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On spikes concentrating on line-segments to a semilinear Neumann problem

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A B S T R A C T

We consider the following singularly perturbed Neumann problem

$$-\varepsilon^2\Delta u+u-u^p=0 \quad \text{in } \Omega, \qquad u>0 \quad \text{in } \Omega,$$
$$\frac{\partial u}{\partial \nu}=0 \quad \text{on } \partial\Omega,$$

where p is subcritical and Ω is a smooth and bounded domain in $\mathbb{R}^n$. We construct a new class of solutions which consist of large number of spikes concentrating on an interior straight-line intersecting with $\partial\Omega$ orthogonally. Our results show that higher-dimensional concentration can exist *without resonance condition*.

1. Introduction and statement of main results

We consider the following singularly perturbed elliptic problem

$$-\varepsilon^2\Delta u+u-u^p=0 \quad \text{in } \Omega, \qquad u>0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu}=0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ with its unit outer normal ν, $N\geqslant 2$, $1<p<\frac{N+2}{N-2}$ for $N\geqslant 3$, while $p>1$ for $N=2$, and $\varepsilon>0$ is a small parameter.

Even though simple-looking, problem (1.1) has a rich and interesting structure of solutions. For the last fifteen years, it has received considerable attention. In particular, the various concentration

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phenomena exhibited by the solutions of (1.1) seem both mathematically intriguing and scientifically useful. We refer to three survey articles [27,28] and [33] for backgrounds and references.

In the pioneering papers [29,30], Ni and Takagi proved the existence of least energy solutions to (1.1), that is, a solution u_ϵ with minimal energy. Furthermore, they showed in [29,30] that, *for each $\epsilon > 0$ sufficiently small, u_ϵ has a spike at the most curved part of the boundary, i.e., the region where the mean curvature attains maximum value*.

Since the publication of [30], problem (1.1) has received a great deal of attention and significant progress has been made. More specifically, solutions with multiple boundary peaks as well as multiple interior peaks have been established. (See [1,5,7–16,19,20,31,32,34,35] and the references therein.) In particular, it was established in Gui and Wei [13] that *for any two given integers $k \geqslant 0$, $l \geqslant 0$ and $k+l > 0$, problem* (1.1) *has a solution with exactly k interior spikes and l boundary spikes* for every ϵ sufficiently small. Furthermore, Lin, Ni and Wei [21] showed that there are at least $\frac{C_N}{(\epsilon|\log\epsilon|)^N}$ number of interior spikes. (We point out that positive solutions having multiple interior or boundary spikes have been exhibited in many works to a wide variety of semilinear elliptic problems, including Cahn–Hilliard equations. We refer to [4,3,34,35], and the references therein.)

It seems natural to ask if problem (1.1) has solutions which "concentrate" on higher-dimensional sets, e.g. curves, or surfaces. In this regard, we mention that it has been *conjectured* for a long time that *problem* (1.1) *actually possesses solutions which have m-dimensional concentration sets for every* $0 \leqslant m \leqslant N-1$. (See e.g. [27].) Progress in this direction, although still limited, has also been made in [2, 22,24–26]. In particular, we mention the results of Malchiodi and Montenegro [24,25] on the existence of solutions concentrating on the *whole boundary* provided that the sequence ϵ satisfies some gap condition. The latter condition is called *resonance*.

In this paper, we consider solutions concentrating on *interior curves*. Formal arguments show that concentrating curves must have zero mean curvature, i.e., must be geodesics. Malchiodi [22] constructed solutions concentrating on geodesics of the boundary along a subsequence $\epsilon_n \to 0$. If the geodesic is contained inside the domain, then it must be a straight-line. In this regard, we mention that the first work was due to Wei and Yang [36] who proved the existence of spike layer on a line intersecting with the boundary orthogonally. In [36], a geometric condition of *non-degeneracy* was derived. Furthermore, the domain is assumed to be *two-dimensional* and a *resonance* condition was needed, i.e., the existence of solutions was established only along a sequence of $\epsilon \to 0$. (The geometric condition was first derived for line interfaces of Allen–Cahn equation by Kowalczyk [17].)

In all the papers above on higher-dimensional concentrations [22–25], the first approximation solution is the one-dimensional homoclinics and so resonance is inevitable. An interesting question persists: can one remove the resonance condition? We shall prove in this paper that it is possible to remove the resonance condition by using *different higher-dimensional approximate solutions*.

We consider the situation of [36], but now in a general n-dimensional domain. Our aim is to construct other new solutions with large number of spikes concentrating along a straight-line. We generalize the results of [36] in several ways: firstly, we put large number of spikes on the line (the distance between the spikes is $O(\epsilon|\log\epsilon|)$). Because of this, we remove the gap condition in [36]. Our results hold for *all* ϵ small. Secondly, we consider a straight-line in an n-dimensional domain. (We believe that similar idea may be used to remove the resonance conditions in [22–25].)

We assume that Ω contains a segment Γ_0 which intersects orthogonally the boundary of Ω in exactly two points Q_1 and Q_2 and whose length is L. We assume that Γ_0 satisfies some non-degeneracy condition which we describe below.

After rotations and translations we may assume that $Q_1 = 0$ and $Q_2 = (0, L) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and that Γ_0 is described by

$$\Gamma_0 = \big\{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}\colon x' = 0,\ 0 < x_N < L\big\}.$$

We assume that the boundary $\partial\Omega$ near Q_1 and Q_2 is given by the graphs of two smooth functions $G_i : B_{\mathbb{R}^{N-1}}(0,\rho) \to \mathbb{R}$, where $B_{\mathbb{R}^{N-1}}(0,\rho)$ denotes the ball of radius ρ and center 0 in $\mathbb{R}^{N-1}$, for some $\rho > 0$ small. Furthermore, we assume that $G_1(0) = 0$, $G_2(0) = L$, $\nabla G_i(0) = 0$ and $D^2 G_i(0)$ non-degenerate, for $i = 1, 2$. It is not restrictive to further assume that $D^2 G_1(0)$ is in diagonal form, namely $D^2 G_1(0) = \operatorname{diag}(\lambda_1, \ldots, \lambda_{N-1})$, where λ_i represent the principal curvatures of $\partial\Omega$ at Q_1.

A curve C^1-close to Γ_0 with end points on $\partial\Omega$ can be parameterized as follows

$$\gamma(t) = \big(h(t), tG_2\big(h(L)\big) + (L-t)G_1\big(h(0)\big)\big), \quad 0 < t < L,$$

where $h : [0, L] \to \mathbb{R}^{N-1}$ is a smooth function with $h([0, L]) \subseteq B_{\mathbb{R}^{N-1}}(0, \rho)$.

Then the length functional $\mathcal{L}$ is given by

$$|\gamma| = \mathcal{L}(h) = \int_0^L \sqrt{\big|G_2\big(h(L)\big) - G_1\big(h(0)\big)\big|^2 + \big|h'(t)\big|^2}\, dt.$$

It is straightforward to show that Γ_0 is a critical point for $\mathcal{L}$, namely $D\mathcal{L}(0) = 0$, since Γ_0 intersects orthogonally the boundary. Furthermore, the second variation of the length functional at 0 is given by the quadratic form

$$D^2\mathcal{L}(0)[h]^2 = \int_0^L \big|h'(t)\big|^2\, dt + G_2''(0)\big[h(L)\big]^2 - G_1''(0)\big[h(0)\big]^2.$$

The segment Γ_0 is said to be non-degenerate if $D^2\mathcal{L}(0)$ is invertible in the set $H_0^{1,2}((0, L); \mathbb{R}^{N-1})$. This amounts to the fact that the problem

$$-h''(t) = 0 \quad \text{in } (0, L), \qquad G_1''(0)\big[h(0)\big] + h'(0) = 0, \qquad G_1''(0)\big[h(L)\big] + h'(L) = 0$$

has only the trivial solution $h(t) = 0$.

This fact is equivalent to the condition

$$\text{determinant} \begin{bmatrix} I & G_1''(0) \\ I + LG_2''(0) & G_2''(0) \end{bmatrix} \neq 0, \tag{1.2}$$

where I denotes the Identity Matrix of dimension $N-1$.

Indeed, from $h''(t) = 0$ we get that $h(t) = \bar{a}t + \bar{b}$, for some vectors $\bar{a}$ and $\bar{b}$ in $\mathbb{R}^{N-1}$. Thus the boundary conditions give

$$G_1''(0)\bar{b} + \bar{a} = 0, \qquad G_2''(0)[\bar{a}L + \bar{b}] + \bar{a} = 0.$$

Under the condition (1.2), the above system has only the solution $\bar{a} = 0$, $\bar{b} = 0$.

