TRIPLE JUNCTION SOLUTIONS FOR A SINGULARLY PERTURBED NEUMANN PROBLEM∗

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Abstract. We consider the singularly perturbed Neumann problem

ε^2 ∆u − u + u^p = 0, u > 0 in Ω,

where Ω is a smooth and bounded domain in R^2. We construct a class of solutions which consist of large number of spikes concentrated on three line segments with a common endpoint which intersect ∂Ω orthogonally.

Key words. triple junctions, singularly perturbed problems, finite-dimensional reduction

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1. Introduction and statement of main results. We consider the following singularly perturbed elliptic problem:

(1.1) ε^2 ∆u − u + u^p = 0, u > 0 in Ω, \frac{∂u}{∂ν} = 0 on ∂Ω,

where Ω is a smooth bounded domain in R^2 with its unit outer normal ν, p > 1 and ε > 0 is a small parameter.

Problem (1.1) is known as the stationary equation of the Keller–Segel system in chemotaxes [25]. It can also be viewed as a limiting stationary equation for the Geirer–Meinhardt system in biological pattern formation [16]. Even though simple-looking, problem (1.1) has a rich and interesting structure of solutions. For the last fifteen years, it has received considerable attention. In particular, the various concentration phenomena exhibited by the solutions of (1.1) seem both mathematically intriguing and scientifically useful. We refer the reader to the three survey articles [35, 36, 44] for background and references.

In the pioneering papers [37, 38], Ni and Takagi proved the existence of least energy solutions to (1.1), that is, a solution u_ε with minimal energy. Furthermore, they showed in [37, 38] that, for each ε > 0 sufficiently small, u_ε blows up at a boundary point that maximizes the mean curvature of ∂Ω.

Since the publication of [38], problem (1.1) has received a great deal of attention and significant progress has been made. It has been proved that higher energy solutions exist, which concentrates at one or several points of the boundary, or at one or more points in the interior, or a combination of the two effects. See [4, 5, 8, 9, 10, 11, 12, 13, 17, 19, 20, 21, 23, 24, 26, 39, 42, 43, 45, 46] and the references therein. In particular, Lin, Ni, and Wei [26] showed that there are at least ΩC_{N}^{1}(ε) log(ε) interior spikes.
It seems natural to ask whether problem (1.1) has solutions which "concentrate" on higher-dimensional sets, e.g., curves or surfaces. In this regard, we mention that it has been conjectured for a long time that problem (1.1) actually possesses solutions which have m-dimensional concentration sets for every $0 \leq m \leq N - 1$. (See, e.g., [36].) Progress in this direction, although still limited, has also been made in [2, 29, 30, 31, 32, 33]. In particular, we mention the results of Malchiodi and Montenegro [31, 32] on the existence of solutions concentrating on the whole boundary provided that the sequence $\varepsilon$ satisfies some gap condition. The latter condition is called resonance.

In the papers [27, 30, 32], the higher-dimensional concentration set is on the boundary. A natural question is whether there are solutions with higher-dimensional concentration set inside the domain. In this regard, the first result was due to Wei and Yang [47], who proved the existence of layer on the line intersecting with the boundary of a two-dimensional domain orthogonally. In [47] the resonance condition is still required. This result was generalized in [3] to domains of dimensions higher than 2.

By rescaling and taking a limit in (1.1), we obtain the following nonlinear elliptic problem in the whole $\mathbb{R}^N$:

$$
\Delta u - u + u^p = 0, \quad u > 0 \text{ in } \mathbb{R}^N.
$$

Recently several interesting results have been obtained on new entire solutions to

$$
\Delta u - u + u^p = 0, \quad u > 0 \text{ in } \mathbb{R}^2.
$$

Dancer [7] first constructed solutions to (1.3) which decays in one direction and is periodic in another direction. In [14], these periodic solutions are called Dancer’s solutions. Using Dancer’s solutions, del Pino et al. built solutions to (1.3) with an even number of ends whose level sets are governed by the one-dimensional Toda system. On the other hand, in [28] Malchiodi constructed another new class of entire solutions to (1.3) by perturbing a configuration of infinitely many copies of the positive solution $w$ arranged along three rays meeting at a common point, where $w$ is the unique radially symmetric solution of

$$
\begin{cases}
\Delta w - w + w^p = 0, & u > 0 \text{ in } \mathbb{R}^2, \\
w(0) = \max_{y \in \mathbb{R}^2} w(y), & w \to 0 \text{ at } \infty.
\end{cases}
$$

Malchiodi’s solutions are known as triple junction solutions. The question we address in this paper is whether triple junction solutions to (1.1) still exist in a bounded domain.

The answer is yes: indeed, we construct triple junction solutions for problem (1.1) which are obtained as perturbation of a large number of $w$ centered at points arranged along a proper triple junction.

Let us first recall the asymptotic behavior of $w$, solution to (1.4), at infinity. It is known that there exists a constant $c_p > 0$, depending on $p$, such that

$$
\lim_{r \to \infty} e^r r^{\frac{4}{p-2}} w = c_p > 0 \quad \text{and} \quad \lim_{r \to \infty} \frac{w'}{w} = -1,
$$

where we have set $r := |x|$.

Furthermore, the solution $w$ is nondegenerate, namely, the $L^\infty$-kernel of the operator

$$
L_0 := \Delta - 1 + pw^{p-1},
$$
which is nothing but the linearized operator about \( w \), is spanned by the functions
\[
\partial_{x_1}w, \ldots, \partial_{x_N}w,
\]
which naturally belong to this space. We refer the reader to [22, 37].

Next we describe our result.

Assume that \( \Omega \) contains three line segments whose origin is the common endpoint which intersect orthogonally the boundary at exactly three points \( B_1, B_2, B_3 \) on \( \partial \Omega \). (See Figure 1.) We denote by
\[
t_1 = (t_{11}, t_{12}), \quad t_2 = (t_{21}, t_{22}), \quad t_3 = (0, -1)
\]
the unit tangent vectors of the three segments pointing from the origin, and by
\[
n_1 = (t_{12}, -t_{11}), \quad n_2 = (t_{22}, -t_{21}), \quad n_3 = (-1, 0),
\]
respectively, the unit normal vectors. Let \( \bar{L}_1, \bar{L}_2, \bar{L}_3 \) be the lengths of the three segments.

We assume that the mutual angles of the three lines satisfy
\[
t_i \angle t_j > \frac{\pi}{3}.
\]

Near the endpoints \( B_1, B_2, \) and \( B_3 \) of the segments, the boundary \( \partial \Omega \) is described as
\[
xn_1 + (h_1(x) + \bar{L}_1)t_1, \quad xn_2 + (h_2(x) + \bar{L}_2)t_2, \quad xn_3 + (h_3(x) + \bar{L}_3)t_3,
\]
respectively, where the functions \( h_i \)'s are smooth functions defined in intervals which include 0. It is not restrictive to assume that they satisfy \( h_i(0) = h_i'(0) = 0 \) for all \( i = 1, 2, 3 \).

Finally, we denote by \( k_i \) the curvature of \( \partial \Omega \) at the point \( B_i \), namely, \( k_i = -h_i''(0) \).

To state our result we need to introduce a function which, as we will show later, measures the interaction between two bumps \( w \) centered at two distinct points. We define \( \Psi(s) \) to be
\[
\Psi(s) := \int_{\mathbb{R}^2} w(x - se) \text{div}(w^n e) dx,
\]
where \( e \) is a unit vector.
Our result is the following.

**Theorem 1.1.** Assume $\Omega \subset \mathbb{R}^2$ contains three segments $\Gamma_1, \Gamma_2, \Gamma_3$ whose origin is the common endpoint which intersects orthogonally the boundary of $\Omega$ in exactly three points $B_1, B_2,$ and $B_3$ and whose lengths are $L_1, L_2,$ and $L_3$, respectively, and satisfy (1.10).

Assume that $k_i \neq 1/\bar{L}_i$ for at least one index $i$.

