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Multiplicity of solutions to nearly critical elliptic equation in the bounded domain of $\mathbb{R}^3$

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ABSTRACT

We consider the following Dirichlet boundary value problem

$$\begin{cases} -\Delta u = u^{5-\varepsilon} + \lambda u^q, \ u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, $1 < q < 3$, the parameters $\lambda > 0$ and $\varepsilon > 0$. By Lyapunov–Schmidt reduction method and the Mountain Pass Theorem, we prove that in suitable ranges for the parameters λ and ε, problem (0.1) has at least two solutions. Additionally if $2 \le q < 3$, we prove the existence of at least three solutions. Consequently, we prove a non-uniqueness result for a subcritical problem with an increasing nonlinearity.

1. Introduction

We are interested in the following semilinear elliptic boundary value problem

$$\begin{cases} -\Delta u = u^p + \lambda u^q, \ u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, λ is a positive parameter and $p > q > 1$.

Existence and multiplicity of solutions to (1.1) have been studied intensively by many researchers for the exponents p and q in different ranges.

Let us mention the question of existence and multiplicity of solutions to (1.1) for $q = 1$. In the following, $\lambda_1 > 0$ denotes the first eigenvalue of $-\Delta$ under Dirichlet boundary condition.

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(*a*) If $1 < p < 5$, for $0 < \lambda < \lambda_1$, then a solution can be found by the standard constrained minimization procedure thanks to compactness of Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$.

(*b*) If $p \geq 5$, this case is more delicate, since for $p = 5$ the embedding loses compactness while for $p > 5$ Sobolev embedding fails. Pohozaev [16] proved that if Ω is strictly star-shaped, then there is no solution to (1.1) for $\lambda \leq 0$ and $p \geq 5$. For the supercritical case, del Pino, Dolbeault and Musso [9], established existence and multiplicity of solutions to problem (1.1) when p is supercritical but sufficiently close to 5. For $p = 5$, the great contribution to this case was the pioneering work of Brézis and Nirenberg [3]. They obtained that if $q = 1$, (1.1) has a solution if and only if $\lambda \in (\frac{1}{4}\lambda_1, \lambda_1)$ when Ω is a ball. The authors also considered the case $q > 1$. More precisely, if $1 < q \leq 3$, there exists a solution if and only if $\lambda > 0$ is large enough. If $3 < q < 5$, (1.1) has a solution for every $\lambda > 0$. In addition, when Ω is a ball, they gave the following conjecture, which is based on numerical computations.

If $q = 3$, there is some $\tilde{\lambda}$ such that

- for $\lambda > \tilde{\lambda}$, there is a unique solution of (1.1);
- for $\lambda \leq \tilde{\lambda}$, there is no solution of (1.1).

If $1 < q < 3$, there is some $\tilde{\lambda}$ such that

- for $\lambda > \tilde{\lambda}$, there are two solutions of (1.1);
- for $\lambda = \tilde{\lambda}$, there is a unique solution of (1.1);
- for $\lambda < \tilde{\lambda}$, there is no solution of (1.1).

Afterwards, Atkinson and Peletier [1] proved the nonuniqueness of solutions to (1.1) conjectured by Brézis and Nirenberg for $p = 5$ and $1 < q < 3$. For the problem in a ball in $\mathbb{R}^N$, not restricting to integer values of N, they established for $2 < N < 4$, $p = \frac{N+2}{N-2}$ and $1 < q < \frac{6-N}{N-2}$, that there exists some $\tilde{\lambda} > 0$ such that (1.1) has at least two solutions for any $\lambda > \tilde{\lambda}$, and it has no solution for $\lambda < \tilde{\lambda}$. Rey [18] provided another partial answer to above conjecture. He obtained that for $p = 5$ and $2 < q < 3$, $\lambda > 0$ large enough, problem (1.1) has at least $Cat(\Omega) + 1$ solutions, where Ω is any smooth and bounded domain in $\mathbb{R}^3$ and $Cat(\Omega)$ denotes Ljusternik–Schnirelman category of Ω.

The purpose of this paper is to establish multiplicity of solutions to problem (1.1) when p approaches to the critical exponent from below. Namely, we consider

$$\begin{cases} -\Delta u = u^{5-\varepsilon} + \lambda u^q, \ u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, $1 < q < 3$, $\lambda > 0$ and $\varepsilon > 0$. In the following, we write $p = 5-\varepsilon$. It is known that solutions to problem (1.2) correspond to the critical points of the following functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1} - \frac{\lambda}{q+1}\int_\Omega |u|^{q+1}, \quad u \in H_0^1(\Omega). \tag{1.3}$$

In order to state our results, we introduce some notations. Let us consider Green's function $G(x,y)$, solution for any given $y \in \Omega$ of

$$\begin{cases} -\Delta_x G(x,y) = \delta_y(x) & \text{in } \Omega; \\ G(x,y) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

and its regular part $H(x,y) = \frac{1}{4\pi|x-y|} - G(x,y)$. Then $H(x,y)$ satisfies

$$\begin{cases} -\Delta_x H(x,y) = 0 & \text{in } \Omega; \\ H(x,y) = \frac{1}{4\pi|x-y|} & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

The Robin's function of Ω is defined as $R(x) = H(x,x)$. So $R(x)$ is smooth, $R(x) \to +\infty$ as $x \to \partial\Omega$, and it is positive by the maximum principle. Thus $R(x)$ has a minimum in Ω, and hence it has at least one critical point $\xi_0 \in \Omega$.

Our main results can be stated as follows.

Theorem 1.1. *Let $1 < q < 3$, we have:*

(i) For any $\varepsilon_0 > 0$ small, there exists $\lambda_0 = \lambda_0(q, \varepsilon_0, \Omega) > 0$, such that problem (1.2) has a mountain pass solution u_1, for all $\lambda \geq \lambda_0$, and $0 \leq \varepsilon \leq \varepsilon_0$, that satisfies $J(u_1) < \frac{\sqrt{3}}{4}\pi^2$.

(ii) For any given $\lambda > 0$, there exists $\varepsilon_1 = \varepsilon_1(\lambda) > 0$, such that for $\varepsilon \in (0, \varepsilon_1)$, there exists a large solution u_2 of (1.2) of the form

$$u_2(x) = 3^{\frac{1}{4}} \frac{(\Lambda_* \varepsilon)^{\frac{1}{2}}}{((\Lambda_* \varepsilon)^2 + |x - \xi_*|^2)^{\frac{1}{2}}} \big(1 + o(1)\big), \tag{1.6}$$

satisfying

$$J(u_2) = \frac{\sqrt{3}}{4}\pi^2 - \frac{\sqrt{3}}{16}\pi^2 \varepsilon \log \varepsilon + O(\varepsilon), \tag{1.7}$$

where $\xi_* \to \xi_0$ with $R(\xi_0) = \min_{\xi \in \Omega} R(\xi)$, $\Lambda_* \to \Lambda_0$ with $\Lambda_0 = (128 R(\xi_0))^{-1}$ and $o(1) \to 0$ uniformly in $\bar{\Omega}$ as $\varepsilon \to 0$.

Corollary 1.1. *Given $1 < q < 3$. For all λ sufficiently large there exists $\tilde{\varepsilon}(\lambda) > 0$ such that for any $\varepsilon \in (0, \tilde{\varepsilon})$ there exist two distinct solutions u_1 and u_2 of (1.2), such that $J(u_1) < \frac{\sqrt{3}}{4}\pi^2 < J(u_2)$.*

Remark 1.1. In the case that Ω is the unit ball $B(0,1)$, we have $\xi_0 = 0$ and $R(0) = 1/(4\pi)$.

Theorem 1.2. *Assume that $2 \leq q < 3$, and consider the problem (1.2) with*

$$\lambda = \begin{cases} \bar{\lambda} \varepsilon^{-\frac{3-q}{2}} & \text{if } 2 < q < 3, \\ \bar{\lambda} \varepsilon^{-\frac{1}{2}} |\log(\varepsilon)|^{-1} & \text{if } q = 2. \end{cases} \tag{1.8}$$

Then there exists $\bar{\lambda}_0 > 0$ depending on q, and Ω, such that for any $0 < \bar{\lambda} < \bar{\lambda}_0$, there exist two positive numbers $\Lambda_-^*(\bar{\lambda}) < \Lambda_+^*(\bar{\lambda})$, such that there exists $\tilde{\varepsilon}$ small enough such that for all $\varepsilon \in (0, \tilde{\varepsilon})$ there are two solutions $u_\pm$ of the form

$$u_\pm(x) = 3^{\frac{1}{4}} \frac{(\Lambda_\pm^* \varepsilon)^{\frac{1}{2}}}{((\Lambda_\pm^* \varepsilon)^2 + |x - \xi_\varepsilon|^2)^{\frac{1}{2}}} \big(1 + o(1)\big), \tag{1.9}$$

satisfying

$$J(u_+) > J(u_-) > \frac{\sqrt{3}}{4}\pi^2, \tag{1.10}$$

where $o(1) \to 0$ uniformly as $\varepsilon \to 0$, and $\xi_\varepsilon \to \xi_0$ with $R(\xi_0) = \min_{\xi \in \Omega} R(\xi)$, $\Lambda_\pm^* \to \Lambda_\pm$ as $\varepsilon \to 0$.

From the proof of Theorem 1.2, we actually obtain explicit formulae for the numbers $\bar{\lambda}_0$ and $\Lambda_\pm$. Next we show the case $2 < q < 3$, for $q = 2$, we can proceed similarly.

