DOUBLE-SPIKE SOLUTIONS FOR A CRITICAL INHOMOGENEOUS ELLIPTIC PROBLEM IN DOMAINS WITH SMALL HOLES

SALOMÓN ALARCÓN

Abstract. In this paper we construct solutions which develop two negative spikes as $\varepsilon \to 0^+$ for the problem $-\Delta u = |u|^{\frac{4}{N-2}} u + \varepsilon f(x)$ in Ω, $u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain exhibiting a small hole, with $f \geq 0$, $f \not\equiv 0$. This result extends Theorem 2 in [9] in the sense that no symmetry assumptions on the domain are required.

1. Introduction

This paper deals with the construction of solutions of the problem

$$\begin{cases} -\Delta u = |u|^{p-1}u + \varepsilon f(x) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, which has a small hole, $p = \frac{N+2}{N-2}$ is the critical Sobolev exponent, $f(x)$ is an inhomogeneous perturbation, $f \geq 0$, $f \not\equiv 0$ and $\varepsilon > 0$ is a small parameter.

In the case $1 < p < \frac{N+2}{N-2}$, it is well-known that if $f = 0$, the associated energy functional to problem (1.1) is even and satisfies the (PS) condition in $H_0^1(\Omega)$ which implies the existence of infinitely many nontrivial solutions by standard Ljusternik-Schnirelmann theory. Also known are many results on existence and multiplicity of sign-changing solutions for small and large inhomogeneous perturbation, see [2, 23, 5, 18, 19, 25]; whereas in [16] was proved that (1.1) does not admit any positive solution if $\varepsilon > 0$ is too large.

In the critical case, $p = \frac{N+2}{N-2}$, the embedding $H_0^1(\Omega) \subset L^{p+1}(\Omega)$ is continuous but not compact, so that the (PS) condition does not hold, and serious difficulties in facing the existence question arise. In fact, Pohozaev [17] proved that (1.1) has no solution if $f = 0$ and Ω is strictly star-shaped. In contrast, Brezis and Nirenberg [7] showed that this situation can be reverted introducing suitable additive perturbations. Rey [20] pointed out that the result in [7] implies that if $f \geq 0$, $f \neq 0$ and $f \in H^{-1}(\Omega)$, then at least two positive solutions exist for $\varepsilon > 0$ small enough. Moreover, in [20] was proven that if $f \geq 0$, $f \not\equiv 0$, is sufficiently regular, then at least $\mathrm{cat}(\Omega) + 1$ positive solutions exist for $\varepsilon > 0$ sufficiently small, one of them converging uniformly to 0 while the others concentrate at some special points in Ω, depending on f and the regular part of Green's function of the Laplacian on Ω, as $\varepsilon \to 0$. In parallel to Rey's result in [20], but with a different approach, Tarantello [26] proved that (1.1) admits at least two solutions for $f \not\equiv 0$ satisfying $\|\varepsilon f\|_{H^{-1}(\Omega)} < C_N$, where C_N is an explicit constant; such solutions are positive if $f \geq 0$. The effect of the symmetries in further multiplicity of solutions has been considered in some works. Ali and Castro

[1] proved that the existence result in [7] is optimal for positive solutions in a ball: if Ω is a ball and $f \equiv 1$, problem (1.1) have exactly two positive solutions for all $\varepsilon > 0$ small enough. More recently, Clapp, Kavian and Ruf [10] proved that if Ω is symmetric with respect to 0, $0 \notin \Omega$, and f is even, then at least $\operatorname{cat}(\Omega) + 2$ positive solutions exist provided that $\|\varepsilon f\|_{H^{-1}}$ is sufficiently small. The results in [7, 20, 26, 1, 10] deal with existence of positive solutions to problem (1.1), provided that $f \geq 0$ and $f \not\equiv 0$, where $\varepsilon > 0$ is a small parameter.

Concerning solutions which are not necessarily positive, Clapp, del Pino and Musso [9] showed existence of solutions of (1.1) under certain symmetry assumptions in the domain Ω and the function f. Such solutions develop k negative spikes, for any $k \geq k_0(\Omega)$ where $k_0(\Omega)$ is a sufficiently large number depending of Ω.

In this paper we leave aside any symmetry assumptions on the domain Ω and the perturbation f, and we find solutions to problem (1.1) developing a negative double-spike shape. Besides, we give precise information about the asymptotic profile of the blow-up of these solutions as $\varepsilon \to 0$ and we indicate a clearly delimited region where the spikes are formed.

More precisely, our setting in problem (1.1) is as follows: let us consider the domain

$$\Omega = \mathcal{D} \setminus \overline{B(P,\mu)}, \tag{1.2}$$

where $\mathcal{D}$ is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, $P \in \mathcal{D}$, and $\mu > 0$ is a small number. Let us consider $f \in C^{0,\gamma}(\overline{\Omega})$, for some $0 < \gamma < 1$, such that $\inf_{x \in \Omega} f(x) > 0$ and, by simplicity, we fix $P = 0$. Then our main result is

Theorem 1.1. *There exists a constant $\mu_0 = \mu_0(f, \mathcal{D}) > 0$, such that for each $0 < \mu < \mu_0$ fixed, there exists a number $\varepsilon_0 > 0$ and a family of solutions u_ε of (1.1), for $0 < \varepsilon = \varepsilon_n < \varepsilon_0$, with the following property: u_ε has exactly a pair of local minimum points $(\xi_1^\varepsilon, \xi_2^\varepsilon) \in \Omega^2$ with $k_*\mu < |\xi_i^\varepsilon| < k^*\mu$, $i = 1, 2$, for certain constants k_*, k^* independent of μ, and such that for each small $\delta > 0$,*

$$\inf_{\{|x-\xi_i^\varepsilon|>\delta,\ i=1,2\}} u_\varepsilon(x) \to 0 \quad \text{and} \quad \inf_{\{|x-\xi_i^\varepsilon|<\delta\}} u_\varepsilon(x) \to -\infty, \quad i = 1, 2$$

as $\varepsilon \to 0$.

Indeed we will find that u_ε is a nontrivial solution of (1.1) of the form

$$u_\varepsilon(x) = -\alpha_N \sum_{i=1}^{2} \left\{ \frac{\varepsilon^{\frac{2}{N-2}} \lambda_{i\varepsilon}}{\varepsilon^{\frac{4}{N-2}} \lambda_{i\varepsilon}^2 + |x - \xi_i^\varepsilon|^2} \right\}^{\frac{N-2}{2}} + \varepsilon^{-1} \hat{\phi}(x) + \theta_\varepsilon(x),$$

where $\theta_\varepsilon(x) \to 0$ uniformly as $\varepsilon \to 0$, $\hat{\phi}$ is the unique solution of the problem

$$\begin{cases} -\Delta \hat{\phi}(x) = \varepsilon^2 f(x) & \text{in } \Omega, \\ \hat{\phi} = 0 & \text{on } \partial\Omega, \end{cases}$$

$\alpha_N = (N(N-2))^{\frac{N-2}{4}}$ and the points $\xi_i^\varepsilon \to \xi_i$, up to subsequences, where (ξ_1, ξ_2) is a critical point of the functional

$$\Phi(x,y) = \frac{1}{2} \left\{ \frac{H(x,x) w^2(y) + 2G(x,y) w(x) w(y) + H(y,y) w^2(x)}{G^2(x,y) - H(x,x) H(y,y)} \right\}$$

defined in the region $\{(x,y)\in\Omega^2 : G(x,y)-H^{\frac12}(x,x)H^{\frac12}(y,y)>0, x\neq y\}$. Here G and H are, respectively, the Green's function of the Laplacian on Ω and its regular part, and w is the unique solution of the problem

$$\begin{cases} -\Delta w = f & \text{in } \Omega,\\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Besides, one can identify the limits λ_i of $\lambda_{i\varepsilon}$ as

$$\lambda_i = \left(a_N^{-1} \frac{H(\xi_j,\xi_j)w(\xi_i)+G(\xi_i,\xi_j)w(\xi_j)}{G^2(\xi_i,\xi_j)-H(\xi_i,\xi_i)H(\xi_j,\xi_j)} \right)^{\frac{2}{N-2}},\ i\neq j,\ i,j=1,2,$$

where a_N is an explicit constant, and consider the constants k_*, k^* as follows: k_* is the unique solution in $]1,+\infty[$ of the equation

$$\frac{2^{2-N}}{s^{N-2}} = \frac{(s^2+1)^{N-2}+(s^2-1)^{N-2}}{(s^4-1)^{N-2}}$$

and, $K\leq k^* = k*(\Omega,f)<\infty$ where K is the unique solution in $]1,+\infty[$ of the equation

$$\frac{2^{1-N}}{s^N} = \frac{(s^2-1)^{N-1}+(s^2+1)^{N-1}}{(s^4-1)^{N-1}}.$$

In particular, if f is a constant and Ω is an annulus, then $k^*=K$.

On the other hand, it will be clear from the proof that the small excised domain does not need to be exactly a ball, and we consider this case just for notational simplicity.

The proof of Theorem 1.1 follows a Lyapunov-Schmidt reduction procedure, related with this problem. This method has been used for solving problem (1.1) in the critical case, see [20, 9] and in the slightly supercritical case with $f=0$, see [12, 13], and also [21, 22] for related results.

In the next section we derive some basic estimates for the *reduced energy* associated to this problem. Sections $3-4$ will be devoted to discuss the finite-dimensional reduction scheme which we use for the construction of solutions of (1.1). In Section 5 we introduce an auxiliary function which will be the key in our min-max scheme which we develop in Section 6 to establish finally the Theorem 1.1.

2. Basic estimates in the reduced energy

Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N\geq 3$, and let us consider the expanded domain

$$\Omega_\varepsilon = \varepsilon^{-\frac{2}{N-2}}\Omega, \quad \varepsilon>0.$$

Doing the change of variable

$$v_\varepsilon(x') = -\varepsilon\, u(\varepsilon^{\frac{2}{N-2}}x'), \quad x'\in\Omega_\varepsilon,$$

we note that u solves (1.1) if and only if v_ε solves

$$\begin{cases} \Delta v + |v|^{p-1}v = \varepsilon^{p+1}\tilde f(x') & \text{in } \Omega_\varepsilon,\\ v=0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{2.1}$$

where $p = \frac{N+2}{N-2}$ and $\tilde{f}(x') = f(\varepsilon^{\frac{2}{N-2}} x')$. It is well-known that all positive solutions of equation $\Delta\vartheta + \vartheta^p = 0$ in $\mathbb{R}^N$ are given by the functions

$$\bar{U}_{\lambda,\xi}(x) = \alpha_N \Big(\frac{\lambda}{\lambda^2 + |x-\xi|^2} \Big)^{\frac{N-2}{2}},$$

with $\lambda > 0$, $\xi \in \mathbb{R}^N$ and $\alpha_N = (N(N-2))^{\frac{N-2}{4}}$, see [3, 24, 7, 8]. Since Ω_ε is expanding to the whole $\mathbb{R}^N$ as $\varepsilon \to 0$, and $\varepsilon^{p+1}\tilde{f}(x') \to 0$ uniformly as $\varepsilon \to 0$, it is reasonable to think that, for certain numbers $\lambda_1, \lambda_2 > 0$ and points $\xi_1, \xi_2 \in \Omega$, some solution v_ε of (2.1) becomes

$$v_\varepsilon \sim \bar{U}_{\lambda_1,\xi_1'} + \bar{U}_{\lambda_2,\xi_2'},$$

where $\xi_i' = \varepsilon^{-\frac{2}{N-2}} \xi_i \in \Omega_\varepsilon$, where from now on we use the letter ξ to denote a point in Ω and ξ' to denote a point in Ω_ε.