Then there exist $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$ and for any real number $L_3$ such that

\begin{equation}
|\ln \varepsilon| \leq L_3, \quad \lim_{\varepsilon \to 0} \varepsilon L_3 = 0,
\end{equation}

for some positive constant $c > 0$ which depends on $\Omega$ and on the lengths of the segments, if there exist positive real numbers $L_1, L_2$ and integers $m,n,l$ satisfying the following balancing formula

\begin{equation}
\Psi(L_1)t_1 + \Psi(L_2)t_2 + \Psi(L_3)t_3 = 0
\end{equation}

and

\begin{equation}
\left(m + \frac{1}{2}\right)L_1 = \frac{\bar{L}_1}{\varepsilon}, \quad \left(n + \frac{1}{2}\right)L_2 = \frac{\bar{L}_2}{\varepsilon}, \quad \left(l + \frac{1}{2}\right)L_3 = \frac{\bar{L}_3}{\varepsilon},
\end{equation}

then there exists a solution $u_\varepsilon$ to problem (1.1). Furthermore there exist $m+n+l$ points

\begin{align*}
P_\varepsilon^j & \text{ for } j = 1, \ldots, m, \\ Q_\varepsilon^j & \text{ for } j = 1, \ldots, n,
\end{align*}

and

\begin{align*}
R_\varepsilon^j & \text{ for } j = 1, \ldots, l
\end{align*}

distributed uniformly at distance $\varepsilon L_1, \varepsilon L_2, \varepsilon L_3$, respectively, along the segments $\Gamma_1, \Gamma_2, \Gamma_3$ and a point $O_\varepsilon$ near the origin such that

\begin{equation}
u_\varepsilon(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} \left[ w \left( \frac{x - P_\varepsilon^j}{\varepsilon} \right) + w \left( \frac{x - Q_\varepsilon^j}{\varepsilon} \right) + w \left( \frac{x - R_\varepsilon^j}{\varepsilon} \right) \right] + o(1),
\end{equation}

where $o(1) \to 0$ as $\varepsilon \to 0$ uniformly over compact sets of $\mathbb{R}^2$.

**Remark 1.1.** As a consequence of the balancing condition (1.13) and the choice (1.8)–(1.9), without loss of generality we can assume that the mutual angles of the three lines satisfy that $t_1 \angle t_3 \geq \frac{\pi}{2} + \theta_0, t_2 \angle t_3 \geq \frac{\pi}{2} + \theta_0$, where $\theta_0$ is a constant. Thus we can get that $t_{ij} \neq 0$ for $i,j = 1,2$.

Conditions (1.13)–(1.14) are satisfied under some restrictions on $\bar{L}_i$ or $\varepsilon$. See the remarks below.

**Remark 1.2.** Let $L = L_3 \gg 1$ be given and multiply relation (1.13) against $t_3$ first and then against $n_3$. This gives the system

\begin{align*}
\Psi(L_1)t_1 \cdot t_3 + \Psi(L_2)t_2 \cdot t_3 & = -\Psi(L), \\
\Psi(L_1)t_1 \cdot n_3 + \Psi(L_2)t_2 \cdot n_3 & = 0.
\end{align*}
This system is solvable since
\[
d = (t_1 \cdot t_3)(t_2 \cdot n_3) - (t_2 \cdot t_3)(t_1 \cdot n_3) = t_{12}t_{21} - t_{11}t_{22} \neq 0.
\]

In this case, we thus have
\[
\Psi(L_1) = \frac{t_{21}}{d} \Psi(L), \quad \Psi(L_2) = -\frac{t_{11}}{d} \Psi(L).
\]

Now, since \(\Psi(s) = C_p e^{-s^2/2}(1 + O(s^{-1}))\) as \(s \to \infty\), with \(C_p\) a positive constant, we obtain that
\[
L_1 = L - C_1, \quad L_2 = L - C_2,
\]
where
\[
C_1 = -\log \frac{t_{21}}{d} + O\left(\frac{1}{L}\right), \quad C_2 = -\log \left(-\frac{t_{11}}{d}\right) + O\left(\frac{1}{L}\right).
\]

Then the condition (1.14) becomes
\[
\frac{\bar{L}_1}{L_1} - \frac{2l + 1}{2m + 1} = \frac{\epsilon C_1(2l + 1)}{L_1}, \quad \frac{\bar{L}_3}{L_2} - \frac{2l + 1}{2n + 1} = \frac{\epsilon C_2(2l + 1)}{L_2}.
\]

**Remark 1.3.** Let us consider the case \(C_1 = C_2 = 0\), i.e.,
\[
t_i \angle t_j = \frac{2\pi}{3}
\]

In this case, condition (1.18) is satisfied if the ratios \(\frac{\bar{L}_1}{L_1}, \frac{\bar{L}_2}{L_2}\) are rational numbers of the form \(\frac{2k + 1}{2q + 1}\). In this case, we let \(\frac{\bar{L}_1}{L_1} = \frac{2k + 1}{2q + 1}, \frac{\bar{L}_2}{L_2} = \frac{2k + 1}{2r + 1}\). Then we choose \(2l + 1 = (2r_1 + 1)(2r_2 + 1)(2k + 1), 2m + 1 = (2q_1 + 1)(2q_2 + 1)(2m + 1), 2n + 1 = (2q_2 + 1)(2r_1 + 1)(2k + 1)\), where \(1 \ll k < \frac{C}{\log c}\). Then (1.12)–(1.14) are satisfied.

**Remark 1.4.** We consider another case, \(C_1 = C_2 \neq 0\), i.e., \(L_1 = L_2 \neq L_3\). (We may assume that \(C_1 = C_2 > 0\).) In this case, we assume that \(L_1 = L_2\). Then we choose \(m = n\) and \((m, l)\) such that
\[
\frac{\bar{L}_3}{L_1} - \frac{2l + 1}{2m + 1} = \frac{\epsilon C_1(2l + 1)}{L_1},
\]

which is possible by the following choices: we may always choose a sequence of integers \(m, l \to +\infty\) such that \(\frac{L_1}{L_3} - \frac{2l + 1}{2m + 1} \sim -\frac{4}{m}\). Then (1.20) is satisfied if we choose a sequence \(\epsilon_{(m, l)} \sim \frac{1}{m} \to \) satisfying (1.18).

If \(C_1 \neq C_2\), condition (1.18) is more complicated. Some conditions on the ratio \(\frac{C_1}{C_2}\) are needed.

By the above remark, we now have the following corollary.

**Corollary 1.2.** Assume \(\Omega \subset \mathbb{R}^2\) contains three segments \(\Gamma_1, \Gamma_2, \Gamma_3\) whose origin is the common endpoint which intersects orthogonally the boundary of \(\Omega\) in exactly three points \(B_1, B_2, B_3\) and whose lengths are \(L_1, L_2, L_3\), respectively, and satisfy (1.10). Assume that at least one \(k_i \neq 1/L_i\). Furthermore suppose
\[
h_i \angle t_j = \frac{2\pi}{3}
\]
and that the ratios $\frac{L_i}{L_j}$ are rational numbers of the form $\frac{2r+1}{2q+1}$. Then problem (1.1) has a triple junction solution for all $\epsilon$ sufficiently small.


\begin{equation}
\Delta U - \nabla W(U) = 0, \quad U : \mathbb{R}^2 \to \mathbb{R}^2,
\end{equation}

and Gui and Schatzman [18] generalized to symmetric quadruple layered solutions. See also Alama, Bronsard, and Gui [1]. As far as we know, Theorem 1.1 and Corollary 1.2 are the first results on the construction of triple junctions in bounded domains. We believe that solutions concentrating on more complex networks may also exist.

2. Ansatz and sketch of the proof. By the scaling $x = \frac{z}{\epsilon}$, problem (1.1) becomes

\begin{equation}
\Delta u - u + u^p = 0, \quad u > 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega,
\end{equation}

where $\Omega = \{ z : z \in \Omega \}$.

We consider a large number $L_i$, and $L_1, L_2, L_3 \in \mathbb{R}$ for which the following condition holds:

\begin{equation}
|L_i - L_j| \leq C_0
\end{equation}

for $i = 1, 2, 3$, where $C_0$ is a positive constant.

Define

\begin{equation}
\begin{cases}
O = (\alpha, \beta), \\
P_i = (L_1 i + a_i) t_1 + L_1 b_i n_1, \\
Q_j = (L_2 j + c_j) t_2 + L_2 d_j n_2, \\
R_k = (L_3 k + e_k) t_3 + L_3 f_k n_3
\end{cases}
\end{equation}

for $i = 1, \ldots, m$, $j = 1, \ldots, n$, $k = 1, \ldots, l$, where the vectors $t_i$ and $n_i$, $i = 1, 2, 3$, are defined in (1.8) and (1.9).

We will assume that all the $\alpha, \beta, a_i, b_i, c_j, d_j, e_k, f_k$ are uniformly bounded as $\epsilon \to 0$. It will be convenient to adopt the following notation:

\begin{align*}
a &= (a_1, a_2, \ldots, a_m), & b &= (b_1, b_2, \ldots, b_m), \\
c &= (c_1, c_2, \ldots, c_n), & d &= (d_1, d_2, \ldots, d_n), \\
e &= (e_1, e_2, \ldots, e_l), & f &= (f_1, f_2, \ldots, f_l)
\end{align*}

We thus assume

\begin{equation}
\|a, b, c, d, e, f, \alpha, \beta\| \leq c
\end{equation}

for some fixed $c > 0$.