Let us consider the function

$$f(\Lambda) = 128R(\xi_0)b\Lambda^{-\frac{3-q}{2}} - b\Lambda^{-\frac{5-q}{2}}, \quad \text{where } b = \frac{1}{8}\frac{\pi^{1/2}}{5-q}\frac{(q+1)}{3^{\frac{q-1}{4}}}\frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q-2}{2})}.$$

We call $\bar{\lambda}_0$ the maximum value of $f(\Lambda)$, that is

$$\bar{\lambda}_0 := \max_{\Lambda>0} f(\Lambda) = b\big(128R(\xi_0)\big)^{\frac{5-q}{2}}\left(\frac{3-q}{5-q}\right)^{\frac{5-q}{2}}\frac{2}{3-q},$$

which is attained at Λ_0, given by

$$\Lambda_0 = \frac{5-q}{3-q}\big(128R(\xi_0)\big)^{-1}.$$

Thus for any $\bar{\lambda}$, such that $0 < \bar{\lambda} < \bar{\lambda}_0$, the equation $\bar{\lambda} = f(\Lambda)$ has exactly two solutions satisfying

$$\big(128R(\xi_0)\big)^{-1} < \Lambda_-(\bar{\lambda}) < \Lambda_0 < \Lambda_+(\bar{\lambda}). \tag{1.11}$$

Note that the solutions of (1.2) in the (λ, u) space can be identified with a set in the (λ, m)-plane, where $m = u(\xi_0) = \|u\|_\infty$. This gives an interpretation of our results in terms of a bifurcation diagram for positive solutions and $\varepsilon > 0$ small.

Consequently, the result in Theorem 1.2 can be portrayed as representing approximately the upper turning point as

$$P^\varepsilon \sim \big(\bar{\lambda}_0\varepsilon^{-\frac{3-q}{2}}, 3^{\frac{1}{4}}(\Lambda_0)^{-\frac{1}{2}}\varepsilon^{-\frac{1}{2}}\big),$$

while the set is itself near this point approximated by the graph

$$\lambda = \varepsilon^{-\frac{3-q}{2}} f\big(3^{\frac{1}{2}}\big(\varepsilon m^2\big)^{-1}\big) \quad \text{for } m \sim \varepsilon^{-\frac{1}{2}}.$$

The next corollary, gives the existence of at least three solution for (1.2).

Corollary 1.2. *Given $2 < q < 3$. For any $\varepsilon > 0$ sufficiently small there exists $\lambda = \lambda(\varepsilon)$ large such that there exist three distinct solutions u_1 and $u_\pm$ of (1.2), such that $J(u_1) < \frac{\sqrt{3}}{4}\pi^2 < J(u_-) < J(u_+)$.*

We regard $\varepsilon > 0$ as a small parameter, solution u_2 in Theorem 1.1 will be constructed by Lyapunov–Schmidt reduction procedure. This method has been used broadly by many authors to study existence and multiplicity of bubble solutions to elliptic equations, which was first developed by Bahri and Coron [2]. We refer to the survey of del Pino and Musso [8], also we can see [7,11,13–15,19,20] and the references therein.

Finally, we mention some contributions to the elliptic equation with two powers in the whole space $\mathbb{R}^N$. J. Campos [5] considered the existence of bubble-tower solutions to

$$-\Delta u = u^{\frac{N+2}{N-2}\pm\varepsilon} + u^q, \quad u > 0 \text{ in } \mathbb{R}^N; \qquad u(z) \to 0 \quad \text{as } |z| \to \infty, \tag{1.12}$$

where $\frac{N}{N-2} < q < \frac{N+2}{N-2}$ with $N \geq 3$ and $\varepsilon > 0$ small. These solutions behave like a superposition of "bubbles" of different blow-up orders centered at the origin. Recently, Dávila, del Pino and Guerra [6] studied nonuniqueness of positive solution of the following problem

$$-\Delta u + u = u^p + \lambda u^q, \quad u > 0 \text{ in } \mathbb{R}^3; \qquad u(z) \to 0 \quad \text{as } |z| \to \infty. \tag{1.13}$$

More precisely, the authors obtained at least three solutions to problem (1.13) if $1 < q < 3$, $\lambda > 0$ is sufficiently large and fixed, and $p < 5$ is close enough to 5.

2. Energy asymptotic expansion

We recall that, according to [4], the functions

$$w_{\mu,\xi}(x) = 3^{\frac{1}{4}} \frac{\mu^{\frac{1}{2}}}{(\mu^2 + |x-\xi|^2)^{\frac{1}{2}}} \quad \mu > 0,\ \xi \in \mathbb{R}^3,$$

are the only radial solutions of the problem

$$-\Delta w = w^5, \quad w > 0 \text{ in } \mathbb{R}^3. \tag{2.1}$$

As $\xi \in \Omega$ and μ goes to zero, these functions provide us with approximate solutions to the problem that we are interested in. However, in view of the Dirichlet boundary condition, the approximate solution needs to be improved.

From now on we assume that $\xi \in \Omega$ and is far from the boundary of Ω, that is, there exists $\delta > 0$ such that

$$d(\xi, \partial\Omega) \geq \delta. \tag{2.2}$$

Let $U_{\mu,\xi}(x)$ be the unique solution of

$$-\Delta U_{\mu,\xi} = w_{\mu,\xi}^5 \quad \text{in } \Omega; \qquad U_{\mu,\xi} = 0 \quad \text{on } \partial\Omega. \tag{2.3}$$

We have the following estimates.

Lemma 2.1. *Let $d(\xi, \partial\Omega) \geq \delta$ with some $\delta > 0$, for $\mu > 0$ small enough, one has*

(a) $0 < U_{\mu,\xi}(x) \leq w_{\mu,\xi}(x)$,

(b) $U_{\mu,\xi}(x) = w_{\mu,\xi}(x) - 4\pi 3^{\frac{1}{4}} \mu^{\frac{1}{2}} H(x,\xi) + O(\mu^{\frac{5}{2}})$.

Proof. By the maximum principle, we obtain (a). Now we define

$$D(x) = U_{\mu,\xi}(x) - w_{\mu,\xi}(x) + 4\pi 3^{\frac{1}{4}} \mu^{\frac{1}{2}} H(x,\xi).$$

Observe that for $x \in \partial\Omega$, as $\mu \to 0$,

$$D(x) = 3^{\frac{1}{4}} \mu^{\frac{1}{2}} \left[\frac{1}{|x-\xi|} - \frac{1}{(\mu^2 + |x-\xi|^2)^{\frac{1}{2}}} \right] \sim \mu^{\frac{5}{2}} |x-\xi|^{-3}.$$

Then $D(x)$ satisfies

$$-\Delta D = 0 \quad \text{in } \Omega; \qquad D = O\big(\mu^{\frac{5}{2}}\big) \quad \text{as } \mu \to 0 \text{ on } \partial\Omega. \tag{2.4}$$

Therefore (b) follows from the maximum principle. $\square$

In the following we write $U = U_{\mu,\xi}$, we now compute the energy expansion $J(U)$, where $J(U)$ is given by (1.3).

Lemma 2.2. *Let $d(\xi,\partial\Omega)\ge\delta$, assume that $\mu>0$ is small enough, then if $2<q<3$,*

$$J(U)=a_0+a_1\mu R(\xi)-a_2\varepsilon\log\mu+a_3\varepsilon-\lambda a_4\mu^{\frac{5-q}{2}}+O\big(\lambda\mu^{\frac{q+1}{2}}\big)+O\big(\mu^2\big)+o(\varepsilon). \tag{2.5}$$

If $q=2$,

$$J(U)=a_0+a_1\mu R(\xi)-a_2\varepsilon\log\mu+a_3\varepsilon+\lambda a_5\mu^{\frac{3}{2}}\log\mu+O\big(\lambda\mu^{\frac{3}{2}}\big)+O\big(\mu^2\big)+o(\varepsilon). \tag{2.6}$$

If $1<q<2$,

$$J(U)=a_0+a_1\mu R(\xi)-a_2\varepsilon\log\mu+a_3\varepsilon-\lambda a_6\mu^{\frac{q+1}{2}}+O\big(\lambda\mu^{\frac{5-q}{2}}\big)+O\big(\mu^2\big)+o(\varepsilon), \tag{2.7}$$

where $o(\varepsilon)$ is uniform in the C^1-sense on the point ξ satisfying (2.2) as $\varepsilon\to0$, a_i, $i=0,1,\dots,6$, are some positive constants.

Proof. We write $J(U)=J_5(U)+(J_p(U)-J_5(U))+J_\lambda(U)$, where

$$J_p(U)=\frac{1}{2}\int\limits_\Omega|\nabla U|^2-\frac{1}{p+1}\int\limits_\Omega U^{p+1}\quad\text{and}\quad J_\lambda(U)=-\frac{\lambda}{q+1}\int\limits_\Omega U^{q+1}.$$

Since U satisfies $-\Delta U=w^5_{\mu,\xi}$ in Ω and $U=0$ on $\partial\Omega$, we write $U=\pi_{\mu,\xi}+w_{\mu,\xi}$, then we have

$$\begin{aligned}J_5(U)&=\frac{1}{2}\int\limits_\Omega w^5_{\mu,\xi}U-\frac{1}{6}\int\limits_\Omega U^6\\&=\frac{1}{3}\int\limits_\Omega w^6_{\mu,\xi}-\frac{1}{2}\int\limits_\Omega w^5_{\mu,\xi}\pi_{\mu,\xi}-\frac{1}{6}\int\limits_\Omega\big[(\pi_{\mu,\xi}+w_{\mu,\xi})^6-w^6_{\mu,\xi}-6w^5_{\mu,\xi}\pi_{\mu,\xi}\big]\\&:=I-II+\mathcal{R}.\end{aligned} \tag{2.8}$$

By the mean value theorem, we find

$$\mathcal{R}=-5\int\limits_\Omega\int\limits_0^1(w_{\mu,\xi}+t\pi_{\mu,\xi})^4\pi^2_{\mu,\xi}(1-t)\,dtdx=O\big(\mu^2\big).$$

We now expand other terms in the right hand side of (2.8).

