepsilon). \tag{5.12}$$

Moreover,

$$\int_{\Omega_{\varepsilon\mu_j}}\frac{8}{(1+|y|^2)^2}\big(H(\xi_j+\varepsilon\mu_j y,\xi_j) - H(\xi_j,\xi_j)\big) = O(\varepsilon). \tag{5.13}$$

Therefore from (5.10)–(5.13) and (2.19), we have

$$\int_\Omega |\nabla PU_{\mu_j,\xi_j}(x)|^2dx = -16\pi + 64\pi^2 H(\xi_j,\xi_j) - 32\pi\log\varepsilon - 16\pi\log(8\mu_j^2) + 16\pi\log(8) + O\left(\frac{1}{\gamma^p}\right)$$

Please cite this article in press as: S. Deng, et al., Multiple blow-up solutions for an exponential nonlinearity with potential in $\mathbb{R}^2$, Nonlinear Analysis (2014), http://dx.doi.org/10.1016/j.na.2014.10.034

$$= -16\pi + 64\pi^2 H(\xi_j,\xi_j) - 32\pi\log\varepsilon + 16\pi\log(8) - 16\pi\log a(\xi_j) + O\left(\frac{1}{\gamma^p}\right)$$
$$- 16\pi\left[\frac{2(p-1)}{2-p}(1-\log 8) + \frac{8\pi}{2-p}\left(H(\xi_j,\xi_j) + \sum_{i\neq j}^{K} G(\xi_i,\xi_j)\right)\right]. \quad (5.14)$$

Now, we calculate that

$$\sum_{l\neq j}\int_\Omega \nabla PU_{\mu_l,\xi_l}\nabla PU_{\mu_j,\xi_j}\,dx = \sum_{l\neq j}\int_\Omega \varepsilon^2 a(\xi_l)e^{U_{\mu_l,\xi_l}}PU_{\mu_j,\xi_j}$$
$$= \sum_{l\neq j}\int_\Omega \frac{8\varepsilon^2\mu_l^2}{(\varepsilon^2\mu_l^2+|x-\xi_l|^2)^2}\left(\log\frac{8\mu_j^2}{(\varepsilon^2\mu_j^2+|x-\xi_j|^2)^2 a(\xi_j)} + 8\pi H(x,\xi_j) - \log\frac{8\mu_j^2}{a(\xi_j)} + O(\mu_j^2\varepsilon^2)\right)$$
$$= \sum_{l\neq j}\int_{\Omega_{\varepsilon\mu_l}} \frac{8}{(1+|z|^2)^2}\left(\log\frac{1}{(\varepsilon^2\mu_j^2+|\varepsilon\mu_l z+\xi_l-\xi_j|^2)^2} + 8\pi H(\xi_l+\varepsilon\mu_l z,\xi_j)\right) + O(\mu_j^2\varepsilon^2)$$
$$= \sum_{l\neq j}\int_{\Omega_{\varepsilon\mu_l}} \frac{8}{(1+|z|^2)^2} 8\pi G(\xi_l,\xi_j) + O(\mu_j^2\varepsilon^2)$$
$$= 64\pi^2\sum_{l\neq j} G(\xi_l,\xi_j) + O(\mu_j^2\varepsilon^2). \quad (5.15)$$

Thus, from (5.9), (5.14) and (5.15) we have

$$\frac{1}{2}\int_\Omega |\nabla U_\lambda(x)|^2\,dx = \frac{1}{p^2\gamma^{2(p-1)}}\left\{-8K\pi - 16K\pi\log\varepsilon + 8K\pi\log(8) - 8K\pi\frac{2(p-1)}{2-p}(1-\log 8)\right.$$
$$\left. - 8\pi\sum_{j=1}^{K}\log a(\xi_j) - \frac{32\pi^2 p}{2-p}\left(\sum_{j=1}^{K} H(\xi_j,\xi_j) + \sum_{i\neq j}^{K} G(\xi_i,\xi_j)\right) + O\left(\frac{1}{|\log\varepsilon|}\right)\right\}. \quad (5.16)$$

Finally, let us evaluate the second term in the energy

$$\frac{\lambda}{p}\int_\Omega a(x)e^{(U_\lambda)^p}\,dx = \frac{\lambda}{p}\int_\Omega a(x)e^{\gamma^p\left(1+\frac{1}{p\gamma^p}(V_\lambda)(\frac{x}{\varepsilon})\right)^p}\,dx$$
$$= \frac{\lambda}{p}\sum_{j=1}^{K}\int_{B(\xi_j,\tilde\delta)} a(x)e^{\gamma^p\left(1+\frac{1}{p\gamma^p}(V_\lambda)(\frac{x}{\varepsilon})\right)^p}\,dx + \frac{\lambda}{p}\int_{\Omega\setminus\bigcup_{j=1}^{K}B(\xi_j,\tilde\delta)} a(x)e^{\gamma^p\left(1+\frac{1}{p\gamma^p}(V_\lambda)(\frac{x}{\varepsilon})\right)^p}\,dx$$
$$:= I + II. \quad (5.17)$$

