MULTIPLE SOLUTIONS FOR A CRITICAL NONHOMOGENEOUS ELLIPTIC PROBLEM IN DOMAINS WITH SMALL HOLES

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Abstract. We consider the problem: $-\Delta u = |u|^{\frac{4}{N-2}}u + \varepsilon f(x)$ in Ω, $u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain which exhibits small holes, $f \geq 0$, $f \not\equiv 0$ and $\varepsilon > 0$ is small. Using the reduction method and a min-max scheme worked out with topological arguments, we construct multiple solutions by gluing negative double-spike patterns located near each of the holes.

1. Introduction

In this paper we construct solutions which are not necessarily positive to the following problem

$$\begin{cases} -\Delta u &= |u|^{p-1}u + \varepsilon f(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth and bounded domain of $\mathbb{R}^N$, $N \geq 3$, which has m small holes, $p = \frac{N+2}{N-2}$ is the critical Sobolev exponent, $f(x)$ is a nonhomogeneous perturbation, $f \geq 0$, $f \not\equiv 0$ and $\varepsilon > 0$ is a small parameter.

It is known that if $f \equiv 0$, problem (1.1) is a delicate matter of treating from a variational viewpoint because the P.S. condition fails. In fact, in this case Pohozaev [23] proved that (1.1) has no solution if Ω is star-shaped. On the other hand, in a recognized paper, Brezis and Nirenberg [8] showed that the previous situation may be reversed introducing suitable nonhomogeneous perturbations. Since then, in the case $f \geq 0$, $f \not\equiv 0$, many results about existence and multiplicity of positive

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solutions of (1.1) have arisen under the assumption that $\varepsilon > 0$ is small enough, see for instance [8, 27, 30, 2, 21, 11].

Concerning solutions which are not necessarily positive, we know two works. Recently, under certain symmetry assumptions on Ω and f, Clapp *et al.* [10] have proved existence and multiplicity of solutions of (1.1) which develop k negative spikes, for any $k \geq k_0(\Omega)$, supposes that $\varepsilon > 0$ is sufficiently small. In special, they proved that if Ω is an anullus of fixed size and $f \geq 0$, $f \not\equiv 0$, then the number of solutions of (1.1) tends to infinite as ε goes to 0, which are negative if support of f is compact in Ω. More recently, the author [1] constructed a solution of (1.1) which develop a negative double-spike shape as $\varepsilon \to 0$.

Motivated by the above results, we leave aside any symmetry assumption on the domain Ω and the perturbation f, and we construct multiple solutions of (1.1) by gluing negative double-spike patterns located near each of the holes of Ω, provided that $\varepsilon > 0$ is small enough. More precisely, our setting in problem (1.1) is as follows: let $\mathcal{D}$ be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, and let us consider points $P_1, P_2, \ldots, P_m$ in $\mathcal{D}$ and smooth domains $\mathfrak{D}_i$ such that $\mathfrak{D}_i \subset \overline{B(P_i, \mu)} \subset \mathcal{D}$, where $\mu > 0$ is a fixed small number. Let us consider the domain

$$\Omega = \mathcal{D} \setminus \bigcup_{i=1}^{m} \overline{\mathfrak{D}}_i,$$

a function $f \in C^{0,\gamma}(\overline{\Omega})$, for some $0 < \gamma < 1$, such that $\inf_{x\in\Omega} f(x) > 0$, and the unique solution w to the problem

$$\begin{cases} -\Delta w = f & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Our main result is

Theorem 1.1. *Let $1 \leq k \leq m$ be fixed. Assume that $\varepsilon = \varepsilon_n \to 0$ as $n \to +\infty$. Then, up to subsequences, there exist positive numbers $\lambda_{ij\varepsilon}$, points ξ_{ij}^{ε} in Ω and nontrivial solutions u_ε of (1.1) of the form*

$$u_\varepsilon(x) = -\alpha_N \sum_{i=1}^{k} \sum_{j=1}^{2} \left(\frac{\varepsilon^{\frac{2}{N-2}} \lambda_{ij\varepsilon}}{\varepsilon^{\frac{4}{N-2}} \lambda_{ij\varepsilon}^2 + |x - \xi_{ij}^{\varepsilon}|^2} \right)^{\frac{N-2}{2}} + \varepsilon w(x) + \theta_\varepsilon(x), \tag{1.3}$$

where $\alpha_N = (N(N-2))^{\frac{N-2}{4}}$ *and* $\theta_\varepsilon(x) \to 0$ *uniformly in* Ω *as* $\varepsilon \to 0$. *In particular, (1.1) has at least* $2^m - 1$ *different solutions.*

The proof actually will allow us to identify the points ξ_{ij}^ε as follows: let G denote Green's function for the Laplace operator with Dirichlet boundary condition on Ω and let H its regular part, then $\xi_{ij}^\varepsilon \to \xi_{ij}$ as $\varepsilon \to 0$, with (ξ_{i1}, ξ_{i2}) being a critical point of the functional

$$\Phi(x,y) = \frac{H(x,x)w^2(y) + 2G(x,y)w(x)w(y) + H(y,y)w^2(x)}{G^2(x,y) - H(x,x)H(y,y)}$$

defined on a suitable subset of

$$\{(x,y) \in \Omega^2 \cap \mathcal{A}_i^2 : G(x,y) - H^{\frac{1}{2}}(x,x)H^{\frac{1}{2}}(y,y) > 0\},$$

where $\mathcal{A}_i = \{x \in \mathbb{R}^N : \mu\rho_1^* < |x - P_i| < \mu\rho_2^*\}$, with $1 < \rho_1^* < \rho_2^*$ being explicit constants independent of μ and P_i. Also we will identify the limits λ_{ij} of $\lambda_{ij\varepsilon}$ as follows

$$\lambda_{ij} = \left(a_N^{-1} \frac{H(\xi_{ij}, \xi_{ij})w(\xi_{il}) + G(\xi_{ij}, \xi_{il})w(\xi_{ij})}{G^2(\xi_{ij}, \xi_{il}) - H(\xi_{ij}, \xi_{ij})H(\xi_{il}, \xi_{il})} \right)^{\frac{2}{N-2}},$$

for $j, l = 1, 2$; $j \neq l$ and $i = 1, 2, \ldots, k$, where a_N is an explicit constant. On the other hand, it will be clear from the proof that f not need to be strictly positive in the whole Ω, we will consider this case just for simplifying calculates.

The proof of Theorem 1.1 is based on a Lyapunov-Schmidt reduction procedure related to problem (1.1). In dealing with positive solution, Rey [27] use the *reduction method* in the critical case, which was more recently devised by del Pino *et al.* [14, 15] in the slightly supercritical case, with $f \equiv 0$. In dealing with solutions which are not necessarily positive, this procedure also already was used in the critical case, see [10, 22, 1]. Also see [18, 19, 28, 31] for some related works with the procedure in other contexts. In essence, here we extend the results of [1] but saving now the serious technical difficulties that arise at once of isolating the different pairs of spikes for avoiding undesirable interactions between points associated with different holes.

The influence of *small holes* in the domain on the appearance of positive solutions of problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

i.e. problem (1.1) with $f \equiv 0$, has been studied extensively in the literature. Coron [12] began these studies for $p = \frac{N+2}{N-2}$ finding via variational methods that (1.4) admits a positive solution under the assumption that Ω is a domain with a small hole. This result was extended notably by Bahri and Coron [5] to domains which possess a non-trivial topology. Rey [24] established existence of multiple solutions if Ω exhibits several small holes, while that in [14] were constructed multi-peak solutions in the slightly supercritical case. Recently, del Pino and Wei [16] have proved that (1.4) has at least one positive solution for any $p > \frac{N+2}{N-2}$, except for some strictly increasing unbounded sequence of values of p, supposed that $\Omega = \mathcal{D} \setminus B(P, \delta)$, for some $P \in \mathcal{D}$ and δ small enough. On the other hand, also have arisen recent results concerning to sign-changing solutions of (1.4). In [22], Musso and Pistoia consider $p = \frac{N+2}{N-2}$, $\Omega = \mathcal{D} \setminus B(0, \varepsilon)$ being symmetric respect to the origin for constructing sign-changing solutions with multiple blow up at the origin as $\varepsilon \to 0$, whereas that in [13], Dancer and Wei extended the result in [16] and showed that given any positive integer m, (1.4) has a sign-changing solution for any $p > \frac{N+2}{N-2}$, except for some strictly increasing unbounded sequence of values of p, which has exactly $m+1$ nodal domains.

This paper is arranged as follows. Sections $2-4$ are devoted to discuss the finite-dimensional reduction scheme used for the construction of solutions of (1.1), whereas in Section 5 the proof of Theorem 1.1 is finished by means of a min-max characterization which uses topological arguments.

2. Basic estimates

In this section, we assume that $\varepsilon > 0$ is small enough and that Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$, and let us consider the expanded domain

$$\Omega_\varepsilon = \varepsilon^{-\frac{2}{N-2}}\Omega.$$

Introducing the change of variable $v_\varepsilon(x') = -\varepsilon\, u(\varepsilon^{\frac{2}{N-2}} x')$, for $x' \in \Omega_\varepsilon$, we note that u solves (1.1) if and only if v_ε solves

$$\begin{cases} \Delta v + |v|^{p-1} v = \varepsilon^{p+1} \tilde{f}(x') & \text{in } \Omega_\varepsilon, \\ v = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{2.1}$$

where $p = \frac{N+2}{N-2}$ and $\tilde{f}(x') = f(\varepsilon^{\frac{2}{N-2}} x')$. Besides, it is known that

$$\bar{U}_{\lambda,\xi}(x) = \alpha_N \Big(\frac{\lambda}{\lambda^2 + |x-\xi|^2} \Big)^{\frac{N-2}{2}},$$

with $\lambda > 0$, $\xi \in {\rm I\!R}^N$ and $\alpha_N = (N(N-2))^{\frac{N-2}{4}}$, are the only positive solutions of equation $\Delta \vartheta + \vartheta^p = 0$ in ${\rm I\!R}^N$, see [3, 29, 7, 9]. Hence, if we consider the orthogonal projections onto $H_0^1(\Omega_\varepsilon)$ of the functions $\bar{U}_{\lambda,\xi'}$, which we denote from now on by $U_{\lambda,\xi'}$, and we put

$$V(x') = \sum_{i=1}^K U_{\lambda_i,\xi_i'}(x'), \quad x' \in \Omega_\varepsilon,$$

it turns out natural to look for solutions of (2.1) of the form

$$v(x') = V(x') + \tilde{\eta}(x'), \quad x' \in \Omega_\varepsilon,$$

which for suitable points ξ' and scalars λ will have the remainder term $\tilde{\eta}$ of small order all over Ω_ε. Since solutions of (2.1) correspond to stationary points of its associated energy functional J_ε defined by

$$J_\varepsilon(v) = \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla v|^2 - \frac{1}{p+1} \int_{\Omega_\varepsilon} |v|^{p+1} + \varepsilon^{p+1} \int_{\Omega_\varepsilon} \tilde{f} v, \tag{2.2}$$

our first goal is to estimate $J_\varepsilon(V)$.