From [11], we know that a better approximation to v_ε should be obtained by using the orthogonal projections onto $H_0^1(\Omega_\varepsilon)$ of the functions $\bar{U}_{\lambda,\xi'}$, denoted by $U_{\lambda,\xi'}$, namely the unique solution of the problem

$$\begin{cases} -\Delta U_{\lambda,\xi'} = \bar{U}^p_{\lambda,\xi'} & \text{in } \Omega_\varepsilon, \\ U_{\lambda,\xi'} = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

In other words, $U_{\lambda,\xi'} = \bar{U}_{\lambda,\xi'} - \bar{\upsilon}_{\lambda,\xi'}$, where $\bar{\upsilon}_{\lambda,\xi'}$ solves

$$\begin{cases} -\Delta \bar{\upsilon}_{\lambda,\xi'} = 0 & \text{in } \Omega_\varepsilon, \\ \bar{\upsilon}_{\lambda,\xi'} = \bar{U}_{\lambda,\xi'} & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Hence, if we consider $\bar{U} = \bar{U}_{1,0}$, we obtain

$$\bar{\upsilon}_{\lambda,\xi'}(x') = \varepsilon^2 \lambda^{\frac{N-2}{2}} H(\varepsilon^{\frac{2}{N-2}} x', \xi) \int_{\mathbb{R}^N} \bar{U}^p + o(\varepsilon^2) \tag{2.2}$$

and, away from $x' = \xi'$,

$$U_{\lambda,\xi'}(x') = \varepsilon^2 \lambda^{\frac{N-2}{2}} G(\varepsilon^{\frac{2}{N-2}} x', \xi) \int_{\mathbb{R}^N} \bar{U}^p + o(\varepsilon^2) \tag{2.3}$$

uniformly for x' on each compact subset of Ω_ε, where G and H are, respectively, the Green's function of the Laplacian with the Dirichlet boundary condition on Ω and its regular part. Now, to simplify notation, we consider the following function

$$V(x') = U_1(x') + U_2(x'), \quad x' \in \Omega_\varepsilon,$$

where $U_i = U_{\lambda_i,\xi_i'}$, $i=1,2$, and we put $\vec{\xi} = (\xi_1, \xi_2) \in \Omega^2$ and $\vec{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{R}^2_+$. Then, we look for solutions of the problem (2.1) of the form

$$v(x') = V(x') + \tilde{\eta}(x'), \quad x' \in \Omega_\varepsilon, \tag{2.4}$$

which for suitable points ξ and scalars λ will have the remainder term $\tilde{\eta}$ of small order all over Ω_ε. Since solutions of (2.1) correspond to stationary points of its associated energy functional J_ε defined by

$$J_\varepsilon(v) = \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla v|^2 - \frac{1}{p+1} \int_{\Omega_\varepsilon} |v|^{p+1} + \varepsilon^{p+1} \int_{\Omega_\varepsilon} \tilde{f} v, \tag{2.5}$$

we have that if a solution of the form (2.4) exists, then we should have $J_\varepsilon(v) \sim J_\varepsilon(V)$ and the corresponding points $(\vec{\xi}, \vec{\lambda})$ in the definition of V also should be "approximately stationary" for the finite-dimensional functional

$(\vec{\xi}, \vec{\lambda}) \mapsto J_\varepsilon(V)$. Thus, our first goal is to estimate $J_\varepsilon(V)$. In order to establish the expansion, we consider the function w which corresponds to the unique solution in $C^{0,\gamma}(\Omega)$ of the problem

$$\begin{cases} -\Delta w = f & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

and we make the following choice of the points and the parameters: we fix $\delta > 0$ and we relabel the parameters λ_i's as

$$\lambda_i = (a_N^{-1}\Lambda_i)^{\frac{2}{N-2}}, \quad i = 1, 2,$$

where $a_N = \int_{\mathbb{R}^N} \bar{U}^p$ and $\Lambda_i \in]\delta, \delta^{-1}[$, for $i = 1, 2$. We also define the set

$$\mathcal{M}_\delta = \{(\vec{\xi}, \vec{\Lambda}) : |\xi_1 - \xi_2| > \delta,\ \operatorname{dist}(\xi_i, \partial\Omega) > \delta;\ i = 1, 2\}, \tag{2.7}$$

where $\vec{\xi} = (\xi_1, \xi_2) \in \Omega^2$ and $\vec{\Lambda} = (\Lambda_1, \Lambda_2) \in]\delta, \delta^{-1}[^2$.

Lemma 2.1. *Let $\delta > 0$ given. The following expansion holds*

$$J_\varepsilon(V) = 2C_N + \varepsilon^2 \Phi(\vec{\xi}, \vec{\Lambda}) + o(\varepsilon^2)$$

uniformly in the C^1-sense, with respect to $(\vec{\xi}, \vec{\Lambda})$ in $\mathcal{M}_\delta$. Here

$$C_N = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla \bar{U}|^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} \bar{U}^{p+1} \tag{2.8}$$

and the function Φ is defined by

$$\Phi(\vec{\xi}, \vec{\Lambda}) = \frac{1}{2}\left\{ \sum_{i=1}^{2} \Lambda_i^2 H(\xi_i, \xi_i) - 2\Lambda_1\Lambda_2 G(\xi_1, \xi_2) \right\} + \sum_{i=1}^{2} \Lambda_i w(\xi_i). \tag{2.9}$$

The proof of the previous lemma is based in (2.2), (2.3) and some estimates established in [4], and follows the general lineaments used to prove Lemma 3.2 of [12] and Proposition 1 of [9], therefore is omitted.

3. The finite-dimensional reduction

We first introduce some notation to be used in what follows. For functions u, v defined in Ω_ε we set

$$\langle u, v \rangle = \int_{\Omega_\varepsilon} uv.$$

Let us fix a small number $\delta > 0$ and consider points $(\vec{\xi'}, \vec{\Lambda})$ in

$$\mathcal{M}_\delta^\varepsilon = \left\{(\vec{\xi'}, \vec{\Lambda}) \in \Omega_\varepsilon^2 \times]\delta, \delta^{-1}[^2 : |\xi_1' - \xi_2'| > \delta_\varepsilon,\ \operatorname{dist}(\xi_i', \partial\Omega_\varepsilon) > \delta_\varepsilon;\ i = 1, 2\right\}, \tag{3.1}$$

where $\delta_\varepsilon = \delta\varepsilon^{-\frac{2}{N-2}}$, $\vec{\xi'} = (\xi_1', \xi_2')$ and $\vec{\Lambda} = (\Lambda_1, \Lambda_2)$. Since all solutions ϑ of the problem $\Delta\vartheta + p\bar{U}_{\Lambda,0}^{p-1}\vartheta = 0$ in $\mathbb{R}^N$ which satisfy $|\vartheta(x)| < C|x|^{2-N}$ belong to $\operatorname{span}\left\{\frac{\partial \bar{U}_{\Lambda,0}}{\partial x_j}, \frac{\partial \bar{U}_{\Lambda,0}}{\partial \Lambda}\right\}_{j=1,\dots,N+1}$, see [8], it is convenient to consider, for $i = 1, 2$, the following functions:

$$\bar{Z}_{ij}(x') = \frac{\partial \bar{U}_i}{\partial \xi_{ij}'}(x'),\ j = 1, \dots, N, \quad \bar{Z}_{i(N+1)}(x') = \frac{\partial \bar{U}_i}{\partial \Lambda_i}(x'),$$

and their respective $H_0^1(\Omega_\varepsilon)$-projections Z_{ij}, namely the unique solutions of

$$\begin{cases} \Delta Z_{ij} = \Delta \bar{Z}_{ij} & \text{in } \Omega_\varepsilon, \\ Z_{ij} = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

In order to simplify notation, we will denote

$$V = U_1 + U_2 \quad \text{and} \quad \bar{V} = \bar{U}_1 + \bar{U}_2.$$

We start studying a linear problem which is the basis for the reduction of (2.1): given $h \in L^\infty(\overline{\Omega}_\varepsilon)$, find a function η and constants c_{ij} such that

$$\begin{cases} \Delta\eta + p|V|^{p-1}\eta = h + \sum_{i,j} c_{ij} U_i^{p-1} Z_{ij} & \text{in } \Omega_\varepsilon, \\ \eta = 0 & \text{on } \partial\Omega_\varepsilon, \\ \langle \eta, U_i^{p-1} Z_{ij} \rangle = 0 & \forall i, j. \end{cases} \tag{3.2}$$

We want to prove that this problem is uniquely solvable with uniform bounds in certain appropriate norms. In other words, we want study the linear operator L_ε associated to (3.2), namely

$$L_\varepsilon(\eta) = \Delta\eta + p|V|^{p-1}\eta, \tag{3.3}$$

under the previous orthogonality conditions. In order to this goal, we introduce the following L^∞-norms with weight. Let $\omega_i = \left(1 + |x' - \xi_i'|^2\right)^{-\frac{N-2}{2}}$ be, $i = 1, 2$; for a function θ defined in Ω_ε, we consider the norms

$$\|\theta\|_* = \|\, (\omega_1 + \omega_2)^{-\sigma}\, \theta(x')\|_\infty + \|\, (\omega_1 + \omega_2)^{-\sigma-1}\, \nabla\theta(x')\|_\infty$$

where $\sigma = \frac{1}{2}$ if $3 \le N \le 6$, $\sigma = \frac{2}{N-2}$ if $N \ge 7$, and

$$\|\theta\|_{**} = \|\, (\omega_1 + \omega_2)^{-\varsigma}\, \theta(x')\|_\infty,$$

where $\varsigma = \frac{p}{2}$ if $3 \le N \le 6$, $\varsigma = \frac{4}{N-2}$ if $N \ge 7$. These norms are similar to those defined in [9] for $N \ge 7$, but for $3 \le N \le 6$ we have modified them, something apparently necessary in that case, since $p \ge 2$. Now, we study the invertibility of the linear operator L_ε defined in (3.3). Hence, also is important to understand its differentiability in the variables $(\vec{\xi'}, \vec{\Lambda}) \in \mathcal{M}_\delta^\varepsilon$.

Proposition 3.1. *Assume that $(\vec{\xi'}, \vec{\Lambda}) \in \mathcal{M}_\delta^\varepsilon$. Then there exist $\varepsilon_0 > 0$ and $C > 0$, such that for all $0 < \varepsilon < \varepsilon_0$ and for all $h \in C^\alpha(\overline{\Omega}_\varepsilon)$, the problem (3.2) admits an unique solution $\eta \equiv M_\varepsilon(h)$. Moreover, the map $(\vec{\xi'}, \vec{\Lambda}, h) \mapsto \eta \equiv M_\varepsilon(h)$ is of class C^1 and satisfies*

$$\|\eta\|_* \le C\|h\|_{**} \quad \text{and} \quad \|\nabla_{(\vec{\xi'}, \vec{\Lambda})}\, \eta\|_* \le C\|h\|_{**}.$$

The proof of this proposition follows from a slight variation of the arguments in the proof of Propositions 4.1 and 4.2 in [12] with the necessary modifications in [14] so that we omit it. In what follows, C represents a generic positive constant which is independent of ε and of the particular points $(\vec{\xi'}, \vec{\Lambda}) \in \mathcal{M}_\delta^\varepsilon$.