Let $N = 2$. In this case $G_1''(0) = \lambda_1$ and $G_2''(0) = -\mu_1$, where λ_1 and μ_1 denote the curvatures of $\partial\Omega$ respectively at P_1 and P_2. Then condition (1.2) becomes

$$\lambda_1 + \mu_1 - L\lambda_1\mu_1 \neq 0.$$

Let $N = 3$, $G_1''(0) = \left[\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right]$, with $\lambda_1\lambda_2 \neq 0$, and $G_2''(0) = \left[\begin{smallmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{smallmatrix}\right]$, with $g_{11}g_{22} - g_{12}^2 \neq 0$. The condition (1.2) becomes in this case

$$[\lambda_1 - g_{11} + L\lambda_1 g_{11}][\lambda_2 - g_{22} + L\lambda_2 g_{22}] - g_{12}^2\big[1 - (\lambda_1 + \lambda_2)L + \lambda_1\lambda_2 L^2\big] \neq 0.$$

Let ω be the unique solution to

$$\begin{cases} \Delta w - w + w^p = 0, \quad w > 0 \quad \text{in } \mathbb{R}^N, \\ w(0) = \max_{y \in \mathbb{R}^N} w(y), \quad w \to 0 \quad \text{at } \infty. \end{cases} \tag{1.3}$$

The existence of ω is standard and follows from well-known arguments in the calculus of variation while the uniqueness follows from results of Kwong [18]. ω is also *non-degenerate*, we refer to Appendix C of [30].

Our main result states that under the non-degeneracy condition (1.2), we can put large number of spikes (at a distance $O(\epsilon|\log\epsilon|)$) on the Γ_0. More precisely we have

Theorem 1.1. *Assume Ω contains a segment Γ_0 which intersects orthogonally the boundary of Ω in exactly two points Q_1 and Q_2 and whose length is L. We assume that Γ_0 satisfies the non-degeneracy condition* (1.2) *described above. There exists $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$ and for any integer k with*

$$\lim_{\varepsilon \to 0} k = \infty \tag{1.4}$$

and

$$k \leqslant \frac{\alpha}{\varepsilon|\ln\varepsilon|} \tag{1.5}$$

where $\alpha > 0$ is a constant depending on Ω, on N and on the length L of the segment Γ_0, then there exists a solution u_ε to problem (1.1)*. Furthermore there exist k points Q_j^ε uniformly distributed along the curve Γ_0 such that*

$$u_\varepsilon(x) = \sum_{j=1}^{k} \omega\left(\frac{x - Q_j^\varepsilon}{\varepsilon}\right) + o(1) \tag{1.6}$$

where $o(1) \to 0$ as $\varepsilon \to 0$ uniformly over compacts of $\mathbb{R}^N$. Moreover,

$$\operatorname{dist}\left(Q_1^\varepsilon, Q_1\right) \sim \frac{L}{2\varepsilon k}, \qquad \operatorname{dist}\left(Q_k^\varepsilon, Q_2\right) \sim \frac{L}{2\varepsilon k}$$

and

$$\operatorname{dist}\left(Q_j^\varepsilon, Q_{j+1}^\varepsilon\right) \sim \frac{L}{\varepsilon k}, \quad \textit{for } 1 \leqslant j < k.$$

Remark 1.1. In [16], Kowalczyk proved the existence of *fixed number of spikes* on a line intersecting with the boundary, under a geometric condition which is different from here. On the other hand, in [6], D'Aprile and Pistoia proved the following result: assume that Ω is a two-dimensional *convex* domain. Let L be a line intersecting with the boundary orthogonally. Then for any K fixed and large, there exist K spikes on the line. The results of [6] is a corollary of Theorem 1.1. In fact since the domain is *convex*, the *non-degeneracy* condition holds automatically. Our Theorem 1.1 allows any non-degenerate line-segment inside the domain, convex or not.

Remark 1.2. If we rescale the domain Ω to $\frac{\Omega}{\epsilon}$, then the distance between the spikes is $O(\log\frac{1}{\epsilon})$. We are arranging k-copies of ω on a long line-segment. The main difficulty is to show that this arrangement is *non-degenerate*. We notice that similar idea has been used by Malchiodi in constructing new *entire* solutions to

$$\Delta u - u + u^p = 0, \quad u > 0 \quad \text{in } \mathbb{R}^2. \tag{1.7}$$

Malchiodi [23] has recently constructed positive (infinite energy) solutions of (1.7) by perturbing a configuration of infinitely many copies of the positive solution ω arranged along three rays meeting at a common point. It is an interesting question if there corresponds to a tripe junction solutions in a bounded domain.

2. Ansatz and sketch of the proof

By the scaling $x = \varepsilon z$, problem (1.1) becomes

$$-\Delta u + u - u^p = 0 \quad \text{in } \Omega_\varepsilon, \qquad u > 0 \quad \text{in } \Omega_\varepsilon, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega_\varepsilon, \tag{2.1}$$

where $\Omega_\varepsilon = \{\frac{x}{\varepsilon}\colon x \in \Omega\}$. In these expanding variables, the segment Γ_0 becomes $\Gamma_0^\varepsilon := \{(x', x_N)\colon x' = 0, 0 < x_N < \frac{L}{\varepsilon}\}$.

Let $\ell > 0$ be a real number and k an integer so that

$$k\ell = \frac{L}{\varepsilon}. \tag{2.2}$$

Observe that under condition (1.5) we have that $\ell \to \infty$ as $\varepsilon \to 0$.

Define

$$P_j = \left[\left(j - \frac{1}{2}\right)\ell + a_{Nj}\right]\mathrm{e}_N + \ell\bar{\mathrm{a}}_j, \quad j = 1, \ldots, k, \tag{2.3}$$

where $\mathrm{e}_N = (0, \ldots, 0, 1)$ and $\bar{\mathrm{a}}_j = (a_{1j}, a_{2j}, \ldots, a_{N-1j}, 0) \in \mathbb{R}^N$, $a_{Nj} \in \mathbb{R}$. The points

$$P_j - a_{Nj}\mathrm{e}_N - \ell\bar{\mathrm{a}}_j$$

are k points distributed along the scaled segment $\frac{\Gamma_0}{\varepsilon}$ at constant distance ℓ one from the other. Let us define the vectors a_j, $j = 1, \ldots, k$, to be

$$\mathrm{a}_j = (a_{1j}, a_{2j}, \ldots, a_{Nj}), \quad \text{for all } j = 1, \ldots, k. \tag{2.4}$$

We will assume that the vectors a_j are uniformly bounded, as $\varepsilon \to 0$, namely

$$\|\mathrm{a}_j\| \leqslant \delta \quad \text{for all } j = 1, \ldots, k. \tag{2.5}$$

We will denote by $\mathcal{P}$ the set of all points P_j, namely

$$\mathcal{P} = \{P_j\colon j = 1, \ldots, k\}. \tag{2.6}$$

Let us define the function

$$U(x) = \sum_{j=1}^{k} U_j(x), \quad \text{with } U_j(x) = \left[\omega_j(x) - \varphi_j(x)\right] \tag{2.7}$$

where

$$\omega_j(x) = \omega(x - P_j)$$

and

$$-\Delta\varphi_j + \varphi_j = 0 \quad \text{in } \Omega_\varepsilon, \qquad \frac{\partial\varphi_j}{\partial\nu} = \frac{\partial\omega(x - P_j)}{\partial\nu} \quad \text{on } \partial\Omega_\varepsilon.$$

Next lemma, whose proof is contained in [21], provides a qualitative description of the function φ_j.

Lemma 2.1. *Assume that $M|\ln\varepsilon| \leqslant d(P_j, \partial\Omega_\varepsilon) \leqslant \frac{\delta}{\varepsilon}$, for some constant M depending on N and a constant $\delta > 0$ sufficiently small. Then*

$$\varphi_j(x) = -\big(1 + o(1)\big)\omega\big(x - P_j^*\big) + o\big(\varepsilon^{N+1}\big) \tag{2.8}$$

where $P_j^* = P_j + 2d(P_j, \partial\Omega_\varepsilon)\nu_{\bar{P}_j}$, $\nu_{\bar{P}_j}$ denotes the unit normal at $\bar{P}_j$ on $\partial\Omega_\varepsilon$, and $\bar{P}_j$ is the unique point on $\partial\Omega_\varepsilon$ such that $d(P_j, \bar{P}_j) = d(P_j, \partial\Omega_\varepsilon)$.