Let us fix a small number $\delta > 0$ and relabel the parameters λ_i's into the Λ_i's given by

$$\Lambda_i = a_N \lambda_i^{\frac{N-2}{2}}, \quad i = 1, 2, \dots, K,$$

where $a_N = \int_{{\rm I\!R}^N} \bar{U}^p$, with $\bar{U} = \bar{U}_{1,0}$, and $\Lambda_i \in]\delta, \delta^{-1}[$. Arguing as in [20, 26, 6], we fix the set

$$\mathcal{M}_\delta = \{ (\vec{\xi}, \vec{\Lambda}) : |\xi_i - \xi_j| > \delta \text{ if } i \neq j, \text{ and } \operatorname{dist}(\xi_i, \partial\Omega) > \delta \}, \tag{2.3}$$

where $\vec{\xi} = (\xi_1, \xi_2, \dots, \xi_K) \in \Omega^K$ and $\vec{\Lambda} = (\Lambda_1, \Lambda_2, \dots, \Lambda_K) \in]\delta, \delta^{-1}[^K$. Then we obtain the next result

Proposition 2.1. *Given $\delta > 0$ small, the following expansion holds*

$$J_\varepsilon(V) = KC_N + \varepsilon^2\Psi(\vec{\xi},\vec{\Lambda}) + o(\varepsilon^2)$$

uniformly in the C^1-sense, with respect to $(\vec{\xi},\vec{\Lambda})$ in $\mathcal{M}_\delta$. Here

$$C_N = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\bar{U}|^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} \bar{U}^{p+1} \tag{2.4}$$

and the function Ψ is defined by

$$\Psi(\vec{\xi},\vec{\Lambda}) = \frac{1}{2}\sum_{i=1}^{K}\Lambda_i^2 H(\xi_i,\xi_i) - \sum_{i<j}\Lambda_i\Lambda_j G(\xi_i,\xi_j) + \sum_{i=1}^{K}\Lambda_i w(\xi_i), \tag{2.5}$$

where w is the unique solution to the problem (1.2).

Proof. The proof of this result is based in the arguments used to prove Lemma 3.2 of [15] and Proposition 1 of [10], so we only sketch it.

For notational simplicity we put $U_{\lambda_i,\xi_i'}(x') = U_i(x')$, for $x' \in \Omega_\varepsilon$, and obtain the following basic estimates which are essentially contained in [4, 6]:

$$\int_{\Omega_\varepsilon} |\nabla U_i|^2 = \int_{\mathbb{R}^N} |\nabla\bar{U}|^2 - \varepsilon^2\Lambda_i^2 H(\xi_i,\xi_i) + o(\varepsilon^2), \tag{2.6}$$

$$\int_{\Omega_\varepsilon} \nabla U_i \nabla U_j = \varepsilon^2\Lambda_i\Lambda_j G(\xi_i,\xi_j) + o(\varepsilon^2), \quad i \neq j, \tag{2.7}$$

$$\int_{\Omega_\varepsilon} \left(V^{p+1} - \sum_{l=1}^{K} U_l^{p+1}\right) = 2\varepsilon^2(p+1)\Lambda_i\Lambda_j G(\xi_i,\xi_j) + o(\varepsilon^2), \quad i \neq j, \tag{2.8}$$

and

$$\int_{\Omega_\varepsilon} U_i^{p+1} = \int_{\mathbb{R}^N} \bar{U}^{p+1} - \varepsilon^2(p+1)\Lambda_i^2 H(\xi_i,\xi_i) + o(\varepsilon^2). \tag{2.9}$$

On the other hand, away from $x' = \xi_i'$, we have that

$$U_i(x') = \varepsilon^2\Lambda_i G(\varepsilon^{\frac{2}{N-2}}x',\xi_i) + o(\varepsilon^2),$$

uniformly on each compact subset of Ω_ε, then straightforward calculates lead to

$$\varepsilon^{p+1}\int_{\Omega_\varepsilon} V\tilde{f} = \varepsilon^2\sum_{i=1}^{K}\Lambda_i w(\xi_i) + o(\varepsilon^2). \tag{2.10}$$

In all previous estimates the quantity $o(\varepsilon^2)$ is actually of this size in the C^1-norm as function of $(\vec{\xi}, \vec{\Lambda}) \in \mathcal{M}_\delta$. Hence, since

$$
\begin{aligned}
J_\varepsilon(V) &= \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla V|^2 - \frac{1}{p+1}\int_{\Omega_\varepsilon} V^{p+1} + \varepsilon^{p+1}\int_{\Omega_\varepsilon} V\tilde{f} \\
&= \frac{1}{2}\sum_{i=1}^{K}\int_{\Omega_\varepsilon} |\nabla U_i|^2 + \sum_{i<j}\int_{\Omega_\varepsilon} \nabla U_i \nabla U_j + \varepsilon^{p+1}\int_{\Omega_\varepsilon} V\tilde{f} \\
&\quad -\frac{1}{p+1}\int_{\Omega_\varepsilon}\Big(\sum_{i=1}^{K} U_i^{p+1}\Big) - \frac{1}{p+1}\int_{\Omega_\varepsilon}\Big(V^{p+1} - \sum_{i=1}^{K} U_i^{p+1}\Big),
\end{aligned}
$$

the result follows from estimates (2.6)–(2.10). □

3. The finite-dimensional reduction

Let us fix a small number $\delta > 0$ and consider points $(\vec{\xi'}, \vec{\Lambda})$ in

$$
\mathcal{M}_\delta^\varepsilon = \{(\vec{\xi'}, \vec{\Lambda}) : |\xi_i' - \xi_j'| > \delta_\varepsilon \text{ if } i \neq j, \text{ and } \operatorname{dist}(\xi_i', \partial\Omega_\varepsilon) > \delta_\varepsilon\}, \tag{3.1}
$$

where $\vec{\xi'} = (\xi_1', \xi_2', \dots, \xi_K') \in \Omega_\varepsilon^K$, $\vec{\Lambda} = (\Lambda_1, \Lambda_2, \dots, \Lambda_K) \in]\delta, \delta^{-1}[^K$ and $\delta_\varepsilon = \delta\varepsilon^{-\frac{2}{N-2}}$. Since solutions of problem $\Delta\vartheta + p\bar{U}_{\Lambda,0}^{p-1}\vartheta = 0$ in $\mathbb{R}^N$ satisfying $|\vartheta(x)| < C|x|^{2-N}$ belong to $\operatorname{span}\{\frac{\partial \bar{U}_{\Lambda,0}}{\partial x_l}, \frac{\partial \bar{U}_{\Lambda,0}}{\partial \Lambda}\}_{l=1,\dots,N}$, see [25], it is convenient to consider, for each $i = 1, 2, \dots, K$, the following functions:

$$
\bar{Z}_{il}(x') = \frac{\partial \bar{U}_i}{\partial \xi_{il}'}(x'),\ l = 1, \dots, N \quad \text{and} \quad \bar{Z}_{i(N+1)}(x') = \frac{\partial \bar{U}_i}{\partial \Lambda_i}(x'),
$$

and their respective $H_0^1(\Omega_\varepsilon)$-projections Z_{il}. Also, for functions u, v defined in Ω_ε we put $\langle u, v\rangle = \int_{\Omega_\varepsilon} uv$, and consider the next problem: find a function $\tilde{\eta}$ such that

$$
\left\{
\begin{aligned}
\Delta(V + \tilde{\eta}) + |V + \tilde{\eta}|^{p-1}(V + \tilde{\eta}) - \varepsilon^{p+1}\tilde{f} &= \sum_{i,l} c_{il} U_i^{p-1} Z_{il} \quad \text{in } \Omega_\varepsilon, \\
\tilde{\eta} &= 0 \quad \text{on } \partial\Omega_\varepsilon, \\
\langle \tilde{\eta}, U_i^{p-1} Z_{il}\rangle &= -\langle \phi, U_i^{p-1} Z_{il}\rangle \quad \forall i, l,
\end{aligned}
\right. \tag{3.2}
$$

for certain constants c_{il}, $i = 1, 2, \dots, K$; $l = 1, \dots, N+1$, where ϕ solves

$$
\left\{
\begin{aligned}
-\Delta\phi &= \varepsilon^{p+1}\tilde{f} \quad \text{in } \Omega_\varepsilon, \\
\phi &= 0 \quad \text{on } \partial\Omega_\varepsilon.
\end{aligned}
\right. \tag{3.3}
$$

Note that $V+\tilde{\eta}$ is a solution of (2.1) if the scalars c_{il} in (3.2) are all zero. Also, we note that the partial differential equation in (3.2) is equivalent to

$$\Delta\eta + p|V|^{p-1}\eta = -N_\varepsilon(\eta) - R_\varepsilon + \sum_{i,l} c_{il}U_i^{p-1}Z_{il} \quad \text{in } \Omega_\varepsilon,$$

where $\eta = \tilde{\eta} + \phi$,

$$N_\varepsilon(\eta) = |V+\eta-\phi|^{p-1}(V+\eta-\phi)_+ - |V|^{p-1}V - p|V|^{p-1}(\eta-\phi) \tag{3.4}$$

and

$$R_\varepsilon = |V|^{p-1}V - \sum_{i=1}^{K}\bar{U}_i^p - p|V|^{p-1}\phi. \tag{3.5}$$

A first step to solve (3.2) consists of dealing with the following problem: given $h \in L^\infty(\overline{\Omega}_\varepsilon)$, find a function η and constants c_{ij} such that

$$\begin{cases} \Delta\eta + p|V|^{p-1}\eta = h + \sum_{i,l} c_{il}U_i^{p-1}Z_{il} & \text{in } \Omega_\varepsilon, \\ \eta = 0 & \text{on } \partial\Omega_\varepsilon, \\ \langle \eta, U_i^{p-1}Z_{il}\rangle = 0 & \forall i,l. \end{cases} \tag{3.6}$$

Hence, we study the linear operator L_ε associated to (3.6), namely

$$L_\varepsilon(\eta) = \Delta\eta + p|V|^{p-1}\eta,$$

under the previous orthogonality conditions introducing suitable L^∞-norms with weight: for a function θ defined in Ω_ε, we consider the norms

$$\|\theta\|_* = \left\| \left(\sum_{i=1}^K \omega_i\right)^{-\sigma} \theta(x') \right\|_\infty + \left\| \left(\sum_{i=1}^K \omega_i\right)^{-\sigma-1} \nabla\theta(x') \right\|_\infty,$$

where $\omega_i = \left(1+|x'-\xi_i'|^2\right)^{-\frac{N-2}{2}}$, $\sigma = \frac{1}{2}$ if $3 \le N \le 6$, $\sigma = \frac{2}{N-2}$ if $N \ge 7$, and

$$\|\theta\|_{**} = \left\| \left(\sum_{i=1}^K \omega_i\right)^{-\varsigma} \theta(x') \right\|_\infty,$$

where $\varsigma = \frac{p}{2}$ if $3 \le N \le 6$ and $\varsigma = \frac{4}{N-2}$ if $N \ge 7$. Then, we have

Proposition 3.1. *Assume that $(\vec{\xi'}, \vec{\Lambda}) \in \mathcal{M}_\delta^\varepsilon$. Then there exist $\varepsilon_0 > 0$ and $C > 0$, such that for all $0 < \varepsilon < \varepsilon_0$ and for all $h \in C^\alpha(\overline{\Omega}_\varepsilon)$, the problem (3.6) admits an* *unique solution* $\eta \equiv M_\varepsilon(h)$. *Moreover, the map* $(\vec{\xi'}, \vec{\Lambda}, h) \mapsto \eta \equiv M_\varepsilon(h)$ *is of class* C^1 *and satisfies*

$$\|\eta\|_* \leq C\|h\|_{**} \quad \textit{and} \quad \|\nabla_{(\vec{\xi'},\vec{\Lambda})}\,\eta\|_* \leq C\|h\|_{**}.$$

The proof of this result is a slight variation of the arguments used to prove Propositions 4.1 and 4.2 in [15], so we omit it. In what follows, $C > 0$ represents a generic constant which is independent of ε and of the particular points $(\vec{\xi'}, \vec{\Lambda}) \in \mathcal{M}^\varepsilon_\delta$.

A second step to solve (3.2) consists in finding a function φ such that for certain constants c_{il}, $i = 1, 2, \ldots, K$; $l = 1, \ldots, N+1$, solves

$$\begin{cases} \Delta(V+\tilde{\eta}) + |V+\tilde{\eta}|^{p-1}(V+\tilde{\eta})_+ - \varepsilon^{p+1}\tilde{f} = \displaystyle\sum_{i,l} c_{il}U_i^{p-1}Z_{il} & \text{in } \Omega_\varepsilon, \\ \varphi = 0 & \text{on } \partial\Omega_\varepsilon, \\ \langle \varphi, U_i^{p-1}Z_{il}\rangle = 0 & \forall i, l, \end{cases} \tag{3.7}$$

where $\tilde{\eta} = \psi + \varphi - \phi$, with ϕ satisfying (3.3), and the function ψ is chosen as

$$\psi = -M_\varepsilon(R_\varepsilon), \tag{3.8}$$

where M_ε is defined as in Proposition 3.1 and R_ε is given by (3.5).

Lemma 3.2. *Let* ψ *be as in (3.8). Then*

$$\|\psi\|_* \leq C\varepsilon^2.$$

Proof. Bearing in mind that: $\sum_{i=1}^K \big(\bar{U}_i(x') - U_i(x')\big) = C\varepsilon^2 + o(\varepsilon^2)$, for $(\vec{\xi'}, \vec{\Lambda}) \in \mathcal{M}^\varepsilon_\delta$, we obtain

$$\left|\left(\sum_{i=1}^K \omega_i\right)^{-\sigma}\left(|V|^{p-1}V - \sum_{i=1}^K \bar{U}_i^p\right)\right| \leq C\varepsilon^2,$$

where $\sigma = \frac{p}{2}$ if $3 \leq N \leq 6$ and $\sigma = \frac{2}{N-2}$ if $N \geq 7$. On the complement of those regions, $|\,|V|^{p-1}V - \sum_{i=1}^K \bar{U}_i^p| \leq C\varepsilon^{2p}$, hence

$$\left\|\,|V|^{p-1}V - \sum_{i=1}^K \bar{U}_i^p\right\|_{**} \leq C\varepsilon^2. \tag{3.9}$$

Now, from definition of ϕ in (3.3), we have that $\|\phi\|_\infty = O(\varepsilon^{p+1})$. Therefore

$$\|\,|V|^{p-1}\phi\|_{**} \leq C\varepsilon^2. \tag{3.10}$$

So the result follows from (3.9), (3.10) and the Proposition 3.1. $\square$

Lemma 3.3. *There exists $C > 0$ such that for all $\varepsilon > 0$ small enough and $\|\varphi\|_* \leq \frac{1}{4}$ one has*

$$\|N_\varepsilon(\psi+\varphi)\|_{**} \leq \begin{cases} C\big(\|\varphi\|_*^2 + \varepsilon\|\varphi\|_* + \varepsilon^{p+1}\big) & \text{if } 3 \leq N \leq 6, \\ C\big(\varepsilon^{2(p-2)}\|\varphi\|_*^2 + \varepsilon^{p^2-3p+2}\|\varphi\|_*^p + \varepsilon^{p^2-p+2}\big) & \text{if } N \geq 7. \end{cases}$$

Proof. Note that $\|\phi\|_* \leq C\varepsilon^p$ if $3 \leq N \leq 6$, $\|\phi\|_* \leq C\varepsilon^2$ if $N \geq 7$. Then considering $\eta = \psi + \varphi$ we have that $\|\eta\|_* < 1$. Also we note that from (3.4) one has

$$N_\varepsilon(\eta) = C|V + \bar{t}(\eta-\phi)|^{p-2}(\eta-\phi)^2, \quad \bar{t} \in]0,1[. \tag{3.11}$$

Hence, for $3 \leq N \leq 6$, it is easy to check that $\|N_\varepsilon(\eta)\|_* \leq C\|\eta-\phi\|_*^2$. On the other hand, for $N \geq 7$, if $|\eta| \leq \frac{1}{2}\big(\sum_{i=1}^K \omega_i\big)$ we use again (3.11) and we obtain

$$\left|\left(\sum_{i=1}^K \omega_i\right)^{-\frac{4}{N-2}} N_\varepsilon(\eta)\right| \leq C\varepsilon^{\frac{6-N}{N-2}}\|\eta-\phi\|_*^2.$$

In another case we obtain directly from (3.4) that

$$\left|\left(\sum_{i=1}^K \omega_i\right)^{-\frac{4}{N-2}} N_\varepsilon(\eta)\right| \leq C\varepsilon^{\frac{6-N}{N-2}\cdot\frac{2}{N-2}}\|\eta-\phi\|_*^p.$$

Combining previous estimates the result follows. $\square$

Now, we deal with the following problem

$$\begin{cases} \Delta\varphi + pV^{p-1}\varphi = -N_\varepsilon(\eta) + \sum_{i,l} c_{il}U_i^{p-1}Z_{il} & \text{in } \Omega_\varepsilon, \\ \varphi = 0 & \text{on } \partial\Omega_\varepsilon, \\ \langle \varphi, U_i^{p-1}Z_{il}\rangle = 0 & \forall i, l, \end{cases} \tag{3.12}$$

where $\eta = \psi + \varphi$ and ψ is the function defined in (3.8).

Proposition 3.4. *Assume that $(\vec{\xi'}, \vec{\Lambda}) \in \mathcal{M}_\delta^\varepsilon$. Then there exists $C > 0$, such that for all $\varepsilon > 0$ small enough there exists an unique solution $\varphi = \varphi(\vec{\xi'}, \vec{\Lambda})$ to problem (3.12). Moreover, the map $(\vec{\xi'}, \vec{\Lambda}) \mapsto \varphi(\vec{\xi'}, \vec{\Lambda})$ is of class C^1 for the $\|\cdot\|_*$-norm and it satisfies*

$$||\varphi||_* \leq C\varepsilon^2 \quad \textit{and} \quad \|\nabla_{(\vec{\xi'},\vec{\Lambda})}\varphi\|_* \leq C\varepsilon^2.$$

Proof. We argue in a similar way as in the proof of Proposition 3.3 in [1] or of Proposition 3 in [10]. Thus, here we only give the main ideas of the proof.

Let us consider the function

$$\begin{aligned} A_\varepsilon :\ \ \mathcal{F}_r &\to H_0^1(\Omega_\varepsilon) \\ \varphi &\to A_\varepsilon(\varphi) = -M_\varepsilon\left(N_\varepsilon(\psi+\varphi)\right), \end{aligned}$$

where $\mathcal{F}_r = \{\varphi \in H_0^1(\Omega_\varepsilon) : \|\varphi\|_* \le r\varepsilon^2\}$, M_ε is the operator defined in Proposition 3.1 and $\psi = -M_\varepsilon(R_\varepsilon)$. For a suitable $r = r(N) > 0$ and using the previous lemmas one shows that A_ε is a contraction, therefore there is a fixed point in $\mathcal{F}_r$ for A_ε, noting that this is equivalent to solving (3.12).

Concerning differentiability properties, we have the following relation

$$B(\vec{\xi'}, \vec{\Lambda}, \varphi) \equiv \varphi + M_\varepsilon\big(N_\varepsilon(\psi+\varphi)\big) = 0.$$

We see that

$$D_\varphi B(\vec{\xi'}, \vec{\Lambda}, \varphi)[\theta] = \theta + M_\varepsilon\big(\theta\, D_\varphi N_\varepsilon(\psi+\varphi)\big) \equiv \theta + \tilde{M}(\theta),$$

and check

$$\|\tilde{M}(\theta)\|_* \le C\varepsilon\|\theta\|_*.$$

This implies that for ε small, the linear operator $D_\varphi B(\vec{\xi'}, \vec{\Lambda}, \varphi)$ is invertible in the space of the continuous functions in Ω_ε with bounded $\|\cdot\|_*$-norm, with uniformly bounded inverse depending continuously on its parameters. Then, applying the implicit function theorem we obtain that $\varphi(\vec{\xi'}, \vec{\Lambda})$ is a C^1-function in L^∞_*. Besides, we get

$$\frac{\partial}{\partial_{\xi'_{il}}}\varphi = -\big(D_\varphi B(\vec{\xi'}, \vec{\Lambda}, \varphi)\big)^{-1}\Big(\frac{\partial}{\partial_{\xi'_{il}}} B(\vec{\xi'}, \vec{\Lambda}, \varphi)\Big),$$

and using the first part of this proposition, the estimates in the previous lemmas, Proposition 3.1 and the fact that $(\vec{\xi}, \vec{\Lambda}) \in \mathcal{M}_\delta^\varepsilon$, we conclude

$$\Big\|\frac{\partial}{\partial_{\xi'_{il}}}\varphi\Big\|_* \le C\varepsilon^2.$$

Similarly, the differentiability of B with respect to $\vec{\Lambda}$ is analyzed. □

4. The reduced functional

Let $(\vec{\xi'}, \vec{\Lambda})$ be in $\mathcal{M}_\delta^\varepsilon$ as in (3.1) and let us consider the function φ given by Proposition 3.4, which is the only one solution of the problem (3.7) with $\tilde{\eta} = \psi + \varphi(\vec{\xi'}, \vec{\Lambda}) - \phi$, where ψ solves (3.8) and ϕ solves (3.3). Note that if $c_{il} = 0$ for all i, l, then a solution of (1.1) is

$$u(x) = -\varepsilon^{-1} v(\varepsilon^{-\frac{2}{N-2}} x), \qquad x \in \Omega,$$

where $v = V + \psi + \varphi(\vec{\xi'}, \vec{\Lambda}) - \phi$. Hence, u will be a critical point of

$$I_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{p+1} \int_\Omega |u|^{p+1} - \varepsilon \int_\Omega f u,$$

while v will be one of J_ε given by (2.2). Then it is convenient to consider the following functions defined in Ω:

$$\hat{U}_i(x) = -\varepsilon^{-1} U_i(\varepsilon^{-\frac{2}{N-2}} x) = -U_{\lambda_{i\varepsilon}, \xi_i}(x), \qquad \hat{\psi}(x) = -\varepsilon^{-1} \psi(\varepsilon^{-\frac{2}{N-2}} x),$$

$$\hat{\varphi}(\vec{\xi}, \vec{\Lambda})(x) = -\varepsilon^{-1} \varphi(\vec{\xi'}, \vec{\Lambda})(\varepsilon^{-\frac{2}{N-2}} x) \qquad \text{and } \hat{\phi}(x) = -\varepsilon^{-1} \phi(\varepsilon^{-\frac{2}{N-2}} x) = -\varepsilon w(x).$$

Note that $\hat{U}_i = -U_{\lambda_{i\varepsilon}, \xi_i}$, where $\lambda_{i\varepsilon} = (a_N^{-1} \Lambda_i \varepsilon)^{\frac{2}{N-2}} \in \mathbb{R}_+$ and $\vec{\xi} = \varepsilon^{\frac{2}{N-2}} \vec{\xi'}$, with $(\vec{\xi}, \vec{\Lambda}) \in \mathcal{M}_\delta$ defined by (2.3), and $\hat{\phi} = -\varepsilon w$, being w the solution of (1.2). Now, we put $\hat{U} = -\sum_{i=1}^K U_{\lambda_{i\varepsilon}, \xi_i}$ and consider the functional

$$\mathcal{I}(\vec{\xi}, \vec{\Lambda}) \equiv I_\varepsilon\big(\hat{U} + \hat{\psi} + \hat{\varphi}(\vec{\xi}, \vec{\Lambda}) - \hat{\phi}\big). \tag{4.1}$$

