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Blowup and self-similar solutions for two-component drift–diffusion systems

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ABSTRACT

We discuss asymptotic properties of solutions of two-component parabolic drift–diffusion systems coupled through an elliptic equation in two space dimensions. In particular, conditions for finite time blowup versus the existence of forward self-similar solutions are studied.

1. Introduction

Drift–diffusion systems are widely used for the modeling of various phenomena in continuum mechanics and biology. For instance, the system of evolution equations describing the interaction of charged particles in the mean field approximation, known as the Nernst–Planck–Debye–Hückel system (see (8) below), is used in plasma physics, electrolyte theory as well as semiconductor modeling [1,2]. Two-component generalizations of the classical Keller–Segel system in chemotaxis theory (e.g. [3]) are used to describe two species interacting via a diffused sensitivity agent which can be either a chemoattractant or a chemorepellent for each of the populations. The interaction of massive particles of two kinds through the gravitational potential generated by themselves can also be modeled by such a mean field model; see (7) below [4,5]. Also, the system (9) below has been introduced in [6]; cf. [7,5].

A general class of multicomponent systems of many species with the densities u_i, $i = 1, \dots, k$, interacting via several sensitivity agents of densities v_j, $j = 1, \dots, \ell$, has been studied in [8]. These populations may also collaborate or compete with each other. Wolansky proposed the following system of parabolic equations:

$$\nu_i \frac{\partial}{\partial t} u_i = \Delta u_i - \sum_j \vartheta_{ij} \nabla \cdot (u_i \nabla v_j), \tag{1}$$

$$\sigma_j \frac{\partial}{\partial t} v_j = \Delta v_j - \alpha v_j + \sum_j \gamma_{ij} u_i + f_j, \tag{2}$$

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describing the diffusion of species, their interactions, and the production and diffusion of sensitivity agents. In the case of bounded domains, the system (1)–(2) is to be supplemented with appropriate boundary conditions (either Neumann or no-flux for u_i's, and either Neumann or free for v_j's). He showed that under an algebraic condition of the absence of conflicts between species, the system (1)–(2) considered in two space dimensions has a variational structure, and the steady states can be studied using generalizations of the Moser–Trudinger inequality; see [9,8].

We will consider two-component systems with $i = 1, 2$, interacting through one sensitivity agent, $j = 1$, which diffuses instantaneously: $\sigma_j = 0$, so (2) is an elliptic equation. Our analysis will be carried out in two space dimensions, in the whole space $\mathbb{R}^2$, which is critical for a balance of diffusion and drift properties. In particular, due to the scaling properties, blowup conditions are expected to be expressed in terms of critical masses (or charges), i.e. the L^1 norms of u_i. Moreover, the existence of integrable self-similar solutions can be proved in some range of masses. Thus, our object of study is the parabolic–elliptic system

$$\frac{\partial}{\partial t}u_1 = \nabla \cdot (\nabla u_1 + \vartheta_1 u_1 \nabla v), \tag{3}$$

$$\frac{\partial}{\partial t}u_2 = \nabla \cdot (\nabla u_2 + \vartheta_2 u_2 \nabla v), \tag{4}$$

$$-\Delta v = \gamma_1 u_1 + \gamma_2 u_2, \tag{5}$$

where $u_1, u_2 \geq 0$ are densities of two species, and the masses (or charges) are

$$\int_{\mathbb{R}^2} u_i \, dx = M_i, \quad i = 1, 2. \tag{6}$$

Here, for simplicity, $\vartheta_i, \gamma_i \in \{-1, 1\}$ but we may consider more general cases $\vartheta_i, \gamma_i \in \mathbb{R} \setminus \{0\}$ in essentially the same way.