Now, we are ready to begin the finite-dimensional reduction. We want to solve the following nonlinear problem: find a function $\tilde{\eta}$ such that for certain constants c_{ij}, $i = 1, 2$, $j = 1, \ldots, N+1$, one has

$$\begin{cases} \Delta(V+\tilde{\eta}) + |V+\tilde{\eta}|^{p-1}(V+\tilde{\eta}) - \varepsilon^{p+1}\tilde{f} = \sum_{i,j} c_{ij} U_i^{p-1} Z_{ij} & \text{in } \Omega_\varepsilon, \\ \tilde{\eta} = 0 & \text{on } \partial\Omega_\varepsilon, \\ \langle \tilde{\eta}, U_i^{p-1} Z_{ij} \rangle = -\langle \phi, U_i^{p-1} Z_{ij} \rangle & \forall i, j, \end{cases} \tag{3.4}$$

where ϕ solves the problem

$$\begin{cases} -\Delta\phi = \varepsilon^{p+1}\tilde{f} & \text{in } \Omega_\varepsilon, \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{3.5}$$

Note that $V + \tilde{\eta}$ is a solution of (2.1) if the scalars c_{ij} in (3.4) are all zero. Also, we note that the partial differential equation in (3.4) is equivalent in Ω_ε to:

$$\Delta\eta + p|V|^{p-1}\eta = -N_\varepsilon(\eta) - R_\varepsilon + \sum_{i,j} c_{ij} U_i^{p-1},$$

where $\eta = \tilde{\eta} - \phi$,

$$N_\varepsilon(\eta) = |V+\eta-\phi|^{p-1}(V+\eta-\phi)_+ - |V|^{p-1}V - p|V|^{p-1}(\eta-\phi) \tag{3.6}$$

and

$$R_\varepsilon = |V|^{p-1}V - \bar{U}_1^p - \bar{U}_2^p - p|V|^{p-1}\phi. \tag{3.7}$$

A first step to solve (3.4) consists of dealing with the following nonlinear problem: find a function φ such that for certain constants c_{ij}, $i = 1, 2$, $j = 1, \dots, N+1$, solves

$$\begin{cases} \Delta(V+\tilde{\eta}) + |V+\tilde{\eta}|^{p-1}(V+\tilde{\eta})_+ - \varepsilon^{p+1}\tilde{f} = \sum_{i,j} c_{ij} U_i^{p-1} Z_{ij} & \text{in } \Omega_\varepsilon, \\ \varphi = 0 & \text{on } \partial\Omega_\varepsilon, \\ \langle \varphi, U_i^{p-1} Z_{ij} \rangle = 0 & \forall i, j, \end{cases} \tag{3.8}$$

where $\tilde{\eta} = \psi + \varphi - \phi$, with ϕ satisfying (3.5), and the function ψ is chosen as

$$\psi = -M_\varepsilon(R_\varepsilon) \tag{3.9}$$

where M_ε is defined as in Proposition 3.1 and R_ε is given by (3.7). Actually, it is easy to check that for points $(\vec{\xi'}, \vec{\Lambda}) \in \mathcal{M}_\delta^\varepsilon$ one has

$$\|\psi\|_* \le C\varepsilon^2.$$

Now, in (3.8) we rewrite the equation of our interest as

$$\Delta\varphi + p|V|^{p-1}\varphi = -N_\varepsilon(\eta) - (\Delta\psi + p|V|^{p-1}\psi + R_\varepsilon) + \sum_{i,j} c_{ij} U_i^{p-1} Z_{ij}$$

where $\eta = \psi + \varphi$.

Lemma 3.2. *Assume that $(\vec{\xi'}, \vec{\Lambda}) \in \mathcal{M}_\delta^\varepsilon$. Then there exists $C > 0$ such that for all $\varepsilon > 0$ small enough and $\|\varphi\|_* \le \frac{1}{4}$ one has*

$$\|N_\varepsilon(\psi+\varphi)\|_{**} \le \begin{cases} C\big(\|\varphi\|_*^2 + \varepsilon\|\varphi\|_* + \varepsilon^{p+1}\big) & \text{if } 3 \le N \le 6, \\ C\big(\varepsilon^{2(p-2)}\|\varphi\|_*^2 + \varepsilon^{p^2-3p+2}\|\varphi\|_*^p + \varepsilon^{p^2-p+2}\big) & \text{if } N \ge 7. \end{cases}$$

Proof. Note that $\|\phi\|_* \le C\varepsilon^p$ if $3 \le N \le 6$, $\|\phi\|_* \le C\varepsilon^2$ if $N \ge 7$ and $\|\psi\|_* \le C\varepsilon^2$. Since $\|\psi + \varphi\|_* \le \|\psi\|_* + \|\varphi\|_*$, then for $\eta = \psi + \varphi$ we have that $\|\eta\|_* < 1$. Also we note that

$$N_\varepsilon(\eta) = C|V + \bar{t}(\eta - \phi)|^{p-2}(\eta - \phi)^2, \tag{3.10}$$

with $\bar{t} \in]0, 1[$. Hence, if $3 \le N \le 6$ then

$$|(\omega_1 + \omega_2)^{-\frac{p}{2}} N_\varepsilon(\eta)| \le C(\omega_1 + \omega_2)^{\frac{p}{2}-1}\|\eta - \phi\|_*^2 \le C\|\eta - \phi\|_*^2.$$

On the other hand, for $N \geq 7$, if $|\eta| \leq \frac{1}{2}(\omega_1 + \omega_2)$ we use again (3.10) and we obtain

$$|(\omega_1+\omega_2)^{-\frac{4}{N-2}} N_\varepsilon(\eta)| \leq C(\omega_1+\omega_2)^{\frac{6-N}{n-2}} \|\eta-\phi\|_*^2 \leq C\varepsilon^{\frac{6-N}{N-2}} \|\eta-\phi\|_*^2.$$

In another case we obtain directly from (3.6) that

$$|(\omega_1+\omega_2)^{-\frac{4}{N-2}} N_\varepsilon(\eta)| \leq C|(\omega_1+\omega_2)^{-\frac{4}{N-2}} (\eta-\phi)^p| \leq C\varepsilon^{\frac{6-N}{N-2}\cdot\frac{2}{N-2}} \|\eta-\phi\|_*^p.$$

Combining previous estimates the result follows. ∎

Now, we deal with the following problem

$$\begin{cases} \Delta\varphi + pV^{p-1}\varphi = -N_\varepsilon(\eta) + \sum_{i,j} c_{ij} U_i^{p-1} Z_{ij} & \text{in } \Omega_\varepsilon, \\ \varphi = 0 & \text{on } \partial\Omega_\varepsilon, \\ \langle \varphi, U_i^{p-1} Z_{ij} \rangle = 0 & \forall i,j, \end{cases} \tag{3.11}$$

where $\eta = \psi + \varphi$ and ψ is the function defined in (3.9).

Proposition 3.3. *Assume that $(\vec{\xi'}, \vec{\Lambda}) \in \mathcal{M}_\delta^\varepsilon$. Then there exists $C > 0$, such that for all $\varepsilon > 0$ small enough there exists an unique solution $\varphi = \varphi(\vec{\xi'}, \vec{\Lambda})$ to problem (3.11). Moreover, the map $(\vec{\xi'}, \vec{\Lambda}) \mapsto \varphi(\vec{\xi'}, \vec{\Lambda})$ is of class C^1 for the $\|\cdot\|_*$-norm and it satisfies*

$$||\varphi||_* \leq C\varepsilon^2 \quad \textit{and} \quad \|\nabla_{(\vec{\xi'},\vec{\Lambda})}\varphi\|_* \leq C\varepsilon^2.$$

Proof. Let us set

$$\mathcal{F}_r = \{\varphi \in H_0^1(\Omega_\varepsilon) : \|\varphi\|_* \leq r\varepsilon^2\},$$

with $r > 0$ a constant to be fixed later. We define the map $A_\varepsilon : \mathcal{F}_r \to H_0^1(\Omega_\varepsilon)$ as

$$A_\varepsilon(\varphi) = -M_\varepsilon\big(N_\varepsilon(\psi+\varphi)\big)$$

where M_ε is the operator defined in Proposition 3.1. Since $\psi = -M_\varepsilon(R_\varepsilon)$, solving (3.11) is equivalent to finding a fixed point φ for A_ε. From Proposition 3.1 and Lemma 3.2, we deduce that if $\varphi \in \mathcal{F}_r$ and $\varepsilon > 0$ is small enough, then

$$\|A_\varepsilon(\varphi)\|_* \leq r\varepsilon^2$$

for a suitable choice of $r = r(N)$ which we consider fixed from now on. Note that for $\varphi_1, \varphi_2 \in \mathcal{F}_r$ we have from Lemma 3.2

$$\|A_\varepsilon(\varphi_1) - A_\varepsilon(\varphi_2)\|_* \leq C\|N_\varepsilon(\psi+\varphi_1) - N_\varepsilon(\psi+\varphi_2)\|_{**} \leq C\varepsilon^p\|\varphi_1-\varphi_2\|_*,$$

for all $N \geq 3$. It follows that, for $\varepsilon > 0$ small enough, the map A_ε is a contraction $\|\cdot\|_*$ in $\mathcal{F}_r$. Therefore, A_ε has a fixed point in $\mathcal{F}_r$.

Concerning differentiability properties, let us recall that $\eta = \psi + \varphi$ is defined by the relation

$$B(\vec{\xi'}, \vec{\Lambda}, \eta) \equiv \eta + M_\varepsilon\big(N_\varepsilon(\psi+\varphi)\big) = 0.$$

We see that

$$D_\eta B(\vec{\xi'}, \vec{\Lambda}, \eta)[\theta] = \theta + M_\varepsilon\big(\theta\, D_\eta N_\varepsilon(\psi+\varphi)\big) \equiv \theta + \tilde{M}(\theta),$$

and check

$$\|\tilde{M}(\theta)\|_* \leq C\varepsilon\|\theta\|_*.$$

This implies that for ε small, the linear operator $D_\eta B(\vec{\xi'},\vec{\Lambda},\eta)$ is invertible in the space of the continuous functions in Ω_ε with bounded $\|\cdot\|_*$-norm, with uniformly bounded inverse depending continuously on its parameters.

