SIAM J. MATH. ANAL.
Vol. 43, No. 6, pp. 2519–2541

TRIPLE JUNCTION SOLUTIONS FOR A SINGULARLY PERTURBED NEUMANN PROBLEM*

WEIWEI AO†, MONICA MUSSO‡, AND JUNCHENG WEI†

Abstract. We consider the singularly perturbed Neumann problem $\varepsilon^2\Delta u - u + u^p = 0$, $u > 0$ in Ω, $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, where $p > 1$ and Ω is a smooth and bounded domain in $\mathbb{R}^2$. We construct a class of solutions which consist of large number of spikes concentrated on three line segments with a common endpoint which intersect $\partial\Omega$ orthogonally.

Key words. triple junctions, singularly perturbed problems, finite-dimensional reduction

AMS subject classifications. 35J25, 35J20, 35B33, 35B40

DOI. 10.1137/100812100

1. Introduction and statement of main results.

We consider the following singularly perturbed elliptic problem:

$$\varepsilon^2\Delta u - u + u^p = 0\ , \quad u > 0 \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$ with its unit outer normal ν, $p > 1$ and $\varepsilon > 0$ is a small parameter.

Problem (1.1) is known as the stationary equation of the Keller–Segel system in chemotaxes [25]. It can also be viewed as a limiting stationary equation for the Geirer–Meinhardt system in biological pattern formation [16]. Even though simple-looking, problem (1.1) has a rich and interesting structure of solutions. For the last fifteen years, it has received considerable attention. In particular, the various concentration phenomena exhibited by the solutions of (1.1) seem both mathematically intriguing and scientifically useful. We refer the reader to the three survey articles [35, 36, 44] for background and references.

In the pioneering papers [37, 38], Ni and Takagi proved the existence of least energy solutions to (1.1), that is, a solution u_ϵ with minimal energy. Furthermore, they showed in [37, 38] that, for each $\epsilon > 0$ sufficiently small, u_ϵ blows up at a boundary point that maximizes the mean curvature of $\partial\Omega$.

Since the publication of [38], problem (1.1) has received a great deal of attention and significant progress has been made. It has been proved that higher energy solutions exist, which concentrates at one or several points of the boundary, or at one or more points in the interior, or a combination of the two effects. See [4, 5, 8, 9, 10, 11, 12, 13, 17, 19, 20, 21, 23, 24, 26, 39, 42, 43, 45, 46] and the references therein. In particular, Lin, Ni, and Wei [26] showed that there are at least $\frac{C_N}{(\epsilon|\log\epsilon|)^N}$ interior spikes.

*Received by the editors October 18, 2010; accepted for publication (in revised form) August 18, 2011; published electronically November 15, 2011.
http://www.siam.org/journals/sima/43-6/81210.html

†Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong (wwao@math.cuhk.edu.hk, wei@math.cuhk.edu.hk). The research of the third author was supported by a General Research Grant from RGC of Hong Kong.

‡Departamento de Matemática, Pontificia Universidad Catolica de Chile, Avda. Vicuña Mackenna 4860, Macul, Chile (mmusso@math.upl.cl). This author's research was partly supported by Fondecyt grant 1080099, Chile, and CAPDE-Anillo ACT-125.

It seems natural to ask whether problem (1.1) has solutions which "concentrate" on higher-dimensional sets, e.g., curves or surfaces. In this regard, we mention that it has been *conjectured* for a long time that *problem* (1.1) *actually possesses solutions which have m-dimensional concentration sets for every* $0 \leq m \leq N-1$. (See, e.g., [36].) Progress in this direction, although still limited, has also been made in [2, 29, 30, 31, 32, 33]. In particular, we mention the results of Malchiodi and Montenegro [31, 32] on the existence of solutions concentrating on the *whole boundary* provided that the sequence ε satisfies some gap condition. The latter condition is called *resonance*.

In the papers [27, 30, 32], the higher-dimensional concentration set is on the boundary. A natural question is whether there are solutions with higher-dimensional concentration set inside the domain. In this regard, the first result was due to Wei and Yang [47], who proved the existence of layer on the line intersecting with the boundary of a two-dimensional domain orthogonally. In [47] the resonance condition is still required. This result was generalized in [3] to domains of dimensions higher than 2.

By rescaling and taking a limit in (1.1), we obtain the following nonlinear elliptic problem in the whole $\mathbb{R}^N$:

$$\Delta u - u + u^p = 0, \quad u > 0 \text{ in } \mathbb{R}^N. \tag{1.2}$$

Recently several interesting results have been obtained on new entire solutions to

$$\Delta u - u + u^p = 0, \quad u > 0 \text{ in } \mathbb{R}^2. \tag{1.3}$$

Dancer [7] first constructed solutions to (1.3) which decays in one direction and is periodic in another direction. In [14], these periodic solutions are called *Dancer's solutions*. Using Dancer's solutions, del Pino et al. built solutions to (1.3) with an even number of ends whose level sets are governed by the one-dimensional Toda system. On the other hand, in [28] Malchiodi constructed another new class of *entire* solutions to (1.3) by perturbing a configuration of infinitely many copies of the positive solution w arranged along three rays meeting at a common point, where w is the unique radially symmetric solution of

$$\begin{cases} \Delta w - w + w^p = 0, \quad u > 0 \text{ in } \mathbb{R}^2, \\ w(0) = \max_{y \in \mathbb{R}^2} w(y), \quad w \to 0 \text{ at } \infty. \end{cases} \tag{1.4}$$

Malchiodi's solutions are known as triple junction solutions. The question we address in this paper is whether triple junction solutions to (1.1) still exist in a bounded domain.

The answer is yes: indeed, we construct triple junction solutions for problem (1.1) which are obtained as perturbation of a large number of w centered at points arranged along a proper triple junction.

Let us first recall the asymptotic behavior of w, solution to (1.4), at infinity. It is known that there exists a constant $c_p > 0$, depending on p, such that

$$\lim_{r \to \infty} e^r \, r^{\frac{1}{2}} \, w = c_p > 0 \quad \text{and} \quad \lim_{r \to \infty} \frac{w'}{w} = -1\,, \tag{1.5}$$

where we have set $r := |x|$.

Furthermore, the solution w is *nondegenerate*, namely, the L^∞-kernel of the operator

$$L_0 := \Delta - 1 + p\, w^{p-1}\,, \tag{1.6}$$

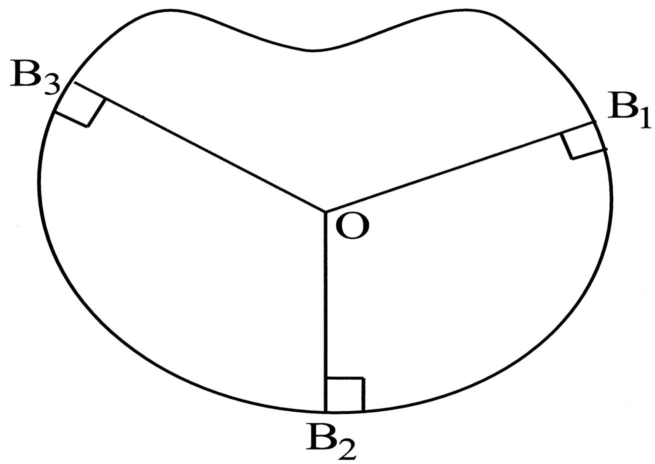

FIG. 1. *Triple junctions.*

which is nothing but the linearized operator about w, is spanned by the functions

$$\partial_{x_1} w, \ldots, \partial_{x_N} w\,, \tag{1.7}$$

which naturally belong to this space. We refer the reader to [22, 37].

Next we describe our result.

Assume that Ω contains three line segments whose origin is the common endpoint which intersect orthogonally the boundary at exactly three points B_1, B_2, B_3 on $\partial\Omega$. (See Figure 1.) We denote by

$$\mathbf{t}_1 = (t_{11}, t_{12}), \quad \mathbf{t}_2 = (t_{21}, t_{22}), \quad \mathbf{t}_3 = (0, -1) \tag{1.8}$$

the unit tangent vectors of the three segments pointing from the origin, and by

$$\mathbf{n}_1 = (t_{12}, -t_{11}), \quad \mathbf{n}_2 = (t_{22}, -t_{21}), \quad \mathbf{n}_3 = (-1, 0), \tag{1.9}$$

respectively, the unit normal vectors. Let $\bar{L}_1$, $\bar{L}_2$, $\bar{L}_3$ be the lengths of the three segments.

We assume that the mutual angles of the three lines satisfy

$$\mathbf{t}_i \angle \mathbf{t}_j > \frac{\pi}{3}. \tag{1.10}$$

Near the endpoints B_1, B_2, and B_3 of the segments, the boundary $\partial\Omega$ is described as

$$x\mathbf{n}_1 + (h_1(x) + \bar{L}_1)\mathbf{t}_1, \quad x\mathbf{n}_2 + (h_2(x) + \bar{L}_2)\mathbf{t}_2, \quad x\mathbf{n}_3 + (h_3(x) + \bar{L}_3)\mathbf{t}_3,$$

respectively, where the functions h_i's are smooth functions defined in intervals which include 0. It is not restrictive to assume that they satisfy $h_i(0) = h_i'(0) = 0$ for all $i = 1, 2, 3$.

Finally, we denote by k_i the curvature of $\partial\Omega$ at the point B_i, namely, $k_i = -h_i''(0)$.

To state our result we need to introduce a function which, as we will show later, measures the interaction between two bumps w centered at two distinct points. We define $\Psi(s)$ to be

$$\Psi(s) := \int_{\mathbb{R}^2} w(x - s\mathbf{e}) \mathrm{div}(w^p \mathbf{e}) dx, \tag{1.11}$$

where $\mathbf{e}$ is a unit vector.

Our result is the following.

THEOREM 1.1. *Assume $\Omega \subset \mathbb{R}^2$ contains three segments Γ_1, Γ_2, Γ_3 whose origin is the common endpoint which intersects orthogonally the boundary of Ω in exactly three points B_1, B_2, and B_3 and whose lengths are $\bar{L}_1$, $\bar{L}_2$, and $\bar{L}_3$, respectively, and satisfy* (1.10).

Assume that $k_i \neq 1/\bar{L}_i$ for at least one index i.

