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Multiple solutions for a non-homogeneous elliptic equation at the critical exponent

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We consider the equation $-\Delta u = |u|^{4/(N-2)}u + \varepsilon f(x)$ under zero Dirichlet boundary conditions in a bounded domain Ω in $\mathbb{R}^N$ exhibiting certain symmetries, with $f \geqslant 0$, $f \neq 0$. In particular, we find that the number of sign-changing solutions goes to infinity for radially symmetric f, as $\varepsilon \to 0$ if Ω is a ball. The same is true for the number of negative solutions if Ω is an annulus and the support of f is compact in Ω.

1. Introduction

This paper is concerned with the existence of multiple solutions of the problem

$$\left.\begin{aligned} -\Delta u &= |u|^{p-1}u + \varepsilon f(x) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \qquad (\Xi_\varepsilon)$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 3$, p is the critical Sobolev exponent $p = (N+2)/(N-2)$ and $f(x)$ is a non-homogeneous perturbation.

If $1 < p < (N+2)/(N-2)$ and $f = 0$, the associated energy functional is even and satisfies the Palais–Smale (PS) condition in $H_0^1(\Omega)$. Standard Ljusternik–Schnirelmann theory then yields the existence of infinitely many non-trivial solutions. On the other hand, when $p = (N+2)/(N-2)$, PS no longer holds and this poses an essential difficulty to the existence question. In fact, when $f = 0$ and the domain Ω is strictly star shaped, it is shown in [13] that no non-trivial solution exists. In [6], Brézis and Nirenberg showed that the presence of the non-homogeneous term may restore solvability. As pointed out in [16], the result in [6] implies that if $f \neq 0$, $f \geqslant 0$, $f \in H^{-1}(\Omega)$, then at least two positive solutions exist for all small ε, while no positive solution exists if ε is sufficiently large. This result was improved by Rey [16] and by Tarantello [20] in two different directions. In [16], it is found that, for $f \geqslant 0$ sufficiently regular with $f \neq 0$ at least $\mathrm{cat}(\Omega)+1$, positive solutions exist, where $\mathrm{cat}(\Omega)$ denotes the Ljusternik–Schnirelmann category of Ω. One of these solutions approaches zero while the others develop single-spike shape at some points in Ω as $\varepsilon \to 0$. The spike-shape solutions resemble $U_\lambda(x-\xi)$ for some $\xi \in \Omega$ and $\lambda > 0$ very small, depending on ε, where

$$U_\lambda(x) = \alpha_N \left(\frac{\lambda}{\lambda^2 + |x|^2} \right)^{(N-2)/2}, \tag{1.1}$$

with $\alpha_N = (N(N-2))^{(N-2)/4}$. We recall that the above are the unique positive solutions up to translations of the equation

$$\Delta u + u^{(N+2)/(N-2)} = 0 \quad \text{in } \mathbb{R}^N \tag{1.2}$$

(see [3,7,18]). On the other hand, in [20], the result in [6] is improved by establishing that at least two solutions exist, provided that $f \neq 0$, $\|\varepsilon f\|_{H^{-1}(\Omega)} < C_N$, where C_N is an explicit constant. These solutions are positive if $f \geqslant 0$.

Ali and Castro [1] showed that the existence result in [6] is optimal for positive solutions in a ball; if Ω is a ball and $f \equiv 1$, problem (Ξ_ε) admits exactly two positive solutions for all sufficiently small ε. Since positive solutions must be radial in that case, their analysis is carried out by means of analysis of the associated ordinary differential equation. One purpose of this paper is to show that the situation is drastically different in the case of sign-changing solutions in a ball centred at 0; for $f \geqslant 0$, $f \not\equiv 0$ radially symmetric, a large number of (non-radial) solutions appears as $\varepsilon \to 0$. More precisely, for any integer k sufficiently large, a solution exists developing *negative* spike shape at the k vertices of a regular polygon centred at 0, with a positive spike at the origin. This result holds true in more generality, including, for instance, the case of a solid of revolution in $\mathbb{R}^3$, which is also symmetric in the coordinate of the rotation axis. Let us state precisely the assumptions we will make in the domain Ω and the non-homogeneous term f. We write $x = (z, x_3, \dots, x_N) = (z, x')$ for a point in $\mathbb{R}^N = \mathbb{C} \times \mathbb{R}^{N-2}$. Assume that the domain Ω in $\mathbb{R}^N$, $N \geqslant 3$, and the non-homogeneous term f satisfy the following properties.

(H1) If $(z, x') \in \Omega$, then $(\mathrm{e}^{\mathrm{i}\theta} z, x') \in \Omega$ for all $\theta \in [0, 2\pi]$.

(H2) If $(z, x_3, \dots, x_i, \dots, x_N) \in \Omega$, then $(z, x_3, \dots, -x_i, \dots, x_N) \in \Omega$ for each $i = 3, \dots, N$.

(H3) $f \in L^\infty(\Omega)$ is non-negative in Ω, has the form $f = f(|z|, x')$ and is even in each variable x_i for $i = 3, \dots, N$.

We will find solutions exhibiting spikes at the vertices of a regular polygon. More precisely, for $k \in \mathbb{N}$, we write

$$P_{jk} = (\mathrm{e}^{2\pi \mathrm{i} j/k}, 0), \quad j = 1, \dots, k. \tag{1.3}$$

THEOREM 1.1. *Assume that Ω satisfies (H1), (H2) and, additionally, that $0 \in \Omega$. Let f satisfy (H3). Then there is a $k_0(\Omega)$ such that, for each $k \geqslant k_0$, the following holds. If ε_n is any sequence with $\varepsilon_n \to 0$, then there is a subsequence of ε_n labelled* *the same way, positive numbers* λ_+, λ_-, ρ *and solutions* u_n *of* Ξ_ε *for* $\varepsilon = \varepsilon_n$ *of the form*

$$u_n(x) = \left[-\sum_{j=1}^{k} U_{\lambda_n^+}(x - \rho P_{jk}) + U_{\lambda_n^-}(x)\right](1 + o(1)), \tag{1.4}$$

where $o(1) \to 0$ *uniformly in* Ω *as* $n \to \infty$ *and*

$$\lambda_n^{\pm} = \varepsilon_n^{2/(N-2)} \lambda^{\pm}.$$

Here, U_λ *is defined by (1.1) and* P_{jk} *by (1.3).*

From the result in [16], we know that the presence of non-trivial topology in the domain Ω induces higher multiplicity of single-spike solutions. The additional effect of symmetries in the multiplicity question has recently been studied in [8]. For instance, if Ω is symmetric with respect to 0, $0 \notin \Omega$, and $f \geqslant 0$ is even, then at least $\operatorname{cat}(\Omega) + 2$ positive solutions exist, provided that $\|\varepsilon f\|_{H^{-1}}$ is small enough. More symmetries induce higher multiplicity of positive solutions; among other results, we find that if Ω is an annulus

$$A_\delta = \{x \in \mathbb{C}^N : 1 < |x| < 1 + \delta\}$$

and f is non-negative radially symmetric, $f \neq 0$, then the number of positive solutions goes to infinity as $\delta \to 0$ and $\varepsilon \to 0$. Positive solutions exhibiting spike shape at the vertices of a k-regular polygon indeed exist for any $k \geqslant 1$ in this situation.