First we observe that

$$II = \lambda\Theta_\lambda(\xi) \quad (5.18)$$

with $\Theta_\lambda(\xi)$ a function, uniformly bounded, as $\lambda\to 0$. On the other hand,

$$I = \frac{1}{p^2\gamma^{2(p-1)}}\sum_{j=1}^{K}\int_{B(\xi_j',\tilde\delta/\varepsilon)} a(\varepsilon y)e^{\gamma^p\left[\left(1+\frac{1}{p\gamma^p}(V_\lambda)(y)\right)^p-1\right]}\,dy$$
$$= \frac{1}{p^2\gamma^{2(p-1)}}\sum_{j=1}^{K}\int_{B(\xi_j',\tilde\delta/\varepsilon)} a(\varepsilon y)e^{\left\{\widetilde w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)\right\}}\left(1+O\left(\frac{1}{\gamma^p}\right)\right)dy$$
$$= \frac{1}{p^2\gamma^{2(p-1)}}\sum_{j=1}^{K}\int_{B\left(0,\frac{\tilde\delta}{\mu_j\varepsilon}\right)} \frac{8}{(1+|y|^2)^2}\left(1+O\left(\frac{1}{\gamma^p}\right)\right)dy$$
$$= \frac{1}{p^2\gamma^{2(p-1)}}8K\pi\left(1+|\log\varepsilon|^{-1}\Theta_\lambda(\xi)\right), \quad (5.19)$$

with $\Theta_\lambda(\xi)$ a function, uniformly bounded, as $\lambda\to 0$. From (5.17)–(5.19) we get

$$\frac{\lambda}{p}\int_\Omega a(x)e^{(U_\lambda)^p}\,dx = \frac{1}{p^2\gamma^{2(p-1)}}8K\pi\left(1+|\log\varepsilon|^{-1}\Theta_\lambda(\xi)\right). \quad (5.20)$$

Therefore, from (5.8), (5.16), (5.20) and (1.7) we get that (5.4) holds.

Let us now prove the validity of (5.5). Fix $m \in \{1, \dots, K\}$ and $l \in \{1, 2\}$. We have

$$\partial_{(\xi_m)_l}\mathcal{J}_\lambda(\xi) = \frac{1}{p^2\gamma^{2(p-1)}}\varepsilon^{-1}\left[\sum_{i=1}^{2}\sum_{j=1}^{K} c_{ij}\int_{\Omega_\varepsilon}\eta_j Z_{ij}\partial_{(\xi'_m)_l}V_\lambda\right]\left(1+O\left(\frac{1}{\gamma^p}\right)\right). \tag{5.21}$$

On the one hand, if we multiply equation in (4.1) against $\partial_{(\xi'_m)_l}V_\lambda$, we get

$$\int_{\Omega_\varepsilon}(\Delta v_\xi + g(v_\xi))\partial_{(\xi'_m)_l}V_\lambda = \sum_{i=1}^{2}\sum_{j=1}^{K} c_{ij}\int_{\Omega_\varepsilon}\eta_j Z_{ij}\partial_{(\xi'_m)_l}V_\lambda$$

where $v_\xi = (V_\lambda + \phi)(y, \xi') = (V_\lambda + \phi)(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon})$. On the other hand, we have that

$$\partial_{(\xi_m)_l}U_\lambda(x) = \frac{\varepsilon^{-1}}{p\gamma^{p-1}}\partial_{(\xi'_m)_l}V_\lambda\left(\frac{x}{\varepsilon}\right).$$

Putting together these information, we have that

$$\partial_{(\xi_m)_l}\mathcal{J}_\lambda(\xi) = \left(\int_\Omega\left[\Delta(U_\lambda+\tilde\phi) + \lambda a(x)(U_\lambda+\tilde\phi)^{p-1}e^{(U_\lambda+\tilde\phi)^p}\right]\partial_{(\xi_m)_l}U_\lambda\right)(1+o(1)).$$