We will denote by $Y$ the set of all points, namely,

\begin{equation}
Y = \{ z : z = O, P_i, Q_j, R_k, i = 1, \ldots, m, j = 1, \ldots, n, k = 1, \ldots, l \}.
\end{equation}

Let us define the function

\begin{equation}
U(x) = \sum_{z \in Y} U_z(x), \text{ with } U_z(x) = w_z(x) - \varphi_z(x),
\end{equation}
where
\[ w_z(x) = w(x - z) \]
and
\[ -\Delta \varphi_z + \varphi_z = 0 \quad \text{in} \quad \Omega_z, \quad \frac{\partial \varphi_z}{\partial \nu} = \frac{\partial w(x - z)}{\partial \nu} \quad \text{on} \quad \partial \Omega_z. \]

The next lemma, whose proof is contained in [26], provides a qualitative description of the function \( \varphi_z \).

**Lemma 2.1.** Assume that \( M | \ln \varepsilon | \leq d(P, \partial \Omega_z) \leq \frac{\delta}{2} \) for some constant \( M \) depending on \( N \) and a constant \( \delta > 0 \) sufficiently small. Then
\[ \varphi_P(x) = -(1+o(1))w(x - P^*) + o(\varepsilon^3), \]
where \( P^* = P + 2d(P, \partial \Omega_z)\nu_{\bar{P}}, \nu_{\bar{P}} \) denotes the unit outer normal of \( \partial \Omega_z \) at \( \bar{P} \), and \( \bar{P} \) is the unique point on \( \partial \Omega_{\varepsilon} \) such that \( d(P, \bar{P}) = d(P, \partial \Omega_{\varepsilon}) \).

We look for a solution of (2.1) of the form \( u = U + \phi \). We set
\[ L(\phi) = -\Delta \phi + \phi - pU^{p-1}\phi, \]
\[ E = U^p - \sum_{z \in Y} w_z^p, \]
and
\[ N(\phi) = (U + \phi)^p - U^p - pU^{p-1}\phi. \]
Problem (2.1) gets rewritten as
\[ L(\phi) = E + N(\phi) \quad \text{in} \quad \Omega_{\varepsilon}, \quad \frac{\partial \phi}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega_{\varepsilon}. \]

Consider a cutoff function \( \chi \in C^\infty(-\infty, \infty) \) such that
\[ \chi(s) \equiv 1 \quad \text{for} \quad s \leq -1, \quad \chi(s) \equiv 0 \quad \text{for} \quad s \geq 0. \]
We fix a constant \( \zeta > 0 \) (independent of \( L \)) so that the balls of radius \( \frac{L - \zeta}{2} \), centered at different points of \( Y \), are mutually disjoint for all \( L \) large enough. We define the compactly supported functions
\[ Z_z(x) := \chi(2|x - z| - L + \zeta) \nabla w(x - z) \]
for \( z \in Y \). Observe that, by construction (in fact given the choice of \( \zeta \)), we have
\[ \int_{\Omega_{\varepsilon}} Z_{z_1}(x)Z_{z_2}(x) \, dx = 0 \]
if \( z_1 \neq z_2 \).

Consider the following intermediate nonlinear projected problem: Given the points in (2.3), find a function \( \phi \) in some proper space and constant vectors \( c_z \) such that
\[ \begin{align*}
L(\phi) &= E + N(\phi) + \sum_{z \in Y} c_z Z_z \quad \text{in} \quad \Omega_{\varepsilon}, \\
\frac{\partial \phi}{\partial \nu} &= 0 \quad \text{on} \quad \partial \Omega_{\varepsilon}, \\
\int_{\Omega_{\varepsilon}} \phi Z_z &= 0 \quad \text{for} \quad z \in Y.
\end{align*} \]
We show unique solvability of problem (2.14) by means of a fixed point argument. Furthermore we prove that the solution \( \phi \) depends smoothly on the points \( z \).

To do so, in section 3 we develop a solvability theory for the linear projected problem

\[
\begin{aligned}
L \phi &= h + \sum_{z \in Y} c_z Z_z \quad \text{in} \quad \Omega_{\varepsilon}, \\
\frac{\partial \phi}{\partial \nu} &= 0 \quad \text{on} \quad \partial \Omega_{\varepsilon}, \\
\int_{\Omega_{\varepsilon}} \phi Z_z &= 0 \quad \text{for} \quad z \in Y
\end{aligned}
\]

for a given right-hand side \( h \) in some proper space. Roughly speaking, the linear operator \( L \) is a superposition of the linear operators

\[
L_j \phi = \Delta \phi - \phi + p a^{\beta/(\beta-1)}(x - z)\phi, \quad z \in Y.
\]

Once we have the unique solvability of problem (2.14), which is proved in section 4, it is clear that \( u = U + \phi \) is indeed an exact solution to our original problem (1.1), with the qualitative properties described in Theorem 1.1, if we can prove that the constants \( c_z \) appearing in (2.14) are all zero. This can be done adjusting properly the parameters \( a, b, c, d, e, f, \alpha, \beta \), as will be shown in section 5, where the proof of Theorem 1.1 will be also given.

3. Linear theory. Our main result in this section states bounded solvability of problem (2.15), uniformly in small \( \varepsilon \), points \( z \) belonging to \( Y \) given by (2.5), uniformly separated from each other at distance \( O(L) \). Indeed we assume that the points \( z \) given by (2.3) satisfy constraints (2.4).

Given \( 0 < \eta < 1 \), consider the norms

\[
\| h \|_* = \sup_{x \in \Omega_{\varepsilon}} \left| \sum_z e^{\eta|x-z|} h(x) \right|,
\]

where \( z \in Y \) with \( Y \) defined in (2.5).

**Proposition 3.1.** Let \( c > 0 \) be fixed. There exist positive numbers \( \eta \in (0, 1), \varepsilon_0 \), and \( C \), such that for all \( 0 \leq \varepsilon \leq \varepsilon_0 \), for all integers \( m, n, l \) and positive real numbers \( L_i \) given by (1.13) and satisfying (1.14), for any points \( z, z \in Y \) in (2.5) given by (2.3) and satisfying (2.4), there is a unique solution \( (\phi, c_z) \) to problem (2.15). Furthermore

\[
\| \phi \|_* \leq C \| h \|_*.
\]

The proof of the above proposition, which we postpone to the end of this section, is based on the Fredholm alternative theorem for compact operators and an a priori bound for solution to (2.15) that we state (and prove) next.

**Proposition 3.2.** Let \( c > 0 \) be fixed. Let \( h \) with \( \| h \|_* \) bounded and assume that \( (\phi, c_z) \) is a solution to (2.15). Then there exist positive numbers \( \varepsilon_0 \) and \( C \), such that for all \( 0 \leq \varepsilon \leq \varepsilon_0 \), for all integers \( m, n, l \) and positive real numbers \( L_1, L_2, L_3 \) given by (1.13) and satisfying (1.14), for any points \( z, z \in Y \) given by (2.3) and satisfying (2.4), one has

\[
\| \phi \|_* \leq C \| h \|_*.
\]

**Proof.** We argue by contradiction. Assume there exists \( \phi \) solution to (2.15) and

\[
\| h \|_* \to 0, \quad \| \phi \|_* = 1.
\]
We prove that

\[(3.4)\]
\[c_z \to 0.\]

Multiply the equation in (2.15) with any of the components of the function \(Z_z\) defined in (2.12), which, with abuse of notation, we still denote by \(Z_z\), and integrate in \(\Omega_\varepsilon\). We get

\[
\int_{\Omega_\varepsilon} L\phi Z_z(x) = \int_{\Omega_\varepsilon} hZ_z + c_z \int_{\Omega_\varepsilon} Z_z^2,
\]

since (2.13) holds true. Given the exponential decay at infinity of \(\partial_x w\) and the definition of \(Z_z\) in (2.12), we get

\[(3.5)\]
\[\int_{\Omega_\varepsilon} Z_z^2 = \int_{\mathbb{R}^N} (\nabla w)^2 + O(e^{-\delta L}) \quad \text{as} \quad L \to \infty \]

for some \(\delta > 0\). On the other hand

\[
\left| \int_{\Omega_\varepsilon} hZ_z \right| \leq C \|h\| \int_{\Omega_\varepsilon} \nabla w(x - z)e^{-\eta|x - z|} \leq C \|h\|.
\]

Here and in what follows, \(C\) stands for a positive constant independent of \(\varepsilon\), as \(\varepsilon \to 0\) (or equivalently independent of \(L\) as \(L \to \infty\)). Finally, if we write \(\tilde{Z}_z(x) = \nabla w(x - z)\) and \(\chi = \chi(2|x - z| - L + \zeta)\), we have

\[
\int_{\Omega_\varepsilon} L\phi Z_z(x) = \int_{B(z, L - \frac{\zeta}{2})} [\Delta \tilde{Z}_z - \tilde{Z}_z + pw^{p-1}(x - z)\tilde{Z}_z] \chi \phi
\]

\[
+ \int_{\partial B(z, \frac{\zeta}{2})} \phi \nabla (\chi(2|x - z| - L + \zeta) \tilde{Z}_z) \cdot n
\]

\[
- \int_{B(z, \frac{\zeta}{2})} \phi (\tilde{Z}_z \Delta \chi + 2\nabla \chi Z_z)
\]

\[
+ p \int_{B(p_j, \frac{\zeta}{2})} (U^{p-1} - u^{p-1}(x - z)) \phi \tilde{Z}_z \chi.
\]

Next we estimate all the terms of the previous formula.