The following theorem reveals a rather surprising dual version of the above result. We find that, in an annulus of fixed size, the number of solutions of Ξ_ε goes to infinity for $f \geqslant 0$, $f \neq 0$, as $\varepsilon \to 0$. These solutions are *negative* if the support of f is compact in Ω.

THEOREM 1.2. *Assume that* Ω *satisfies (H1), (H2) and, additionally, that* $0 \notin \Omega$*. Then there is a* $k_0(\Omega)$ *such that, for each* $k \geqslant k_0$*, the following holds. If* f *satisfies (H3) and* $\varepsilon_n \to 0$*, then, passing to a subsequence, there exist positive numbers* λ*,* ρ *and non-trivial solutions* u_n *of* Ξ_ε *for* $\varepsilon = \varepsilon_n$ *of the form*

$$u_n(x) = -\left[\sum_{j=1}^{k} U_{\lambda_n}(x - \rho P_{jk})\right](1 + o(1)), \tag{1.5}$$

where $o(1) \to 0$ *uniformly in* Ω *as* $n \to \infty$ *and*

$$\lambda_n^{\pm} = \varepsilon_n^{2/(N-2)} \lambda^{\pm}.$$

Moreover, if the support of f *is compact, then* u_n *is negative in* Ω*.*

The proofs of theorems 1.1 and 1.2, to which we devote the rest of this paper, follow a Lyapunov–Schmidt reduction procedure, related to that in [16], recently devised for the study of the slightly supercritical problem

$$\begin{aligned} -\Delta u &= u^{(N+2)/(N-2)+\varepsilon} && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

in [9–11]. In particular, a result similar to that in theorem 1.2 is found for the above problem in [11].

Finally, we should mention that the subcritical case $1 < p < (N+2)/(N-2)$ has been extensively considered in the literature. While it is shown in [12] that no positive solution exists for large ε, several results implying the existence of multiple or infinitely many sign-changing solutions are available (for small and also large non-homogeneous perturbations) (see [2,4,14,15,17,19]).

2. Ansatz and expansion of its associated energy

Let f and ε_n be as in the assumptions of theorem 1.1. In order to construct the solutions predicted in theorem 1.1, it is convenient to introduce the change of variables

$$v(y) = -\varepsilon u(\varepsilon^{2/(N-2)} y), \tag{2.1}$$

where, for notational convenience, we drop the subindex n from ε_n. Then u is a solution to problem $(\varXi_\varepsilon)$ if and only if v solves

$$\left.\begin{aligned} \Delta v + |v|^{4/(N-2)} v - \varepsilon^{2N/(N-2)} \tilde{f}(y) = 0 \quad \text{in } \varOmega_\varepsilon, \\ u \in H_0^1(\varOmega_\varepsilon), \end{aligned}\right\} \tag{2.2}$$

where $\varOmega_\varepsilon$ is the rescaled domain given by

$$\varOmega_\varepsilon = \varepsilon^{-2/(N-2)} \varOmega,$$

while $\tilde{f}(y) = f(\varepsilon^{-2/(N-2)} y)$.

Letting $\varepsilon \to 0$ in (2.2), the limiting equation becomes

$$\Delta v + |v|^{4/(N-2)} v = 0 \quad \text{in } \mathbb{R}^N, \tag{2.3}$$

whose positive solutions are all given by

$$\bar{V}_{\lambda,\xi'}(y) = \alpha_N \left(\frac{\lambda}{\lambda^2 + |y - \xi'|^2} \right)^{(N-2)/2}, \tag{2.4}$$

where $\alpha_N = (N(N-2))^{(N-2)/4}$, $\xi' \in \mathbb{R}^N$ and $\lambda > 0$. We also write $\bar{V} = \bar{V}_{1,0}$.

Let us consider $(k+1)$-tuples of points and numbers,

$$\xi = (\xi_0, \xi_1, \dots, \xi_k) \in \varOmega^{k+1}, \qquad \lambda = (\lambda_0, \lambda_1, \dots, \lambda_k) \in \mathbb{R}_+^{k+1}.$$

We set

$$\xi_i' = \varepsilon^{-2/(N-2)} \xi_i \in \varOmega_\varepsilon \quad \text{and} \quad \xi' = (\xi_0', \dots, \xi_k') \in \varOmega_\varepsilon^{k+1}.$$

In order to find the solutions predicted by theorem 1.1, it is then natural to look for solutions to (2.2), in the class of functions that respect the symmetries of $\varOmega_\varepsilon$, which, at a first approximation, look like

$$v \sim \sum_{i=1}^{k} (\bar{V}_{\lambda_i,\xi_i'} - \bar{V}_{\lambda_0,\xi_0'})$$

for appropriate choice of points ξ_i' and parameters λ_i. In order to take into account the boundary conditions in problem (2.2), a better approximation is then given by the projections of the functions $\bar{V}_{\lambda_i,\xi_i'}$ onto $H_0^1(\Omega_\varepsilon)$. More precisely, we define by $V_{\lambda_i,\xi_i'}$ the unique solution of the problem

$$\left.\begin{aligned} -\Delta V_{\lambda_i,\xi_i'} &= \bar{V}_{\lambda_i,\xi_i}^{(N+2)/(N-2)} && \text{in } \Omega_\varepsilon, \\ V_{\lambda_i,\xi_i} &= 0 && \text{on } \partial\Omega_\varepsilon. \end{aligned}\right\} \tag{2.5}$$

For notational convenience, we write

$$V_i = V_{\lambda_i,\xi_i'}, \quad V^+ = \sum_{i=1}^{k} V_i, \quad V^- = V_{\lambda_0,\xi_0'} \quad \text{and} \quad V = V^+ - V^-. \tag{2.6}$$

We then look for a solution of (2.2) of the form

$$v(y) = V(y) + \phi(y),$$

where ϕ represents a lower-order term.

Let $p = (N+2)/(N-2)$. The functional associated to (2.2) is given by

$$J_\varepsilon(v) = \frac{1}{2}\int_{\Omega_\varepsilon} |Dv|^2 \,\mathrm{d}y - \frac{1}{p+1}\int_{\Omega_\varepsilon} |v|^{p+1} + \varepsilon^{p+1}\int_{\Omega_\varepsilon} \tilde{f}(y)v(y)\,\mathrm{d}y. \tag{2.7}$$

We will work out the asymptotic expansion for the energy functional J_ε at the function V, assuming that the points ξ_i and the parameters λ_i satisfy certain conditions.

We make the following choice for the points and the parameters. For a given $\delta > 0$, we consider points ξ_i and parameters λ_i such that

$$\operatorname{dist}(\xi_i, \partial\Omega) > \delta, \qquad |\xi_i - \xi_j| > \delta, \qquad \delta < \lambda_i < \delta^{-1}. \tag{2.8}$$

The advantage of this constraint on points and parameters is the validity of an expansion of $J_\varepsilon(V)$ in terms of Green's function and of its regular part of the Laplacian with Dirichlet boundary conditions on Ω. We denote by $G(x,y)$ Green's function of Ω, namely, the solution of

$$\begin{aligned} \Delta_x G(x,y) &= \delta_0(x-y), && x \in \Omega, \\ G(x,y) &= 0, && x \in \partial\Omega, \end{aligned}$$

where $\delta_0(x)$ denotes the Dirac mass at the origin and $H(x,y)$ its regular part, namely,

$$H(x,y) = \Gamma(x-y) - G(x,y) \quad \forall (x,y) \in \Omega \times \Omega, \tag{2.9}$$

where Γ is the fundamental solution of the Laplacian, $\Gamma(x) = b_N|x|^{2-N}$. In order to state the expansion, we denote

$$\gamma(\xi) = \int_\Omega f(x)G(x,\xi)\,\mathrm{d}x. \tag{2.10}$$

In other words, γ solves

$$\begin{aligned} -\Delta\gamma &= f && \text{in } \Omega, \\ \gamma &= 0 && \text{on } \partial\Omega. \end{aligned}$$

We observe that, since $f \in L^\infty(\Omega)$, then $\gamma \in C^{1,\alpha}(\Omega)$ for any $\alpha < 1$.