Furthermore, since $\|\tilde\phi\|_{L^\infty(\Omega)} \le \frac{C}{\gamma^{p-1}}\|\phi\|_{L^\infty(\Omega_\varepsilon)}$, by definition of U_λ we have that

$$(U_\lambda+\tilde\phi)(x) = U_\lambda(x)\left(1+O\left(\frac{1}{\gamma^p}\right)\right) \quad \text{in } \Omega.$$

Hence, by means of integration by parts, and the boundary conditions satisfied by U_λ, we get that

$$\partial_{(\xi_m)_l}\mathcal{J}_\lambda(\xi) = \left(\int_\Omega\left[\Delta U_\lambda + \lambda a(x)U_\lambda^{p-1}e^{U_\lambda^p}\right]\partial_{(\xi_m)_l}U_\lambda\right)\left(1+O\left(\frac{1}{\gamma^p}\right)\right),$$

where $O(1)$ here denotes a smooth function of the points ξ, which is uniformly bounded as $\lambda \to 0$. We thus conclude that

$$\partial_{(\xi_m)_l}\mathcal{J}_\lambda(\xi) = \left(\int_\Omega\left[-\nabla U_\lambda\nabla\partial_{(\xi_m)_l}U_\lambda + \lambda a(x)U_\lambda^{p-1}e^{U_\lambda^p}\partial_{(\xi_m)_l}U_\lambda\right]\right)\left(1+O\left(\frac{1}{\gamma^p}\right)\right).$$

Computations analogous to the ones we performed to get expansion (5.4) give us the validity of (5.5). This concludes the proof of the lemma. □

6. Proof of the main results

6.1. Proof of Theorem 1.2

Proof of Theorem 1.2. According to Lemma 5.1, we have a solution to (1.1) if we find a critical point ξ^λ of $\mathcal{J}_\lambda(\xi)$, it is equivalent to finding a critical point of the function $\widetilde{\mathcal{I}}(\xi) : \mathcal{O} \to \mathbb{R}$ defined by

$$\widetilde{\mathcal{I}}(\xi) = \frac{2-p}{32\pi^2}\left[-\lambda^{-1}\varepsilon^{\frac{2(2-p)}{p}}\mathcal{J}_\lambda(\xi) + \frac{8K\pi}{2-p}[-2+\log 8] - \frac{16K\pi}{p}\log\varepsilon\right].$$

From Lemma 5.2, we have

$$\widetilde{\mathcal{I}}(\xi) = \Phi^p_{a,K}(\xi) + o(1), \tag{6.1}$$

where $o(1) \to 0$ uniformly for any points $(\xi_1, \dots, \xi_K) \in \mathcal{O}$, and $\Phi^p_{a,K}(\xi)$ defined by (1.6). By assumption that $\xi^* = (\xi^*_1, \dots, \xi^*_K)$ is a $\mathcal{C}^0$ stable critical point of $\Phi^p_{a,K}$, by Definition 1.1, there exists a critical point $\xi^*_\lambda \in \mathcal{O}$ of $\widetilde{\mathcal{I}}$ such that $\widetilde{\mathcal{I}}(\xi^*_\lambda) \to \widetilde{\mathcal{I}}(\xi^*)$. Moreover, up to a subsequence, $\xi^*_\lambda \to \xi^\lambda$ as $\lambda \to 0$, with $\Phi^p_{a,K}(\xi^\lambda) = \Phi^p_{a,K}(\xi^*)$.

Furthermore, expansion (1.8) follows from (1.7) and (5.20), while (1.9) holds as a direct consequence of the construction of U_λ. Expansion (1.10) is a consequence of (5.4). □

6.2. Proof of Theorem 1.3

Proof of Theorem 1.3. According to the result of Theorem 1.2, the proof of Theorem 1.3 reduces to show that, for K as in assumption (A_2), the function $\Phi^p_{a,K}$ has a nontrivial critical values in some open set $\mathcal{O}$, compactly contained in $(\Omega \setminus Z)^K$.

This fact has already been established in [6] under some minor modifications. For completeness, we recall here the principal ingredients employed to characterize a topological nontrivial critical value of $\Phi^p_{a,K}$ in some set $\mathcal{O}$, compactly contained in $(\Omega\setminus Z)^K$. We refer the reader to [6] for a complete proof of each step.

