Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 256 (2009) 275–306
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Multipeak solutions to the Bahri–Coron problem in domains with a shrinking hole ☆

Mónica Clapp [a], Monica Musso [b,c], Angela Pistoia [d,*]

[a] *Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, C.U., 04510 México DF, Mexico*
[b] *Departamento de Matemática, Pontificia Universidad Católica de Chile, Avenida Vicuna Mackenna 4860, Macul, Santiago, Chile*
[c] *Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy*
[d] *Dipartimento di Metodi e Modelli Matematici, Università di Roma "La Sapienza", via A. Scarpa 16, 00161 Roma, Italy*

Received 14 April 2008; accepted 13 June 2008

Available online 28 August 2008

Communicated by J.-M. Coron

Abstract

We construct positive and sign changing multipeak solutions to the pure critical exponent problem in a bounded domain with a shrinking hole, having a peak which concentrates at some point inside the shrinking hole (i.e. outside the domain) and one or more peaks which concentrate at interior points of the domain. These are, to our knowledge, the first multipeak solutions in a domain with a single small hole.

Keywords: Nonlinear elliptic boundary value problem; Pure critical exponent; Multipeak solutions

☆ The second and the third authors are supported by the M.I.U.R. National Project "Metodi variazionali e topologici nello studio di fenomeni non lineari" (Italy). The second author is supported by Fondecyt 1040936 (Chile). The first author is supported by CONACYT 58049 and PAPIIT IN105106 (Mexico).

* Corresponding author.
E-mail addresses: mclapp@math.unam.mx (M. Clapp), mmusso@mat.puc.cl (M. Musso), pistoia@dmmm.uniroma1.it (A. Pistoia).

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2008.06.034

1. Introduction

In this paper we investigate the existence of solutions, both positive and sign changing, to the problem

$$\begin{cases} -\Delta u = |u|^{\frac{4}{N-2}} u & \text{in } \Omega \setminus \varepsilon(\omega + \xi_0), \\ u = 0 & \text{on } \partial\big(\Omega \setminus \varepsilon(\omega + \xi_0)\big), \end{cases} \tag{1}$$

where Ω is a connected bounded smooth domain in $\mathbb{R}^N$, $\xi_0 \in \Omega$, ω is a closed bounded neighborhood of 0 in $\mathbb{R}^N$ with smooth boundary, $N \geqslant 3$, and $\varepsilon > 0$ is small enough.

The exponent of the nonlinearity is $2^* - 1$ where $2^* := \frac{2N}{N-2}$ is the so-called critical Sobolev exponent. This problem has a rich geometric structure: it is invariant under the group of Möbius transformations; in particular, it is invariant under dilations. This fact is responsible for the lack of compactness of the Sobolev embedding $H_0^1(D) \hookrightarrow L^{2^*}(D)$ even when D is a bounded domain, and produces a dramatic change in the behavior of this problem with respect to the subcritical one. Indeed, whereas for $q \in (2, 2^*)$ problem

$$-\Delta u = |u|^{q-2} u \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D, \tag{2}$$

has infinitely many solutions in every bounded smooth domain D of $\mathbb{R}^N$, for $q = 2^*$ Pohožaev [20] showed that it has only the trivial solution if D is strictly starshaped. Moreover, for $q = 2^*$ this problem does not have a nontrivial least energy solution unless $D = \mathbb{R}^N$. Solvability for $q = 2^*$ is, thus, a difficult issue.

There are some well known existence results for $q = 2^*$. The first one was given by Kazdan and Warner [13] who showed that, if D is an annulus, then (2) has infinitely many radial solutions. Later, without any symmetry assumption, Coron [10] proved the existence of a positive solution to (2) if D is annular shaped, i.e.

$$\big\{x \in \mathbb{R}^N\colon\ 0 < R_1 < |x| < R_2\big\} \subset D \quad \text{and} \quad 0 \notin D, \tag{3}$$

and R_2/R_1 is small enough. Substantial improvement was obtained by Bahri and Coron [3] (see also [2]) who showed that, if the reduced homology of D with coefficients in $\mathbb{Z}_2$ is nontrivial, then problem (2) has at least one positive solution.

Concerning Coron's result, an interesting issue is the study of the asymptotic behavior of Coron's solution for R_2 fixed and $R_1 \to 0$, in other words, when D has a small hole whose diameter tends to zero. If the hole is the ball of radius R_1, then the solution found by Coron concentrates around the hole and it converges, in the sense of measure, to a Dirac delta centered at the center of the hole as $R_1 \to 0$. We refer the reader to [14,15,21] where the study of existence of positive multipeak solutions to (2) in domains with several small circular holes and their asymptotic behavior as the size of the holes goes to zero has been carried out.

Recently, Clapp and Weth [9] extended Coron's result. They showed that, if D has a small enough hole, then (2) has at least two solutions. But nothing can be said about the sign of the second one. Existence of sign changing solutions for symmetric domains with a small hole was first shown by Clapp and Weth [8]. Musso and Pistoia [16] proved that, if the domain has certain symmetries and a small spherical hole, then the number of sign changing solutions becomes arbitrarily large as the radius of the sphere goes to zero. Recently, Clapp and Pacella [7] considered

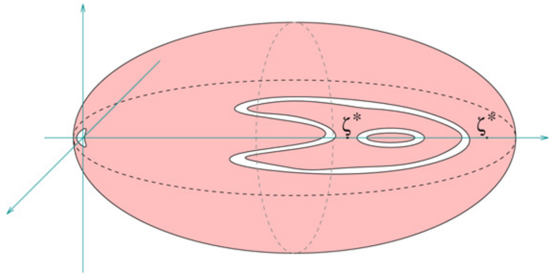

Fig. 1.

annular shaped domains D which are invariant under a finite group Γ of orthogonal transformations of $\mathbb{R}^N$ and established the existence of multiple sign changing solutions even if the hole is large provided the cardinality of the minimal Γ-orbit of D is also large. Finally, if the domain D has two small holes, then Musso and Pistoia [18] proved that problem (2) has at least one pair of sign changing solutions.

Results obtained so far suggest that solutions to problem (1) should concentrate at points outside the domain. In this paper we shall construct positive and sign changing multipeak solutions to (1) having a peak which concentrates at some point inside the shrinking hole $\varepsilon(\omega + \xi_0)$ (i.e. outside the domain) and one or more peaks which concentrate at interior points of the domain $\Omega \setminus \varepsilon(\omega + \xi_0)$, for certain points $\xi_0 \in \Omega$. These are, to our knowledge, the first known solutions to problem (1) exhibiting this kind of concentration behavior, and the first multipeak solutions in a domain with a single small hole.

Our first three results concern existence of positive multipeak solutions. Set

$$A_N := \left[N(N-2)\right]^{\frac{N}{2}} \int_{\mathbb{R}^N} \left(1+|y|^2\right)^{-(N+2)/2} dy.$$

Theorem 1. *Assume that $\partial\Omega$ is not connected. There exists $\rho_0 > 0$, depending only on Ω, such that, for each point $\xi_0 \in \Omega$ with $\operatorname{dist}(\xi_0, \partial\Omega) \leqslant \rho_0$, there exist $\zeta^* \in \Omega \setminus \{\xi_0\}$ and $\varepsilon_0 > 0$ with the following property: for every $\varepsilon \in (0, \varepsilon_0)$ there is a positive solution u_ε to problem* (1) *satisfying*

$$|\nabla u_\varepsilon|^2 \, dx \rightharpoonup A_N\left(\delta_{\xi_0} + \delta_{\zeta^*}\right) \quad \text{in the sense of measures, as } \varepsilon \to 0.$$

Under some symmetry assumptions on the domain, we obtain multiplicity of positive multipeak solutions. The domains considered in Theorems 2 and 3 are illustrated by Figs. 1 and 2, respectively.

Theorem 2. *Assume that, for some $n \leqslant N$,*

$$(x_1, \ldots, x_n, x_{n+1}, \ldots, x_N) \in \Omega \quad \Leftrightarrow \quad (x_1, \ldots, x_n, -x_{n+1}, \ldots, -x_N) \in \Omega, \tag{4}$$

$$(x_1, \ldots, x_n, x_{n+1}, \ldots, x_N) \in \omega \quad \Leftrightarrow \quad (x_1, \ldots, x_n, -x_{n+1}, \ldots, -x_N) \in \omega. \tag{5}$$

There exists $\rho_0 > 0$, depending only on Ω, such that, for each $\xi_0 \in \Omega \cap (\mathbb{R}^n \times \{0\})$ with $\operatorname{dist}(\xi_0, \partial\Omega) \leqslant \rho_0$ and each connected component C of $\Omega \cap (\mathbb{R}^n \times \{0\})$ with nonconnected bound-

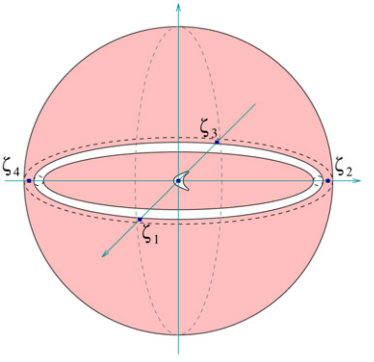

Fig. 2.

ary, such that $\xi_0 \notin C$ *if* $n = 1$, *there exist* $\zeta_C^* \in C \setminus \{\xi_0\}$ *and* $\varepsilon_0 > 0$ *with the following property*: *for every* $\varepsilon \in (0, \varepsilon_0)$ *and every* C *there is a positive solution* $u_{C,\varepsilon}$ *to problem* (1) *satisfying*

$$|\nabla u_{C,\varepsilon}|^2\,dx \rightharpoonup A_N(\delta_{\xi_0} + \delta_{\zeta_C^*}) \quad \text{in the sense of measures, as } \varepsilon \to 0.$$

Theorem 3. *Let* $\Omega := B(0,1) \setminus T(r,\rho)$, *where*

$$B(0,1) := \{x \in \mathbb{R}^N \colon |x| < 1\},$$
$$T(r,\rho) := \{x \in \mathbb{R}^N \colon \operatorname{dist}(x, S(r)) < \rho\},$$
$$S(r) := \{(x_1, x_2, 0, \dots, 0) \in \mathbb{R}^N \colon x_1^2 + x_2^2 = r^2\}, \quad r \in \left(\frac{1}{2}, 1\right).$$

Let $\xi_0 = 0$ *and assume that* ω *satisfies*

$$(x_1, x_2, x_3, \dots, x_N) \in \omega \quad \Leftrightarrow \quad (x_1, x_2, -x_3, \dots, -x_N) \in \omega.$$

Then, for each integer $k \geqslant 1$ *there exists* $\rho_0 \in (\frac{1}{2}, 1)$ *such that, if* $r + \rho \in (\rho_0, 1)$, *there exist* $r_* \in (r + \rho, 1)$ *and* $\varepsilon_0 > 0$ *with the following property*: *for every* $\varepsilon \in (0, \varepsilon_0)$ *there is a positive solution* u_ε *to problem* (1) *satisfying*

$$|\nabla u_\varepsilon|^2\,dx \rightharpoonup A_N\left(\delta_0 + \sum_{j=0}^{k-1} \delta_{\zeta_j}\right) \quad \text{in the sense of measures, as } \varepsilon \to 0,$$

where $\zeta_j := r_*(\cos\frac{2\pi j}{k}, \sin\frac{2\pi j}{k}, 0, \dots, 0)$.

Concerning existence of sign changing multipeak solutions to (1), we prove the following two results.

Theorem 4. *Assume that $x \in \Omega$ iff $-x \in \Omega$ and let $\xi_0 = 0$. Then there exist $\zeta^* \in \Omega \setminus \{0\}$ and $\varepsilon_0 > 0$ such that, for each $\varepsilon \in (0, \varepsilon_0)$, there is a pair $\pm u_\varepsilon$ of sign changing solutions to problem* (1) *satisfying*

$$|\nabla u_\varepsilon|^2 \rightharpoonup A_N(\delta_0 - \delta_{\zeta^*} - \delta_{-\zeta^*}) \quad \text{in the sense of measures, as } \varepsilon \to 0.$$

Theorem 5. *Let $\Omega := B(0,1)$ and $\xi_0 = 0$. If N is odd, assume that ω satisfies*

$$(x_1, x_2, x_3, \ldots, x_N) \in \omega \quad \text{iff} \quad (x_1, x_2, -x_3, \ldots, -x_N) \in \omega.$$

Then, for every integer $k \geqslant 1$ there exist $r_* \in (0,1)$ and $\varepsilon_0 > 0$ such that, for each $\varepsilon \in (0, \varepsilon_0)$, there exists a pair $\pm u_\varepsilon$ of sign changing solutions to problem (1) *satisfying*

$$|\nabla u_\varepsilon|^2 \, dx \rightharpoonup A_N\left(\delta_0 - \sum_{j=0}^{k-1} \delta_{\zeta_j}\right) \quad \text{in the sense of measures, as } \varepsilon \to 0,$$

where $\zeta_j := r_*(\cos \frac{2\pi j}{k}, \sin \frac{2\pi j}{k}, 0, \ldots, 0)$.

One may ask whether the solutions given by the above results are solely created by the topology of Ω or whether there is really an effect of the hole. In other words, do these solutions persist for $\varepsilon = 0$? The answer is that, in general, they do not persist. In fact, Ben Ayed, El Mehdi and Hammami [4] showed that for thin annuli the least energy of a positive solution goes to infinity as the width of the annulus goes to zero. In particular, a thin annulus does not have 2-peak solutions, so the solutions provided by Theorem 1 for small ε blow up as $\varepsilon \to 0$.

This paper is organized as follows. In Section 2, we describe the construction of a first approximation for a solution to problem (1) and we give the scheme of the proof of our results, which is based in a finite-dimensional reduction. Section 3 is devoted to the proof of our main results. In particular, we state and prove a general existence result for solutions to problem (1) under some general symmetry assumptions. This result, together with a topological lemma, are the tools to construct positive and sign changing solutions to (1), as asserted in the previous theorems. In Section 4 we give the expansion of the energy functional associated to the problem at the ansatz. Finally, Section 5 is devoted to the study of the associated nonlinear problem which provides the finite-dimensional reduction.

2. An approximate solution and scheme of the proof

In this section we describe a first approximation of the solution to problem (1). To simplify notation, we shall assume from now on that $\xi_0 = 0$.