Then there exist $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$ and for any real number L_3 such that

$$c|\ln \varepsilon| \leq L_3, \quad \lim_{\varepsilon \to 0} \varepsilon L_3 = 0, \tag{1.12}$$

for some positive constant $c > 0$ which depends on Ω and on the lengths of the segments, if there exist positive real numbers L_1, L_2 and integers m, n, l satisfying the following balancing formula

$$\Psi(L_1)\boldsymbol{t}_1 + \Psi(L_2)\boldsymbol{t}_2 + \Psi(L_3)\boldsymbol{t}_3 = \boldsymbol{0} \tag{1.13}$$

and

$$\left(m + \frac{1}{2}\right) L_1 = \frac{\bar{L}_1}{\varepsilon}, \quad \left(n + \frac{1}{2}\right) L_2 = \frac{\bar{L}_2}{\varepsilon}, \quad \left(l + \frac{1}{2}\right) L_3 = \frac{\bar{L}_3}{\varepsilon}, \tag{1.14}$$

then there exists a solution u_ε to problem (1.1). *Furthermore there exist $m + n + l$ points*

$$P_j^\varepsilon \quad for \quad j = 1, \ldots, m, \quad Q_j^\varepsilon \quad for \quad j = 1, \ldots, n,$$

and

$$R_j^\varepsilon \quad for \quad j = 1, \ldots, l$$

distributed uniformly at distance εL_1, εL_2, εL_3, respectively, along the segments Γ_1, Γ_2, Γ_3 and a point O^ε near the origin such that

$$u_\varepsilon(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} \left[w\left(\frac{x - P_i^\varepsilon}{\varepsilon}\right) + w\left(\frac{x - Q_j^\varepsilon}{\varepsilon}\right) + w\left(\frac{x - R_k^\varepsilon}{\varepsilon}\right) \right] + w\left(\frac{x - O^\varepsilon}{\varepsilon}\right) + o(1), \tag{1.15}$$

where $o(1) \to 0$ as $\varepsilon \to 0$ uniformly over compact sets of $\mathbb{R}^2$.

Remark 1.1. As a consequence of the balancing condition (1.13) and the choice (1.8)–(1.9), without loss of generality we can assume that the mutual angles of the three lines satisfy that $\mathbf{t}_1 \angle \mathbf{t}_3 \geq \frac{\pi}{2} + \theta_0$, $\mathbf{t}_2 \angle \mathbf{t}_3 \geq \frac{\pi}{2} + \theta_0$, where θ_0 is a constant. Thus we can get that $t_{ij} \neq 0$ for $i, j = 1, 2$.

Conditions (1.13)–(1.14) are satisfied under some restrictions on $\bar{L}_i$ or ϵ. See the remarks below.

Remark 1.2. Let $L = L_3 \gg 1$ be given and multiply relation (1.13) against $\mathbf{t}_3$ first and then against $\mathbf{n}_3$. This gives the system

$$\Psi(L_1)\mathbf{t}_1 \cdot \mathbf{t}_3 + \Psi(L_2)\mathbf{t}_2 \cdot \mathbf{t}_3 = -\Psi(L),$$

$$\Psi(L_1)\mathbf{t}_1 \cdot \mathbf{n}_3 + \Psi(L_2)\mathbf{t}_2 \cdot \mathbf{n}_3 = 0.$$

This system is solvable since

$$d = (\mathbf{t}_1 \cdot \mathbf{t}_3)(\mathbf{t}_2 \cdot \mathbf{n}_3) - (\mathbf{t}_2 \cdot \mathbf{t}_3)(\mathbf{t}_1 \cdot \mathbf{n}_3) = t_{12}t_{21} - t_{11}t_{22} \neq 0.$$

In this case, we thus have

$$\Psi(L_1) = \frac{t_{21}}{d}\Psi(L), \qquad \Psi(L_2) = -\frac{t_{11}}{d}\Psi(L).$$

Now, since $\Psi(s) = C_p e^{-s} s^{-\frac{1}{2}}(1 + O(s^{-1}))$ as $s \to \infty$, with C_p a positive constant, we obtain that

$$L_1 = L - C_1, \qquad L_2 = L - C_2, \tag{1.16}$$

where

$$C_1 = -\log\frac{t_{21}}{d} + O\left(\frac{1}{L}\right), \qquad C_2 = -\log\left(-\frac{t_{11}}{d}\right) + O\left(\frac{1}{L}\right). \tag{1.17}$$

Then the condition (1.14) becomes

$$\frac{\bar{L}_3}{\bar{L}_1} - \frac{2l+1}{2m+1} = \frac{\epsilon C_1(2l+1)}{\bar{L}_1}, \qquad \frac{\bar{L}_3}{\bar{L}_2} - \frac{2l+1}{2n+1} = \frac{\epsilon C_2(2l+1)}{\bar{L}_2}. \tag{1.18}$$

Remark 1.3. Let us consider the case $C_1 = C_2 = 0$, i.e.,

$$\mathbf{t}_i \angle \mathbf{t}_j = \frac{2\pi}{3}. \tag{1.19}$$

In this case, condition (1.18) is satisfied if the ratios $\frac{\bar{L}_3}{\bar{L}_1}, \frac{\bar{L}_3}{\bar{L}_2}$ are rational numbers of the form $\frac{2r+1}{2q+1}$. In this case, we let $\frac{\bar{L}_3}{\bar{L}_1} = \frac{2r_1+1}{2q_1+1}, \frac{\bar{L}_3}{\bar{L}_2} = \frac{2r_2+1}{2q_2+1}$. Then we choose $2l+1 = (2r_1+1)(2r_2+1)(2k+1), 2m+1 = (2q_1+1)(2r_2+1)(2k+1), 2n+1 = (2q_2+1)(2r_1+1)(2k+1)$, where $1 \ll k < \frac{C}{\varepsilon|\lg\varepsilon|}$. Then (1.12)–(1.14) are satisfied.

Remark 1.4. We consider another case, $C_1 = C_2 \neq 0$, i.e., $L_1 = L_2 \neq L_3$. (We may assume that $C_1 = C_2 > 0$.) In this case, we assume that $\bar{L}_1 = \bar{L}_2$. Then we choose $m = n$ and (m, l) such that

$$\frac{\bar{L}_3}{\bar{L}_1} - \frac{2l+1}{2m+1} = -\frac{\epsilon C_1(2l+1)}{\bar{L}_1}, \tag{1.20}$$

which is possible by the following choices: we may always choose a sequence of integers $m, l \to +\infty$ such that $\frac{\bar{L}_3}{\bar{L}_1} - \frac{2l+1}{2m+1} \sim -\frac{1}{m}$. Then (1.20) is satisfied if we choose a sequence $\epsilon_{(m,l)} \sim \frac{1}{m^2} \to$ satisfying (1.18).

If $C_1 \neq C_2$, condition (1.18) is more complicated. Some conditions on the ratio $\frac{C_1}{C_2}$ are needed.

By the above remark, we now have the following corollary.

COROLLARY 1.2. *Assume $\Omega \subset \mathbb{R}^2$ contains three segments Γ_1, Γ_2, Γ_3 whose origin is the common endpoint which intersects orthogonally the boundary of Ω in exactly three points B_1, B_2, and B_3 and whose lengths are $\bar{L}_1$, $\bar{L}_2$, and $\bar{L}_3$, respectively, and satisfy* (1.10). *Assume that at least one $k_i \neq 1/\bar{L}_i$. Furthermore suppose*

$$\boldsymbol{t}_i \angle \boldsymbol{t}_j = \frac{2\pi}{3} \tag{1.21}$$

and that the ratios $\frac{\bar{L}_i}{L_j}$ *are rational numbers of the form* $\frac{2r+1}{2q+1}$. *Then problem* (1.1) *has a triple junction solution for all* ϵ *sufficiently small.*

Triple junctions have appeared in many phase transition problems. In general they appear in vector-valued minimization problems. See Sternberg [40] and Sternberg and Ziemer [41]. Bronsard, Gui, and Schatzman [6] constructed symmetric layered solutions for the vectoral Allen–Cahn equation

$$\Delta U - \nabla W(U) = 0, \quad U : \mathbb{R}^2 \to \mathbb{R}^2, \tag{1.22}$$

and Gui and Schatzman [18] generalized to symmetric quadruple layered solutions. See also Alama, Bronsard, and Gui [1]. As far as we know, Theorem 1.1 and Corollary 1.2 are the first results on the construction of triple junctions in bounded domains. We believe that solutions concentrating on more complex networks may also exist.

2. Ansatz and sketch of the proof. By the scaling $x = \varepsilon z$, problem (1.1) becomes

$$\Delta u - u + u^p = 0\ , \quad u > 0 \quad \text{in} \quad \Omega_\varepsilon, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega_\varepsilon, \tag{2.1}$$

where $\Omega_\varepsilon = \{\frac{x}{\varepsilon} : x \in \Omega\}$.

We consider a large number L, and $L_1, L_2, L_3 \in \mathbb{R}$ for which the following condition holds:

$$|L_i - L| \le C_0 \tag{2.2}$$

for $i = 1, 2, 3$, where C_0 is a positive constant.

Define

$$\begin{cases} O = (\alpha, \beta), \\ P_i = (L_1 i + a_i)\mathbf{t}_1 + L_1 b_i \mathbf{n}_1, \\ Q_j = (L_2 j + c_j)\mathbf{t}_2 + L_2 d_j \mathbf{n}_2, \\ R_k = (L_3 k + e_k)\mathbf{t}_3 + L_3 f_k \mathbf{n}_3 \end{cases} \tag{2.3}$$

for $i = 1, \dots, m$, $j = 1, \dots, n$, $k = 1, \dots, l$, where the vectors $\mathbf{t}_i$ and $\mathbf{n}_i$, $i = 1, 2, 3$, are defined in (1.8) and (1.9).

We will assume that all the $\alpha, \beta, a_i, b_i, c_j, d_j, e_k, f_k$ are uniformly bounded as $\varepsilon \to 0$. It will be convenient to adopt the following notation:

$$\begin{aligned} \mathbf{a} &= (a_1, a_2, \dots, a_m), \quad \mathbf{b} = (b_1, b_2, \dots, b_m), \\ \mathbf{c} &= (c_1, c_2, \dots, c_n), \quad \mathbf{d} = (d_1, d_2, \dots, d_n), \\ \mathbf{e} &= (e_1, e_2, \dots, e_l), \quad \mathbf{f} = (f_1, f_2, \dots, f_l). \end{aligned}$$

We thus assume

$$\|\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \alpha, \beta\| \le c \tag{2.4}$$

for some fixed $c > 0$.