From the assumptions (A_1) and (A_2), without loss of generality we write

$$a(x)=\prod_{s=1}^m |x-q_s|^{2\alpha_s}.$$

Then we have

$$\begin{aligned}\Phi^p_{a,K}(\xi) &= \sum_{j=1}^K H(\xi_j,\xi_j)+\sum_{i\neq j}^K G(\xi_i,\xi_j)-\frac{(2-p)}{p}\sum_{j=1}^K\sum_{s=1}^m \frac{\alpha_s}{2\pi}\log\frac{1}{|\xi_j-q_s|}\\ &= \sum_{j=1}^K H(\xi_j,\xi_j)+\sum_{i\neq j}^K G(\xi_i,\xi_j)-\frac{(2-p)}{p}\sum_{j=1}^K\sum_{s=1}^m \alpha_s G(\xi_j,q_s)+\underbrace{\frac{(2-p)}{p}\sum_{j=1}^K\sum_{s=1}^m \alpha_s H(\xi_j,q_s)}_{O(1)}.\end{aligned} \tag{6.2}$$

Define the set

$$\mathcal{M}:=\left\{\xi=(\xi_1,\ldots,\xi_K)\in(\Omega\setminus Z)^K:\xi_i\neq\xi_j \text{ if } i\neq j\right\}.$$

Define the set

$$\mathcal{D}=\left\{\xi\in\mathcal{M}\mid \Psi(\xi):=\sum_{j=1}^K H(\xi_j,\xi_j)-\frac{(2-p)}{p}\sum_{j=1}^K\sum_{s=1}^m\alpha_s G(\xi_j,q_s)-\sum_{i\neq j}^K G(\xi_i,\xi_j)+O(1)>-M\right\} \tag{6.3}$$

where $M>0$ is a sufficiently large number to be chosen. We have that $\mathcal{D}$ is compactly contained in $\mathcal{M}$.

From (1.11), we write set $\{1,2,\ldots,K\}=I_1\cup I_2\cup\cdots\cup I_m$ where

$$\begin{aligned}&I_1=\{1,\ldots,K_1\},\\ &I_2=\{K_1+1,\ldots,K_1+K_2\},\\ &\ldots\\ &I_s=\{K_1+\cdots+K_{s-1}+1,\ldots,K_1+\cdots+K_{s-1}+K_s\},\\ &\ldots\\ &I_m=\{K_1+\cdots+K_{m-1}+1,\ldots,K\}.\end{aligned}$$

Let us fix angles θ_q $(q\in Z)$ and a number $\delta\in(0,\frac{\pi}{2})$ sufficiently small such that the cones

$$\left\{q+\rho e^{i(\theta_q+\theta)}:\rho\geq 0,\ \theta\in[-\delta,\delta]\right\},\quad q\in Z \tag{6.4}$$

are disjoint from one another. Moreover, we assume

$$\mathrm{dist}(q,\partial\Omega)>2\delta\quad\forall q\in Z,\qquad |q_i-q_j|>4\delta\quad\forall q_i,q_j\in Z,\ i\neq j. \tag{6.5}$$

Now we define K-tuple

$$\xi_0=(\xi_1^0,\ldots,\xi_K^0)$$

by

$$\xi_j^0=q_s+\frac{3}{2}\delta e^{i\left(\theta_{q_s}+j\frac{\delta}{K}\right)}\quad\forall j\in I_s,\ s=1,\ldots,m.$$

Let us set an annulus with radii δ and 2δ centered in q_s, that is

$$U_s:=\left\{\xi\in\mathbb{R}^2:\delta<|\xi-q_s|<2\delta\right\},$$

and consider the K-tuple $\xi=(\xi_1,\ldots,\xi_K)$ belongs to the open set

$$\left\{\xi\in U_1^{K_1}\times\cdots\times U_m^{K_m}:|\xi_i-\xi_j|>M^{-1}\ \forall i\neq j\right\}. \tag{6.6}$$

The choice of δ in (6.4) and (6.5) implies that $\xi_i^0\neq\xi_j^0$ for $i\neq j$, then we have that ξ_0 belongs to (6.6) provided that M is sufficiently large. Then we define

$$W:=\text{the connectedness of (6.6) containing } \xi_0$$

$$\mathcal{K}:=\bar{W},\qquad \mathcal{K}_0=\left\{\xi\in\mathcal{K}:\min_{i\neq j}|\xi_i-\xi_j|=M^{-1}\right\}.$$

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From these facts, we get that

(P1) *$\mathcal{D}$ is an open set, $\mathcal{K}$ and $\mathcal{K}_0$ are compact sets, $\mathcal{K}$ is connected and*

$$\mathcal{K}_0 \subset \mathcal{K} \subset \mathcal{D} \subset \bar{\mathcal{D}} \subset \mathcal{M}.$$

Let us define $\mathcal{F}$ to be the class of all continuous maps $\eta : \mathcal{K} \to \mathcal{D}$ with the property that there exists a continuous homotopy $\Gamma : [0, 1] \times \mathcal{K} \to \mathcal{D}$ such that

$$\Gamma(0,\cdot) = \mathrm{id}, \qquad \Gamma(1,\cdot) = \eta, \qquad \Gamma(t,\xi) = \xi \quad \forall t \in [0,1],\ \forall \xi \in \mathcal{K}_0.$$