Since

\[\Delta \tilde{Z}_z - \tilde{Z}_z + pw^{p-1}(x - P_j)\tilde{Z}_z = 0\]

we get that the first term is 0. Furthermore, using the estimates in (1.5), we have

\[
\left| \int_{\partial B(z, \frac{\zeta}{2})} \phi \nabla (\chi(2|x - z| - L + \zeta) \tilde{Z}_z) \cdot n \right|
\]

\[
\leq C \|\phi\| \int_{\partial B(z, \frac{\zeta}{2})} e^{-(1+N)|x - z|} |x - z|^{-\frac{N}{2}} dx
\]

\[
\leq Ce^{-(1+N)\delta L} \|\phi\|.
\]
for some proper $\xi > 0$. Using again (1.5), the third integral can be estimated as follows:

$$
\left| \int_{B(z, \frac{L-\xi}{2})} \phi \tilde{Z}_{\xi} \Delta \chi (2|x-z| - L + \xi) + 2\nabla \chi (2|x-z| - L + \xi) \nabla \tilde{Z}_z \right|
\leq C \|\phi\| \int_{\frac{L-\xi}{2}}^{\frac{L+\xi}{2}} e^{-s(1+\eta)s} s^\frac{3}{2} ds \leq C e^{-\left(1+\frac{1}{2}\right)\|\phi\|},
$$

again for some $\xi > 0$. Finally, we observe that in $B(z, \frac{L-\xi}{2})$

$$
|U^{p-1}(x) - w^{p-1}(x-z)| \leq w^{p-2}(x-z) \left[ \sum_{x_i \neq z} w(x-x_i) \right].
$$

Having this, we conclude that

$$
\left| p \int_{B(z, \frac{L-\xi}{2})} (U^{p-1}(x) - w^{p-1}(x-z)) \phi \tilde{Z}_{\xi} \chi (2|x-z| - L + \xi) \right|
\leq C e^{-\left(1+\frac{1}{2}\right)\|\phi\|},
$$

for a proper $\xi > 0$, depending on $N$ and $p$. We thus conclude that

$$
(3.7) \quad |c_z| \leq C \left[ e^{-\left(1+\frac{1}{2}\right)\|\phi\|} + \|h\| \right].
$$

Thus we get the validity of (3.4), since we are assuming $\|\phi\| = 1$ and $\|h\| \to 0$.

Now let $\eta \in (0,1)$. It is easy to check that the function

$$
W := \sum_{z \in Y} e^{-\eta|z-z|}
$$

satisfies

$$
L W \leq \frac{1}{2} (\eta^2 - 1) W
$$

in $\Omega \setminus \cup_{z \in Y} B(z, \rho)$ provided $\rho$ is fixed large enough (independently of $L$). Hence the function $W$ can be used as a barrier to prove the pointwise estimate

$$
(3.8) \quad |\phi|(x) \leq C \left( \|L \phi\| + \sum_{z} \|\phi\|_{L^\infty(\partial B(z, \rho))} \right) W(x)
$$

for all $x \in \Omega \setminus \cup_{z \in Y} B(z, \rho)$.

Granted these preliminary estimates, the proof of the result goes by contradiction. Let us assume there exist a sequence of $L$ tending to $\infty$ and a sequence of solutions of (2.15) for which the inequality is not true. The problem being linear, we can reduce to the case where we have a sequence $L^{(n)}$ tending to $\infty$ and sequences $h^{(n)}$, $\phi^{(n)}$, $e^{(n)}$ such that

$$
\|h^{(n)}\| \to 0 \quad \text{and} \quad \|\phi^{(n)}\| = 1.
$$
But (3.4) implies that we also have
\[ |c^{(n)}| \to 0. \]

Then (3.8) implies that there exists \( z^{(n)} \in Y \) (see (2.5) for the definition of \( Y \)) such that
\[ \|\phi^{(n)}\|_{L^\infty(B(z^{(n)},\rho))} \geq C \]
for some fixed constant \( C > 0 \). Using elliptic estimates together with Ascoli–Arzelà’s theorem, we can find a sequence \( z^{(n)} \) and extract from the sequence \( \phi^{(n)}(\cdot - z^{(n)}) \) a subsequence which will converge (on compact) to \( \phi_\infty \), a solution of
\[ \left( \Delta - 1 + pw^{p-1} \right) \phi_\infty = 0 \]
in \( \mathbb{R}^2 \), which is bounded by a constant times \( e^{-\eta |x|} \), with \( \eta > 0 \). Moreover, recall that \( \phi^{(n)} \) satisfies the orthogonality conditions in (2.15). Therefore, the limit function \( \phi_\infty \) also satisfies
\[ \int_{\mathbb{R}^2} \phi_\infty \nabla w \, dx = 0. \]

But the solution \( w \) being nondegenerate, this implies that \( \phi_\infty \equiv 0 \), which is certainly in contradiction with (3.9), which implies that \( \phi_\infty \) is not identically equal to 0.

Having reached a contradiction, this completes the proof of the proposition.

We can now prove Proposition 3.1.

**Proof of Proposition 3.1.** Consider the space
\[ \mathcal{H} = \left\{ u \in H^1(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} uZ_z = 0, \quad z \in Y \right\}. \]

Notice that problem (2.15) in \( \phi \) gets rewritten as
\[ \phi + K(\phi) = \tilde{h} \quad \text{in} \quad \mathcal{H}, \]
where \( \tilde{h} \) is defined by duality and \( K : \mathcal{H} \to \mathcal{H} \) is a linear compact operator. Using Fredholm’s alternative, showing that (3.10) has a unique solution for each \( \tilde{h} \) is equivalent to showing that the equation has a unique solution for \( \tilde{h} = 0 \), which in turn follows from Proposition 3.2. The estimate (3.2) follows directly from Proposition 3.2. This concludes the proof of Proposition 3.1.

In the following, if \( \phi \) is the unique solution given by Proposition 3.1, we set
\[ \phi = A(h). \]
Estimate (3.2) implies
\[ \|A(h)\|_* \leq C\|h\|_. \]

**4. The nonlinear projected problem.** For small \( \varepsilon \), large \( L \), and fixed points \( z \in Y \) (see (2.5), (2.4), and (2.3)), we show solvability in \( \phi \), \( c_z \) of the nonlinear projected problem
\[
\begin{cases}
L(\phi) = E + N(\phi) + \sum_{z \in Y} c_z Z_z & \text{in } \Omega_\varepsilon, \\
\frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial \Omega_\varepsilon, \\
\int_{\Omega_\varepsilon} \phi Z_z = 0 & \text{for } z \in Y.
\end{cases}
\]
We have the validity of the following result.
Proposition 4.1. Let $c > 0$ be fixed. There exist positive numbers $\varepsilon_0, C,$ and $\xi > 0$ such that for all $\varepsilon \leq \varepsilon_0$, for all integers $m,n,l$ and positive real numbers $L_i$ given by (1.13) and satisfying (1.14), for any points $z, z \in Y$ given by (2.3) and satisfying (2.4), there is a unique solution $(\phi, c_z)$ to problem (2.14). This solution depends continuously on the parameters of the construction (namely, $a, b, c, d, e, f, \alpha, \beta$) and furthermore

$$
\|\phi\|_* \leq C e^{-\frac{(1+\varepsilon_0)}{2} L}.
$$

Proof. The proof relies on the contraction mapping theorem in the $\| \cdot \|_*$-norm introduced above. Observe that $\phi$ solves (2.14) if and only if

$$
\phi = A (E + N(\phi)),
$$

where $A$ is the operator introduced in (3.11). In other words, $\phi$ solves (2.14) if and only if $\phi$ is a fixed point for the operator

$$
T(\phi) := A (E + N(\phi)).
$$

Given $r > 0$, define

$$
B = \left\{ \phi \in C^2(\Omega) : \|\phi\|_* \leq r e^{-\frac{(1+\varepsilon_0)}{2} L}, \int_{\Omega_\varepsilon} \phi Z_z = 0 \right\}.
$$

We will prove that $T$ is a contraction mapping from $B$ in itself.