We will denote by Y the set of all points, namely,

$$Y = \{z : z = O, P_i, Q_j, R_k, i = 1, \dots, m, j = 1, \dots, n, k = 1, \dots, l\}. \tag{2.5}$$

Let us define the function

$$U(x) = \sum_{z \in Y} U_z(x), \text{ with } U_z(x) = w_z(x) - \varphi_z(x), \tag{2.6}$$

where

$$w_z(x) = w(x-z)$$

and

$$-\Delta\varphi_z + \varphi_z = 0 \quad \text{in} \quad \Omega_\varepsilon, \qquad \frac{\partial \varphi_z}{\partial \nu} = \frac{\partial w(x-z)}{\partial \nu} \quad \text{on} \quad \partial\Omega_\varepsilon.$$

The next lemma, whose proof is contained in [26], provides a qualitative description of the function φ_z.

LEMMA 2.1. *Assume that* $M|\ln\varepsilon| \le d(P, \partial\Omega_\varepsilon) \le \frac{\delta}{\varepsilon}$ *for some constant* M *depending on* N *and a constant* $\delta > 0$ *sufficiently small. Then*

$$\varphi_P(x) = -(1+o(1))w(x-P^*) + o(\varepsilon^3), \tag{2.7}$$

where $P^* = P + 2d(P, \partial\Omega_\varepsilon)\nu_{\bar{P}}$, $\nu_{\bar{P}}$ *denotes the unit outer normal of* $\partial\Omega_\varepsilon$ *at* $\bar{P}$, *and* $\bar{P}$ *is the unique point on* $\partial\Omega_\varepsilon$ *such that* $d(P,\bar{P}) = d(P,\partial\Omega_\varepsilon)$.

We look for a solution of (2.1) of the form $u = U + \phi$. We set

$$L(\phi) = -\Delta\phi + \phi - pU^{p-1}\phi, \tag{2.8}$$

$$E = U^p - \sum_{z\in Y} w_z^p, \tag{2.9}$$

and

$$N(\phi) = (U+\phi)^p - U^p - pU^{p-1}\phi. \tag{2.10}$$

Problem (2.1) gets rewritten as

$$L(\phi) = E + N(\phi) \quad \text{in} \quad \Omega_\varepsilon, \qquad \frac{\partial\phi}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega_\varepsilon.$$

Consider a cutoff function $\chi \in C^\infty(-\infty,\infty)$ such that

$$\chi(s) \equiv 1 \quad \text{for} \quad s \le -1, \qquad \chi(s) \equiv 0 \quad \text{for} \quad s \ge 0\,. \tag{2.11}$$

We fix a constant $\zeta > 0$ (independent of L) so that the balls of radius $\frac{L-\zeta}{2}$, centered at different points of Y, are mutually disjoint for all L large enough. We define the compactly supported functions

$$Z_z(x) := \chi\left(2|x-z| - L + \zeta\right)\, \nabla w(x-z) \tag{2.12}$$

for $z \in Y$. Observe that, by construction (in fact given the choice of ζ), we have

$$\int_{\Omega_\varepsilon} Z_{z_1}(x) Z_{z_2}(x)\, dx = 0 \tag{2.13}$$

if $z_1 \ne z_2$.

Consider the following intermediate nonlinear projected problem: Given the points in (2.3), find a function ϕ in some proper space and constant vectors c_z such that

$$\begin{cases} L(\phi) = E + N(\phi) + \sum_{z\in Y} c_z Z_z \quad \text{in} \quad \Omega_\varepsilon, \\ \frac{\partial\phi}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \phi Z_z = 0 \quad \text{for} \quad z \in Y. \end{cases} \tag{2.14}$$

We show unique solvability of problem (2.14) by means of a fixed point argument. Furthermore we prove that the solution ϕ depends smoothly on the points z.

To do so, in section 3 we develop a solvability theory for the linear projected problem

$$\begin{cases} L\phi = h + \sum_{z\in Y} c_z Z_z \quad \text{in} \quad \Omega_\varepsilon, \\ \frac{\partial \phi}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \phi Z_z = 0 \quad \text{for} \quad z \in Y \end{cases} \tag{2.15}$$

for a given right-hand side h in some proper space. Roughly speaking, the linear operator L is a superposition of the linear operators

$$L_j\phi = \Delta\phi - \phi + pw^{p-1}(x-z)\phi, \quad z \in Y.$$

Once we have the unique solvability of problem (2.14), which is proved in section 4, it is clear that $u = U + \phi$ is indeed an exact solution to our original problem (1.1), with the qualitative properties described in Theorem 1.1, if we can prove that the constants c_z appearing in (2.14) are all zero. This can be done adjusting properly the parameters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \alpha, \beta$, as will be shown in section 5, where the proof of Theorem 1.1 will be also given.

3. Linear theory. Our main result in this section states bounded solvability of problem (2.15), uniformly in small ε, in points z belonging to Y given by (2.5), uniformly separated from each other at distance $O(L)$. Indeed we assume that the points z given by (2.3) satisfy constraints (2.4).

Given $0 < \eta < 1$, consider the norms

$$\|h\|_* = \sup_{x\in\Omega_\varepsilon} \left| \sum_z e^{\eta|x-z|} h(x) \right|, \tag{3.1}$$

where $z \in Y$ with Y defined in (2.5).

PROPOSITION 3.1. *Let $c > 0$ be fixed. There exist positive numbers $\eta \in (0,1)$, ε_0, and C, such that for all $0 \le \varepsilon \le \varepsilon_0$, for all integers m, n, l and positive real numbers L_i given by* (1.13) *and satisfying* (1.14), *for any points z, $z \in Y$ in* (2.5) *given by* (2.3) *and satisfying* (2.4), *there is a unique solution (ϕ, c_z) to problem* (2.15). *Furthermore*

$$\|\phi\|_* \le C\|h\|_*. \tag{3.2}$$

The proof of the above proposition, which we postpone to the end of this section, is based on the Fredholm alternative theorem for compact operators and an a priori bound for solution to (2.15) that we state (and prove) next.

PROPOSITION 3.2. *Let $c > 0$ be fixed. Let h with $\|h\|_*$ bounded and assume that (ϕ, c_z) is a solution to* (2.15). *Then there exist positive numbers ε_0 and C, such that for all $0 \le \varepsilon \le \varepsilon_0$, for all integers m, n, l and positive real numbers L_1, L_2, L_3 given by* (1.13) *and satisfying* (1.14), *for any points z, $z \in Y$ given by* (2.3) *and satisfying* (2.4), *one has*

$$\|\phi\|_* \le C\|h\|_*. \tag{3.3}$$

Proof. We argue by contradiction. Assume there exists ϕ solution to (2.15) and

$$\|h\|_* \to 0, \quad \|\phi\|_* = 1.$$

We prove that

$$c_z \to 0. \tag{3.4}$$

Multiply the equation in (2.15) with any of the components of the function Z_z defined in (2.12), which, with abuse of notation, we still denote by Z_z, and integrate in Ω_ε. We get

$$\int_{\Omega_\varepsilon} L\phi Z_z(x) = \int_{\Omega_\varepsilon} hZ_z + c_z \int_{\Omega_\varepsilon} Z_z^2,$$

since (2.13) holds true. Given the exponential decay at infinity of $\partial_{x_i} w$ and the definition of Z_z in (2.12), we get

$$\int_{\Omega_\varepsilon} Z_z^2 = \int_{\mathbb{R}^N} (\nabla w)^2 + O(e^{-\delta L}) \quad \text{as} \quad L \to \infty \tag{3.5}$$

for some $\delta > 0$. On the other hand

$$\left| \int_{\Omega_\varepsilon} hZ_z \right| \le C\|h\|_* \int_{\Omega_\varepsilon} \nabla w(x-z) e^{-\eta|x-z|} \le C\|h\|_*.$$

Here and in what follows, C stands for a positive constant independent of ε, as $\varepsilon \to 0$ (or equivalently independent of L as $L \to \infty$). Finally, if we write $\tilde{Z}_z(x) = \nabla w(x-z)$ and $\chi = \chi\,(2|x-z| - L + \zeta)$, we have

$$\begin{aligned} -\int_{\Omega_\varepsilon} L\phi Z_z(x) = & \int_{B(z,\frac{L-\zeta}{2})} [\Delta \tilde{Z}_z - \tilde{Z}_z + p w^{p-1}(x-z)\tilde{Z}_z]\chi\phi \\ & + \int_{\partial B(z,\frac{L-\zeta}{2})} \phi \nabla(\chi\,(2|x-z| - L + \zeta)\,\tilde{Z}_z) \cdot \mathtt{n} \\ & - \int_{B(z,\frac{L-\zeta}{2})} \phi(\tilde{Z}_z \Delta\chi + 2\nabla\chi Z_z) \\ & + p \int_{B(P_j,\frac{L-\zeta}{2})} \left(U^{p-1} - w^{p-1}(x-z)\right) \phi \tilde{Z}_z \chi. \end{aligned} \tag{3.6}$$

Next we estimate all the terms of the previous formula.

Since

$$\Delta \tilde{Z}_z - \tilde{Z}_z + p w^{p-1}(x-P_j)\tilde{Z}_z = 0$$

we get that the first term is 0. Furthermore, using the estimates in (1.5), we have

$$\begin{aligned} & \left| \int_{\partial B(z,\frac{L-\zeta}{2})} \phi \nabla(\chi\,(2|x-z| - L + \zeta)\,\tilde{Z}_z) \cdot \mathtt{n} \right| \\ & \le C\|\phi\|_* \int_{\partial B(zj,\frac{L-\zeta}{2})} e^{-(1+\eta)|x-z|} |x-z|^{-\frac{1}{2}}\, dx \\ & \le C e^{-(1+\xi)\frac{L}{2}} \|\phi\|_* \end{aligned}$$

for some proper $\xi > 0$. Using again (1.5), the third integral can be estimated as follows:

$$\left| \int_{B(z,\frac{L-\zeta}{2})} \phi(\tilde{Z}_z \Delta\chi\,(2|x-z|-L+\zeta) + 2\nabla\chi\,(2|x-z|-L+\zeta)\,\nabla\tilde{Z}_z) \right| \leq C\|\phi\|_* \int_{\frac{L-\zeta}{2}-1}^{\frac{L-\zeta}{2}} e^{-(1+\eta)s} s^{\frac{1}{2}} ds \leq C e^{-(1+\xi)\frac{L}{2}} \|\phi\|_*,$$

again for some $\xi > 0$. Finally, we observe that in $B(z, \frac{L-\zeta}{2})$

$$|U^{p-1}(x) - w^{p-1}(x-z)| \leq w^{p-2}(x-z) \left[\sum_{x_i \neq z} w(x-x_i) \right].$$

Having this, we conclude that

$$\left| p \int_{B(z,\frac{L-\zeta}{2})} \left(U^{p-1}(x) - w^{p-1}(x-z)\right) \phi \tilde{Z}_z \chi\,(2|x-z|-L+\zeta) \right| \leq C e^{-(1+\xi)\frac{L}{2}} \|\phi\|_*$$

for a proper $\xi > 0$, depending on N and p. We thus conclude that

$$|c_z| \leq C \left[e^{-(1+\xi)\frac{L}{2}} \|\phi\|_* + \|h\|_* \right]. \tag{3.7}$$

Thus we get the validity of (3.4), since we are assuming $\|\phi\|_* = 1$ and $\|h\|_* \to 0$.