$$I=\frac{1}{3}\bigg(\int\limits_{\mathbb{R}^3}3^{\frac{3}{2}}\frac{1}{(1+|z|^2)^3}\,dz-\int\limits_{\mathbb{R}^3\setminus\frac{\Omega-\xi}{\mu}}3^{\frac{3}{2}}\frac{1}{(1+|z|^2)^3}\,dz\bigg)=a_0+O\big(\mu^3\big),$$

where $a_0=\frac{\sqrt{3}\pi^2}{4}$. Moreover from Lemma 2.1, we have

$$\begin{aligned}II&=\frac{1}{2}\mu^{\frac{1}{2}}\int\limits_{\frac{\Omega-\xi}{\mu}}3^{\frac{5}{4}}\frac{1}{(1+|z|^2)^{\frac{5}{2}}}\pi_{\mu,\xi}(\mu z+\xi)\,dz\\&=\frac{1}{2}\int\limits_{\frac{\Omega-\xi}{\mu}}w^5_{1,0}(z)\big[-4\pi3^{\frac{1}{4}}\mu\big[R(\xi)+O\big(\mu|z|\big)+o(\mu)\big]+O\big(\mu^3\big)\big]dz\\&=-\mu R(\xi)a_1+O\big(\mu^2\big),\end{aligned}$$

where $a_1 = 2\pi 3^{\frac{1}{4}} \int_{\mathbb{R}^3} w_{1,0}^5(z)\,dz = 8\sqrt{3}\pi^2$. Thus we get

$$J_5(U) = a_0 + a_1\mu R(\xi) + O(\mu^2).$$

On the other hand, we have

$$\begin{aligned} J_p(U) - J_5(U) &= \varepsilon\left[\frac{1}{6}\int_\Omega U^6 \log U - \frac{1}{36}\int_\Omega U^6\right] + o(\varepsilon) \\ &= \varepsilon\left[\frac{1}{6}\int_\Omega w_{\mu,\xi}^6 \log w_{\mu,\xi} - \frac{1}{36}\int_\Omega w_{\mu,\xi}^6 + O(\mu\log\mu)\right] + o(\varepsilon) \\ &= (-a_2\log\mu + a_3)\varepsilon + o(\varepsilon), \end{aligned} \tag{2.9}$$

where $a_2 = \frac{1}{12}\int_{\mathbb{R}^3} w_{1,0}^6(z)\,dz = \frac{\sqrt{3}\pi^2}{16}$ and $a_3 = \frac{1}{36}\int_{\mathbb{R}^3} w_{1,0}^6(z)[6\log(w_{1,0}(z)) - 1]\,dz$.

Finally we compute $J_\lambda(U)$. If $2 < q < 3$,

$$\begin{aligned} J_\lambda(U) &= -\frac{\lambda}{q+1}\int_\Omega U^{q+1}\,dx = -\frac{\lambda}{q+1}\int_\Omega w_{\mu,\xi}^{q+1}\,dx + O\big(\lambda\mu^{\frac{q+1}{2}}\big) \\ &= -\lambda a_4 \mu^{\frac{5-q}{2}} + O\big(\lambda\mu^{\frac{q+1}{2}}\big), \end{aligned} \tag{2.10}$$

where $a_4 = \frac{1}{q+1}\int_{\mathbb{R}^3} w_{1,0}^{q+1}(z)\,dz = \frac{3^{\frac{q+1}{4}}\pi^{\frac{3}{2}}\Gamma(\frac{q-2}{2})}{(q+1)\Gamma(\frac{q+1}{2})}$. If $q = 2$,

$$J_\lambda(U) = -\frac{\lambda}{3}\mu^{\frac{3}{2}}\int_{\frac{\Omega-\xi}{\mu}} 3^{\frac{3}{4}}\frac{1}{(1+|z|^2)^{\frac{3}{2}}}\,dz + O\big(\lambda\mu^{\frac{3}{2}}\big) = \lambda a_5\mu^{\frac{3}{2}}\log\mu + O\big(\lambda\mu^{\frac{3}{2}}\big), \tag{2.11}$$

where $a_5 = 2\pi 3^{-\frac{1}{4}}$, here we use the fact $\int_0^a \frac{r^2}{(1+r^2)^{3/2}}dr = \log(a+\sqrt{1+a^2}) - \frac{a}{\sqrt{1+a^2}}$. If $1 < q < 2$,

$$\begin{aligned} J_\lambda(U) &= -\frac{\lambda}{q+1}\int_\Omega \big[w_{\mu,\xi}(x) - 4\pi 3^{\frac{1}{4}}\mu^{\frac{1}{2}}H(x,\xi) + O\big(\mu^{\frac{5}{2}}\big)\big]^{q+1} \\ &= -\mu^{\frac{q+1}{2}}\frac{\lambda}{q+1}\int_\Omega \left\{3^{\frac{1}{4}}\left[\frac{1}{(\mu^2+|x-\xi|^2)^{\frac{1}{2}}} - \frac{1}{|x-\xi|}\right] + 4\pi 3^{\frac{1}{4}}G(x,\xi) + O\big(\mu^2\big)\right\}^{q+1} \\ &= -\lambda\mu^{\frac{q+1}{2}}a_6 + O\big(\lambda\mu^{\frac{5-q}{2}}\big), \end{aligned} \tag{2.12}$$

where

$$a_6 = \frac{1}{q+1}\big(4\pi 3^{\frac{1}{4}}\big)^{q+1}\int_\Omega G^{q+1}(x,\xi)\,dx. \tag{2.13}$$

From (2.9)–(2.12), we obtain C^0-estimate of the energy expansion. By the same way, C^1-estimate also holds. □

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. First we note that (1.2) has a mountain pass solution u_1 for any $\varepsilon > 0$. Next we build solution u_2 of (1.2) when $\varepsilon > 0$ small enough by using Lyapunov–Schmidt reduction procedure.

3.1. Approximate solution and the linearized problem

If u is a solution of (1.2), taking the change of variables

$$v(y)=\varepsilon^{\kappa}u(\varepsilon y),\quad \kappa=\frac{2}{p-1},\ y\in\Omega_\varepsilon,$$

where $\Omega_\varepsilon=\frac{\Omega}{\varepsilon}$. Then $v(y)$ satisfies

$$\begin{cases}-\Delta v=f_\varepsilon(v),\ v>0 & \text{in } \Omega_\varepsilon;\\ v=0 & \text{on } \partial\Omega_\varepsilon,\end{cases}\tag{3.1}$$

where and in the following we denote $f_\varepsilon(v)=v^p+\lambda\varepsilon^\alpha v^q$ with $\alpha=\frac{2(p-q)}{p-1}$.

Define the function

$$V(y)\equiv V_{\Lambda,\xi'}(y)=\varepsilon^{\frac{1}{2}}U_{\mu,\xi}(\varepsilon y),\quad \Lambda=\frac{\mu}{\varepsilon},\ \xi'=\frac{\xi}{\varepsilon},\ y\in\Omega_\varepsilon,\tag{3.2}$$

where $U_{\mu,\xi}$ is the solution of (2.3). Then $V(y)$ satisfies

$$-\Delta V(y)=w^5_{\Lambda,\xi'}(y)\quad \text{in } \Omega_\varepsilon;\qquad V(y)=0\quad \text{on } \partial\Omega_\varepsilon.\tag{3.3}$$

Note that assumption (2.2) is equivalent to

$$d\big(\xi',\partial\Omega_\varepsilon\big)\geq\frac{\delta}{\varepsilon}.\tag{3.4}$$

We assume that

$$\hat{\delta}<\Lambda<\frac{1}{\hat{\delta}},\tag{3.5}$$

with $\hat{\delta}>0$ small but fixed.

From Lemma 2.1, for ξ' and Λ satisfying (3.4) and (3.5), we have

$$0<V(y)\leq w_{\Lambda,\xi'}(y)\quad \text{in } \Omega_\varepsilon.\tag{3.6}$$

$$V(y)=w_{\Lambda,\xi'}(y)-4\pi 3^{\frac{1}{4}}\Lambda^{\frac{1}{2}}\varepsilon H\big(\varepsilon y,\varepsilon\xi'\big)+O\big(\varepsilon^3\big)\quad \text{in } \Omega_\varepsilon,\ \text{as } \varepsilon\to 0.\tag{3.7}$$

We next look for a solution of (3.1) of the form

$$v(y)=V(y)+\phi(y),$$

where V is given by (3.2) and ϕ is a small term. We can rewrite (3.1) as

$$\begin{cases}L_\varepsilon(\phi)=N(\phi)+R & \text{in } \Omega_\varepsilon;\\ \phi=0 & \text{on } \partial\Omega_\varepsilon,\end{cases}\tag{3.8}$$

where

$$L_\varepsilon(\phi)=-\Delta\phi-f'_\varepsilon(V)\phi,\qquad N(\phi)=f_\varepsilon(V+\phi)-f_\varepsilon(V)-f'_\varepsilon(V)\phi,\qquad R=\Delta V+f_\varepsilon(V).$$

We first consider the linearized problem at V and we invert it in an orthogonal space. More precisely, we consider the following problem: $h\in L^\infty(\Omega_\varepsilon)$ being given, find a solution ϕ which satisfies

$$\begin{cases} -\Delta\phi - (5-\varepsilon)V^{4-\varepsilon}\phi - \lambda q\varepsilon^{\alpha}V^{q-1}\phi = h + \sum_{i=0}^{3} c_i w_{\Lambda,\xi'}^4 Z_i & \text{in } \Omega_\varepsilon; \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon; \\ \int_{\Omega_\varepsilon} \phi w_{\Lambda,\xi'}^4 Z_i = 0 \quad i = 0,1,2,3, \end{cases} \tag{3.9}$$

for some numbers c_i $(i = 0,1,2,3)$, where Z_i are defined by

$$Z_0 = \frac{\partial V}{\partial \Lambda}, \qquad Z_i = \frac{\partial V}{\partial \xi_i'}, \quad i = 1,2,3.$$

Then $Z_i (i = 0,1,2,3)$ satisfy

$$-\Delta Z_i = 5w_{\Lambda,\xi'}^4 \tilde{Z}_i \quad \text{in } \Omega_\varepsilon; \qquad Z_i = 0 \quad \text{on } \partial\Omega_\varepsilon,$$

with $\tilde{Z}_0 = \frac{\partial w_{\Lambda,\xi'}}{\partial \Lambda}$, and $\tilde{Z}_i = \frac{\partial w_{\Lambda,\xi'}}{\partial \xi_i'}$ for $i = 1,2,3$.