However, if some of ϑ_i, γ_i are allowed to vanish, the classical single-component Keller–Segel system [3,10,11] arises for

$$\vartheta_1 = -1, \qquad \vartheta_2 = 0, \qquad \gamma_1 = 1, \qquad \gamma_2 = 0$$

(note that the equation for u_2 decouples), while

$$\vartheta_1 = 1, \qquad \vartheta_2 = 0, \qquad \gamma_1 = 1, \qquad \gamma_2 = 0$$

corresponds to a model of a cloud of identically charged particles [1]. Here $\vartheta_i\gamma_i < 0$ means that the particles of the ith species attract each other, while $\vartheta_i\gamma_i > 0$ signifies their mutual repulsion. Finally, $(\vartheta_1\gamma_1)(\vartheta_2\gamma_2) < 0$ results in a conflict of interest; cf. [8].

A priori, there are sixteen possible choices of signs for ϑ_i, γ_i. Clearly, exchanging the variables u_1, u_2 results in equivalent systems. Similarly, changing simultaneously the sign of all the parameters $\vartheta_1, \vartheta_2, \gamma_1, \gamma_2$ also gives the equivalent systems, denoted below by $\sim$.

Proceeding in this way we obtain the following list of the sixteen quadruples $\langle \vartheta_1, \vartheta_2, \gamma_1, \gamma_2 \rangle$ corresponding to six genuinely different systems, four of them without conflicts of interest, called here

$$\text{gravitational: } \langle -1, -1, 1, 1 \rangle \sim \langle 1, 1, -1, -1 \rangle, \tag{7}$$

$$\text{electric: } \langle 1, -1, 1, -1 \rangle \sim \langle -1, 1, -1, 1 \rangle, \tag{8}$$

$$\text{K–O}: \ \langle -1, 1, 1, -1 \rangle \sim \langle 1, -1, -1, 1 \rangle, \tag{9}$$

$$\text{repulsive: } \langle -1, -1, -1, -1 \rangle \sim \langle 1, 1, 1, 1 \rangle, \tag{10}$$

and also two kinds of systems with conflicts of interest:

$$\text{mixed: } \langle -1, -1, 1, -1 \rangle \sim \langle -1, -1, -1, 1 \rangle \sim \langle 1, 1, -1, 1 \rangle \sim \langle 1, 1, 1, -1 \rangle, \tag{11}$$

$$\text{uniform: } \langle -1, 1, 1, 1 \rangle \sim \langle 1, -1, 1, 1 \rangle \sim \langle 1, -1, -1, -1 \rangle \sim \langle -1, 1, -1, -1 \rangle. \tag{12}$$

The names that we coined for them are related to the type of interaction potential generated by the different components in the system. The system (7) corresponds to particles of two different masses that attract each other through the Newtonian potential. The system (8) describes particles of opposite charges interacting through the Coulombic potential. The system (9) is similar to (8) but the potential generated by the particles acts in the opposite direction compared to (8). The system (10) corresponds to particles of two different kinds that repel each other. The systems (8), (9) are well known, and are called Debye–Hückel and Kurokiba–Ogawa systems, respectively. For short, we call them simply electric and K–O systems, respectively.

2. Blowup of solutions

In this section we formulate simple algebraic criteria in terms of either masses or charges M_1, M_1 which are sufficient for a finite time blowup of nonnegative solutions of the Cauchy problem for the systems without conflicts of interest, with prescribed initial values $u_1(\cdot, 0)$ and $u_2(\cdot, 0)$ that satisfy (6). Formally, mass conservation and positivity preserving properties are satisfied for (3)–(5). We refer the readers to [6,7,1,12] for the rigorous proofs of those properties for the systems (8) and (9), respectively. These proofs extend easily to the general situation.

We note that in some systems, solutions may blow up for suitably chosen initial data that do not even satisfy such a general sufficient condition in terms of masses; see [13, Theorems 5 and 6].