Now, let us consider the differentiability with respect to the $\vec{\xi'}$ variable and by simplicity we write $\frac{\partial}{\partial\xi'_{ij}} = \partial_{\xi'_{ij}}$. Then

$$\partial_{\xi'_{ij}} B(\vec{\xi'},\vec{\Lambda},\eta) = \partial_{\xi'_{ij}} M_\varepsilon(N_\varepsilon(\psi+\varphi)) + M_\varepsilon(\partial_{\xi'_{ij}} N_\varepsilon(\psi+\varphi)) + M_\varepsilon(D_\eta N_\varepsilon(\psi+\varphi)\partial_{\xi'_{ij}}\psi).$$

It is clear that all expressions which define to $\partial_{\xi'_{ij}} B(\vec{\xi'},\vec{\Lambda},\eta)$ depend continuously on their parameters. Applying the implicit function theorem we obtain that $\varphi(\vec{\xi'},\vec{\Lambda})$ is a C^1-function in L^∞_*. Besides, we get

$$\partial_{\xi'_{ij}}\varphi = -\big(D_\eta B(\vec{\xi'},\vec{\Lambda},\eta)\big)^{-1}\big(\partial_{\xi'_{ij}} B(\vec{\xi'},\vec{\Lambda},\eta)\big),$$

and using the first part of this proposition, the estimates in the previous lemmas, Proposition 3.1 and the fact that $(\vec{\xi},\vec{\Lambda}) \in \mathcal{M}^\varepsilon_\delta$, we conclude

$$\|\partial_{\xi'_{ij}}\varphi\|_* \le C\big(\|N_\varepsilon(\psi+\varphi)\|_{**} + \|\partial_{\xi'_{ij}} N_\varepsilon(\psi+\varphi)\|_{**} + \|D_\eta N_\varepsilon(\psi+\varphi)\partial_{\xi'_{ij}}\psi\|_{**} \le C\varepsilon^2$$

Similarly, we can analyze differentiability of B with respect to $\vec{\Lambda}$. This finishes the proof. ■

4. The reduced functional

Now we are ready to solve the full problem. Let us consider $(\vec{\xi'},\vec{\Lambda}) \in \mathcal{M}^\varepsilon_\delta$ whit $\mathcal{M}^\varepsilon_\delta$ defined by (3.1). All estimates obtained below will be uniform on these points. Let $\varphi = \varphi(\vec{\xi'},\vec{\Lambda})$ be the unique solution, given by Proposition 3.3, of the problem (3.8) with $\tilde{\eta} = \psi + \varphi - \phi$, where φ solves (3.9) and ϕ solves (3.5). Note that if $\vec{\xi} = \varepsilon^{\frac{2}{N-2}}\vec{\xi'} \in \Omega^2$ and $\vec{\lambda} = (\lambda_1,\lambda_2) \in \mathbb{R}^2_+$ so that $c_{ij} = 0$ for all i,j, then a solution of (1.1) is

$$u(x) = -\varepsilon^{-1} v(\varepsilon^{-\frac{2}{N-2}}x), \qquad x \in \Omega,$$

where $v = V + \psi + \varphi(\vec{\xi'},\vec{\Lambda}) - \phi$. Hence, u will be a critical point of

$$I_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1} - \varepsilon\int_\Omega fu.$$

while v will be one of J_ε given by (2.5). Then it is convenient to consider the following functions defined in Ω:

$$\hat{U}_i(x) = \varepsilon^{-1}U_i(\varepsilon^{-\frac{2}{N-2}}x) = U_{\lambda^\varepsilon_i,\xi_i}(x), \qquad \hat{\psi}(x) = \varepsilon^{-1}\psi(\varepsilon^{-\frac{2}{N-2}}x),$$

$$\hat{\varphi}(\vec{\xi},\vec{\Lambda})(x) = \varepsilon^{-1}\varphi(\vec{\xi'},\vec{\Lambda})(\varepsilon^{-\frac{2}{N-2}}x) \quad \text{and} \quad \hat{\phi}(x) = \varepsilon^{-1}\phi(\varepsilon^{-\frac{2}{N-2}}x).$$

Note that $\hat{U}_i = U_{\lambda_{i\varepsilon},\xi_i}$ where $\lambda_{i\varepsilon} = (c_N\Lambda_i^2\varepsilon)^{\frac{2}{N-2}} \in \mathbb{R}_+$ and $\vec{\xi} = \varepsilon^{\frac{2}{N-2}}\vec{\xi'}$, with $(\vec{\xi},\vec{\Lambda}) \in \mathcal{M}_\delta$ defined by (2.7). Now, let us put $\hat{U} = \hat{U}_1 + \hat{U}_2$. Consider now the functional

$$\mathcal{I}(\vec{\xi},\vec{\Lambda}) \equiv I_\varepsilon\big(\hat{U} + \hat{\psi} + \hat{\varphi}(\vec{\xi},\vec{\Lambda}) - \hat{\phi}\big). \tag{4.1}$$

It is easy to check that

$$\mathcal{I}(\vec{\xi},\vec{\lambda}) = J_\varepsilon\big(V + \psi + \varphi(\vec{\xi'},\vec{\Lambda}) - \phi\big).$$

Then, putting $\tilde{\eta} = \psi + \varphi(\vec{\xi}', \vec{\Lambda}) - \phi$, one shows that $DJ_\varepsilon(V + \tilde{\eta})\,[\vartheta] = 0$ for all $\vartheta \in H_\varepsilon$, where $H_\varepsilon = \{\vartheta \in H_0^1(\Omega_\varepsilon) : \langle \vartheta, V_i^{p-1} Z_{ij}\rangle = 0 \ \ \forall i, j\}$. Also one has

$$\frac{\partial V}{\partial \xi'_{lk}} = Z_{lk} + o(1) \quad \forall l, k; \qquad \frac{\partial V}{\partial \Lambda_{l(N+1)}} = Z_{l(N+1)} + o(1) \quad \forall l,$$

with $o(1) \to 0$ in the $\|\cdot\|_*$-norm as $\varepsilon \to 0$. Then from Proposition 3.3 we obtain the following basic result:

Lemma 4.1. *The function $u = \hat{U} + \hat{\psi} + \hat{\varphi}(\vec{\xi}, \vec{\Lambda}) - \hat{\phi}$ is a solution of the problem (1.1) if only if $(\vec{\xi}, \vec{\Lambda})$ is a critical point of $\mathcal{I}$.*

Next step is then to give an asymptotic estimate for $\mathcal{I}(\vec{\xi}, \vec{\Lambda})$. Put

$$\sigma_f = \int_\Omega f(x) w(x)\, dx, \tag{4.2}$$

where w is the solution of (2.6). Then

Proposition 4.2. *The following expansion holds:*

$$\mathcal{I}(\vec{\xi}, \vec{\Lambda}) = 2C_N + \varepsilon^2 \big\{\Phi(\vec{\xi}, \vec{\Lambda}) + \sigma_f\big\} + o(\varepsilon^2)\theta(\vec{\xi}, \vec{\Lambda}) \tag{4.3}$$

uniformly in the C^1-sense with respect to $(\vec{\xi}, \vec{\Lambda}) \in \mathcal{M}_\delta$, where θ is a bounded uniformly function independently of $\varepsilon > 0$. Here C_N is the constant given by (2.8) and Φ is the function given by (2.9).

Proof. The first step to achieve our goal is to prove that

$$\mathcal{I}(\vec{\xi}, \vec{\Lambda}) - I_\varepsilon(\hat{V} + \hat{\psi} - \hat{\phi}) = o(\varepsilon^2) \tag{4.4}$$

and

$$\nabla_{(\vec{\xi}, \vec{\Lambda})}\big(\mathcal{I}(\vec{\xi}, \vec{\Lambda}) - I_\varepsilon(\hat{V} + \hat{\psi} - \hat{\phi})\big) = o(\varepsilon^2). \tag{4.5}$$

Let us set $\vartheta = V + \psi - \phi$ and notice that

$$\begin{aligned} \mathcal{I}(\vec{\xi}, \vec{\Lambda}) - I_\varepsilon(\hat{V} + \hat{\psi} - \hat{\phi}) \ &= \ -\int_0^1 t\left(\int_{\Omega_\varepsilon} N_\varepsilon(\psi + \varphi)\varphi\right) dt \\ &\quad + \int_0^1 t\left(\int_{\Omega_\varepsilon} p\big(|V|^{p-1} - |\vartheta + t\varphi|^{p-1}\big)\varphi^2\right) dt. \end{aligned}$$

Now, differentiating with respect to the $\vec{\xi}$ variable, we obtain

$$\begin{aligned} D_{\vec{\xi}}\big(\mathcal{I}(\vec{\xi}, \vec{\Lambda}) - I_\varepsilon(\hat{\vartheta})\big) &= -\varepsilon^{-\frac{2}{N-2}} \int_0^1 t \int_{\Omega_\varepsilon} p \nabla_{\vec{\xi}'}\big[|\vartheta + t\varphi|^{p-1}\varphi^2 - |V|^{p-1}\varphi^2\big]\, dt \\ &\quad - \varepsilon^{-\frac{2}{N-2}} \int_{\Omega_\varepsilon} \nabla_{\vec{\xi}'}\big(N_\varepsilon(\psi + \varphi)\varphi\big). \end{aligned}$$

Keeping in mind that $\|N_\varepsilon(\psi + \varphi)\|_* + \|\varphi\|_* + \|\psi\|_* + \|\nabla_{\xi'_i}\varphi\|_* + \|\nabla_{\xi'_1}\psi\|_* \le O(\varepsilon^2)$, we get that (4.4) and (4.5) hold true.

A second step is to prove that

$$I_\varepsilon(\hat{V} + \hat{\psi} - \hat{\phi}) - I_\varepsilon(\hat{V} - \hat{\phi}) = o(\varepsilon^2) \tag{4.6}$$

and

$$\nabla_{(\vec{\xi}, \vec{\Lambda})}\big(I_\varepsilon(\hat{V} + \hat{\psi} - \hat{\phi}) - I_\varepsilon(\hat{V} - \hat{\phi})\big) = o(\varepsilon^2). \tag{4.7}$$

Put $\eta = V - \phi$ and, by the fundamental calculus theorem, note that

$$I_\varepsilon(\hat{\eta}+\hat{\psi}) - I_\varepsilon(\hat{\eta}) = \int_0^1 (1-t)\Big(\int_{\Omega_\varepsilon} p|\eta + t\psi|^{p-1}\psi^2 - \int_{\Omega_\varepsilon}|\nabla\psi|^2\Big)\,dt + \int_{\Omega_\varepsilon}\big(|V|^p - |\eta|^p - p|V|^{p-1}\phi\big)\psi + \int_{\Omega_\varepsilon} R_\varepsilon\psi. \tag{4.8}$$

Now, differentiating with respect to $\vec{\xi}$ variables, we obtain

$$\begin{aligned} D_{\vec{\xi}}\big(I_\varepsilon(\hat{\eta}+\hat{\psi}) - I_\varepsilon(\hat{\eta})\big) &= \varepsilon^{-\frac{2}{N-2}}\int_0^1(1-t)\int_{\Omega_\varepsilon}\nabla_{\vec{\xi}'}\big(p|\eta+t\psi|^{p-1}\psi^2 - |\nabla\psi|^2\big)\,dt \\ &\quad + \varepsilon^{-\frac{2}{N-2}}\int_{\Omega_\varepsilon}\nabla_{\vec{\xi}'}\big(|V|^p - |\eta|^p - p|V|^{p-1}\phi\big)\psi \\ &\quad + \varepsilon^{-\frac{2}{N-2}}\int_{\Omega_\varepsilon}\big(|V|^p - |\eta|^p - p|V|^{p-1}\phi\big)\nabla_{\vec{\xi}'}\psi \\ &\quad + \varepsilon^{-\frac{2}{N-2}}\int_{\Omega_\varepsilon}\nabla_{\vec{\xi}'}R_\varepsilon\psi + \varepsilon^{-\frac{2}{N-2}}\int_{\Omega_\varepsilon}R_\varepsilon\nabla_{\vec{\xi}'}\psi. \end{aligned}$$

Since $\|R_\varepsilon\|_{**} + \|\nabla_{\xi_i'}R_\varepsilon\|_{**} + \|\phi\|_\infty + \|\psi\|_* + \|\nabla_{\xi_i'}\psi\|_* \le O(\varepsilon^2)$ and $\|\phi\|_* \le O(\varepsilon^p)$ if $3 \le N \le 6$, $\|\phi\|_* \le O(\varepsilon^2)$ if $N \ge 7$, one has that (4.6) and (4.7) hold.