Our next aim is to prove that problem (3.9) has a unique solution with uniform bounds in some appropriate norms. For f a function in Ω_ε, we define the following weighted L^∞-norms

$$\|f\|_* = \sup_{y\in\Omega_\varepsilon} \left(1 + |y-\xi'|^2\right)^{\frac{\theta-2}{2}} |f(y)|, \tag{3.10}$$

and

$$\|f\|_{**} = \sup_{y\in\Omega_\varepsilon} \left(1 + |y-\xi'|^2\right)^{\frac{\theta}{2}} |f(y)|, \tag{3.11}$$

where θ satisfies

$$2 < \theta < 3. \tag{3.12}$$

Observe that the first norm $\|\cdot\|_*$ is equivalent to $\|w_{\Lambda,\xi'}^{-(\theta-2)} f\|_\infty$ and the second norm $\|\cdot\|_{**}$ is equivalent to $\|w_{\Lambda,\xi'}^{-\theta} f\|_\infty$ uniformly with respect to Λ and ξ'.

We have the following result.

Proposition 3.1. *Let $\lambda > 0$ be fixed and ξ', Λ satisfy (3.4), (3.5), then there exists $\varepsilon_0 > 0$ and a constant $C > 0$ independent of ε, such that for all $0 < \varepsilon < \varepsilon_0$ and all $h \in L^\infty(\Omega_\varepsilon)$ with $\|h\|_{**} < +\infty$, problem (3.9) has a unique solution $\phi := T_\varepsilon(h)$ with $\|\phi\|_* < +\infty$. Moreover,*

$$\|\phi\|_* \le C\|h\|_{**}, \qquad |c_i| \le C\|h\|_{**}. \tag{3.13}$$

The argument of its proof follows from the ideas of del Pino et al. in [10] and Rey et al. in [19].

We first prove a priori estimate for solutions of the following problem

$$\begin{cases} -\Delta\phi - (5-\varepsilon)V^{4-\varepsilon}\phi = h + \sum_{i=0}^{3} c_i w_{\Lambda,\xi'}^4 Z_i & \text{in } \Omega_\varepsilon; \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon; \\ \int_{\Omega_\varepsilon} \phi w_{\Lambda,\xi'}^4 Z_i = 0 \quad i = 0,1,2,3. \end{cases} \tag{3.14}$$

Lemma 3.1. *Under the assumptions of Proposition 3.1. Then there exists $C > 0$ such that if $\varepsilon > 0$ is sufficiently small, for any h, ϕ satisfying (3.14), we have*

$$\|\phi\|_* \le C\|h\|_{**}, \qquad |c_i| \le C\|h\|_{**}.$$

The proof follows from the following lemma.

Lemma 3.2. *Assume ϕ_ε solves (3.14) for $h = h_\varepsilon$. If $\|h_\varepsilon\|_{**} \to 0$ as $\varepsilon \to 0$, then $\|\phi_\varepsilon\|_* \to 0$.*

Proof. For $0 < \rho < \theta - 2$, we define

$$\|f\|_\rho = \sup_{y\in\Omega_\varepsilon} \left(1 + \left|y - \xi'\right|^2\right)^{\frac{\theta-2-\rho}{2}} \left|f(y)\right|.$$

Claim. $\|\phi_\varepsilon\|_\rho \to 0$ *as* $\varepsilon \to 0$.

Indeed, by contradiction, we may assume that $\|\phi_\varepsilon\|_\rho = 1$. Multiplying the first equation in (3.14) by Z_j and integrating on Ω_ε, we get

$$\int_{\Omega_\varepsilon} \left(-\Delta Z_j - (5-\varepsilon)V^{4-\varepsilon} Z_j\right)\phi_\varepsilon - \int_{\Omega_\varepsilon} h_\varepsilon Z_j = \sum_{i=0}^{3} c_i \int_{\Omega_\varepsilon} w_{\Lambda,\xi'}^4 Z_i Z_j.$$

By the definition of Z_j and (3.7), we can find

$$\int_{\Omega_\varepsilon} \left(-\Delta Z_j - (5-\varepsilon)V^{4-\varepsilon} Z_j\right)\phi_\varepsilon = o\left(\|\phi_\varepsilon\|_\rho\right).$$

Moreover, we have

$$\int_{\Omega_\varepsilon} h_\varepsilon Z_j \le \|h_\varepsilon\|_{**} \int_{\Omega_\varepsilon} w_{\Lambda,\xi'}^\theta \left(\tilde{Z}_j + O(\varepsilon)\right) = O\left(\|h_\varepsilon\|_{**}\right),$$

and

$$\int_{\Omega_\varepsilon} w_{\Lambda,\xi'}^4 Z_i Z_j = \delta_{ij}\left(\gamma_i + o(1)\right),$$

where δ_{ij} is Kronecker's delta and γ_i, $(i = 0, 1, 2, 3)$ are strictly positive constants. Consequently, we find

$$c_i = O\left(\|h_\varepsilon\|_{**}\right) + o\left(\|\phi_\varepsilon\|_\rho\right). \tag{3.15}$$

In particular, $c_i = o(1)$ as $\varepsilon \to 0$.

Moreover, the first equation in (3.14) can be written as

$$\phi_\varepsilon(x) = \int_{\Omega_\varepsilon} G_\varepsilon(x,y)\left[(5-\varepsilon)V^{4-\varepsilon}(y)\phi_\varepsilon(y) + h_\varepsilon(y) + \sum_{i=0}^{3} c_i w_{\Lambda,\xi'}^4(y) Z_i(y)\right] dy, \tag{3.16}$$

where $G_\varepsilon(x,y)$ is the Green's function of $-\Delta$ in Ω_ε with Dirichlet boundary condition, which satisfies

$$G_\varepsilon(x,y) = \varepsilon G(\varepsilon x, \varepsilon y) \le \frac{C}{|x-y|}.$$

In the following, we use the basic estimate, which was proved in the Appendix B in [20]: for any $0 < \sigma < 1$, there is a constant $C > 0$ such that

$$\int\limits_{\mathbb{R}^3} \frac{1}{|z-y|} \frac{1}{(1+|y|)^{2+\sigma}}\, dy \le \frac{C}{(1+|z|)^\sigma}.$$

Hence we have

$$\begin{aligned}
\Bigg|\int\limits_{\Omega_\varepsilon} G_\varepsilon(x,y) V^{4-\varepsilon}(y)\phi_\varepsilon(y)\, dy\Bigg| &\le C \int\limits_{\Omega_\varepsilon} \frac{1}{|x-y|} \big|w_{\Lambda,\xi'}^{4-\varepsilon}(y)\phi_\varepsilon(y)\big|\, dy \\
&\le C\|\phi_\varepsilon\|_\rho \int\limits_{\Omega_\varepsilon} \frac{1}{|x-y|} \frac{1}{(1+|y-\xi'|^2)^{\frac{1}{2}(4-\varepsilon)}} \frac{1}{(1+|y-\xi'|^2)^{\frac{\theta-2-\rho}{2}}}\, dy \\
&\le C\|\phi_\varepsilon\|_\rho \int\limits_{\Omega_\varepsilon} \frac{1}{|(x-\xi')-(y-\xi')|} \frac{1}{(1+|y-\xi'|)^{2+\theta-2}} \frac{1}{(1+|y-\xi'|)^{2-\rho-\varepsilon}}\, dy \\
&\le C\|\phi_\varepsilon\|_\rho \int\limits_{\mathbb{R}^3} \frac{1}{|(x-\xi')-(y-\xi')|} \frac{1}{(1+|y-\xi'|)^{2+\theta-2}}\, dy \\
&\le C\|\phi_\varepsilon\|_\rho \big(1+|x-\xi'|^2\big)^{-\frac{\theta-2}{2}}, \qquad (3.17)
\end{aligned}$$

$$\begin{aligned}
\Bigg|\int\limits_{\Omega_\varepsilon} G_\varepsilon(x,y) h_\varepsilon(y)\, dy\Bigg| &\le C\|h_\varepsilon\|_{**} \int\limits_{\Omega_\varepsilon} \frac{1}{|x-y|} \frac{1}{(1+|y-\xi'|^2)^{\frac{\theta}{2}}}\, dy \\
&\le C\|h_\varepsilon\|_{**} \int\limits_{\mathbb{R}^3} \frac{1}{|(x-\xi')-(y-\xi')|} \frac{1}{(1+|y-\xi'|)^{2+\theta-2}}\, dy \\
&\le C\|h_\varepsilon\|_{**} \big(1+|x-\xi'|^2\big)^{-\frac{\theta-2}{2}}, \qquad (3.18)
\end{aligned}$$

and

$$\begin{aligned}
\Bigg|\int\limits_{\Omega_\varepsilon} G_\varepsilon(x,y) w_{\Lambda,\xi'}^4(y) Z_i(y)\, dy\Bigg| &\le C \int\limits_{\Omega_\varepsilon} \frac{1}{|x-y|} \frac{1}{(1+|y-\xi'|^2)^{\frac{5}{2}}}\, dy \\
&\le C \int\limits_{\Omega_\varepsilon} \frac{1}{|x-y|} \frac{1}{(1+|y-\xi'|)^{2+\theta-2}} \frac{1}{(1+|y-\xi'|)^{5-\theta}}\, dy \\
&\le C\big(1+|x-\xi'|^2\big)^{-\frac{\theta-2}{2}}. \qquad (3.19)
\end{aligned}$$

Then from (3.16)–(3.19), we get

$$\big|\phi_\varepsilon(x)\big| \le C\big(\|\phi_\varepsilon\|_\rho + \|h_\varepsilon\|_{**} + |c_i|\big)\big(1+|x-\xi'|^2\big)^{-\frac{\theta-2}{2}}, \qquad (3.20)$$

which yields that

$$\big(1+|x-\xi'|^2\big)^{\frac{\theta-2-\rho}{2}} \big|\phi_\varepsilon(x)\big| \le C\big(1+|x-\xi'|^2\big)^{-\frac{\rho}{2}}. \qquad (3.21)$$

Moreover, $\|\phi_\varepsilon\|_\rho = 1$ and (3.21) imply that there exist $R > 0$, $\gamma > 0$ independent of ε such that

$$\|\phi_\varepsilon\|_{L^\infty(B_R(\xi'))} > \gamma. \tag{3.22}$$

Set $\bar{\phi}_\varepsilon(y) = \phi_\varepsilon(y - \xi')$, by elliptic estimates, passing to a subsequence of $(\bar{\phi}_\varepsilon)_\varepsilon$, still denote $(\bar{\phi}_\varepsilon)_\varepsilon$, such that $(\bar{\phi}_\varepsilon)_\varepsilon$ converges uniformly on any compact set of $\mathbb{R}^3$ to a nontrivial solution of

$$-\Delta\bar{\phi} = 5w_{\Lambda,0}^4\bar{\phi} \quad \text{for some } \Lambda > 0.$$

It is well known that [17],

$$\bar{\phi} = \alpha_0 \frac{\partial w_{\Lambda,0}}{\partial \Lambda} + \sum_{i=1}^{3} \alpha_i \frac{\partial w_{\Lambda,0}}{\partial y_i}.$$

Recall that

$$\int_{\Omega_\varepsilon} \phi_\varepsilon w_{\Lambda,\xi'}^4 Z_i = 0, \quad \text{for } i = 0, 1, 2, 3.$$

By dominated convergence, we have

$$\alpha_0 \int_{\mathbb{R}^3} \left(\frac{\partial w_{\Lambda,0}}{\partial \Lambda}\right)^2 w_{\Lambda,0}^4 = 0 \quad \text{and} \quad \alpha_i \int_{\mathbb{R}^3} \left(\frac{\partial w_{\Lambda,0}}{\partial y_i}\right)^2 w_{\Lambda,0}^4 = 0, \quad \text{for } i = 1, 2, 3.$$

So $\alpha_i = 0$ for $i = 0, 1, 2, 3$ and $\bar{\phi} = 0$, this contradicts (3.22). Therefore we get $\|\phi_\varepsilon\|_\rho \to 0$ as $\varepsilon \to 0$. Finally, from (3.15) and (3.20), we have

$$\|\phi_\varepsilon\|_* \le C\big(\|h_\varepsilon\|_{**} + \|\phi_\varepsilon\|_\rho\big).$$

Hence $\|\phi_\varepsilon\|_* \to 0$ as $\varepsilon \to 0$. We complete the proof. □

Lemma 3.3. *Let $\lambda > 0$ be fixed and ξ', Λ satisfy (3.4), (3.5). There exists $C > 0$ such that if $\varepsilon > 0$ is sufficiently small, for any h, ϕ satisfying (3.9), we have*

$$\|\phi\|_* \le C\|h\|_{**}, \qquad |c_i| \le C\|h\|_{**}.$$

Proof. We claim that $\|V^{q-1}\phi\|_{**} \le C\varepsilon^{q-3}\|\phi\|_*$. Since $V \le w_{\Lambda,\xi'}$, we only need to show that

$$\big\|w_{\Lambda,\xi'}^{q-1}\phi\big\|_{**} \le C\varepsilon^{q-3}\|\phi\|_*.$$

In fact,

$$\begin{aligned}
\big\|w_{\Lambda,\xi'}^{q-1}\phi\big\|_{**} &= \sup_{y\in\Omega_\varepsilon} \big(1 + |y-\xi'|^2\big)^{\frac{\theta}{2}} \big|w_{\Lambda,\xi'}(y)\big|^{q-1} \big|\phi(y)\big| \\
&\le \|\phi\|_* \sup_{y\in\Omega_\varepsilon} \big(1 + |y-\xi'|^2\big) \big|w_{\Lambda,\xi'}(y)\big|^{q-1} \le \|\phi\|_* \sup_{y\in\Omega_\varepsilon} \big(1 + |y-\xi'|^2\big)^{1-\frac{q-1}{2}} \\
&\le \|\phi\|_* \sup_{y\in\Omega_\varepsilon} |y-\xi'|^{3-q} \le C\varepsilon^{q-3}\|\phi\|_*.
\end{aligned}$$

By the first estimate in Lemma 3.1, we get

$$\|\phi\|_* \le C\|h\|_{**} + C\varepsilon^\alpha \left\|V^{q-1}\phi\right\|_{**} \le C\|h\|_{**} + C\varepsilon^{\alpha+q-3}\|\phi\|_*.$$

Recall that $\alpha = \frac{5-q}{2} + O(\varepsilon)$, we have that $\alpha + q - 3 > 0$. Thus we get $\|\phi\|_* \le C\|h\|_{**}$.

Similarly, we can get $|c_i| \le C\|h\|_{**}$. □

Proof of Proposition 3.1. By Lemma 3.3, we get the estimates in (3.13). Now we prove existence and uniqueness of solution to (3.9). Consider the Hilbert space

$$H = \left\{\phi \in H_0^1(\Omega_\varepsilon): \int_{\Omega_\varepsilon} \phi w_{\Lambda,\xi'}^4 Z_i = 0,\ i = 0, 1, 2, 3\right\}$$

with inner product

$$\langle \phi, \psi\rangle = \int_{\Omega_\varepsilon} \nabla\phi\nabla\psi.$$

Then problem (3.9) is equivalent to find $\phi \in H$ such that

$$\langle \phi, \psi\rangle = \int_{\Omega_\varepsilon} \left[(5-\varepsilon)V^{4-\varepsilon}\phi + \lambda q\varepsilon^\alpha V^{q-1}\phi + h\right]\psi, \quad \text{for } \forall\psi \in H. \tag{3.23}$$

By the Riesz representation theorem, (3.23) is equivalent to solve

$$\phi = K(\phi) + \tilde{h} \tag{3.24}$$

with $\tilde{h} \in H$ depending linearly on h, and $K : H \to H$ being a compact operator. Fredholm's alternative guarantees that there is a unique solution to problem (3.24) for any h provided that

$$\phi = K(\phi) \tag{3.25}$$

has only the zero solution in H. (3.25) is equivalent to problem (3.9) with $h = 0$. If $h = 0$, the first estimate in (3.13) implies that $\phi = 0$. This completes the proof. □

For later purpose, it is important to understand the differentiability of the operator T_ε with respect to Λ, ξ'. Consider the L_*^∞ (resp. L_{**}^∞) functions defined on Ω_ε with $\|\cdot\|_*$ norm (resp. $\|\cdot\|_{**}$ norm). We have the following result.

Proposition 3.2. *Under the conditions of Proposition 3.1, the map $(\Lambda, \xi') \mapsto T_\varepsilon(h)$ is C^1 with respect to Λ, ξ' in the considered region and the L_*^∞ norm. Moreover,*

$$\left\|\partial_\Lambda T_\varepsilon(h)\right\|_* \le C\|h\|_{**}, \qquad \left\|\partial_{\xi'} T_\varepsilon(h)\right\|_* \le C\|h\|_{**}. \tag{3.26}$$

Proof. T_ε is C^1 with respect to Λ and ξ' follows from the smoothness of K and $\tilde{h}$, which occur in the implicit definition (3.24) of $\phi = T_\varepsilon(h)$, with respect to these variables. Differentiating (3.9) with respect to ξ_k' $(k = 1, 2, 3)$, set $\phi = T_\varepsilon(h)$, $Y = \partial_{\xi_k'}\phi$ and $d_i = \partial_{\xi_k'} c_i$, $k = 1, 2, 3$, then Y satisfies

$$\begin{cases} -\Delta Y - (5-\varepsilon)V^{4-\varepsilon}Y - \lambda q\varepsilon^\alpha V^{q-1}Y = \bar{h} + \sum_{i=0}^{3} d_i w_{\Lambda,\xi'}^4 Z_i & \text{in } \Omega_\varepsilon; \\ Y = 0 \quad \text{on } \partial\Omega_\varepsilon; \qquad \int_{\Omega_\varepsilon} \left[\phi\partial_{\xi_k'}\left(w_{\Lambda,\xi'}^4 Z_i\right) + Y w_{\Lambda,\xi'}^4 Z_i\right] = 0, \quad i = 0, \cdots, 3, \end{cases} \tag{3.27}$$

where

$$\bar{h} = (5-\varepsilon)(4-\varepsilon)V^{3-\varepsilon}Z_i\phi + \lambda q(q-1)\varepsilon^\alpha V^{q-2}Z_i\phi + \sum_{i=0}^{3} c_i\partial_{\xi'_k}\big(w^4_{\Lambda,\xi'}Z_i\big).$$

Set $\eta = Y - \sum_{j=0}^{3} b_jZ_j$, where $b_j \in \mathbb{R}$ is chosen such that $\int_{\Omega_\varepsilon} \eta w^4_{\Lambda,\xi'}Z_i = 0$, that is, b_j satisfies

$$\sum_{j=0}^{3} b_j \int_{\Omega_\varepsilon} w^4_{\Lambda,\xi'}Z_iZ_j = \int_{\Omega_\varepsilon} Yw^4_{\Lambda,\xi'}Z_i. \tag{3.28}$$

Since this system is almost diagonal, it has a unique solution and we have

$$|b_j| \le C\|\phi\|_*. \tag{3.29}$$

Moreover, η satisfies

$$\begin{cases} -\Delta\eta - (5-\varepsilon)V^{4-\varepsilon}\eta - \lambda q\varepsilon^\alpha V^{q-1}\eta = g + \sum_{i=0}^{3} d_iw^4_{\Lambda,\xi'}Z_i & \text{in } \Omega_\varepsilon; \\ \eta = 0 \quad \text{on } \partial\Omega_\varepsilon; \qquad \int_{\Omega_\varepsilon} \eta w^4_{\Lambda,\xi'}Z_i = 0 \quad i = 0,1,2,3, \end{cases} \tag{3.30}$$

with

$$g = \sum_{j=0}^{3} b_j\big[-\Delta Z_j - (5-\varepsilon)V^{4-\varepsilon}Z_j - \lambda q\varepsilon^\alpha V^{q-1}Z_j\big] + \bar{h}.$$

By Proposition 3.1, we have that $\eta = T_\varepsilon(g)$ and $\|\eta\|_* \le C\|g\|_{**}$.

By simple calculations, we have

$$\begin{aligned} \|g\|_{**} &\le \sum_{j=0}^{3} |b_j|\big\|-\Delta Z_j - (5-\varepsilon)V^{4-\varepsilon}Z_j - \lambda q\varepsilon^\alpha V^{q-1}Z_j\big\|_{**} \\ &\quad + C\big\|V^{3-\varepsilon}Z_i\phi\big\|_{**} + C\varepsilon^\alpha\big\|V^{q-2}Z_i\phi\big\|_{**} + \sum_{i=0}^{3}|c_i|\big\|\partial_{\xi'_k}\big(w^4_{\Lambda,\xi'}Z_i\big)\big\|_{**} \\ &\le C\big(|b_j| + \|\phi\|_* + |c_i|\big) \le C\|h\|_{**}, \end{aligned}$$

here we use $|b_j| \le C\|\phi\|_*$, $\|\phi\|_* \le C\|h\|_{**}$ and $|c_i| \le C\|h\|_{**}$.

Thus

$$\|\eta\|_* \le C\|h\|_{**}. \tag{3.31}$$

By (3.29), (3.31) and $\|Z_j\|_* \le C$, we obtain that

$$\|\partial_{\xi'_k}\phi\|_* \le \sum_{j=0}^{3}|b_j|\|Z_j\|_* + \|\eta\|_* \le C\|h\|_{**}.$$

Similarly, we can get the estimate for $\|\partial_\Lambda\phi\|_*$ in (3.26). □

3.2. The nonlinear problem

In this subsection, our purpose is to study the nonlinear problem. First, we estimate $\|R\|_{**}$, $\|\partial_\Lambda R\|_{**}$ and $\|\partial_{\xi'} R\|_{**}$.

Lemma 3.4. *Assume $1 < q < 3$, let $\lambda > 0$ be fixed and ξ', Λ satisfy (3.4), (3.5), then choosing $2 < \theta < 3$ appropriately in the norms (3.10), (3.11), there exists a constant $C > 0$ independent of ξ', Λ, such that*

$$\|R\|_{**} \le C\varepsilon, \qquad \|\partial_\Lambda R\|_{**} \le C\varepsilon, \qquad \|\partial_{\xi'} R\|_{**} \le C\varepsilon, \tag{3.32}$$

for $\varepsilon > 0$ small enough.

Proof. Recall that $R = V^{5-\varepsilon} - w^5_{\Lambda,\xi'} + \lambda\varepsilon^\alpha V^q$. By (3.7), $V = w_{\Lambda,\xi'} + O(\varepsilon)$. Consequently,

$$\left|V^{5-\varepsilon} - w^5_{\Lambda,\xi'}\right| \le \left|V^{5-\varepsilon} - w^{5-\varepsilon}_{\Lambda,\xi'}\right| + \left|w^{5-\varepsilon}_{\Lambda,\xi'} - w^5_{\Lambda,\xi'}\right| \le C\varepsilon\left(w^{4-\varepsilon}_{\Lambda,\xi'} + w^5_{\Lambda,\xi'}|\log w_{\Lambda,\xi'}|\right).$$

Thus for $2 < \theta < 3$,

$$\left\|V^{5-\varepsilon} - w^5_{\Lambda,\xi'}\right\|_{**} \le C\left\|w^{-\theta}_{\Lambda,\xi'}\left(V^{5-\varepsilon} - w^5_{\Lambda,\xi'}\right)\right\|_\infty \le C\varepsilon \sup_{\Omega_\varepsilon} w^{-\theta}_{\Lambda,\xi'}\left(w^{4-\varepsilon}_{\Lambda,\xi'} + w^5_{\Lambda,\xi'}|\log w_{\Lambda,\xi'}|\right) \le C\varepsilon.$$

Moreover,

$$\left\|\lambda\varepsilon^\alpha V^q\right\|_{**} \le C\lambda\varepsilon^\alpha\left\|w^{-\theta}_{\Lambda,\xi'} V^q\right\|_\infty \le C\lambda\varepsilon^\alpha \sup_{\Omega_\varepsilon}\left|w^{q-\theta}_{\Lambda,\xi'}\right| \le \begin{cases} C\lambda\varepsilon^\alpha & \text{if } q > \theta; \\ C\lambda\varepsilon^{\alpha+q-\theta} & \text{if } q \le \theta. \end{cases}$$

Note that $\alpha = \frac{5-q}{2} + O(\varepsilon) > 1$ for $1 < q < 3$. We choose $2 < \theta < \frac{3+q}{2}$, so $\alpha + q - \theta > 1$. Therefore we get the first estimate in (3.32). Furthermore

$$\partial_\Lambda R = (5-\varepsilon)V^{4-\varepsilon}Z_0 - 5w^4_{\Lambda,\xi'}\tilde{Z}_0 + \lambda q\varepsilon^\alpha V^{q-1}Z_0,$$

and

$$\partial_{\xi'_i} R = (5-\varepsilon)V^{4-\varepsilon}Z_i - 5w^4_{\Lambda,\xi'}\tilde{Z}_i + \lambda q\varepsilon^\alpha V^{q-1}Z_i, \quad i = 1,2,3.$$

By similar computations as $\|R\|_{**}$, we can get the rest estimates in (3.32). □

Now we consider the following problem

$$-\Delta\phi - (5-\varepsilon)V^{4-\varepsilon}\phi - \lambda q\varepsilon^\alpha V^{q-1}\phi = N(\phi) + R + \sum_{i=0}^{3} c_i w^4_{\Lambda,\xi'} Z_i \quad \text{in } \Omega_\varepsilon; \tag{3.33}$$

with

$$\phi = 0 \quad \text{on } \partial\Omega_\varepsilon, \quad \text{and} \quad \int_{\Omega_\varepsilon} \phi w^4_{\Lambda,\xi'} Z_i = 0 \quad i = 0,1,2,3. \tag{3.34}$$

Proposition 3.3. *There exists $C > 0$ independent of ξ', Λ satisfying (3.4), (3.5), such that for $\varepsilon > 0$ small enough, there exists a unique solution $\phi = \phi(\Lambda, \xi')$ of problem (3.33)–(3.34), satisfying*

$$\|\phi\|_* \le C\varepsilon. \tag{3.35}$$

Proof. By Proposition 3.1, problem (3.33)–(3.34) can be written as the fixed point problem

$$\phi = T_\varepsilon\big(N(\phi)+R\big) := A_\varepsilon(\phi).$$

Define

$$\mathcal{F}_M = \big\{\phi \in H_0^1(\Omega_\varepsilon)\cap L^\infty(\Omega_\varepsilon) : \|\phi\|_* \le M\varepsilon\big\}$$

with $M>0$ large but fixed which will be chosen later. Then A_ε sends $\mathcal{F}_M$ into itself.

Indeed, by Proposition 3.1, we have

$$\big\|A_\varepsilon(\phi)\big\|_* = \big\|T_\varepsilon\big(N(\phi)+R\big)\big\|_* \le C\big(\big\|N(\phi)\big\|_{**} + \|R\|_{**}\big). \tag{3.36}$$

Moreover,

$$\begin{aligned}
\big\|N(\phi)\big\|_{**} &= \left\|\int_0^1 \big[f_\varepsilon'(V+t\phi) - f_\varepsilon'(V)\big]\phi\,dt\right\|_{**} \\
&\le C\left\|w_{\Lambda,\xi'}^{-2}\int_0^1 \big|f_\varepsilon'(V+t\phi)-f_\varepsilon'(V)\big|\,dt\right\|_\infty \|\phi\|_* \\
&\le C\big(\big\|w_{\Lambda,\xi'}^{-2}\big[\big(V+|\phi|\big)^{4-\varepsilon} - V^{4-\varepsilon}\big]\big\|_\infty + \lambda\varepsilon^\alpha\big\|w_{\Lambda,\xi'}^{-2}\big[\big(V+|\phi|\big)^{q-1} - V^{q-1}\big]\big\|_\infty\big)\|\phi\|_*.
\end{aligned} \tag{3.37}$$

Since

$$\begin{aligned}
\big\|w_{\Lambda,\xi'}^{-2}\big[(V+|\phi|)^{4-\varepsilon} - V^{4-\varepsilon}\big]\big\|_\infty &\le C\big\|w_{\Lambda,\xi'}^{-2}\big(w_{\Lambda,\xi'}^{3-\varepsilon}|\phi| + |\phi|^{4-\varepsilon}\big)\big\|_\infty \\
&\le C\big\|w_{\Lambda,\xi'}^{\theta-1-\varepsilon}\big\|_\infty\|\phi\|_* + C\big\|w_{\Lambda,\xi'}^{(\theta-2)(4-\varepsilon)-2}\big\|_\infty\|\phi\|_*^{4-\varepsilon} \\
&\le C\varepsilon^{\theta-1-\varepsilon}\|\phi\|_* + C\varepsilon^{\min\{(\theta-2)(4-\varepsilon)-2,0\}}\|\phi\|_*^{4-\varepsilon}.
\end{aligned} \tag{3.38}$$

On the other hand, by Lemma 2.2 in [12], we have

$$\big||V+\phi|^{q-1} - |V|^{q-1}\big| \le C\begin{cases} |V|^{q-2}|\phi| + |\phi|^{q-1} & \text{if } 2\le q<3; \\ \min\{|V|^{q-2}|\phi|, |\phi|^{q-1}\} & \text{if } 1<q<2. \end{cases}$$

Thus for $1<q<2$,

$$\begin{aligned}
\big\|w_{\Lambda,\xi'}^{-2}\big[\big(V+|\phi|\big)^{q-1} - V^{q-1}\big]\big\|_\infty &\le C\min\big\{\|w_{\Lambda,\xi'}\|_\infty^{q-4+\theta-2}\|\phi\|_*, \|w_{\Lambda,\xi'}\|_\infty^{(\theta-2)(q-1)-2}\|\phi\|_*^{q-1}\big\} \\
&\le C\min\big\{\varepsilon^{q+\theta-6}\|\phi\|_*, \varepsilon^{(\theta-2)(q-1)-2}\|\phi\|_*^{q-1}\big\}.
\end{aligned} \tag{3.39}$$

For $2\le q<3$,

$$\begin{aligned}
\big\|w_{\Lambda,\xi'}^{-2}\big[\big(V+|\phi|\big)^{q-1} - V^{q-1}\big]\big\|_\infty &\le C\big\|w_{\Lambda,\xi'}^{-2}\big[w_{\Lambda,\xi'}^{q-2}|\phi| + |\phi|^{q-1}\big]\big\|_\infty \\
&\le C\varepsilon^{q+\theta-6}\|\phi\|_* + C\varepsilon^{(\theta-2)(q-1)-2}\|\phi\|_*^{q-1}.
\end{aligned} \tag{3.40}$$

From (3.37)–(3.40), if $1<q<3$, for $\phi\in\mathcal{F}_M$, then we have

$$\big\|N(\phi)\big\|_{**} \le C\varepsilon^\tau\|\phi\|_* \quad \text{with some } \tau>0. \tag{3.41}$$

Thus by (3.32), (3.36) and (3.41), we find for $\phi \in \mathcal{F}_M$,

$$\|A_\varepsilon(\phi)\|_* \le C(\varepsilon^\tau\|\phi\|_* + \varepsilon) \le C(M\varepsilon^\tau + 1)\varepsilon.$$

Choosing M large such that $C(M\varepsilon^\tau + 1) \le M$. It implies that $A_\varepsilon(\mathcal{F}_M) \subset \mathcal{F}_M$.