Lemma 2.1. *Suppose that $\langle u_1, u_2\rangle$ is a solution of (3)–(5) with $\vartheta_1\gamma_2 = \vartheta_2\gamma_1$ (hence without conflicts of interest), with the initial data $0 \le u_1(\cdot, 0), u_2(\cdot, 0) \in L^1(\mathbb{R}^2, (1 + |x|^2)\,\mathrm{d}x)$, and that the terms*

$$m_i(t) = \int_{\mathbb{R}^2} |x|^2 u_i(x, t)\,\mathrm{d}x, \quad i = 1, 2, \tag{13}$$

denote the second moments of the measures $u_i\,\mathrm{d}x$. Then the following differential equation holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_1(t) + m_2(t)) = 4(M_1 + M_2) + \frac{1}{2\pi}\left(\vartheta_1\gamma_1 M_1^2 + (\vartheta_1\gamma_2 + \vartheta_2\gamma_1)M_1M_2 + \vartheta_2\gamma_2 M_2^2\right). \tag{14}$$

Proof. Multiplying (3) and (4) by $|x|^2$, and summing them, we get after integration by parts

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(m_1(t) + m_2(t)) &= -2\int(\nabla u_1 + \nabla u_2)\cdot x\,\mathrm{d}x - 2\int(\vartheta_1 u_1(x,t) + \vartheta_2 u_2(x,t))\nabla v(x,t)\cdot x\,\mathrm{d}x \\ &= 4(M_1 + M_2) - 2\iint(\vartheta_1 u_1(x,t) + \vartheta_2 u_2(x,t)) \times (\gamma_1 u_1(y,t) \\ &\quad + \gamma_2 u_2(y,t))\frac{-1}{2\pi}\frac{(x-y)\cdot x}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

since $E(z) = -\frac{1}{2\pi}\log|z|$ is the fundamental solution of $-\Delta$ in $\mathbb{R}^2$. Next, we symmetrize the double integral, exchanging the variables x and y to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(m_1(t) + m_2(t)) &= 4(M_1 + M_2) + \frac{\vartheta_1\gamma_1}{2\pi}\iint u_1(x,t)u_1(y,t)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \frac{\vartheta_1\gamma_2 + \vartheta_2\gamma_1}{2\pi}\iint u_1(x,t)u_2(y,t)\,\mathrm{d}x\,\mathrm{d}y + \frac{\vartheta_2\gamma_2}{2\pi}\iint u_2(x,t)u_2(y,t)\,\mathrm{d}x\,\mathrm{d}y \\ &= 4(M_1 + M_2) + \frac{1}{2\pi}\left(\vartheta_1\gamma_1 M_1^2 + (\vartheta_1\gamma_2 + \vartheta_2\gamma_1)M_1M_2 + \vartheta_2\gamma_2 M_2^2\right);\end{aligned}$$

note that $(x - y)\cdot x + (y - x)\cdot y = |x - y|^2$ and $\vartheta_1\gamma_2 = \vartheta_2\gamma_1$. □

Remark. In the above simple *virial* computation (similar to that in [14]) moments of order 2 can be replaced (in any dimension) by moments of lower order, as was done in [12], and the assumption on the existence of suitable moments can be removed by taking a cutoff of the weight function $|x|^2$ in the two-dimensional case; see [6]. Blowup phenomena for systems (7) and (9) in the higher dimensional case are studied in [15].

Proposition 2.2. *If the initial data satisfy the condition*

$$8\pi(M_1 + M_2) + \vartheta_1\gamma_1 M_1^2 + (\vartheta_1\gamma_2 + \vartheta_2\gamma_1)M_1M_2 + \vartheta_2\gamma_2 M_2^2 < 0, \tag{15}$$

then any nonnegative solution $\langle u_1, u_2\rangle$ of the Cauchy problem for (3)–(5) with $\vartheta_1\gamma_2 = \vartheta_2\gamma_1$ (and hence without conflicts of interest) cannot be global in time.