In [6], the following facts are proven:

(P2)

$$\Phi^* := \sup_{\eta\in\mathcal{F}} \min_{\xi\in\mathcal{K}} \Phi^p_{a,K}(\eta(\xi)) < \min_{\xi\in\mathcal{K}_0} \Phi^p_{a,K}(\xi).$$

(P3) *For every $\xi \in \partial\mathcal{D}$ such that $\Phi^p_{a,K}(\xi) = \Phi^*$, $\partial\mathcal{D}$ is smooth at ξ and there exists a vector τ_ξ tangent to $\partial\mathcal{D}$ at ξ so that $\tau_\xi \cdot \nabla\Phi^p_{a,K}(\xi) \neq 0$.*

Under (P1)–(P3), a critical point $\xi\,\mathcal{D}$ of $\Phi^p_{a,K}$ with $\Phi^p_{a,K}(\xi) = \Phi^*$ exists, as a standard deformation argument involving the gradient flow of $\Phi^p_{a,K}$ shows. This finishes the proof of Theorem 1.3. □

6.3. Proof of Theorem 1.4

Proof of Theorem 1.4. According to Theorem 1.2, the proof of Theorem 1.4 reduces to show that function $\Phi^p_{a,K}$ has a $\mathcal{C}^0$-critical point. For $a(x) = |x|^{2\alpha}$ and $\Omega = B$ is the unit ball in $\mathbb{R}^2$. Following the approach in [12], we obtain that this holds.

Indeed, for $\rho \in (0, 1)$, we set

$$\xi_{j,\rho} = \left(\rho\cos\frac{2\pi(j-1)\pi}{K}, \rho\sin\frac{2\pi(j-1)\pi}{K}\right) \quad \text{for any } j = 1,\dots,K.$$

Then by symmetry, we have

$$\Phi^p_{a,K}(\xi_\rho) = K\left[H(\xi_{1,\rho},\xi_{1,\rho}) + \sum_{i=2}^{K} G(\xi_{1,\rho},\xi_{i,\rho}) + \frac{(2-p)\alpha}{2p\pi}\log\rho\right].$$

Thus it is equivalent to find a $\mathcal{C}^0$-critical point of

$$F(\rho) = H(\xi_{1,\rho},\xi_{1,\rho}) + \sum_{i=2}^{K} G(\xi_{1,\rho},\xi_{i,\rho}) + \frac{(2-p)\alpha}{2p\pi}\log\rho.$$

In the unit ball of $\mathbb{R}^2$ we have

$$G(x,y) = \frac{1}{2\pi}\log\frac{1}{|x-y|} - \frac{1}{2\pi}\log\frac{1}{\sqrt{|x|^2|y|^2+1-2(x,y)}},$$

$$H(x,x) = -\frac{1}{2\pi}\log\frac{1}{1-|x|^2}.$$

Hence

$$F(\rho) = \frac{1}{2}\log(1-\rho^2) + \frac{1}{2\pi}\left(\frac{2-p}{p}\alpha - (K-1)\right)\log\rho + \frac{1}{2\pi}\sum_{i=2}^{K}\log\frac{\sqrt{\rho^4+1-2\rho^2(\xi_1^*,\xi_i^*)}}{|\xi_1^*-\xi_i^*|}.$$

Here

$$\xi_j^* = \left(\cos\frac{2\pi(j-1)\pi}{K}, \sin\frac{2\pi(j-1)\pi}{K}\right) \quad \text{for any } j = 1,\dots,K.$$

If $\frac{2-p}{p}\alpha - (K-1) > 0$, that is $K < \frac{2-p}{p}\alpha + 1$, we find that

$$\lim_{\rho\to 1^-} F(\rho) = \lim_{\rho\to 0^+} F(\rho) = -\infty.$$

Then there exists $\rho_0 \in (0, 1)$ such that

$$F(\rho_0) = \max_{\rho\in(0,1)} F(\rho),$$

and ρ_0 is a $\mathcal{C}^0$-critical point of $F(\rho)$. This completes the proof of Theorem 1.4. □

Acknowledgments

The research of the first author has been partly supported by Postdoctoral Fondecyt Grant 3140403 and Fundamental Research Funds for the Central Universities XDJK2015C043. The research of the second author has been partly supported by Mecesup Grant 0711. The research of the third author has been partly supported by Fondecyt Grant 1120151.

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