Let δ be a positive real number and z be a point in $\mathbb{R}^N$. The basic element to construct a solution to problem (1) is the so called *standard bubble* $U_{\delta,z}$ defined by

$$U_{\delta,z}(x) = \alpha_N \frac{\delta^{\frac{N-2}{2}}}{(\delta^2 + |x-z|^2)^{\frac{N-2}{2}}}, \quad \delta > 0,\ z \in \mathbb{R}^N,$$

with $\alpha_N := [N(N-2)]^{\frac{N-2}{4}}$. It is well known (see [1,6,24]) that these functions are the positive solutions of the equation $-\Delta u = u^p$ in $\mathbb{R}^N$, where $p := \frac{N+2}{N-2}$. These are the basic cells to build an actual solution of (1) after we perform a suitable correction to fit in the boundary condition. To this purpose, we replace $U_{\delta,z}$ by its projection $P_\varepsilon U_{\delta,z}$ onto $H_0^1(\Omega\setminus\varepsilon\omega)$, defined by

$$\begin{cases} -\Delta P_\varepsilon U_{\delta,z} = U_{\delta,z}^p & \text{in } \Omega\setminus\varepsilon\omega,\\ P_\varepsilon U_{\delta,z} = 0 & \text{on } \partial(\Omega\setminus\varepsilon\omega).\end{cases}$$

We will look for a solution to (1) of the form

$$u = V_{\lambda,\zeta} + \phi, \quad V_{\lambda,\zeta} := P_\varepsilon U_{\mu,\xi} + \sum_{j=1}^{k} \nu_j P_\varepsilon U_{\lambda_j,\zeta_j}, \tag{6}$$

where $V_{\lambda,\zeta}$ represents the leading term and ϕ is a lower order term. Here $\nu_j = \pm 1$, $\lambda = (\mu, \lambda_1, \dots, \lambda_k) \in (0,\infty)^{k+1}$ and $\zeta = (\xi, \zeta_1, \dots, \zeta_k) \in \Omega^{k+1}$. We will choose points $\xi, \zeta_j \in \Omega$ and parameters $\mu, \lambda_j \in (0,\infty)$, $j = 1, \dots, k$, as follows:

$$\mu := d\sqrt{\varepsilon}, \quad \eta < d < \eta^{-1} \quad \text{and} \quad \xi := \mu\tau, \quad \tau \in \mathbb{R}^N,\ |\tau| < \eta, \tag{7}$$

and for $j = 1, \dots, k$,

$$\lambda_j := \Lambda_j\sqrt{\varepsilon}, \quad \eta < \Lambda_j < \eta^{-1}, \tag{8}$$

$$|\zeta_j| > 2\eta, \qquad \operatorname{dist}(\zeta_j, \partial\Omega) > 2\eta, \qquad |\zeta_j - \zeta_s| > 2\eta \quad \text{if } j \neq s, \tag{9}$$

for some positive small fixed η. Set $\bar{\Lambda} := (\Lambda_1, \dots, \Lambda_k) \in (0,\infty)^k$ and $\bar{\zeta} := (\zeta_1, \dots, \zeta_k) \in \Omega^k$.

In terms of ϕ, problem (1) becomes

$$\begin{cases} L_{\lambda,\zeta}(\phi) = N_{\lambda,\zeta}(\phi) + R_{\lambda,\zeta} & \text{in } \Omega\setminus\varepsilon\omega,\\ \phi = 0 & \text{on } \partial(\Omega\setminus\varepsilon\omega),\end{cases} \tag{10}$$

where

$$\begin{aligned} L_{\lambda,\zeta}(\phi) &:= -\Delta\phi - f'(V_{\lambda,\zeta})\phi,\\ N_{\lambda,\zeta}(\phi) &:= f(V_{\lambda,\zeta}+\phi) - f(V_{\lambda,\zeta}) - f'(V_{\lambda,\zeta})\phi,\\ R_{\lambda,\zeta} &:= f(V_{\lambda,\zeta}) + \Delta V_{\lambda,\zeta}. \end{aligned}$$

Here $f(u) := |u|^{\frac{4}{N-2}}u$. We denote by $\widehat{\mathcal{K}}_{\lambda_j,\zeta_j}$ the kernel of the operator $-\Delta - pU_{\lambda_j,\zeta_j}^{p-1}$ on $L^2(\mathbb{R}^N)$, and consider the spaces

$$\mathcal{K}_{\lambda,\zeta} := \operatorname{span}\left\{ f'(V_{\lambda,\zeta})P_\varepsilon\theta:\ \theta \in \bigcup_{j=0}^{k} \widehat{\mathcal{K}}_{\lambda_j,\zeta_j} \right\},$$

$$\mathcal{K}_{\lambda,\zeta}^{\perp} := \left\{ \phi \in H_0^1(\Omega\setminus\varepsilon\omega):\ \int_{\Omega\setminus\varepsilon\omega} \phi\psi = 0 \text{ for all } \psi \in \mathcal{K}_{\lambda,\zeta} \right\},$$

where $P_\varepsilon\theta$ denotes the orthogonal projection of θ onto $H_0^1(\Omega \setminus \varepsilon\omega)$, i.e. $\Delta P_\varepsilon\theta = \Delta\theta$ in $\Omega \setminus \varepsilon\omega$, $P_\varepsilon\theta = 0$ on $\partial(\Omega \setminus \varepsilon\omega)$.

To prove the existence of a solution to (10), we first solve the problem

$$(\wp_{\lambda,\zeta}) \quad \begin{cases} L_{\lambda,\zeta}(\phi) = N_{\lambda,\zeta}(\phi) + R_{\lambda,\zeta} + \psi, \\ \psi \in \mathcal{K}_{\lambda,\zeta}, \\ \phi \in \mathcal{K}^{\perp}_{\lambda,\zeta}. \end{cases} \tag{11}$$

Now, in order to solve this problem we recall [5] that $\widehat{\mathcal{K}}_{\delta,z}$ has dimension $N+1$ and is spanned by the functions

$$Z^0_{\delta,z}(x) := \frac{\partial U_{\delta,z}}{\partial\delta}(x) = \alpha_N \frac{N-2}{2}\delta^{(N-4)/2}\frac{|x-z|^2-\delta^2}{(\delta^2+|x-z|^2)^{N/2}}, \quad x \in \mathbb{R}^N,$$

$$Z^i_{\delta,z}(x) := \frac{\partial U_{\delta,z}}{\partial z_i}(x) = -\alpha_N(N-2)\delta^{(N-2)/2}\frac{x_i-z_i}{(\delta^2+|x-z|^2)^{N/2}}, \quad x \in \mathbb{R}^N,$$

for $i = 1, \dots, N$. So solving problem $(\wp_{\lambda,\zeta})$ in (11) is equivalent to finding ϕ and coefficients c^i_j, $i = 0, \dots, N$, $j = 0, \dots, k$, such that

$$\begin{cases} L_{\lambda,\zeta}(\phi) = N_{\lambda,\zeta}(\phi) + R_{\lambda,\zeta} + \sum_{i,j} c^i_j f'(V_{\lambda,\zeta}) P_\varepsilon Z^i_{\lambda_j,\zeta_j} & \text{in } \Omega \setminus \varepsilon\omega, \\ \phi = 0 & \text{on } \partial(\Omega \setminus \varepsilon\omega), \\ \int_{\Omega\setminus\varepsilon\omega} \phi f'(V_{\lambda,\zeta}) P_\varepsilon Z^i_{\lambda_j,\zeta_j} = 0 & i = 0, \dots, N,\ j = 0, \dots, k. \end{cases} \tag{12}$$

For technical reasons, it is useful to scale the problem. Let

$$\Omega_\varepsilon := \frac{\Omega \setminus \varepsilon\omega}{\sqrt{\varepsilon}} \quad \text{and} \quad y = \frac{x}{\sqrt{\varepsilon}} \in \Omega_\varepsilon.$$

Then u is a solution to (1) if and only if the function $\hat{u}(y) := \varepsilon^{\frac{1}{p-1}} u(\sqrt{\varepsilon} y)$ solves the problem

$$\begin{cases} -\Delta v = |v|^{\frac{4}{N-2}} v & \text{in } \Omega_\varepsilon, \\ v = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{13}$$

In this expanded variables, the solution we are looking for looks like $\hat{u}(y) = \hat{V}_{\lambda,\zeta} + \hat{\phi}(y)$, where

$$\hat{V}_{\lambda,\zeta}(y) := \varepsilon^{\frac{1}{p-1}} V_{\frac{\lambda}{\sqrt{\varepsilon}},\frac{\zeta}{\sqrt{\varepsilon}}}(\sqrt{\varepsilon} y) \quad \text{and} \quad \hat{\phi}(y) := \varepsilon^{\frac{1}{p-1}} \phi(\sqrt{\varepsilon} y), \quad y \in \Omega_\varepsilon.$$

Now, in terms of $\hat{\phi}$, problem (12) becomes

$$\begin{cases} \hat{L}_{\lambda,\zeta}(\hat{\phi}) = \hat{N}_{\lambda,\zeta}(\hat{\phi}) + \hat{R}_{\lambda,\zeta} + \sum_{i,j} c_j^i f'(\hat{V}_{\lambda,\zeta})\hat{Z}_j^i & \text{in } \Omega_\varepsilon, \\ \hat{\phi} = 0 & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \hat{\phi} f'(\hat{V}_{\lambda,\zeta})\hat{Z}_j^i = 0 & i = 0, \dots, N,\ j = 0, \dots, k, \end{cases} \tag{14}$$

where $\hat{Z}_j^i(y) := \varepsilon^{\frac{1}{p-1}} P_\varepsilon Z^i_{\frac{\lambda_j}{\sqrt{\varepsilon}}, \frac{\zeta_j}{\sqrt{\varepsilon}}}(\sqrt{\varepsilon} y)$ and

$$\hat{L}_{\lambda,\zeta}(\hat{\phi}) := -\Delta\hat{\phi} - f'(\hat{V}_{\lambda,\zeta})\hat{\phi},$$
$$\hat{N}_{\lambda,\zeta}(\hat{\phi}) := f(\hat{V}_{\lambda,\zeta} + \hat{\phi}) - f(\hat{V}_{\lambda,\zeta}) - f'(\hat{V}_{\lambda,\zeta})\hat{\phi},$$
$$\hat{R}_{\lambda,\zeta} := f(\hat{V}_{\lambda,\zeta}) - \sum_{j=0}^{k} f(U_{\frac{\lambda_j}{\sqrt{\varepsilon}}, \frac{\zeta_j}{\sqrt{\varepsilon}}}).$$

We point out that $\hat{\phi}$ solves (14) if and only if ϕ solves (12). The solution to problem (14) will be obtained as a fixed point of a certain contraction map, which will be defined thanks to the solvability of the following linear problem. Fix points and parameters as in (7)–(9). Given a function h, we consider the problem of finding $\hat{\phi}$ such that for certain real numbers c_j^i the following is satisfied

$$\begin{cases} \hat{L}_{\lambda,\zeta}(\hat{\phi}) = h + \sum_{i,j} c_j^i f'(\hat{V}_{\lambda,\zeta})\hat{Z}_j^i & \text{in } \Omega_\varepsilon, \\ \hat{\phi} = 0 & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \hat{\phi} f'(\hat{V}_{\lambda,\zeta})\hat{Z}_j^i = 0 & i = 0, \dots, N,\ j = 0, \dots, k. \end{cases} \tag{15}$$

In order to perform an invertibility theory for $\hat{L}_{\lambda,\zeta}$ subject to the above orthogonality conditions, we introduce $L_*^\infty(\Omega_\varepsilon)$ and $L_{**}^\infty(\Omega_\varepsilon)$ to be, respectively, the spaces of functions defined on Ω_ε with finite $\|\cdot\|_*$-norm (respectively $\|\cdot\|_{**}$-norm), where

$$\|\psi\|_* = \sup_{x\in\Omega_\varepsilon} \left[\left| w^{-\beta}(x)\psi(x) \right| + \left| w^{-(\beta + \frac{1}{N-2})}(x) D\psi(x) \right| \right],$$

with

$$w(x) = \left(1 + |x - \xi'|^2\right)^{-\frac{N-2}{2}} + \sum_j \left(1 + \left|x - \zeta_j'\right|^2\right)^{-\frac{N-2}{2}},$$

$\beta = 1$ if $N = 3$ and $\beta = \frac{2}{N-2}$ if $N \geqslant 4$. Similarly we define, for any dimension $N \geqslant 3$,

$$\|\psi\|_{**} = \sup_{x\in\Omega_\varepsilon} \left| w^{-\frac{4}{N-2}}(x)\psi(x) \right|.$$

The operator $\hat{L}_{\lambda,\zeta}$ is indeed uniformly invertible with respect to the above weighted L^∞-norm, for all ε small enough. This fact is established in the next proposition. We refer the reader to [11,17] for the proof.

Proposition 6. *Let $\eta > 0$ be fixed. There are numbers $\varepsilon_0 > 0$, $C > 0$, such that, for points and parameters satisfying* (7)–(9)*, problem* (15) *admits a unique solution $\hat{\phi} =: T_{\lambda,\zeta}(h)$ for all $0 < \varepsilon < \varepsilon_0$ and all $h \in C^\alpha(\bar{\Omega}_\varepsilon)$. Moreover,*

$$\left\| T_{\lambda,\zeta}(h) \right\|_* \leqslant C\|h\|_{**}, \qquad \left\| \partial_{d,\bar{\Lambda},\tau,\bar{\zeta}} T_{\lambda,\zeta}(h) \right\|_* \leqslant C\|h\|_{**} \tag{16}$$

and

$$|c_i| \leqslant C\|h\|_{**}. \tag{17}$$

The solvability of problem (14) is established in the following proposition.

Proposition 7. *Let $\eta > 0$ be fixed. There are numbers $\varepsilon_0 > 0$, $C > 0$, such that, for points and parameters satisfying* (7)–(9) *there exists a unique solution $\hat{\phi} = \hat{\phi}(d, \bar{\Lambda}, \tau, \bar{\zeta})$ to problem* (14)*, such that the map $(d, \bar{\Lambda}, \tau, \bar{\zeta}) \to \hat{\phi}(d, \bar{\Lambda}, \tau, \bar{\zeta})$ is of class C^1 for the $\|\cdot\|_*$-norm and*

$$\|\hat{\phi}\|_* \leqslant C\varepsilon^{\frac{N-2}{2}}, \tag{18}$$

$$\|\nabla_{(d,\bar{\Lambda},\tau,\bar{\zeta})}\hat{\phi}\|_* \leqslant C\varepsilon^{\frac{N-2}{2}}. \tag{19}$$

The proof of the previous proposition will be postponed to Section 5. Here we just mention that the size of $\hat{\phi}$ and its derivatives, given in (18) and (19), is strictly related to the size of $\|\hat{R}_{\lambda,\zeta}\|_{**}$, which turns out to be of order $\varepsilon^{\frac{N-2}{2}}$ in all the different existence results we obtain, as shown in the proof of Proposition 7.