Proof. Indeed, under the assumption (15), by Lemma 2.1 the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_1(t) + m_2(t)) \le -\varepsilon$$

holds with a strictly positive ε, so the sum of moments of nonnegative functions u_1, u_2 becomes negative in a finite time. This is a contradiction with the existence of globally defined nonnegative solutions of (3)–(5). □

Corollary 2.3. *For the gravitational system (7), condition (15) becomes*

$$8\pi < M_1 + M_2. \tag{16}$$

For the K–O system (9) the sufficient condition for a blowup reads (cf. [6])

$$8\pi < \frac{(M_1 - M_2)^2}{M_1 + M_2}. \tag{17}$$

For the electric (8) and repulsive (10) systems, the sufficient blowup condition (15) cannot be satisfied for any nonnegative initial data.

It is interesting to compare the (sufficient) conditions for blowup with (necessary) conditions for the global in time existence of solutions. We aim at doing this for special scale invariant solutions in the next section, and we will show that these conditions are in a sense complementary. For arbitrary solutions the questions of global in time existence for various single-component and two-component systems have been studied in, e.g., [11] (for the Keller–Segel system), [6,7] (K–O), [5,4] (gravitational and K–O), [1] (single Debye), and [16] (a version of gravitational system).

3. Self-similar solutions

It is easy to see that the scaling $u_i(x,t) \mapsto \lambda^2 u_i(\lambda x, \lambda^2 t)$ for each $\lambda > 0$ leaves invariant the system (3)–(5), and preserves the L^1 norms of u_i's in two space dimensions. Thus, it is natural to look for solutions which are scale invariant, i.e. are of the form

$$u_i(x,t) = \frac{1}{t} U_i\left(\frac{x}{\sqrt{t}}\right), \quad i = 1, 2, \qquad v(x,t) = V\left(\frac{x}{\sqrt{t}}\right)$$

for some functions U_i, V of two variables. Moreover, to simplify our considerations we assume that the U_i's are radially symmetric (for some systems like Keller–Segel and Debye–Hückel ones it can be done with no loss of generality; see e.g. [11,2]). It is standard to find that the *cumulated densities*

$$\Phi_i(y) = \frac{1}{2\pi}\int_{B(0,y)} U_i \,\mathrm{d}x, \qquad \Psi(y) = \frac{1}{2\pi}\int_{B(0,y)} V \,\mathrm{d}x$$

(cf. related calculations in [10,17]) satisfy the system of two ordinary differential equations

$$\Phi_1''(y) + \frac{1}{4}\Phi_1' - \frac{\vartheta_1}{2y}\Phi_1'(\gamma_1\Phi_1 + \gamma_2\Phi_2) = 0, \tag{18}$$

$$\Phi_2''(y) + \frac{1}{4}\Phi_2' - \frac{\vartheta_2}{2y}\Phi_2'(\gamma_1\Phi_1 + \gamma_2\Phi_2) = 0, \tag{19}$$

with $' = \frac{\mathrm{d}}{\mathrm{d}y}$; note that Ψ is eliminated from the system, resulting in (18)–(19).

The self-similar solutions can be interpreted as solutions of the Cauchy problem issuing from the initial data $u_i(\cdot,0) = M_i\delta_0$, $i = 1, 2$, if, of course, we are able to show the uniqueness of the solutions of the Cauchy problem with such singular initial data. The advantage of that formulation is that we now have a system of two ordinary differential equations of order 2 with singular coefficients, but this is no longer nonlocal. The system (18)–(19) is supplemented with the boundary conditions at 0 and at ∞:

$$\Phi_i(0) = 0, \qquad \Phi_i(\infty) = \frac{M_i}{2\pi}, \quad i = 1, 2. \tag{20}$$

First, it is useful to recall results for a single equation related to each of (18), (19).

Proposition 3.1. (i) *The boundary value problem*

$$\begin{aligned} &Z'' + \frac{1}{4}Z' + \frac{1}{2y}Z'Z = 0, \\ &Z(0) = 0, \qquad Z(\infty) = N > 0, \end{aligned} \tag{21}$$

has a unique solution $Z = Z(y) > 0$ $(y > 0)$ for each $N \in (0, 4)$.

(ii) *Moreover, if $Z(y) \geq 0$ satisfies*

$$Z'' + \frac{1}{4}Z' + \frac{1}{2y}Z'Z \leq 0, \qquad Z(0) = 0, \tag{22}$$

then $\limsup_{y\to\infty} Z(y) < 4$.

(iii) *The boundary value problem*

$$\begin{aligned} &W'' + \frac{1}{4}W' - \frac{1}{2y}W'W = 0, \\ &W(0) = 0, \qquad W(\infty) = N > 0, \end{aligned} \tag{23}$$

has a unique solution for each $N > 0$. Moreover, $0 \leq W'(0) < \frac{1}{2}$ holds for any solution $W \geq 0$ which exists for all $y \geq 0$.

Proof. (i) The problem (21) has been studied in connection with the existence of self-similar solutions for the Keller–Segel model in [18]. We recall some properties of (21) important for further applications of Proposition 3.1 while the detailed proofs can be inferred from [18, Section 4].

First, note that $Z(\infty) > 0$ implies $Z(y) > 0$ for each $y > 0$ and $Z'(0) > 0$. Despite the coefficient $\frac{1}{y}$ being singular at $y = 0$ in (21), this equation enjoys the uniqueness of solutions for the Cauchy problem, i.e. with the conditions $Z(y_0) = 0$, $Z'(y_0) = z_0 > 0$ for each fixed $y_0 \geq 0$; cf. [18, p. 1579]. Moreover, $Z'(0) \in (0, \infty)$ and $Z(\infty) \in (0, 4)$ are in bijective correspondence, i.e. different values of $Z'(0)$ correspond to different values of $Z(\infty)$, and for each $N \in (0, 4)$ there is a unique value of the initial slope $Z'(0) > 0$ such that $Z(\infty) = N$.

(ii) The inequality (22) with $Z \geq 0, Z' \geq 0$ leads to

$$(4yZ')' - 4Z' + yZ' + 2Z'Z \leq 0.$$

This, after an integration over $(0, \infty)$, results in

$$-4Z(\infty) + \int_0^\infty yZ'(y)\,\mathrm{d}y + Z^2(\infty) \leq 0,$$

so $Z(\infty) < 4$. We used above the monotonicity of $Z' \geq 0$ and its integrability on $(0, \infty)$ leading to $\lim_{y\to\infty} yZ'(y) = 0$. In fact, $(N - Z(y))$ and $yZ'(y)$ decay exponentially to 0 as $y \to \infty$; see Lemma 4.1 in [18] and its proof.

(iii) The system (23) corresponds to the single-component electric system studied in, e.g., [2,19]. The crucial observation is that: if $W'(0) < \frac{1}{2}$ then W is concave. Then, from

$$\frac{W''(y)}{W'(y)} = \frac{1}{2}\left(\frac{W(y)}{y} - \frac{1}{2}\right)$$

one infers that $\lim_{y\to\infty} W(y)$ exists. The existence of solutions is established via the Banach contraction fixed point theorem for

$$W(y) = 4W'(0)\left(1 - \mathrm{e}^{-y/4}\right) + \frac{1}{2}\int_0^y \left(\mathrm{e}^{-t/4}\int_0^t \mathrm{e}^{s/4}s^{-1}W(s)W'(s)\,\mathrm{d}s\right)\mathrm{d}t.$$

Here, again $(N - W(y))$ and $yW'(y)$ decay to 0 exponentially as $y \to \infty$: cf. [19, Lemma 4.2]. □

A similar approach can be applied to the system (18)–(19) (either a construction of solutions via the shooting method with the parameters $a_i = \Phi'(0)$ or a direct analysis of the Cauchy problem; cf. [18,17] for related systems) but, of course, the range of parameters M_1, M_2 for which there exist self-similar solutions is not *a priori* known.

Remark. An obvious observation (which applies to the electric, K–O and also mixed systems) is that:

If $\gamma_1\gamma_2 < 0$, then for each $M_1 = M_2 = M > 0$ there is a self-similar solution with

$$\Phi_1(y) = \Phi_2(y) = \frac{M}{2\pi}\left(1 - \mathrm{e}^{-y/4}\right)$$

describing two species that do not interact on average, i.e. in the mean field approximation used in those models.

The following proposition gives some necessary conditions for the existence of self-similar solutions with prescribed masses.

Proposition 3.2. *If $\langle U, V\rangle$ is a self-similar solution of the system (3)–(5) and $\vartheta_1\gamma_2 = \vartheta_2\gamma_1$ (and hence there is no conflict of interest in the system), then*

$$8\pi(M_1 + M_2) + \vartheta_1\gamma_1 M_1^2 + \vartheta_2\gamma_2 M_2^2 + (\vartheta_1\gamma_2 + \vartheta_2\gamma_1)M_1M_2 > 0. \tag{24}$$

Remark. The condition (24) – necessary for the existence of a radial global in time self-similar solution – is complementary to (15) – sufficient for the blowup.

Proof. The system (18)–(19) is equivalent to

$$4y\Phi_1'' + y\Phi_1' - 2\vartheta_1\Phi_1'(\gamma_1\Phi_1 + \gamma_2\Phi_2) = 0, \tag{25}$$

$$4y\Phi_2'' + y\Phi_2' - 2\vartheta_2\Phi_2'(\gamma_1\Phi_1 + \gamma_2\Phi_2) = 0, \tag{26}$$

which, in turn, implies

$$4y(\Phi_1 + \Phi_2)'' + y(\Phi_1 + \Phi_2)' - 2(\vartheta_1\Phi_1' + \vartheta_2\Phi_2')(\gamma_1\Phi_1 + \gamma_2\Phi_2) = 0 \tag{27}$$

with the boundary conditions

$$(\Phi_1+\Phi_2)(0)=0, \qquad (\Phi_1+\Phi_2)(\infty)=\frac{1}{2\pi}(M_1+M_2) \tag{28}$$

following from (20). Integrating (27) over $[0,\infty)$ we formally get

$$4y(\Phi_1+\Phi_2)'\,|_0^\infty -4\int_0^\infty(\Phi_1+\Phi_2)'\,\mathrm{d}y + y\left((\Phi_1+\Phi_2)-\frac{1}{2\pi}(M_1+M_2)\right)\Big|_0^\infty$$
$$+\int_0^\infty\left(\frac{1}{2\pi}(M_1+M_2)-(\Phi_1+\Phi_2)\right)\mathrm{d}y-\vartheta_1\gamma_1\Phi_1^2\,|_0^\infty-\vartheta_2\gamma_2\Phi_2^2\,|_0^\infty-2\vartheta_1\gamma_2\int_0^\infty\Phi_1'\Phi_2\,\mathrm{d}y$$
$$-2\vartheta_2\gamma_1\int_0^\infty\Phi_1\Phi_2'\,\mathrm{d}y=0.$$

Observe that $\int_0^\infty(\Phi_1+\Phi_2)'\,\mathrm{d}y=\frac{1}{2\pi}(M_1+M_2)$, $\int_0^\infty(\Phi_1'\Phi_2+\Phi_1\Phi_2')\,\mathrm{d}y=\frac{1}{4\pi^2}M_1M_2$.

Since the quantities $\int_0^\infty\left(\frac{1}{2\pi}M_1-\Phi_i\right)\mathrm{d}y$ – equivalent to the second moments of the densities U_i – are positive, we arrive at the conclusion (24) if, of course, $\vartheta_1\gamma_2=\vartheta_2\gamma_1$. To justify this formal calculation let us observe that for nondecreasing solutions Φ_1, Φ_2 of (25)–(26) which satisfy (28) we have

$$\int_0^\infty\left(\frac{M_i}{2\pi}-\Phi_i(y)\right)\mathrm{d}y=\int_0^\infty y\Phi_i'(y)\,\mathrm{d}y<\infty$$

(the moments of the Φ_i's are finite and $y\Phi_i'(y)\to 0$ as $y\to\infty$). This is a specific property of self-similar solutions because even for the Keller–Segel system there exist (radially symmetric steady state) solutions with infinite second moment such that $y\Phi'(y)$ does not decay to 0.

Indeed, from (27) we infer that $Z=\Phi_1+\Phi_2$ satisfies $Z'\ge 0$ and

$$\left|4yZ''+yZ'\right|\le 2Z'Z.$$

Therefore, $|4y\log Z'+y|\le\frac{1}{4\pi}(M_1+M_2)|\log y|+C$, and

$$0\le Z'(y)\mathrm{e}^{y/4}\le C\,(y|\log y|+y+1)$$

so $Z'(y)$ and $\left(\frac{1}{4\pi}(M_1+M_2)-Z(y)\right)$ decay to 0 exponentially as $y\to\infty$. □

Corollary 3.3. *For the gravitational system (7) the condition $M_1+M_2<8\pi$ is a necessary and sufficient condition for the existence of self-similar solutions.*

Proof. The necessary condition (24) reads $8\pi(M_1+M_2)>(M_1+M_2)^2$. As we have already computed in the proof of Proposition 3.2, $Z=\Phi_1+\Phi_2$ satisfies (21) which proves that $M_1+M_2<8\pi$ is the optimal range of existence of self-similar solutions. Having Z, we see that the *linear* equation for $\Phi_1\Phi_1''+\frac{1}{4}\Phi_1'+\frac{1}{2y}\Phi_1'Z=0$ can be uniquely solved with any prescribed $0\le\Phi_1(\infty)<Z(\infty)$. Finally, $\Phi_2=Z-\Phi_1$. □

Remark. It can be shown using the constructions in [11] that radial self-similar solutions with $M_1>0$, $M_2>0$ and $M_1+M_2<8\pi$ majorize radial rearrangements (of suitable translations) of solutions of the Cauchy problem with $u_1(\cdot,0)$, $u_2(\cdot,0)$ of mass M_1, M_2, respectively. Thus, the global in time solutions exist if and only if $M_1+M_2<8\pi$; cf. also [16] for a related problem. Of course, the classical result for the one-component Keller–Segel model is a particular case of the above with $M_2\equiv 0$.

For Keller–Segel [11,18] and electric systems ([19,20,2] in higher dimensions and [1] in bounded two-dimensional domains) our conditions for blowup and for the existence of self-similar solutions are complementary. Moreover, in the case of the whole space $\mathbb{R}^2$, self-similar solutions determine the (intermediate) asymptotics of global radially symmetric solutions which, in turn, control the global in time existence of general solutions. We expect a similar property to hold in certain cases here, except for K–O systems, as the results in [13,4] show.

Proposition 3.4. *The necessary condition for the existence of self-similar solutions for the K–O system (9) reads $|M_1-M_2|<8\pi$.*

Proof. The condition (24) gives $(M_1-M_2)^2<8\pi(M_1+M_2)$. The more restrictive condition $|M_1-M_2|<8\pi$ is a consequence of the following reasoning. Suppose that $\Phi_1'(0)>\Phi_2'(0)$; then $Z=\Phi_1-\Phi_2$ satisfies $Z(0)=0$, $Z'(0)>0$ and $Z''+\frac{1}{4}Z'+\frac{1}{2y}Z(\Phi_1'+\Phi_2')=0$ with $Z>0$, $Z'\le\Phi_1'+\Phi_2'$. By Proposition 3.1(ii) we have $\lim_{y\to\infty}Z(y)<4$ which means that $M_1-M_2<8\pi$. Similarly, if $\Phi_2'(0)>\Phi_1'(0)$, then $Z=\Phi_2-\Phi_1$ again satisfies $Z(0)=0$, $Z'(0)>0$ and $Z''+\frac{1}{4}Z'+\frac{1}{2y}Z(\Phi_1'+\Phi_2')=0$. This also leads to $Z(\infty)<4$, and thus finishes the proof. □

Proposition 3.5. *For the systems (8) and (10) (for which the condition (24) is not restrictive at all), self-similar solutions exist for each pair of nonnegative $\langle M_1, M_2\rangle$.*

Proof. For the system (10) the function $W = \Phi_1 + \Phi_2$ satisfies Eq. (23), and therefore, the boundary value problem with $W(0) = 0$ and $W(\infty) = \frac{1}{2\pi}(M_1 + M_2)$ has a unique positive solution W by Proposition 3.1(iii). Then, the *linear* equation

$$\Phi_1'' + \frac{1}{4}\Phi_1' - \frac{1}{2y}\Phi_1' W = 0$$

has a solution with $\Phi_1(0) = 0$ and a prescribed value of $\Phi_1(\infty) = \frac{M_1}{2\pi} \le W(\infty)$.

For the system (8) with $M_1 > M_2$ (no loss of generality), the function $W = \Phi_1 - \Phi_2$ satisfies

$$W'' + \frac{1}{4}W' - \frac{1}{2y}\left(\Phi_1' + \Phi_2'\right) W = 0.$$

Thus, by comparison arguments, W with the prescribed $W(\infty) = \frac{1}{2\pi}(M_1 - M_2)$ does exist by Proposition 3.1(iii). Indeed, this is a subsolution of Eq. (23) with prescribed $W(0) = 0$ and $W(\infty)$, since $W' \le \Phi_1' + \Phi_2'$, and so

$$W'' + \frac{1}{4}W' - \frac{1}{2y}W'W \ge 0.$$

Finally, the *linear* equation

$$\Phi_1'' + \frac{1}{4}\Phi_1' - \frac{1}{2y}\Phi_1' W = 0$$

is solved with the conditions $\Phi_1(0) = 0$ and $\Phi_1(\infty) = \frac{M_1}{2\pi}$. The positivity properties of Φ_1', Φ_2' are easy to check. □

Numerical studies of self-similar solutions performed in [20] for the problem (8) suggest some nontrivial restrictions on the range of values of $\Phi_1'(0)$, $\Phi_2'(0)$ for which those special solutions do exist.

Like in Proposition 3.4, we can check that the following result holds true.

Proposition 3.6. *For the mixed systems (11) (i.e. those with conflicts of interest and such that $\vartheta_1\vartheta_2 > 0$ and $\gamma_1\gamma_2 < 0$), the self-similar solutions exist if and only if the one-sided bounds are satisfied:*

(i) *for $\vartheta_1\gamma_1 < 0$: $0 \le M_1 < M_2 + 8\pi$;*
(ii) *for $\vartheta_1\gamma_1 > 0$: $0 \le M_2 < M_1 + 8\pi$.*

Proof. The function $Z = \Phi_1 - \Phi_2$ satisfies the equation

$$Z'' + \frac{1}{4}Z' - \frac{\vartheta_1\gamma_1}{2y}ZZ' = 0.$$

As a consequence, in the case (i), by Proposition 3.1(i) and (iii), such a Z exists if and only if $Z(\infty) \in (-\infty, 4)$.

In the case (ii), similarly, $Z = \Phi_2 - \Phi_1$, which satisfies

$$Z'' + \frac{1}{4}Z' + \frac{\vartheta_1\gamma_1}{2y}ZZ' = 0,$$

exists if and only if $Z(\infty) \in (-\infty, 4)$.

To reconstruct Φ_1 we solve then the linear problem

$$\Phi_1'' + \frac{1}{4}\Phi_1' \pm \frac{1}{2y}\Phi_1' Z = 0$$

with $\Phi_1(0) = 0$ and $\Phi_1(\infty) = \frac{1}{2\pi}M_1$. Finally, we put $\Phi_1 = \Phi_1 \mp Z$, depending on whether we are considering case (i) or (ii). □

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