Next we show that A_ε is a contraction map. For $\phi_1, \phi_2 \in \mathcal{F}_M$,

$$\begin{aligned}\|A_\varepsilon(\phi_1) - A_\varepsilon(\phi_2)\|_* &\le C\|N(\phi_1) - N(\phi_2)\|_{**} \\ &= C\|[f'_\varepsilon(V + t\phi_1 + (1-t)\phi_2) - f'_\varepsilon(V)](\phi_1 - \phi_2)\|_{**} \\ &\le C\|w^{-\theta}_{\Lambda,\xi'}[f'_\varepsilon(V + \tilde{\phi}) - f'_\varepsilon(V)](\phi_1 - \phi_2)\|_\infty \\ &\le C\|w^{-2}_{\Lambda,\xi'}[f'_\varepsilon(V + \tilde{\phi}) - f'_\varepsilon(V)]\|_\infty\|\phi_1 - \phi_2\|_*,\end{aligned}$$

where $\tilde{\phi} = t\phi_1 + (1-t)\phi_2 \in \mathcal{F}_M$ for $t \in (0,1)$. It can be easily checked that

$$\|A_\varepsilon(\phi_1) - A_\varepsilon(\phi_2)\|_* \le C\varepsilon^\tau\|\phi_1 - \phi_2\|_*, \quad \text{with some } \tau > 0.$$

This yields that A_ε has a unique fixed point in $\mathcal{F}_M$. Hence problem (3.33)–(3.34) has a unique solution ϕ such that $\|\phi\|_* \le C\varepsilon$, for some $C > 0$. □

Proposition 3.4. *The solution $\phi(\Lambda, \xi')$ constructed in Proposition 3.3 is C^1 with respect to Λ and ξ' in the considered region. Moreover,*

$$\|\partial_\Lambda\phi\|_* \le C\varepsilon, \qquad \|\partial_{\xi'}\phi\|_* \le C\varepsilon. \tag{3.42}$$

Proof. We write

$$B(\Lambda, \xi', \phi) = \phi - T_\varepsilon(N(\phi) + R), \tag{3.43}$$

we have

$$B(\Lambda, \xi', \phi) = 0, \tag{3.44}$$

and

$$\partial_\phi B(\Lambda, \xi', \phi)[\psi] = \psi - \partial_\phi[T_\varepsilon(N(\phi) + R)]\psi = \psi - T_\varepsilon[\partial_\phi(N(\phi))\psi]. \tag{3.45}$$

By a direct calculation, we get

$$\|T_\varepsilon[\partial_\phi(N(\phi))\psi]\|_* \le C\|\partial_\phi(N(\phi))\psi\|_{**} \le C\|w^{-2}_{\Lambda,\xi'}\partial_\phi(N(\phi))\|_\infty\|\psi\|_* \le C\varepsilon^\tau\|\psi\|_*,$$

with $\tau > 0$. Therefore

$$\|\partial_\phi B(\Lambda, \xi', \phi)[\psi]\|_* \le (1 + C\varepsilon^\tau)\|\psi\|_*.$$

It follows that for $\varepsilon > 0$ small enough, $\partial_\phi B(\Lambda, \xi', \phi)$ is invertible in $\|\cdot\|_*$ with uniformly bounded inverse. It also depends continuously on its parameters. Let us differentiate (3.43) with respect to ξ' and by (3.45), we have

$$\partial_{\xi'}B(\Lambda, \xi', \phi) = -(\partial_{\xi'}T_\varepsilon)(N(\Lambda, \xi', \phi) + R) - T_\varepsilon((\partial_{\xi'}N)(\Lambda, \xi', \phi) + \partial_{\xi'}R), \tag{3.46}$$

where all the previous expressions depend continuously on their parameters. Hence the implicit function theorem implies that $\phi = \phi(\Lambda, \xi')$ is C^1 with respect to Λ, ξ' in the considered region.

Moreover, differentiating (3.44) with respect to ξ', we get

$$\partial_{\xi'}\phi = -\big(\partial_\phi B\big(\Lambda, \xi', \phi\big)\big)^{-1}\partial_{\xi'} B\big(\Lambda, \xi', \phi\big).$$

By (3.46), (3.26) and (3.13), we get

$$\|\partial_{\xi'}\phi\|_* \le C\big(\big\|N(\phi)\big\|_{**} + \|R\|_{**} + \big\|(\partial_{\xi'}N)\big(\Lambda, \xi', \phi\big)\big\|_{**} + \|\partial_{\xi'}R\|_{**}\big) \le C\varepsilon.$$

Similarly, we can get $\|\partial_\Lambda \phi\|_* \le C\varepsilon$. □

3.3. The reduced functional

We have solved the nonlinear problem (3.33)–(3.34). In order to find a solution to problem (3.1), we need to find Λ and ξ' such that

$$c_i\big(\Lambda, \xi'\big) = 0 \quad \text{for } i = 0, 1, 2, 3. \tag{3.47}$$

The energy functional to problem (3.1) is given by

$$I(v) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla v|^2 - \frac{1}{p+1}\int_{\Omega_\varepsilon} |v|^{p+1} - \lambda\frac{\varepsilon^\alpha}{q+1}\int_{\Omega_\varepsilon} |v|^{q+1}$$

Set

$$\mathcal{I}\big(\Lambda, \xi'\big) = I\big(V_{\Lambda,\xi'}(y) + \phi_{\Lambda,\xi'}(y)\big), \tag{3.48}$$

where $V_{\Lambda,\xi'}$ is defined in (3.2) and $\phi_{\Lambda,\xi'}$ is solved by Proposition 3.3. We have the following fact.

Lemma 3.5. *Let ξ' and Λ satisfy (3.4) and (3.5). Then $\mathcal{I}(\Lambda, \xi')$ is of class C^1. Moreover, for all $\varepsilon > 0$ sufficiently small, the function $v(y) = V_{\Lambda,\xi'}(y) + \phi_{\Lambda,\xi'}(y)$ is a solution to problem (3.1) if and only if (Λ, ξ') is a critical point of $\mathcal{I}(\Lambda, \xi')$.*

Proof. As a consequence of Proposition 3.4, we can get the map $(\Lambda, \xi') \mapsto \mathcal{I}(\Lambda, \xi')$ is of class C^1. For $k \in \{1, 2, 3\}$, we have

$$\begin{aligned}\partial_{\xi'_k}\mathcal{I}\big(\Lambda, \xi'\big) &= DI(V_{\Lambda,\xi'} + \phi_{\Lambda,\xi'})\left[\frac{\partial V_{\Lambda,\xi'}}{\partial \xi'_k} + \frac{\partial \phi_{\Lambda,\xi'}}{\partial \xi'_k}\right] \\ &= \sum_{i=0}^{3} c_i \int_{\Omega_\varepsilon} w^4_{\Lambda,\xi'} Z_i\left[\frac{\partial V_{\Lambda,\xi'}}{\partial \xi'_k} + \frac{\partial \phi_{\Lambda,\xi'}}{\partial \xi'_k}\right] = \sum_{i=0}^{3} c_i \int_{\Omega_\varepsilon} w^4_{\Lambda,\xi'} Z_i Z_k\big(1 + o(1)\big),\end{aligned}$$

here we use the fact that $\|\partial_{\xi'_k}\phi_{\Lambda,\xi'}\|_* = O(\varepsilon)$. Similarly, we find

$$\partial_\Lambda \mathcal{I}\big(\Lambda, \xi'\big) = \sum_{i=0}^{3} c_i \int_{\Omega_\varepsilon} w^4_{\Lambda,\xi'} Z_i Z_0\big(1 + o(1)\big),$$

where $o(1) \to 0$ as $\varepsilon \to 0$ uniformly for the norm $\|\cdot\|_*$. It defines an almost diagonal linear equation system for c_i. Thus (Λ, ξ') is a critical point of $\mathcal{I}(\Lambda, \xi')$ if and only if $c_i = 0$ for $i = 0, 1, 2, 3$. This ends the proof of lemma. □

Lemma 3.6. *As $\varepsilon \to 0$, we have the following expansion*

$$\mathcal{I}\big(\Lambda, \xi'\big) - I(V_{\Lambda,\xi'}) = o(\varepsilon),$$

where $o(\varepsilon)$ is in the C^1- sense uniformly on ξ', Λ satisfying (3.4)and (3.5).