To do so, we claim that

$$
\|E\|_* \leq C e^{-\frac{(1+\varepsilon_0)}{2} L}
$$

and

$$
\|N(\phi)\|_* \leq C \left[ \|\phi\|_*^2 + \|\phi\|_*^p \right]
$$

for some fixed function $C$ independent of $L$ as $L \to \infty$. We postpone the proof of the estimates above until the end of the proof of this proposition. Assuming the validity of (4.4) and (4.5), and taking into account (3.12), we have for any $\phi \in B$

$$
\|T(\phi)\|_* \leq C \left[ \|E + N(\phi)\|_* \right] \leq C \left[ e^{-\frac{(1+\varepsilon_0)}{2} L} + r^2 e^{-\frac{(1+\varepsilon_0)}{2} L} + r^p e^{-\frac{p(1+\varepsilon_0)}{2} L} \right]
$$

$$
\leq r e^{-\frac{p(1+\varepsilon_0)}{2} L}
$$

for a proper choice of $r$ in the definition of $B$, since $p > 1$.

Take now $\phi_1$ and $\phi_2$ in $B$. Then it is straightforward to show that

$$
\|T(\phi_1) - T(\phi_2)\|_* \leq C \|N(\phi_1) - N(\phi_2)\|_*
$$

$$
\leq C \left[ \|\phi_1\|_*^{\min(1,p-1)} + \|\phi_2\|_*^{\min(1,p-1)} \right] \|\phi_1 - \phi_2\|_*
$$

$$
\leq o(1) \|\phi_1 - \phi_2\|_*.
$$

This means that $T$ is a contraction mapping from $B$ into itself.
To conclude the proof of this proposition we are left to show the validity of (4.4) and (4.5). We start with (4.4).

Fix \( z \in Y \) and consider the region \(|x - z| \leq \frac{L}{E+\sigma}\), where \( \sigma \) is a small positive number to be chosen later. In this region the error \( E \), whose definition is in (2.9), can be estimated in the following way (see (1.5)):

\[
|E(x)| \leq C \left[ \sum_{z \neq z} w(x - x_i) + \sum_{x_i \neq z} w^p(x - x_i) \right] \\
\leq C w^{p-1}(x - z) \sum_{x_i \neq z} e^{-\left(\frac{1}{2} + \frac{\sigma}{2(p+1)}\right) L} \\
\leq C w^{p-1}(x - z) e^{-\left(\frac{1}{2} + \frac{\sigma}{2(p+1)}\right) L} e^{-\frac{\sigma}{2(p+1)} L} \\
\leq C w^{p-1}(x - z) e^{-\frac{1 + \sigma}{2} L}
\]

(4.6)

for a proper choice of \( \xi > 0 \).

Consider now the region \(|x - z| > \frac{L}{E+\sigma}\) for all \( j \). Since \( 0 < \mu < p - 1 \), we write \( \mu = p - 1 - M \). From the definition of \( E \), we get in the region under consideration

\[
|E(x)| \leq C \left[ \sum_{z \neq z} w^p(x - z) \right] \leq C \left[ \sum_{z \neq z} e^{-\mu|x-z|} \right] e^{-(p-\mu)i \theta} \\
\leq \left[ \sum_{z \neq z} e^{-\mu|x-z|} \right] e^{-\frac{1 + \sigma}{2} L} \leq \left[ \sum_{z} e^{-\mu|x-z|} \right] e^{-\frac{1 + \sigma}{2} L}
\]

(4.7)

for some \( \xi > 0 \) if we chose \( M \) and \( \sigma \) small enough. From (4.6) and (4.7) we get (4.4).

We now prove (4.5). Let \( \phi \in \mathcal{B} \). Then

\[
|N(\phi)| \leq |(U + \phi)^p - U^p - pU^{p-1}\phi| \leq C(\phi^2 + |\phi|^p).
\]

Thus we have

\[
\left| \sum_j e^{\gamma j|x-P_j|} N(\phi) \right| \leq C \|\phi\|_* (|\phi| + |\phi|^{p-1}) \\
\leq C (\|\phi\|^2 + \|\phi\|^p).
\]

This gives (4.5).

A direct consequence of the fixed point characterization of \( \phi \) given above together with the fact that the error term \( E \) depends continuously (in the \( * \)-norm) on the parameters \((a, b, c, d, e, f, \alpha, \beta)\) is that the map

\[
(a, b, c, d, e, f, \alpha, \beta) \rightarrow \phi
\]

into the space \( C(\overline{\Omega}_z) \) is continuous (in the \( * \)-norm). This concludes the proof of the proposition.

Given points \( z \in Y \) satisfying constraint (2.4), Proposition 4.1 guarantees the existence (and gives estimates) of a unique solution \( \phi, c_z, z \in Y \), to problem (2.14).
It is clear then that the function \( u = U + \phi \) is an exact solution to our problem (1.1), with the required properties stated in Theorem 1.1 if we show that there exists a configuration for the points \( z \) that gives all the constants \( c_z \) in (2.14) equal to zero. In order to do so we first need to find the correct conditions on the points to get \( c_z = 0 \). This condition is naturally given by projecting in \( L^2(\Omega) \) the equation in (2.14) into the space spanned by \( Z_z \), namely, by multiplying the equation in (2.14) by \( Z_z \) and integrating all over \( \Omega_z \). We will do it in detail in the next and final section.

5. Error estimates and the proof of Theorem 1.1. The first aim of this section is to evaluate the \( L^2(\Omega) \) projection of the error term \( E \) in (2.9) against the elements \( Z_z \) in (2.12) for any \( z \in Y \) in (2.5).

Let us introduce the following notation.

Let \( P_m^* = P_m + 2d(P_m, \partial \Omega_x)\nu_{\bar{P}_m} \), where \( \nu_{\bar{P}_m} \) denotes the outer unit normal at \( \bar{P}_m \) on \( \partial \Omega_x \) and \( \bar{P}_m \) is the unique point on \( \partial \Omega_x \) such that \( d(P_m, \bar{P}_m) = d(P_m, \partial \Omega_x) \). In an analogous way we define \( Q_n^*, Q_n \) and \( R^*_i, R_i \).

Thus there exist three coordinates \( x_1, x_2, \) and \( x_3 \) such that

\[
P_m = L_1x_1n_1 + \left( \frac{\bar{L}_1 + h_1(\epsilon L_1x_1)}{\epsilon} \right) t_1, \quad Q_n = L_2x_2n_2 + \left( \frac{\bar{L}_2 + h_2(\epsilon L_2x_2)}{\epsilon} \right) t_2,
\]
and

\[
R_i = L_3x_3n_3 + \left( \frac{\bar{L}_3 + h_3(\epsilon L_3x_3)}{\epsilon} \right) t_3.
\]

More explicitly, the coordinates \( x_i \) are defined as solutions of the system

\[
\begin{aligned}
L_1(x_1 - b_m) + \left( \frac{\bar{L}_1}{2} + h_1(\epsilon L_1x_1) - a_m \right) h_1'(\epsilon L_1x_1) = 0, \\
L_2(x_2 - a_n) + \left( \frac{\bar{L}_2}{2} + h_2(\epsilon L_2x_2) - c_n \right) h_2'(\epsilon L_2x_2) = 0, \\
L_3(x_3 - f_l) + \left( \frac{\bar{L}_3}{2} + h_3(\epsilon L_3x_3) - e_l \right) h_3'(\epsilon L_3x_3) = 0.
\end{aligned}
\]

We have the validity of the following lemma.