Finally, only we need hold the following two estimates

$$I_\varepsilon(\hat{V} - \hat{\phi}) - I_\varepsilon(\hat{V}) = \varepsilon^2\sigma_f + o(\varepsilon^2), \tag{4.9}$$

where σ_f is given by (4.2), and

$$D_{(\vec{\xi},\vec{\Lambda})}\big(I_\varepsilon(\hat{V} - \hat{\phi}) - I_\varepsilon(\hat{V})\big) = o(\varepsilon^2). \tag{4.10}$$

Now, we have that

$$I_\varepsilon(\hat{V} - \hat{\phi}) - I_\varepsilon(\hat{V}) = \int_0^1 \Big(\int_{\Omega_\varepsilon}|\nabla\phi|^2 - \int_{\Omega_\varepsilon} p|V - t\phi|^{p-1}\phi^2\Big)dt + \int_{\Omega_\varepsilon}\big(\bar{U}_1^p + \bar{U}_2^p - |V - t\phi|^p\big)\phi. \tag{4.11}$$

Note that

$$\int_0^1 t\int_{\Omega_\varepsilon}|\nabla\phi|^2\,dt = \int_{\Omega_\varepsilon}|\nabla\phi|^2 = \varepsilon^{p+1}\int_{\Omega_\varepsilon}\tilde{f}\phi = \varepsilon^2\int_\Omega fw = \varepsilon^2\sigma_f,$$

and since $\|\phi\|_\infty \le O(\varepsilon^{p+1})$, we have that

$$\Big|\int_{\Omega_\varepsilon} p|V - t\phi|^{p-1}\phi^2\Big| \le C\varepsilon^4\int_{\Omega_\varepsilon}(\omega_1 + \omega_2)^{p-1} \le o(\varepsilon^2).$$

On the other hand, it is not difficult to check that

$$\Big|\int_{\Omega_\varepsilon}\Big(\sum_{i=1}^2 \bar{U}_i^p - |V - t\phi|^p\Big)\phi\Big| = \Big|\int_{\Omega_\varepsilon}R_\varepsilon\phi + \int_{\Omega_\varepsilon}\big(|V|^p - |V - t\phi|^p - p|V|^{p-1}\phi\big)\phi\Big| \le o(\varepsilon^2).$$

The above estimates hold (4.9). Now, from (4.11) we get

$$\begin{aligned} D_{\vec{\xi}}\big(I_\varepsilon(\hat{V} - \hat{\phi}) - I_\varepsilon(\hat{V})\big) &= \varepsilon^{-\frac{2}{N-2}}\int_0^1 t\int_{\Omega_\varepsilon} p|V - t\phi|^{p-2}\nabla_{\vec{\xi}'}V\phi^2\,dt \\ &\quad + \varepsilon^{-\frac{2}{N-2}}\int_{\Omega_\varepsilon}\nabla_{\vec{\xi}'}\big(\bar{U}_1^p + \bar{U}_2^p - |V - t\phi|^p\big)\phi, \end{aligned}$$

but since $\|\phi\|_\infty \leq O(\varepsilon^{p+1})$, it is easy to check that (4.10) is truth. Similarly we hold results for the differentiability with respect to $\vec{\Lambda}$. ■

Remark 4.3. Lemma 2.1 and previous proposition yield

$$\nabla_{(\vec{\xi},\vec{\Lambda})}\mathcal{I}(\vec{\xi},\vec{\Lambda}) = \varepsilon^2\nabla_{(\vec{\xi},\vec{\Lambda})}\Phi(\vec{\xi},\vec{\Lambda}) + o(\varepsilon^2)\nabla_{(\vec{\xi},\vec{\Lambda})}\theta(\vec{\xi},\vec{\Lambda}), \tag{4.12}$$

uniformly with respect to $(\vec{\xi},\vec{\Lambda}) \in \mathcal{M}_\delta$, where θ and $\nabla_{(\vec{\xi},\vec{\Lambda})}\theta$ are bounded uniformly functions, independently of all $\varepsilon > 0$ small. □

5. An auxiliary function on the exterior domain

In this section we consider the domain Ω defined in (1.2) with $P = 0$, $\mu > 0$ small and fixed and we assume that $f \in C^{0,\gamma}(\overline{\Omega})$, for some $0 < \gamma < 1$, with $\min_{x\in\Omega} f(x) = \alpha > 0$. Let w be the unique solution in $C^{2,\gamma}(\overline{\Omega})$ of problem (2.6), then it is easy to check that $w_\mu(x) = \mu^{-2}w(\mu x)$ is the unique $C^{2,\gamma}(\overline{\mu^{-1}\Omega})$ solution of the problem

$$\begin{cases} -\Delta w_\mu = \hat{f} & \text{in} \quad \mu^{-1}\Omega \\ w_\mu = 0 & \text{on} \quad \partial(\mu^{-1}\Omega), \end{cases}$$

where $\hat{f}(x) = f(\mu x)$ for $x \in (\mu^{-1}\Omega)$.

Now, we consider the exterior domain

$$E = \mathbb{R}^N \setminus \overline{B(0,1)}$$

and we denote by G_E and H_E, respectively, the Green's function on E and its regular part. By convenience, in the set:

$$\mathbf{V} = \left\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : G_E(x,y) - H_E^{\frac{1}{2}}(x,x)H_E^{\frac{1}{2}}(y,y) > 0\right\} \cap (\mu^{-1}\Omega)$$

we define the function:

$$\varPhi_E(x,y) = \frac{1}{2}\left\{\frac{H_E(x,x)w_\mu^2(y) + 2G_E(x,y)w_\mu(x)w_\mu(y) + H_E(y,y)w_\mu^2(x)}{G_E^2(x,y) - H_E(x,x)H_E(y,y)}\right\}.$$

Then, if x and y are variable vectors whose magnitudes remain constant and we differentiate $\varPhi_E$ with respect to the angle θ formed between them, we obtain

$$\frac{\partial}{\partial\theta}\varPhi_E(x,y) = F(x,y,\theta)\sin\theta$$

for $0 < \theta < \pi$. Since $F(x,y,\theta) > 0$ for all $\theta \in]0,\pi[$, $(x,y) \in \mathbf{V}$, we have that for given magnitudes $|x|$ and $|y|$, $\varPhi_E$ maximizes its value when $\theta = \pi$, is to say when x and y have opposite directions. In the rest of this section we assume that this is the situation.

5.1. **A first step to the auxiliary function: a radial case.**

In this subsection we consider a fixed constant $T > 0$ and the domain

$$\Omega := \mathcal{A}_\mu = \{x \in \mathbb{R}^N : 1 < |x| < \mu^{-1}\} \quad \text{and} \quad f \equiv 1.$$

We write $R := R(\mu, T) = \mu^{-1}T$ so that $w_\mu \in C^{2,\gamma}\left(\overline{\mathcal{A}_\mu}\right)$ is defined by

$$w_\mu(x) := W_R(x) = \frac{1}{2N}\left\{\frac{R^2-1}{R^{2-N}-1}|x|^{2-N} - |x|^2 + R^{2-N}\frac{1-R^N}{R^{2-N}-1}\right\}.$$

From the maximum principle we have that W_R is strictly positive in $\mathcal{A}_\mu$. Besides, it achieves its maximum value in

$$x_\mu^* \in \mathbb{R}^N \text{ such that } |x_\mu^*| = R_\mu^* = \left(\frac{(N-2)R^{N-2}(R^2-1)}{2(R^{N-2}-1)} \right)^{\frac{1}{N}}.$$

Note that $R_\mu^* \to +\infty$ as $\mu \to 0$. Now we consider an unitary vector $\mathbf{e}$ and we put $x = s\mathbf{e}$, $y = -t\mathbf{e}$ with $s, t > 1$. Then

$$\begin{aligned} 2\beta_N \Phi_E(x,y) &:= 2\beta_N \Phi_R(x,y) \\ &= 2\beta_N \tilde{\Phi}_R(s,t) \\ &= \frac{\frac{\tilde{W}_R^2(t)}{(s^2-1)^{N-2}} + 2\left\{ \frac{1}{(s+t)^{N-2}} - \frac{1}{(st+1)^{N-2}} \right\} \tilde{W}_R^2(s)\tilde{W}_R^2(t) + \frac{\tilde{W}_R^2(s)}{(t^2-1)^{N-2}}}{\left(\frac{1}{(s+t)^{N-2}} - \frac{1}{(st+1)^{N-2}} \right)^2 - \frac{1}{[(s^2-1)(t^2-1)]^{N-2}}}, \end{aligned}$$

where $\tilde{W}_R(r) = W_R(r\mathbf{e})$, for $1 < r < R$.

Remark 5.1. We define in $]1,+\infty[\times]1,+\infty[$ the following function:

$$\tilde{\Psi}(s,t) = \frac{1}{(s+t)^{N-2}} - \frac{1}{(st+1)^{N-2}} - \frac{1}{[(s^2-1)(t^2-1)]^{\frac{N-2}{2}}}.$$

From (5.1), it is easy to check that we can choose μ_0 small enough such that for all $0 < \mu < \mu_0$ there are $1 < k_* < K < R_{\mu_0}^*$ independent of μ, verifying $\tilde{\Psi}(k_*, k_*) = 0$, $\tilde{\Psi}(K,K) = \max_{(x,y)\in E} \tilde{\Psi}(|x|,|y|)$. Moreover, k_* is the unique solution in $]1,+\infty[$ of the equation

$$\frac{2^{2-N}}{s^{N-2}} = \frac{(s^2+1)^{N-2} + (s^2-1)^{N-2}}{(s^4-1)^{N-2}}$$

and K is the only one solution in $]1,+\infty[$ of

$$\frac{2^{1-N}}{s^N} = \frac{(s^2+1)^{N-1} + (s^2-1)^{N-1}}{(s^4-1)^{N-1}}. \square$$

Now, it is not difficult to prove

Lemma 5.2. *The function $\tilde{\Phi}_R$ achieves only one minimum value at a critical point of the form $(\rho_R, \rho_R) \in]k_*, K[^2$.*

5.2. **General case.** Let $\mathcal{D}$ the smooth and bounded domain in $\mathbb{R}^N$, $N \geq 3$, which define to Ω in (1.2). In this subsection we consider the values m, M as follows: m is the radius of the biggest ball centered at the origin contained in $\mathcal{D}$ and M is the radius of the smallest ball centered at the origin containing to $\mathcal{D}$. Let w be the unique solution $C^{2,\gamma}(\overline{\Omega})$ of the problem (2.6). By the maximum principle, we check that

$$z_m(x) \leq w(x) \leq z_M(x), \quad \forall\, \mu < |x| < m,$$

where $z_m(x) = \alpha\mu^2 W_{R_1}(\mu^{-1}x)$ and $z_M(x) = \beta\mu^2 W_{R_2}(\mu^{-1}x)$, with $R_1 = \mu^{-1}m$ and $R_2 = \mu^{-1}M$. Hence,

$$\Phi_{R_1}(\mu^{-1}x, \mu^{-1}y) \leq \Phi_E(\mu^{-1}x, \mu^{-1}y) \leq \Phi_{R_2}(\mu^{-1}x, \mu^{-1}y), \quad \forall\, \mu < |x|, |y| < m.$$