PROPOSITION 2.1. *Given δ and choosing*

$$\lambda_j = (a_N^{-1}\Lambda_j)^{2/(N-2)}$$

with $a_N = \int_{\mathbb{R}^N} \bar{V}^p \, dx$, we have

$$J_\varepsilon(V) = (k+1)S_N + \varepsilon^2 \psi_k(\xi, \Lambda) + o(\varepsilon^2)$$

uniformly in the C^1 sense with respect to (ξ, Λ) satisfying (2.8).

The constant S_N is here given by

$$S_N = \frac{1}{2}\int_{\mathbb{R}^N} |D\bar{V}|^2 \, dx - \frac{1}{p+1}\int_{\mathbb{R}^N} \bar{V}^{p+1} \, dx$$

and the function ψ_k is defined by

$$\psi_k(\xi, \Lambda) = \frac{1}{2}\left[\sum_{j=0}^{k} H(\xi_j, \xi_j)\Lambda_j^2 - 2\sum_{i<j,\, i\neq 0} G(\xi_i, \xi_j)\Lambda_i\Lambda_j + 2\sum_{i\neq 0} G(\xi_0, \xi_i)\Lambda_i\Lambda_0\right] + \sum_{j=1}^{k}\gamma(\xi_j)\Lambda_j - \gamma(\xi_0)\Lambda_0. \tag{2.11}$$

Proof. We write

$$J_\varepsilon(V) = \hat{J}_\varepsilon(V) + \varepsilon^{p+1}\int_{\Omega_\varepsilon} \tilde{f}(y)V(y) \, dy. \tag{2.12}$$

The expansion of $\hat{J}_\varepsilon(V)$ follows from the arguments developed in [5,10]. Given (2.8), we have that

$$\hat{J}_\varepsilon(V) = (k+1)S_N + \frac{1}{2}\left[\sum_{j=0}^{k} H(\xi_j, \xi_j)\Lambda_j^2 - 2\sum_{i<j,\, i\neq 0} G(\xi_i, \xi_j)\Lambda_i\Lambda_j + 2\sum_{i\neq 0} G(\xi_0, \xi_i)\Lambda_i\Lambda_0\right]\varepsilon^2 + o(\varepsilon^2) \tag{2.13}$$

uniformly in the C^1 sense with respect to points and parameters that satisfy (2.8).

On the other hand, taking into account that, away from $x = \xi_i$,

$$V_{\lambda_i, \xi_i}(\varepsilon^{-2/(N-2)}x) = G(x, \xi_i)\lambda_i^{(N-2)/2}\varepsilon^2 \int_{\mathbb{R}^N} \bar{V}^p + o(\varepsilon^2) \tag{2.14}$$

uniformly on each compact subset of Ω, a direct computation yields

$$\begin{aligned}
\varepsilon^{p+1}&\int_{\Omega_\varepsilon} \tilde{f}(y)V(y) \, dy \\
&= \varepsilon^{p+1}\sum_{i=1}^{k}\int_{\Omega_\varepsilon} \tilde{f}(y)V_i(y) \, dy - \varepsilon^{p+1}\int_{\Omega_\varepsilon} \tilde{f}(y)V_0(y) \\
&= \sum_{i=1}^{k}\int_{\Omega} f(x)V_i(\varepsilon^{-2/(N-2)}x) \, dx - \int_{\Omega} f(x)V_0(\varepsilon^{-2/(N-2)}x) \, dx
\end{aligned}$$

$$= \varepsilon^2 \sum_{i=1}^{k} \lambda_i^{(N-2)/2} \left(\int_\Omega \bar{V}^p \, \mathrm{d}x \right) \int_\Omega f(x) G(x, \xi_i) \, \mathrm{d}x$$
$$- \varepsilon^2 \lambda_0^{(N-2)/2} \left(\int_\Omega \bar{V}^p \, \mathrm{d}x \right) \int_\Omega f(x) G(x, \xi_0) \, \mathrm{d}x + o(\varepsilon^2)$$
$$= \varepsilon^2 \left(\sum_{i=1}^{k} \gamma(\xi_i) \varLambda_i - \gamma(\xi_0) \varLambda_0 \right) + o(\varepsilon^2). \tag{2.15}$$

This concludes the proof of proposition 2.1. □

3. The finite-dimensional reduction

In this section we consider the problem of finding a function ϕ that, for certain constants c_{ij}, solves

$$\left.\begin{aligned} \Delta(V+\phi) + |V+\phi|^{p-1}(V+\phi) - \varepsilon^{p+1} \tilde{f}(y) &= \sum_{ij} c_{ij} V_i^{p-1} Z_{ij} \quad \text{in } \varOmega_\varepsilon, \\ \phi &= 0 \quad \text{on } \partial\varOmega_\varepsilon, \\ \int_{\varOmega_\varepsilon} \phi V_i^{p-1} Z_{ij} &= 0 \quad \text{for all } i, j, \end{aligned}\right\} \tag{3.1}$$

where the functions Z_{ij} are defined as the $H_0^1(\varOmega_\varepsilon)$-projection of the function $\bar{Z}_{ij}$, where

$$\bar{Z}_{ij} = \frac{\partial \bar{V}_i}{\partial y_j}, \quad j = 1, \dots, N,$$
$$\bar{Z}_{iN+1} = \frac{\partial \bar{V}_i}{\partial \lambda_i} = (x - \xi_i^{'}) \cdot \nabla \bar{V}_i + (N-2)\bar{V}_i,$$

namely, $Z_{ij} \in H_0^1(\varOmega_\varepsilon)$ satisfies the equation $\Delta Z_{ij} = \Delta \bar{Z}_{ij}$.

A first step to solve (3.1) consists of dealing with the following linear problem. Given $h \in L^\infty(\bar{\varOmega}_\varepsilon)$, find a function ϕ and constants c_{ij} such that

$$\left.\begin{aligned} \Delta\phi + p|V|^{p-1}\phi &= h + \sum_{i,j} c_{ij} V_i^{p-1} Z_{ij} \quad \text{in } \varOmega_\varepsilon, \\ \phi &= 0 \quad \text{on } \partial\varOmega_\varepsilon, \\ \int_{\varOmega_\varepsilon} V_i^{p-1} Z_{ij} \phi &= 0 \quad \text{for all } i, j. \end{aligned}\right\} \tag{3.2}$$

In order to study the invertibility of the linear operator L_ε associated to (3.2), namely,

$$L_\varepsilon(\phi) = \Delta\phi + p|V|^{p-1}\phi$$

under the previous orthogonality condition, it is useful to introduce convenient norms that depend on the points ξ'. For a function ψ defined in $\varOmega_\varepsilon$, we define

$$\|\psi\|_* = \sup_{x \in \varOmega_\varepsilon} \left| \left(\sum_{j=0}^{k} \frac{1}{1 + |x - \xi_j'|} \right)^{-\beta} \psi(x) \right|,$$

where $\beta = 1$ if $N = 3$, $\beta = 2$ if $N \geqslant 4$, and, for any dimension $N \geqslant 3$,

$$\|\psi\|_{**} = \sup_{x \in \Omega_\varepsilon} \left| \left(\sum_{j=0}^{k} \frac{1}{1 + |x - \xi_j'|} \right)^{-4} \psi(x) \right|.$$

Concerning the solvability of (3.2), a slight modification of the results obtained in [9–11] yields the following result.