**Lemma 5.1.** Let us define

\[
\kappa_i = - (\log \Psi)'(L_i).
\]

The following expansions hold true:

\[
\int_{\omega_z} EZ_{P_i} dx = - \Psi(L_1)[\kappa_1(a_{i+1} - 2a_i + a_{i-1})t_1 - (b_{i+1} - 2b_i + b_{i-1})n_1] + e^{-\delta L_1} A + \Psi(L_1) \mathcal{N}
\]

for \( i = 2, \ldots, m - 1, \)

\[
\int_{\omega_z} EZ_{Q_j} dx = - \Psi(L_2)[\kappa_2(c_{j+1} - 2c_j + c_{j-1})t_2 - (d_{j+1} - 2d_j + d_{j-1})n_2] + e^{-\delta L_2} A + \Psi(L_2) \mathcal{N}
\]

for \( j = 2, \ldots, n - 1, \)

\[
\int_{\Omega_z} EZ_{R_k} dx = - \Psi(L_3)[\kappa_3(e_{k+1} - 2e_k + e_{k-1})t_3 - (f_{k+1} - 2f_k + f_{k-1})n_3] + e^{-\delta L_3} A + \Psi(L_3) \mathcal{N}
\]

for \( k = 2, \ldots, m - 1. \)
for $k = 2, \ldots, l - 1$,

$$
\int_{w_k} EZ_{P_k} dx = -\Psi(L_1) \left[ \kappa_1 (a_2 - 2a_1 + (\alpha, \beta) t_1) t_1 - \left( b_2 - 2b_1 + \frac{(\alpha, \beta) n_3}{L_1} n_1 \right) \right] + e^{-\delta L_1} A + \Psi(L_1) N,
$$

(5.5)

$$
\int_{w_k} EZ_{Q_k} dx = -\Psi(L_2) \left[ \kappa_2 (c_2 - 2c_1 + (\alpha, \beta) t_2) t_2 - \left( d_2 - 2d_1 + \frac{(\alpha, \beta) n_2}{L_2} n_2 \right) \right] + e^{-\delta L_2} A + \Psi(L_2) N,
$$

(5.6)

$$
\int_{w_k} EZ_{R_k} dx = -\Psi(L_3) \left[ \kappa_3 (e_2 - 2e_1 + (\alpha, \beta) t_3) t_3 - \left( f_2 - 2f_1 + \frac{(\alpha, \beta) n_3}{L_3} n_3 \right) \right] + e^{-\delta L_3} A + \Psi(L_3) N,
$$

(5.7)

$$
\int_{w_k} EZ_{P_m} dx = -\Psi(L_1) \left[ \kappa_1 (a_{m-1} - 3a_m + \frac{2h_1(L_1 \varepsilon x_1)}{\varepsilon}) t_1 - (b_{m-1} - 3b_m + 2x_1) n_1 \right] + e^{-\delta L_1} A + \Psi(L_1) N,
$$

(5.8)

$$
\int_{w_k} EZ_{Q_m} dx = -\Psi(L_2) \left[ \kappa_2 (c_{m-1} - 3c_m + \frac{2h_2(L_2 \varepsilon x_2)}{\varepsilon}) t_2 - (d_{m-1} - 3d_m + 2x_2) n_2 \right] + e^{-\delta L_2} A + \Psi(L_2) N,
$$

(5.9)

$$
\int_{w_k} EZ_{R_m} dx = -\Psi(L_3) \left[ \kappa_3 (e_{m-1} - 3e_m + \frac{2h_3(L_3 \varepsilon x_3)}{\varepsilon}) t_3 - (f_{m-1} - 3f_m + 2x_3) n_3 \right] + e^{-\delta L_3} A + \Psi(L_3) N,
$$

(5.10)

$$
\int_{\Omega_x} EZ_{O} dx = \Psi(L_1) \left[ \kappa_1 (a_1 - (\alpha, \beta) t_1) t_1 + \left( b_1 - \frac{(\alpha, \beta) n_1}{L_1} n_1 \right) \right] + \Psi(L_2) \left[ \kappa_2 (c_1 - (\alpha, \beta) t_2) t_2 + \left( d_1 - \frac{(\alpha, \beta) n_2}{L_2} n_2 \right) \right] + \Psi(L_3) \left[ \kappa_3 (e_1 - (\alpha, \beta) t_3) t_3 + \left( f_1 - \frac{(\alpha, \beta) n_3}{L_3} n_3 \right) \right] + e^{-\delta L} A + \Psi(L) N.
$$

(5.11)

Furthermore,

$$
\int L(\phi) Z_z = e^{-\delta L} A, \quad z \in Y,
$$

(5.12)
where $\delta > 1$, $A = A(a, b, c, d, e, f, x, \alpha, \beta)$ and $N = N(a, b, c, d, e, f, x, \alpha, \beta)$ denote the smooth vector-valued functions (which vary from line to line), uniformly bounded as $L \to \infty$, and the Taylor expansion of $N$ with respect to $a, b, c, d, e, f, x, \alpha, \beta$ does not involve any constant or any linear terms.

**Proof.** Observe that, given $e \in \mathbb{R}^2$ with $|e| = 1$ and $a \in \mathbb{R}^N$, a direct consequence of estimates (1.5) is that the following expansion holds:

$$
\int N(\phi)Z_z = e^{-\delta L}A \quad z \in Y,
$$

(5.13)

Estimates (5.2)–(5.4) are by now standard; see, for instance, [34]. For completeness, we show

$$
\int_{w_\ast} EZ_{p_i}(x)dx = \int_{R^2} w(x - P_{i-1})p_\ast^{-1}(x - P_i)\nabla w(x - P_i)dx
\quad + \int_{R^2} w(x - P_{i+1})p_\ast^{-1}(x - P_i)\nabla w(x - P_i)dx + e^{-\delta L_1}A
$$

$$
= -\Psi(L_1)(\kappa_1(a_{i+1} - 2a_i + a_{i-1})n_1 - (b_{i+1} - 2b_i + b_{i-1})m_1)
\quad + e^{-\delta L_1}A + \Psi(L_1)N
$$

(5.15)

for $i = 2, \ldots, m - 1$.

Similarly we can get the two equations for $Q_j, R_k$.

Concerning estimates (5.5)–(5.7), a direct use of (5.14) gives

$$
\int_{\Omega_\ast} EZ_{p_i}dx = \int_{R^2} w(x - O)p_\ast^{-1}(x - P_i)\nabla w(x - P_i)dx
\quad + \int_{R^2} w(x - P_2)p_\ast^{-1}(x - P_i)\nabla w(x - P_i)dx + e^{-\delta L_2}A
$$

$$
= -\Psi(L_1)\left(\kappa_1(a_2 - 2a_1 + (\alpha, \beta)t_1)t_1 - (b_2 - 2b_1)n_1 - \frac{(\alpha, \beta)m_1}{L_1}n_1\right)
\quad + e^{-\delta L_2}A + \Psi(L_1)N.
$$

Similarly we can get the two equations for $Q_1, R_1$.

To compute (5.8)–(5.10), we use the result of Lemma 2.1. Given (5.1) we obtain that

\[\begin{cases}
P_m^* - P_m = 2(x_1 - b_m)L_1n_1 + (L_1 + \frac{2h_1(L_1x_1)}{x_1}) - 2a_m)t_1, \\
Q_n^* - Q_n = 2(x_2 - d_n)L_2n_2 + (L_2 + \frac{2h_2(L_2x_2)}{x_2}) - 2\tilde{c}_n)t_2, \\
R_2^* - R_1 = 2(x_3 - f_1)L_3n_3 + (L_3 + \frac{2h_3(L_3x_3)}{x_3}) - 2e_1)t_3.
\end{cases}\]

(5.15)
Thus a direct use of Lemma 2.1 gives estimate (5.8) as follows:

\[
\int_{\Omega} EW_{P_m} dx = \int_{\Omega} p w^{p-1} (x - P_m) \nabla w(x - P_m) w(x - P_m - 1) dx \\
+ \int_{\Omega} p w^{p-1} (x - P_m) \nabla w(x - P_m) w(x - P_1^*) + e^{-\delta L} A \\
= -\Psi(L_1) \left( \kappa_1 \left( 3a_m - a_{m-1} - \frac{2h_1(L_1 \varepsilon x_1)}{\varepsilon} \right) t_1 \\
+ (2x_1 + b_{m-1} - 3b_m) n_1 \right) + e^{-\delta L} A + \Psi(L_1) N.
\]

In the same way we get the equations for \( Q_n, R_l \).

Finally, expansion (5.11) is given by

\[
\int_{\Omega} EZ_O(x) dx = \int_{\Omega} p w^{p-1} (x - O) w(x - P_1) \nabla w(x - O) dx \\
+ \int_{w_1} p w^{p-1} (x - O) w(x - Q_1) \nabla w(x - O) dx \\
+ \int_{w_1} p w^{p-1} (x - O) w(x - R_1) \nabla w(x - O) dx + e^{-\delta L} A \\
= -\Psi(L_1) \left( t_1 - \kappa_1 (a_1 - (\alpha, \beta) t_1) t_1 + \frac{L_1 b_1 - (\alpha, \beta) n_1}{L_1} n_1 \right) \\
- \Psi(L_2) \left( t_2 - \kappa_2 (c_1 - (\alpha, \beta) t_2) t_2 + \frac{L_2 d_1 - (\alpha, \beta) n_2}{L_2} n_2 \right) \\
- \Psi(L_3) \left( t_3 - \kappa_3 (e_1 - (\alpha, \beta) t_3) t_3 + \frac{L_3 f_1 - (\alpha, \beta) n_3}{L_3} n_3 \right) \\
+ e^{-\delta L} A + N \Psi(L).
\]

The proof of (5.12) follows the line of the proof of Proposition 3.2 (see formula (3.6) and the subsequent estimates, together with (4.2)).