Now let $\eta \in (0,1)$. It is easy to check that the function

$$W := \sum_{z \in Y} e^{-\eta\,|\cdot - z|}$$

satisfies

$$L\,W \leq \frac{1}{2}\,(\eta^2 - 1)\,W$$

in $\Omega_\varepsilon \setminus \cup_{z\in Y} B(z,\rho)$ provided ρ is fixed large enough (independently of L). Hence the function W can be used as a barrier to prove the pointwise estimate

$$|\phi|(x) \leq C \left(\|L\,\phi\|_* + \sum_z \|\phi\|_{L^\infty(\partial B(z,\rho))} \right) W(x) \tag{3.8}$$

for all $x \in \Omega_\varepsilon \setminus \cup_{z\in Y} B(z,\rho)$.

Granted these preliminary estimates, the proof of the result goes by contradiction. Let us assume there exist a sequence of L tending to ∞ and a sequence of solutions of (2.15) for which the inequality is not true. The problem being linear, we can reduce to the case where we have a sequence $L^{(n)}$ tending to ∞ and sequences $h^{(n)}$, $\phi^{(n)}, c^{(n)}$ such that

$$\|h^{(n)}\|_* \to 0 \quad \text{and} \quad \|\phi^{(n)}\|_* = 1.$$

But (3.4) implies that we also have

$$|c^{(n)}| \to 0\,.$$

Then (3.8) implies that there exists $z^{(n)} \in Y$ (see (2.5) for the definition of Y) such that

$$\|\phi^{(n)}\|_{L^\infty(B(z^{(n)},\rho))} \geq C \tag{3.9}$$

for some fixed constant $C > 0$. Using elliptic estimates together with Ascoli–Arzelà's theorem, we can find a sequence $z^{(n)}$ and extract from the sequence $\phi^{(n)}(\cdot - z^{(n)})$ a subsequence which will converge (on compact) to ϕ_∞, a solution of

$$\left(\Delta - 1 + p\,w^{p-1}\right)\,\phi_\infty = 0$$

in $\mathbb{R}^2$, which is bounded by a constant times $e^{-\eta\,|x|}$, with $\eta > 0$. Moreover, recall that $\phi^{(n)}$ satisfies the orthogonality conditions in (2.15). Therefore, the limit function ϕ_∞ also satisfies

$$\int_{\mathbb{R}^2} \phi_\infty\, \nabla w\, dx = 0\,.$$

But the solution w being nondegenerate, this implies that $\phi_\infty \equiv 0$, which is certainly in contradiction with (3.9), which implies that ϕ_∞ is not identically equal to 0.

Having reached a contradiction, this completes the proof of the proposition.

We can now prove Proposition 3.1.

Proof of Proposition 3.1. Consider the space

$$\mathcal{H} = \left\{ u \in H^1(\Omega_\varepsilon)\,:\, \int_{\Omega_\varepsilon} uZ_z = 0, \quad z \in Y \right\}.$$

Notice that problem (2.15) in ϕ gets rewritten as

$$\phi + K(\phi) = \bar{h} \quad \text{in} \quad \mathcal{H}, \tag{3.10}$$

where $\bar{h}$ is defined by duality and $K : \mathcal{H} \to \mathcal{H}$ is a linear compact operator. Using Fredholm's alternative, showing that (3.10) has a unique solution for each $\bar{h}$ is equivalent to showing that the equation has a unique solution for $\bar{h} = 0$, which in turn follows from Proposition 3.2. The estimate (3.2) follows directly from Proposition 3.2. This concludes the proof of Proposition 3.1.

In the following, if ϕ is the unique solution given by Proposition 3.1, we set

$$\phi = \mathcal{A}(h). \tag{3.11}$$

Estimate (3.2) implies

$$\|\mathcal{A}(h)\|_* \leq C\|h\|_*. \tag{3.12}$$

4. The nonlinear projected problem. For small ε, large L, and fixed points $z \in Y$ (see (2.5), (2.4), and (2.3)), we show solvability in ϕ, c_z of the nonlinear projected problem

$$\begin{cases} L(\phi) = E + N(\phi) + \sum_{z\in Y} c_z Z_z \quad \text{in} \quad \Omega_\varepsilon, \\ \frac{\partial\phi}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \phi Z_z = 0 \quad \text{for} \quad z \in Y. \end{cases} \tag{4.1}$$

We have the validity of the following result.

PROPOSITION 4.1. *Let $c > 0$ be fixed. There exist positive numbers ε_0, C, and $\xi > 0$ such that for all $\varepsilon \le \varepsilon_0$, for all integers m, n, l and positive real numbers L_i given by* (1.13) *and satisfying* (1.14)*, for any points z, $z \in Y$ given by* (2.3) *and satisfying* (2.4)*, there is a unique solution (ϕ, c_z) to problem* (2.14)*. This solution depends continuously on the parameters of the construction (namely, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$, $\boldsymbol{f}$, α, β) and furthermore*

$$\|\phi\|_* \le Ce^{-\frac{(1+\xi)}{2}L}. \tag{4.2}$$

Proof. The proof relies on the contraction mapping theorem in the $\|\cdot\|_*$-norm introduced above. Observe that ϕ solves (2.14) if and only if

$$\phi = \mathcal{A}\left(E + N(\phi)\right), \tag{4.3}$$

where $\mathcal{A}$ is the operator introduced in (3.11). In other words, ϕ solves (2.14) if and only if ϕ is a fixed point for the operator

$$T(\phi) := \mathcal{A}\left(E + N(\phi)\right).$$

Given $r > 0$, define

$$\mathcal{B} = \left\{\phi \in C^2(\Omega_\varepsilon) \,:\, \|\phi\|_* \le re^{-\frac{(1+\xi)}{2}L}, \int_{\Omega_\varepsilon} \phi Z_z = 0\right\}.$$

We will prove that T is a contraction mapping from $\mathcal{B}$ in itself.

To do so, we claim that

$$\|E\|_* \le Ce^{-\frac{(1+\xi)}{2}L} \tag{4.4}$$

and

$$\|N(\phi)\|_* \le C\left[\|\phi\|_*^2 + \|\phi\|_*^p\right] \tag{4.5}$$

for some fixed function C independent of L as $L \to \infty$. We postpone the proof of the estimates above until the end of the proof of this proposition. Assuming the validity of (4.4) and (4.5), and taking into account (3.12), we have for any $\phi \in \mathcal{B}$

$$\begin{aligned}\|T(\phi)\|_* &\le C\left[\|E + N(\phi)\|_*\right] \le C\left[e^{-\frac{(1+\xi)}{2}L} + r^2 e^{-(1+\xi)L} + r^p e^{-\frac{p(1+\xi)}{2}L}\right] \\ &\le re^{-\frac{(1+\xi)}{2}L}\end{aligned}$$

for a proper choice of r in the definition of $\mathcal{B}$, since $p > 1$.

Take now ϕ_1 and ϕ_2 in $\mathcal{B}$. Then it is straightforward to show that

$$\begin{aligned}\|T(\phi_1) - T(\phi_2)\|_* &\le C\|N(\phi_1) - N(\phi_2)\|_* \\ &\le C\left[\|\phi_1\|_*^{\min(1,p-1)} + \|\phi_2\|_*^{\min(1,p-1)}\right]\|\phi_1 - \phi_2\|_* \\ &\le o(1)\|\phi_1 - \phi_2\|_*.\end{aligned}$$

This means that T is a contraction mapping from $\mathcal{B}$ into itself.

To conclude the proof of this proposition we are left to show the validity of (4.4) and (4.5). We start with (4.4).

Fix $z \in Y$ and consider the region $|x-z| \leq \frac{L}{2+\sigma}$, where σ is a small positive number to be chosen later. In this region the error E, whose definition is in (2.9), can be estimated in the following way (see (1.5)):

$$|E(x)| \leq C\left[w^{p-1}(x-z)\sum_{x_i\neq z} w(x-x_i) + \sum_{x_i\neq z} w^p(x-x_i)\right]$$
$$\leq Cw^{p-1}(x-z)\sum_{x_i\neq z} e^{-(\frac{1}{2}+\frac{\sigma}{2(2+\sigma)})L}$$
$$\leq Cw^{p-1}(x-z)e^{-(\frac{1}{2}+\frac{\sigma}{4(2+\sigma)})L}\, e^{-\frac{\sigma}{4(2+\sigma)}L}$$
$$\leq Cw^{p-1}(x-z)e^{-\frac{1+\xi}{2}L} \tag{4.6}$$

for a proper choice of $\xi > 0$.

Consider now the region $|x-z| > \frac{L}{2+\sigma}$ for all j. Since $0 < \mu < p-1$, we write $\mu = p-1-M$. From the definition of E, we get in the region under consideration

$$|E(x)| \leq C\left[\sum_z w^p(x-z)\right] \leq C\left[\sum_z e^{-\mu|x-z|}\right] e^{-(p-\mu)\frac{L}{2+\sigma}} \tag{4.7}$$
$$\leq \left[\sum_z e^{-\mu|x-z|}\right] e^{-\frac{1+M}{2+\sigma}L} \leq \left[\sum_z e^{-\mu|x-z|}\right] e^{-\frac{1+\xi}{2}L}$$

for some $\xi > 0$ if we chose M and σ small enough. From (4.6) and (4.7) we get (4.4).

We now prove (4.5). Let $\phi \in \mathcal{B}$. Then

$$|N(\phi)| \leq |(U+\phi)^p - U^p - pU^{p-1}\phi| \leq C(\phi^2 + |\phi|^p). \tag{4.8}$$

Thus we have

$$\left|\sum_j e^{\eta|x-P_j|} N(\phi)\right| \leq C\|\phi\|_* \left(|\phi| + |\phi|^{p-1}\right)$$
$$\leq C(\|\phi\|_*^2 + \|\phi\|_*^p).$$

This gives (4.5).

A direct consequence of the fixed point characterization of ϕ given above together with the fact that the error term E depends continuously (in the $*$-norm) on the parameters $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \alpha, \beta)$ is that the map

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \alpha, \beta) \to \phi$$

into the space $C(\bar{\Omega}_\varepsilon)$ is continuous (in the $*$-norm). This concludes the proof of the proposition. ∎

Given points $z \in Y$ satisfying constraint (2.4), Proposition 4.1 guarantees the existence (and gives estimates) of a unique solution ϕ, c_z, $z \in Y$, to problem (2.14).