We look for a solution of (2.1) of the form $u = U + \phi$. We set

$$L(\phi) = -\Delta\phi + \phi - pU^{p-1}\phi, \tag{2.9}$$

$$E = U^p - \sum_{j=1}^{k} \omega_j^p \tag{2.10}$$

and

$$N(\phi) = (U + \phi)^p - U^p - pU^{p-1}\phi. \tag{2.11}$$

Problem (2.1) gets re-written as

$$L(\phi) = E + N(\phi) \quad \text{in } \Omega_\varepsilon, \qquad \frac{\partial\phi}{\partial\nu} = 0 \quad \text{on } \partial\Omega_\varepsilon.$$

Consider a cut off function $\chi \in C^\infty(0,\infty)$ such that

$$\chi(s) \equiv 1 \quad \text{for } s \leqslant -1, \quad \text{and} \quad \chi(s) \equiv 0 \quad \text{for } s \geqslant 0. \tag{2.12}$$

We fix a constant $\zeta > 0$ (independent of ℓ) so that the balls of radius $\frac{\ell-\zeta}{2}$, centered at different points of $\mathcal{P}$ are mutually disjoint, for all ℓ large enough. We define the compactly supported functions

$$Z_{ji}(x) := \chi\big(2|x - P_j| - \ell + \zeta\big)\partial_{x_i} w(x - P_j) \tag{2.13}$$

for $j = 1, \dots, k$ and $i = 1, \dots, N$. Observe that, by construction (in fact given the choice of ζ), we have

$$\int_{\Omega_\varepsilon} Z_{ij}(x) Z_{rs}(x)\, dx = 0, \tag{2.14}$$

if $i \neq r$ or if $j \neq s$.

Consider the following intermediate nonlinear projected problem: given the points P_j in (2.3), satisfying (2.5), find a function ϕ in some proper space and numbers c_{ji} such that

$$\begin{cases} L(\phi) = E + N(\phi) + \sum_{j=1}^{k}\sum_{i=1}^{N} c_{ji} Z_{ji} & \text{in } \Omega_\varepsilon, \\ \frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \phi Z_{ji} = 0 & \text{for } j = 1, \dots, k,\ i = 1, \dots, N. \end{cases} \tag{2.15}$$

We show unique solvability of problem (2.15) by means of a fixed point argument. Furthermore we prove that the solution ϕ depends smoothly on the points P_j.

To do so, in Section 3 we develop a solvability theory for the linear projected problem

$$\begin{cases} L\phi = h + \sum_{j=1}^{k}\sum_{i=1}^{N} c_{ji} Z_{ji} & \text{in } \Omega_\varepsilon, \\ \frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \phi Z_{ji} = 0 & \text{for } j = 1, \dots, k,\ i = 1, \dots, N, \end{cases} \tag{2.16}$$

for a given right-hand side h in some proper space. Roughly speaking, the linear operator L is a super position of the linear operators

$$L_j\phi = \Delta\phi - \phi + p\omega^{p-1}(x - P_j)\phi, \quad P_j \in \mathcal{P}.$$

Once we have the unique solvability of problem (2.15), which is proved in Section 4, it is clear that $u = U + \phi$ is indeed an exact solution to our original problem (1.1), with the qualitative properties described in Theorem 1.1, if we can prove that the constants c_{ji} appearing in (2.15) are all zero. This can be done adjusting properly the parameters a_j, $j = 1, \dots, k$, as will be shown in Section 5, where the proof of Theorem 1.1 will be also given.

3. Linear theory

Our main result in this section states bounded solvability of problem (2.16), uniformly in small ε, in points P_j, uniformly separated from each other at distance $O(\ell)$. Indeed we assume that the points P_j given by (2.3) satisfy constraints (2.5).

Given $0 < \eta < 1$, consider the norms

$$\|h\|_* = \sup_{x\in\Omega_\varepsilon}\left|\sum_j e^{\eta|x-P_j|} h(x)\right| \tag{3.1}$$

where $P_j \in \mathcal{P}$ with $\mathcal{P}$ defined in (2.6).

Proposition 3.1. *Let $\delta > 0$ be fixed. There exist positive numbers $\eta \in (0,1)$, ε_0 and C, such that for all $\varepsilon \leqslant \varepsilon_0$, for all integers k and positive real numbers ℓ given by* (2.2) *and satisfying* (1.5)*, for any points P_j, $j = 1, \dots, k$, given by* (2.3) *and satisfying* (2.5)*, there is a unique solution (ϕ, c_{ji}) to problem* (2.16)*. Furthermore*

$$\|\phi\|_* \leqslant C\|h\|_*. \tag{3.2}$$

The proof of the above proposition, which we postpone to the end of this section, is based on Fredholm alternative theorem for compact operator and an a priori bound for solution to (2.16) that we state (and prove) next.

Proposition 3.2. *Let $\delta > 0$ be fixed. Let h be with $\|h\|_*$ bounded and assume that (ϕ, c_{ji}) is a solution to* (2.16)*. Then there exist positive numbers ε_0 and C, such that for all $\varepsilon \leqslant \varepsilon_0$, for all integers k and positive real numbers ℓ given by* (2.2) *and satisfying* (1.5)*, for any points P_j, $j = 1, \ldots, k$, given by* (2.3) *and satisfying* (2.5)*, one has*

$$\|\phi\|_* \leqslant C\|h\|_*. \tag{3.3}$$

Proof. We argue by contradiction. Assume there exist ϕ solution to (2.16) and

$$\|h\|_* \to 0, \qquad \|\phi\|_* = 1.$$

We prove that

$$c_{ji} \to 0. \tag{3.4}$$

Multiplying the equation in (2.16) against Z_{ji} and integrating in Ω_ε, we get

$$\int_{\Omega_\varepsilon} L\phi Z_{ji}(x) = \int_{\Omega_\varepsilon} hZ_{ji} + c_{ji}\int_{\Omega_\varepsilon} Z_{ji}^2,$$

since (2.14) holds true. Given the exponential decay at infinity of $\partial_{x_i}\omega$ and the definition of Z_{ji} in (2.13), we get

$$\int_{\Omega_\varepsilon} Z_{ji}^2 = \int_{\mathbb{R}^N} (\partial_{x_1}\omega)^2 + O\left(e^{-\delta\ell}\right) \quad \text{as } \ell \to \infty, \tag{3.5}$$

for some $\delta > 0$. On the other hand

$$\left|\int_{\Omega_\varepsilon} hZ_{ji}\right| \leqslant C\|h\|_* \int_{\Omega_\varepsilon} \partial_{x_i}\omega(x - P_j)e^{-\eta|x-P_j|} \leqslant C\|h\|_*.$$

Here and in what follows, C stands for a positive constant independent of ε, as $\varepsilon \to 0$ (or equivalently independent of ℓ as $\ell \to \infty$). Finally, if we write $\tilde{Z}_{ji}(x) = \partial_{x_i}\omega(x - P_j)$ and $\chi = \chi(2|x - P_j| - \ell + \zeta)$, we have

$$\begin{aligned}
-\int_{\Omega_\varepsilon} L\phi Z_{ji}(x) = {} & \int_{B(P_j, \frac{\ell-\zeta}{2})} \left[\Delta\tilde{Z}_{ji} - \tilde{Z}_{ji} + p\omega^{p-1}(x - P_j)\tilde{Z}_{ji}\right]\chi\left(2|x - P_j| - \ell + \zeta\right)\phi \\
& + \int_{\partial B(P_j, \frac{\ell-\zeta}{2})} \phi\nabla\left(\chi\left(2|x - P_j| - \ell + \zeta\right)\tilde{Z}_{ji}\right)\cdot \mathrm{n} \\
& - \int_{B(P_j, \frac{\ell-\zeta}{2})} \phi\left(\tilde{Z}_{ji}\Delta\chi\left(2|x - P_j| - \ell + \zeta\right) + 2\nabla\chi\left(2|x - P_j| - \ell + \zeta\right)\nabla\tilde{Z}_j\right)
\end{aligned}$$

$$+ p \int\limits_{B(P_j, \frac{\ell-\zeta}{2})} \big(U^{p-1} - \omega^{p-1}(x - P_j)\big)\phi \tilde{Z}_{ji} \chi\big(2|x - P_j| - \ell + \zeta\big). \tag{3.6}$$

Next we estimate all the terms of the previous formula.

Since

$$\Delta \tilde{Z}_{ji} - \tilde{Z}_{ji} + p\omega^{p-1}(x - P_j)\tilde{Z}_{ji} = 0$$

we get the first term is 0. Furthermore, we have

$$\begin{aligned} &\Bigg| \int\limits_{\partial B(P_j, \frac{\ell-\zeta}{2})} \phi \nabla\big(\chi\big(2|x - P_j| - \ell + \zeta\big)\tilde{Z}_{ji}\big) \cdot \mathrm{n} \Bigg| \\ &\quad \leqslant C\|\phi\|_* \int\limits_{\partial B(P_j, \frac{\ell-\zeta}{2})} e^{-(1+\eta)|x-P_j|}|x - P_j|^{-\frac{N-1}{2}}\, dx \\ &\quad \leqslant Ce^{-(1+\xi)\frac{\ell}{2}}\|\phi\|_* \end{aligned}$$

for some proper $\xi > 0$. The third integral can be estimated as follows

$$\begin{aligned} &\Bigg| \int\limits_{B(P_j, \frac{\ell-\zeta}{2})} \phi\big(\tilde{Z}_{ji}\Delta\chi\big(2|x - P_j| - \ell + \zeta\big) + 2\nabla\chi\big(2|x - P_j| - \ell + \zeta\big)\nabla\tilde{Z}_{ji}\big) \Bigg| \\ &\quad \leqslant C\|\phi\|_* \int_{\frac{\ell-\zeta}{2}-1}^{\frac{\ell-\zeta}{2}} e^{-(1+\eta)s}s^{\frac{N-1}{2}}\, ds \leqslant Ce^{-(1+\xi)\frac{\ell}{2}}\|\phi\|_*, \end{aligned}$$

again for some $\xi > 0$. Finally, we observe that in $B(P_j, \frac{\ell-\zeta}{2})$

$$\big|U^{p-1}(x) - \omega^{p-1}(x - P_j)\big| \leqslant \omega^{p-2}(x - P_j)\Bigg[\sum_{x_i \neq P_j} \omega(x - x_i)\Bigg].$$

Having this, we conclude that

$$\Bigg| p \int\limits_{B(P_j, \frac{\ell-\zeta}{2})} \big(U^{p-1}(x) - \omega^{p-1}(x - P_j)\big)\phi\tilde{Z}_{ji}\chi\big(2|x - P_j| - \ell + \zeta\big) \Bigg| \leqslant Ce^{-(1+\xi)\frac{\ell}{2}}\|\phi\|_*$$

for a proper $\xi > 0$, depending on N and p. We thus conclude that

$$|c_{ji}| \leqslant C\big[e^{-(1+\xi)\frac{\ell}{2}}\|\phi\|_* + \|h\|_*\big]. \tag{3.7}$$

Thus we get the validity of (3.4), since we are assuming $\|\phi\|_* = 1$ and $\|h\|_* \to 0$.

Let now $\eta \in (0,1)$. It is easy to check that the function

$$W := \sum_{j=1}^{k} e^{-\eta|\cdot - P_j|}$$

satisfies

$$LW \leqslant \frac{1}{2}\big(\eta^2 - 1\big)W,$$

in $\Omega_\varepsilon \setminus \bigcup_{j=1,\dots,k} B(P_j,\rho)$ provided ρ is fixed large enough (independently of ℓ). Hence the function W can be used as a barrier to prove the pointwise estimate

$$|\phi|(x) \leqslant C\left(\|L\phi\|_* + \sum_j \|\phi\|_{L^\infty(\partial B(P_j,\rho))}\right)W(x), \tag{3.8}$$

for all $x \in \Omega_\varepsilon \setminus \bigcup_j B(P_j,\rho)$.

Granted these preliminary estimates, the proof of the result goes by contradiction. Let us assume there exist a sequence of ℓ tending to ∞ and a sequence of solutions of (2.16) for which the inequality is not true. The problem being linear, we can reduce to the case where we have a sequence $\ell^{(n)}$ tending to ∞ and sequences $h^{(n)}$, $\phi^{(n)}, c^{(n)}$ such that

$$\big\|h^{(n)}\big\|_* \to 0 \quad \text{and} \quad \big\|\phi^{(n)}\big\|_* = 1.$$

But (3.4) implies that we also have

$$\big\|c^{(n)}\big\|_* \to 0.$$

Then (3.8) implies that there exists $P^{(n)} \in \mathcal{P}$ (see (2.6) for the definition of $\mathcal{P}$) such that

$$\big\|\phi^{(n)}\big\|_{L^\infty(B(P^{(n)},\rho))} \geqslant C, \tag{3.9}$$

for some fixed constant $C > 0$. Using elliptic estimates together with Ascoli–Arzela's theorem, we can find a sequence $P^{(n)}$ and we can extract, from the sequence $\phi^{(n)}(\cdot - P^{(n)})$ a subsequence which will converge (on compact) to ϕ_∞ a solution of

$$\big(\Delta - 1 + pw^{p-1}\big)\phi_\infty = 0,$$

in $\mathbb{R}^N$, which is bounded by a constant times $e^{-\eta|x|}$, with $\eta > 0$. Moreover, recall that $\phi^{(n)}$ satisfies the orthogonality conditions in (2.16). Therefore, the limit function ϕ_∞ also satisfies

$$\int_{\mathbb{R}^N} \phi_\infty \nabla w \, dx = 0.$$

But the solution w being non-degenerate, this implies that $\phi_\infty \equiv 0$, which is certainly in contradiction with (3.9) which implies that ϕ_∞ is not identically equal to 0.

Having reached a contradiction, this completes the proof of the proposition. $\square$

We can now prove Proposition 3.1.

Proof of Proposition 3.1. Consider the space

$$\mathcal{H}=\left\{u\in H^1(\Omega_\varepsilon):\ \int\limits_{\Omega_\varepsilon} uZ_{ji}=0,\ j=1,\dots,k,\ i=1,\dots,N\right\}.$$

Notice that the problem (2.16) in ϕ gets re-written as

$$\phi+K(\phi)=\bar{h}\quad \text{in } \mathcal{H} \tag{3.10}$$

where $\bar{h}$ is defined by duality and $K:\mathcal{H}\to\mathcal{H}$ is a linear compact operator. Using Fredholm's alternative, showing that Eq. (3.10) has a unique solution for each $\bar{h}$ is equivalent to showing that the equation has a unique solution for $\bar{h}=0$, which in turn follows from Proposition 3.2. The estimate (3.2) follows directly from Proposition 3.2. This concludes the proof of Proposition 3.1.

In the following, if ϕ is the unique solution given by Proposition 3.1, we set

$$\phi=\mathcal{A}(h). \tag{3.11}$$

Estimate (3.2) implies

$$\big\|\mathcal{A}(h)\big\|_*\leqslant C\|h\|_*. \qquad \Box \tag{3.12}$$

4. The nonlinear projected problem

For small ε, large ℓ, and fixed points P_j given by (2.3) satisfying constraints (2.5) we show solvability in ϕ, c_{ji} of the nonlinear projected problem

$$\begin{cases} L(\phi)=E+N(\phi)+\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{k}c_{ji}Z_{ji} & \text{in } \Omega_\varepsilon,\\ \dfrac{\partial\phi}{\partial\nu}=0 & \text{on } \partial\Omega_\varepsilon,\\ \int\limits_{\Omega_\varepsilon}\phi Z_{ji}=0 & \text{for } j=1,\dots,k,\ i=1,\dots,N. \end{cases} \tag{4.1}$$

We have the validity of the following result.

Proposition 4.1. *Let $\delta>0$ be fixed. There exist positive numbers ε_0, C, and $\xi>0$ such that for all $\varepsilon\leqslant\varepsilon_0$, for all integers k and positive real numbers ℓ given by* (2.2) *and satisfying* (1.5)*, for any points P_j, $j=1,\dots,k$, given by* (2.3) *and satisfying* (2.5)*, there is a unique solution (ϕ,c_{ji}) to problem* (2.15)*. This solution depends continuously on the parameters of the construction* (*namely* a_j, $j=1,\dots,k$) *and furthermore*

$$\|\phi\|_*\leqslant Ce^{-\frac{(1+\xi)}{2}\ell}. \tag{4.2}$$

Proof. The proof relies on the contraction mapping theorem in the $\|\cdot\|_*$-norm above introduced. Observe that ϕ solves (2.15) if and only if

$$\phi=\mathcal{A}\big(E+N(\phi)\big) \tag{4.3}$$

where $\mathcal{A}$ is the operator introduced in (3.11). In other words, ϕ solves (2.15) if and only if ϕ is a fixed point for the operator

$$T(\phi) := \mathcal{A}\big(E + N(\phi)\big).$$

Given $r > 0$, define

$$\mathcal{B} = \left\{\phi \in C^2(\Omega_\varepsilon):\ \|\phi\|_* \leqslant r e^{-\frac{(1+\xi)}{2}\ell},\ \int\limits_{\Omega_\varepsilon} \phi Z_{ji} = 0\right\}.$$

We will prove that T is a contraction mapping from $\mathcal{B}$ into itself.