The proof of (5.13) follows from estimates (4.5) and (4.2).

For any integer \( k \) let us now define the following \( k \times k \) matrix:

\[
(5.16) \quad T := \begin{pmatrix}
2 & -1 & 0 & \ldots & 0 \\
-1 & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & 2 & -1 \\
0 & \ldots & 0 & -1 & 2
\end{pmatrix} \in \mathcal{M}_{k \times k}.
\]

It is well known that the matrix \( T \) is invertible and its inverse is the matrix whose entries are given by

\[
(T^{-1})_{ij} = \min(i, j) - \frac{ij}{k + 1}.
\]
We define the vectors $S^\perp$ and $S^\uparrow$ by
\begin{equation}
T S^\perp := \begin{pmatrix}
0 \\
\vdots \\
0 \\
1 \\
0 \\
0
\end{pmatrix} \in \mathbb{R}^k, \quad T S^\uparrow := \begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix} \in \mathbb{R}^k.
\end{equation}

It is immediate to check that
\begin{equation}
S^\perp := \begin{pmatrix}
\kappa x \\
\kappa y \\
\kappa z \\
1 \\
\kappa x \\
\kappa y \\
\kappa z \\
1
\end{pmatrix} \in \mathbb{R}^k, \quad S^\uparrow := \begin{pmatrix}
\kappa x \\
\kappa y \\
\kappa z \\
1 \\
\kappa x \\
\kappa y \\
\kappa z \\
1
\end{pmatrix} \in \mathbb{R}^k.
\end{equation}

With this in mind we have that the above lemma gives the validity of the following.

**Lemma 5.2.** The coefficients $c_\alpha$ in problem (4.1) are all equal to 0 if and only if the parameters $a, b, c, d, e, f, \alpha, \beta$ are solutions of the nonlinear system
\begin{equation}
\begin{cases}
\begin{align*}
a &= \frac{(2h_1(x_1, x_2))}{\varepsilon} - a_{1n} + \alpha \beta t_1 S^\uparrow + e^{-\delta L} A + N \in \mathbb{R}^m, \\
b &= (2x_1 - b_{1n}) S^\perp + \frac{(\alpha \beta n_1 L_1)}{L_1} S^\perp + e^{-\delta L} A + N \in \mathbb{R}^m, \\
c &= (2h_2(x_2, x_3))\frac{e}{\varepsilon} - c_1 + \alpha \beta t_2 S^\uparrow + e^{-\delta L} A + N \in \mathbb{R}^2, \\
d &= (2x_2 - d_{1n}) S^\perp + \frac{(\alpha \beta n_2 L_2)}{L_2} S^\perp + e^{-\delta L} A + N \in \mathbb{R}^2, \\
e &= (2h_3(x_3, x_1))\frac{e}{\varepsilon} - e_1 + \alpha \beta t_3 S^\uparrow + e^{-\delta L} A + N \in \mathbb{R}^l, \\
f &= (2x_3 - f_{1n}) S^\perp + \frac{(\alpha \beta n_3 L_3)}{L_3} S^\perp + e^{-\delta L} A + N \in \mathbb{R}^l,
\end{align*}
\end{cases}
\end{equation}

where $\delta > 0$ and $x_1, x_2,$ and $x_3$ are given by (5.1). Furthermore $\alpha, \beta$ satisfy
\begin{equation}
\begin{aligned}
-\Psi(L_1) \left( -\kappa_1 (a_1 - (\alpha, \beta) t_1) t_1 + \left( b_1 - \frac{(\alpha, \beta) n_1 L_1}{L_1} \right) n_1 \right) \\
-\Psi(L_2) \left( -\kappa_2 (c_1 - (\alpha, \beta) t_2) t_2 + \left( d_1 - \frac{(\alpha, \beta) n_2 L_2}{L_2} \right) n_2 \right) \\
-\Psi(L_3) \left( -\kappa_3 (e_1 - (\alpha, \beta) t_3) t_3 + \left( f_1 - \frac{(\alpha, \beta) n_3 L_3}{L_3} \right) n_3 \right) \\
+ e^{-\delta L} A + N \Psi(L) = 0.
\end{aligned}
\end{equation}

In this last formula $\delta_1 > 1$. In the above formula we have denoted by $A = A(a, b, c, d, e, f, \alpha, \beta)$ and $N = N(a, b, c, d, e, f, \alpha, \beta)$ smooth vector-valued functions (which vary from line to line), uniformly bounded as $L \to \infty$ and the Taylor expansion of $N$ with respect to $a, b, c, d, e, f, \alpha, \beta$ does not involve any constant or any linear term.

Given the result of the above lemma, we are left to show that (5.19)–(5.20) have a solution $(a, b, c, d, e, f, \alpha, \beta)$ with
\begin{equation}

\|(a, b, c, d, e, f, \alpha, \beta)\| \leq c
\end{equation}

for some positive $c$, small and independent of $\varepsilon$. 

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We first observe that using the assumptions that \( h_i(0) = h'_i(0) = 0 \) for all \( i = 1, 2, 3 \), from the equations (5.1) satisfied by \( x_1, x_2, x_3 \), we get that

\[
\begin{aligned}
x_1 &= (1 - \frac{L_1 h''(0)}{2m+1}) b_m + e^{-\delta L} A + N, \\
x_2 &= (1 - \frac{L_2 h''(0)}{2n+1}) d_n + e^{-\delta L} A + N, \\
x_3 &= (1 - \frac{L_3 h''(0)}{2t+1}) f_t + e^{-\delta L} A + N
\end{aligned}
\]  
(5.21)

for some constant \( \delta > 0 \).

On the other hand, using the expressions for \( S^1 \) and \( S^2 \) given by (5.18), from the first two equations in (5.19) we get that

\[
\begin{aligned}
a_1 &= \frac{2h(eL_1 x_1)}{2m+1} + \frac{2m-1}{2m+1} (\alpha, \beta) \cdot t_1 + e^{-\delta L} A + N, \\
b_1 &= \frac{2eL_1 x_1}{2m+1} + \frac{2m-1}{2m+1} (\alpha, \beta) \cdot n_1 + e^{-\delta L} A + N, \\
a_m &= \frac{2mL_1 x_1}{2m+1} + \frac{1}{2m+1} (\alpha, \beta) \cdot t_1 + e^{-\delta L} A + N, \\
b_m &= \frac{2mL_1 x_1}{2m+1} + \frac{1}{2m+1} (\alpha, \beta) \cdot n_1 + e^{-\delta L} A + N.
\end{aligned}
\]  
(5.22)

In a very similar way, from the last four equations in (5.19) we get the expressions of \( c_1, c_n, d_1, d_n, e_1, e_t, f_1, f_t \).

Using (5.21) and (5.22), from (5.20) we can get that the parameters \( \alpha, \beta \) satisfy the system

\[
\begin{aligned}
B \alpha - C \beta &= \frac{2\Psi(L_1)}{2m+1} t_{12} x_1 + \frac{2\Psi(L_2)}{2n+1} t_{22} x_2 - \frac{2\Psi(L_3)}{2k+1} x_3 + e^{-\delta L} A + N \Psi(L), \\
C \alpha - D \beta &= \frac{2\Psi(L_1)}{2m+1} t_{11} x_1 + \frac{2\Psi(L_2)}{2n+1} t_{21} x_2 + e^{-\delta L} A + N \Psi(L)
\end{aligned}
\]  
(5.23)

for some \( \delta > 1 \), where the constants \( B, C, \) and \( D \) are given by

\[
\begin{aligned}
B &= \frac{2\Psi(L_1)}{(2m+1)L_1} t_{12}^2 + \frac{2\Psi(L_2)}{(2n+1)L_2} t_{22}^2 + \frac{2\Psi(L_3)}{(2k+1)L_3} - \frac{L_1}{2m+1} l_{11} - \frac{L_2}{2n+1} l_{22}, \\
C &= \frac{2\Psi(L_1)}{(2m+1)L_1} t_{11}^2 + \frac{2\Psi(L_2)}{(2n+1)L_2} t_{21}^2 - \frac{\Psi(L_1)}{2m+1} l_{11}^2 - \frac{\Psi(L_2)}{2n+1} l_{22}^2, \\
D &= \frac{2\Psi(L_1)}{(2m+1)L_1} t_{12} - \frac{2\Psi(L_2)}{(2n+1)L_2} - \frac{\Psi(L_1)}{2m+1} l_{12} - \frac{\Psi(L_2)}{2n+1} l_{21}, \\
E &= \frac{2\Psi(L_1)}{(2m+1)L_1} t_{11} - \frac{2\Psi(L_2)}{(2n+1)L_2} t_{22} - \frac{\Psi(L_1)}{2m+1} l_{12} - \frac{\Psi(L_2)}{2n+1} l_{21}.
\end{aligned}
\]  
(5.24)

Recall that the numbers \( t_{ij} \) are the components of the vectors \( t_1 \) and \( t_2 \) in (1.8). A direct computation shows that the system in \( \alpha \) and \( \beta \) is uniquely solvable, since

\[
C^2 - BD = -\frac{\Psi(L_1)\Psi(L_3)}{(2m+1)(2l+1)} t_{11}^2 - \frac{\Psi(L_2)\Psi(L_3)}{(2n+1)(2l+1)} t_{21}^2
\]  
(5.25)

given the fact that we have already observed that it is not restrictive to assume that all \( t_{ij} \neq 0 \) for \( i, j = 1, 2 \).

One can check that

\[
\begin{aligned}
\alpha &= \frac{1}{C - BD} \left( \frac{2\Psi(L_1)}{2m+1} t_{11} x_1 + C \frac{2\Psi(L_2)}{2n+1} t_{21} x_2 - D \frac{2\Psi(L_1)}{2m+1} t_{12} x_1 - D \frac{2\Psi(L_2)}{2n+1} t_{22} x_2 + D \frac{2\Psi(L_3)}{2k+1} x_3, \\
\beta &= \frac{1}{C - BD} \left( \frac{2\Psi(L_1)}{2m+1} t_{11} x_1 + B \frac{2\Psi(L_2)}{2n+1} t_{21} x_2 - C \frac{2\Psi(L_1)}{2m+1} t_{12} x_1 - C \frac{2\Psi(L_2)}{2n+1} t_{22} x_2 + C \frac{2\Psi(L_3)}{2k+1} x_3.
\end{aligned}
\]  
(5.26)
Replacing these values of $\alpha$ and $\beta$, together with (5.21), (5.22) and in the corresponding equations for the parameters $c_1, c_n, d_1, d_n, e_1, e_t, f_1, f_t$, we obtain that the whole problem is reduced to the solvability of the following nonlinear system in the variables $b_1, b_m, d_1, d_n, f_1, f_t$:

\begin{equation}
\begin{pmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4
\end{pmatrix}
(b_1, d_1, f_1, b_m, d_n, f_t)^t = e^{-\delta L}A + \mathbf{N},
\end{equation}

where

\[
T_1 = -I_{3 \times 3}, \\
T_3 = O_{3 \times 3},
\]

and

\[
T_4 = \left( 1 + \frac{2n}{2^{n+1}} L_1 h''_1 + \frac{A_2}{B_1} + \frac{2n}{2^{n+1}} L_2 h''_2 + \frac{B_2}{C_1} + \frac{2n}{2^{n+1}} L_3 h''_3 + C_3 \right),
\]

with the constants $A_j, B_j, C_j$ defined as follows:

\begin{equation}
\begin{cases}
A_1 = \frac{2\Psi(L_1)}{(c^2 - \nu_1)(2m+1)L_1} (B_1 t_{11}^2 - 2C_1 t_{12} + D t_{12}^2)(1 - \frac{L_1 h''_1(0)}{2m+1}), \\
A_2 = \frac{2\Psi(L_2)}{(c^2 - \nu_1)(2m+1)L_1} (B_2 t_{11} - C_1 t_{12} + D t_{12}^2)(1 - \frac{L_2 h''_1(0)}{2m+1}), \\
A_3 = \frac{2\Psi(L_3)}{(c^2 - \nu_1)(2m+1)L_1} (C_1 - D t_{12})(1 - \frac{L_3 h''_1(0)}{2m+1}), \\
B_1 = \frac{2\Psi(L_1)}{(c^2 - \nu_1)(2m+1)L_2} (B_1 t_{11} - C_1 t_{12} + D t_{12}^2)(1 - \frac{L_1 h''_2(0)}{2m+1}), \\
B_2 = \frac{2\Psi(L_2)}{(c^2 - \nu_1)(2m+1)L_2} (B_2 t_{12} + D t_{12}^2)(1 - \frac{L_2 h''_2(0)}{2m+1}), \\
B_3 = \frac{2\Psi(L_3)}{(c^2 - \nu_1)(2m+1)L_2} (C_1 t_{12} - D t_{12})(1 - \frac{L_3 h''_2(0)}{2m+1}), \\
C_1 = \frac{2\Psi(L_1)}{(c^2 - \nu_1)(2m+1)L_3} (B_1 t_{11} - C_1 t_{12})(1 - \frac{L_1 h''_3(0)}{2m+1}), \\
C_2 = \frac{2\Psi(L_2)}{(c^2 - \nu_1)(2m+1)L_3} (B_2 t_{12})(1 - \frac{L_2 h''_3(0)}{2m+1}), \\
C_3 = \frac{2\Psi(L_3)}{(c^2 - \nu_1)(2m+1)L_3} D(1 - \frac{L_3 h''_3(0)}{2m+1}).
\end{cases}
\end{equation}

In order to solve (5.27), we need to compute the determinant of the matrix \( \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \).

Set $H_i = 1 + \bar{L}_i h''_i(0)$. We write

\begin{equation}
\det \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} = \sum_{j=1}^{4} \Delta_j,
\end{equation}

where

\begin{equation}
\Delta_1 = \left( H_1 - \frac{L_1 h''_1(0)}{2m+1} \right) \left( H_2 - \frac{L_2 h''_2(0)}{2n+1} \right) \left( H_3 - \frac{L_3 h''_3(0)}{2l+1} \right),
\end{equation}

\begin{equation}
\Delta_2 = A_1 \left( H_2 - \frac{L_2 h''_2(0)}{2n+1} \right) \left( H_3 - \frac{L_3 h''_3(0)}{2l+1} \right) + B_2 \left( H_1 - \frac{L_1 h''_1(0)}{2m+1} \right) \left( H_3 - \frac{L_3 h''_3(0)}{2l+1} \right).
\end{equation}
\[ (5.31) \quad \Delta_3 = (C_3B_2 - C_2B_3) \left( H_1 - \frac{\bar{L}_1h''_1(0)}{2m+1} \right) + (A_1C_3 - A_3C_1) \left( H_2 - \frac{\bar{L}_2h''_2(0)}{2n+1} \right) + (A_1B_2 - A_2B_1) \left( H_3 - \frac{\bar{L}_3h''_3(0)}{2l+1} \right), \]

and

\[ (5.32) \quad \Delta_4 = \det \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}. \]

Using the expression of the constants \( A_j, B_j, \) and \( C_j \) in (5.28), an involved but direct computation gives us that

\[ \Delta_4 = 0. \]

On the other hand, we observe the following: From the definition of the constant \( A_1, B_2, \) and \( C_3 \) in (5.28) and from the expression of \( C^2 - BD \) in (5.25), we get that

\[ (5.34) \quad A_1, B_2, C_3 = M(t_1, \bar{L}_1, C_0) \frac{1}{L}, \]

where \( M \) is uniformly bounded from below away from zero as \( \varepsilon \to 0 \) (or equivalently as \( L \to \infty \)).

Furthermore, from the definition of

\[ A_1C_3 - C_1A_3 = \frac{\psi(L_1)\psi(L_3)}{(2m+1)(2l+1)(BD-C^2)L_1L_3} t_{11}^2, \]

we get that \( |A_1C_3 - C_1A_3| \geq c_0 \frac{1}{L^2} \) as \( \varepsilon \to 0 \), where \( c_0 > 0 \). In fact we get

\[ C_3B_2 - B_3C_2, A_1C_3 - C_1A_3, A_1B_2 - B_1A_2 = N(t_1, \bar{L}_1, C_0) \frac{1}{L^2}, \]

where \( N \) is uniformly bounded from below away from zero as \( L \to \infty \).

We thus conclude that under the assumption that at least one \( H_i \) is nonzero, the nonlinear system (5.27) can be uniquely solved by fixed point theorem of contraction mapping.

REFERENCES


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