Since the function $\tilde{\Psi}(s,s)$ defined in Remark 2 is decreasing in its diagonal for values of s greater that K and goes to 0, then is not difficult to show that the system

$$\frac{\tilde{\Phi}_{R_1}(s,s)}{\tilde{\Phi}_{R_2}(K,K)} \geq 1 \quad \text{and} \quad s \geq K$$

posses solution, we say k^*, when we have chosen $\mu > 0$ sufficiently small but fixed. Indeed, if we put $\beta = \max_{x\in\Omega} f(x)$ and $(\alpha m^2 - \beta M^2)K^{N-2} + \beta M^2 \neq 0$ then we can chose in the limit for μ

$$k^* = \max\left\{K, \left\{\Big(\frac{\alpha m^2 K^{N-2}}{(\alpha m^2 - \beta M^2)\,K^{N-2} + \beta M^2}\Big)_+\right\}^{\frac{1}{N-2}}\right\}.$$

If $(\alpha m^2 - \beta M^2)K^{N-2} + \beta M^2 = 0$, we change K by a value a few greater that K in the definition of k^*. Then the following lemma is obtained

Lemma 5.3. *The function $\Phi_E(x,y)$ achieves a relative minimum value in a critical point (x_μ, y_μ) with x_μ and y_μ having opposite directions, and $(|x_\mu|, |y_\mu|) \in]k_*, k^*[^2$. Moreover, $|x_\mu|$ and $|y_\mu|$ belong to a compact region fully contained in $]k_*, k^*[^2$, which is independent of all $\mu > 0$ small enough.*

Let

$$\mathbf{Q} = \{(x,y) \in \mathbf{V} \times \mathbf{V} : k_* < |x|, |y| < k^*\},$$

We define the following value

$$c_\mu = \Phi_E(x_\mu, y_\mu) = \min_{(x,y)\in\mathbf{Q}} \Phi_E(x,y). \tag{5.1}$$

Let $\delta_\mu > 0$ a suitable small value such that the level set

$$\{(x,y) \in \mathbf{Q} : \Phi_E(x,y) = \delta_\mu\}$$

is a closed curve and that $\nabla\Phi_E(x,y)$ does not vanish on it. Let us set

$$\Upsilon_\mu = \{(x,y) \in \mathbf{Q} : \Phi_E(x,y) < \delta_\mu\}. \tag{5.2}$$

Thus, on this region we have that $\Phi_E(x,y) < \delta_\mu$ and if $(x,y) \in \partial\Upsilon_\mu$ then one of the following two situations happen: either there is a tangential direction τ to $\partial\Upsilon_\mu$ such that $\nabla\Phi_E(x,y)\cdot\tau \neq 0$; or x and y lie in opposite directions, $\Phi_E(x,y) = \delta_\mu$ and $\nabla\Phi_E(x,y) \neq 0$, being points orthogonally outwards to Υ_μ. Moreover, for $\mu_0 > 0$ small enough fixed

$$\Upsilon_{\hat{\mu}} \subset\subset \Upsilon_\mu \subset\subset \mathbf{Q} \quad \text{for all} \quad 0 < \hat{\mu} < \mu < \mu_0. \tag{5.3}$$

Let us consider now the exterior domain

$$E_\mu = \mathbb{R}^N \setminus \overline{B(0,\mu)}.$$

and we denote by G_μ and H_μ, respectively, the Green's function on E_μ and its regular part, then $G_\mu(x,y) = \mu^{2-N} G_E(\mu^{-1}x, \mu^{-1}y)$ and $H_\mu(x,y) = \mu^{2-N} H_E(\mu^{-1}x, \mu^{-1}y)$. In particular, if we put

$$\Sigma_\Omega^\mu = \mu\Upsilon_\mu, \tag{5.4}$$

with Υ_μ defined by (5.2), then Σ_Ω^μ corresponds precisely to the set where $\Phi_E(\mu^{-1}x, \mu^{-1}y) < \delta_\mu$, with δ_μ defined by (5.2). Moreover, since

$$G(x,y) = G_\mu(x,y) + O(1) \quad \forall (x,y) \in \mu\mathbf{Q},$$

where the quantity $O(1)$ is bounded independently of all small μ, in the C^1-sense, and the same is true for the function H, we have that the function

$$\Phi(x,y) = \frac{1}{2}\left\{\frac{H(x,x)w^2(y) + 2G(x,y)w(x)w(y) + H(y,y)w^2(x)}{G^2(x,y) - H(x,x)H(y,y)}\right\} \tag{5.5}$$

satisfies in the region $\mu\mathbf{Q}$ the following relation

$$\Phi(x,y) = \mu^{N+2}\Phi_E(\mu^{-1}x, \mu^{-1}y) + o(1) \tag{5.6}$$

where the quantity $o(1)$ is bounded independently of all small number $\mu > 0$ in the C^1-sense. Besides, $o(1) \to 0$ as $\mu \to 0$.

6. The min-max and the proof of the main result

In this section $\mu > 0$ is a fixed small enough number and Ω is the domain given in (1.2) with $P = 0$. According to the results previously obtained, (4.1) and (4.12), our problem reduces to that of finding a critical point for

$$\Phi(\vec{\xi}, \vec{\Lambda}) = \frac{1}{2}\left\{\sum_{i=1}^{2}\Lambda_i^2 H(\xi_i,\xi_i) - 2\Lambda_1\Lambda_2 G(\xi_1,\xi_2)\right\} + \sum_{i=1}^{2}\Lambda_i w(\xi_i), \tag{6.1}$$

where $\vec{\xi} = (\xi_1,\xi_2) \in \Omega^2$ and $\vec{\Lambda} = (\Lambda_1,\Lambda_2) \in \mathbb{R}^2_+$. Here we consider the function Φ defined over the class $\Sigma^\mu_\Omega \times \mathbb{R}^2_+$, where Σ^μ_Ω is defined by (5.4). Indeed Φ has some singularities on this class which we can avoid by replacing the term $G(\xi_1,\xi_2)$ in (6.1) by

$$G_{|M}(\xi_1,\xi_2) = \begin{cases} G(\xi_1,\xi_2) & \text{if } G(\xi_1,\xi_2) \le M, \\ M & \text{if } G(\xi_1,\xi_2) > M, \end{cases} \tag{6.2}$$

where M is a big number. Hence, we can work with the functional modified, which by simplicity we still call Φ.

For every $\vec{\xi} \in \Sigma^\mu_\Omega$ we choose $d(\vec{\xi}) = \big(d_1(\vec{\xi}), d_2(\vec{\xi})\big) \in \mathbb{R}^2$ being a vector which defines a negative direction of the associated quadratic form with Φ. Such direction exists since $G^2(x,y) - H(x,x)H(y,y) > 0$ over Σ^μ_Ω. More precisely, for fixed $\vec{\xi_0} \in \Sigma^\mu_\Omega$, the function

$$\Phi(\vec{\xi_0}, \vec{d}) = \frac{1}{2}\left\{\sum_{i=1}^{2} d_i^2 H(\xi_{0,i},\xi_{0,i}) - 2d_1d_2 G(\xi_{0,1},\xi_{0,2})\right\} + \sum_{i=1}^{2} d_i w(\xi_{0,i}),$$

regarded as a function of $\vec{d} = (d_1,d_2)$ only, with $d_1, d_2 > 0$, has a unique critical point $\bar{\mathbf{d}}(\vec{\xi_0}) = \big(\bar{d}_1(\vec{\xi_0}), \bar{d}_2(\vec{\xi_0})\big)$ given by

$$\bar{d}_i(\vec{\xi_0}) = \frac{H(\xi_{0,j},\xi_{0,j})w(\xi_{0,i}) + G(\xi_{0,i},\xi_{0,2})w(\xi_{0,j})}{G^2(\xi_{0,i},\xi_{0,j}) - H(\xi_{0,i},\xi_{0,i})H(\xi_{0,j},\xi_{0,j})}, \quad i,j = 1,2,\ i \neq j.$$

In particular,

$$\Phi\big(\vec{\xi_0}, \bar{\mathbf{d}}(\vec{\xi_0})\big) = \Phi\big(\vec{\xi_0}\big) \tag{6.3}$$

where Φ_Ω is the function given by (5.5). Then we simply choose $d(\vec{\xi}) = \bar{\mathbf{d}}(\vec{\xi})$. Let x_μ and y_μ the points given by (5.1). From now on we consider $\hat{\rho}_\mu = |x_\mu|$ and $\bar{\rho}_\mu = |y_\mu|$. Put

$$\mathbf{S} = \{(x,y) \in \mathbf{Q}^2 : (|x|,|y|) = (\mu\hat{\rho}_\mu, \mu\bar{\rho}_\mu)\}.$$

Let $\mathcal{K}$ be the class of all continuous functions

$$\kappa : \mathbf{S} \times I_0 \times [0,1] \to \Sigma_\Omega^\mu \times \mathbb{R}_+^2$$

such that
1) $\kappa(\vec{\xi}, \sigma_0, t) = \big(\vec{\xi}, \sigma_0 d(\vec{\xi})\big)$ and $\kappa(\vec{\xi}, \sigma_0^{-1}, t) = \big(\vec{\xi}, \sigma_0^{-1} d(\vec{\xi})\big)$ for all $\vec{\xi} \in \mathbf{S}$, $t \in [0,1]$.
2) $\kappa(\vec{\xi}, \sigma, 0) = \big(\vec{\xi}, \sigma d(\vec{\xi})\big)$ for all $(\vec{\xi}, \sigma) \in \mathbf{S} \times I_0$, where $I_0 = [\sigma_0, \sigma_0^{-1}]$, being σ_0 a small number to be chosen later.

Then we define the min-max value as

$$c(\Omega) = \inf_{\kappa \in \mathcal{K}} \sup_{(\vec{\xi},\sigma) \in \mathbf{S} \times I_0} \Phi\big(\kappa(\vec{\xi}, \sigma, 1)\big). \tag{6.4}$$

In what follows we will prove that $c(\Omega)$ is a critical value of Φ.

Lemma 6.1. *For all sufficiently small $\mu > 0$, the following estimate holds:*

$$c(\Omega) \leq \mu^{N+2} c_\mu + o(1)$$

where $o(1) \to 0$ as $\mu \to 0$, and c_μ is the value defined in (5.1).

Proof. For all $t \in [0,1]$, we consider the test path defined as $\kappa(\vec{\xi}, \sigma, t) = \big(\vec{\xi}, \sigma d(\vec{\xi})\big)$. Maximizing $\Phi\big(\vec{\xi}, \sigma d(\vec{\xi})\big)$ in the variable σ, we note that this maximum value is attained at $\sigma = 1$, because our choice of the vector $d(\vec{\xi})$. Hence, from (6.3), we have that

$$\max_{\sigma \in I_0} \Phi\big(\vec{\xi}, \sigma d(\vec{\xi})\big) = \Phi\big(\vec{\xi}, d(\vec{\xi})\big).$$

On the other hand, by definition of $\mathbf{S}$, we see that

$$\mathit{\Phi}_E(\mu^{-1}\xi_1, \mu^{-1}\xi_2) = c_\mu.$$

Then the conclusion is immediate from (5.6) and the definition of $c(\Omega)$. ∎

In order to prove that $c(\Omega)$ is indeed a critical point of Φ we need an intersection lemma. The idea behind this result is the topological continuation of the set of solution of an equation (see [15]). For every $(\vec{\xi}, \sigma, t) \in \mathbf{S} \times I_0 \times [0,1]$ we denote $\kappa(\vec{\xi}, \sigma, t) = \big(\tilde{\xi}(\xi, \sigma, t), \tilde{\Lambda}(\vec{\xi}, \sigma, t)\big) \in \Sigma_\Omega^\mu \times \mathbb{R}_+^2$, with $\tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2)$, $\tilde{\Lambda} = (\tilde{\Lambda}_1, \tilde{\Lambda}_2)$ and we define the set

$$\mathbf{M} = \{(\vec{\xi}, \sigma) \in \mathbf{S} \times I_0 : \tilde{\Lambda}_1(\vec{\xi}, \sigma, 1) \cdot \tilde{\Lambda}_2(\vec{\xi}, \sigma, 1) = 1\}.$$

The following lemma has been probed by Del Pino, Felmer and Musso in Lema 6.2 of [13], therefore here the proof is omitted.