It is clear then that the function $u = U + \phi$ is an exact solution to our problem (1.1), with the required properties stated in Theorem 1.1 if we show that there exists a configuration for the points z that gives all the constants c_z in (2.14) equal to zero. In order to do so we first need to find the correct conditions on the points to get $c_z = 0$. This condition is naturally given by projecting in $L^2(\Omega_\varepsilon)$ the equation in (2.14) into the space spanned by Z_z, namely, by multiplying the equation in (2.14) by Z_z and integrating all over Ω_ε. We will do it in detail in the next and final section.

5. Error estimates and the proof of Theorem 1.1. The first aim of this section is to evaluate the $L^2(\Omega_\varepsilon)$ projection of the error term E in (2.9) against the elements Z_z in (2.12) for any $z \in Y$ in (2.5).

Let us introduce the following notation.

Let $P_m^* = P_m + 2d(P_m, \partial\Omega_\varepsilon)\nu_{\bar{P}_m}$, where $\nu_{\bar{P}_m}$ denotes the outer unit normal at $\bar{P}_m$ on $\partial\Omega_\varepsilon$ and $\bar{P}_m$ is the unique point on $\partial\Omega_\varepsilon$ such that $d(P_m, \bar{P}_m) = d(P_m, \partial\Omega_\varepsilon)$. In an analogous way we define Q_n^*, $\bar{Q}_n$ and R_l^*, $\bar{R}_l$.

Thus there exist three coordinates x_1, x_2, and x_3 such that

$$\bar{P}_m = L_1 x_1 \mathbf{n}_1 + \left(\frac{\bar{L}_1 + h_1(\epsilon L_1 x_1)}{\epsilon}\right)\mathbf{t}_1, \quad \bar{Q}_n = L_2 x_2 \mathbf{n}_2 + \left(\frac{\bar{L}_2 + h_2(\epsilon L_2 x_2)}{\epsilon}\right)\mathbf{t}_2,$$

and

$$\bar{R}_l = L_3 x_3 \mathbf{n}_3 + \left(\frac{\bar{L}_3 + h_3(\epsilon L_3 x_3)}{\epsilon}\right)\mathbf{t}_3.$$

More explicitly, the coordinates x_i are defined as solutions of the system

$$\left\{ \begin{array}{c} L_1(x_1 - b_m) + (\frac{L_1}{2} + \frac{h_1(L_1 \epsilon x_1)}{\epsilon} - a_m)h_1'(L_1\epsilon x_1) = 0, \\ L_2(x_2 - d_n) + (\frac{L_2}{2} + \frac{h_2(L_2 \epsilon x_2)}{\epsilon} - c_n)h_2'(L_2\epsilon x_2) = 0, \\ L_3(x_3 - f_l) + (\frac{L_3}{2} + \frac{h_3(L_3 \epsilon x_3)}{\epsilon} - e_l)h_3'(L_3\epsilon x_3) = 0. \end{array} \right. \tag{5.1}$$

We have the validity of the following lemma.

LEMMA 5.1. *Let us define*

$$\kappa_i = -(\log \Psi)'(L_i).$$

The following expansions hold true:

$$\begin{aligned} \int_{w_\varepsilon} E Z_{P_i} dx = {} & -\Psi(L_1)[\kappa_1(a_{i+1} - 2a_i + a_{i-1})\boldsymbol{t}_1 - (b_{i+1} - 2b_i + b_{i-1})\boldsymbol{n}_1] \\ & + e^{-\delta L_1} A + \Psi(L_1)\boldsymbol{N} \end{aligned} \tag{5.2}$$

for $i = 2, \ldots, m-1$,

$$\begin{aligned} \int_{w_\varepsilon} E Z_{Q_j} dx = {} & -\Psi(L_2)[\kappa_2(c_{j+1} - 2c_j + c_{j-1})\boldsymbol{t}_2 - (d_{j+1} - 2d_j + d_{j-1})\boldsymbol{n}_2] \\ & + e^{-\delta L_2} A + \Psi(L_2)\boldsymbol{N} \end{aligned} \tag{5.3}$$

for $j = 2, \ldots, n-1$,

$$\begin{aligned} \int_{\Omega_\varepsilon} E Z_{R_k} dx = {} & -\Psi(L_3)[\kappa_3(e_{k+1} - 2e_k + e_{k-1})\boldsymbol{t}_3 - (f_{k+1} - 2f_k + f_{k-1})\boldsymbol{n}_3] \\ & + e^{-\delta L_3} A + \Psi(L_3)\boldsymbol{N} \end{aligned} \tag{5.4}$$

for $k = 2, \ldots, l-1$,

$$\int_{w_\varepsilon} EZ_{P_1} dx = -\Psi(L_1)\left[\kappa_1(a_2 - 2a_1 + (\alpha,\beta)\boldsymbol{t}_1)\boldsymbol{t}_1 - \left(b_2 - 2b_1 + \frac{(\alpha,\beta)\boldsymbol{n}_1}{L_1}\boldsymbol{n}_1\right)\right]$$

$$+ e^{-\delta L_1}A + \Psi(L_1)\boldsymbol{N}, \tag{5.5}$$

$$\int_{w_\varepsilon} EZ_{Q_1} dx = -\Psi(L_2)\left[\kappa_2(c_2 - 2c_1 + (\alpha,\beta)\boldsymbol{t}_2)\boldsymbol{t}_2 - \left(d_2 - 2d_1 + \frac{(\alpha,\beta)\boldsymbol{n}_2}{L_2}\boldsymbol{n}_2\right)\right]$$

$$+ e^{-\delta L_2}A + \Psi(L_2)\boldsymbol{N}, \tag{5.6}$$

$$\int_{w_\varepsilon} EZ_{R_1} dx = -\Psi(L_3)\left[\kappa_3(e_2 - 2e_1 + (\alpha,\beta)\boldsymbol{t}_3)\boldsymbol{t}_3 - \left(f_2 - 2f_1 + \frac{(\alpha,\beta)\boldsymbol{n}_3}{L_3}\boldsymbol{n}_3\right)\right]$$

$$+ e^{-\delta L_3}A + \Psi(L_3)\boldsymbol{N}, \tag{5.7}$$

$$\int_{w_\epsilon} EZ_{P_m} dx = -\Psi(L_1)\left[\kappa_1\left(a_{m-1} - 3a_m + \frac{2h_1(L_1\varepsilon x_1)}{\varepsilon}\right)\boldsymbol{t}_1 - (b_{m-1} - 3b_m + 2x_1)\boldsymbol{n}_1\right]$$

$$+ e^{-\delta L_1}A + \Psi(L_1)\boldsymbol{N}, \tag{5.8}$$

$$\int_{w_\epsilon} EZ_{Q_n} dx = -\Psi(L_2)\left[\kappa_2\left(c_{n-1} - 3c_n + \frac{2h_2(L_2\varepsilon x_2)}{\varepsilon}\right)\boldsymbol{t}_2 - (d_{n-1} - 3d_n + 2x_2)\boldsymbol{n}_2)\right]$$

$$+ e^{-\delta L_2}A + \Psi(L_2)\boldsymbol{N}, \tag{5.9}$$

$$\int_{w_\epsilon} EZ_{R_l} dx = -\Psi(L_3)\left[\kappa_3\left(e_{l-1} - 3e_l + \frac{2h_3(L_3\varepsilon x_3)}{\varepsilon}\right)\boldsymbol{t}_3 - (f_{l-1} - 3f_l + 2x_3)\boldsymbol{n}_3\right]$$

$$+ e^{-\delta L_3}A + \Psi(L_3)\boldsymbol{N}, \tag{5.10}$$

$$\begin{aligned}\int_{\Omega_\varepsilon} EZ_O dx = \;& \Psi(L_1)\left[\kappa_1(a_1 - (\alpha,\beta)\boldsymbol{t}_1)\boldsymbol{t}_1 + \left(b_1 - \frac{(\alpha,\beta)\boldsymbol{n}_1}{L_1}\right)\boldsymbol{n}_1\right] \\ & + \Psi(L_2)\left[\kappa_2(c_1 - (\alpha,\beta)\boldsymbol{t}_2)\boldsymbol{t}_2 + \left(d_1 - \frac{(\alpha,\beta)\boldsymbol{n}_2}{L_2}\right)\boldsymbol{n}_2\right] \\ & + \Psi(L_3)\left[\kappa_3(e_1 - (\alpha,\beta)\boldsymbol{t}_3)\boldsymbol{t}_3 + \left(f_1 - \frac{(\alpha,\beta)\boldsymbol{n}_3}{L_3}\right)\boldsymbol{n}_3\right] \\ & + e^{-\delta L}A + \Psi(L)\boldsymbol{N}.\end{aligned} \tag{5.11}$$

Furthermore,

$$\int L(\phi)Z_z = e^{-\delta L}A, \quad z \in Y, \tag{5.12}$$

and

$$\int N(\phi)Z_z = e^{-\delta L}A \quad z \in Y, \tag{5.13}$$

where $\delta > 1, A = A(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}, \boldsymbol{x}, \alpha, \beta)$ *and* $\boldsymbol{N} = \boldsymbol{N}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}, \boldsymbol{x}, \alpha, \beta)$ *denote the smooth vector-valued functions (which vary from line to line), uniformly bounded as* $L \to \infty$*, and the Taylor expansion of* $\boldsymbol{N}$ *with respect to* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}, \boldsymbol{x}, \alpha, \beta$ *does not involve any constant or any linear terms.*

Proof. Observe that, given $\mathsf{e} \in \mathbb{R}^2$ with $|\mathsf{e}| = 1$ and $\mathsf{a} \in \mathbb{R}^N$, a direct consequence of estimates (1.5) is that the following expansion holds:

$$\Psi(|\tilde{L}\mathsf{e} + \mathsf{a}|)\frac{(\tilde{L}\mathsf{e} + \mathsf{a})}{|\tilde{L}\mathsf{e} + \mathsf{a}|} = \Psi(\tilde{L})\left(\mathsf{e} - \tilde{\kappa}\,\mathsf{a}^{||} + \frac{1}{\tilde{L}}\,\mathsf{a}^{\perp} + O(|\mathsf{a}|^2)\right) \tag{5.14}$$

as $\tilde{L} \to \infty$, where $\tilde{\kappa} = -(\log\Psi)'(\tilde{L})$. Here, we have decomposed $\mathsf{a} = \mathsf{a}^{||} + \mathsf{a}^{\perp}$, where $\mathsf{a}^{||}$ is collinear to e and $\mathsf{a}^{\perp}$ is orthogonal to e. See also [34].