Looking back at (14), we conclude that, in the expanded variable, the function $\hat{V}_{\lambda,\zeta} + \hat{\phi}$ is an actual solution to (13), or equivalently that the function $V_{\lambda,\zeta} + \phi$ in (6) is an actual solution to our original problem (1), if we show that, for a proper election of $(d, \bar{\Lambda}, \tau, \bar{\zeta})$, the constants c_j^i are all zero. This reduces our problem to a finite-dimensional one.

Let $J_\varepsilon : H_0^1(\Omega \setminus \varepsilon\omega) \to \mathbb{R}$ be the energy functional given by

$$J_\varepsilon(u) = \frac{1}{2} \int\limits_{\Omega\setminus\varepsilon\omega} |Du|^2 - \frac{1}{p+1} \int\limits_{\Omega\setminus\varepsilon\omega} |u|^{p+1}. \tag{20}$$

It is well known that critical points of J_ε are solutions to (1).

We introduce the function $J_\varepsilon^* : (0,\infty)^{k+1} \times \mathbb{R}^N \times \Omega^k \to \mathbb{R}$ given by

$$J_\varepsilon^*(d, \bar{\Lambda}, \tau, \bar{\zeta}) := J_\varepsilon(V_{\lambda,\zeta} + \phi)$$

where ϕ is the unique solution to problem $(\wp_{\lambda,\zeta})$ in (11) given by Proposition 7.

Using standard tools one can prove the following results.

Lemma 8. *$u_\varepsilon = V_{\lambda,\zeta} + \phi$ is a solution of problem* (1)*, i.e. $c_j^i = 0$ in* (12) *for all i, j, if and only if $(d, \bar{\Lambda}, \tau, \bar{\zeta})$ is a critical point of J_ε^*.*

A direct consequence of estimates (18) and (19) is the following expansion.

Proposition 9. *Let $\eta > 0$ be fixed and assume* (7)–(9) *hold true. Then we have the following expansion*

$$J_\varepsilon^*(d, \bar{\Lambda}, \tau, \bar{\zeta}) = J_\varepsilon(V_{\lambda,\zeta}) + o\big(\varepsilon^{\frac{N-2}{2}}\big), \tag{21}$$

where, as ε goes to zero, the term $o(\varepsilon^{\frac{N-2}{2}})$ is C^1-uniform over all $(d, \bar{\Lambda}, \tau, \bar{\zeta})$'s satisfying (7)–(9).

Finally, we conclude this section with the asymptotic expansion of the main part of the energy $J_\varepsilon(V_{\lambda,\zeta})$, which will be obtained in Section 4.

The expansion of $J_\varepsilon(V_{\lambda,\zeta})$ is given in terms of the Green function of the Laplace operator vanishing at the boundary $\partial\Omega$, defined by

$$G(x, y) = \kappa_N\left(\frac{1}{|x-y|^{N-2}} - H(x, y)\right), \tag{22}$$

with $\kappa_N := \frac{1}{(N-2)|\partial B|}$, where $|\partial B|$ denotes the surface area of the unit sphere in $\mathbb{R}^N$. The function H denotes the regular part of the Green function, which for all $y \in \Omega$ satisfies

$$\Delta H(x, y) = 0 \quad \text{in } \Omega, \qquad H(x, y) = \kappa_N \frac{1}{|x-y|^{N-2}}, \quad x \in \partial\Omega. \tag{23}$$

The function $H(x, x)$ is called the Robin function of Ω at x. It is useful to point out the following properties of G and H:

$$0 \leqslant G(x, y) \leqslant \kappa_N \frac{1}{|x-y|^{N-2}} \quad \text{for any } x, y \in \Omega, \tag{24}$$

$$\lim_{x \to \partial\Omega} H(x, x) = +\infty \tag{25}$$

and

$$H(x, x) \geqslant \min_{x \in \Omega} H(x, x) =: H_\Omega > 0. \tag{26}$$

Proposition 10. *Let $\eta > 0$ be fixed and assume that* (7)–(9) *hold. Then we have the following expansion*

$$J_\varepsilon(V_{\lambda,\zeta}) = (k+1)a_1 - \varepsilon^{\frac{N-2}{2}} \Psi(d, \bar{\Lambda}, \tau, \bar{\zeta}) + o\big(\varepsilon^{\frac{N-2}{2}}\big), \tag{27}$$

where Ψ is defined by

$$\begin{aligned} \Psi(d, \bar{\Lambda}, \tau, \bar{\zeta}) := {} & F(\tau)\frac{1}{d^{N-2}} + a_2 H(0,0) d^{N-2} \\ & + a_2\Bigg[\sum_{j=1}^{k} H(\zeta_j, \zeta_j)\Lambda_j^{N-2} - \sum_{\substack{j,s=1 \\ s \neq j}}^{k} \nu_j \nu_s G(\zeta_j, \zeta_s) \Lambda_j^{\frac{N-2}{2}} \Lambda_s^{\frac{N-2}{2}}\Bigg] \\ & - 2a_2 \sum_{j=1}^{k} \nu_j G(0, \zeta_j) \Lambda_j^{\frac{N-2}{2}} d^{\frac{N-2}{2}}, \end{aligned} \tag{28}$$

where

$$F(\tau) := \alpha_N^{p+1} c_\omega \frac{1}{(1+|\tau|^2)^{\frac{N-2}{2}}} \int_{\mathbb{R}^N} \frac{1}{|y+\tau|^{N-2}} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\, dy,$$

a_1 *and* a_2 *are positive constants and, as* ε *goes to zero, the term* $o(\varepsilon^{\frac{N-2}{2}})$ *is* C^1*-uniform over all* $(d, \bar{\Lambda}, \tau, \bar{\zeta})$*'s satisfying* (7)–(9).

Roughly speaking, we may say that any critical point of Ψ stable with respect C^1-perturbation generates a solution to (1) which has a positive blow-up point at the origin and k positive (if $\nu_j = +1$) or negative (if $\nu_j = -1$) blow-up points $\zeta_j \in \Omega \setminus \{0\}$.

3. Multipeak solutions

Let Γ be a closed subgroup of the group $O(N)$ of orthogonal transformations of $\mathbb{R}^N$. We denote by

$$\Gamma x := \{\gamma x\colon\ \gamma \in \Gamma\}$$

the Γ-orbit of $x \in \mathbb{R}^N$. A subset X of $\mathbb{R}^N$ is said to be Γ-invariant if $\Gamma x \subset X$ for every $x \in X$, and a function $u : X \to \mathbb{R}$ is Γ-invariant if it is constant on every Γ-orbit of X.

The Green function satisfies the following.

Lemma 11. *If* Ω *is* Γ*-invariant then*

$$G(\gamma x, \gamma y) = G(x, y) \quad \textit{and} \quad H(\gamma x, \gamma y) = H(x, y),$$

for all $x, y \in \Omega$*,* $\gamma \in \Gamma$*.*

Proof. Fix $x \in \Omega$, $\gamma \in \Gamma$. The map $y \mapsto H(x, \gamma^{-1} y)$ is harmonic and, since $\gamma y \in \partial\Omega$ for every $y \in \partial\Omega$, it satisfies

$$H\big(x, \gamma^{-1} y\big) = \frac{1}{|x - \gamma^{-1} y|^{N-2}} = \frac{1}{|\gamma x - y|^{N-2}} \quad \forall y \in \partial\Omega.$$

Therefore,

$$H(\gamma x, y) = H\big(x, \gamma^{-1} y\big) \quad \forall x, y \in \Omega,\ \gamma \in \Gamma.$$

This proves our claim. □

Let Γ be a group of the form $\Gamma := \Gamma_1 \times \Gamma_2$, where Γ_1 is a closed subgroup of $O(n)$ and Γ_2 is a closed subgroup of $O(m)$, $n + m = N$, acting on $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$ by

$$(\gamma_1, \gamma_2)(y, z) := (\gamma_1 y, \gamma_2 z) \quad \forall \gamma_1 \in \Gamma_1,\ \gamma_2 \in \Gamma_2,\ y \in \mathbb{R}^n,\ z \in \mathbb{R}^m.$$

From now on, we assume that these groups have the following properties:

(I) Γ_1 is a finite group which acts freely on $\mathbb{R}^n \setminus \{0\}$, that is, $\gamma y \neq y$ for every $\gamma \in \Gamma_1$, $y \in \mathbb{R}^n$.
(II) Γ_2 acts without fixed points on $\mathbb{R}^m \setminus \{0\}$, that is, for every $z \in \mathbb{R}^m \setminus \{0\}$ there exists $\gamma \in \Gamma_2$ such that $\gamma z \neq z$.

To simplify notation we write Γ_1 for the subgroup $\Gamma_1 \times \{1\}$ of Γ and Γ_2 for the subgroup $\{1\} \times \Gamma_2$ of Γ. Property (II) implies that the fixed point space of the Γ_2-action on $\mathbb{R}^N$ is

$$\{x \in \mathbb{R}^N\colon \gamma x = x \ \forall \gamma \in \Gamma_2\} = \mathbb{R}^n \times \{0\}, \tag{29}$$

thus

$$\Gamma y = \Gamma_1 y \quad \forall y \in \mathbb{R}^n \times \{0\},$$

and, since Γ_1 acts freely on $\mathbb{R}^n \setminus \{0\}$, its cardinality $\#\Gamma y$ is the order $|\Gamma_1|$ of the group Γ_1.

For $\zeta \in (\Omega \setminus \{0\}) \cap (\mathbb{R}^n \times \{0\})$ we define

$$\alpha(\zeta) := H(\zeta, \zeta) - \sum_{\gamma \in \Gamma_1 \setminus \{1\}} G(\zeta, \gamma\zeta).$$

Set

$$\Omega_1 := \{\zeta \in (\Omega \setminus \{0\}) \cap (\mathbb{R}^n \times \{0\})\colon \alpha(\zeta) \neq 0\},$$

and let $\varphi : \Omega_1 \to \mathbb{R}$ be defined by

$$\varphi(\zeta) := H(0,0) - \frac{|\Gamma_1| G^2(0,\zeta)}{\alpha(\zeta)}.$$

By Lemma 11, both α and φ are Γ_1-invariant, that is,

$$\begin{aligned} \alpha(\gamma\zeta) &= \alpha(\zeta) \quad \text{for all } \gamma \in \Gamma_1,\ \zeta \in (\Omega \setminus \{0\}) \cap (\mathbb{R}^n \times \{0\}), \\ \varphi(\gamma\zeta) &= \varphi(\zeta) \quad \text{for all } \gamma \in \Gamma_1,\ \zeta \in \Omega_1. \end{aligned} \tag{30}$$

The following holds.

Theorem 12. *Assume that Ω is Γ-invariant and ω is Γ_2-invariant, and let $\zeta^* \in \Omega_1$ be a C^1-stable critical point of φ.*

(i) *If $\alpha(\zeta^*) > 0$ and $\varphi(\zeta^*) > 0$, then there exists $\varepsilon_0 > 0$ such that, for each $\varepsilon \in (0, \varepsilon_0)$, there is a positive Γ_2-invariant solution u_ε to problem* (1) *which satisfies*

$$|\nabla u_\varepsilon|^2\, dx \rightharpoonup A_N\Big(\delta_0 + \sum_{\gamma \in \Gamma_1} \delta_{\gamma\zeta^*}\Big) \quad \textit{in the sense of measures, as } \varepsilon \to 0.$$

(ii) *If $\alpha(\zeta^*) < 0$, then there exists $\varepsilon_0 > 0$ such that, for each $\varepsilon \in (0, \varepsilon_0)$, there is a sign changing Γ_2-invariant solution u_ε to problem* (1) *which satisfies*

$$|\nabla u_\varepsilon|^2\,dx \rightharpoonup A_N\left(\delta_0 - \sum_{\gamma\in\Gamma_1}\delta_{\gamma\zeta^*}\right) \quad \textit{in the sense of measures, as } \varepsilon \to 0.$$

Proof. We look for a Γ_2-invariant solution to problem (1) of the form

$$u = V + \phi, \qquad V := P_\varepsilon U_{\mu,\xi} + \sum_{\gamma\in\Gamma_1} \nu P_\varepsilon U_{\lambda,\gamma\zeta} \tag{31}$$

with $\nu \in \{1, -1\}$, and $\mu, \lambda \in (0, \infty)$, $\xi, \zeta \in \Omega \cap (\mathbb{R}^n \times \{0\})$, such that conditions (7)–(9) hold with $\lambda_j = \lambda$ and $\zeta_j = \gamma_j\zeta$, that is,

$$\mu := d\sqrt{\varepsilon}, \qquad \lambda := \Lambda\sqrt{\varepsilon}, \quad \eta < d,\ \Lambda < \eta^{-1}, \tag{32}$$

$$\xi := \mu\tau, \quad \tau \in \mathbb{R}^N,\ |\tau| < \eta, \tag{33}$$

$$|\zeta| > 2\eta, \qquad \operatorname{dist}(\zeta, \partial\Omega) > 2\eta, \qquad |\zeta - \gamma\zeta| > 2\eta \quad \forall\gamma \in \Gamma_1,\ \gamma \neq 1, \tag{34}$$

for some $\eta > 0$. In this case, by Lemma 11, the function Ψ defined in (28) reduces to

$$\begin{aligned}\Psi(d, \Lambda, \tau, \zeta) := {} & F(\tau)\frac{1}{d^{N-2}} + a_2 H(0,0) d^{N-2} \\ & + a_2\left[kH(\zeta,\zeta) - k\sum_{\gamma\in\Gamma_1\setminus\{1\}} G(\zeta,\gamma\zeta)\right]\Lambda^{N-2} \\ & - 2a_2\nu k G(0,\zeta)\Lambda^{\frac{N-2}{2}} d^{\frac{N-2}{2}},\end{aligned}$$

where $k := |\Gamma_1|$ and, abusing notation, we have set $\Lambda := (\Lambda, \dots, \Lambda)$ and $\zeta := (\zeta, \gamma_2\zeta, \dots, \gamma_k\zeta)$ for some chosen ordering of the elements of $\Gamma_1 = \{\gamma_1 := 1, \gamma_2, \dots, \gamma_k\}$. We will now show that, for some $\eta > 0$, the restriction of Ψ to the set

$$\mathcal{S}_\eta := \left\{(d, \bar\Lambda, \tau, \bar\zeta) \in (0,\infty)^{k+1} \times \left(\mathbb{R}^n \times \{0\}\right) \times \left(\Omega \cap \left(\mathbb{R}^n \times \{0\}\right)\right)^k\colon \text{(32)–(34) hold}\right\}$$

has a critical point which is stable with respect to C^1-perturbation. The claim will then follow from Propositions 10, 9, and Lemma 13 below.