PROPOSITION 3.1. *Assume that constraints (2.8) hold. Then there are numbers* $\varepsilon_0 > 0$, $C > 0$, *such that, for all* $0 < \varepsilon < \varepsilon_0$ *and all* $h \in C^\alpha(\bar{\Omega}_\varepsilon)$, *problem (3.2) admits a unique solution* $\phi \equiv L_\varepsilon(h)$. *Furthermore, the map*

$$(\xi', \lambda, h) \to L_\varepsilon(h) = \phi$$

is of class C^1 *and satisfies*

$$\|\phi\|_* \leqslant C\|h\|_{**} \tag{3.3}$$

and

$$\|\nabla_{\xi',\lambda}\phi\|_* \leqslant C\|h\|_{**}. \tag{3.4}$$

Here and in the rest of this paper, we denote by C a generic constant that is independent of ε and of the particular ξ_i, λ_i chosen satisfying (2.8).

We are now in a position to solve problem (3.1). The first equation in (3.1) can be written in the following form,

$$L_\varepsilon(\phi) = -N_\varepsilon(\phi) - R_\varepsilon - \tilde{F}_\varepsilon + \sum_{i,j} c_{ij} V_i^{p-1} Z_{ij}, \tag{3.5}$$

where

$$N_\varepsilon(\phi) = |V + \phi|^{p-1}(V + \phi) - |V|^{p-1}V - p|V|^{p-1}\phi, \tag{3.6}$$

$$R_\varepsilon = |V|^{p-1}V - \left(\sum_{i=1}^{k} \bar{V}_i^p - \bar{V}_0^p \right), \tag{3.7}$$

$$\tilde{F}_\varepsilon = \varepsilon^{p+1} \tilde{f}(y). \tag{3.8}$$

For small $\varepsilon > 0$ and $\|\phi\|_* \leqslant \frac{1}{4}$, the following estimates hold (see [10]):

$$\|N_\varepsilon(\phi)\|_{**} \leqslant C \begin{cases} C\|\phi\|_*^2 & \text{if } N \leqslant 6, \\ C(\varepsilon^{2(2\beta-1)}\|\phi\|_*^2 + \varepsilon^{-2(2-p)\beta}\|\phi\|_*^p) & \text{if } N > 6. \end{cases} \tag{3.9}$$

Now, taking into account the fact that

$$\bar{V}_{\lambda_i,\xi_i'}(x) - V_i(x) = C\varepsilon^2 + o(\varepsilon^2)$$

for $|x - \xi_i'| < \delta\varepsilon^{-2/(N-2)}$ and $\delta < \lambda_i < \delta^{-1}$, we have

$$\left| \left(\sum_{i=1}^{k} \frac{1}{1 + |x - \xi_i'|} \right)^{-4} R_\varepsilon \right| \leqslant C\varepsilon^2.$$

In the complement of these regions, $|R_\varepsilon| \leqslant C\varepsilon^{2(N+2)/(N-2)}$, and hence we get

$$\|R_\varepsilon\|_{**} \leqslant C\varepsilon^2. \tag{3.10}$$

Finally, since $f \in L^\infty(\Omega)$, a direct computation yields

$$\|\tilde{F}_\varepsilon\|_{**} \leqslant C\varepsilon^2. \tag{3.11}$$

The following result holds.

PROPOSITION 3.2. *Assume that relations (2.8) hold. Then there is a constant $C > 0$ such that, for all $\varepsilon > 0$ small enough, there exists a unique solution $\phi = \phi(\xi', \lambda)$ to problem (3.1) of the form $\phi = \bar{\phi} + \tilde{\phi}$, with $\tilde{\phi} = -L_\varepsilon^{-1}(R_\varepsilon)$. Furthermore, the map $(\xi', \lambda) \mapsto \bar{\phi}(\xi', \lambda)$ is of class C^1 for the $\|\cdot\|_*$ norm and it satisfies*

$$\|\bar{\phi}\|_* \leqslant C\varepsilon^2.$$

Moreover,

$$\|D_{\xi',\lambda}\bar{\phi}\|_* \leqslant C\varepsilon^2.$$

Proof. Problem (3.1) is equivalent to solving a fixed point problem. Indeed, ϕ is a solution of (3.1) if and only if

$$\phi = L_\varepsilon^{-1}(N_\varepsilon(\phi + \tilde{\phi}) + R_\varepsilon + \tilde{F}_\varepsilon) =: A_\varepsilon(\phi).$$

Thus we need to prove that the operator A_ε defined above is a contraction in a proper region. Let us consider the set

$$\mathcal{F}_r = \{\phi : \|\phi\|_* \leqslant r\varepsilon^2\},$$

with r a positive number to be fixed later. From proposition 3.1 and estimates (3.9), (3.10), (3.11), we get

$$\|A_\varepsilon(\phi)\|_* \leqslant C\|N_\varepsilon(\phi) + R_\varepsilon + \tilde{F}_\varepsilon\|_{**} \leqslant \begin{cases} r\varepsilon^2 & \text{if } N \leqslant 6, \\ C(\varepsilon^{4\beta+2} + \varepsilon^{2p\beta+2} + \varepsilon^2) \leqslant r\varepsilon^2 & \text{if } N > 6 \end{cases}$$

for all small ε, provided that r is chosen large enough, but independent of ε. Thus A_ε maps $\mathcal{F}_r$ into itself for this choice of r. Moreover, A_ε turns out to be a contraction mapping in this region. This follows from the fact that N_ε defines a contraction in the $\|\cdot\|_*$ norm, which can be proved in a straightforward way.

Concerning now the differentiability of the function $\phi(\xi', \lambda)$, let us write

$$B(\xi', \lambda, \phi) := \phi - T_\varepsilon(N_\varepsilon(\phi) + R_\varepsilon + \tilde{F}_\varepsilon).$$

Of course, we have $B(\xi', \lambda, \phi) = 0$. Now we write

$$D_\phi B(\xi', \lambda, \phi)[\theta] = \theta - T_\varepsilon(\theta D_\phi(N_\varepsilon(\phi))) =: \theta + M(\theta).$$

It is not hard to check that the following estimate holds:

$$\|M(\theta)\|_* \leqslant C\varepsilon\|\theta\|_*.$$

It follows that, for small ε, the linear operator $D_\phi B(\xi', \lambda, \phi)$ is invertible, with uniformly bounded inverse, in $\mathcal{C}_*$, the Banach space of continuous functions in Ω_ε

with bounded $\|\cdot\|_*$ norm. It also depends continuously on its parameters. Let us differentiate with respect to ξ' (analogous arguments give the differentiability with respect to λ). We have

$$D_{\xi'}B(\xi',\lambda,\phi) = -(D_{\xi'}T_\varepsilon)(N_\varepsilon(\phi)+R_\varepsilon+\tilde{F}_\varepsilon) - T_\varepsilon((D_{\xi'}N_\varepsilon)(\xi',\phi)+D_{\xi'}R_\varepsilon),$$

where all the previous expressions depend continuously on their parameters. Hence the implicit function theorem yields that $\phi(\xi',\lambda)$ is a C^1 function into $\mathcal{C}_*$. Moreover, we have

$$D_{\xi'}\phi = -(D_\phi B(\xi',\lambda,\phi))^{-1}[D_{\xi'}B(\xi',\lambda,\phi)],$$

so that

$$\|D_{\xi'}\phi\|_* \leqslant C(\|N_\varepsilon(\phi)+R_\varepsilon+\tilde{F}_\varepsilon\|_* + \|D_{\xi'}N_\varepsilon(\xi',\lambda,\phi)\|_*) \leqslant C\varepsilon^2.$$

This concludes the proof of proposition 3.2. □

Given the unique solvability of (3.1), problem (2.2) admits a solution of the desired form if the points ξ_i and the parameters λ_i are chosen so that

$$c_{ij}(\xi,\lambda) = 0. \tag{3.12}$$

Observe now that, integrating (3.1) against Z_{ij}, we obtain an 'almost diagonal' system, which can be written in the form

$$DJ_\varepsilon(V+\phi)[Z_{ij}] = 0, \tag{3.13}$$

where J_ε is the functional introduced in (2.7). In fact, this system is equivalent to (3.12).

Let us now write

$$I_\varepsilon(\xi,\lambda) = J_\varepsilon(V+\phi).$$

We claim that (3.13), and hence (3.12), are equivalent to

$$\nabla I_\varepsilon(\xi,\lambda) = 0. \tag{3.14}$$

In fact, observe that

$$\frac{\partial(V+\phi)}{\partial\xi_{ij}} = \varepsilon^{-2/(N-2)}(\alpha_i Z_{ij}+o(1)), \qquad \frac{\partial(V+\phi)}{\partial\lambda_i} = \alpha_i Z_{i(N+1)} + o(1),$$

with $\alpha_i=-1$ for $i=0$, $\alpha_i=1$ for $i\neq 0$ and $o(1)\to 0$ uniformly on Ω_ε as $\varepsilon\to 0$.

Each term $o(1)$ can be written as the sum of a function that belongs to the space spanned by the Z_{ij} and a function η that satisfies

$$\int_{\Omega_\varepsilon} \eta V_i^{p-1} Z_{ij} = 0$$

for all i,j. Again, from (3.1), we get $DJ_\varepsilon(V+\phi)[\eta]=0$. Hence, for certain numbers β_{ij}, we get

$$\nabla I_\varepsilon(\xi,\lambda) = DJ_\varepsilon(V+\phi)[\nabla(V+\phi)] = \sum_{ij}\beta_{ij}DJ_\varepsilon(V+\phi)[Z_{ij}] = 0,$$

which proves the equivalence between (3.12) and (3.14).

The next step will be to show that solving (3.14) reduces to finding critical points of the leading part of $J_\varepsilon(V+\phi)$, namely, $J_\varepsilon(V)$. This result is established in the following lemma, the proof of which can be found in [10,11].

LEMMA 3.3. *Let ϕ be the function given by proposition 3.2. Then the following expansion holds,*

$$I_\varepsilon(\xi,\lambda) = J_\varepsilon(V) + o(\varepsilon^2),$$

where the term $o(\varepsilon^2)$ is uniform in the C^1 sense over all points satisfying constraint (2.8), for given $\delta > 0$.

4. Proof of theorem 1.1

According to the results of the previous sections, the final step to establish theorem 1.1 consists of finding critical points $\xi = (\xi_0,\dots,\xi_k)$ and $\lambda = (\lambda_0,\dots,\lambda_k)$ of the function

$$I_\varepsilon(\xi,\Lambda) = J_\varepsilon(V+\phi),$$

where $\lambda_i = (a_N^{-1}\Lambda_i)^{2/(N-2)}$ as in proposition 2.1.

We will now see that the symmetry of the domain and of the functions V, f let us look for critical points of I_ε of the very special form

$$\xi_0 = 0, \quad \xi_j = \rho P_j, \quad \Lambda_j = \Lambda \quad \forall j = 1,\dots,k. \tag{4.1}$$

Let us set

$$\mathcal{I}_\varepsilon(\rho,\Lambda_0,\Lambda) = I_\varepsilon((0,\rho(P_1,\dots,P_k)),(\Lambda_0,\Lambda(1,\dots,1))). \tag{4.2}$$

We have the following result.

LEMMA 4.1. *Under the assumptions of theorem 1.1, if (ρ,Λ_0,Λ) is a critical point of $\mathcal{I}_\varepsilon$, then $(\xi,\Lambda) = ((0,\rho(P_1,\dots,P_k)),(\Lambda_0,\Lambda(1,\dots,1)))$ is a critical point of I_ε.*

Proof. Observe that the functions V and $\tilde f$ are even with respect to each of the variables $y_3,\dots,y_N$ in Ω_ε and they are invariant under rotations in the plane spanned by the first two coordinates. Since ϕ solves (3.1), it follows that $\phi(y_1,\dots,y_N)$ shares the same properties with V and $\tilde f$. Hence, since (3.1) is uniquely solvable, $c_{ij} = 0$ automatically for all $0 \leqslant i \leqslant k$ and $2 \leqslant j \leqslant N$ and $c_{01} = 0$.

As a consequence, only the term

$$\sum_{j=1}^{k} c_{j1} V_j^{p-1} Z_{j1} + \sum_{j=0}^{k} c_{j(N+1)} V_j^{p-1} Z_{j(N+1)}$$

appears in the right-hand side of the first equation in (3.1).

Using again the invariance of ϕ under rotations in the (y_1,y_2)-plane, the previous summation reduces to

$$\sum_{j=1}^{k} (c_1 V_j^{p-1}\tilde Z_j + c_2 V_j^{p-1} Z_{j(N+1)}) + c_3 V_0^{p-1} Z_{0(N+1)},$$

where

$$\tilde Z_j = Z_{j1}\cos\left(\frac{2\pi j}{k}\right) + Z_{j2}\sin\left(\frac{2\pi j}{k}\right)$$

and $c_i = c_i(\rho, \Lambda_0, \Lambda)$, $i = 1, 2, 3$. Therefore, finding critical points of I_ε of the form

$$(\xi, \Lambda) = (0, \rho(P_1, \dots, P_k), \Lambda_0, \Lambda(1, \dots, 1))$$

reduces to solving $c_i(\rho, \Lambda_0, \Lambda) = 0$ for $i = 1, 2, 3$.

On the other hand, these relations are equivalent to saying that $(\rho, \Lambda_0, \Lambda)$ is a critical point of $\mathcal{I}_\varepsilon$. In fact, observe first that

$$\begin{aligned} \frac{\partial}{\partial \rho}(V + \phi) &= \sum Z_{ij} - o(1) \quad \text{for } i = 1, \dots, k,\ j = 1, \dots, N, \\ \frac{\partial}{\partial \Lambda}(V + \phi) &= \sum_{l=1}^{k} Z_{l(N+1)} + o(1), \\ \frac{\partial}{\partial \Lambda_0}(V + \phi) &= Z_{0(N+1)} + o(1), \end{aligned}$$

with $o(1) \to 0$ uniformly as $\varepsilon \to 0$. Now, $\nabla \mathcal{I}_\varepsilon = 0$ is equivalent to

$$\begin{aligned} DJ_\varepsilon(V + \phi)\left[\frac{\partial}{\partial \rho}(V + \phi)\right] &= DJ_\varepsilon(V + \phi)\left[\frac{\partial}{\partial \Lambda}(V + \phi)\right] \\ &= DJ_\varepsilon(V + \phi)\left[\frac{\partial}{\partial \Lambda_0}(V + \phi)\right] \\ &= 0. \end{aligned} \tag{4.3}$$

Using again the observation that each term $o(1)$ can be written as the sum of a function that belongs to the space spanned by the Z_{ij} and a function η that satisfies $\int_{\Omega_\varepsilon} \eta V_i^{p-1} Z_{ij} = 0$ for all i, j, equation (4.3) reads as the system

$$\sum_{i=1}^{3} (\delta_{ij} + o(1)) c_i = 0 \quad \text{for } j = 1, 2, 3.$$

Hence $c_1 = c_2 = c_3 = 0$. □

We are now in a position to prove theorem 1.1.

Proof of theorem 1.1. According to lemma 4.1, we need to find a critical point $(\rho, \Lambda_0, \Lambda)$ of the function $\mathcal{I}_\varepsilon(\rho, \Lambda_0, \Lambda)$ defined in (4.2).

Now, from proposition 2.1 and lemma 3.3, we get

$$\mathcal{I}_\varepsilon(\rho, \Lambda_0, \Lambda) = (k + 1) S_N + \varepsilon^2 \Psi_k(\rho, \Lambda_0, \Lambda) + o(\varepsilon^2), \tag{4.4}$$

where

$$\Psi_k(\rho, \Lambda_0, \Lambda) = \psi_k(0, \rho(P_1, \dots, P_k), \Lambda_0, \Lambda(1, \dots, 1))$$

(see (2.11)).

Let $[0, R)$ be the maximal range for ρ. We claim that Ψ_k has a critical point $(\rho_k, \Lambda_{0k}, \Lambda_k) \in (0, R) \times \mathbb{R}_+^2$ for any k large.

We may write Ψ_k in a more compact way,

$$\Psi_k(\rho, \Lambda_0, \Lambda) = \tfrac{1}{2} L^t M_k(\rho) L + L^t \gamma_k(\rho), \tag{4.5}$$

where

$$L = \begin{bmatrix} \Lambda \\ \Lambda_0 \end{bmatrix}, \qquad M_k(\rho) = \begin{bmatrix} N_k(\rho) & G(0,\rho P_1) \\ G(0,\rho P_1) & H(0,0)/k \end{bmatrix}, \tag{4.6}$$

with

$$N_k(\rho) = H(\rho P_1, \rho P_1) - \sum_{j \neq 1} G(\rho P_1, \rho P_j) \tag{4.7}$$

and

$$\gamma_k(\rho) = \begin{bmatrix} \gamma(\rho P_1) \\ \gamma(0)/k \end{bmatrix}, \tag{4.8}$$

with γ defined as in (2.10).

First observe that $\nabla_{(\Lambda_0,\Lambda)}\Psi_k = 0$ amounts to the relation

$$L(\rho) = -M_k^{-1}(\rho)\gamma_k(\rho),$$

i.e.

$$\begin{bmatrix} \Lambda(\rho) \\ \Lambda_0(\rho) \end{bmatrix} = -\frac{1}{\det M_k(\rho)} \begin{bmatrix} \dfrac{H(0,0)}{k}\gamma(\rho P_1) + G(0,\rho P_1)\dfrac{\gamma(0)}{k} \\ -G(0,\rho P_1) - N_k(\rho)\dfrac{\gamma(0)}{k} \end{bmatrix}, \tag{4.9}$$

where $N_k(\rho)$ is given by (4.7). The previous expression makes sense for values of ρ such that $\det M_k(\rho) \neq 0$.

Consider now

$$\tilde{\Psi}_k(\rho) = \Psi_{k_{|\{\nabla_{(\Lambda_0,\Lambda)}\Psi_k = 0\}}}(\rho).$$

An easy computation yields

$$\tilde{\Psi}_k(\rho) = -\tfrac{1}{2}\gamma_k^t(\rho)M_k^{-1}(\rho)\gamma_k(\rho) = -\frac{1}{2\det M_k(\rho)}\psi_k(\rho), \tag{4.10}$$

where

$$\psi_k(\rho) = H(0,0)\gamma^2(\rho P_1) + 2G(0,\rho P_1)\gamma(0)\gamma(\rho P_1) + N_k(\rho)\frac{\gamma^2(0)}{k}. \tag{4.11}$$

The key observation to show that Ψ_k has an admissible critical point for any k sufficiently large is the following. There exists $\hat{\rho} > 0$, $k_0 \in \mathbb{N}$ such that

$$\psi_k(\rho) < 0 \quad \text{for all } \rho \in [0,\hat{\rho}] \text{ for all } k \geqslant k_0. \tag{4.12}$$

In fact, observe that, for $\rho \to 0$, $H(\rho P_1, \rho P_1)$, $\gamma(\rho P_1)$ are bounded quantities. Moreover, from the properties of Green's function, there exists $\delta > 0$ such that, for $0 < \rho < \delta$ and $j \neq 1$, we have

$$G(\rho P_1, \rho P_j) \geqslant \frac{b_N}{\rho^{N-2}|P_1 - P_k|^{N-2}} - O(1), G(\rho P_1, 0) \leqslant \frac{b_N}{\rho^{N-2}} + O(1), \tag{4.13}$$

where $O(1)$ denote quantities uniformly bounded and positive in $[0,\delta]$.

Hence we get

$$\begin{aligned}
&\frac{1}{k}\sum_{j\neq 1} G(\rho P_1, \rho P_j) \\
&\qquad \geqslant \frac{1}{k}\sum_{j\neq 1}\left(\frac{b_N}{\rho^{N-2}|P_1-P_k|^{N-2}} - O(1)\right) \\
&\qquad \geqslant \frac{1}{k}\sum_{j\neq 1}\left(\frac{b_N k^{N-2}}{(2\pi j)^{N-2}\rho^{N-2}} - O(1)\right) \\
&\qquad \geqslant \begin{cases} \dfrac{b_N}{(2\pi)^{N-2}\rho^{N-2}} k^{N-3}\left(\dfrac{1}{N-3} - \dfrac{1}{(N-3)k^{N-3}}\right) - \dfrac{k-1}{k} O(1) & \text{if } N > 3, \\ \dfrac{b_N}{(2\pi)^{N-2}\rho^{N-2}} \log k - \dfrac{k-1}{k} O(1) & \text{if } N = 3. \end{cases}
\end{aligned} \tag{4.14}$$

The previous remarks, together with (4.13) and (4.14) imply (4.12). Now, a direct computation gives that

$$\psi_k(\rho) < 0 \quad \Rightarrow \quad \det M_k(\rho) < 0.$$

In particular, we have, for $0 \leqslant \rho \leqslant \hat{\rho}$ and $k \geqslant k_0$,

$$\det M_k(\rho) < 0. \tag{4.15}$$

Using the properties of Green's function and its regular part, one easily see that, for any k,

$$\lim_{\rho\to 0^+} \det M_k(\rho) = -\infty \quad \text{and} \quad \lim_{\rho\to R^-} \det M_k(\rho) = +\infty.$$

Then, for any k, there exists $\hat{\rho}_k$, $0 < \hat{\rho}_k < R$ with the property that

$$\det M_k(\rho) < 0 \quad \text{for } 0 < \rho < \hat{\rho}_k \quad \text{and} \quad \det M_k(\hat{\rho}_k) = 0. \tag{4.16}$$

As a consequence, $\hat{\rho} < \hat{\rho}_k$ for any k large enough and an easy computation gives

$$\psi_k(\hat{\rho}_k) > 0. \tag{4.17}$$

We now have the tools to show that $\tilde{\Psi}_k(\rho)$ has a minimum in $(0, \hat{\rho}_k)$, with negative value, for any k large enough. In fact, for k large enough, equations (4.10), (4.11), (4.12) and (4.15) imply that

$$\lim_{\rho\to 0^+} \tilde{\Psi}_k(\rho) = 0 \quad \text{and} \quad \tilde{\Psi}_k(\rho) < 0 \quad \text{for } \rho \sim 0^+.$$

On the other hand, equations (4.10), (4.11), (4.16) and (4.17) yield

$$\lim_{\rho\to\hat{\rho}_k^-} \tilde{\Psi}_k(\rho) = +\infty.$$

We denote by c_k and ρ_k, respectively, the minimum value and the minimum point of $\tilde{\Psi}_k$ in $(0, \hat{\rho}_k)$, that is,

$$c_k = \tilde{\Psi}_k(\rho_k) = \min_{\rho\in(0,\hat{\rho}_k)} \tilde{\Psi}_k(\rho) < 0.$$

We can then conclude that $(\rho_k, \Lambda_0(\hat\rho_k), \Lambda(\hat\rho_k))$ (see (4.9)) is a critical point for $\tilde\Psi_k$. We may conclude that this critical point is admissible after we check that $L(\rho_k) \in \mathbb{R}^2_+$.

The fact that $\Lambda_1(\hat\rho_k) > 0$ is a direct consequence of (4.9) and $\det M_k(\hat\rho_k) < 0$.

On the other hand, since $\det M_k(\rho_k) < 0$ and $\tilde\Psi_k(\rho_k) < 0$, we have $\psi_k(\rho_k) < 0$ and hence

$$\begin{aligned}\Lambda_0(\rho_k) &= -\frac{1}{\det M_k(\rho_k)}\left(-G(0,\rho_k P_1)\gamma(\rho_k P_1) - N_k(\rho_k)\frac{\gamma(0)}{k}\right)\\ &> -\frac{1}{\det M_k(\rho_k)}\left(G(0,\rho_k P_1)\gamma(\rho_k P_1) + H(0,0)\frac{\gamma^2(\rho_k)}{\gamma(0)}\right)\\ &> 0.\end{aligned}$$

To conclude the proof of theorem 1.1, we need to show that this critical point persists under small C^1 perturbation. In fact, since (4.4) holds, this implies that $\mathcal{I}_\varepsilon(\rho, \Lambda_0, \Lambda)$ has itself a critical point $(\rho_k^\varepsilon, \Lambda_{0k}^\varepsilon, \Lambda_k^\varepsilon)$ close to $(\rho_k, \Lambda_0(\rho_k), \Lambda_1(\rho_k))$.

Let $a > 0$ and define

$$D_a = (\rho_k - a, \rho_k + a) \times (\Lambda_0(\rho_k) - a, \Lambda_0(\rho_k) + a) \times (\Lambda(\rho_k) - a, \Lambda(\rho_k) + a).$$

Since ρ_k is a non-degenerate minimum of $\tilde\Psi_k$ and from the definition of the function Ψ_k, we can choose, by continuity, a small enough so that the following relations hold true:

$$\frac{\partial}{\partial\rho}\Psi_k(\rho_k - a, \Lambda_0, \Lambda) < 0, \qquad \frac{\partial}{\partial\rho}\Psi_k(\rho_k + a, \Lambda_o, \Lambda) > 0$$
$$\text{for all } (\Lambda_0, \Lambda) \in [\Lambda_0(\rho_k) - a, \Lambda(\rho_k) + a] \times [\Lambda(\rho_k) - a, \Lambda(\rho_k) + a]. \tag{4.18}$$

On the other hand, the point $(\Lambda_0(\rho_k), \Lambda_1(\rho_k))$ is a saddle point for the function $(\Lambda_0, \Lambda) \to \Psi_k(\rho_k, \Lambda_0, \Lambda)$. It follows then that the local degree $\deg(\nabla\Psi_k, D_a, 0)$ is well defined and different from 0. On the other hand, since $\nabla\mathcal{I}_\varepsilon = \varepsilon^2\nabla\Psi_k + o(\varepsilon^2)$ uniformly in D_a as a consequence of (4.4), for all sufficiently small ε, we also have $\deg(\nabla\mathcal{I}_\varepsilon, D_a, 0) \neq 0$. This gives the existence of a critical point for $\mathcal{I}_\varepsilon$ and it concludes the proof of theorem 1.1. □

5. Proof of theorem 1.2

The proof of theorem 1.2 follows the same scheme as the proof of theorem 1.1. We work in the expanded domain

$$\Omega_\varepsilon = \varepsilon^{-1/(N-2)}\Omega, \quad \varepsilon > 0,$$

and now, since we are looking for multi-peak positive solutions of (2.2), we fix an integer k and we set up the ansatz

$$v = \sum_{i=1}^{k} V_j + \phi, \tag{5.1}$$

where ϕ is a lower-order term, $V_j = V_{\lambda_j, \xi'_j}$ are the functions defined in (2.6) for parameters $\lambda_j \in \mathbb{R}^+$ and points $\xi'_j \in \Omega_\varepsilon$. Observe that, in our new problem, the negative peak at the origin in the ansatz (2.6) is neglected. We denote again by V the leading part of (5.1), namely, $V = \sum_{j=1}^{k} V_j$.

Proof of theorem 1.2. In order to carry out the construction of a solution with the form (5.1), we again introduce the intermediate problem (3.1) for ϕ. With the same arguments used in § 3, we obtain the solvability and the estimates for ϕ contained in proposition 3.2.

Hence a solution with the desired form exists if points ξ and scalars λ can be chosen so that the $k(N+1) \times k(N+1)$ system of equations

$$c_{ij}(\boldsymbol{\xi}, \boldsymbol{\lambda}) = 0 \quad \text{for all } i, j \tag{5.2}$$

is satisfied. This system turns out to be equivalent to finding critical points

$$(\boldsymbol{\xi}, \boldsymbol{\Lambda}) = (\xi_1, \dots, \xi_k, \Lambda_1, \dots, \Lambda_k),$$

with Λ_i defined as in proposition 2.1, of

$$I_\varepsilon(\boldsymbol{\xi}, \boldsymbol{\Lambda}) = J_\varepsilon(V + \phi)$$

(see (2.7)) and, since lemma 3.3 still holds, we have

$$I_\varepsilon(\boldsymbol{\xi}, \boldsymbol{\Lambda}) = J_\varepsilon(V) + o(\varepsilon^2)$$

uniformly in the C^1 sense with respect to $(\boldsymbol{\xi}, \boldsymbol{\Lambda})$ satisfying

$$\operatorname{dist}(\xi_i, \partial\Omega) > \delta, \qquad |\xi_i - \xi_j| > \delta, \qquad \delta < \lambda_i < \delta^{-1} \tag{5.3}$$

for all $i = 1, \dots, k$, $i \neq j$, for a given small δ.