Estimates (5.2)–(5.4) are by now standard; see, for instance, [34]. For completeness, we show

$$\begin{aligned}\int_{w_\varepsilon} EZ_{P_i}(x)dx &= \int_{R^2} w(x - P_{i-1})pw^{p-1}(x - P_i)\nabla w(x - P_i)dx \\ &\quad + \int_{R^2} w(x - P_{i+1})pw^{p-1}(x - P_i)\nabla w(x - P_i)dx + e^{-\delta L_1}A \\ &= -\Psi(L_1)(\kappa_1(a_{i+1} - 2a_i + a_{i-1})\mathbf{t}_1 - (b_{i+1} - 2b_i + b_{i-1})\mathbf{n}_1) \\ &\quad + e^{-\delta L_1}A + \Psi(L_1)\mathbf{N}\end{aligned}$$

for $i = 2, \ldots, m-1$.

Similarly we can get the two equations for Q_j, R_k.

Concerning estimates (5.5)–(5.7), a direct use of (5.14) gives

$$\begin{aligned}\int_{\Omega_\varepsilon} EZ_{P_1}dx &= \int_{R^2} w(x - O)pw^{p-1}(x - P_1)\nabla w(x - P_1)dx \\ &\quad + \int_{R^2} w(x - P_2)pw^{p-1}(x - P_1)\nabla w(x - P_1)dx + e^{-\delta L_1}A \\ &= -\Psi(L_1)\left(\kappa_1(a_2 - 2a_1 + (\alpha,\beta)\mathbf{t}_1)\mathbf{t}_1 - (b_2 - 2b_1)\mathbf{n}_1 - \frac{(\alpha,\beta)\mathbf{n}_1}{L_1}\mathbf{n}_1\right) \\ &\quad + e^{-\delta L_2}A + \Psi(L_1)\mathbf{N}.\end{aligned}$$

Similarly we can get the two equations for Q_1, R_1.

To compute (5.8)–(5.10), we use the result of Lemma 2.1. Given (5.1) we obtain that

$$\begin{cases} P_m^* - P_m = 2(x_1 - b_m)L_1\mathbf{n}_1 + (L_1 + \frac{2h_1(\epsilon L_1 x_1)}{\epsilon} - 2a_m)\mathbf{t}_1, \\ Q_n^* - Q_n = 2(x_2 - d_n)L_2\mathbf{n}_2 + (L_2 + \frac{2h_2(\epsilon L_2 x_2)}{\epsilon} - 2c_n)\mathbf{t}_2, \\ R_l^* - R_l = 2(x_3 - f_l)L_3\mathbf{n}_3 + (L_3 + \frac{2h_3(\epsilon L_3 x_3)}{\epsilon} - 2e_l)\mathbf{t}_3. \end{cases} \tag{5.15}$$

Thus a direct use of Lemma 2.1 gives estimate (5.8) as follows:

$$
\begin{aligned}
\int_{w_\epsilon} EZ_{P_m}dx &= \int_{\Omega_\varepsilon} pw^{p-1}(x-P_m)\nabla w(x-P_m)w(x-P_{m-1}) \\
&\quad + \int_{\Omega_\varepsilon} pw^{p-1}(x-P_m)\nabla w(x-P_m)w(x-P_m^*) + e^{-\delta L_1}A \\
&= -\Psi(L_1)\left(\kappa_1\left(3a_m - a_{m-1} - \frac{2h_1(L_1\epsilon x_1)}{\epsilon}\right)\mathbf{t}_1\right. \\
&\quad \left. + (2x_1 + b_{m-1} - 3b_m)\mathbf{n}_1\right) + e^{-\delta L_1}A + \Psi(L_1)\mathbf{N}.
\end{aligned}
$$

In the same way we get the equations for Q_n, R_l.

Finally, expansion (5.11) is given by

$$
\begin{aligned}
\int_{\Omega_\varepsilon} EZ_O(x)dx &= \int_{\Omega_\varepsilon} pw^{p-1}(x-O)w(x-P_1)\nabla w(x-O)dx \\
&\quad + \int_{w_\epsilon} pw^{p-1}(x-O)w(x-Q_1)\nabla w(x-O)dx \\
&\quad + \int_{w_\epsilon} pw^{p-1}(x-O)w(x-R_1)\nabla w(x-O)dx + e^{-\delta L}A \\
&= -\Psi(L_1)\left(\mathbf{t}_1 - \kappa_1(a_1 - (\alpha,\beta)\mathbf{t}_1)\mathbf{t}_1 + \frac{L_1 b_1 - (\alpha,\beta)\mathbf{n}_1}{L_1}\mathbf{n}_1\right) \\
&\quad - \Psi(L_2)\left(\mathbf{t}_2 - \kappa_2(c_1 - (\alpha,\beta)\mathbf{t}_2)\mathbf{t}_2 + \frac{L_2 d_1 - (\alpha,\beta)\mathbf{n}_2}{L_2}\mathbf{n}_2\right) \\
&\quad - \Psi(L_3)\left(\mathbf{t}_3 - \kappa_3(e_1 - (\alpha,\beta)\mathbf{t}_3)\mathbf{t}_3 + \frac{L_3 f_1 - (\alpha,\beta)\mathbf{n}_3}{L_3}\mathbf{n}_3\right) \\
&\quad + e^{-\delta L}A + \mathbf{N}\Psi(L).
\end{aligned}
$$

The proof of (5.12) follows the line of the proof of Proposition 3.2 (see formula (3.6) and the subsequent estimates, together with (4.2)).

The proof of (5.13) follows from estimates (4.5) and (4.2). ∎

For any integer k let us now define the following $k \times k$ matrix:

$$
T := \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix} \in \mathcal{M}_{k\times k}\,. \tag{5.16}
$$

It is well known that the matrix T is invertible and its inverse is the matrix whose entries are given by

$$
(T^{-1})_{ij} = \min(i,j) - \frac{ij}{k+1}\,.
$$

We define the vectors $S^{\downarrow}$ and $S^{\uparrow}$ by

$$T\,S^{\downarrow} := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^k, \qquad T\,S^{\uparrow} := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^k . \tag{5.17}$$

It is immediate to check that

$$S^{\downarrow} := \begin{pmatrix} \frac{1}{k+1} \\ \frac{2}{k+1} \\ \vdots \\ \frac{k-1}{k+1} \\ \frac{k}{k+1} \end{pmatrix} \in \mathbb{R}^k, \qquad S^{\uparrow} := \begin{pmatrix} \frac{k}{k+1} \\ \frac{k-1}{k+1} \\ \vdots \\ \frac{2}{k+1} \\ \frac{1}{k+1} \end{pmatrix} \in \mathbb{R}^k . \tag{5.18}$$

With this in mind we have that the above lemma gives the validity of the following.

Lemma 5.2. *The coefficients c_z in problem* (4.1) *are all equal to* 0 *if and only if the parameters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}, \alpha, \beta$ are solutions of the nonlinear system*

$$\begin{cases} \boldsymbol{a} = (\frac{2h_1(\varepsilon L_1 x_1)}{\epsilon} - a_m)S^{\downarrow} + (\alpha,\beta).\boldsymbol{t}_1 S^{\uparrow} + e^{-\delta L}A + \boldsymbol{N} \in \mathbb{R}^m, \\ \boldsymbol{b} = (2x_1 - b_m)S^{\downarrow} + \frac{(\alpha,\beta).\boldsymbol{n}_1}{L_1} S^{\uparrow} + e^{-\delta L}A + \boldsymbol{N} \in \mathbb{R}^m, \\ \boldsymbol{c} = (\frac{2h_2(\varepsilon L_2 x_2)}{\epsilon} - c_n)S^{\downarrow} + (\alpha,\beta).\boldsymbol{t}_2 S^{\uparrow} + e^{-\delta L}A + \boldsymbol{N} \in \mathbb{R}^2, \\ \boldsymbol{d} = (2x_2 - d_n)S^{\downarrow} + \frac{(\alpha,\beta).\boldsymbol{n}_2}{L_2} S^{\uparrow} + e^{-\delta L}A + \boldsymbol{N} \in \mathbb{R}^2, \\ \boldsymbol{e} = (\frac{2h_3(\varepsilon L_3 x_3)}{\epsilon} - e_l)S^{\downarrow} + (\alpha,\beta).\boldsymbol{t}_3 S^{\uparrow} + e^{-\delta L}A + \boldsymbol{N} \in \mathbb{R}^l, \\ \boldsymbol{f} = (2x_3 - f_l)S^{\downarrow} + \frac{(\alpha,\beta).\boldsymbol{n}_3}{L_3} S^{\uparrow} + e^{-\delta L}A + \boldsymbol{N} \in \mathbb{R}^l, \end{cases} \tag{5.19}$$

where $\delta > 0$ and x_1, x_2, and x_3 are given by (5.1). *Furthermore α, β satisfy*

$$\begin{aligned} &- \Psi(L_1)\left(-\kappa_1(a_1 - (\alpha,\beta).\boldsymbol{t}_1)\boldsymbol{t}_1 + \left(b_1 - \frac{(\alpha,\beta).\boldsymbol{n}_1}{L_1}\right)\boldsymbol{n}_1\right) \\ &- \Psi(L_2)\left(-\kappa_2(c_1 - (\alpha,\beta).\boldsymbol{t}_2)\boldsymbol{t}_2 + \left(d_1 - \frac{(\alpha,\beta).\boldsymbol{n}_2}{L_2}\right)\boldsymbol{n}_2\right) \\ &- \Psi(L_3)\left(-\kappa_3(e_1 - (\alpha,\beta).\boldsymbol{t}_3)\boldsymbol{t}_3 + \left(f_1 - \frac{(\alpha,\beta).\boldsymbol{n}_3}{L_3}\right)\boldsymbol{n}_3\right) \\ &+ e^{-\delta_1 L}A + \boldsymbol{N}\Psi(L) = 0. \end{aligned} \tag{5.20}$$

In this last formula $\delta_1 > 1$. In the above formula we have denoted by $A = A(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}, \alpha, \beta)$ and $\boldsymbol{N} = \boldsymbol{N}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}, \alpha, \beta)$ smooth vector-valued functions (which vary from line to line), uniformly bounded as $L \to \infty$ and the Taylor expansion of $\boldsymbol{N}$ with respect to $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}, \alpha, \beta$ does not involve any constant or any linear term.

Given the result of the above lemma, we are left to show that (5.19)–(5.20) have a solution ($\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$, α, β) with

$$\|(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \alpha, \beta)\| \le c$$

for some positive c, small and independent of ε.