It is easy to check that, if

$$\frac{\nu G(0,\zeta)}{\alpha(\zeta)} > 0 \quad \text{and} \quad \varphi(\zeta) := H(0,0) - \frac{kG^2(0,\zeta)}{\alpha(\zeta)} > 0, \tag{35}$$

there exist unique $d(\zeta), \Lambda(\zeta) > 0$, $\tau(\zeta) \in \mathbb{R}^n$, such that

$$\nabla_{(d,\bar\Lambda,\tau)}\Psi\big(d(\zeta), \Lambda(\zeta), \tau(\zeta), \zeta\big) = 0. \tag{36}$$

In fact, $\tau(\zeta) = 0$,

$$d(\zeta)=\left[\frac{F(0)}{a_2}\frac{\alpha(\zeta)}{H(0,0)\alpha(\zeta)-kG^2(0,\zeta)}\right]^{\frac{1}{2(N-2)}},$$

$$\Lambda(\zeta)=\left[\frac{\nu G(0,\zeta)}{\alpha(\zeta)}\right]^{\frac{2}{N-2}}d(\zeta).$$

If follows from (24) and (26) that conditions (35) hold if, either $\nu=1$, $\alpha(\zeta)>0$ and $\varphi(\zeta)>0$, or if $\nu=-1$ and $\alpha(\zeta)<0$. An easy computation shows that

$$\Psi\big(d(\zeta),\Lambda(\zeta),\tau(\zeta),\zeta\big)=2\big(a_2F(0)k\big)^{1/2}\varphi(\zeta)^{1/2}.$$

Therefore, if ζ^* is a C^1-stable critical point of φ satisfying (35) then, by (30), $\gamma\zeta^*$ is a C^1-stable critical point of φ for all $\gamma\in\Gamma_1$ and, by (36), $(d(\zeta^*),\Lambda(\zeta^*),0,\zeta^*)$ is a critical point of the restriction of Ψ to the set $\mathcal{S}_\eta$ for some $\eta>0$. Moreover, since $D^2_{(d,\bar\Lambda,\tau)}\Psi(d(\zeta),\Lambda(\zeta),\tau(\zeta),\zeta)$ is invertible, the critical point $(d(\zeta^*),\Lambda(\zeta^*),0,\zeta^*)$ is C^1-stable. This concludes the proof. □

Lemma 13. *If $(d,\bar\Lambda,\tau,\bar\zeta)$ is a critical point of the restriction*

$$J_\varepsilon^*\,|_{(0,\infty)^{k+1}\times(\mathbb{R}^n\times\{0\})\times(\Omega\cap(\mathbb{R}^n\times\{0\}))^k}$$

then $u=V_{\lambda,\zeta}+\phi$ is a Γ_2-invariant solution to problem (1).

Proof. It suffices to show that J_ε^* is Γ_2-invariant with respect to the Γ_2-action on $(0,\infty)^{k+1}\times\mathbb{R}^N\times\Omega^k$ given by $\gamma(d,\bar\Lambda,\tau,\bar\zeta):=(d,\bar\Lambda,\gamma\tau,\gamma\bar\zeta)$, where $\gamma\bar\zeta:=(\gamma\zeta_1,\ldots,\gamma\zeta_k)$. Indeed, property (II) implies that the fixed point set of this action is $(0,\infty)^{k+1}\times(\mathbb{R}^n\times\{0\})\times(\Omega\cap(\mathbb{R}^n\times\{0\}))^k$. Therefore, by the principle of symmetric criticality [19,25], we conclude that, if $(d,\bar\Lambda,\tau,\bar\zeta)$ is a critical point of the restriction

$$J_\varepsilon^*\,|_{(0,\infty)^{k+1}\times(\mathbb{R}^n\times\{0\})\times(\Omega\cap(\mathbb{R}^n\times\{0\}))^k},$$

then it is a critical point of J_ε^*. The claim now follows from Lemma 8.

To prove that J_ε^* is Γ_2-invariant first observe that, since Ω and ω are Γ_2-invariant, the domain $\Omega\setminus\varepsilon\omega$ is Γ_2-invariant for every $\varepsilon>0$, and one has an action of Γ_2 on $H_0^1(\Omega\setminus\varepsilon\omega)$ given by $(\gamma u)(x):=u(\gamma^{-1}x)$. This action preserves the Sobolev and the L^{p+1} norms, i.e.

$$\int\limits_{\Omega\setminus\varepsilon\omega}\nabla(\gamma u)\nabla(\gamma v)=\int\limits_{\Omega\setminus\varepsilon\omega}\nabla u\nabla v\quad\text{and}\quad\int\limits_{\Omega\setminus\varepsilon\omega}|\gamma u|^{p+1}=\int\limits_{\Omega\setminus\varepsilon\omega}|u|^{p+1}$$

for all $\gamma\in\Gamma_2$, $u,v\in H_0^1(\Omega\setminus\varepsilon\omega)$. Therefore, the functional J_ε defined in (20) is Γ_2-invariant with respect to this action, i.e.

$$J_\varepsilon(\gamma u)=J_\varepsilon(u)\quad\text{for all }\gamma\in\Gamma_2,\ u\in H_0^1(\Omega\setminus\varepsilon\omega).\tag{37}$$

Secondly, we claim that for any $\gamma\in\Gamma_2$

$$(\phi,\psi)\text{ solves }(\wp_{\lambda,\zeta})\quad\Leftrightarrow\quad(\gamma\phi,\gamma\psi)\text{ solves }(\wp_{\lambda,\gamma\zeta}),\tag{38}$$

where problems $(\wp_{\lambda,\zeta})$ and $(\wp_{\lambda,\gamma\zeta})$ are defined in (11), and $\gamma\zeta := (\gamma\xi, \gamma\zeta_1, \dots, \gamma\zeta_k)$. Indeed, first notice that

$$U_{\lambda_j,\gamma\zeta_j}(x) = U_{\lambda_j,\zeta_j}\big(\gamma^{-1}x\big) =: \gamma U_{\lambda_j,\zeta_j}(x) \quad \text{for all } \gamma \in \Gamma_2, \tag{39}$$

$j = 0, \dots, k$, $\zeta_0 := \xi$. Since

$$\begin{aligned}\int\limits_{\mathbb{R}^N} \nabla(\gamma\theta)\nabla(\gamma v) &= \int\limits_{\mathbb{R}^N} \nabla(\theta)\nabla(v) = p\int\limits_{\mathbb{R}^N} U_{\lambda_j,\zeta_j}^{p-1}\theta v \\ &= p\int\limits_{\mathbb{R}^N} (\gamma U_{\lambda_j,\zeta_j})^{p-1}(\gamma\theta)(\gamma v) \quad \text{for all } v \in C_c^\infty\big(\mathbb{R}^N\big),\end{aligned}$$

we have that $\theta \in \widehat{\mathcal{K}}_{\lambda_j,\zeta_j}$ iff $\gamma\theta \in \widehat{\mathcal{K}}_{\lambda_j,\gamma\zeta_j}$. Similar arguments show that $\psi \in \mathcal{K}_{\lambda,\zeta}$ iff $\gamma\psi \in \mathcal{K}_{\lambda,\gamma\zeta}$, that $\phi \in \mathcal{K}_{\lambda,\zeta}^{\perp}$ iff $\gamma\phi \in \mathcal{K}_{\lambda,\gamma\zeta}^{\perp}$, and that

$$L_{\lambda,\zeta}(\phi) = N_{\lambda,\zeta}(\phi) + R_{\lambda,\zeta} + \psi$$

holds iff

$$L_{\lambda,\gamma\zeta}(\gamma\phi) = N_{\lambda,\gamma\zeta}(\gamma\phi) + R_{\lambda,\gamma\zeta} + \gamma\psi$$

holds. Therefore (38) follows.

This allows us to conclude J_ε^* is Γ_2-invariant. Indeed, since the solution (ϕ, ψ) to problem $(\wp_{\lambda,\zeta})$ in (11) is unique, (38) guarantees that $(\gamma\phi, \gamma\psi)$ is the unique solution to problem $(\wp_{\lambda,\gamma\zeta})$. It follows from (37) and (39) that

$$\begin{aligned}J_\varepsilon^*(d, \bar{\Lambda}, \gamma\tau, \gamma\bar{\zeta}) &= J_\varepsilon(V_{\lambda,\gamma\zeta} + \gamma\phi) = J_\varepsilon\big(\gamma(V_{\lambda,\zeta} + \phi)\big) \\ &= J_\varepsilon(V_{\lambda,\zeta} + \phi) = J_\varepsilon^*(d, \bar{\Lambda}, \tau, \bar{\zeta}) \quad \text{for all } \gamma \in \Gamma_2,\end{aligned}$$

as claimed. □

To prove Theorem 2 we need the following topological lemma.

Lemma 14. *Let D be a connected bounded smooth domain in $\mathbb{R}^n$, $n \geqslant 2$, with nonconnected boundary. Then there exists a point $x_0 \in \mathbb{R}^n \setminus \overline{D}$ with the following property: for every $v \in \mathbb{R}^n$, $v \neq 0$, there exist $t_2 > t_1 > 0$ such that $x_0 + t_1 v$ and $x_0 + t_2 v$ are in different components of ∂D, and $x_0 + tv \in D$ for every $t \in (t_1, t_2)$.*

Proof. Let $K_1, \dots, K_k$ be the connected components of ∂D, $k \geqslant 2$. Then K_j is an $(n-1)$-dimensional compact connected submanifold of $\mathbb{R}^n$. By Alexander's and Poincaré's duality theorems [23, Chapter 6, Section 2, Theorems 16 and 18],

$$\widetilde{H}_0\big(\mathbb{R}^n \setminus K_j; \mathbb{Z}\big) \cong H^{n-1}(K_j; \mathbb{Z}) \cong H_0(K_j; \mathbb{Z}) \cong \mathbb{Z},$$

where $H_*(\cdot;\mathbb{Z})$ and $H^*(\cdot;\mathbb{Z})$ denote singular homology and cohomology with integer coefficients. Hence, $\mathbb{R}^n \setminus K_j$ has precisely two connected components D_j and U_j, with D_j bounded and U_j unbounded. Note that, if $D_j \cap D_s \neq \emptyset$ and $j \neq s$ then, either $\overline{D_j} \subset D_s$, or $\overline{D_s} \subset D_j$. Now, since D is bounded, it must be contained in one of the D_j's and, since D is connected, such D_j is unique. So, after reordering, we conclude that

$$D = D_1 \setminus (\overline{D_2} \cup \cdots \cup \overline{D_k}),$$
$$\overline{D_j} \subset D_1 \quad \text{for all } j = 2, \ldots, k,$$
$$\overline{D_j} \cap \overline{D_s} = \emptyset \quad \text{for all } j, s = 2, \ldots, k,\ j \neq s.$$

Let $x_0 \in D_2$ and let $v \in \mathbb{R}^n$, $v \neq 0$. Define

$$t_1 := \max\{t > 0\colon x_0 + tv \in \overline{D_2}\} \quad \text{and} \quad t_2 := \min\{t > t_1\colon x_0 + tv \in \partial D\}.$$

It is easy to check that they have the desired properties. □

Proof of Theorem 2. Let $\Gamma_1 := \{1\}$, and $\Gamma_2 := \{1, -1\}$ acting by multiplication on $\mathbb{R}^{N-n}$. We will prove that the function $\varphi : (\Omega \setminus \{0\}) \cap (\mathbb{R}^n \times \{0\}) \to \mathbb{R}$ defined by

$$\varphi(\zeta) := H(0,0) - \frac{G^2(0,\zeta)}{H(\zeta,\zeta)}$$

has a critical point of mountain pass type $\zeta^* \in C$, which is stable with respect to C^1-perturbations, such that $\varphi(\zeta^*) > 0$ if 0 is close enough to $\partial\Omega$. Note that, in this case, $\alpha(\zeta) := H(\zeta,\zeta) > 0$ for all ζ. The claim then follows from Theorem 12.

First note that, since Ω is Γ-invariant and $\mathbb{R}^n \times \{0\}$ is the fixed point set of the Γ-action on $\mathbb{R}^N$, the normal to $\partial\Omega$ at each point $x \in \partial\Omega \cap (\mathbb{R}^n \times \{0\})$ lies in $\mathbb{R}^n \times \{0\}$. Hence, $\Omega \cap (\mathbb{R}^n \times \{0\})$ is a bounded smooth domain in $\mathbb{R}^n \times \{0\}$. Consider the function $f : (\overline{\Omega} \setminus \{0\}) \cap (\mathbb{R}^n \times \{0\}) \to \mathbb{R}$ defined by

$$f(\zeta) := \begin{cases} \frac{G^2(0,\zeta)}{H(\zeta,\zeta)} & \text{if } \zeta \in (\Omega \setminus \{0\}) \cap (\mathbb{R}^n \times \{0\}), \\ 0 & \text{if } \zeta \in \partial\Omega \cap (\mathbb{R}^n \times \{0\}). \end{cases}$$

Note that $f(\zeta) \to 0$ as $\operatorname{dist}(\zeta, \partial\Omega) \to 0$. Let C_0 be the connected component of $\Omega \cap (\mathbb{R}^n \times \{0\})$ containing 0. We consider two cases.

Case 1: $C \neq C_0$. Fix two points $\xi_1, \xi_2 \in \partial C$ in different connected components of ∂C, and consider the set

$$\Theta := \big\{\sigma \in C^0\big([0,1], \overline{C}\big)\colon \sigma(0) = \xi_1,\ \sigma(1) = \xi_2\big\}.$$

It is not difficult to check that there exists $\zeta^* \in C$ such that

$$f(\zeta^*) = \inf_{\sigma \in \Theta} \max_{t \in [0,1]} f\big(\sigma(t)\big)$$

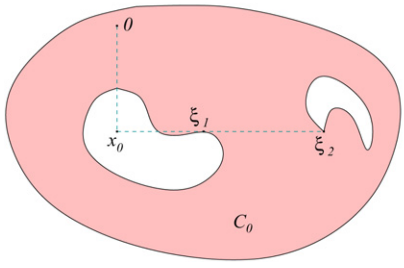

Fig. 3.

and ζ^* is a critical point of mountain pass type of the function f which is stable with respect to C^1-perturbation (see [12]). Therefore, ζ^* is a C^1-stable critical point of the function φ.

Now, let us estimate $f(\zeta^*)$. Since $\Omega \cap (\mathbb{R}^n \times \{0\})$ is a bounded smooth domain in $\mathbb{R}^n \times \{0\}$, we have that $\overline{C} \cap \overline{C_0} = \emptyset$. Hence,

$$r_C := \operatorname{dist}(C, C_0) > 0.$$

By (24) and (26), there is a constant $a := \frac{\kappa_N^2}{H_\Omega} > 0$ such that, for any $\zeta \in C$,

$$f(\zeta) = \frac{G^2(0,\zeta)}{H(\zeta,\zeta)} \leqslant a|\zeta|^{-2(N-2)} \leqslant a r_C^{-2(N-2)}.$$

In particular,

$$f(\zeta^*) \leqslant a r_C^{-2(N-2)}.$$

Therefore, by (25), there exists $\rho_0 > 0$, depending only on Ω, such that

$$\varphi(\zeta^*) \geqslant H(0,0) - a r_C^{-2(N-2)} > 0$$

if $\operatorname{dist}(0, \partial\Omega) < \rho_0$.

Case 2: $C = C_0$. Let $x_0 \in (\mathbb{R}^n \times \{0\}) \setminus \overline{C_0}$ be as in Lemma 14, choose $v \in \mathbb{R}^n \times \{0\}$, $v \neq 0$, orthogonal to x_0, and let $t_2 > t_1 > 0$ be such that $\xi_1 := x_0 + t_1 v$ and $\xi_2 := x_0 + t_2 v$ lie in different components of ∂C_0 and $x_0 + tv \in C_0$ for every $t \in (t_1, t_2)$, see Fig. 3.

Consider the set

$$\Theta := \big\{\sigma \in C^0\big([0,1], \overline{C_0} \setminus \{0\}\big)\colon \sigma(0) = \xi_1,\ \sigma(1) = \xi_2\big\}.$$

As in the previous case, there exists $\zeta^* \in C$ such that

$$f(\zeta^*) = \inf_{\sigma\in\Theta} \max_{t\in[0,1]} f\big(\sigma(t)\big),$$

ζ^* is a C^1-stable critical point of the function f and, hence, also of φ.

To estimate $f(\zeta^*)$, set

$$r_0 := \operatorname{dist}(x_0, C_0) > 0,$$

and consider the path $\tau \in \Theta$ given by $\tau(t) := (1-t)\xi_1 + t\xi_2$, $t \in [0,1]$. From (24) and (26), since x_0 is orthogonal to v, we obtain that

$$f\bigl(\tau(t)\bigr) = \frac{G^2(0,\tau(t))}{H(\tau(t),\tau(t))} \leqslant a\bigl|\tau(t)\bigr|^{-2(N-2)} \leqslant a|x_0|^{-2(N-2)} \leqslant a r_0^{-2(N-2)}$$

with $a := \frac{\kappa_N^2}{H_\Omega} > 0$ and, consequently,

$$f(\zeta^*) \leqslant \max_{t\in[0,1]} f\bigl(\tau(t)\bigr) \leqslant a r_0^{-2(N-2)}.$$

So, by (25), there exists $\rho_0 > 0$, depending only on Ω, such that

$$\varphi(\zeta^*) \geqslant H(0,0) - a r_0^{-2(N-2)} > 0$$

if $\operatorname{dist}(0, \partial\Omega) < \rho_0$.

This concludes the proof. □

Remark 15. Observe that Theorem 2 remains true if instead of (4) and (5) we assume that Ω and ω are Γ_2-invariant for some closed subgroup Γ_2 of $O(N-n)$ satisfying property (II) above.

Proof of Theorem 1. Since Ω is assumed to be connected, Theorem 1 follows from Theorem 2 taking $n = N$. □

Proof of Theorem 3. Let $\Gamma_1 := \{e^{2\pi ij/k} \in \mathbb{C}\colon\ j = 0,\dots,k-1\}$, acting on $\mathbb{R}^2 \equiv \mathbb{C}$ by complex multiplication, and let $\Gamma_2 := \{1,-1\}$, acting by multiplication on $\mathbb{R}^{N-2}$. For every $\zeta \in \mathbb{C}$ with $|\zeta| \in (\frac{1}{2}, 1)$ using (24) we obtain

$$G(0,\zeta) \leqslant \kappa_N \frac{1}{|\zeta|} \leqslant 2\kappa_N =: c_1,$$

and, for $j = 1,\dots,k-1$,

$$G\bigl(\zeta, e^{2\pi ij/k}\zeta\bigr) \leqslant \kappa_N \frac{1}{|\zeta - e^{2\pi ij/k}\zeta|} \leqslant \frac{2\kappa_N}{|1 - e^{2\pi i/k}|} =: c_2.$$

The Robin function H depends on r and ρ. Nevertheless it is not difficult to check that

$$\lim_{r+\rho\to 1} H(0,0)\Bigl[\min_{|\zeta|\in(r+\rho,1)} H(\zeta,\zeta) - (k-1)c_2\Bigr] = +\infty.$$

Consequently, there exists $\rho_0 \in (\frac{1}{2}, 1)$ such that, if $r + \rho \in (\rho_0, 1)$ then

$$\alpha(\zeta) = H(\zeta,\zeta) - \sum_{j=1}^{k-1} G\bigl(\zeta, e^{2\pi ij/k}\zeta\bigr)$$
$$\geqslant \min_{|\zeta|\in(r+\rho,1)} H(\zeta,\zeta) - (k-1)c_2 > 0 \quad \text{for all } |\zeta| \in (r+\rho,1),$$

and

$$\varphi(\zeta) = H(0,0) - \frac{kG^2(0,\zeta)}{\alpha(\zeta)}$$
$$\geqslant H(0,0) - \frac{kc_1^2}{\min_{|\zeta|\in(r+\rho,1)} H(\zeta,\zeta) - (k-1)c_2}$$
$$> 0 \quad \text{for all } |\zeta| \in (r+\rho,1).$$

Let $C := \{\zeta \in \Omega \cap (\mathbb{R}^2 \times \{0\}):\ |\zeta| \in (r+\rho,1)\}$. Arguing as in the proof of Theorem 2, we prove that the function

$$f(\zeta) := \begin{cases} \frac{kG^2(0,\zeta)}{\alpha(\zeta)} & \text{if } \zeta \in C, \\ 0 & \text{if } \zeta \in \partial C \end{cases}$$

has a critical point $\zeta^* \in C$ which is stable with respect to C^1-perturbation. Therefore, ζ^* is a C^1-stable critical point of φ with $\alpha(\zeta^*) > 0$ and $\varphi(\zeta^*) > 0$. Since Ω is $O(2)$-invariant we may take $\zeta^* := \rho_*(1,0,\dots,0)$. The result now follows from Theorem 12. □

Theorems 4 and 5 are special cases of the following result.

Theorem 16. *Assume that Ω is Γ-invariant and ω is Γ_2-invariant, and that $|\Gamma_1| \geqslant 2$. Then there exists $\varepsilon_0 > 0$ such that, for each $\varepsilon \in (0,\varepsilon_0)$, there exists a pair $\pm u_\varepsilon$ of Γ_2-invariant sign changing solutions to problem* (1) *satisfying*

$$|\nabla u_\varepsilon|^2\,dx \rightharpoonup A_N\Bigl(\delta_0 - \sum_{\gamma\in\Gamma_1} \delta_{\gamma\zeta^*}\Bigr) \quad \textit{in the sense of measures, as } \varepsilon \to 0,$$

for some $\zeta^* \in (\Omega \setminus \{0\}) \cap (\mathbb{R}^n \times \{0\})$.

Proof. Since Γ_1 acts without fixed points on $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n:\ |\zeta| = 1\}$, one has that

$$\min_{\zeta\in\mathbb{S}^{n-1}} \min_{\gamma\in\Gamma_1\setminus\{1\}} |\zeta - \gamma\zeta| = a_0 > 0.$$

Hence, for every $\gamma \in \Gamma_1 \setminus \{1\}$ and every $\zeta \in (\Omega \setminus \{0\}) \cap (\mathbb{R}^n \times \{0\})$, we obtain that

$$G(\zeta,\gamma\zeta) \leqslant \frac{\kappa_N}{|\zeta - \gamma\zeta|^{N-2}} \leqslant \frac{\kappa_N}{a_0^{N-2}|\zeta|^{N-2}}$$

and, therefore,

$$\alpha(\zeta) := H(\zeta,\zeta) - \sum_{\gamma\in\Gamma_1\setminus\{1\}} G(\zeta,\gamma\zeta) \geqslant H(\zeta,\zeta) - \frac{\kappa_N}{a_0^{N-2}|\zeta|^{N-2}}.$$

This, together with (25), implies that

$$\lim_{|\zeta|\to 0} \alpha(\zeta) = -\infty \quad \text{and} \quad \lim_{\zeta\to\partial\Omega} \alpha(\zeta) = +\infty.$$

Let

$$\mathcal{O} := \big\{\zeta \in \big(\Omega \setminus \{0\}\big) \cap \big(\mathbb{R}^n \times \{0\}\big)\colon \alpha(\zeta) < 0\big\}.$$

Then $\mathcal{O}$ is open in $\mathbb{R}^n$ and

$$\inf_{\zeta\in\mathcal{O}} \varphi(\zeta) = \inf_{\zeta\in\mathcal{O}} \left[H(0,0) - \frac{|\Gamma_1| G^2(0,\zeta)}{\alpha(\zeta)} \right] \geqslant H(0,0).$$

Since

$$\varphi(\zeta) \to +\infty \quad \text{as } \operatorname{dist}(\zeta, \partial\mathcal{O}) \to 0,$$

there exists $\zeta^* \in \mathcal{O}$ such that

$$\varphi(\zeta^*) = \inf_{\zeta\in\mathcal{O}} \varphi(\zeta).$$

ζ^* is a C^1-stable critical point of φ with $\varphi(\zeta^*) < 0$. The result now follows from Theorem 12. □

Proof of Theorem 4. Apply Theorem 16 with $n = N$, $\Gamma_1 = \{-1, 1\}$ acting by multiplication on $\mathbb{R}^N$, and $\Gamma_2 = \{1\}$. □

Proof of Theorem 5. If N is odd, apply Theorem 16 with $n = 2$, $\Gamma_1 := \{e^{2\pi ij/k} \in \mathbb{C}\colon\ j = 0, \ldots, k-1\}$ acting on $\mathbb{R}^2 \equiv \mathbb{C}$ by complex multiplication, and $\Gamma_2 := \{1, -1\}$ acting by multiplication on $\mathbb{R}^{N-2}$. If N is even, apply Theorem 16 with $n = N$, $\Gamma_1 := \{e^{2\pi ij/k} \in \mathbb{C}\colon\ j = 0, \ldots, k-1\}$ acting on $\mathbb{R}^N \equiv \mathbb{C}^{N/2}$ by complex multiplication, and $\Gamma_2 := \{1\}$. □

4. The expansion of the energy

This section is devoted to prove Proposition 10.

First, we describe the asymptotic expansion of the projection of the standard bubble centered at a point which is inside the hole of our domain. The following result holds (see [17, Lemma 2.1]).

Lemma 17. *Problem*

$$\begin{cases} -\Delta u = 0 & \text{in } \mathbb{R}^N \setminus \omega, \\ u = 1 & \text{on } \partial\omega, \\ u \in \mathcal{D}^{1,2}\big(\mathbb{R}^N \setminus \omega\big) \end{cases} \tag{40}$$

has a unique solution φ_ω. Moreover,

$$\frac{c_1}{|x|^{N-2}} \leqslant \varphi_\omega(x) \leqslant \frac{c_2}{|x|^{N-2}} \quad \forall x \in \mathbb{R}^N \setminus \omega$$

for some positive constants c_1, c_2. Furthermore,

$$\lim_{|x| \to +\infty} |x|^{N-2} \varphi_\omega(x) = c_\omega$$

with

$$c_\omega = \frac{1}{(N-2)|S^{N-1}|} \int\limits_{\mathbb{R}^N \setminus \omega} \left|\nabla \varphi_\omega(x)\right|^2 dx.$$

Observe that, if $\omega = B(0,1)$ then $\varphi_\omega(x) = \frac{1}{|x|^{N-2}}$. The following expansion holds (see [17, Lemma 2.2]).

Lemma 18. *Let*

$$R^\varepsilon_{d,\tau}(x) := P_\varepsilon U_{\mu,\xi}(x) - U_{\mu,\xi}(x) + \alpha_N \mu^{\frac{N-2}{2}} H(x,\xi) + \alpha_N \frac{1}{\mu^{\frac{N-2}{2}} (1+|\tau|^2)^{\frac{N-2}{2}}} \varphi_\omega\left(\frac{x}{\varepsilon}\right).$$

Then there exists a positive constant c such that for any $x \in \Omega \setminus \varepsilon\omega$

$$\left|R^\varepsilon_{d,\tau}(x)\right| \leqslant c\varepsilon^{\frac{N-2}{4}} \left(\frac{\varepsilon^{\frac{N-1}{2}}}{|x|^{N-2}} + \varepsilon\right) \quad \text{if } N \geqslant 4, \tag{41}$$

$$\left|R^\varepsilon_{d,\tau}(x)\right| \leqslant c\varepsilon^{\frac{1}{4}} \left(\frac{\varepsilon}{|x|} + \sqrt{\varepsilon}\right) \quad \text{if } N = 3, \tag{42}$$

$$\left|\partial_d R^\varepsilon_{d,\tau}(x)\right| \leqslant c\varepsilon^{\frac{N-2}{4}} \left(\frac{\varepsilon^{\frac{N-1}{2}}}{|x|^{N-2}} + \varepsilon\right) \quad \text{if } N \geqslant 4, \tag{43}$$

$$\left|\partial_d R^\varepsilon_{d,\tau}(x)\right| \leqslant c\varepsilon^{\frac{1}{4}} \left(\frac{\varepsilon}{|x|} + \sqrt{\varepsilon}\right) \quad \text{if } N = 3 \tag{44}$$

$$\left|\partial_{\tau_i} R^\varepsilon_{d,\tau}(x)\right| \leqslant c\varepsilon^{\frac{N}{4}} \left(\frac{\varepsilon^{\frac{N-2}{2}}}{|x|^{N-2}} + \varepsilon^{\frac{N-3}{2}}\right) \quad \text{if } N \geqslant 3, \tag{45}$$

Proof. The function $R := R^\varepsilon_{d,\tau}$ solves $-\Delta R = 0$ in $\Omega \setminus \varepsilon\omega$ with

$$R(x) = \alpha_N \left[-\frac{\mu^{\frac{N-2}{2}}}{(\mu^2 + |x-\xi|^2)^{\frac{N-2}{2}}} + \frac{\mu^{\frac{N-2}{2}}}{|x-\xi|^{N-2}} + \frac{1}{\mu^{\frac{N-2}{2}} (1+|\tau|^2)^{\frac{N-2}{2}}} \varphi_\omega\left(\frac{x}{\varepsilon}\right)\right], \quad x \in \partial\Omega,$$

$$R(x) = \alpha_N \left[-\frac{\mu^{\frac{N-2}{2}}}{(\mu^2 + |x-\xi|^2)^{\frac{N-2}{2}}} + \mu^{\frac{N-2}{2}} H(x,\xi) + \frac{1}{\mu^{\frac{N-2}{2}} (1+|\tau|^2)^{\frac{N-2}{2}}}\right], \quad x \in \partial\varepsilon\omega.$$

Therefore (41) and (42) follow, because

$$\varepsilon^{-\frac{N-2}{4}} R(x) = O\big(\varepsilon + \varepsilon^{\frac{N-2}{2}}\big), \quad x \in \partial\Omega \quad \text{and} \quad \varepsilon^{-\frac{N-2}{4}} R(x) = O\big(\varepsilon^{-\frac{N-3}{2}}\big), \quad x \in \partial\varepsilon\omega.$$

The function $R_d(x) = \partial_d R^{\varepsilon}_{d,\tau}(x)$ solves $-\Delta R_d = 0$ in $\Omega \setminus \varepsilon\omega$ with

$$\begin{aligned}
R_d(x) &= \alpha_N \frac{N-2}{2} \mu^{\frac{N-4}{2}} \varepsilon^{\frac{1}{2}} \bigg[\frac{\mu^2 - |x-\xi|^2}{(\mu^2 + |x-\xi|^2)^{\frac{N}{2}}} - 2\mu \frac{(x-\xi,\tau)}{(\mu^2 + |x-\xi|^2)^{\frac{N}{2}}} \\
&\quad + \frac{1}{|x-\xi|^{N-2}} + 2\mu \frac{(x-\xi,\tau)}{|x-\xi|^{N}} - \frac{1}{\mu^{N-2}(1+|\tau|^2)^{\frac{N-2}{2}}} \varphi_\omega\left(\frac{x}{\varepsilon}\right)\bigg], \quad x \in \partial\Omega, \\
R_d(x) &= \alpha_N \frac{N-2}{2} \mu^{\frac{N-4}{2}} \varepsilon^{\frac{1}{2}} \bigg[\frac{\mu^2 - |x-\xi|^2}{(\mu^2 + |x-\xi|^2)^{\frac{N}{2}}} - 2\mu \frac{(x-\xi,\tau)}{(\mu^2 + |x-\xi|^2)^{\frac{N}{2}}} \\
&\quad + H(x,\xi) + \frac{2\mu}{N-2}\big(\nabla_y H(x,\xi),\tau\big) - \frac{1}{\mu^{N-2}(1+|\tau|^2)^{\frac{N-2}{2}}}\bigg], \quad x \in \partial\varepsilon\omega.
\end{aligned}$$

Therefore (43) and (44) follow, because

$$\varepsilon^{-\frac{N-2}{4}} R_d(x) = O\big(\varepsilon + \varepsilon^{\frac{N-2}{2}}\big), \quad x \in \partial\Omega \quad \text{and} \quad \varepsilon^{-\frac{N-2}{4}} R_d(x) = O\big(\varepsilon^{-\frac{N-3}{2}}\big), \quad x \in \partial\varepsilon\omega.$$

The function $R_i(x) = \partial_{\tau_i} R^{\varepsilon}_{d,\tau}(x)$ solves $-\Delta R_i = 0$ in $\Omega \setminus \varepsilon\omega$ with

$$\begin{aligned}
R_i(x) &= \alpha_N (N-2) \mu^{\frac{N}{2}} \bigg[\frac{(x-\xi)_i}{(\mu^2 + |x-\xi|^2)^{\frac{N}{2}}} - \frac{(x-\xi)_i}{|x-\xi|^{\frac{N}{2}}} - \frac{1}{\mu^{N-1}} \frac{\tau_i}{(1+|\tau|^2)^{\frac{N}{2}}} \varphi_\omega\left(\frac{x}{\varepsilon}\right)\bigg], \\
&\quad x \in \partial\Omega, \\
R_i(x) &= \alpha_N (N-2) \mu^{\frac{N}{2}} \bigg[\frac{(x-\xi)_i}{(\mu^2 + |x-\xi|^2)^{\frac{N}{2}}} + \frac{\partial_{y_i} H(x,\xi)}{N-2} - \frac{1}{\mu^{N-1}} \frac{\tau_i}{(1+|\tau|^2)^{\frac{N}{2}}}\bigg], \\
&\quad x \in \partial\varepsilon\omega.
\end{aligned}$$

Therefore (45) follows, because

$$\varepsilon^{-\frac{N}{4}} R_i(x) = O\big(\varepsilon + \varepsilon^{\frac{N-3}{2}}\big), \quad x \in \partial\Omega \quad \text{and} \quad \varepsilon^{-\frac{N}{4}} R_i(x) = O\big(\varepsilon^{-\frac{N-2}{2}}\big), \quad x \in \partial\varepsilon\omega.$$

This finishes the proof. □

The asymptotic expansion of the projection of the standard bubble centered at a point inside the domain is, by now, a standard fact. We refer the reader to [22]. We state the result in the following.

Lemma 19. *Let $\eta > 0$ be fixed. If* (8) *and* (9) *hold, then the following facts hold true. Let*

$$r^{\varepsilon}_{\Lambda,\zeta}(x) := P_\varepsilon U_{\lambda,\zeta}(x) - U_{\lambda,\zeta}(x) + \alpha_N \lambda^{\frac{N-2}{2}} H(x,\zeta).$$

Then, for any $x \in \Omega \setminus \varepsilon\omega$,

$$0 \leqslant r^{\varepsilon}_{\Lambda,\zeta}(x) \leqslant c\lambda^{\frac{N+2}{2}}, \tag{46}$$

for some positive and fixed constant c. Furthermore, for any $x \in \Omega \setminus \varepsilon\omega$

$$\left|\partial_\Lambda r^{\varepsilon}_{\Lambda,\zeta}(x)\right| \leqslant c\varepsilon^{\frac{N+2}{4}} \quad \text{if } N \geqslant 4, \qquad \left|\partial_\Lambda r^{\varepsilon}_{\Lambda,\zeta}(x)\right| \leqslant c\varepsilon^{\frac{3}{4}} \quad \text{if } N = 3 \tag{47}$$

and for $i = 1, \dots, N$

$$\left|\partial_{\zeta_i} r^{\varepsilon}_{\Lambda,\zeta}(x)\right| \leqslant c\varepsilon^{\frac{N+2}{4}}, \tag{48}$$

for some positive and fixed constant c.

We have now all the elements needed to perform the expansion (27).

Proof of Proposition 10. For the sake of simplicity, we will prove estimate (27) when $k = 1$. Let $\eta > 0$ be fixed and assume (7)–(9) hold with λ, ζ instead of λ_1, ζ_1. We will prove that

$$\begin{aligned} &J_\varepsilon(P_\varepsilon U_{\mu,\xi} \pm P_\varepsilon U_{\lambda,\zeta}) \\ &\quad = 2a_1 - F(\tau)\left(\frac{\varepsilon}{\mu}\right)^{N-2}\big(1 + o(1)\big) \\ &\qquad - a_2\big(H(0,0)\mu^{N-2} + H(\zeta,\zeta)\lambda^{N-2} \mp 2G(0,\zeta)\lambda^{\frac{N-2}{2}}\mu^{\frac{N-2}{2}}\big)\big(1 + o(1)\big), \end{aligned} \tag{49}$$

uniformly in the C^1-sense for $(\tau, \zeta, d, \Lambda)$ satisfying (7)–(9). The constants that appear in (49) are given by

$$a_1 := \alpha_N^{p+1} \int_{\mathbb{R}^N} \frac{1}{(1 + |y|^2)^N}\, dy, \tag{50}$$

$$a_2 := \frac{1}{2}\alpha_N^{p+1} \int_{\mathbb{R}^N} \frac{1}{(1 + |y|^2)^{\frac{N+2}{2}}}\, dy, \tag{51}$$

We have that

$$\begin{aligned} &J_\varepsilon(P_\varepsilon U_{\mu,\xi} \pm P_\varepsilon U_{\lambda,\zeta}) \\ &\quad = \frac{1}{2}\int_{\Omega\setminus\varepsilon\omega} \left|\nabla(P_\varepsilon U_{\mu,\xi} \pm P_\varepsilon U_{\lambda,\zeta})\right|^2 - \frac{1}{p+1}\int_{\Omega\setminus\varepsilon\omega} |P_\varepsilon U_{\mu,\xi} \pm P_\varepsilon U_{\lambda,\zeta}|^{p+1} \\ &\quad = \frac{1}{2}\int_{\Omega\setminus\varepsilon\omega} P_\varepsilon U_{\mu,\xi} U^p_{\mu,\xi} + \frac{1}{2}\int_{\Omega\setminus\varepsilon\omega} P_\varepsilon U_{\lambda,\zeta} U^p_{\lambda,\zeta} \pm \int_{\Omega\setminus\varepsilon\omega} P_\varepsilon U_{\lambda,\zeta} U^p_{\mu,\xi} \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{p+1}\int_{\Omega\setminus\varepsilon\omega}|P_\varepsilon U_{\mu,\xi}\pm P_\varepsilon U_{\lambda,\zeta}|^{p+1}\\
&=\frac{1}{N}\int_{\Omega\setminus\varepsilon\omega}U_{\mu,\xi}^{p+1}+\frac{1}{2}\int_{\Omega\setminus\varepsilon\omega}(P_\varepsilon U_{\mu,\xi}-U_{\mu,\xi})U_{\mu,\xi}^{p}\\
&\quad+\frac{1}{N}\int_{\Omega\setminus\varepsilon\omega}U_{\lambda,\zeta}^{p+1}+\frac{1}{2}\int_{\Omega\setminus\varepsilon\omega}(P_\varepsilon U_{\lambda,\zeta}-U_{\lambda,\zeta})U_{\lambda,\zeta}^{p}\pm\int_{\Omega\setminus\varepsilon\omega}P_\varepsilon U_{\lambda,\zeta}U_{\mu,\xi}^{p}\\
&\quad-\frac{1}{p+1}\int_{\Omega\setminus\varepsilon\omega}\big(|P_\varepsilon U_{\mu,\xi}\pm P_\varepsilon U_{\lambda,\zeta}|^{p+1}-U_{\mu,\xi}^{p+1}-U_{\lambda,\zeta}^{p+1}\big).
\end{aligned}
\tag{52}
$$

Now, setting $x=\mu y$ we obtain

$$
\begin{aligned}
\int_{\Omega\setminus\varepsilon\omega}U_{\mu,\xi}^{p+1}&=\alpha_N^{p+1}\int_{\Omega\setminus\varepsilon\omega}\frac{\mu^N}{(\mu^2+|x-\xi|^2)^N}\,dx\\
&=\alpha_N^{p+1}\int_{\frac{\Omega\setminus\varepsilon\omega}{\mu}}\frac{1}{(1+|y-\tau|^2)^N}\,dy\\
&=\alpha_N^{p+1}\int_{\mathbb{R}^N}\frac{1}{(1+|y|^2)^N}\,dy+O\left(\left(\frac{\varepsilon}{\mu}\right)^N+\mu^N\right).
\end{aligned}
\tag{53}
$$

By Lemma 18 we have that

$$
\begin{aligned}
&\int_{\Omega\setminus\varepsilon\omega}(P_\varepsilon U_{\mu,\xi}-U_{\mu,\xi})U_{\mu,\xi}^{p}\,dx\\
&=-\alpha_N^{p+1}\int_{\Omega\setminus\varepsilon\omega}\left(\mu^{\frac{N-2}{2}}H(x,\xi)+\frac{1}{\mu^{\frac{N-2}{2}}(1+|\tau|^2)^{\frac{N-2}{2}}}\varphi_\omega\left(\frac{x}{\varepsilon}\right)\right)\frac{\mu^{\frac{N+2}{2}}}{(\mu^2+|x-\xi|^2)^{\frac{N+2}{2}}}\,dx\\
&\quad+\int_{\Omega\setminus\varepsilon\omega}R_{d,\tau}^{\varepsilon}U_{\mu,\xi}^{p}\,dx.
\end{aligned}
\tag{54}
$$

Now, setting $x-\xi=\mu y$ we have

$$
\begin{aligned}
&\int_{\Omega\setminus\varepsilon\omega}\mu^{\frac{N-2}{2}}H(x,\xi)\frac{\mu^{\frac{N+2}{2}}}{(\mu^2+|x-\xi|^2)^{\frac{N+2}{2}}}\,dx\\
&\quad=\int_{\frac{\Omega\setminus\varepsilon\omega-\xi}{\mu}}\mu^{N-2}H(\mu y+\xi,\xi)\frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\,dy
\end{aligned}
$$

$$= \mu^{N-2} H(0,0)\Biggl(\int\limits_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\,dy + o(1)\Biggr). \tag{55}$$

Moreover, we get

$$\begin{aligned}
&\frac{1}{\mu^{\frac{N-2}{2}}(1+|\tau|^2)^{\frac{N-2}{2}}} \int\limits_{\Omega\setminus\varepsilon\omega} \varphi_\omega\biggl(\frac{x}{\varepsilon}\biggr) \frac{\mu^{\frac{N+2}{2}}}{(\mu^2+|x-\xi|^2)^{\frac{N+2}{2}}}\,dx \\
&= \frac{1}{(1+|\tau|^2)^{\frac{N-2}{2}}} \int\limits_{\frac{\Omega\setminus\varepsilon\omega-\xi}{\mu}} \varphi_\omega\biggl(\frac{\mu}{\varepsilon}(y+\tau)\biggr) \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\,dy \\
&= \biggl(\frac{\varepsilon}{\mu}\biggr)^{N-2} \frac{1}{(1+|\tau|^2)^{\frac{N-2}{2}}} \int\limits_{\frac{\Omega\setminus\varepsilon\omega-\xi}{\mu}} f_\varepsilon(y) \frac{1}{|y+\tau|^{N-2}} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\,dy \\
&= \biggl(\frac{\varepsilon}{\mu}\biggr)^{N-2} \Biggl(c_\omega \frac{1}{(1+|\tau|^2)^{\frac{N-2}{2}}} \int\limits_{\mathbb{R}^N} \frac{1}{|y+\tau|^{N-2}} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\,dy + o(1)\Biggr).
\end{aligned} \tag{56}$$

Here we have set $f_\varepsilon(y) := (\frac{\mu}{\varepsilon})^{N-2}|y+\tau|^{N-2}\varphi_\omega(\frac{\mu}{\varepsilon}(y+\tau))$ and applied Lebesgue's dominated convergence theorem and Lemma 17. Therefore

$$\begin{aligned}
&\int\limits_{\Omega\setminus\varepsilon\omega} (P_\varepsilon U_{\mu,\xi} - U_{\mu,\xi}) U_{\mu,\xi}^p\,dx \\
&\quad = -\alpha_N^{p+1} c_3 H(0,0)\mu^{N-2}\bigl(1+o(1)\bigr) - F(\tau)\biggl(\frac{\varepsilon}{\mu}\biggr)^{N-2}\bigl(1+o(1)\bigr),
\end{aligned} \tag{57}$$

where

$$c_3 := \int\limits_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\,dy$$

and

$$F(\tau) := \alpha_N^{p+1} c_\omega \frac{1}{(1+|\tau|^2)^{\frac{N-2}{2}}} \int\limits_{\mathbb{R}^N} \frac{1}{|y+\tau|^{N-2}} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\,dy.$$

A standard computation proves

$$\int\limits_{\Omega\setminus\varepsilon\omega} U_{\lambda,\zeta}^{p+1} = \alpha_N^{p+1} \int\limits_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^N}\,dy + O\bigl(\lambda^N\bigr) = \alpha_N^{p+1} c_1 + O\bigl(\lambda^N\bigr) \tag{58}$$

and also

$$\begin{aligned}
&\int\limits_{\Omega\setminus\varepsilon\omega} (P_\varepsilon U_{\lambda,\zeta} - U_{\lambda,\zeta}) U_{\lambda,\zeta}^p \, dx \\
&\quad = -\alpha_N^{p+1} \int\limits_{\Omega\setminus\varepsilon\omega} \lambda^{\frac{N-2}{2}} H(x,\zeta) \frac{\lambda^{\frac{N+2}{2}}}{(\lambda^2 + |x-\zeta|^2)^{\frac{N+2}{2}}} \, dx + \int\limits_{\Omega\setminus\varepsilon\omega} r_{\varepsilon,\lambda} U_{\lambda,\zeta}^p \, dx \\
&\quad = -\alpha_N^{p+1} \lambda^{N-2} H(\zeta,\zeta) \biggl(\int\limits_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}} \, dy + o(1) \biggr) \\
&\quad = -\alpha_N^{p+1} c_3 \lambda^{N-2} H(\zeta,\zeta) \bigl(1 + o(1)\bigr).
\end{aligned} \tag{59}$$

Now we have to estimate the interaction. Setting $\mu y = x - \xi$ we obtain

$$\begin{aligned}
&\int\limits_{\Omega\setminus\varepsilon\omega} U_{\mu,\xi}^p P_\varepsilon U_{\lambda,\zeta} \, dx \\
&\quad = \alpha_N^{p+1} \int\limits_{\Omega\setminus\varepsilon\omega} \frac{\mu^{\frac{N+2}{2}}}{(\mu^2 + |x-\xi|^2)^{\frac{N+2}{2}}} \\
&\qquad \times \biggl[\frac{\lambda^{\frac{N-2}{2}}}{(\lambda^2 + |x-\zeta|^2)^{\frac{N-2}{2}}} - \lambda^{\frac{N-2}{2}} H(x,\zeta) + r_{\varepsilon,\lambda}(x) \biggr] dx \\
&\quad = \alpha_N^{p+1} \lambda^{\frac{N-2}{2}} \mu^{\frac{N-2}{2}} \int\limits_{\frac{\{\Omega\setminus\varepsilon\omega\}-\xi}{\mu}} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}} \\
&\qquad \times \biggl[\frac{1}{(\lambda^2 + |\mu y + \xi - \zeta|^2)^{\frac{N-2}{2}}} - H(\mu y + \xi, \zeta) \biggr] dy \\
&\qquad + \alpha_N^{p+1} \mu^{\frac{N-2}{2}} \int\limits_{\frac{\{\Omega\setminus\varepsilon\omega\}-\xi}{\mu}} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}} r_{\varepsilon,\mu}(\mu y + \xi) \, dy \\
&\quad = \alpha_N^{p+1} \lambda^{\frac{N-2}{2}} \mu^{\frac{N-2}{2}} G(\zeta,\xi) \int\limits_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}} + o\bigl(\lambda^{\frac{N-2}{2}} \mu^{\frac{N-2}{2}}\bigr) \\
&\quad = \alpha_N^{p+1} c_3 \lambda^{\frac{N-2}{2}} \mu^{\frac{N-2}{2}} G(\zeta,\xi) \bigl(1 + o(1)\bigr).
\end{aligned} \tag{60}$$

It remains to estimate the term

$$\frac{1}{p+1} \int\limits_{\Omega\setminus\varepsilon\omega} \bigl(|P_\varepsilon U_{\mu,\xi} \pm P_\varepsilon U_{\lambda,\zeta}|^{p+1} - U_{\mu,\xi}^{p+1} - U_{\lambda,\zeta}^{p+1} \bigr).$$

Let $\eta > 0$ be fixed such that $B(0,\eta) \cap B(\zeta,\eta) = \emptyset$. If ε is small enough then $\varepsilon\omega \subset B(0,\eta)$ and we can write

$$\int\limits_{\Omega\setminus\varepsilon\omega} \left(|P_\varepsilon U_{\mu,\xi} \pm P_\varepsilon U_{\lambda,\zeta}|^{p+1} - U_{\mu,\xi}^{p+1} - U_{\lambda,\zeta}^{p+1}\right)$$

$$= \int\limits_{B(0,\eta)\setminus\varepsilon\omega} \dots + \int\limits_{B(\zeta,\eta)} \dots + \int\limits_{\Omega\setminus\{B(0,\eta)\cup B(\zeta,\eta)\}} \dots. \tag{61}$$

It is easy to check that

$$\int\limits_{\Omega\setminus\{B(0,\eta)\cup B(\zeta,\eta)\}} \dots = O\Bigg(\int\limits_{\Omega\setminus\{B(0,\eta)\cup B(\zeta,\eta)\}} \left(U_{\mu,\xi}^{p+1} + U_{\lambda,\zeta}^{p+1}\right)\Bigg)$$
$$= O\left(\mu^N + \lambda^N\right). \tag{62}$$

Via a Taylor expansion we have, for some $t \in [0,1]$,

$$\int\limits_{B(0,\eta)\setminus\varepsilon\omega} \left(|P_\varepsilon U_{\mu,\xi} \pm P_\varepsilon U_{\lambda,\zeta}|^{p+1} - U_{\mu,\xi}^{p+1} - U_{\lambda,\zeta}^{p+1}\right) dx$$
$$= \int\limits_{B(0,\eta)\setminus\varepsilon\omega} \left(\left|U_{\mu,\xi} + (P_\varepsilon U_{\mu,\xi} - U_{\mu,\xi} \pm P_\varepsilon U_{\lambda,\zeta})\right|^{p+1} - U_{\mu,\xi}^{p+1}\right) dx - \int\limits_{B(0,\eta)\setminus\varepsilon\omega} U_{\lambda,\zeta}^{p+1}\, dx$$
$$= (p+1) \int\limits_{B(0,\eta)\setminus\varepsilon\omega} U_{\mu,\xi}^{p}(P_\varepsilon U_{\mu,\xi} - U_{\mu,\xi} \pm P_\varepsilon U_{\lambda,\zeta})\, dx$$
$$+ \frac{p(p+1)}{2} \int\limits_{B(0,\eta)\setminus\varepsilon\omega} \left|U_{\mu,\xi} + t(P_\varepsilon U_{\mu,\xi} - U_{\mu,\xi} \pm P_\varepsilon U_{\lambda,\zeta})\right|^{p}$$
$$\times (P_\varepsilon U_{\mu,\xi} - U_{\mu,\xi} \pm P_\varepsilon U_{\lambda,\zeta})\, dx - \int\limits_{B(0,\eta)\setminus\varepsilon\omega} U_{\lambda,\zeta}^{p+1}\, dx. \tag{63}$$

Setting again $\mu y = x - \xi$, we have that

$$\int\limits_{B(0,\eta)\setminus\varepsilon\omega} U_{\mu,\xi}^{p} P_\varepsilon U_{\lambda,\zeta}\, dx$$
$$= \alpha_N^{p+1} \int\limits_{B(0,\eta)\setminus\varepsilon\omega} \frac{\mu^{\frac{N+2}{2}}}{(\mu^2 + |x-\xi|^2)^{\frac{N+2}{2}}}$$
$$\times \left[\frac{\lambda^{\frac{N-2}{2}}}{(\lambda^2 + |x-\zeta|^2)^{\frac{N-2}{2}}} - \lambda^{\frac{N-2}{2}} H(x,\zeta) + r_{\varepsilon,\lambda}(x)\right] dx$$
$$= \alpha_N^{p+1} \lambda^{\frac{N-2}{2}} \mu^{\frac{N-2}{2}} \int\limits_{\frac{\{B(0,\eta)\setminus\varepsilon\omega\}-\xi}{\mu}} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}$$

$$\times\left[\frac{1}{(\lambda^2+|\mu y+\xi-\zeta|^2)^{\frac{N-2}{2}}}-H(\mu y+\xi,\zeta)\right]dy$$

$$+\alpha_N^{p+1}\mu^{\frac{N-2}{2}}\int\limits_{\frac{\{B(0,\eta)\setminus\varepsilon\omega\}-\xi}{\mu}}\frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}r_{\varepsilon,\mu}(\mu y+\xi)\,dy$$

$$=\alpha_N^{p+1}\lambda^{\frac{N-2}{2}}\mu^{\frac{N-2}{2}}G(\zeta,\xi)\int\limits_{\mathbb{R}^N}\frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}+o\big(\lambda^{\frac{N-2}{2}}\mu^{\frac{N-2}{2}}\big)$$

$$=\alpha_N^{p+1}c_3\lambda^{\frac{N-2}{2}}\mu^{\frac{N-2}{2}}G(\zeta,\xi)\big(1+o(1)\big). \tag{64}$$

The term $\int_{B(0,\eta)\setminus\varepsilon\omega}U_{\mu,\xi}^p(P_\varepsilon U_{\mu,\xi}-U_{\mu,\xi})\,dx$ was estimated in (57). The remaining terms are of lower order.

In a similar way, via a Taylor expansion we have, for some $t\in[0,1]$,

$$\int\limits_{B(\zeta,\eta)}\big(|P_\varepsilon U_{\mu,\xi}\pm P_\varepsilon U_{\lambda,\zeta}|^{p+1}-U_{\mu,\xi}^{p+1}-U_{\lambda,\zeta}^{p+1}\big)\,dx$$

$$=\int\limits_{B(\zeta,\eta)}\big(\big|U_{\lambda,\zeta}+(P_\varepsilon U_{\lambda,\zeta}-U_{\lambda,\zeta}\pm P_\varepsilon U_{\mu,\xi})\big|^{p+1}-U_{\lambda,\zeta}^{p+1}\big)\,dx-\int\limits_{B(\zeta,\eta)}U_{\mu,\xi}^{p+1}\,dx$$

$$=(p+1)\int\limits_{B(\zeta,\eta)}U_{\lambda,\zeta}^{p}(P_\varepsilon U_{\lambda,\zeta}-U_{\lambda,\zeta}\pm P_\varepsilon U_{\mu,\xi})\,dx$$

$$+\frac{p(p+1)}{2}\int\limits_{B(\zeta,\eta)}\big|U_{\lambda,\zeta}+t(P_\varepsilon U_{\lambda,\zeta}-U_{\lambda,\zeta}\pm P_\varepsilon U_{\mu,\xi})\big|^{p}$$

$$\times(P_\varepsilon U_{\lambda,\zeta}-U_{\lambda,\zeta}\pm P_\varepsilon U_{\mu,\xi})\,dx-\int\limits_{B(\zeta,\eta)}U_{\mu,\xi}^{p+1}\,dx. \tag{65}$$

Setting now $\lambda y=x-\zeta$, we have

$$\int\limits_{B(\zeta,\eta)}U_{\lambda,\zeta}^{p}P_\varepsilon U_{\mu,\xi}\,dx$$

$$=\alpha_N^{p+1}\int\limits_{B(\zeta,\eta)}\frac{\lambda^{\frac{N+2}{2}}}{(\lambda^2+|x-\zeta|^2)^{\frac{N+2}{2}}}$$

$$\times\left[\frac{\mu^{\frac{N-2}{2}}}{(\mu^2+|x-\xi|^2)^{\frac{N-2}{2}}}-\mu^{\frac{N-2}{2}}H(x,\xi)-\frac{1}{\mu^{\frac{N-2}{2}}}\varphi_\omega\left(\frac{x}{\varepsilon}\right)+R_{\varepsilon,\mu}(x)\right]dx$$

$$=\alpha_N^{p+1}\lambda^{\frac{N-2}{2}}\mu^{\frac{N-2}{2}}\int\limits_{B(0,\eta/\lambda)}\frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}$$

$$\times\left[\frac{1}{(\mu^2+|\lambda y+\zeta-\xi|^2)^{\frac{N-2}{2}}}-H(\lambda y+\zeta,\xi)\right]dy$$

$$-\alpha_N^{p+1}\lambda^{\frac{N-2}{2}}\int\limits_{B(0,\eta/\lambda)}\frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\left[\frac{1}{\mu^{\frac{N-2}{2}}}\varphi_\omega\left(\frac{\lambda y+\zeta}{\varepsilon}\right)+R_{\varepsilon,\mu}(\lambda y+\zeta)\right]dy$$

$$=\alpha_N^{p+1}\lambda^{\frac{N-2}{2}}\mu^{\frac{N-2}{2}}G(\zeta,\xi)\int\limits_{\mathbb{R}^N}\frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}+o\big(\lambda^{\frac{N-2}{2}}\mu^{\frac{N-2}{2}}\big)$$

$$=\alpha_N^{p+1}c_3\lambda^{\frac{N-2}{2}}\mu^{\frac{N-2}{2}}G(\zeta,\xi)\big(1+o(1)\big). \tag{66}$$

The term $\int_{B(\zeta,\eta)}U_{\lambda,\zeta}^p(P_\varepsilon U_{\lambda,\zeta}-U_{\lambda,\zeta})\,dx$ was estimated in (59).

Collecting all the previous estimates, we get that expansion (49) holds true uniformly for (d,Λ,τ,ζ) satisfying (7)–(9).

Arguing in a similar way and using estimates (44), (47) and (48), we prove that the expansion holds true also uniformly in the C^1-sense. This proves our claim. □

5. The associated nonlinear problem

This section is devoted to prove Proposition 7.

First, we estimate the $\|\ \|_{**}$-norm of $\hat{N}_{\lambda,\zeta}(\vartheta)$. It is convenient, and sufficient for our purposes, to assume $\|\vartheta\|_*<1$. In order to estimate $\|\hat{N}_{\lambda,\zeta}(\vartheta)\|_{**}$ we need to distinguish two cases: $N\leqslant 6$ and $N>6$.

If $N\leqslant 6$, then $p\geqslant 2$ and we can estimate

$$\left|w^{-\frac{4}{N-2}}\hat{N}_{\lambda,\zeta}(\vartheta)\right|\leqslant Cw^{(p-2)\beta+2\beta-\frac{4}{N-2}}\|\vartheta\|_*^2,$$

hence

$$\left\|\hat{N}_{\lambda,\zeta}(\vartheta)\right\|_{**}\leqslant C\|\vartheta\|_*^2.$$

Assume now that $N>6$. If $|\vartheta|\geqslant\frac{1}{2}|\hat{V}_{\lambda,\zeta}|$, we see directly that $|\hat{N}_{\lambda,\zeta}(\vartheta)|\leqslant C|\vartheta|^p$ and hence

$$\left|w^{-\frac{4}{N-2}}\hat{N}_{\lambda,\zeta}(\vartheta)\right|\leqslant Cw^{p-2}\|\vartheta\|_*^p\leqslant C\varepsilon^{-\frac{N-6}{2}}\|\vartheta\|_*^p.$$

Let us consider now the case $|\vartheta|\leqslant\frac{1}{2}|\hat{V}_{\lambda,\zeta}|$. In the region where $\operatorname{dist}(y,\partial\Omega_\varepsilon)\geqslant\delta\varepsilon^{-\frac{1}{2}}$ for some $\delta>0$, one has that $\hat{V}_{\lambda,\zeta}(y)\geqslant\alpha_\delta w(y)$ for some $\alpha_\delta>0$; hence in this region, we have

$$\left|w^{-\frac{4}{N-2}}\hat{N}_{\lambda,\zeta}(\vartheta)\right|\leqslant Cw^{2\beta-1}\|\vartheta\|_*^2\leqslant C\varepsilon^{(2\beta-1)\frac{N-2}{2}}\|\vartheta\|_*^2.$$

On the other hand, when $\operatorname{dist}(y,\partial\Omega_\varepsilon)\leqslant\delta\varepsilon^{-\frac{1}{2}}$, the following facts occur: $w(y),\hat{V}_{\lambda,\zeta}(y)=O(\varepsilon^{\frac{N-2}{2}})$, and

$$\hat{V}_{\lambda,\zeta}(y)=C\varepsilon^{\frac{N-1}{2}}\operatorname{dist}(y,\partial\Omega_\varepsilon)\big(1+o(1)\big)\quad\text{as } y\to\partial\Omega_\varepsilon.$$

This second assertion is a consequence of the fact that the Green function of the domain Ω vanishes linearly with respect to $\operatorname{dist}(x, \partial\Omega)$ as $x \to \partial\Omega$. These two facts imply that, if $\operatorname{dist}(y, \partial\Omega_\varepsilon) \leqslant \delta\varepsilon^{-\frac{1}{2}}$, then

$$\begin{aligned}\left|w^{-\frac{4}{N-2}}\hat{N}_{\lambda,\zeta}(\vartheta)\right| &\leqslant w^{-\frac{4}{N-2}}|\hat{V}_{\lambda,\zeta}|^{p-2}|\vartheta|^2 \\ &\leqslant Cw^{-\frac{4}{N-2}}\left(\varepsilon^{\frac{N-1}{2}}\operatorname{dist}(y, \partial\Omega_\varepsilon)\right)^{p-2}\operatorname{dist}(y, \partial\Omega_\varepsilon)2\left|D\vartheta(y)\right|^2 \\ &\leqslant Cw^{-\frac{4}{N-2}+2\beta+\frac{2}{N-2}}\varepsilon^{\frac{N-1}{2}(p-2)-\frac{p}{2}}\|\vartheta\|_*^2 \leqslant C\varepsilon^{-\frac{N-4}{2}}\|\vartheta\|_*^2.\end{aligned}$$

Combining these relations we get

$$\left\|\hat{N}_{\lambda,\zeta}(\vartheta)\right\|_{**} \leqslant \begin{cases} C\|\vartheta\|_*^2 & \text{if } N \leqslant 6,\\ C(\varepsilon^{-\frac{N-4}{2}}\|\vartheta\|_*^2 + \varepsilon^{p-2}\|\vartheta\|_*^p) & \text{if } N > 6.\end{cases}$$

Next we estimate the term $\hat{R}_{\lambda,\zeta}$. In the region $|y - \frac{\zeta_i}{\sqrt{\varepsilon}}| > \frac{\delta}{\sqrt{\varepsilon}}$, for any $i = 0, 1, \ldots, k$ and some positive small δ, direct computations show that $|\hat{R}_{\lambda,\zeta}| \leqslant C\varepsilon^{\frac{N+2}{2}}$. Assume now that $|y - \frac{\zeta_i}{\sqrt{\varepsilon}}| \leqslant \frac{\delta}{\sqrt{\varepsilon}}$ for some $i = 0, 1, \ldots, k$. Then, in this region, using either Lemma 18 or Lemma 19, we get

$$|\hat{R}_{\lambda,\zeta}| \leqslant C\varepsilon^{\frac{N-2}{2}}U^{p-1}_{\frac{\lambda_i}{\sqrt{\varepsilon}},\frac{\zeta_i}{\sqrt{\varepsilon}}}.$$

Using the boundedness of $\frac{\lambda_i}{\sqrt{\varepsilon}}$, we conclude that

$$\|\hat{R}_{\lambda,\zeta}\|_{**} \leqslant C\varepsilon^{\frac{N-2}{2}}. \tag{67}$$

Now, we are in position to prove that problem (14) has a unique solution $\hat{\phi} = \tilde{\phi} + \tilde{\psi}$, with $\tilde{\psi} := T_{\lambda,\zeta}(\hat{R}_{\lambda,\zeta})$ (see Proposition 6), having the required properties.

Problem (14) is equivalent to solving a fixed point problem. Indeed, $\hat{\phi} = \tilde{\phi} + \tilde{\psi}$ is a solution of (14) if and only if

$$\tilde{\phi} = T_{\lambda,\zeta}\left(\hat{N}_{\lambda,\zeta}(\tilde{\phi} + \tilde{\psi})\right) =: A_{\lambda,\zeta}(\tilde{\phi}),$$

because $\tilde{\psi} = T_{\lambda,\zeta}(R)$. We shall prove that the operator $A_{\lambda,\zeta}$ defined above is a contraction inside a properly chosen region.

First observe that, from the definition of $\tilde{\psi}$, from (67) and from Proposition 6, we infer that

$$\|\tilde{\psi}\|_{**} \leqslant C\left(|\lambda\log\varepsilon| + \varepsilon^{\frac{N-2}{2}}\right)$$

and for $\|\vartheta\|_* \leqslant 1$,

$$\left\|\hat{N}_{\lambda,\zeta}(\tilde{\psi} + \vartheta)\right\|_{**} \leqslant \begin{cases} C(\|\vartheta\|_*^2 + \varepsilon^{N-2}) & \text{if } N \leqslant 6,\\ C(\varepsilon^{-\frac{N-4}{2}}\|\vartheta\|_*^2 + \varepsilon^{p-2}\|\vartheta\|_*^p + \varepsilon^{\frac{N}{2}}) & \text{if } N > 6.\end{cases} \tag{68}$$

Let us set

$$\mathcal{F}_\varepsilon := \left\{\vartheta \in H_0^1(\Omega_\varepsilon)\colon \|\vartheta\|_* \leqslant \delta\varepsilon^{\frac{N-2}{2}}\right\}.$$

From Proposition 6 and (68) we conclude that, for ε sufficiently small and any $\vartheta \in \mathcal{F}_\varepsilon$ we have

$$\left\|A_{\lambda,\zeta}(\vartheta)\right\|_* \leqslant \varepsilon^{\frac{N-2}{2}}.$$

Now we will show that the map $A_{\lambda,\zeta}$ is a contraction for any ε small enough. That will imply that $A_{\lambda,\zeta}$ has a unique fixed point in $\mathcal{F}_\varepsilon$ and, hence, that problem (14) has a unique solution.

For any ϑ_1, ϑ_2 in $\mathcal{F}_\varepsilon$ we have

$$\left\|A_{\lambda,\zeta}(\vartheta_1) - A_{\lambda,\zeta}(\vartheta_2)\right\|_* \leqslant C\left\|\hat{N}_{\lambda,\zeta}(\tilde{\psi} + \vartheta_1) - \hat{N}_{\lambda,\zeta}(\tilde{\psi} + \vartheta_2)\right\|_{**},$$

hence we just need to check that $\hat{N}_{\lambda,\zeta}$ is a contraction in its corresponding norms. By definition of $\hat{N}_{\lambda,\zeta}$

$$D_\vartheta \hat{N}_{\lambda,\zeta}(\vartheta) = p\left[f'(\hat{V}_{\lambda,\zeta} + \vartheta) - f'(\hat{V}_{\lambda,\zeta})\right].$$

Hence we get

$$\left|\hat{N}_{\lambda,\zeta}(\tilde{\psi} + \vartheta_1) - \hat{N}_{\lambda,\zeta}(\tilde{\psi} + \vartheta_2)\right| \leqslant C\hat{V}_{\lambda,\zeta}^{p-2}|\bar{\vartheta}||\vartheta_1 - \vartheta_2|$$

for some $\bar{\vartheta}$ in the segment joining $\tilde{\psi} + \vartheta_1$ and $\tilde{\psi} + \vartheta_2$. Hence, we get for small enough $\|\bar{\vartheta}\|_*$,

$$\omega^{-\frac{4}{N-2}}\left|\hat{N}_{\lambda,\zeta}(\tilde{\psi} + \vartheta_1) - \hat{N}_{\lambda,\zeta}(\tilde{\psi} + \vartheta_2)\right| \leqslant C\varepsilon^{p-2+2\beta}\|\bar{\vartheta}\|_*\|\vartheta_1 - \vartheta_2\|_*.$$

We conclude that there exists $c \in (0, 1)$ such that

$$\left\|\hat{N}_{\lambda,\zeta}(\tilde{\psi} + \vartheta_1) - \hat{N}_{\lambda,\zeta}(\tilde{\psi} + \vartheta_2)\right\|_{**} \leqslant c\|\vartheta_1 - \vartheta_2\|_*.$$

Arguing like in [11], we obtain the estimate (19). This concludes the proof.

References

[1] T. Aubin, Problemes isoperimetriques et espaces de Sobolev, J. Differential Geom. 11 (1976) 573–598.

[2] A. Bahri, Critical Points at Infinity in Some Variational Problems, Pitman Res. Notes Math. Ser., vol. 182, Longman, Harlow, 1989.

[3] A. Bahri, J.M. Coron, On a nonlinear elliptic equation involving the critical Sobolev exponent: The effect of the topology of the domain, Comm. Pure Appl. Math. 41 (1988) 253–294.

[4] M. Ben Ayed, K. El Mehdi, M. Hammami, A nonexistence result for Yamabe type problems on thin annuli, Ann. Inst. H. Poincaré Anal. Non Linéaire 19 (2002) 715–744.

[5] G. Bianchi, H. Egnell, A note on the Sobolev inequality, J. Funct. Anal. 100 (1991) 18–24.

[6] L. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behaviour of semilinear elliptic equations with critical Sobolev growth, Comm. Pure Appl. Math. 42 (1989) 271–297.

[7] M. Clapp, F. Pacella, Multiple solutions to the pure critical exponent problem in domains with a hole of arbitrary size, Math. Z. 259 (2008) 575–589.

[8] M. Clapp, T. Weth, Minimal nodal solutions of the pure critical exponent problem on a symmetric domain, Calc. Var. Partial Differential Equations 21 (2004) 1–14.

[9] M. Clapp, T. Weth, Two solutions of the Bahri–Coron problems in punctured domains via the fixed point transfer, Commun. Contemp. Math. 10 (2008) 81–101.
[10] J.M. Coron, Topologie et cas limite des injections de Sobolev, C. R. Acad. Sci. Paris Ser. I Math. 299 (1984) 209–212.
[11] M. del Pino, P. Felmer, M. Musso, Two-bubble solutions in the super-critical Bahri–Coron's problem, Calc. Var. Partial Differential Equations 16 (2) (2003) 113–145.
[12] H. Hofer, The topological degree at a critical point of mountain-pass type, in: Nonlinear Functional Analysis and Its Applications, part 1, Berkeley, CA, 1983, in: Proc. Sympos. Pure Math., vol. 45, Amer. Math. Soc., Providence, RI, 1986, pp. 501–509.
[13] J. Kazdan, F.W. Warner, Remarks on some quasilinear elliptic equations, Comm. Pure Appl. Math. 28 (1975) 567–597.
[14] R. Lewandowski, Little holes and convergence of solutions of $-\Delta u = u^{(N+2)/(N-2)}$, Nonlinear Anal. 14 (1990) 873–888.
[15] G. Li, S. Yan, J. Yang, An elliptic problem with critical growth in domains with shrinking holes, J. Differential Equations 198 (2004) 275–300.
[16] M. Musso, A. Pistoia, Sign changing solutions to a nonlinear elliptic problem involving the critical Sobolev exponent in pierced domains, J. Math. Pures Appl. 86 (2006) 510–528.
[17] M. Musso, A. Pistoia, Persistence of Coron's solutions in nearly critical problems, Ann. Sc. Norm. Super. Pisa Cl. Sci. 6 (2) (2007) 331–357.
[18] M. Musso, A. Pistoia, Sign changing solutions to a Bahri–Coron's problems in pierced domains, Adv. Differential Equations 21 (1) (2008) 295–306.
[19] R. Palais, The principle of symmetric criticality, Comm. Math. Phys. 69 (1979) 19–30.
[20] S.I. Pohožaev, On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Dokl. Akad. Nauk SSSR 165 (1965) 36–39 (in Russian).
[21] O. Rey, Sur un probléme variationnel non compact: l'effect de petits trous dans le domain, C. R. Acad. Sci. Paris Sér. I Math. 308 (1989) 349–352.
[22] O. Rey, The role of the Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent, J. Funct. Anal. 89 (1990) 1–52.
[23] E.H. Spanier, Algebraic Topology, McGraw–Hill, New York, 1966.
[24] G. Talenti, Best constants in Sobolev inequality, Ann. Mat. Pura Appl. 110 (1976) 353–372.
[25] M. Willem, Minimax Theorems, Progr. Nonlin. Differential Equations Appl., vol. 24, Birkhäuser Boston, Boston, MA, 1996.