Now, taking into account the symmetry of the problem, we look for critical points of the very special form

$$\xi_j = \rho P_j, \quad \Lambda_j = \Lambda \quad \forall j = 1, \dots, k.$$

We call (a, b) the interval of values for ρ. Arguing as in the proof of lemma 4.1, we get that, under the assumptions of theorem 1.2, if (ρ, Λ) is a critical point of

$$\mathcal{I}_\varepsilon(\rho, \Lambda) = I_\varepsilon(\rho(P_1, \dots, P_k), \Lambda(1, \dots, 1)), \tag{5.4}$$

then $(\boldsymbol{\xi}, \boldsymbol{\Lambda}) = (\rho(P_1, \dots, P_k), \Lambda(1, \dots, 1))$ is a critical point for I_ε.

From lemma 3.3 and proposition 2.1, we have

$$\mathcal{I}_\varepsilon(\rho, \Lambda) = kS_N + \varepsilon^2 \Psi_k(\rho, \Lambda) + o(\varepsilon^2) \tag{5.5}$$

uniformly in ε, in the C^1 sense, on compact subsets of $(a, b) \times (0, +\infty)$, where

$$\Psi_k(\rho, \Lambda) = k\{\tfrac{1}{2}\Lambda^2 F_k(\rho) + \gamma(\rho)\Lambda\}, \tag{5.6}$$

with

$$F_k(\rho) = H(\xi_1, \xi_1) - \sum_{i=2}^{k} G(\xi_1, \xi_i),$$

and $\xi_j = \rho P_j$.

Under the given assumptions, there exist numbers $a < a' < b' < b$ such that $F_k(\rho) < 0$ for $\rho \in (a', b')$, for any k sufficiently large. In fact, since the Robin function $H(\xi, \xi)$ tends to $+\infty$ as ξ approaches $\partial\Omega$, it follows that, for any integer k,

$$\lim_{\rho \to a} F_k(\rho) = \lim_{\rho \to b} F_k(\rho) = +\infty.$$

On the other hand, if $\xi_j = \frac{1}{2}(a+b)P_j$, then

$$G(\xi_1, \xi_2) = b_N |\xi_1 - \xi_2|^{2-N} + O(1),$$

where the quantity $O(1)$ is bounded independently of k, and hence

$$G(\xi_1, \xi_2) \geqslant A k^{N-2}$$

for all large k, with A independent of k. Now, $H(\xi_1, \xi_1) \leqslant B$ with B independent of k. It follows that

$$F_k(\tfrac{1}{2}(a+b)) \leqslant k(B - Ak^{N-2}) < 0$$

for all sufficiently large k.

In particular, we can choose $a' < b'$ such that F_k has a negative minimum in (a', b') and that $F_k'(a') < 0$, $F_k'(b') > 0$ and $F_k(\rho) < 0$ hold for all $\rho \in (a', b')$. Then, if δ is fixed and sufficiently small, we see that the following relations hold:

$$\frac{\partial}{\partial \Lambda} \Psi_k(\rho, \delta) > 0, \quad \frac{\partial}{\partial \Lambda} \Psi_k(\rho, \delta^{-1}) < 0 \quad \text{for all } \rho \in [a', b'], \tag{5.7}$$

$$\frac{\partial}{\partial \rho} \Psi_k(a', \Lambda) > 0, \qquad \frac{\partial}{\partial \rho} \Psi_k(b', \Lambda) < 0 \quad \text{for all } \Lambda \in [\delta, \delta^{-1}]. \tag{5.8}$$

Let us set $\mathcal{R} = (a', b') \times (\delta, \delta^{-1})$ and let (d_1, d_2) be the centre point of this rectangle. We consider the homotopy

$$H_t(\rho, \Lambda) = t \nabla \Psi_k(\rho, \Lambda) + (1-t)(\rho - d_1, -(\Lambda - d_2)), \quad t \in [0, 1].$$

Then, from (5.7) and (5.8), we see that the degree $\deg(H_t, \mathcal{R}, 0)$ is well defined and constant for $t \in [0, 1]$. It follows then that $\deg(\nabla \Psi_k, \mathcal{R}, 0) = 1$. Since $\nabla \mathcal{I}_\varepsilon$ is a small uniform perturbation of $\nabla \Psi_k$ on $\mathcal{R}$, we conclude that $\deg(\nabla \mathcal{I}_\varepsilon, \mathcal{R}, 0) = 1$ for all sufficiently small ε. Hence a critical point $(\rho_\varepsilon, \mu_\varepsilon) \in \mathcal{R}$ of $\mathcal{I}_\varepsilon$ indeed exists for all sufficiently small ε and the existence part of the theorem is thus concluded.

It only remains to establish that if f is compactly supported in Ω, then the solutions v_ε found here are positive. To prove this, we claim first that if

$$v_\varepsilon(y) = V(y) + \phi(y) \leqslant 0,$$

then y needs to be close to $\partial\Omega_\varepsilon$. In fact, we claim that, given $\delta > 0$,

$$\operatorname{dist}(y, \partial\Omega_\varepsilon) \leqslant \delta \varepsilon^{-2/(N-2)}$$

for all sufficiently small ε. Let us assume the opposite holds for some $\delta > 0$. Then it is easy to see that

$$V(y) \geqslant C_\delta \sum_{i=1}^{k} (1 + |y - \xi_i'|)^{2-N}$$

for some $C_\delta > 0$. On the other hand, we recall that $\|\phi\|_* = O(\varepsilon^2)$. In other words,

$$|\phi(y)| \leqslant C\varepsilon^2(|y - \xi_i'| + 1)^{-\beta},$$

where $\beta = 2$ for $N \geqslant 4$ and $\beta = 1$ for $N = 3$. Besides, for all i,

$$|y - \xi_i'| \leqslant C\varepsilon^{-2/(N-2)}.$$

Combining these facts, we see that $v_\varepsilon(y) > 0$, which is impossible, and the claim is proved. Thus, if the support of f is compact and we set

$$\Omega_\varepsilon^- = \{y \in \Omega_\varepsilon \mid (V + \phi)(y) \leqslant 0\},$$

then $v_\varepsilon = V + \phi$ satisfies, in this set,

$$\Delta v + |v|^p = 0$$

for all small ε. Using the fact that $|v_\varepsilon| \leqslant C\varepsilon^2$ in this region, and the equation, we get

$$\int_{\Omega_\varepsilon^-} |\nabla v_\varepsilon|^2 \leqslant C\varepsilon^{2(p-1)} \int_{\Omega_\varepsilon^-} v_\varepsilon^2.$$

But Poincaré's inequality in this domain yields

$$C\varepsilon^{4/(N-2)} \int_{\Omega_\varepsilon^-} v_\varepsilon^2 \leqslant \int_{\Omega_\varepsilon^-} |\nabla v_\varepsilon|^2.$$

Since $4/(N-2) = p - 1$, we conclude that $v_\varepsilon \equiv 0$ in this set. Hence $v_\varepsilon > 0$ in Ω_ε and the desired result then follows. □

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