We first observe that using the assumptions that $h_i(0) = h_i'(0) = 0$ for all $i = 1, 2, 3$, from the equations (5.1) satisfied by x_1, x_2, x_3, we get that

$$\begin{cases} x_1 = (1 - \frac{\bar{L}_1 h_1''(0)}{2m+1}) b_m + e^{-\delta L} A + \mathbf{N}, \\ x_2 = (1 - \frac{\bar{L}_2 h_2''(0)}{2n+1}) d_n + e^{-\delta L} A + \mathbf{N}, \\ x_3 = (1 - \frac{\bar{L}_3 h_2''(0)}{2l+1}) f_l + e^{-\delta L} A + \mathbf{N} \end{cases} \tag{5.21}$$

for some constant $\delta > 0$.

On the other hand, using the expressions for $S^{\uparrow}$ and $S^{\downarrow}$ given by (5.18), from the first two equations in (5.19) we get that

$$\begin{cases} a_1 = \frac{2h_1(\varepsilon L_1 x_1)}{(2m+1)\epsilon} + \frac{2m-1}{2m+1}(\alpha, \beta).\mathbf{t}_1 + e^{-\delta L} A + \mathbf{N}, \\ b_1 = \frac{2x_1}{2m+1} + \frac{2m-1}{(2m+1)L_1}(\alpha, \beta).\mathbf{n}_1 + e^{-\delta L} A + \mathbf{N}, \\ a_m = \frac{2m h_1(\varepsilon L_1 x_1)}{(2m+1)\epsilon} + \frac{1}{2m+1}(\alpha, \beta).\mathbf{t}_1 + e^{-\delta L} A + \mathbf{N}, \\ b_m = \frac{2m x_1}{2m+1} + \frac{1}{(2m+1)L_1}(\alpha, \beta).\mathbf{n}_1 + e^{-\delta L} A + \mathbf{N}. \end{cases} \tag{5.22}$$

In a very similar way, from the last four equations in (5.19) we get the expressions of $c_1, c_n, d_1, d_n, e_1, e_l, f_1, f_l$.

Using (5.21) and (5.22), from (5.20) we can get that the parameters α, β satisfy the system

$$\begin{cases} B\alpha - C\beta = \frac{2\Psi(L_1)}{2m+1} t_{12} x_1 + \frac{2\Psi(L_2)}{2n+1} t_{22} x_2 - \frac{2\Psi(L_3)}{2k+1} x_3 + e^{-\delta L} A + \mathbf{N}\Psi(L), \\ C\alpha - D\beta = \frac{2\Psi(L_1)}{2m+1} t_{11} x_1 + \frac{2\Psi(L_2)}{2n+1} t_{21} x_2 + e^{-\delta L} A + \mathbf{N}\Psi(L) \end{cases} \tag{5.23}$$

for some $\delta > 1$, where the constants B, C, and D are given by

$$\begin{aligned} B &= \frac{2\Psi(L_1)}{(2m+1)L_1} t_{12}^2 + \frac{2\Psi(L_2)}{(2n+1)L_2} t_{22}^2 + \frac{2\Psi(L_3)}{(2k+1)L_3} - \frac{\Psi(L_1)}{2m+1} t_{11}^2 - \frac{\Psi(L_2)}{2n+1} t_{21}^2, \\ C &= \frac{2\Psi L_1}{(2m+1)L_1} t_{11} t_{12} + \frac{2\Psi(L_2)}{(2n+1)L_2} t_{21} t_{22} + \frac{\Psi(L_1)}{2m+1} t_{11} t_{12} + \frac{\Psi(L_2)}{2n+1} t_{21 t_{22}}, \\ D &= \frac{2\Psi(L_1)}{(2m+1)L_1} t_{11}^2 + \frac{2\Psi(L_2)}{(2n+1)L_2} t_{21}^2 - \frac{\Psi(L_1)}{2m+1} t_{12}^2 - \frac{\Psi(L_2)}{2n+1} t_{22}^2 - \frac{\Psi(L_3)}{2k+1}. \end{aligned} \tag{5.24}$$

Recall that the numbers t_{ij} are the components of the vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ in (1.8). A direct computation shows that the system in α and β is uniquely solvable, since

$$\begin{aligned} C^2 - BD = &-\frac{\Psi(L_1)\Psi(L_3)}{(2m+1)(2l+1)} t_{11}^2 - \frac{\Psi(L_2)\Psi(L_3)}{(2n+1)(2l+1)} t_{21}^2 \\ &-\frac{\Psi(L_1)\Psi(L_2)}{(2m+1)(2n+1)} (t_{11} t_{22} - t_{12} t_{21})^2 \neq 0, \end{aligned} \tag{5.25}$$

given the fact that we have already observed that it is not restrictive to assume that all $t_{ij} \neq 0$ for $i, j = 1, 2$.

One can check that

(5.26)

$$\begin{cases} \alpha = \frac{1}{C^2 - BD}(C \frac{2\Psi(L_1)}{2m+1}) t_{11} x_1 + C \frac{2\Psi(L_2}{2n+1} t_{21} x_2 - D \frac{2\Psi(L_1)}{2m+1} t_{12} x_1 - D \frac{2\Psi(L_2)}{2n+1} t_{22} x_2 + D \frac{2\Psi(L_3)}{2k+1} x_3, \\ \beta = \frac{1}{C^2 - BD}(B \frac{2\Psi(L_1)}{2m+1}) t_{11} x_1 + B \frac{2\Psi(L_2)}{2n+1} t_{21} x_2 - C \frac{2\Psi(L_1)}{2m+1} t_{12} x_1 - C \frac{2\Psi(L_2)}{2n+1} t_{22} x_2 + C \frac{2\Psi(L_3)}{2k+1} x_3. \end{cases}$$

Replacing these values of α and β, together with (5.21), (5.22) and in the corresponding equations for the parameters $c_1, c_n, d_1, d_n, e_1, e_l, f_1, f_l$, we obtain that the whole problem is reduced to the solvability of the following nonlinear system in the variables $b_1, b_m, d_1, d_n, f_1, f_l$:

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} (b_1, d_1, f_1, b_m, d_n, f_l)^t = e^{-\delta L} A + \mathbf{N}, \tag{5.27}$$

where

$$T_1 = -I_{3\times 3},$$
$$T_3 = O_{3\times 3},$$

and

$$T_4 = \begin{pmatrix} 1 + \frac{2m}{2m+1}\bar{L}_1 h_1'' + A_1 & A_2 & A_3 \\ B_1 & 1 + \frac{2n}{2n+1}\bar{L}_2 h_2'' + B_2 & B_3 \\ C_1 & C_2 & 1 + \frac{2l}{2l+1}\bar{L}_3 h_3'' + C_3 \end{pmatrix},$$

with the constants A_j, B_j, and C_j defined as follows:

$$\begin{cases} A_1 = \frac{2\Psi(L_1)}{(C^2-BD)(2m+1)L_1}(Bt_{11}^2 - 2Ct_{11}t_{12} + Dt_{12}^2)(1 - \frac{\bar{L}_1 h_1''(0)}{2m+1}), \\ A_2 = \frac{2\Psi(L_2)}{(C^2-BD)(2n+1)L_1}(Bt_{21}t_{11} - Ct_{11}t_{22} - Ct_{21}t_{12} + Dt_{12}t_{22})(1 - \frac{\bar{L}_2 h_2''(0)}{2n+1}), \\ A_3 = \frac{2\Psi(L_3)}{(C^2-BD)(2k+1)L_1}(Ct_{11} - Dt_{12})(1 - \frac{\bar{L}_3 h_3''(0)}{2l+1}), \\ B_1 = \frac{2\Psi(L_1)}{(C^2-BD)(2m+1)L_2}(Bt_{11}t_{21} - Ct_{12}t_{21} - Ct_{11}t_{22} + Dt_{12}t_{22})(1 - \frac{\bar{L}_1 h_1''(0)}{2m+1}), \\ B_2 = \frac{2\Psi(L_2)}{(C^2-BD)(2n+1)L_2}(Bt_{21}^2 - 2Ct_{21}t_{22} + Dt_{22}^2)(1 - \frac{\bar{L}_2 h_2''(0)}{2n+1}), \\ B_3 = \frac{2\Psi(L_3)}{(C^2-BD)(2k+1)L_2}(Ct_{21} - Dt_{22})(1 - \frac{\bar{L}_3 h_3''(0)}{2l+1}), \\ C_1 = \frac{2\Psi(L_1)}{(C^2-BD)(2m+1)L_3}(Ct_{11} - Dt_{12})(1 - \frac{\bar{L}_1 h_1''(0)}{2m+1}), \\ C_2 = \frac{2\Psi(L_2)}{(C^2-BD)(2n+1)L_3}(Ct_{21} - Dt_{22})(1 - \frac{\bar{L}_2 h_2''(0)}{2n+1}), \\ C_3 = \frac{2\Psi(L_3)}{(C^2-BD)(2k+1)L_3}D(1 - \frac{\bar{L}_3 h_3''(0)}{2l+1}). \end{cases} \tag{5.28}$$

In order to solve (5.27), we need to compute the determinant of the matrix $\left(\begin{smallmatrix} T_1 & T_2 \\ T_3 & T_4 \end{smallmatrix}\right)$.

Set $H_i = 1 + \bar{L}_i h_i''(0)$. We write

$$\det \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} = \sum_{j=1}^{4} \Delta_j, \tag{5.29}$$

where

$$\Delta_1 = \left(H_1 - \frac{\bar{L}_1 h_1''(0)}{2m+1}\right)\left(H_2 - \frac{\bar{L}_2 h_2''(0)}{2n+1}\right)\left(H_3 - \frac{\bar{L}_3 h_3''(0)}{2l+1}\right), \tag{5.30}$$

$$\Delta_2 = A_1\left(H_2 - \frac{\bar{L}_2 h_2''(0)}{2n+1}\right)\left(H_3 - \frac{\bar{L}_3 h_3''(0)}{2l+1}\right) + B_2\left(H_1 - \frac{\bar{L}_1 h_1''(0)}{2m+1}\right)\left(H_3 - \frac{\bar{L}_3 h_3''(0)}{2l+1}\right)$$

$$+ C_3 \left(H_1 - \frac{\bar{L}_1 h_1''(0)}{2m+1} \right) \left(H_2 - \frac{\bar{L}_2 h_2''(0)}{2n+1} \right), \tag{5.31}$$

$$\Delta_3 = (C_3 B_2 - C_2 B_3) \left(H_1 - \frac{\bar{L}_1 h_1''(0)}{2m+1} \right) + (A_1 C_3 - A_3 C_1) \left(H_2 - \frac{\bar{L}_2 h_2''(0)}{2n+1} \right)$$

$$+ (A_1 B_2 - A_2 B_1) \left(H_3 - \frac{\bar{L}_3 h_3''(0)}{2l+1} \right), \tag{5.32}$$

and

$$\Delta_4 = \det \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}. \tag{5.33}$$

Using the expression of the constants A_j, B_j, and C_j in (5.28), an involved but direct computation gives us that

$$\Delta_4 = 0.$$

On the other hand, we observe the following: From the definition of the constant A_1, B_2, and C_3 in (5.28) and from the expression of $C^2 - BD$ in (5.25), we get that

$$A_1, B_2, C_3 = M(\mathbf{t}_i, \bar{L}_i, C_0) \frac{1}{L}, \tag{5.34}$$

where M is uniformly bounded from below away from zero as $\varepsilon \to 0$ (or equivalently as $L \to \infty$).