Lemma 6.2. *For every open neighborhood W of $\mathbf{M}$ in $\mathbf{S} \times I_0$, the projection $g : W \to \mathbf{S}$ induces a monomorphism in cohomology, that is*

$$g^* : H^*(\mathbf{S}) \to H^*(W)$$

is injective.

Proposition 6.3. *There is a constant $A > 0$ such that*

$$\sup_{(\vec{\xi},\sigma) \in \mathbf{S} \times I_0} \Phi\big(\kappa(\vec{\xi}, \sigma, 1)\big) \geq -A, \quad \forall \kappa \in \mathcal{K}.$$

Proof. Note that $\vec{\xi} \in \Sigma_\Omega^\mu$ implies that $\xi_i \in B(0,\mu k^*)\setminus B(0,\mu k_*)$, for $i=1,2$, with $\hat{\rho}_\mu, \bar{\rho}_\mu \in]k_*, k^*[$ for any μ sufficiently small. Thus, we can find a number $\delta_0 > 0$ such that if $|\xi_1 - \xi_2| < \delta_0$, then $\xi_1 \cdot \xi_2 > 0$. Let $A_0 > 0$ be such that $G(x,y) \geq A_0$ implies that $|x-y| < \delta_0$.

We argue by contradiction. Let us assume that for certain $\kappa \in \mathcal{K}$

$$\Phi\big(\kappa(\vec{\xi},\sigma,1)\big) \leq -A_0 \quad \forall(\vec{\xi},\sigma) \in \mathbf{S} \times I_0.$$

This implies that for all $(\vec{\xi},\sigma) \in \mathbf{M}$, $(\tilde{\xi},\tilde{\sigma}) = \big(\tilde{\xi}(\vec{\xi},\sigma,1), \tilde{\Lambda}(\vec{\xi},\sigma,1)\big)$, we have

$$2G(\tilde{\xi_1},\tilde{\xi_2}) - \big(\tilde{\Lambda}_i^2 H(\tilde{\xi_1},\tilde{\xi_1}) + 2\tilde{\Lambda}_1 w(\tilde{\xi_1} + H(\tilde{\xi_2},\tilde{\xi_2}) + 2\tilde{\Lambda}_2 w(\tilde{\xi_2})\big) \geq 2A_0$$

and since $H(\tilde{\xi_i},\tilde{\xi_i}) > 0$ and $w(\tilde{\xi_i}) > 0$, we conclude that if we take a small neighborhood W of $\mathbf{M}$ in $\mathbf{S} \times I_0$, then for every $(\vec{\xi},\sigma) \in W$ one has

$$G\big(\tilde{\xi}(\xi,\sigma,1)\big) \geq A_0.$$

Hence $|\tilde{\xi_1} - \tilde{\xi_2}| < \delta_0$. Let us fix points $\zeta_i \in \mathbb{R}^N$, $i=1,2$, such that $|\zeta_1| = \hat{\rho}_\mu$ and $|\zeta_2| = \bar{\rho}_\mu$, then $\vec{\zeta} = (\zeta_1,\zeta_2) \in \mathbf{S}$. Denoting $\kappa_1 = \kappa(\,\cdot\,,1)$, we see that because of the above conclusion $\kappa_1(W) \subset \big(\Sigma_\Omega^\mu \setminus T(\vec{\zeta})\big) \times \mathbb{R}_+^2$, where $T(\vec{\zeta}) = \big\{(t_1\zeta_1, t_2\zeta_2) : t_1, t_2 \in]k, K[\big\}$.

Consider the map $s : \Sigma_\Omega^\mu \times \mathbb{R}_+^2 \to \mathbf{S}$ defined componentwise as $s(\vec{\xi},\vec{\Lambda}) = \mu(\hat{\rho}_\mu \xi_1/|\xi_1|, \bar{\rho}_\mu \xi_2/|\xi_2|)$. Then $\kappa_0^* \circ s^* : H^*(\mathbf{S}) \to H^*(\mathbf{S}\times I_0)$, where $\kappa_0 = \kappa(\,\cdot\,,0)$ is an isomorphism. By the homotopy axiom we deduce then that $\kappa_1^* \circ s^*$ is also an isomorphism. We consider the following commutative diagram:

$$\begin{array}{ccccc}
H^*(\mathbf{S}\times I_0) & \overset{\kappa_1^*}{\longleftarrow} & H^*(\Sigma_\Omega^\mu \times \mathbb{R}_+^2) & \overset{\kappa^*}{\longleftarrow} & H^*(\mathbf{S}) \\
{\scriptstyle i_1^*}\downarrow & & {\scriptstyle i_2^*}\downarrow & & {\scriptstyle i_3^*}\downarrow \\
H^*(W) & \overset{\tilde{\kappa}_1^*}{\longleftarrow} & H^*(\kappa_1(W)) & \overset{\tilde{s}^*}{\longleftarrow} & H^*(\mathbf{S}\setminus\{\vec{\zeta}\}),
\end{array}$$

where i_1, i_2 and i_3 are inclusion maps, $\tilde{\kappa}_1 = \kappa_1|_W$ y $\tilde{s} = s|_{\kappa_1(W)}$. From Lemma 6.2 we have that i_1^* is a monomorphism which is a contradiction with the fact that $H^{2N}(\mathbf{S}\setminus\{\vec{\zeta}\}) = 0$. Thus, the result follows. ■

In order to prove that the min-max number (6.4) is a critical value of Φ, we need care about the fact the domain in which Φ is defined is not necessarily closed for the gradient flow of Φ. The following lemma appears in this direction.

Lemma 6.4. *Assume that $\mu > 0$ is a small enough number. Let $(\xi^n, \Lambda^n) \in \Sigma_\Omega^\mu \times \mathbb{R}_+^2$ be a sequence such that*

$$\nabla_{\vec{\Lambda}} \Phi(\vec{\xi_n}, \vec{\Lambda}_n) \to 0. \tag{6.5}$$

Then each component of $\vec{\Lambda}_n$ is bounded above and below by positive constants.

Proof. Note that $\overline{\Sigma_\Omega^\mu} \subset\subset \Omega$. Hence $w(\xi_i) > 0$, $i=1,2$, for all $\vec{\xi} \in \overline{\Sigma_\Omega^\mu}$. We put $\vec{\xi_n} = (\xi_{1,n}, \xi_{2,n})$ and $\vec{\Lambda}_n = (\Lambda_{1,n}, \Lambda_{2,n})$. Then (6.5) is equivalent to

$$\Lambda_{i,n} H(\xi_{i,n},\xi_{i,n}) - \Lambda_{j,n} G(\xi_{i,n},\xi_{j,n}) + w(\xi_{i,n}) \to 0; \quad i,j = 1,2, \quad i \neq j.$$

It is clear that $|\vec{\Lambda}_n| \to 0$ or $\Lambda_{i,n} \to 0$ and $\Lambda_{j,n} \to C$, with C different of zero and $i \neq j$, cannot happen. Hence, we can suppose that $|\vec{\Lambda}_n| \to +\infty$. Since H and G remain uniformly controlled, (μ is fixed) we easily see that

$\Lambda_{1,n} \to +\infty$ and $\Lambda_{2,n} \to +\infty$. We put $\tilde{\Lambda}_{i,n} = \frac{\Lambda_{i,n}}{|\vec{\Lambda}_n|}$, for $i = 1, 2$, and passing to a subsequence, if necessary, we may assume that this sequence it approaches a nonzero vector $(\hat{\Lambda}_1, \hat{\Lambda}_2)$ with $\hat{\Lambda}_i \neq 0$ for $i = 1, 2$. It follows that

$$\tilde{\Lambda}_{i,n} H(\xi_{i,n}, \xi_{i,n}) - \tilde{\Lambda}_{j,n} G(\xi_{1,n}, \xi_{2,n}) + \frac{w(\xi_{i,n})}{|\vec{\Lambda}_n|} \to 0; \quad i, j = 1, 2, \quad i \neq j.$$

For a suitable subsequence, we obtain for some $(\bar{\xi}_1, \bar{\xi}_2) \in \overline{\Sigma^\mu_\Omega}$ the system

$$\frac{\hat{\Lambda}_1}{\hat{\Lambda}_2} = \frac{G(\bar{\xi}_1, \bar{\xi}_2)}{H(\bar{\xi}_1, \bar{\xi}_1)} \quad \text{and} \quad \frac{\hat{\Lambda}_2}{\hat{\Lambda}_1} = \frac{G(\bar{\xi}_1, \bar{\xi}_2)}{H(\bar{\xi}_2, \bar{\xi}_2)}.$$

Hence

$$G^2(\bar{\xi}_1, \bar{\xi}_2) - H(\bar{\xi}_1, \bar{\xi}_1) H(\bar{\xi}_2, \bar{\xi}_2) = 0$$

which is a contradiction, since the quantity on the left hand side in the previous equality is strictly positive when $\mu > 0$ is chosen sufficiently small. This finishes the proof. ■

Proposition 6.5. *Let us assume that $\mu > 0$ is an small enough number. Then the functional Φ satisfies the (PS) condition in the region $\Sigma^\mu_\Omega \times \mathbb{R}^2_+$ at the level $c(\Omega)$ given in (6.4).*

Proof. Let us consider a sequence $(\vec{\xi}_n, \vec{\Lambda}_n) \in \Sigma^\mu_\Omega \times \mathbb{R}^2_+$ such that

$$\nabla_{\vec{\Lambda}} \Phi(\vec{\xi}_n, \vec{\Lambda}_n) \to 0 \quad \text{and} \quad \nabla^\tau_{\vec{\xi}} \Phi(\vec{\xi}_n, \vec{\Lambda}_n) \to 0,$$

where $\nabla^\tau_{\vec{\xi}} \Phi$ corresponds to the tangential gradient of Φ to $\partial \Sigma^\mu_\Omega \times \mathbb{R}^2_+$ in case that $\vec{\xi}_n$ it is approaching to $\partial \Sigma^\mu_\Omega$ or the full gradient in otherwise. From the previous lemma, the components of $\vec{\Lambda}_n$ are bounded above and below by positive constants, so that we may assume, passing to a subsequence if necessary, that $(\vec{\xi}_n, \vec{\Lambda}_n) \to (\vec{\xi}_0, \vec{\Lambda}_0) \in \overline{\Sigma^\mu_\Omega} \times \mathbb{R}^2_+$ and $\Phi(\vec{\xi}_n, \vec{\Lambda}_n) \to c(\Omega)$. Then

$$\nabla_{\vec{\Lambda}} \Phi(\vec{\xi}_0, \vec{\Lambda}_0) = 0.$$

Observe that if $\vec{\xi}_0 \in \mathrm{int}(\Sigma^\mu_\Omega)$ then $\vec{\xi}_0$ is a critical point of Φ. We assume the opposite, this is that $\vec{\xi}_0 \in \partial \Sigma^\mu_\Omega$. Then

$$\mathit{\Phi}_E(\mu^{-1} \xi_{0,1}, \mu^{-1} \xi_{0,2}) = \delta_\mu.$$

Firstly we note that $\nabla_{\vec{\Lambda}} \Phi(\vec{\xi}_0, \vec{\Lambda}_0) = 0$, then $\vec{\Lambda}_0$ satisfies

$$\Lambda_{0,i} = \frac{H(\xi_{0,j}, \xi_{0,j}) w(\xi_{0,i}) + G(\xi_{0,i}, \xi_{0,j}) w(\xi_{0,j})}{G^2(\xi_{0,i}, \xi_{0,j}) - H(\xi_{0,i}, \xi_{0,i}) H(\xi_{0,j}, \xi_{0,j})}, \quad i, j = 1, 2, \ i \neq j.$$