To do so, we claim that

$$\|E\|_* \leqslant C e^{-\frac{(1+\xi)}{2}\ell} \tag{4.4}$$

and

$$\big\|N(\phi)\big\|_* \leqslant C\big[\|\phi\|_*^2 + \|\phi\|_*^p\big], \tag{4.5}$$

for some fixed function C independent of ℓ, as $\ell \to \infty$. We postpone the proof of the estimates above to the end of the proof of this proposition. Assuming the validity of (4.4) and (4.5) and taking into account (3.12), we have for any $\phi \in \mathcal{B}$

$$\begin{aligned}\big\|T(\phi)\big\|_* &\leqslant C\big[\big\|E + N(\phi)\big\|_*\big] \leqslant C\big[e^{-\frac{(1+\xi)}{2}\ell} + r^2 e^{-(1+\xi)\ell} + r^p e^{-\frac{p(1+\xi)}{2}\ell}\big] \\ &\leqslant r e^{-\frac{(1+\xi)}{2}\ell}\end{aligned}$$

for a proper choice of r in the definition of $\mathcal{B}$, since $p > 1$.

Take now ϕ_1 and ϕ_2 in $\mathcal{B}$. Then it is straightforward to show that

$$\begin{aligned}\big\|T(\phi_1) - T(\phi_2)\big\|_* &\leqslant C\big\|N(\phi_1) - N(\phi_2)\big\|_* \\ &\leqslant C\big[\|\phi_1\|_*^{\min(1,p-1)} + \|\phi_2\|_*^{\min(1,p-1)}\big]\|\phi_1 - \phi_2\|_* \\ &\leqslant o(1)\|\phi_1 - \phi_2\|_*.\end{aligned}$$

This means that T is a contraction mapping from $\mathcal{B}$ into itself.

To conclude the proof of this proposition we are left to show the validity of (4.4) and (4.5). We start with (4.4).

Fix $j \in \{1, \dots, k\}$ and consider the region $|x - P_j| \leqslant \frac{\ell}{2+\sigma}$, where σ is a small positive number to be chosen later. In this region the error E, whose definition is in (2.10), can be estimated in the following way

$$
\begin{aligned}
\left|E(x)\right| &\leqslant C\left[\omega^{p-1}(x-P_j)\sum_{P_i\neq P_j}\omega(x-P_i)+\sum_{P_i\neq P_j}\omega^p(x-P_i)\right]\\
&\leqslant C\omega^{p-1}(x-P_j)\sum_{P_i\neq P_j}e^{-(\frac{1}{2}+\frac{\sigma}{2(2+\sigma)})\ell}\\
&\leqslant C\omega^{p-1}(x-P_j)e^{-(\frac{1}{2}+\frac{\sigma}{4(2+\sigma)})\ell}e^{-\frac{\sigma}{4(2+\sigma)}\ell}\\
&\leqslant C\omega^{p-1}(x-P_j)e^{-\frac{1+\xi}{2}\ell}
\end{aligned}
\tag{4.6}
$$

for a proper choice of $\xi > 0$.

Consider now the region $|x-P_j| > \frac{\ell}{2+\sigma}$, for all j. Since $0<\mu<p-1$, we write $\mu = p-1-M$. From the definition of E, we get in the region under consideration

$$
\begin{aligned}
\left|E(x)\right| &\leqslant C\left[\sum_j \omega^p(x-P_j)\right]\leqslant C\left[\sum_j e^{-\mu|x-P_j|}\right]e^{-(p-\mu)\frac{\ell}{2+\sigma}}\\
&\leqslant \left[\sum_j e^{-\mu|x-P_j|}\right]e^{-\frac{1+M}{2+\sigma}\ell}\leqslant \left[\sum_j e^{-\mu|x-P_j|}\right]e^{-\frac{1+\xi}{2}\ell}
\end{aligned}
\tag{4.7}
$$

for some $\xi > 0$, if we chose M and σ small enough. From (4.6) and (4.7) we get (4.4).

We now prove (4.5). Let $\phi \in \mathcal{B}$. Then

$$
\left|N(\phi)\right| \leqslant \left|(U+\phi)^p - U^p - pU^{p-1}\phi\right| \leqslant C\left(\phi^2+|\phi|^p\right). \tag{4.8}
$$

Thus we have

$$
\begin{aligned}
\left|\sum_j e^{\eta|x-P_j|}N(\phi)\right| &\leqslant C\|\phi\|_*\left(|\phi|+|\phi|^{p-1}\right)\\
&\leqslant C\left(\|\phi\|_*^2+\|\phi\|_*^p\right).
\end{aligned}
$$

This gives (4.5).

A direct consequence of the fixed point characterization of ϕ given above together with the fact that the error term E depends continuously (in the $*$-norm) on the parameters a_j, $j=1,\dots,k$, is that the map $(\mathsf{a}_1,\dots,\mathsf{a}_k)\to\phi$ into the space $C(\bar{\Omega}_\varepsilon)$ is continuous (in the $*$-norm). This concludes the proof of the proposition. $\square$

Given points P_j defined by (2.3), satisfying constraint (2.5), Proposition 4.1 guarantees the existence (and gives estimates) of a unique solution ϕ, c_{ji}, $j=1,\dots,k$, $i=1,\dots,N$, to problem (2.15). It is clear then that the function $u=U+\phi$ is an exact solution to our problem (1.1), with the required properties stated in Theorem 1.1 if we show that there exists a configuration for the points P_j that gives all the constants c_{ji} in (2.15) equal to zero. In order to do so we first need to find the correct conditions on the points to get $c_{ji}=0$. This condition is naturally given by projecting in $L^2(\Omega_\varepsilon)$ the equation in (2.15) into the space spanned by Z_{ji}, namely by multiplying the equation in (2.15) by Z_{ji} and integrate all over Ω_ε. We will do it in detail in the next final section.

5. Projection of the error and proof of Theorem 1.1

Define the following $k \times k$ matrix

$$T := \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix} \in \mathcal{M}_{k \times k}. \tag{5.1}$$

The inverse of T is the matrix whose entries are given by

$$\left(T^{-1}\right)_{ij} = \min(i, j) - \frac{ij}{k+1}.$$

We define the vectors $S^{\downarrow}$ and $S^{\uparrow}$ by

$$TS^{\downarrow} := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^k, \qquad TS^{\uparrow} := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^k. \tag{5.2}$$

It is immediate to check that

$$S^{\downarrow} := \begin{pmatrix} \frac{1}{k+1} \\ \frac{2}{k+1} \\ \vdots \\ \frac{k-1}{k+1} \\ \frac{k}{k+1} \end{pmatrix} \in \mathbb{R}^k, \qquad S^{\uparrow} := \begin{pmatrix} \frac{k}{k+1} \\ \frac{k-1}{k+1} \\ \vdots \\ \frac{2}{k+1} \\ \frac{1}{k+1} \end{pmatrix} \in \mathbb{R}^k. \tag{5.3}$$

We will reorder the parameters a_{ij}, for $i = 1, \dots, N$ and $j = 1, \dots, k$ in the following way: for any $j = 1, \dots, N$,

$$\mathsf{a}_j^* = \begin{pmatrix} a_{j1} \\ a_{j2} \\ \vdots \\ a_{jk} \end{pmatrix} \in \mathbb{R}^k. \tag{5.4}$$

Proposition 5.1. *Let ϕ be the solution of* (2.15) *which has been obtained in Proposition* 4.1. *The coefficients c_{ij} are all equal to* 0 *if and only if the vectors a_j^* defined in* (5.4) *are solutions of the nonlinear system*

$$\mathsf{a}_j^* = \left(1 + \frac{L}{k}\lambda_j\right) a_{1j} S^{\uparrow} + \left[\left(1 - \frac{L}{k} g_{jj}\right) a_{kj} - \frac{L}{k} \sum_{i \neq j} g_{ij} a_{ki}\right] S^{\downarrow} + e^{-\delta_2 \ell} A + Q \in \mathbb{R}^k$$

for any $j = 1, \dots, N-1$

$$\mathrm{a}_N^* = -a_{1N} S^{\uparrow} - a_{kN} S^{\downarrow} + e^{-\delta_2 \ell} A + Q \in \mathbb{R}^k \tag{5.5}$$

where $\delta_2 > 0$, $A = A(\mathrm{a}_1, \dots, \mathrm{a}_k)$ *and* $Q = Q(\mathrm{a}_1, \dots, \mathrm{a}_k)$ *denote smooth vector valued functions* (*which vary from line to line*), *uniformly bounded as* $\varepsilon \to 0$ (*or equivalently as* $\ell \to \infty$) *and the Taylor expansion of* Q *with respect to* $\mathrm{a}_1, \dots, \mathrm{a}_k$ *does not involve any constant nor any linear term. Here the vectors* a_j, $j = 1, \dots, k$, *are defined in* (2.4) *and they satisfy constraints* (2.5).

Proof. Observe that all c_{ij} are zero if and only if

$$\int_{\Omega_\varepsilon} \big(L\phi + E + N(\phi)\big) Z_{ij} \, dx = 0 \quad \text{for all } i, j.$$

Using the lemmas below, it is easy to check that this reduces to the solvability of a nonlinear system in a_j^*, that can be written in the desired form using the inverse of the matrices T.

Observe that the norms of the inverses of the matrix T blow up at most linearly in k, as $k \to \infty$. Under our assumptions (1.5), this can be absorbed since the error tends to 0 exponentially fast in terms of ℓ. $\square$

Define $\kappa := 1 + \frac{N-1}{2\ell}$.

Lemma 5.1. *The following expansions hold*

$$-\frac{e^{\ell} \ell^{\frac{N-1}{2}}}{C_{N,p}} \int_{\Omega_\varepsilon} E Z_{1j} \, dx = \left(a_{2j} - a_{1j} + \frac{L}{k} \lambda_j a_{1j} \right) + e^{-\delta_2 \ell} A + Q, \tag{5.6}$$

for $j = 1, \dots, N-1$, *and*

$$-\frac{e^{\ell} \ell^{\frac{N-1}{2}}}{C_{N,p}} \int_{\Omega_\varepsilon} E Z_{1N} \, dx = \kappa (3a_{1N} - a_{2N}) + e^{-\delta_2 \ell} A + Q. \tag{5.7}$$

For $h = 2, \dots, k-1$

$$-\frac{e^{\ell} \ell^{\frac{N-1}{2}}}{C_{N,p}} \int_{\Omega_\varepsilon} E Z_{hj} \, dx = (a_{h-1,j} - 2a_{hj} + a_{h+1,j}) + e^{-\delta_2 \ell} A + Q, \tag{5.8}$$

for $j = 1, \dots, N-1$, *and*

$$-\frac{e^{\ell} \ell^{\frac{N-1}{2}}}{C_{N,p}} \int_{\Omega_\varepsilon} E Z_{hN} \, dx = \kappa (-a_{h-1,N} + 2a_{h,N} - a_{h+1,N}) + e^{-\delta_2 \ell} A + Q. \tag{5.9}$$

Finally

$$-\frac{e^{\ell} \ell^{\frac{N-1}{2}}}{C_{N,p}} \int_{\Omega_\varepsilon} E Z_{kj} \, dx = \left(a_{k-1,j} - a_{kj} - \frac{L}{k} G_2''(0)[\bar{\mathrm{a}}_k][\mathrm{e}_j] \right) + e^{-\delta_2 \ell} A + Q, \tag{5.10}$$

for $j = 1, \dots, N-1$, *and*

$$-\frac{e^{\ell}\ell^{\frac{N-1}{2}}}{C_{N,p}}\int\limits_{\Omega_\varepsilon} EZ_{kN}\,dx = \kappa(3a_{k,N}-a_{k-1,N}) + e^{-\delta_2\ell}A + Q, \tag{5.11}$$

where $\delta_2 > 0$, $A = A(\mathrm{a}_1,\dots,\mathrm{a}_k)$, $Q = Q(\mathrm{a}_1,\dots,\mathrm{a}_k)$ denote smooth vector valued functions (which vary from line to line), uniformly bounded as $\ell \to \infty$ and the Taylor expansion of Q with respect to $\mathrm{a}_1,\dots,\mathrm{a}_k$ does not involve any constant nor any linear term.

Proof. Given $P \in \mathcal{P}$, we would like to estimate

$$\int\limits_{\Omega_\varepsilon} EZ_{ij}\,dx.$$

An important estimate that we will use several times to compute the above expression is the following: There exists a constant $C_{N,p} > 0$ only depending on N and p such that the following expansion holds

$$p\int\limits_{\mathbb{R}^N} w(\cdot - y)w^{p-1}\partial_{x_j}\omega\,dx = -C_{N,p}\psi\big(|y|\big)y\cdot \mathrm{e}_j + O\big(e^{-\delta_3|y|}\big), \tag{5.12}$$

where $\delta_3 > 1$ is a constant which depends on p and N. In (5.12) the function ψ is defined as follows: for all $s > 0$,

$$\psi(s) := e^{-s}s^{-\frac{N+1}{2}}.$$

The proof of (5.12) is by now standard, we refer to [23] and [21] for details.

Observe that, given $\mathrm{e} \in \mathbb{R}^N$ with $|\mathrm{e}| = 1$ and $\mathrm{a} \in \mathbb{R}^N$, the following expansion holds

$$\psi\big(|\tilde{\ell}\mathrm{e}+\mathrm{a}|\big)(\tilde{\ell}\mathrm{e}+\mathrm{a}) = e^{-\tilde{\ell}}\tilde{\ell}^{-\frac{N-1}{2}}\left(\mathrm{e}-\tilde{\kappa}\mathrm{a}^{\parallel}+\frac{1}{\tilde{\ell}}\mathrm{a}^{\perp}\right) + e^{-\tilde{\ell}}\tilde{\ell}^{-\frac{N-1}{2}}O\big(|\mathrm{a}|^2\big) \tag{5.13}$$

as $\tilde{\ell} \to \infty$. Here, we have decomposed $\mathrm{a} = \mathrm{a}^{\parallel} + \mathrm{a}^{\perp}$ where $\mathrm{a}^{\parallel}$ is collinear to e and $\mathrm{a}^{\perp}$ is orthogonal to e.

We have all the elements now to proceed in the computations of estimates (5.6)–(5.11).

Estimates (5.6) and (5.7). Observe that, given the structure of U, the fact that the function w decays exponentially and the result in Lemma 2.1, we can write using Taylor's expansion

$$\int\limits_{\Omega_\varepsilon} EZ_{1j}\,dx = p\int\limits_{\Omega_\varepsilon}\omega_1^{p-1}[-\varphi_1+\omega_2]Z_{1j}\,dx + e^{-\delta_3\ell}A \tag{5.14}$$

where $\delta_3 > 1$ and $A = A(\mathrm{a}_1,\dots,\mathrm{a}_k)$ is uniformly bounded as $\ell \to \infty$ for vectors a_j satisfying (2.5).

Let $\bar{P}_1$ be the only point on $\partial\Omega_\varepsilon$ such that $d(P_1,\partial\Omega_\varepsilon) = d(P_1,\bar{P}_1)$ and define P_1^* to be $P_1^* - P_1 = 2(\bar{P}_1 - P_1)$. Again Lemma 2.1 gives that $\varphi_1(x) = -\omega(x-P_1^*) + o(\varepsilon^{N+1})$. Then, since $P_1 = \frac{\ell}{2}\mathrm{e}_N + a_{N1}\mathrm{e}_N + \ell\bar{\mathrm{a}}_1$ with $\bar{\mathrm{a}}_1 = (a_{11}, a_{21},\dots,a_{N-1,1})$ and $|a_{i1}| \leqslant \delta$, we have

$$P_1^* - P_1 = -(\ell + 2a_{N1})\mathrm{e}_N + \ell^2\big(G_1''(0)[\varepsilon\bar{\mathrm{a}}_1],0\big) + \big(O\big(\varepsilon^2\ell^2\big), O(\varepsilon\ell)\big).$$

Thus we get from (5.12) and (5.13), for $j = 1,\dots,N-1$,

$$
\begin{aligned}
-p\int_{\Omega_\varepsilon}\omega_1^{p-1}\varphi_1 Z_{1j}\,dx &= p\int_{\Omega_\varepsilon}\omega_1^{p-1}\omega\big(x-P_1^*\big)Z_{1j}\,dx\\
&= -C_{N,p}\ell^{-\frac{N-1}{2}}e^{-\ell}\big[\varepsilon\ell D^2G_1(0)[\bar{\mathrm{a}}_1]\cdot \mathrm{e}_j\big]\\
&\quad + e^{-\delta_3\ell}A+\ell^{-\frac{N+1}{2}}e^{-\ell}Q\\
&= -C_{N,p}\ell^{-\frac{N-1}{2}}e^{-\ell}\left[\frac{L}{k}\lambda_j a_{1j}\right]+e^{-\delta_3\ell}A+\ell^{-\frac{N+1}{2}}e^{-\ell}Q
\end{aligned}
\tag{5.15}
$$

while for $j=N$, we have

$$
\begin{aligned}
-p\int_{\Omega_\varepsilon}\omega_1^{p-1}\varphi_1 Z_{1N}\,dx &= p\int_{\Omega_\varepsilon}\omega_1^{p-1}\omega\big(x-P_1^*\big)Z_{1j}\,dx\\
&= -C_{N,p}\ell^{-\frac{N-1}{2}}e^{-\ell}[-1+2a_{1N}\kappa]+e^{-\delta_3\ell}A+\ell^{-\frac{N+1}{2}}e^{-\ell}Q.
\end{aligned}
$$

In the above computations we use the fact that by definition $\frac{L}{\varepsilon}=k\ell$.