Proof. For notation simplicity, we write $V_{\Lambda,\xi'}$ by V, and $\phi_{\Lambda,\xi'}$ by ϕ. By the Taylor expansion and the fact that $DI(V_{\Lambda,\xi'} + \phi_{\Lambda,\xi'})[\phi] = 0$, we have

$$\begin{aligned}\mathcal{I}\big(\Lambda, \xi'\big) - I(V_{\Lambda,\xi'}) = I(V+\phi) - I(V) &= \int_0^1 D^2 I(V+t\phi)[\phi,\phi]t\,dt \\ &\le C\int_{\Omega_\varepsilon}\big|V^{p-1} - (V+\phi)^{p-1}\big|\phi^2\,dy + C\lambda\varepsilon^\alpha\int_{\Omega_\varepsilon}\big|V^{q-1} - (V+\phi)^{q-1}\big|\phi^2\,dy \\ &\quad + \int_{\Omega_\varepsilon}|R||\phi|\,dy + \int_{\Omega_\varepsilon}\big|N(\phi)\big||\phi|\,dy,\end{aligned}$$

and since $\|R\|_{**} \le C\varepsilon$, $\|N(\phi)\|_{**} \le C\varepsilon^\tau\|\phi\|_*$ and $\|\phi\|_* \le C\varepsilon$, we get

$$\mathcal{I}\big(\Lambda, \xi'\big) - I(V) = o(\varepsilon),$$

where $o(\varepsilon)$ is uniform in the C^1-sense for ξ', Λ satisfying (3.4), (3.5). By a similar way, we can obtain

$$D_{(\Lambda,\xi')}\big(\mathcal{I}\big(\Lambda, \xi'\big) - I(V)\big) = o(\varepsilon).$$

This ends the proof of lemma. □

Proof of Theorem 1.1. We first prove (i). We follow the proof in [3], where the case $p = 5$ is considered. Taking ε_0 fix and small, we can rewrite the proof of Corollary 2.4 in [3] for variable p in the range $[5-\varepsilon_0, 5]$ and clearly choose a λ_0 that depends only on ε_0, q and Ω. Then [3, Theorem 2.1], also holds for $p \in [5-\varepsilon_0, 5]$, and existence of a solution u_1 of problem (1.2), follows for all $\lambda \ge \lambda_0$. In [3, Remark 2.2], they prove that for this problem, we have $J(u_1) < \frac{\sqrt{3}}{4}\pi^2$.

Now we prove (ii). By Lemma 3.5, we know that $u(\varepsilon y) = \varepsilon^{-\kappa}(V_{\Lambda,\xi'}(y) + \phi_{\Lambda,\xi'}(y))$ is a solution to problem (1.2) if and only if (Λ, ξ') is a critical point of $\mathcal{I}(\Lambda, \xi')$. So we have to show existence of the critical point of $\mathcal{I}(\Lambda, \xi')$.

It is easy to check that

$$I(V_{\Lambda,\xi'}) = \varepsilon^{2\kappa-1}J\big(\varepsilon^{\frac{1}{2}-\kappa}U\big) = J(U) + o(\varepsilon), \tag{3.49}$$

since $2\kappa - 1 = O(\varepsilon)$. This together with Lemma 3.6 and Lemma 2.2, and recalling that $\mu = \Lambda\varepsilon$, we have that for $1 < q < 3$,

$$\mathcal{I}\big(\Lambda, \xi'\big) = a_0 + \varepsilon\varphi(\Lambda, \xi) - a_2\varepsilon\log\varepsilon + a_3\varepsilon + o(\varepsilon), \tag{3.50}$$

where

$$\varphi(\Lambda, \xi) = a_1\Lambda R(\xi) - a_2\log\Lambda,$$

with constants $a_1, a_2 > 0$ being given in Lemma 2.2, and $o(\varepsilon)$ is uniform in the C^1- sense for ξ', Λ in the considered region.

Define

$$\widetilde{\mathcal{I}}(\Lambda, \xi') = \frac{1}{\varepsilon}\mathcal{I}(\Lambda, \xi') - \frac{a_0}{\varepsilon} - a_3.$$

Then we have

$$\widetilde{\mathcal{I}}(\Lambda, \xi') = \varphi(\Lambda, \xi) + o(1), \tag{3.51}$$

where $\xi' = \frac{\xi}{\varepsilon}$ and $o(1)$ is in the C^1- sense uniformly on ξ', Λ satisfying (3.4), (3.5). Since the function $R(\xi)$ has at least one critical point, denoted by ξ_0, with $R(\xi_0) > 0$, then (Λ_0, ξ_0), with $\Lambda_0 = \frac{a_2}{a_1 R(\xi_0)}$, is a nondegenerate critical point of $\varphi(\Lambda, \xi)$. It follows that the local degree $\deg(\nabla\varphi(\Lambda, \xi), \mathcal{O}, 0)$ is well defined and is nonzero, where $\mathcal{O}$ is arbitrary small neighborhood of (Λ_0, ξ_0). So $\deg(\nabla\widetilde{\mathcal{I}}(\Lambda, \xi'), \mathcal{O}, 0) \neq 0$ for $\varepsilon > 0$ small enough. Hence we find a critical point (Λ_*, ξ'_*) of $\widetilde{\mathcal{I}}(\Lambda, \xi')$, such that $(\Lambda_*, \xi'_*) \to (\Lambda_0, \xi'_0)$ with $\xi'_0 = \frac{\xi_0}{\varepsilon}$ as $\varepsilon \to 0$. Then (Λ_*, ξ'_*) is also a critical point of $\mathcal{I}(\Lambda, \xi')$. Thus we get that

$$u_2(x) = \varepsilon^{-\kappa}(V_{\Lambda_*, \xi'_*} + \phi_{\Lambda_*, \xi'_*})\left(\frac{x}{\varepsilon}\right)$$

is the solution of problem (1.2). Recall that $\kappa = \frac{2}{p-1} = \frac{1}{2} + \frac{1}{8}\varepsilon + o(\varepsilon)$, then by above construction and Lemma 2.2, we can get (1.6) and (1.7). □

4. Proof of Theorem 1.2

Proof of Theorem 1.2. For $2 < q < 3$, taking $\lambda = \bar{\lambda}\varepsilon^{-\frac{3-q}{2}}$ and $\mu = \Lambda\varepsilon$ in the energy expansion (2.5), we have

$$J(U) = a_0 + \varepsilon\tilde{\Psi}(\Lambda, \xi) - a_2\varepsilon \log \varepsilon + a_3\varepsilon + o(\varepsilon), \tag{4.1}$$

where

$$\tilde{\Psi}(\Lambda, \xi) = a_1 \Lambda R(\xi) - a_2 \log \Lambda - \bar{\lambda} a_4 \Lambda^{\frac{5-q}{2}}.$$

First, $\xi \mapsto \tilde{\Psi}(\Lambda, \xi)$ has a minimum point ξ_0, that is $\partial_\xi \tilde{\Psi}(\Lambda, \xi)|_{\xi=\xi_0} = 0$. On the other hand, the function

$$\Lambda \mapsto \varphi(\Lambda) := \tilde{\Psi}(\Lambda, \xi_0) = a_1 \Lambda R(\xi_0) - a_2 \log \Lambda - \bar{\lambda} a_4 \Lambda^{\frac{5-q}{2}}$$

has two non-degenerate critical points $\Lambda_- < \Lambda_+$ for each $0 < \bar{\lambda} < \bar{\lambda}_0$. In fact

$$\varphi'(\Lambda) = a_1 R(\xi_0) - a_2 \Lambda^{-1} - \bar{\lambda}\frac{5-q}{2} a_4 \Lambda^{\frac{3-q}{2}}$$

and this can be written,

$$\varphi'(\Lambda) = \frac{5-q}{2} a_4 \Lambda^{\frac{3-q}{2}} \left[f(\Lambda) - \bar{\lambda}\right], \quad \text{where } f(\Lambda) = \frac{2}{5-q}\frac{a_1}{a_4} R(\xi_0) \Lambda^{-\frac{3-q}{2}} - \frac{2}{5-q}\frac{a_2}{a_4}\Lambda^{-\frac{5-q}{2}}$$

is negative for small and large Λ. The function $\varphi'(\Lambda)$ takes positive values if and only if $0 < \bar{\lambda} < \bar{\lambda}_0$ where $\bar{\lambda}_0 = \max_{\Lambda>0} f(\Lambda)$. In this case, the equation $\varphi'(\Lambda) = 0$ has two positive solutions $\Lambda_\pm(\bar{\lambda})$.

Finally, if we define $b = \frac{2}{5-q}\frac{a_2}{a_4}$ and use the expressions of a_1, a_2, we obtain that $f(\Lambda) = 128R(\xi_0)b\Lambda^{-\frac{3-q}{2}} - b\Lambda^{-\frac{5-q}{2}}$.

Note also that $\tilde{\Psi}(\Lambda_-, \xi_0) < \tilde{\Psi}(\Lambda_+, \xi_0)$ and so $J(u_-) < J(u_+)$ for sufficiently small $\varepsilon > 0$. □

Proof of Corollary 1.1. Choose any ε_0 small. By Theorem 1.1 (i), there exists λ_0, such that for any $\lambda > \lambda_0$ there exists a solution u_1 for any $\varepsilon \in [0, \varepsilon_0]$. Now for any fix $\hat{\lambda} > \lambda_0$ there exists a solution u_2 for $\varepsilon \in (0, \varepsilon_1)$. Then for this $\hat{\lambda}$ there are two solutions u_1 and u_2 of (1.2) for any $\varepsilon \in (0, \min\{\varepsilon_1, \varepsilon_0\})$. □

Proof of Corollary 1.2. Choose any ε_0 small. By Theorem 1.1 (i), there exists λ_0, such that for all $\lambda > \lambda_0$ there is a solution u_1 for any $\varepsilon \in [0, \varepsilon_0]$. By Theorem 1.2, choosing $\bar{\lambda} \in (0, \overline{\lambda_1})$ and $\varepsilon < \varepsilon_2 \le \tilde{\varepsilon}$ such that $\lambda = \bar{\lambda}\varepsilon^{-\frac{3-q}{2}} > \lambda_0$, then there exist two solutions u_- and u_+ for any $\varepsilon \in (0, \varepsilon_2)$. Then for $\varepsilon \in (0, \min\{\varepsilon_2, \varepsilon_0\})$, there are three solutions u_1 and u_- and u_+ of (1.2), with the desired energy properties. □

Remark 4.1 (*The case* $q = 2$). In this case, the expansion of the energy is given by formula (2.6). We can rewrite the proof of Theorem 1.2 using this expansion with $\lambda = \bar{\lambda}\varepsilon^{-\frac{1}{2}}|\log\varepsilon|^{-1}$ and $\mu = \Lambda\varepsilon$. Then, we obtain

$$J(U) = a_0 - a_2\varepsilon\log\varepsilon + \varepsilon\tilde{\Psi}(\Lambda, \xi) + a_3\varepsilon + o(\varepsilon) \tag{4.2}$$

where

$$\tilde{\Psi}(\Lambda, \xi) = a_1\Lambda R(\xi) - a_2\log\Lambda - \bar{\lambda}a_5\Lambda^{\frac{3}{2}}.$$

We note that this expansion is similar to (4.1), that is, changing a_4 for a_5, and so Theorem 1.2 follows.

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