Then, we have the following basic result

Lemma 4.1. *The function $u = -\sum_{i=1}^K U_{\lambda_{i\varepsilon}, \xi_i} + \varepsilon w + \hat{\psi} + \hat{\varphi}(\vec{\xi}, \vec{\Lambda})$ is a solution of problem (1.1) if and only if $(\vec{\xi}, \vec{\Lambda})$ is a critical point of functional $\mathcal{I}$ given by (4.1).*

Proof. It is easy to check that $\mathcal{I}(\vec{\xi}, \vec{\Lambda}) = J_\varepsilon\big(V + \psi + \varphi(\vec{\xi'}, \vec{\Lambda}) - \phi\big)$. Also, putting $\tilde{\eta} = \psi + \varphi(\vec{\xi'}, \vec{\Lambda}) - \phi$, it is not difficult to show that $DJ_\varepsilon(V + \tilde{\eta})\,[\vartheta] = 0$ for all $\vartheta \in \{\vartheta \in H_0^1(\Omega_\varepsilon) : \langle \vartheta, V_i^{p-1} Z_{il} \rangle = 0 \quad \forall i, l\}$. On the other hand,

$$\frac{\partial V}{\partial \xi'_{il}} = Z_{il} + o(1) \quad \forall i, l; \qquad \frac{\partial V}{\partial \Lambda_i} = Z_{i(N+1)} + o(1) \quad \forall i,$$

with $o(1) \to 0$ in the $\|\cdot\|_*$-norm as $\varepsilon \to 0$. Then the result follows from Proposition 3.4. $\square$

Next step is then to give an asymptotic estimate for $\mathcal{I}(\vec{\xi},\vec{\Lambda})$. Put

$$\sigma_f = \int_\Omega f(x)w(x)\,dx, \tag{4.2}$$

where w is the solution of (1.2). Then

Proposition 4.2. *The following expansions hold:*

$$\mathcal{I}(\vec{\xi},\vec{\Lambda}) = KC_N + \varepsilon^2\big(\Psi(\vec{\xi},\vec{\Lambda}) + \sigma_f\big) + o(\varepsilon^2)\theta(\vec{\xi},\vec{\Lambda}) \tag{4.3}$$

and

$$\nabla_{(\vec{\xi},\vec{\Lambda})}\mathcal{I}(\vec{\xi},\vec{\Lambda}) = \varepsilon^2\nabla_{(\vec{\xi},\vec{\Lambda})}\Psi(\vec{\xi},\vec{\Lambda}) + o(\varepsilon^2)\nabla_{(\vec{\xi},\vec{\Lambda})}\theta(\vec{\xi},\vec{\Lambda}), \tag{4.4}$$

uniformly with respect to $(\vec{\xi},\vec{\Lambda}) \in \mathcal{M}_\delta$, where θ and $\nabla_{(\vec{\xi},\vec{\Lambda})}\theta$ are uniformly bounded functions, independently of all $\varepsilon > 0$ small. Here C_N is the constant given by (2.4) and Ψ is the function given by (2.5).

Proof. The first step to achieve our goal is to prove that

$$\mathcal{I}(\vec{\xi},\vec{\Lambda}) - I_\varepsilon(\hat{V} + \hat{\psi} - \hat{\phi}) = o(\varepsilon^2) \tag{4.5}$$

and

$$\nabla_{\vec{\xi}}\big(\mathcal{I}(\vec{\xi},\vec{\Lambda}) - I_\varepsilon(\hat{V} + \hat{\psi} - \hat{\phi})\big) = o(\varepsilon^2). \tag{4.6}$$

Let us set $\vartheta = V + \psi - \phi$ and notice that

$$\begin{aligned}\mathcal{I}(\vec{\xi},\vec{\Lambda}) - I_\varepsilon(\hat{V} + \hat{\psi} - \hat{\phi}) \;=\; & -\int_0^1 t\Big(\int_{\Omega_\varepsilon} N_\varepsilon(\psi+\varphi)\varphi\Big)\,dt \\ & + \int_0^1 t\Big(\int_{\Omega_\varepsilon} p\big(|V|^{p-1} - |\vartheta + t\varphi|^{p-1}\big)\varphi^2\Big)\,dt.\end{aligned}$$

Differentiating with respect to $\vec{\xi}$ variables we obtain

$$\begin{aligned}D_{\vec{\xi}}\big(\mathcal{I}(\vec{\xi},\vec{\Lambda}) - I_\varepsilon(\hat{\vartheta})\big) \;=\; & -\varepsilon^{-\frac{2}{N-2}}\int_0^1 t\int_{\Omega_\varepsilon} p\nabla_{\vec{\xi}'}\big[|\vartheta + t\varphi|^{p-1}\varphi^2 - |V|^{p-1}\varphi^2\big]\,dt \\ & -\varepsilon^{-\frac{2}{N-2}}\int_{\Omega_\varepsilon}\nabla_{\vec{\xi}'}\big(N_\varepsilon(\psi+\varphi)\varphi\big),\end{aligned}$$

and bearing in mind that $\|\varphi\|_* + \|\psi\|_* + \|\nabla_{\xi_i'}\varphi\|_* + \|\nabla_{\xi_i'}\psi\|_* \le O(\varepsilon^2)$, we get

$$\mathcal{I}(\vec{\xi},\vec{\Lambda}) - I_\varepsilon(\hat{V} + \hat{\psi} - \hat{\phi}) = \begin{cases} O(\varepsilon^4|\log\varepsilon|) & \text{if } 3 \le N \le 6 \text{ or } N = 8, \\ O(\varepsilon^4) & \text{if } N = 7, \\ O(\varepsilon^{2+\frac{6}{N-2}}) & \text{if } N \ge 9, \end{cases}$$

and

$$D_{\vec{\xi}}\big(\mathcal{I}(\vec{\xi},\vec{\Lambda})-I_\varepsilon(\hat{\vartheta})\big)=\begin{cases}O(\varepsilon^4) & \text{if } 3\le N\le 7,\\ O(\varepsilon^4|\log\varepsilon|) & \text{if } N=8,\\ O(\varepsilon^{2+\frac{6}{N-2}}) & \text{if } N\ge 9.\end{cases}$$

Therefore (4.5) and (4.6) hold.

The next step is to prove that

$$I_\varepsilon(\hat{V}+\hat{\psi}-\hat{\phi})-I_\varepsilon(\hat{V}-\hat{\phi})=o(\varepsilon^2) \tag{4.7}$$

and

$$\nabla_{\vec{\xi}}\big(I_\varepsilon(\hat{V}+\hat{\psi}-\hat{\phi})-I_\varepsilon(\hat{V}-\hat{\phi})\big)=o(\varepsilon^2). \tag{4.8}$$

Put $\eta=V-\phi$ and, by the fundamental calculus theorem, note that

$$\begin{aligned}I_\varepsilon(\hat{\eta}+\hat{\psi})-I_\varepsilon(\hat{\eta})= &\int_0^1(1-t)\Big(\int_{\Omega_\varepsilon}p|\eta+t\psi|^{p-1}\psi^2-\int_{\Omega_\varepsilon}|\nabla\psi|^2\Big)\,dt\\ &+\int_{\Omega_\varepsilon}\big(|V|^p-|\eta|^p-p|V|^{p-1}\phi\big)\psi+\int_{\Omega_\varepsilon}R_\varepsilon\psi.\end{aligned}$$

Now, differentiating with respect to $\vec{\xi}$ variables we get

$$\begin{aligned}D_{\vec{\xi}}\big(I_\varepsilon(\hat{\eta}+\hat{\psi})-I_\varepsilon(\hat{\eta})\big)= &\ \varepsilon^{-\frac{2}{N-2}}\int_0^1(1-t)\int_{\Omega_\varepsilon}\nabla_{\vec{\xi}'}\big(p|\eta+t\psi|^{p-1}\psi^2-|\nabla\psi|^2\big)\,dt\\ &+\varepsilon^{-\frac{2}{N-2}}\int_{\Omega_\varepsilon}\nabla_{\vec{\xi}'}\big(|V|^p-|\eta|^p-p|V|^{p-1}\phi\big)\psi\\ &+\varepsilon^{-\frac{2}{N-2}}\int_{\Omega_\varepsilon}\big(|V|^p-|\eta|^p-p|V|^{p-1}\phi\big)\nabla_{\vec{\xi}'}\psi\\ &+\varepsilon^{-\frac{2}{N-2}}\int_{\Omega_\varepsilon}\nabla_{\vec{\xi}'}R_\varepsilon\psi+\varepsilon^{-\frac{2}{N-2}}\int_{\Omega_\varepsilon}R_\varepsilon\nabla_{\vec{\xi}'}\psi.\end{aligned}$$

Since $\|R_\varepsilon\|_{**}+\|\nabla_{\xi_i'}R_\varepsilon\|_{**}+\|\psi\|_*+\|\nabla_{\xi_i'}\psi\|_*\le O(\varepsilon^2)$, $\|\phi\|_\infty\le O(\varepsilon^2)$ and $\|\phi\|_*\le O(\varepsilon^p)$ if $3\le N\le 6$, $\|\phi\|_*\le O(\varepsilon^2)$ if $N\ge 7$, one has that

$$I_\varepsilon(\hat{\eta}+\hat{\psi})-I_\varepsilon(\hat{\eta})=\begin{cases}O(\varepsilon^4|\log\varepsilon|) & \text{if } 3\le N\le 6 \text{ or } N=8,\\ O(\varepsilon^{4+\frac{8}{N-2}}) & \text{if } N=7,\\ O(\varepsilon^{2+\frac{6}{N-2}}) & \text{if } N\ge 9,\end{cases}$$

and

$$D_{\vec{\xi}}\big(\mathcal{I}(\vec{\xi},\vec{\Lambda})-I_\varepsilon(\hat{\vartheta})\big)=\begin{cases}O(\varepsilon^4) & \text{if } 3\le N\le 5,\\ O(\varepsilon^4|\log\varepsilon|) & \text{if } N=6 \text{ or } N=8,\\ O(\varepsilon^{4+\frac{8}{N-2}}) & \text{if } N=7,\\ O(\varepsilon^{2+\frac{6}{N-2}}) & \text{if } N\ge 9.\end{cases}$$

It follows that (4.7) and (4.8) yield.

Now we hold the following two estimates

$$I_\varepsilon(\hat{V}-\hat{\phi})-I_\varepsilon(\hat{V})=\varepsilon^2\sigma_f+o(\varepsilon^2), \tag{4.9}$$

where σ_f is given by (4.2), and

$$D_{\vec{\xi}}\big(I_\varepsilon(\hat{V}-\hat{\phi})-I_\varepsilon(\hat{V})\big)=o(\varepsilon^2). \tag{4.10}$$

Note that

$$\begin{aligned}I_\varepsilon(\hat{V}-\hat{\phi})-I_\varepsilon(\hat{V}) &=\int_0^1\Big(\int_{\Omega_\varepsilon}|\nabla\phi|^2-\int_{\Omega_\varepsilon}p|V-t\phi|^{p-1}\phi^2\Big)dt\\ &\quad+\int_{\Omega_\varepsilon}\Big(\sum_{i=1}^K\bar{U}_i^p-|V-t\phi|^p\Big)\phi.\end{aligned}$$

Besides, from (4.2) one has that

$$\int_0^1 t\int_{\Omega_\varepsilon}|\nabla\phi|^2\,dt=\int_{\Omega_\varepsilon}|\nabla\phi|^2=\varepsilon^{p+1}\int_{\Omega_\varepsilon}\tilde{f}\phi=\varepsilon^2\int_\Omega fw=\varepsilon^2\sigma_f,$$

and since $\|\phi\|_\infty\le O(\varepsilon^{p+1})$, we get

$$\Big|\int_{\Omega_\varepsilon}p|V-t\phi|^{p-1}\phi^2\Big|\le C\varepsilon^4\int_{\Omega_\varepsilon}(\omega_1+\omega_2)^{p-1}\le o(\varepsilon^2).$$

On the other hand, it is not difficult to check that

$$\Big|\int_{\Omega_\varepsilon}\Big(\sum_{i=1}^K\bar{U}_i^p-|V-t\phi|^p\Big)\phi\Big|=\Big|\int_{\Omega_\varepsilon}R_\varepsilon\phi+\int_{\Omega_\varepsilon}\big(|V|^p-|V-t\phi|^p-p|V|^{p-1}\phi\big)\phi\Big|\le o(\varepsilon^2).$$

Therefore (4.9) holds. Also, note that

$$\begin{aligned}D_{\vec{\xi}}\big(I_\varepsilon(\hat{V}-\hat{\phi})-I_\varepsilon(\hat{V})\big) &=\varepsilon^{-\frac{2}{N-2}}\int_0^1 t\int_{\Omega_\varepsilon}p|V-t\phi|^{p-2}\nabla_{\vec{\xi}'}V\phi^2\,dt\\ &\quad+\varepsilon^{-\frac{2}{N-2}}\int_{\Omega_\varepsilon}\nabla_{\vec{\xi}'}\Big(\sum_{i=1}^K\bar{U}_i^p-|V-t\phi|^p\Big)\phi,\end{aligned}$$

and since $\|\phi\|_\infty \leq O(\varepsilon^{p+1})$, it is easy to check that

$$D_{\vec{\xi}}\big(I_\varepsilon(\hat{V}-\hat{\phi})-I_\varepsilon(\hat{V})\big) = \begin{cases} O(\varepsilon^4) & \text{if } 3\leq N\leq 7,\\ O(\varepsilon^4|\log\varepsilon|) & \text{if } N=8,\\ O(\varepsilon^{2+\frac{6}{N-2}}) & \text{if } N\geq 9.\end{cases}$$

Therefore (4.10) is truth. Similarly we hold results for the differentiability with respect to $\vec{\Lambda}$. □

5. The min-max

Let $\mathcal{D}$ be a bounded domain with smooth boundary in $\mathbb{R}^N$, $N\geq 3$, and $P_1, P_2, \ldots, P_m$ points of $\mathcal{D}$. Let us consider now smooth domains $\mathfrak{D}_i$ such that $\mathfrak{D}_i \subset \overline{B(P_i,\mu)} \subset \mathcal{D}$ and the domain

$$\Omega = \mathcal{D}\setminus\bigcup_{i=1}^m \mathfrak{D}_i.$$

We denote by G and H respectively its Green's function and regular part, and fix $1\leq k\leq m$. According to the results obtained in the previous section, see Lemma 4.1, (4.3) and (4.4), our problem reduces to that of finding a critical point for

$$\Psi(\vec{\xi},\vec{\Lambda}) = \sum_{i=1}^k \psi(\vec{\xi^i},\vec{\Lambda}^i) - R(\vec{\xi},\vec{\Lambda}), \tag{5.1}$$

where $\vec{\xi^i} = (\xi_{i1},\xi_{i2})\in\Omega^2$, $\vec{\Lambda}^i = (\Lambda_{i1},\Lambda_{i2})\in\mathbb{R}^2_+$, $\vec{\xi} = (\vec{\xi^1},\vec{\xi^2},\ldots,\vec{\xi^k})\in\Omega^{2k}$, $\vec{\Lambda} = (\vec{\Lambda}^1,\vec{\Lambda}^2,\ldots,\vec{\Lambda}^k)\in\mathbb{R}^{2k}_+$,

$$\psi(\vec{\xi^i},\vec{\Lambda}^i) = \frac{1}{2}\sum_{j=1}^2 \Lambda_{ij}^2 H(\xi_{ij},\xi_{ij}) - \Lambda_{i1}\Lambda_{i2}G(\xi_{i1},\xi_{i2}) + \sum_{j=1}^2 \Lambda_{ij}w(\xi_{ij}) \tag{5.2}$$

and

$$R(\vec{\xi},\vec{\Lambda}) = \sum_{i<l}\sum_{1\leq j_1,j_2\leq 2}\Lambda_{ij_1}\Lambda_{lj_2}G(\xi_{ij_1},\xi_{lj_2}).$$

It is convenient to recall that the function ψ is well defined in $(\Omega^2\setminus\triangle)\times\mathbb{R}^2_+$, where $\triangle = \{(x,y)\in\Omega^2 : x=y\}$. Hence, in order to avoid the singularity of ψ over $\triangle$, we consider $M>0$ and define

$$G_M(x,y) = \begin{cases} G(x,y) & \text{if } G(x,y)\leq M,\\ M & \text{if } G(x,y)>M.\end{cases} \tag{5.3}$$

Now, we work with the functional modified $\Psi_{M,\rho} : \Omega_\rho^{2k} \times \mathbb{R}_+^{2k} \to \mathbb{R}$ defined by

$$\Psi_{M,\rho}(\vec{\xi},\vec{\Lambda}) = \Psi(\vec{\xi},\vec{\Lambda}) - \sum_{i=1}^{k} \Lambda_{i1}\Lambda_{i2}\big(\, G_M(\xi_{i1},\xi_{i2}) - G(\xi_{i1},\xi_{i2})\,\big),$$

where $\rho > 0$ and $\Omega_\rho = \{\xi \in \Omega : \mathrm{dist}(\xi, \partial\Omega) > \rho\}$ with ρ and M to be specified later. By simplicity notational we write $\Psi = \Psi_{M,\rho}$.

Before defining $\min - \max$ class that we will use for concluding the proof of the Theorem 1.1, we introduce some results and notations preliminary. We start with a result related to the function $\varphi : \Omega^2 \to \mathbb{R}$ defined by

$$\varphi(x,y) = G(x,y) - H^{\frac{1}{2}}(x,x)H^{\frac{1}{2}}(y,y), \tag{5.4}$$

which is key in all what follows. This result is an adaptation of Corollary 2.1 in [15].

Lemma 5.1. *For any fixed value $\rho^* > 1$, there is a $\mu_0 > 0$ such that if $\mathfrak{D}_i$ is any domain contained in $\overline{B(P_i,\mu)}$ and $0 < \mu < \mu_0$, then*

$$\inf_{|P_i - x| = |P_i - y| = \mu\rho^*} \varphi(x,y) > 0, \quad \textit{for all } i = 1,2,\ldots,k.$$

Proof. We consider the function $\varphi_{\mathcal{D}} : \Omega^2 \to \mathbb{R}$ defined by

$$\varphi_{\mathcal{D}}(x,y) = G_{\mathcal{D}}(x,y) - H_{\mathcal{D}}^{\frac{1}{2}}(x,x)H_{\mathcal{D}}^{\frac{1}{2}}(y,y),$$

where $G_{\mathcal{D}}$ and $H_{\mathcal{D}}$ respectively its Green's function and regular part. Since $H_{\mathcal{D}}$ is smooth near each P_i and $G_{\mathcal{D}}$ becomes unbounded, one has

$$\inf_{|P_i - x| = |P_i - y| = \mu\rho^*} \varphi_{\mathcal{D}}(x,y) > 0, \quad \text{for any } \mu > 0 \text{ small enough.}$$

On the other hand, if for a number $r > 0$ we consider the domain

$$\mathcal{D}_r = \mathcal{D} \setminus \bigcup_{i=1}^{k} \overline{B(P_i,r)}$$

and denote by G_r and H_r respectively its Green's function and regular part, then by harmonicity, it is not difficult to check that

$$\lim_{r\to 0} H_r(x,y) = H_{\mathcal{D}}(x,y)$$

uniformly on x, y in compact subsets of $\overline{\mathcal{D}} \setminus \{P_1, P_2, \ldots, P_k\}$. Then for fixed $\mu_0 = \mu_0(\rho^*) > 0$ small enough, one has that H and G become uniformly close to $H_{\mathcal{D}}$ and $G_{\mathcal{D}}$ on $|P_i - x| = |P_i - y| = \mu\rho^*$ if $0 < \mu < \mu_0$. This finishes the proof. $\square$

Now, we define the function $\varphi^* \colon]1, +\infty[^2 \to \mathbb{R}$ as follows

$$\varphi^*(s,t) := \frac{1}{(s+t)^{N-2}} - \frac{1}{(st+1)^{N-2}} - \frac{1}{(s^2-1)^{\frac{N-2}{2}}(t^2-1)^{\frac{N-2}{2}}}. \tag{5.5}$$

Note that there is an only one point of the form $(\rho_1^*, \rho_1^*) \in \,]1, +\infty[^2$ such that

$$\varphi^*(\rho_1^*, \rho_1^*) = 0.$$

Moreover, the function φ^* has a positive maximum global value, attained at a point of the form $(\rho_2, \rho_2) \in]1, +\infty[^2$. Specifically,

$$\varphi^*(\rho_2, \rho_2) = \max_{(s,t) \in \,]1,+\infty[^2} \varphi^*(s,t) > 0.$$

Actually we have that $1 < \rho_1^* < \rho_2$ verify

$$\frac{2^{2-N}}{{\rho_1^*}^{N-2}} = \frac{1}{\left({\rho_1^*}^2 + 1\right)^{N-2}} + \frac{1}{\left({\rho_1^*}^2 - 1\right)^{N-2}}$$

and

$$\frac{2^{1-N}}{\rho_2^N} = \frac{1}{(\rho_2^2+1)^{N-1}} + \frac{1}{(\rho_2^2-1)^{N-1}}.$$

Also we assume that $\inf_{x \in \Omega} f(x) = \alpha > 0$ and consider the following positive constants:

$$\beta = \max_{x \in \Omega} f(x), \quad m_1 = \min_{0 \le i \le m} r_i \quad \text{and} \quad m_2 = \max_{1 \le i \le m} R_i, \tag{5.6}$$

where $r_0 = \frac{1}{2} \min_{i \neq j} |P_i - P_j|$, and for $i = 1, 2, \ldots, m$, r_i is the radius of the biggest ball centered at P_i contained in $\mathcal{D}$, and R_i is the radius of the smallest ball centered at P_i containing to $\mathcal{D}$.

Note that if $(\alpha m_1^2 - \beta m_2^2)\rho_2^{N-2} + \beta m_2^2 \neq 0$,

$$\tilde{\rho}_2 = \left| \frac{\alpha m_1 \rho_2^{N-2}}{(\alpha m_1^2 - \beta m_2^2)\rho_2^{N-2} + \beta m_2^2} \right|^{\frac{1}{N-2}} > 0.$$

In opposite case, if $(\alpha m_1^2 - \beta m_2^2)\rho_2^{N-2} + \beta m_2^2 = 0$, we can replace ρ_2 by a few larger value on the definition of $\tilde{\rho}_2$, so that $\tilde{\rho}_2$ still is a positive constant. Then we choose

$$\rho_2^* = \max\{\rho_2, \tilde{\rho}_2\}.$$

Consider now values $\rho^{*i} \in\,]\rho_1^*, \rho_2^*[\,$, for $i = 1, 2\ldots, k$, and $\mu > 0$ small enough such that the conclusion of Lemma 5.1 remains valid. From now on, we will consider the following manifolds of Ω

$$\mathcal{S}_i = \mu\rho^{*i} S_i^{N-1}, \tag{5.7}$$

where S_i^{N-1} is the unitary sphere centered in P_i and we put

$$\mathcal{N} = \prod_{i=1}^{k} \mathcal{S}_i^2. \tag{5.8}$$

Besides, considering anullus $\mathcal{A}_i = B(P_i, \mu\rho_2^*) \setminus \overline{B(P_i, \mu\rho_1^*)}$ and the set

$$D_\varphi = \{\vec{\xi} \in \Omega_\rho^{2k} : \varphi(\vec{\xi^i}) > \delta^* \text{ for all } i = 1, 2, \ldots, k\},$$

where φ is defined by (5.4) and δ^* is chosen so that $\mathcal{N} \subset D_\varphi$, we restrict the domain of definition of Ψ to

$$D_\varphi^* = \left(\prod_{i=1}^{k} \mathcal{A}_i^2\right) \cap D_\varphi. \tag{5.9}$$

Now, for every $\vec{\xi} \in \mathcal{N}$ we choose $\vec{d}(\vec{\xi}\,) = \big(\vec{d^1}(\vec{\xi}\,), \vec{d^2}(\vec{\xi}\,), \ldots, \vec{d^k}(\vec{\xi}\,)\big) \in \mathbb{R}^{2k}$ being a vector which defines a negative direction of the associated quadratic form with Ψ. More precisely, in agreement to (5.2), for fixed $\vec{\xi^i} \in \mathcal{S}_i^2$, the function

$$\psi(\vec{\xi^i}, \vec{d^i}\,) = \frac{1}{2}\sum_{j=1}^{2} d_{ij}^2 H(\xi_{ij}, \xi_{ij}) - d_{i1}d_{i2}G(\xi_{i1}, \xi_{i2}) + \sum_{j=1}^{2} d_{ij} w(\xi_{ij}),$$

regarded as a function of $\vec{d^i} = (d_{i1}, d_{i2})$ only, with $d_{i1}, d_{i2} > 0$, has a unique critical point $\vec{\mathbf{d}}^i(\vec{\xi^i}) = \big(\bar{d}_{i1}, \bar{d}_{i2}\big)$ given by

$$\bar{d}_{ij} = \frac{H(\xi_{ij}, \xi_{ij})w(\xi_{il}) + G(\xi_{il}, \xi_{ij})w(\xi_{ij})}{G^2(\xi_{il}, \xi_{ij}) - H(\xi_{il}, \xi_{il})H(\xi_{ij}, \xi_{ij})}, \quad l, j = 1, 2,\ l \neq j.$$

In particular,

$$\psi\big(\vec{\xi^i}, \vec{\mathbf{d}}^i(\vec{\xi^i})\big) = \frac{1}{2}\Phi\big(\vec{\xi^i}\big), \tag{5.10}$$

where Φ is the function defined on D_φ^*, given by

$$\Phi(x, y) = \frac{H(x, x)w^2(y) + 2G(x, y)w(x)w(y) + H(y, y)w^2(x)}{G^2(x, y) - H(x, x)H(y, y)}, \tag{5.11}$$

being w the only one solution of (1.2). Then, we simply choose $\vec{d}(\vec{\xi}) = \vec{\mathbf{d}}(\vec{\xi})$, with $\vec{\mathbf{d}}(\vec{\xi}) = \big(\vec{\mathbf{d}}^1(\vec{\xi^1}), \vec{\mathbf{d}}^2(\vec{\xi^2}), \ldots, \vec{\mathbf{d}}^k(\vec{\xi^k})\big) \in \mathbb{R}_+^{2k}$ and easily see that there is a constant $c > 0$ such that $c < \bar{d}_{ij} < c^{-1}$ for all $\vec{\xi} \in \mathcal{N}$.

Consider now the class Γ of all continuous functions

$$\gamma : \mathcal{N} \times I_0^k \times [0,1] \to D_\varphi^* \times \mathrm{I\!R}_+^{2k}$$

defined by

$$\gamma(\vec{\xi}, \vec{\sigma}, t) = \big(\gamma_1(\vec{\xi^1}, \sigma_1, t), \gamma_2(\vec{\xi^2}, \sigma_2, t), \ldots, \gamma_k(\vec{\xi^k}, \sigma_k, t)\big),$$

such that

1. $\gamma_i(\vec{\xi^i}, \sigma_0, t) = \big(\vec{\xi^i}, \sigma_0 \vec{d^i}(\vec{\xi^i})\big)$ and $\gamma_i(\vec{\xi^i}, \sigma_0^{-1}, t) = \big(\vec{\xi^i}, \sigma_0^{-1} \vec{d^i}(\vec{\xi^i})\big)$ for all $\vec{\xi^i} \in \mathcal{S}_i$, $t \in [0,1]$.
2. $\gamma_i(\vec{\xi^i}, \sigma, 0) = \big(\vec{\xi^i}, \sigma \vec{d^i}(\vec{\xi^i})\big)$ for all $(\vec{\xi^i}, \sigma) \in \mathcal{S}_i \times I_0$,

where $I_0 = [\sigma_0, \sigma_0^{-1}]$, being σ_0 a small number to be chosen later. Then we define the min-max value as

$$c(\Omega) = \inf_{\gamma \in \Gamma} \sup_{(\vec{\xi}, \vec{\sigma}) \in \mathcal{N} \times I_0^k} \Psi\big(\gamma(\vec{\xi}, \vec{\sigma}, 1)\big). \tag{5.12}$$

In what follows, we will prove that $c(\Omega)$ is actually a critical value of Ψ. A first step in this direction consists of finding an upper estimate for $c(\Omega)$. Let us consider the exterior domain

$$E = \mathrm{I\!R}^N \setminus \overline{B(0,1)}$$

and denote by G_E and H_E, respectively, the Green's function on E and its regular part. We define the function

$$\Phi_E^i(x,y) = \frac{H_E(x,x)(w_\mu^i)^2(y) + 2G_E(x,y) w_\mu^i(x) w_\mu^i(y) + H_E(y,y)(w_\mu^i)^2(x)}{G_E^2(x,y) - H_E(x,x)H_E(y,y)},$$

for $(x,y) \in D_{\varphi^*}^i$, where $w_\mu^i(x) = \mu^{-2} w\,(\mu x + P_i)$ is a function defined in the set $\Omega_\mu^i = \{x \in \mathrm{I\!R}^N : x = \mu^{-1}(z - P_i),\ z \in \Omega\}$, with w given by (1.2) and

$$D_{\varphi^*}^i = \big\{(x,y) \in (\Omega_\mu^i) \times (\Omega_\mu^i) : \varphi^*(|x|,|y|) > 0\big\},$$

where φ^* is the function defined in (5.5). Since was proved in Lemma 5 of [1], Φ_E^i achieves a relative minimum value in a critical point of the form $(\bar{x}_i, \bar{y}_i)$, with $\bar{x}_i$ and $\bar{y}_i$ having opposite directions, and such that $(|\bar{x}_i|, |\bar{y}_i|) \in \,]\rho_1^*, \rho_2^*[^2$. Actually one has that $|\bar{y}_i| = |\bar{x}_i| + o(\mu) \sim \bar{\rho}_i^*$, with $\rho_1^* + \rho_0 < \bar{\rho}_i^* < \rho_2^* - \rho_0$ for some ρ_0 independents of all $\mu > 0$ small enough, and

$$\min_{(x,y)\in D^i_{\varphi^*}|^{\rho_2^*}_{\rho_1^*}} \Phi_E(x,y) = c_i^*,$$

where $D^i_{\varphi^*}|^{\rho_2^*}_{\rho_1^*} = \{(x,y) \in D^i_{\varphi^*} : \rho_1^* < |x|, |y| < \rho_2^*\}$. Moreover, it is not difficult to check that

$$c_i^* \le \frac{\frac{m_2^4\beta^2}{4N^2}\left(\frac{\left(\frac{1}{\mu}\right)^2-1}{\left(\frac{1}{\mu}\right)^{2-N}-1}(\mu\bar{\rho}_i^*)^{2-N} - (\mu\bar{\rho}_i^*)^2 + \left(\frac{1}{\mu}\right)^{2-N}\frac{1-\left(\frac{1}{\mu}\right)^N}{\left(\frac{1}{\mu}\right)^{2-N}-1}\right)^2}{\frac{\left((\mu\bar{\rho}_i^*)^2+1\right)^{N-2}-(2\mu\bar{\rho}_i^*)^{N-2}}{(2\mu\bar{\rho}_i^*)^{N-2}((\mu\bar{\rho}_i^*)^2+1)^{N-2}} - ((\mu\bar{\rho}_i^*)^2-1)^{2-N}} + \tilde{o}(\mu),$$

where β and m_2 are the definite constants in (5.6), and $\mu^{N+2}\tilde{o}(\mu) \to 0$ as $\mu \to 0$.

Proposition 5.2. *The following estimate holds*

$$c(\Omega) \le \frac{km_2^4\beta^2}{8N^2}\left(\frac{2}{\bar{\rho}^*}\right)^{N-2} + o(1),$$

where $o(1) \to 0$ as $\mu \to 0$, and $\bar{\rho}^* = \min_{1\le i\le k} \bar{\rho}_i^*$.

Proof. For all $t \in [0,1]$, we consider the test path defined componentwise as $\gamma_i(\vec{\xi^i}, \sigma_i, t) = \big(\vec{\xi^i}, \sigma_i\vec{d^i}(\vec{\xi^i})\big)$. Maximizing $\Phi\big(\gamma(\vec{\xi}, \vec{\sigma}, t)\big)$ in the variable $\vec{\sigma}$, we note that this maximum value is attained approximately at $\vec{\sigma} = (1, 1, \dots, 1)$, because of our choice of the vector $\vec{d}(\vec{\xi})$. Besides, since $\vec{d}(\vec{\xi}) = O(\mu^{N-2})$, one has that $R\big(\vec{\xi}, \vec{d}(\vec{\xi})\big) = O(\mu^{N-2})$. Hence, from (5.1), (5.10) and (5.11), we have that

$$\max_{\vec{\sigma}\in I_0^k} \Psi\big(\gamma(\vec{\xi}, \vec{\sigma}, t)\big) = \frac{1}{2}\sum_{i=1}^k \Phi(\vec{\xi^i}) + o(1),$$

where $o(1) \to 0$ as $\mu \to 0$. We note that in D^*_φ one has

$$\Phi(x,y) = \mu^{N+2}\Phi_E^i\big(\mu^{-1}(x-P_i), \mu^{-1}(y-P_i)\big) + o(1),$$

where $o(1) \to 0$ as $\mu \to 0$. In particular, we choose $\rho^{*i} = \bar{\rho}_i^*$ to definite each $\mathcal{S}_i$ in (5.7) and $\mathcal{N}$ in (5.8). Then we obtain

$$\begin{aligned} c(\Omega) &\le \frac{k}{2}\mu^{N+2}\Phi_E^i(\bar{x}, \bar{y}) + o(1) \\ &= \frac{k}{2}\mu^{N+2}c_i^* + o(1), \end{aligned}$$

where $o(1) \to 0$ as $\mu \to 0$. From the estimate previous to this proposition, the result follows. $\square$

For next step, we need an intersection lemma. The idea behind this result is the topological continuation of the set of solution of an equation, see [17]. For every $(\vec{\xi}, \vec{\sigma}, t) \in \mathcal{N} \times I_0^k \times [0,1]$ we denote

$$\gamma(\vec{\xi}, \vec{\sigma}, t) = \big(\vec{\xi}^*(\vec{\xi}, \vec{\sigma}, t), \vec{\Lambda}^*(\vec{\xi}, \vec{\sigma}, t)\big) \in D_\varphi^* \times \mathbb{R}_+^{2k},$$

where $\vec{\xi}^* = \big(\vec{\xi}^{*1}, \vec{\xi}^{*2}, \ldots, \vec{\xi}^{*k}\big) \in \Omega^{2k}$ and $\vec{\Lambda}^* = \big(\vec{\Lambda}^{*1}, \vec{\Lambda}^{*2}, \ldots, \vec{\Lambda}^{*k}\big) \in \mathbb{R}_+^{2k}$, with $\vec{\xi}^{*i} = (\xi_{i1}^*, \xi_{i2}^*) \in \Omega^2$ and $\vec{\Lambda}^{*i} = (\Lambda_{i1}^*, \Lambda_{i2}^*) \in \mathbb{R}_+^2$. Also we define the set

$$\mathbf{M} = \{(\vec{\xi}, \vec{\sigma}) \in \mathcal{N} \times I_0^k : \Lambda_{i1}^*(\vec{\xi}, \vec{\sigma}, 1) \cdot \Lambda_{i2}^*(\vec{\xi}, \vec{\sigma}, 1) = 1\}.$$

Lemma 5.3. *For every open neighborhood W of $\mathbf{M}$ in $\mathcal{N} \times I_0^k$, the projection $g : W \to \mathcal{N}$ induces a monomorphism in cohomology, that is*

$$g^* : H^*(\mathcal{N}) \to H^*(W)$$

is injective.

The proof of this result is almost identical to that found for proving Lemma 6.2 in [14], except minor details, we therefore omit it. Nevertheless, it is suitable to indicate that in this proof one chooses σ_0 small enough in order that certain inclusion is well definite.

Proposition 5.4. *There is a constant $A > 0$, independent of σ_0, such that*

$$\sup_{(\vec{\xi}, \vec{\sigma}) \in \mathcal{N} \times I_0^k} \Psi\big(\gamma(\vec{\xi}, \vec{\sigma}, 1)\big) \geq -A \quad \text{for all } \gamma \in \Gamma.$$

Proof. Note that $\vec{\xi} \in \mathcal{N}$ implies that $\xi_{ij} \in B(P_i, \mu\rho_2^*) \setminus B(P_i, \mu\rho_1^*)$. Thus we can find $\delta_0 > 0$ such that if $|\xi_{i1} - \xi_{i2}| < \delta_0$, then $(\xi_{i1} - P_i) \cdot (\xi_{i2} - P_i) > 0$.

We argue by contradiction. Let $A_0 > 0$ be such that $G(x, y) \geq A_0$ implies $|x - y| < \delta_0$ and let us assume that for certain $\gamma \in \Gamma$

$$\Psi\big(\gamma(\vec{\xi}, \vec{\sigma}, 1)\big) < -kA_0 \quad \text{for all } (\vec{\xi}, \vec{\sigma}) \in \mathcal{N} \times I_0^k.$$

Then for every $(\vec{\xi},\vec{\sigma})\in\mathbf{M}$, $(\vec{\xi}^*,\vec{\Lambda}^*)=\big(\vec{\xi}^*(\vec{\xi},\vec{\sigma},1),\vec{\Lambda}^*(\vec{\xi},\vec{\sigma},1)\big)\in D_\varphi\times\mathbb{R}^{2k}_+$ and

$$\sum_{i=1}^k G(\xi^*_{i1},\xi^*_{i2})-\sum_{i=1}^k\sum_{j=1}^2\Big(\frac{1}{2}\Lambda^{*^2}_{ij}H(\xi^*_{ij},\xi^*_{ij})+\Lambda^*_{ij}w(\xi^*_{ij})\Big)+R(\vec{\xi}^*,\vec{\Lambda}^*)>kA_0.$$

Since $H(\xi^*_{ij},\xi^*_{ij})>0$ and $w(\xi^*_{ij})>0$, we conclude that if W is a small neighborhood of $\mathbf{M}$ contained in $\mathcal{N}\times I_0^k$, then for every $(\vec{\xi},\vec{\sigma})\in W$ one has that $|R(\vec{\xi}^*,\vec{\Lambda}^*)|$ is small compared to $G\big(\vec{\xi}^{*i}(\vec{\xi},\vec{\sigma},1)\big)$. Hence, for every $(\vec{\xi},\vec{\sigma})\in W$ there exists $i\in\{1,2,\dots,k\}$ such that

$$G\big(\vec{\xi}^{*i}(\vec{\xi},\vec{\sigma},1)\big)\geq A_0$$

and then $|\xi^*_{i1}-\xi^*_i|<\delta_0$. Now, we fix points $\zeta_i\in\mathbb{R}^N$ such that $|\zeta_i|=\mu\rho^{*i}$. It is follows that $\vec{\zeta}^i=(P_i+\zeta_i,P_i-\zeta_i)\in\mathcal{S}_i$ and $\vec{\zeta}=(\vec{\zeta}^1,\vec{\zeta}^2,\dots,\vec{\zeta}^k)\in\mathcal{N}$. Denoting $\gamma^1=\gamma(\,\cdot\,,1)$ and putting $T(\vec{\zeta})=\big\{t\vec{\zeta}:t\in\,]\rho_1^*,\rho_2^*[\,\big\}$, we see that because of the above conclusion one has that $\gamma^1(W)\subset\big(D_\varphi\setminus T(\vec{\zeta})\big)\times\mathbb{R}^{2k}_+$.

Let us consider now the map $s:D_\varphi\times\mathbb{R}^{2k}_+\to\mathcal{N}$ defined componentwise as $s_i(\vec{\xi},\vec{\Lambda})=\mu\rho^*\big(\frac{\xi_{i1}}{|\xi_{i1}|},\frac{\xi_{i2}}{|\xi_{i2}|}\big)$. Then $(\gamma^0)^*\circ s^*:H^*(\mathcal{N})\to H^*(\mathcal{N}\times I_0^k)$, where $\gamma^0=\gamma(\,\cdot\,,0)$ is an isomorphism. By the homotopy axiom we deduce that $(\gamma^1)^*\circ s^*$ is also an isomorphism. Now, we consider the following commutative diagram:

$$\begin{array}{ccccc}
H^*(\mathcal{N}\times I_0^k) & \overset{(\gamma^1)^*}{\longleftarrow} & H^*(D_\varphi\times\mathbb{R}^{2k}_+) & \overset{\gamma^*}{\longleftarrow} & H^*(\mathcal{N})\\
i_1^*\downarrow & & i_2^*\downarrow & & i_3^*\downarrow\\
H^*(W) & \overset{(\tilde{\gamma}^1)^*}{\longleftarrow} & H^*(\kappa_1(W)) & \overset{\tilde{s}^*}{\longleftarrow} & H^*(\mathcal{N}\setminus\{\vec{\zeta}\}),
\end{array}$$

where i_1, i_2 and i_3 are inclusion maps, $\tilde{\gamma}^1=\gamma^1|_W$ and $\tilde{s}=s|_{\gamma^1(W)}$. From Lemma 5.3 we have that i_1^* is a monomorphism which is a contradiction with the fact that $H^{2Nk}(\mathcal{N}\setminus\{\vec{\zeta}\})=0$. Thus, the result follows. □

Now, we need to care about the fact that the domain in which Ψ is defined is not necessarily closed for the gradient flow of Ψ.

Proposition 5.5. *Let $(\vec{\xi}_n,\vec{\Lambda}_n)\in D^*_\varphi\times\mathbb{R}^{2k}_+$ be a sequence such that*

$$\nabla_{\vec{\Lambda}}\Psi(\vec{\xi}_n,\vec{\Lambda}_n)\to 0. \tag{5.13}$$

Then each component of $\vec{\Lambda}_n$ is bounded above and below by positive constants.

Proof. From (5.9) note that $D_\varphi^* \subset\subset \Omega^{2k}$. Hence $w(\xi_{ij}) > 0$, for all $\vec{\xi} \in \overline{D_\varphi^*}$. We put $\vec{\xi}_n^i = (\xi_{i1}^n, \xi_{i2}^n)$ and $\vec{\Lambda}_n^i = (\Lambda_{i1}^n, \Lambda_{i2}^n)$. Then (5.13) is equivalent to

$$\Lambda_{ij}^n H(\xi_{ij}^n, \xi_{ij}^n) - \Lambda_{il}^n G(\xi_{ij}^n, \xi_{il}^n) + w(\xi_{ij}^n) - \sum_{st \neq i1, i2} \Lambda_{st}^n G(\xi_{ij}^n, \xi_{st}^n) = o(1).$$

It is clear that $\Lambda_{il}^n \to 0$ or $\Lambda_{ij}^n \to 0$, and $\Lambda_{st}^n \to C_2^{st}$ if $st \neq i1, i2$, with $C_2^{st} \geq 0$, cannot happen. Hence, since that H and G remain uniformly controlled, to suppose that $|\vec{\Lambda}_n| \to +\infty$ implies that if $\Lambda_{ij}^n \to +\infty$, for some $i \in \{1, 2, \ldots, k\}$, then also $\Lambda_{il}^n \to +\infty$, $j, l = 1, 2$ and $j \neq l$. We put $\tilde{\Lambda}_{lj}^n = \frac{\Lambda_{lj}^n}{|\vec{\Lambda}_n|}$, and passing to a subsequence, if necessary, we may assume that this sequence it approaches a nonzero vector $\hat{\vec{\Lambda}}$ with $\hat{\Lambda}_{ij} \neq 0$ if $\Lambda_{ij}^n \to +\infty$ (for $j = 1, 2$). It follows that

$$\tilde{\Lambda}_{ij}^n H(\xi_{ij}^n, \xi_{ij}^n) - \tilde{\Lambda}_{ij}^n G(\xi_{ij}^n, \xi_{il}^n) + \frac{w(\xi_{ij}^n)}{|\vec{\Lambda}_n|} - \sum_{st \neq i1, i2} \tilde{\Lambda}_{st}^n G(\xi_{ij}^n, \xi_{st}^n) \to 0.$$

Then, for a suitable subsequence, we obtain for some $\vec{\bar{\xi}} \in \overline{D_\varphi^*}$ the system

$$\begin{cases} \hat{\Lambda}_{i1} H(\bar{\xi}_{i1}, \bar{\xi}_{i1}) - \hat{\Lambda}_{i2} G(\bar{\xi}_{i1}, \bar{\xi}_{i2}) - \displaystyle\sum_{st \neq i1, i2} \hat{\Lambda}_{st} G(\bar{\xi}_{i1}, \bar{\xi}_{st}) = 0, \\ \hat{\Lambda}_{i2} H(\bar{\xi}_{i2}, \bar{\xi}_{i2}) - \hat{\Lambda}_{i1} G(\bar{\xi}_{i1}, \bar{\xi}_{i2}) - \displaystyle\sum_{st \neq i1, i2} \hat{\Lambda}_{st} G(\bar{\xi}_{i2}, \bar{\xi}_{st}) = 0. \end{cases}$$

Hence, solving for $\hat{\Lambda}_{i1}$, we conclude that

$$\begin{aligned} G^2(\bar{\xi}_{i1}, \bar{\xi}_{i2}) - &H(\bar{\xi}_{i1}, \bar{\xi}_{i1}) H(\bar{\xi}_{i2}, \bar{\xi}_{i2}) = \\ &- \sum_{st \neq i1, i2} \frac{\hat{\Lambda}_{st}}{\hat{\Lambda}_{i2}} \big(G(\bar{\xi}_{i1}, \bar{\xi}_{i2}) G(\bar{\xi}_{i1}, \bar{\xi}_{st}) + H(\bar{\xi}_{i1}, \bar{\xi}_{i1}) G(\bar{\xi}_{i2}, \bar{\xi}_{st}) \big), \end{aligned}$$

which is a contradiction, since the quantity on the left hand side of the previous equality is strictly positive when $\mu > 0$ is chosen sufficiently small. This finishes the proof. $\square$

Let $\delta_*^i > 0$ a suitable small values such that the level set

$$\{\vec{\xi} \in D_\varphi^* : \Phi_E^i\big(\mu^{-1}(\xi_{i1} - P_i), \mu^{-1}(\xi_{i2} - P_i)\big) = \delta_*^i\}$$

is a closed curve and that $\nabla\Phi_E^i\big(\mu^{-1}(\xi_{i1}-P_i),\mu^{-1}(\xi_{i2}-P_i)\big)$ does not vanish on it. Let us set

$$\Upsilon_\mu=\{\vec{\xi}\in D_\varphi^*:\Phi_E^i\big(\mu^{-1}(\xi_{i1}-P_i),\mu^{-1}(\xi_{i2}-P_i)\big)<\delta_*^i\}.$$

Thus, on this region we have that $\Phi_E^i\big(\mu^{-1}(\xi_{i1}-P_i),\mu^{-1}(\xi_{i2}-P_i)\big)<\delta_*^i$ and if $(\xi_{i1},\xi_{i2})\in\partial\Upsilon_\mu$ then one of the following two situations happen: either there is a tangential direction τ to $\partial\Upsilon_\mu$ such that

$$\nabla\Phi_E^i\big(\mu^{-1}(\xi_{i1}-P_i),\mu^{-1}(\xi_{i2}-P_i)\big)\cdot\tau\neq 0,$$

or ξ_{i1} and ξ_{i2} lie in opposite directions, $\Phi_E^i(x,y)=\delta_*^i$ and

$$\nabla\Phi_E^i\big(\mu^{-1}(\xi_{i1}-P_i),\mu^{-1}(\xi_{i2}-P_i)\big)\neq 0,$$

being points orthogonally outwards to Υ_μ. Moreover, if μ_1 and μ_2 are small enough, and $\mu_1<\mu_2$, then $\Upsilon_{\mu_1}\subset\subset\Upsilon_{\mu_2}\subset\subset D_\varphi^*$.

Proposition 5.6. *The functional Ψ satisfies the P.S. condition in the region $\Upsilon_\mu\times\mathbb{R}_+^{2k}$ at the level $c(\Omega)$ given in (5.12).*

Proof. Let us consider a sequence $(\vec{\xi}_n,\vec{\Lambda}_n)\in\Upsilon_\mu\times\mathbb{R}_+^{2k}$ such that

$$\nabla_{\vec{\Lambda}}\Psi(\vec{\xi}_n,\vec{\Lambda}_n)\to 0\quad\text{and}\quad\nabla_{\vec{\xi}}^{\tau}\Psi(\vec{\xi}_n,\vec{\Lambda}_n)\to 0,$$

where $\nabla_{\vec{\xi}}^{\tau}\Psi$ corresponds to the tangential gradient of Ψ to $\partial\Upsilon_\mu\times\mathbb{R}_+^{2k}$ in case that $\vec{\xi}_n$ is approaching to $\partial\Upsilon_\mu$ or the full gradient in otherwise. From the previous lemma, the components of $\vec{\Lambda}_n$ are bounded above and below by positive constants, so that we may assume, passing to a subsequence if necessary, that $(\vec{\xi}_n,\vec{\Lambda}_n)\to(\vec{\xi}_0,\vec{\Lambda}_0)$ for some $(\vec{\xi}_0,\vec{\Lambda}_0)\in\overline{\Upsilon}_\mu\times\mathbb{R}_+^{2k}$ and $\Psi(\vec{\xi}_n,\vec{\Lambda}_n)\to c(\Omega)$. Then

$$\nabla_{\vec{\Lambda}}\Psi(\vec{\xi}_0,\vec{\Lambda}_0)=0.$$

Observe that if $\vec{\xi}_0\in\operatorname{int}(\Upsilon_\mu)$ then $\vec{\xi}_0$ is a critical point of Ψ. We assume the opposite, this is that $\vec{\xi}_0\in\partial\Upsilon_\mu$. Then

$$\Phi_E^i\big(\mu^{-1}(\xi_{i1}^0-P_i),\mu^{-1}(\xi_{i2}^0-P_i)\big)=\delta_*^i.$$

Firstly we note that since $\nabla_{\vec{\Lambda}}\Psi(\vec{\xi_0},\vec{\Lambda}_0)=0$, one has that $\vec{\Lambda}_0$ satisfies

$$\Lambda_{ij}^0=\frac{H(\xi_{il}^0,\xi_{il}^0)w(\xi_{ij}^0)+G(\xi_{ij}^0,\xi_{il}^0)w(\xi_{il}^0)}{G^2(\xi_{ij}^0,\xi_{il}^0)-H(\xi_{ij}^0,\xi_{ij}^0)H(\xi_{il}^0,\xi_{il}^0)}+\theta_{ij},\quad j,l=1,2,\ j\neq l,$$

where the quantity θ_{ij} is of small order. Substituting these values in Ψ, from (5.10) we obtain

$$c(\Omega)=\Psi(\vec{\xi_0},\vec{\Lambda}_0)=\sum_{i=1}^{k}\psi(\vec{\xi}^{0i},\vec{\Lambda}^{0i})-R(\vec{\xi_0},\vec{\Lambda}_0)$$

and then we deduce that

$$c(\Omega)=\sum_{i=1}^{k}\mu^{N+2}\Phi_E^i\big(\mu^{-1}(\xi_{i1}^0-P_i),\mu^{-1}(\xi_{i2}^0-P_i)\big)+\theta(\vec{\xi_0}),$$

where $\theta(\vec{\xi_0})$ is small in the C^1 sense, as $\mu>0$ becomes smaller. Hence, for any tangential direction τ to $\partial\Upsilon_\mu$ we have that $\nabla_{\vec{\xi}}\Psi(\vec{\xi_0},\vec{\Lambda}_0)\cdot\tau\sim 0$. Thus, from the analysis previous to this proposition, the points ξ_{i1}^0,ξ_{i2}^0 are in opposite directions, $\Psi(\vec{\xi_0},\vec{\Lambda}_0)\sim\mu^{N+2}(\delta_*^1+\delta_*^2+\ldots+\delta_*^k)$ and $\nabla_{\vec{\xi}}\Psi(\vec{\xi_0},\vec{\Lambda}_0)$ must be away from 0. Choosing τ parallel to $\nabla_{\vec{\xi}}\Psi(\vec{\xi_0},\vec{\Lambda}_0)$ we obtain that $\nabla_{\vec{\xi}}\Psi(\vec{\xi_0},\vec{\Lambda}_0)\cdot\tau$ must to be away from 0, which is a contradiction. Therefore $\vec{\xi_0}\in\operatorname{int}(\Upsilon_\mu)$, which implies that the P.S. condition holds. $\square$

Now we are in conditions to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Let us consider the domain $\mathbf{\Upsilon}_{\mathbf{a}}^{\mathbf{b}}=\Upsilon_\mu\times[\mathbf{a},\mathbf{b}]^2$ with $\mathbf{a},\mathbf{b}$ to be choose later. Then the functional $\mathcal{I}$ given by (4.1) is well defined on $\mathbf{\Upsilon}_{\mathbf{a}}^{\mathbf{b}}$ except on the set $\Delta_\rho=\{(\vec{\xi},\vec{\Lambda})\in\mathbf{\Upsilon}_{\mathbf{a}}^{\mathbf{b}}:|\xi_1-\xi_2|<\rho\}$. From (4.3) we can extend $\mathcal{I}$ to all $\mathbf{\Upsilon}_{\mathbf{a}}^{\mathbf{b}}$ by extending Ψ as in (5.3), and keep relations (4.3) and (4.4). From Proposition 5.6, Ψ satisfies the P.S. condition over $\mathbf{\Upsilon}_{\mathbf{a}}^{\mathbf{b}}$. Then there exist constants $\mathbf{b}>0$, $c>0$ and $\varrho_0>0$, such that if $0<\varrho<\varrho_0$, and $(\vec{\xi},\vec{\Lambda})\in\mathbf{\Upsilon}_{\mathbf{a}}^{\mathbf{b}}$ satisfying $|\vec{\Lambda}|\geq\mathbf{b}$ and $c(\Omega)-2\varrho\leq\Psi(\vec{\xi},\vec{\Lambda})\leq c(\Omega)+2\varrho$, then $|\nabla\Psi(\vec{\xi},\vec{\Lambda})|\geq c$.

We now use the min-max characterization of $c(\Omega)$ to choose $\gamma\in\Gamma$ so that

$$c(\Omega)\leq\sup_{(\vec{\xi},\sigma)\in\mathcal{N}\times I_0^k}\Psi\big(\gamma(\vec{\xi},\sigma,1)\big)\leq c(\Omega)+\varrho.$$

By making $\mathbf{a}$ small and $\mathbf{b}$ large if necessary, we can assume that

$$\gamma(\vec{\xi},\sigma,1)\in\mathbf{\Upsilon}_{2\mathbf{a}}^{\frac{\mathbf{b}}{2}}\subset\mathbf{\Upsilon}_{\mathbf{a}}^{\mathbf{b}}\ \text{ for all }\ (\vec{\xi},\sigma)\in\mathcal{N}\times I_0^k.$$

Consider now $\eta : \boldsymbol{\Upsilon}_{\mathbf{a}}^{\mathbf{b}} \times [0, +\infty] \to \boldsymbol{\Upsilon}_{\mathbf{a}}^{\mathbf{b}}$ being the solution of the problem

$$\begin{cases} \dot{\eta}(\vec{\xi}, \vec{\Lambda}, t) & = -h\big(\eta(\vec{\xi}, \vec{\Lambda}, t)\big)\nabla\mathcal{I}\big(\eta(\vec{\xi}, \vec{\Lambda}, t)\big), \quad t \geq 0 \\ \eta(\vec{\xi}, \vec{\Lambda}, 0) & = (\vec{\xi}, \vec{\Lambda}). \end{cases}$$

Here the function h is defined in $\boldsymbol{\Upsilon}_{\mathbf{a}}^{\mathbf{b}}$ so that if $\Psi(\vec{\xi}, \vec{\Lambda}) \leq c(\Omega) - 2\varrho$, then $h(\vec{\xi}, \vec{\Lambda}) = 0$, and if $\Psi(\vec{\xi}, \vec{\Lambda}) \geq c(\Omega) - \varrho$, then $h(\vec{\xi}, \vec{\Lambda}) = 1$; satisfying $0 \leq h \leq 1$ for all $(\vec{\xi}, \vec{\Lambda}) \in \boldsymbol{\Upsilon}_{\mathbf{a}}^{\mathbf{b}}$. Hence, by the choice of $\mathbf{a}$ y $\mathbf{b}$, and bearing in mind (4.3) and (4.4), we have that $\eta(\vec{\xi}, \vec{\Lambda}, t) \in \boldsymbol{\Upsilon}_{\mathbf{a}}^{\mathbf{b}}$ for all $t \geq 0$. Then the following min-max value

$$C(\Omega) = \inf_{t \geq 0} \sup_{(\vec{\xi}, \sigma) \in \mathcal{N} \times I_0^k} \mathcal{I}\big(\eta(\gamma(\vec{\xi}, \sigma, 1), t)\big)$$

is a critical value for $\mathcal{I}$. We are always assuming that ε is small enough, to make the errors in (4.3) and (4.4) sufficiently small. Then, considering $\mathcal{M}_\delta$ as in (2.3), for $\delta > 0$ fixed sufficiently small, and $\lambda_{ij\varepsilon} = a_N^{-1} \Lambda_{ij\varepsilon}^{\frac{2}{N-2}}$, from Lemma 4.1 we conclude that there exist $(\vec{\xi}_\varepsilon, \vec{\Lambda}_\varepsilon) \in \mathcal{M}_\delta$ such that problem (1.1) has a nontrivial solution u_ε of the form (1.3). Theorem 1.1 has been proven. $\square$

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