On the other hand, a direct use of (5.12) and (5.13) gives, for $j=1,\dots,N-1$,

$$
p\int_{\Omega_\varepsilon}\omega_1^{p-1}\omega_2 Z_{1j}\,dx = -C_{N,p}\ell^{-\frac{N-1}{2}}e^{-\ell}[a_{2j}-a_{1j}]+e^{-\delta_3\ell}A+\ell^{-\frac{N+1}{2}}e^{-\ell}Q
$$

and, for $j=N$,

$$
\begin{aligned}
p\int_{\Omega_\varepsilon}\omega_1^{p-1}\omega_2 Z_{1N}\,dx &= -C_{N,p}\ell^{-\frac{N-1}{2}}e^{-\ell}\big[1-(a_{2N}-a_{1N})\kappa\big]\\
&\quad + e^{-\delta_3\ell}A+\ell^{-\frac{N+1}{2}}e^{-\ell}Q.
\end{aligned}
$$

Putting together the above computation in (5.14) we get the validity of (5.6) and (5.7).

Estimates (5.8) and (5.9). Let $1<h<k$. In this case, we have

$$
\int_{\Omega_\varepsilon} EZ_{hj}\,dx = p\int_{\Omega_\varepsilon} U_h^{p-1}[\omega_{h-1}+\omega_{h+1}]Z_{hj}\,dx+e^{-\delta_3\ell}A
\tag{5.16}
$$

where $\delta_3>1$ and $A=A(\mathrm{a}_1,\dots,\mathrm{a}_k)$ is uniformly bounded as $\varepsilon\to 0$ for vectors a_j satisfying (2.5). This fact is again a consequence of the exponential decay of ω at infinity, and of the result contained in Lemma 2.1.

In this case, a direct computation gives (5.8) and (5.9).

Estimates (5.10) and (5.11). Arguing as in the case of the proof of estimates (5.6) and (5.7), we get

$$
\int_{\Omega_\varepsilon} EZ_{kj}\,dx = p\int_{\Omega_\varepsilon}\omega_k^{p-1}[-\varphi_k+\omega_{k-1}]Z_{kj}\,dx+e^{-\delta_3\ell}A
\tag{5.17}
$$

where $\delta_3>1$ and $A=A(\mathrm{a}_1,\dots,\mathrm{a}_k)$ is uniformly bounded as $\ell\to\infty$ for vectors a_j satisfying (2.5).

Let $\bar{P}_k$ be the only point on $\partial\Omega_\varepsilon$ such that $d(P_k, \partial\Omega_\varepsilon) = d(P_k, \bar{P}_k)$ and define P_k^* to be $P_1^* - P_k = 2(\bar{P}_k - P_k)$. Again Lemma 2.1 gives that $\varphi_k(x) = -\omega(x - P_k^*) + o(\varepsilon^{N+1})$. Then, since $P_k = (k - \frac{1}{2})\ell \mathrm{e}_N + a_{Nk}\mathrm{e}_N + \ell \bar{\mathrm{a}}_k$ with $\bar{\mathrm{a}}_1 = (a_{11}, a_{21}, \dots, a_{N-11})$ and $|a_{i1}| \leqslant \delta$, we have

$$P_k^* - P_1 = (\ell - 2a_{Nk})\mathrm{e}_N - \ell^2\big(G_2''(0)[\varepsilon\bar{\mathrm{a}}_k], 0\big) + \big(O\big(\varepsilon^2\ell^2\big), O(\varepsilon\ell)\big).$$

Thus, recalling that $\frac{L}{\varepsilon} = k\ell$, we get from (5.12) and (5.13), for $j = 1, \dots, N-1$,

$$\begin{aligned}
-p\int_{\Omega_\varepsilon} \omega_k^{p-1}\varphi_k Z_{kj}\,dx &= p\int_{\Omega_\varepsilon} \omega_k^{p-1}\omega\big(x - P_k^*\big)Z_{kj}\,dx \\
&= -C_{N,p}\ell^{-\frac{N-1}{2}}e^{-\ell}\big[-\varepsilon\ell D^2G_2(0)[\bar{\mathrm{a}}_k\cdot \mathrm{e}_j]\big] \\
&\quad + e^{-\delta_3\ell}A + \ell^{-\frac{N+1}{2}}e^{-\ell}Q \\
&= C_{N,p}\ell^{-\frac{N-1}{2}}e^{-\ell}\left[\frac{L}{k}D^2G_2(0)[\bar{\mathrm{a}}_k\cdot \mathrm{e}_j]\right] \\
&\quad + e^{-\delta_3\ell}A + \ell^{-\frac{N+1}{2}}e^{-\ell}Q
\end{aligned}$$

while for $j = N$, we have

$$\begin{aligned}
-p\int_{\Omega_\varepsilon} \omega_k^{p-1}\varphi_k Z_{kN}\,dx &= p\int_{\Omega_\varepsilon} \omega_1^{p-1}\omega\big(x - P_k^*\big)Z_{kj}\,dx \\
&= -C_{N,p}\ell^{-\frac{N-1}{2}}e^{-\ell}[1 + 2a_{kN}\kappa] \\
&\quad + e^{-\delta_3\ell}A + \ell^{-\frac{N+1}{2}}e^{-\ell}Q.
\end{aligned}$$

On the other hand, a direct use of (5.12) and (5.13) gives, for $j = 1, \dots, N-1$,

$$\begin{aligned}
p\int_{\Omega_\varepsilon} \omega_k^{p-1}\omega_{k-1}Z_{kj}\,dx &= -C_{N,p}\ell^{-\frac{N-1}{2}}e^{-\ell}[a_{(k-1)j} - a_{kj}] \\
&\quad + e^{-\delta_3\ell}A + \ell^{-\frac{N+1}{2}}e^{-\ell}Q
\end{aligned}$$

and, for $j = N$,

$$\begin{aligned}
p\int_{\Omega_\varepsilon} \omega_k^{p-1}\omega_{k-1}Z_{kN}\,dx &= -C_{N,p}\ell^{-\frac{N-1}{2}}e^{-\ell}\big[-1 - (a_{(k-1)N} - a_{kN})\kappa\big] \\
&\quad + e^{-\delta_3\ell}A + \ell^{-\frac{N+1}{2}}e^{-\ell}Q.
\end{aligned}$$

This concludes the proof of (5.10) and (5.11). $\square$

The next result is easier to get. It reads:

Lemma 5.2. *The following expansions hold*

$$e^{\ell}\ell^{\frac{N-1}{2}}\int\limits_{\mathbb{R}^N}(L\phi)Z_{ji}\,dx=e^{-\delta_2\ell}A$$

and

$$e^{\ell}\ell^{\frac{N-1}{2}}\int\limits_{\mathbb{R}^N}N(\phi)Z_{ji}\,dx=e^{-\delta_2\ell}A,$$

where $\delta_2>0$ *and* $A=A(\mathrm{a}_1,\dots,\mathrm{a}_k)$ *denote smooth vector valued functions* (*which vary from line to line*)*, uniformly bounded as* $\ell\to\infty$.

Proof. The proof of the first estimate follows the line of the proof of Proposition 3.2 (see formula (3.6) and the subsequent estimates, together with (4.2)).

The proof of the second estimate follows from estimates (4.5) and (4.2). □

We now explain how (5.5) can be solved. We claim that this system is equivalent to

$$\mathrm{a}_j^*=e^{-\tilde{\delta}_2\ell}A+Q \tag{5.18}$$

where $\tilde{\delta}_2>0$ and $A=A(\mathrm{a}_1,\dots,\mathrm{a}_k)$ and $Q=Q(\mathrm{a}_1,\dots,\mathrm{a}_k)$ satisfy the usual assumptions.

Indeed, by using the explicit expression for $S^{\downarrow}$ and $S^{\uparrow}$ given by (5.3), we get that solving the system (5.5) reduces to find a solution to the following nonlinear system in the $2N$ variables a_{1j}, a_{kj}, for $j=1,\dots,N$

$$(-1+L\lambda_j)a_{1j}+\left(1-\frac{L}{k}g_{jj}\right)a_{kj}-\frac{L}{k}\sum_{i\neq j}g_{ij}a_{ki}=e^{-\delta_3\ell}A+Q,$$
$$\left(1+\frac{L}{k}\lambda_j\right)a_{1j}+(-1-Lg_{jj})a_{kj}-L\sum_{i\neq j}g_{ij}a_{ki}=e^{-\delta_3\ell}A+Q$$

for $j=1,\dots,N-1$, and

$$(2k+1)a_{1N}+a_{kN}=e^{-\delta_3\ell}A+Q,$$
$$a_{1N}+(2k+1)a_{kN}=e^{-\delta_3\ell}A+Q,$$

for some $0<\delta_3\leqslant\delta_2$. This system can be solved provided the following $2(N-1)\times 2(N-1)$ matrix

$$B_k:=\begin{pmatrix}-I+LG_1''(0) & I-\frac{L}{k}G_2''(0)\\ I+\frac{L}{k}G_1''(0) & -I-LG_2''(0)\end{pmatrix}$$

has non-zero determinant. In the above expression I denotes the identity matrix of dimension $N-1$.

Denote by $\mathbf{B}$ the matrix $\left(\begin{smallmatrix}-I+LG_1''(0) & I\\ I & -I-LG_2''(0)\end{smallmatrix}\right)$ and observe that

$$\det\mathbf{B}=(-L)^{N-1}\det\begin{pmatrix}G_1''(0) & I\\ G_2''(0) & I+LG_2''(0)\end{pmatrix}=\text{const}\neq 0$$

since we are assuming the non-degeneracy condition (1.2). Thus we have, since we are assuming (1.4), namely that $k \to \infty$ as $\varepsilon \to 0$,

$$\det B_k = \det \mathbf{B} \times \left(1 + \frac{L}{k} \operatorname{tr}\left(\mathbf{B}^{-1} \begin{pmatrix} 0 & -G_2''(0) \\ G_1''(0) & 0 \end{pmatrix}\right) + O\left(\frac{1}{k^2}\right)\right)$$
$$= \det \mathbf{B}\left(1 + O\left(\frac{1}{k}\right)\right) \neq 0.$$

This completes the proof of the claim.

It is now straightforward to prove, using Browder's fixed point theorem, that

Lemma 5.3. *There exist $C > 0$ and $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$ (or equivalently ℓ large), there exists a solution of (5.18) such that*

$$\left|\mathsf{a}_j^*\right| \leqslant C e^{-\tilde{\delta}_2 \ell} \quad \textit{for all } j = 1, \ldots, N.$$

This last result completes the proof of Theorem 1.1.

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References

[1] N. Alikakos, M. Kowalczyk, Critical points of a singular perturbation problem via reduced energy and local linking, J. Differential Equations 159 (2) (1999) 403–426.
[2] A. Ambrosetti, A. Malchiodi, W.-M. Ni, Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres, Part II, Indiana Univ. Math. J. 53 (2) (2004) 297–329.
[3] P. Bates, G. Fusco, Equilibria with many nuclei for the Cahn–Hilliard equation, J. Differential Equations 160 (2000) 283–356.
[4] P. Bates, E. Dancer, J. Shi, Multi-spike stationary solutions of the Cahn–Hilliard equation in higher-dimension and instability, Adv. Differential Equations 4 (1999) 1–69.
[5] G. Cerami, J. Wei, Multiplicity of multiple interior peak solutions for some singularly perturbed Neumann problems, Int. Math. Res. Not. IMRN 12 (1998) 601–626.
[6] T. D'Aprile, A. Pistoia, On the existence of some new positive interior spike solutions to a semilinear Neumann problem, J. Differential Equations 248 (2010) 556–573.
[7] M. del Pino, P. Felmer, J. Wei, On the role of mean curvature in some singularly perturbed Neumann problems, SIAM J. Math. Anal. 31 (1999) 63–79.
[8] M. del Pino, P. Felmer, J. Wei, On the role of distance function in some singularly perturbed problems, Comm. Partial Differential Equations 25 (2000) 155–177.
[9] M. del Pino, P. Felmer, J. Wei, Multiple peak solutions for some singular perturbation problems, Calc. Var. Partial Differential Equations 10 (2000) 119–134.
[10] E. Dancer, S. Yan, Multipeak solutions for a singularly perturbed Neumann problem, Pacific J. Math. 189 (1999) 241–262.
[11] C. Gui, Multipeak solutions for a semilinear Neumann problem, Duke Math. J. 84 (1996) 739–769.
[12] C. Gui, J. Wei, Multiple interior spike solutions for some singular perturbed Neumann problems, J. Differential Equations 158 (1999) 1–27.
[13] C. Gui, J. Wei, On multiple mixed interior and boundary peak solutions for some singularly perturbed Neumann problems, Canad. J. Math. 52 (2000) 522–538.
[14] C. Gui, J. Wei, M. Winter, Multiple boundary peak solutions for some singularly perturbed Neumann problems, Ann. Inst. H. Poincaré Anal. Non Linéaire 17 (2000) 249–289.
[15] M. Grossi, A. Pistoia, J. Wei, Existence of multipeak solutions for a semilinear Neumann problem via nonsmooth critical point theory, Calc. Var. Partial Differential Equations 11 (2000) 143–175.
[16] M. Kowalczyk, Multiple spike layers in the shadow Gierer–Meinhardt system: existence of equilibria and the quasi-invariant manifold, Duke Math. J. 98 (1) (1999) 59–111.
[17] M. Kowalczyk, On the existence and Morse index of solutions to the Allen–Cahn equation in two dimensions, Ann. Mat. Pura Appl. 184 (1) (2005) 17–52.
[18] M. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^N$, Arch. Ration. Mech. Anal. 105 (1991) 243–266.

[19] Y.-Y. Li, On a singularly perturbed equation with Neumann boundary condition, Comm. Partial Differential Equations 23 (1998) 487–545.
[20] Y.-Y. Li, L. Nirenberg, The Dirichlet problem for singularly perturbed elliptic equations, Comm. Pure Appl. Math. 51 (1998) 1445–1490.
[21] F.H. Lin, W.M. Ni, J.C. Wei, On the number of interior peak solutions for a singularly perturbed Neumann problem, Comm. Pure Appl. Math. 60 (2007) 252–281.
[22] A. Malchiodi, Concentration at curves for a singularly perturbed Neumann problem in three-dimensional domains, Geom. Funct. Anal. 15 (6) (2005) 1162–1222.
[23] A. Malchiodi, Some new entire solutions of semilinear elliptic equations in $\mathbb{R}^N$, Adv. Math. 221 (6) (2009) 1843–1909.
[24] A. Malchiodi, M. Montenegro, Boundary concentration phenomena for a singularly perturbed elliptic problem, Comm. Pure Appl. Math. 55 (2002) 1507–1508.
[25] A. Malchiodi, M. Montenegro, Multidimensional boundary layers for a singularly perturbed Neumann problem, Duke Math. J. 124 (1) (2004) 105–143.
[26] A. Malchiodi, W.-M. Ni, J. Wei, Multiple clustered layer solutions for semilinear Neumann problems on a ball, Ann. Inst. H. Poincaré Anal. Non Linéaire 22 (2) (2005) 143–163.
[27] W.-M. Ni, Diffusion, cross-diffusion, and their spike-layer steady states, Notices Amer. Math. Soc. 45 (1998) 9–18.
[28] W.-M. Ni, Qualitative properties of solutions to elliptic problems, in: Handbook of Differential Equations: Stationary Partial Differential Equations, vol. I, North-Holland, Amsterdam, 2004, pp. 157–233.
[29] W.-M. Ni, I. Takagi, On the shape of least energy solution to a semilinear Neumann problem, Comm. Pure Appl. Math. 41 (1991) 819–851.
[30] W.-M. Ni, I. Takagi, Locating the peaks of least energy solutions to a semilinear Neumann problem, Duke Math. J. 70 (1993) 247–281.
[31] W.-M. Ni, J. Wei, On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems, Comm. Pure Appl. Math. 48 (1995) 731–768.
[32] J. Wei, On the boundary spike layer solutions of singularly perturbed semilinear Neumann problem, J. Differential Equations 134 (1997) 104–133.
[33] J. Wei, Existence and stability of spikes for the Gierer–Meinhardt system, in: M. Chipot (Ed.), Handbook of Differential Equations: Stationary Partial Differential Equations, vol. 5, North-Holland, Amsterdam, 2008, pp. 487–585.
[34] J. Wei, M. Winter, Stationary solutions for the Cahn–Hilliard equation, Ann. Inst. H. Poincaré Anal. Non Linéaire 15 (1998) 459–492.
[35] J. Wei, M. Winter, Multiple boundary spike solutions for a wide class of singular perturbation problems, J. Lond. Math. Soc. 59 (1999) 585–606.
[36] J. Wei, J. Yang, Concentration on lines for a singularly perturbed Neumann problem in two-dimensional domains, Indiana Univ. Math. J. 56 (2007) 3025–3073.