Furthermore, from the definition of

$$A_1 C_3 - C_1 A_3 = \frac{\Psi(L_1)\Psi(L_3)}{(2m+1)(2l+1)(BD - C^2) L_1 L_3} t_{11}^2,$$

we get that $|A_1 C_3 - C_1 A_3| \geq c_0 \frac{1}{L^2}$ as $\epsilon \to 0$, where $c_0 > 0$. In fact we get

$$C_3 B_2 - B_3 C_2, A_1 C_3 - C_1 A_3, A_1 B_2 - B_1 A_2 = N(\mathbf{t}_1, \bar{L}_i, C_0) \frac{1}{L^2},$$

where N is uniformly bounded from below away from zero as $L \to \infty$.

We thus conclude that under the assumption that at least one H_i is nonzero, the nonlinear system (5.27) can be uniquely solved by fixed point theorem of contraction mapping.

REFERENCES

[1] S. Alama, L. Bronsard, and C. Gui, *Stationary layered solutions in R^2 for an Allen-Cahn system with multiple well potential*, Calc. Var. Partial Differential Equations, 5 (1997), pp. 359–390.

[2] A. Ambrosetti, A. Malchiodi, and W. M. Ni, *Singularly perturbed elliptic equations with symmetry: Existence of solutions concentrating on spheres*, II, Indiana Univ. Math. J., 53 (2004), pp. 297–329.

[3] W. Ao, M. Musso, and J. Wei, *On spikes concentrating on line-segments to a semilinear Neumann problem*, J. Differential Equations, 251 (2011), pp. 881–901.

[4] P. Bates, E. N. Dancer, and J. Shi, *Multi-spike stationary solutions of the Cahn-Hilliard equation in higher-dimension and instability*, Adv. Differential Equations, 4 (1999), pp. 1–69.

[5] P. Bates and G. Fusco, *Equilibria with many nuclei for the Cahn-Hilliard equation*, J. Differential Equations, 160 (2000), pp. 283–356.

[6] L. Bronsard, C. Gui, and M. Schatzman, *A three-layered minimizer in R^2 for a variational problem with a symmetric three-well potential*, Comm. Pure Appl. Math., 49 (1996), pp. 677–715.

[7] E. N. Dancer, *New solutions of equations on $\mathbf{R}^n$*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 30 (2001), pp. 535–563.

[8] E. N. Dancer and S. Yan, *Multipeak solutions for a singularly perturbed Neumann problem*, Pacific J. Math., 189 (1999), pp. 241–262.

[9] E. N. Dancer and S. Yan, *Interior and boundary peak solutions for a mixed boundary value problem*, Indiana Univ. Math. J., 48 (1999), pp. 1177–1212.

[10] T. D'Aprile and A. Pistoia, *On the existence of some new positive interior spike solutions to a semilinear Neumann problem*, J. Differential Equations, 248 (2010), pp. 556–573.

[11] M. del Pino, P. L. Felmer, and J. Wei, *On the role of mean curvature in some singularly perturbed Neumann problems*, SIAM J. Math. Anal., 31 (1999), pp. 63–79.

[12] M. del Pino, P. Felmer, and J.-C. Wei, *Multiple peak solutions for some singular perturbation problems*, Calc. Var. Partial Differential Equations, 10 (2000), pp. 119–134.

[13] M. del Pino, P. Felmer, and J. Wei, *On the role of distance function in some singular perturbation problems*, Comm. Partial Differential Equations, 25 (2000), pp. 155–177.

[14] M. del Pino, M. Kowalcyzk, F. Pacard, and J. Wei, *The Toda system and multiple-end solutions of autonomous planar elliptic problems*, Adv. Math., 224 (2010), pp. 1462–1516.

[15] A. Floer and A. Weinstein, *Nonspreading wave packets for the cubic Schrödinger equation with bounded potential*, J. Funct. Anal., 69 (1986), pp. 397–408.

[16] A. Gierer and H. Meinhardt, *A theory of biological pattern formation*, Kybernetik (Berlin), 12 (1972), pp. 30–39.

[17] M. Grossi, A. Pistoia, and J. Wei, *Existence of multipeak solutions for a semilinear Neumann problem via nonsmooth critical point theory*, Calc. Var. Partial Differential Equations, 11 (2000), pp. 143–175.

[18] C. Gui and M. Schatzman, *Symmetric quadruple phase transitions*, Indiana Univ. Math. J., 57 (2008), pp. 781–836.

[19] C. Gui and J. Wei, *Multiple interior peak solutions for some singularly perturbed Neumann problems*, J. Differential Equations, 158 (1999), pp. 1–27.

[20] C. Gui and J. Wei, *On multiple mixed interior and boundary peak solutions for some singularly perturbed Neumann problems*, Canad. J. Math., 52 (2000), pp. 522–538.

[21] C. Gui, J. Wei, and M. Winter, *Multiple boundary peak solutions for some singularly perturbed Neumann problems*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 17 (2000), pp. 47–82.

[22] M. K. Kwong, *Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^N$*, Arch. Rational Mech. Anal., 105 (1991), pp. 243–266.

[23] Y.-Y. Li, *On a singularly perturbed equation with Neumann boundary condition*, Comm. Partial Differential Equations, 23 (1998), pp. 487–545.

[24] Y.-Y. Li and L. Nirenberg, *The Dirichlet problem for singularly perturbed elliptic equations*, Comm. Pure Appl. Math., 51 (1998), pp. 1445–1490.

[25] C. S. Lin, W. M. Ni, and I. Takagi, *Large amplitude stationary solutions to a chemotaxis systems*, J. Differential Equations, 72 (1988), pp. 1–27.

[26] F. H. Lin, W. M. Ni, and J. C. Wei, *On the number of interior peak solutions for a singularly perturbed Neumann problem*, Comm. Pure Appl. Math., 60 (2007), pp. 252–281.

[27] F. Mahmoudi and A. Malchiodi, *Concentration on minimal submanifolds for a singularly perturbed Neumann problem*, Adv. Math., 209 (2007), pp. 460–525.

[28] A. Malchiodi, *Some new entire solutions of semilinear elliptic equations in $\mathbb{R}^N$*, Adv. Math., 221 (2009), pp. 1843–1909.

[29] A. Malchiodi, *Solutions concentrating at curves for some singularly perturbed elliptic problems*, C. R. Math. Acad. Sci. Paris, 338 (2004), pp. 775–780.

[30] A. Malchiodi, *Concentration at curves for a singularly perturbed Neumann problem in three-dimensional domains*, Geom. Funct. Anal., 15 (2005), pp. 1162–1222.

[31] A. Malchiodi and M. Montenegro, *Boundary concentration phenomena for a singularly perturbed elliptic problem*, Comm. Pure Appl. Math., 55 (2002), pp. 1507–1568.

[32] A. Malchiodi and M. Montenegro, *Multidimensional boundary layers for a singularly perturbed Neumann problem*, Duke Math. J., 124 (2004), pp. 105–143.

[33] A. Malchiodi, W. M. Ni, and J. Wei, *Multiple clustered layer solutions for semilinear Neu-*

mann problems on a ball, Ann. Inst. H. Poincaré Anal. Non Linéaire, 22 (2005), pp. 143–163.

[34] M. Musso, F. Pacard, and J. Wei, *Finite-energy sign-changing solutions with dihedral symmetry for the stationary non linear Schrödinger equation*, J. Eur. Math. Soc. (JEMS), to appear.

[35] W. M. Ni, *Diffusion, cross-diffusion, and their spike-layer steady states*, Notices Amer. Math. Soc., 45 (1998), pp. 9–18.

[36] W. M. Ni, *Qualitative properties of solutions to elliptic problems*, in Stationary Partial Differential Equations, Vol. I, Handb. Differ. Equ., North-Holland, Amsterdam, 2004, pp. 157–233.

[37] W. M. Ni and I. Takagi, *On the shape of least energy solution to a semilinear Neumann problem*, Comm. Pure Appl. Math., 41 (1991), pp. 819–851.

[38] W. M. Ni and I. Takagi, *Locating the peaks of least energy solutions to a semilinear Neumann problem*, Duke Math. J., 70 (1993), pp. 247–281.

[39] W. M. Ni and J. Wei, *On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems*, Comm. Pure Appl. Math., 48 (1995), pp. 731–768.

[40] P. Sternberg, *Vector-valued local minimizers of nonconvex variational problems*, Rocky Mountain J. Math., 21 (1991), pp. 799–807.

[41] P. Sternberg and W. Ziemer, *Local minimizers of a three-phase partition problem with triple junctions*, Proc. Roy. Soc. Edinburgh Sect. A, 124 (1994), pp. 1059–1073.

[42] J. Wei, *On the boundary spike layer solutions to a singularly perturbed Neumann problem*, J. Differential Equations, 134 (1997), pp. 104–133.

[43] J. Wei, *On the interior spike layer solutions to a singularly perturbed Neumann problem*, Tohoku Math. J. (2), 50 (1998), pp. 159–178.

[44] J. Wei, *Existence and stability of spikes for the Gierer-Meinhardt system*, in Handbook of Differential Equations: Stationary Partial Differential Equations, Vol. 5, M. Chipot, ed., Elsevier/North-Holland, Amsterdam, 2008, pp. 489–581.

[45] J. Wei and M. Winter, *Stationary solutions for the Cahn-Hilliard equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 15 (1998), pp. 459–492.

[46] J. Wei and M. Winter, *Multi-peak solutions for a wide class of singular perturbation problems*, J. London Math. Soc., 59 (1999), pp. 585–606.

[47] J. Wei and J. Yang, *Concentration on lines for a singularly perturbed Neumann problem in two-dimensional domains*, Indiana Univ. Math. J., 56 (2007), pp. 3025–3074.