Substituting these values in Φ, from (6.3) we obtain

$$c(\Omega) = \Phi(\vec{\xi}_0, \vec{\Lambda}_0) = \mathit{\Phi}(\vec{\xi}_0)$$

and from (5.6) we deduce that

$$c(\Omega) = \mu^{N+2} \mathit{\Phi}_E(\mu^{-1} \xi_{0,1}, \mu^{-1} \xi_{0,2}) + \theta(\vec{\xi}_0),$$

where $\theta(\vec{\xi}_0)$ is small in the C^1 sense, as $\mu > 0$ becomes smaller. Hence, $\nabla_{\vec{\xi}} \Phi(\vec{\xi}_0, \vec{\Lambda}_0) \cdot \tau \sim 0$ for any tangential direction τ to $\partial \Sigma^\mu_\Omega$. Thus, from the analysis in the previous section, we have that $\xi_{0,1}, \xi_{0,2}$ are in opposite directions, $\Phi(\vec{\xi}_0, \vec{\Lambda}_0) \sim \mu^{N+2} \delta_\mu$ and $\nabla_{\vec{\xi}} \Phi(\vec{\xi}_0, \vec{\Lambda}_0)$ must be away from 0. Then choosing τ parallel to $\nabla_{\vec{\xi}}\Phi(\vec{\xi_0},\vec{\Lambda}_0)$ we obtain that $\nabla_{\vec{\xi}}\Phi(\vec{\xi_0},\vec{\Lambda}_0)\cdot\tau$ must to be away from 0, which is a contradiction. Then, the point $\vec{\xi_0}\in \mathrm{int}(\Sigma_\Omega^\mu)$, which implies that the (PS) condition holds and the results follows. ■

Now we are in conditions to complete the proof of Theorem 1.1

Proof of Theorem 1.1. Let us consider the domain $\mathbf{\Sigma_a^b} = \Sigma_\Omega^\mu \times [\mathbf{a},\mathbf{b}]^2$ with $\mathbf{a},\mathbf{b}$ to be choose later. Then the functional $\mathcal{I}$ given by (4.1) is well defined on $\mathbf{\Sigma_a^b}$ except on the set

$$\Delta_\rho = \{\vec{\xi},\vec{\Lambda}) \in \mathbf{\Sigma_a^b} : |\xi_1-\xi_2| < \rho\}.$$

From (4.3) we can extend $\mathcal{I}$ to all $\mathbf{\Sigma_a^b}$ by extending Φ as in (6.2), and keep relations (4.3) and (4.12) over $\mathbf{\Sigma_a^b}$.

From Proposition 6.5, Φ satisfies the (PS) condition. Then there exist constants $\mathbf{b}>0$, $c>0$ and $\varrho_0>0$, such that if $0<\varrho<\varrho_0$, and $(\vec{\xi},\vec{\Lambda})\in\Sigma_\Omega^\mu$ satisfying $|\vec{\Lambda}|\geq \mathbf{b}$ and $c(\Omega)-2\varrho\leq \Phi(\vec{\xi},\vec{\Lambda})\leq c(\Omega)+2\varrho$, then $|\nabla\Phi(\vec{\xi},\vec{\Lambda})|\geq c$.

We now use the min-max characterization of $c(\Omega)$ to choose $\kappa\in\mathcal{K}$ so that

$$c(\Omega)\leq \sup_{(\vec{\xi},\sigma)\in \mathbf{S}\times I_0} \Phi\big(\kappa(\vec{\xi},\sigma,1)\big)\leq c(\Omega)+\varrho.$$

By making $\mathbf{a}$ small and $\mathbf{b}$ large if necessary, we can assume that $\kappa(\vec{\xi},\sigma,1)\in \mathbf{\Sigma_{2a}^{b/2}}\subset \mathbf{\Sigma_a^b}$ for all $(\vec{\xi},\sigma)\in\mathbf{S}\times I_0$.

Consider now $\eta:\mathbf{\Sigma_a^b}\times[0,+\infty]\to\mathbf{\Sigma_a^b}$ being the solution of the equation $\dot{\eta}=-h(\eta)\nabla\mathcal{I}(\eta)$ with initial condition $\eta(\vec{\xi},\vec{\Lambda},0)=(\vec{\xi},\vec{\Lambda})$. Here the function h is defined in $\mathbf{\Sigma_a^b}$ so that $h(\vec{\xi},\vec{\Lambda})=0$ for all $(\vec{\xi},\vec{\Lambda})$ with $\Phi(\vec{\xi},\vec{\Lambda})\leq c(\Omega)-2\varrho$ and $h(\vec{\xi},\vec{\Lambda})=1$ if $\Phi(\vec{\xi},\vec{\Lambda})\geq c(\Omega)-\varrho$, satisfying $0\leq h\leq 1$.

Hence, by the choice of $\mathbf{a}$ y $\mathbf{b}$, and bearing in mind (4.3) and (4.12), we have that $\eta(\vec{\xi},\vec{\Lambda},t)\in\mathbf{\Sigma_a^b}$ for all $t\geq 0$. Then the following min-max value

$$C(\Omega)=\inf_{t\geq 0}\sup_{(\vec{\xi},\sigma)\in\mathbf{S}\times I_0}\mathcal{I}\big(\eta(\kappa(\vec{\xi},\sigma,1),t)\big)$$

is a critical value for $\mathcal{I}$. We are always assuming that ε is small enough, to make the errors in (4.1) sufficiently small. Theorem 1.1 has been proven. ■

Acknowledgement

The author is indebted to the Departamento de Ingeniería Matemática at Universidad de Chile, where this work was carried out, for its support by a fellowship from MECESUP grant UCH0009. The author also thanks to the Professor Manuel Del Pino for useful suggestions and comments.

References

[1] I. Ali and A. Castro, *Positive solutions for a semilinear elliptic problem whit critical exponent*, Nonlinear Anal., **27** (1996), pp. 327-338.

[2] A. Ambrosetti, *A perturbation theorem for superlinear boundary value problems*, Math. Res. Center, Univ. Wisconsin at Madison, Tech. Sum. Report, **1446** (1974).

[3] T. Aubin, *Problèmes isopérimétriques et espaces de Sobolev*, J. Differ. Geometry, **11** (1976), pp. 573-598.

[4] A. Bahri, "Critical points at infinity in some variational problems", Pitman Research Notes in Math. Series 182, Longman 1989.

[5] A. Bahri and H. Berestycki, *A perturbation method in critical point theory and applications*, Trans. Amer. Math. Soc., **267** (1981), pp. 1-32.

[6] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math., **36** (1983), pp. 437-477.
[7] H. Brezis and L. Nirenberg, *A minimization problem with critical exponent and nonzero data*, in *Symmetry in Nature*, Scuola Norm. Sup. Pisa, 1989, pp. 129-140.
[8] L.A. Caffarelli, B. Gidas and J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math., **42** (1989), pp. 271-297.
[9] M. Clapp, M. Del Pino and M. Musso, *Multiple solutions for a non-homogeneous elliptic equation at the critical exponent*, Proc. Royal Soc. Edinburgh Section A, **134** (2004), pp. 69-87.
[10] M. Clapp, O. Kavian and B. Ruf, *Multiple solutions of nonhomogeneous elliptic equations with critical nonlinearity on symmetric domains*, Commun. in Contemp. Math., **5** (2003), pp. 147-169.
[11] J.M. Coron, *Topologie et cas limite des injections de Sobolev*, C.R. Acad. Sc. Paris, **299** Serie I (1984), pp. 209-212.
[12] M. del Pino, P. Felmer and M. Musso, *Two-bubble solutions in the super-critical Bahri-Coron's problem*, Calculus of Variations and PDE, **16** (2003), pp. 113-145.
[13] M. del Pino, P. Felmer and M. Musso, *Multi-peak solution for super-critical elliptic problems in domains with small holes*, J. Diff. Eq., **36** (2002), pp. 511-540.
[14] M. del Pino, P. Felmer and M. Musso, *Erratum: Two-bubble solutions in the super-critical Bahri-Coron's problem*, Calculus of Variations and PDE, **20** (2004), pp. 231-233.
[15] P. Fitzpatrick, I. Massabó and J. Pejsachowicz, *Global several-parameter bifurcation and continuation theorem: a unified approach via Complementing Maps*, Math. Ann., **263** (1983), pp. 61-73.
[16] F. Merle, *Sur la non-existence de solutions positives d'équations elliptiques surlinéaires*, C. R. Acad. Sci. Paris, **306** Serie I (1988), pp. 313-316.
[17] S. Pohozaev, *Eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl., **6** (1965), pp. 1408-1411.
[18] P.H. Rabinowitz, *Multiple critical points of perturbed symmetric functionals*, Trans. Amer. Math. Soc., **272** (1982), pp. 753-770.
[19] P.H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, Reg. Conf. Ser. Math. **65**, Amer. Math. Soc., Providence, RI 1986.
[20] O. Rey, *Concentration of solutions to elliptic equations with critical nonlinearity*, Ann. Inst. Henri Poincaré, Anal. non linéaire, **9** (1992), pp. 201-218.
[21] O. Rey and J. Wei, *Blowing up solution for an elliptic Newmann problem with sub- or supercritical nonlinearity I.* $N = 3$, J. Funct. Anal., **212** (2004), pp. 472-499.
[22] O. Rey and J. Wei, *Blowing up solution for an elliptic Newmann problem with sub- or supercritical nonlinearity II.* $N \geq 4$, Ann. Inst. H. Poincaré, Anal. Nonlinéaire **22** (2005), pp. 459-484.
[23] M. Struwe, *Infinitely many critical points for functionals which are not even and applications to superlinear boundary value problems*, Manuscripta Math. **32** (1980), pp. 335-364.
[24] G. Talenti, *Best constants in Sobolev inequality*, Ann. Mat. Pure Appl., **110** (1976), pp. 353-372.
[25] K. Tanaka, *Morse indices at critical points related to the symmetric mountain pass theorem and applications*, Commun. in Partial Diff. Eq., **14** (1989), pp. 99-128.
[26] G. Tarantello, *On nonhomogeneous elliptic equations involving critical Sobolev exponent*, Ann. Inst. Henri Poincaré, Anal. non linéaire, **9** (1992), pp. 281-304.

S. Alarcón - Departamento de Ingeniería Matemática, Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile.