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AN IMPROVED HARDY INEQUALITY FOR A NONLOCAL OPERATOR

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Abstract. Let $0 < s < 1$ and $1 < p < 2$ be such that $ps < N$ and let Ω be a bounded domain containing the origin. In this paper we prove the following improved Hardy inequality:

given $1 \leqslant q < p$, there exists a positive constant $C \equiv C(\Omega, q, N, s)$ such that

$$\int_{I\!R^N}\int_{I\!R^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \Lambda_{N,p,s}\int_{I\!R^N} \frac{|u(x)|^p}{|x|^{ps}}\,dx \geqslant C\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+qs}}dxdy$$

for all $u \in \mathcal{C}_0^\infty(\Omega)$. Here $\Lambda_{N,p,s}$ is the optimal constant in the Hardy inequality (1.1).

1. Introduction and statement of main results.

In [17], Frank and Seiringer proved the following Hardy inequality: for $p > 1$ with $sp < N$ and for all $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$,

$$\int_{I\!R^N}\int_{I\!R^N} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+ps}}dxdy \geqslant \Lambda_{N,p,s}\int_{I\!R^N} \frac{|\phi(x)|^p}{|x|^{ps}}dx \tag{1.1}$$

where the constant $\Lambda_{N,p,s}$ is given by

$$\Lambda_{N,p,s} = 2\int_0^\infty |1-\sigma^{-\gamma}|^{p-2}(1-\sigma^{-\gamma})\sigma^{N-\beta-1}K(\sigma) \tag{1.2}$$

with

$$K(\sigma) = \int\limits_{|y'|=1} \frac{dH^{n-1}(y')}{|x'-\sigma y'|^{N+ps}}.$$

Moreover they proved that if $p \geqslant 2$, then if we set

$$G_{s,p}(u) \equiv \int_{I\!R^N}\int_{I\!R^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy - \Lambda_{N,p,s}\int_{I\!R^N} \frac{|u(x)|^p}{|x|^{ps}}dx,$$

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there exists a positive constant $C = C(p, N, s)$ such that for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$,

$$G_{s,p}(u) \geqslant C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} w^{\frac{p}{2}}(x) w^{\frac{p}{2}}(y)\, dx\, dy \tag{1.3}$$

where $w(x) = |x|^{-\frac{N-ps}{p}}$ and $v(x) = \dfrac{u(x)}{w(x)}$. The above inequality turns out to be equality for $p = 2$ with $C = 1$.

For $p = 2$, the author in [15], proves the following improved Hardy inequality,

$$G_{s,2}(u) \geqslant C(\Omega, q, N, s)\|u\|^2_{W_0^{\tau,2}(\Omega)} \text{ for all } u \in \mathcal{C}_0^\infty(\Omega) \text{ and all } s/2 < \tau < s. \tag{1.4}$$

See also [2] and [14] for alternative proofs without using the Fourier transform.

Following closely the arguments used in [2] and based on the inequality (1.3), the authors in [1] obtained an improved Hardy inequality stated in the next theorem.

Theorem 1.1. *([1]) Let $N \geqslant 1$, $0 < s < 1$, $p \geqslant 2$ with $N > ps$. Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain, then for all $1 < q < p$, there exists a positive constant $C = C(\Omega, q, N, s)$ such that for all $u \in \mathcal{C}_0^\infty(\Omega)$,*

$$G_{s,p}(u) \geqslant C \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+qs}}\, dx\, dy. \tag{1.5}$$

The first main aim of this note is to extend inequality (1.3) to the range $p < 2$. Precisely we prove the following *ground state inequality.*

Theorem 1.2. *Let $p < 2$, $0 < s < 1$ and $N > ps$. Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain, then for all $1 < q < p$, there exists a positive constant $C = C(\Omega, q, N, s)$ such that for all $u \in \mathcal{C}_0^\infty(\Omega)$,*

$$G_{s,p}(u) \geqslant C \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^p}{|x - y|^{N+qs}}\, w(x)^{\frac{p}{2}} w(y)^{\frac{p}{2}}\, dxdy. \tag{1.6}$$

As a consequence, we get the next improved Hardy inequality.

Theorem 1.3. *Suppose that the hypotheses of Theorem 1.2 hold, then for all $1 < q < p$, there exists a positive constant $C = C(\Omega, q, N, s)$ such that for all $u \in \mathcal{C}_0^\infty(\Omega)$,*

$$G_{s,p}(u) \geqslant C \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+qs}}\, dx\, dy. \tag{1.7}$$

Before closing the introduction we recall the following algebraic inequalities which will be useful throughout the paper. The first result is proved in [17].

Lemma 1.4. *([17]) Assume that $p > 1$, then for all $0 \leqslant t \leqslant 1$ and $a \in \mathbb{C}$, we have*

$$|a - t|^p \geqslant (1 - t)^{p-1}(|a|^p - t). \tag{1.8}$$

The second one is proved in [24].

Lemma 1.5. *([24]) For any $1 < p < 2$ there exists a constant $c \equiv c(p) > 0$ such that for all $a, b \in \mathbb{R}^N$ we have :*

$$|a|^p - |b|^p - p|b|^{p-2}\langle b, a - b\rangle \geqslant c\frac{|a - b|^2}{(|a| + |b|)^{2-p}} \tag{1.9}$$

and

$$|a + b|^p - |b|^p - p|b|^{p-2}\langle b, a\rangle \leqslant c|a|^p. \tag{1.10}$$

The paper is organized as follows. In Section 2 we give some nonlinear tools in order to get the improved Hardy inequality. In particular we define the weighted fractional Sobolev space, we prove a Picone's type inequality in that's space and the weighted Hardy inequality.

In Section 3 we prove the main results of the paper. We begin by proving Lemma 3.1 that can be seen as a ground state representation. As a consequence we get the proof of Theorems 1.2 and 1.3.

2. **Some functional setting.** In this section we will introduce some notations, definitions and prove some intermediate lemmas that will be usefull later to prove the main results of this paper.

Let $1 \leqslant q \leqslant p$, $0 < \beta < \frac{N-qs}{2}$ and $\Omega \subset \mathbb{R}^N$ a smooth bounded domain with $0 \in \Omega$, the weighted Sobolev space $X^{s,p,q,\beta}(\Omega)$ is defined by

$$X^{s,p,q,\beta}(\Omega) := \Big\{ \phi \in L^p(\Omega, \frac{dx}{|x|^{2\beta}}) \, : \, \int_\Omega \int_\Omega \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+qs}} \frac{dxdy}{|x|^\beta |y|^\beta} < +\infty \Big\}.$$

It is not difficult to prove that $X^{s,p,\beta}(\Omega)$ is a Banach space endowed with the norm

$$\|\phi\|_{X^{s,p,q,\beta}(\Omega)} := \Big(\int_\Omega \frac{|\phi(x)|^p dx}{|x|^{2\beta}} \Big)^{\frac{1}{p}} + \Big(\int_\Omega \int_\Omega \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+qs}} \frac{dxdy}{|x|^\beta |y|^\beta} \Big)^{\frac{1}{p}}. \quad (2.11)$$

Now, we define the weighted Sobolev space $X_0^{s,p,q,\beta}(\Omega)$ as the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the previous norm (2.11).

Following the arguments in [3], see also [12], we can prove the next extension result.

Lemma 2.1. *Assume that $\Omega \subset \mathbb{R}^N$ is a smooth domain, then for all $w \in X^{s,p,q,\beta}(\Omega)$, there exists $\tilde{w} \in X^{s,p,q,\beta}(\mathbb{R}^N)$ such that $\tilde{w}_{|\Omega} = w$ and*

$$||\tilde{w}||_{X^{s,p,q,\beta}(\mathbb{R}^N)} \leqslant C ||w||_{X^{s,p,q,\beta}(\Omega)}$$

where $C \equiv C(N, s, p, \Omega) > 0$.

Remark 2.2. As in the case $\beta = 0$ and $q = p$, if Ω is a smooth bounded domain, we can endow $X_0^{s,p,q,\beta}(\Omega)$ with the equivalent norm

$$|||\phi|||_{X_0^{s,p,q,\beta}(\Omega)} := \Big(\int_\Omega \int_\Omega \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+qs}} \frac{dxdy}{|x|^\beta |y|^\beta} \Big)^{\frac{1}{p}}.$$

Now, for $w \in X_0^{s,p,q,\beta}(\Omega)$, we set

$$\tilde{L}(w)(x) \equiv \tilde{L}_{s,p,\beta}(w)(x) = \text{ P.V. } \int_{\mathbb{R}^N} \frac{|w(x)-w(y)|^{p-2}(w(x)-w(y))}{|x-y|^{N+qs}} \frac{dy}{|x|^\beta |y|^\beta}.$$

It is clear that for all $w, v \in X^{s,p,q\beta}(\mathbb{R}^N)$, we have

$$\langle \tilde{L}(w), v \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(x)-w(y)|^{p-2}(w(x)-w(y))(v(x)-v(y))}{|x-y|^{N+qs}} \frac{dxdy}{|x|^\beta |y|^\beta}.$$

The next result that we will be needed later is the following Picone's type inequality.

Lemma 2.3. *(Picone's inequality) Let $w \in X_0^{s,p,q,\beta}(\Omega)$ be such that $w > 0$ in Ω. Assume that $\tilde{L}(w)(x) \geqslant 0$, then for all $u \in \mathcal{C}_0^\infty(\Omega)$, we have*

$$\frac{1}{2}\int\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+qs}}\frac{dxdy}{|x|^\beta|y|^\beta} \geqslant \langle \tilde{L}(w), \frac{|u|^p}{w^{p-1}}\rangle$$

where $Q = \mathbb{R}^N \times \mathbb{R}^N \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$.

Proof. This result has been proven for particular cases of p, q and β. For example, Leonori et al. in [22] treated the case $q = p = 2$ and $\beta = 0$ and then Abdellaoui et al. in [1] proved the result for $q = p \neq 2$. Here, we will give some details for the case $\beta \neq 0$ and $1 < q \leqslant p$.

We set $v(x) = \dfrac{|u(x)|^p}{|w(x)|^{p-1}}$ and $k(x,y) = \dfrac{1}{|x-y|^{N+qs}|x|^\beta|y|^\beta}$, then

$$\begin{aligned}\langle \tilde{L}(w)(w(x)), v(x)\rangle &= \int_\Omega v(x)\int_{\mathbb{R}^N} |w(x)-w(y)|^{p-2}(w(x)-w(y))k(x,y)\,dy\,dx\\ &= \int_\Omega \frac{|u(x)|^p}{|w(x)|^{p-1}}\int_{\mathbb{R}^N} |w(x)-w(y)|^{p-2}(w(x)-w(y))k(x,y)\,dy\,dx.\end{aligned}$$

Since k is symmetric, then we clearly get

$$\langle \tilde{L}(w)(w(x)), v(x)\rangle =$$

$$\frac{1}{2}\int\int_Q \left(\frac{|u(x)|^p}{|w(x)|^{p-1}} - \frac{|u(y)|^p}{|w(y)|^{p-1}}\right)|w(x)-w(y)|^{p-2}(w(x)-w(y))k(x,y)\,dy\,dx.$$

Now, we define $v_1 := \dfrac{u}{w}$ so that the above quantity can be rewritten as

$$\langle \tilde{L}(w)(w(x)), v(x)\rangle =$$

$$\frac{1}{2}\int\int_Q (|v_1(x)|^p w(x) - |v_1(y)|^p w(y))\,|w(x)-w(y)|^{p-2}(w(x)-w(y))k(x,y)\,dy\,dx.$$

We next define

$$\Phi(x,y) := |u(x)-u(y)|^p - (|v_1(x)|^p w(x) - |v_1(y)|^p w(y))\,|w(x)-w(y)|^{p-2}(w(x)-w(y)).$$

It follows that

$$\begin{aligned}\langle \tilde{L}(w)(w(x)), v(x)\rangle &+ \frac{1}{2}\int_Q \Phi(x,y)k(x,y)\,dy\,dx\\ &= \frac{1}{2}\int\int_Q |u(x)-u(y)|^p k(x,y)\,dy\,dx.\end{aligned}$$

We claim that $\Phi \geqslant 0$. Indeed, it is clear, by a symmetry argument, that we can assume $w(x) \geqslant w(y)$. Then, letting $t = w(y)/w(x)$ and $a = u(x)/u(y)$ and using inequality (1.8), we easily obtain the claim: $\Phi(x,y) \geqslant 0$. This proves the lemma. $\square$

Now, we prove the following lemma.

Lemma 2.4. *Fix $0 < \beta < \frac{N-qs}{2}$ and let $w(x) = |x|^{-\gamma}$ with $0 < \gamma < \dfrac{N-qs-2\beta}{p-1}$, then there exists a positive constant $\Lambda(\gamma) > 0$ such that*

$$\tilde{L}(w) = \Lambda(\gamma)\frac{w^{p-1}}{|x|^{qs+2\beta}} \qquad a.e \text{ in } \in \mathbb{R}^N\backslash\{0\}. \tag{2.12}$$

Proof. Using polar coordinates: $r = |x|$ and $\rho = |y|$ and writing $x = rx', y = \rho y'$ with $|x'| = |y'| = 1$, we have that

$$\tilde{L}(w) = \frac{1}{|x|^\beta}\int_0^{+\infty} |r^{-\gamma}-\rho^{-\gamma}|^{p-2}\frac{(r^{-\gamma}-\rho^{-\gamma})\rho^{N-1}}{\rho^\beta r^{N+qs}}\left(\int\limits_{|y'|=1}\frac{dH^{n-1}(y')}{|x'-\frac{\rho}{r}y'|^{N+qs}}\right)d\rho.$$

Now, using the change of variables $\sigma = \dfrac{\rho}{r}$, we clearly get

$$\tilde{L}(w) = \frac{w^{p-1}(x)}{|x|^{qs+2\beta}}\int_0^{+\infty} |1-\sigma^{-\gamma}|^{p-2}(1-\sigma^{-\gamma})\sigma^{N-\beta-1}\left(\int\limits_{|y'|=1}\frac{dH^{n-1}(y')}{|x'-\sigma y'|^{N+qs}}\right)d\sigma.$$

We set

$$K(\sigma) = \int\limits_{|y'|=1}\frac{dH^{n-1}(y')}{|x'-\sigma y'|^{N+qs}},$$

since $x' \in S^{N-1}$, then using a suitable change of variable, we reach that K does not depend on x', we refer to to the classical book of Grafakos [19] to see this fact.

Now as in [18], we obtain that

$$K(\sigma) = 2\frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})}\int_0^\pi \frac{\sin^{N-2}(\theta)}{(1-2\sigma\cos(\theta)+\sigma^2)^{\frac{N+qs}{2}}}d\theta. \tag{2.13}$$

It is clear that, for $\sigma = 0$, we have

$$K(0) = 2\frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})}\int_0^\pi \sin^{N-2}(\theta)d\theta$$

and $K(\sigma) \simeq \sigma^{-N-qs}$ as $\sigma \to \infty$, moveover for $\sigma \to 1$, we can prove that $K(\sigma) \leqslant C|\sigma-1|^{-1-qs}$. Thus we conclude that

$$\tilde{L}(w) = \frac{w^{p-1}(x)}{|x|^{qs+2\beta}}\int_0^{+\infty}\psi(\sigma)\,d\sigma,$$

with

$$\psi(\sigma) = |1-\sigma^{-\gamma}|^{p-2}(1-\sigma^{-\gamma})\sigma^{N-\beta-1}K(\sigma). \tag{2.14}$$

Defining $\Lambda(\gamma) := \displaystyle\int_0^{+\infty}\psi(\sigma)\,d\sigma$, to conclude we just need to show that $0 < \Lambda(\gamma) < \infty$. For this aim, it is convenient to write

$$\Lambda(\gamma) = \int_0^1 \psi(\sigma)\,d\sigma + \int_1^\infty \psi(\sigma)\,d\sigma = I_1 + I_2.$$

Notice that $K(\frac{1}{\xi}) = \xi^{N+qs}K(\xi)$ for any $\xi > 0$, then using the change of variable $\xi = \frac{1}{\sigma}$ in I_1, it holds

$$\Lambda(\gamma) = \int_1^{+\infty} K(\sigma)(\sigma^\gamma-1)^{p-1}\left(\sigma^{N-1-\beta-\gamma(p-1)} - \sigma^{\beta+qs-1}\right)d\sigma. \tag{2.15}$$

As $\sigma \to \infty$, we have

$$K(\sigma)(\sigma^\gamma-1)^{p-1}\left(\sigma^{N-1-\beta-\gamma(p-1)} - \sigma^{\beta+qs-1}\right) \backsimeq \sigma^{-1-\beta-ps} \in L^1((2,\infty)).$$

Now, as $\sigma \to 1$, we get

$$K(\sigma)(\sigma^{\gamma}-1)^{p-1}\left(\sigma^{N-1-\beta-\gamma(p-1)}-\sigma^{\beta+qs-1}\right) \backsimeq (\sigma-1)^{p-1-qs} \in L^1((1,2)).$$

Therefore, combining the above estimates, we get $|\Lambda(\gamma)| < \infty$. On the other hand, since $0 < \gamma < \dfrac{N-qs-2\beta}{p-1}$, then $\Lambda(\gamma) > 0$.

As a summary, we have proved that

$$\tilde{L}(w) = \Lambda(\gamma)\frac{w^{p-1}}{|x|^{qs+2\beta}} \text{ a.e. in } \mathbb{R}^N\backslash\{0\},$$

where clearly the right hand-side $\dfrac{w^{p-1}}{|x|^{qs+2\beta}} \in L^1_{loc}(\mathbb{R}^N)$. This conclude the proof of the desired result. □

As a consequence we have the following weighted Hardy inequality.

Theorem 2.5. *Let $\beta < \frac{N-qs}{2}$, then for all $u \in \mathcal{C}_0^{\infty}(\mathbb{R}^N)$, we have*

$$2\Lambda(\gamma)\int_{I\!R^N}\frac{|u(x)|^p}{|x|^{qs+2\beta}}\,dx \quad\leqslant\quad \int_{I\!R^N}\int_{I\!R^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+qs}}\frac{dx}{|x|^{\beta}}\frac{dy}{|y|^{\beta}}, \tag{2.16}$$

where $\Lambda(\gamma)$ is defined in (2.15).

Proof. Let $u \in \mathcal{C}_0^{\infty}(\mathbb{R}^N)$ and $w(x) = |x|^{-\gamma}$ with $\gamma < \dfrac{N-ps-2\beta}{p-1}$. By Lemma 2.4, we have

$$\tilde{L}(w) = \Lambda(\gamma)\,\frac{w^{p-1}}{|x|^{qs+2\beta}}.$$

Since $\dfrac{w^{p-1}}{|x|^{qs+2\beta}} \in L^1_{loc}(\mathbb{R}^N)$, then using Picone's inequality in Lemma 2.3, it follows that

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+qs}}\frac{dx}{|x|^{\beta}}\frac{dy}{|y|^{\beta}} \geqslant \langle \tilde{L}(w), \frac{|u|^p}{w^{p-1}}\rangle = \Lambda(\gamma)\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{qs+2\beta}}\,dx.$$

Thus we conclude. □

Fix $q \leqslant p$ and define

$$\Lambda_{N,p,g,s,\beta} := \inf_{\{\phi\in\mathcal{C}_0^{\infty}(\mathbb{R}^N)\backslash\{0\}\}}\frac{\displaystyle\int_{I\!R^N}\int_{I\!R^N}\frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+qs}|x|^{\beta}|y|^{\beta}}dxdy}{\displaystyle\int_{\mathbb{R}^N}\frac{|\phi(x)|^p}{|x|^{qs+2\beta}}dx},$$

then, as in [17] (see also [1]), we can prove that $\Lambda_{N,p,s,\gamma} = 2\Lambda(\gamma_0)$ where $\gamma_0 = \frac{N-\beta-qs}{p}$.

In the case where we consider a smooth bounded domain Ω of $\mathbb{R}^N$ with $0 \in \Omega$, we can prove the following Hardy inequality.

Lemma 2.6. *Let Ω be a smooth bounded domain such that $0 \in \Omega$, then there exists a constant $C \equiv C(\Omega, s, p, N) > 0$ such that for all $u \in \mathcal{C}_0^{\infty}(\Omega)$, we have*

$$C\int_{\Omega}\frac{|u(x)|^p}{|x|^{qs+2\beta}}\,dx \leqslant \int_{\Omega}\int_{\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{N+qs}}\frac{dx}{|x|^{\beta}}\frac{dy}{|y|^{\beta}}. \tag{2.17}$$

3. **Proof of the main results.** Let us begin by proving the next Lemma.

Lemma 3.1. *Let $u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ and define $w(x) = |x|^{-\alpha}$ with $\alpha = \frac{N-ps}{p}$. Consider $v(x) = \dfrac{u(x)}{w(x)}$, then for all $1 \leqslant q < p$, we have*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+qs}} w(x)^{\frac{p}{2}} w(y)^{\frac{p}{2}} dxdy \geqslant C \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+qs}} dxdy. \tag{3.18}$$

Proof. The integrand in the left hand side can be written as

$$\begin{aligned} \frac{|v(x)-v(y)|^p}{|x-y|^{N+qs}} w(x)^{\frac{p}{2}} w(y)^{\frac{p}{2}} &= \frac{|w(y)u(x)-w(x)u(y)|^p}{|x-y|^{N+qs}} \frac{1}{(w(x)w(y))^{\frac{p}{2}}} \\ &= \frac{\left|(u(x)-u(y)) - \frac{u(y)}{w(y)}(w(x)-w(y))\right|^p}{|x-y|^{N+qs}} \left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}} \\ &:= f_1(x,y). \end{aligned}$$

In the same way, thanks to the symmetry of $f_1(x,y)$, it immediately follows that

$$\begin{aligned} \frac{|v(x)-v(y)|^p}{|x-y|^{N+qs}} (w(x))^{\frac{p}{2}} (w(y))^{\frac{p}{2}} &= \frac{\left|(u(y)-u(x)) - \frac{u(x)}{w(x)}(w(y)-w(x))\right|^p}{|x-y|^{N+qs}} \left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}} \\ &:= f_2(x,y). \end{aligned}$$

Hence, we can write the integral of the above term as

$$\begin{aligned} H(v) &:= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+qs}} w(x)^{\frac{p}{2}} w(y)^{\frac{p}{2}} dxdy \\ &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_1(x,y)\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_2(x,y)\,dx\,dy. \end{aligned}$$

We define the quantities

$$Q(x,y) := \frac{(w(x)\,w(y))^{\frac{p}{2}}}{w(x)^p + w(y)^p},$$

and

$$D(x,y) \equiv \left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}} + \left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}} \equiv \frac{w(x)^p + w(y)^p}{(w(x)w(y))^{\frac{p}{2}}}.$$

It is clear that $Q(x,y) \leqslant \frac{1}{2}$ and $Q(x,y)D(x,y) = 1$. Thus using (1.9), one easily obtain

$$f_1(x,y) \geqslant C\,Q(x,y)\left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}} \times$$

$$\left[\frac{|u(x)-u(y)|^p}{|x-y|^{N+qs}} + p\frac{|u(x)-u(y)|^{p-2}}{|x-y|^{N+qs}}\left\langle u(x)-u(y), \frac{u(y)}{w(y)}(w(x)-w(y))\right\rangle\right].$$

It then follows that

$$\begin{aligned} f_1(x,y) &\geqslant \Big[CQ(x,y)\left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+qs}}\Big] \\ &- \Big[pCQ(x,y)\left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}}\frac{|u(x)-u(y)|^{p-1}}{|x-y|^{N+qs}}\Big|\frac{u(y)}{w(y)}\Big||(w(x)-w(y))|\Big]. \end{aligned}$$

The same argument applied to f_2 yields

$$\begin{aligned} f_2(x,y) &\geqslant \Big[CQ(x,y)\left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}}\frac{|u(y)-u(x)|^p}{|x-y|^{N+qs}}\Big] \\ &- \Big[pCQ(x,y)\left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}}\frac{|u(x)-u(y)|^{p-1}}{|x-y|^{N+qs}}\Big|\frac{u(x)}{w(x)}\Big||(w(x)-w(y))|\Big]. \end{aligned}$$

Combining the above two estimates we get

$$\begin{aligned} H(v) &\geqslant C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} Q(x,y)\Big(\left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}}+\left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}}\Big)\frac{|u(x)-u(y)|^p}{|x-y|^{N+qs}}\,dx\,dy \\ &- pC\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Big[Q(x,y)\left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}}\frac{|u(x)-u(y)|^{p-1}}{|x-y|^{N+qs}}\Big|\frac{u(y)}{w(y)}\Big||(w(x)-w(y))|\Big]\,dx\,dy \\ &- pC\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Big[Q(x,y)\left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}}\frac{|u(x)-u(y)|^{p-1}}{|x-y|^{N+qs}}\Big|\frac{u(x)}{w(x)}\Big||(w(x)-w(y))|\Big]\,dx\,dy. \end{aligned}$$

Thus

$$\begin{aligned} H(v) &\geqslant C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+qs}}\,dx\,dy \\ &- C_1(p)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\big(h_1(x,y)+h_2(x,y)\big)\,dx\,dy, \end{aligned} \tag{3.19}$$

with

$$h_1(x,y)=Q(x,y)\left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}}\frac{|u(x)-u(y)|^{p-1}}{|x-y|^{N+qs}}\Big|\frac{u(y)}{w(y)}\Big||(w(x)-w(y))|,$$

$$h_2(x,y)=Q(x,y)\left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}}\frac{|u(x)-u(y)|^{p-1}}{|x-y|^{N+qs}}\Big|\frac{u(x)}{w(x)}\Big||(w(x)-w(y))|.$$

Since $h_1(x,y)$ and $h_2(x,y)$ are symmetric functions, we just have to estimate the quantity

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} h_2(x,y)\,dx\,dy.$$

To this aim we use Young's inequality which yields

$$\begin{aligned} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} h_2(x,y)dxdy &\leqslant \varepsilon\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+qs}}\,dx\,dy \\ &+ C(\varepsilon)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G(x,y)\,dx\,dy, \end{aligned} \tag{3.20}$$

with

$$G(x,y)=(Q(x,y))^p\left(\frac{w(x)}{w(y)}\right)^{\frac{p^2}{2}}\Big|\frac{u(x)}{w(x)}\Big|^p\frac{|(w(x)-w(y))|^p}{|x-y|^{N+qs}}.$$

We claim that

$$I := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G(x,y)\,dx\,dy \leqslant C \int\limits_{I\!R^N}\int\limits_{I\!R^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+qs}}\, w(x)^{\frac{p}{2}} w(y)^{\frac{p}{2}}\, dxdy.$$

Indeed, notice that

$$\begin{aligned} I &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x))^p}{|x-y|^{N+qs}} \frac{(w(x))^{p^2-p}|(w(x)-w(y))|^p}{(w(x)^p+w(y)^p)^p}\, dx\,dy, \\ &= \int_{\mathbb{R}^N} u^p(x)\Big[\int_{\mathbb{R}^N} \frac{||x|^\alpha-|y|^\alpha|^p}{(|x|^{\alpha p}+|y|^{\alpha p})^p} \frac{|y|^{\alpha p(p-1)}}{|x-y|^{N+qs}}\, dy\Big]\, dx. \end{aligned}$$

To compute the above integral, we closely follow the arguments used in [18]. We set $y=\rho y'$ and $x = rx'$ with $|x'|=|y'|=1$, then

$$\begin{aligned} I &= \int_{\mathbb{R}^N} u^p(x)\Big[\int_{\mathbb{R}^N} \frac{||x|^\alpha-|y|^\alpha|^p}{(|x|^{\alpha p}+|x|^{\alpha p})^p} \frac{|y|^{\alpha p(p-1)}}{|x-y|^{N+qs}}\, dy\Big]\, dx \\ &= \int_{\mathbb{R}^N} u^p(x)\int_0^\infty \frac{(|r^\alpha-\rho^\alpha|^p\rho^{\alpha p(p-1)+N-1}}{(r^{p\alpha}+\rho^{p\alpha})^p}\Big(\int_{\mathbb{S}^{N-1}} \frac{dy'}{|\rho y'-rx'|^{N+qs}}\Big) d\rho dx. \end{aligned}$$

We set $\rho = r\sigma$, then

$$\begin{aligned} I &= \int_{\mathbb{R}^N} \frac{u^p(x)}{|x|^{qs}}\int_0^\infty \frac{|1-\sigma^\alpha|^p\sigma^{\alpha p(p-1)+N-1}}{(1+\sigma^{\alpha p})^p}\Big(\int_{\mathbb{S}^{N-1}} \frac{dy'}{|\sigma y'-x'|^{N+qs}}\Big) d\sigma dx \\ &= \int_{\mathbb{R}^N} \frac{u^p(x)}{|x|^{qs}}\int_0^\infty \frac{|1-\sigma^\alpha|^p\sigma^{\alpha p(p-1)+N-1}}{(1+\sigma^{\alpha p})^p} K(\sigma) d\sigma dx \\ &= \mu\int_{\mathbb{R}^N} \frac{u^p(x)}{|x|^{qs}} dx, \end{aligned}$$

where

$$\mu = \int_0^\infty \frac{|1-\sigma^\alpha|^p\sigma^{\alpha p(p-1)+N-1}}{(1+\sigma^{\alpha p})^p} K(\sigma) d\sigma.$$

As in the proof of Lemma 2.4, we can prove that $0<\mu<\infty$. Since $u(x) = v(x)|x|^{-(\frac{N-ps}{p})}$, we get

$$I = \mu\int_{\mathbb{R}^N} \frac{|v(x)|^p}{|x|^{N-s(p-q)}}\, dx.$$

Let $\beta_0 = \frac{N-ps}{2}+\frac{(q-p)s}{2}$, then $\beta_0 < \frac{N-qs}{2}$. Applying Lemma 2.4, we obtain that

$$I \leqslant C \int\limits_{I\!R^N}\int\limits_{I\!R^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+qs}}\, w(x)^{\frac{p}{2}} w(y)^{\frac{p}{2}}\, dx\,dy.$$

and the claim follows.

As a direct consequence of the above estimates, we have proved that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+qs}}\, dx\,dy \leqslant C \int\limits_{I\!R^N}\int\limits_{I\!R^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+qs}}\, w(x)^{\frac{p}{2}} w(y)^{\frac{p}{2}}\, dxdy \quad (3.21)$$

which is the desired result. $\square$

Remark 3.2. Let Ω be a bounded domain such that $0 \in \Omega$ and define

$$H_\Omega(v) = \int_\Omega \int_\Omega \frac{|v(x)-v(y)|^p}{|x-y|^{N+qs}}\, w(x)^{\frac{p}{2}} w(y)^{\frac{p}{2}}\, dxdy$$

where $u \in \mathcal{C}_0^\infty(\Omega)$ and $v(x) = \frac{u(x)}{w(x)}$, then using the same arguments as in the proof of the previous lemma and by the extension Lemma 2.1, we can prove that

$$H_\Omega(v) \geqslant C(\Omega) \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+qs}} dxdy. \tag{3.22}$$

Before proving the main results, we need the next algebraic inequality.

Lemma 3.3. *Assume that $1 \leqslant p \leqslant 2$, then for all $0 \leqslant t \leqslant 1$ and $a \in \mathbb{R}$, we have*

$$|a-t|^p - (1-t)^{p-1}(|a|^p - t) \geqslant C_p \frac{|a-1|^2 t}{(|a-t| + |1-t|)^{2-p}} \tag{3.23}$$

for some positive constant C_p depending only on p.

Proof. Clearly the above inequality is true for $t = 0$ and $t = 1$. In the same way we can prove that (3.23) holds if $a = t$. Then we can assume that $0 < t < 1$ and $a \in \mathbb{R}\backslash\{t\}$.

We define $\alpha = \frac{a-t}{1-t}$. We can rewrite (3.23) in the equivalent form

$$\frac{1}{t}\left\{|\alpha|^p - \frac{|\alpha(1-t)+t|^p - t}{1-t}\right\} \geqslant C_p \frac{(\alpha-1)^2}{(|\alpha|+1)^{2-p}} \tag{3.24}$$

We divide the proof in two cases:

The first case: $\alpha > 0$**.** Notice that if (3.24) holds for $\alpha > 1$ and for all $t \in (0,1)$, then (3.24) holds in particular for $z = 1-t$. Thus

$$\frac{1}{1-z}\left\{|\alpha|^p - \frac{|\alpha z + (1-z)|^p - (1-z)}{z}\right\} \geqslant C_p \frac{(\alpha-1)^2}{(|\alpha|+1)^{2-p}}. \tag{3.25}$$

Now, let $\xi \in (0,1)$ and define $\alpha = \frac{1}{\xi}$, then $\alpha > 1$. Substituting α in (3.25) we obtain that

$$\frac{1}{z}\left\{|\xi|^p - \frac{|\xi(1-z)+z|^p - z}{1-z}\right\} \geqslant C_p \frac{(\xi-1)^2}{(|\xi|+1)^{2-p}}.$$

Thus, it suffices in this case to prove (3.24) for $\alpha \geqslant 1$. For this aim, we define

$$h_\alpha(t) = \frac{|\alpha(1-t)+t|^p - t}{1-t}.$$

By a direct computations we obtain that

$$h'_\alpha(t) = \frac{p(1-\alpha)|\alpha(1-t)+t|^{p-2}(\alpha(1-t)+t) - 1}{1-t} + \frac{h_\alpha(t)}{1-t}$$

and

$$h''_\alpha(t) = \tfrac{1}{(1-t)^3}\Big\{|\alpha(1-t)+t|^{p-2}\Big(2(\alpha(1-t)+t)^2 - 2p(\alpha-1)(1-t)(\alpha(1-t)+t) + p(p-1)(\alpha-1)^2(1-t)^2\Big) - 2\Big\}$$

Using the fact that $\alpha(1-t)+t = (\alpha-1)(1-t)+1$, it follows that

$$h''_\alpha(t) = \tfrac{1}{(1-t)^3} \times \Big\{|\alpha(1-t)+t|^{p-2}\Big(-(2-p)(p-1)(\alpha-1)^2(1-t)^2 + 2(2-p)(\alpha-1)(1-t) + 2\Big) - 2\Big\}$$

Since $\alpha(1-t)+t\geqslant 1$, we claim that $h''_\alpha(t)\leqslant 0$. Indeed, setting $\rho=(\alpha-1)(1-t)$, we clearly have that

$$h''_\alpha(t)=\frac{1}{(1-t)^3}\,T(\rho)=\frac{1}{(1-t)^3}\,(\rho+1)^{p-2}\Big(2+2(2-p)\rho-(2-p)(p-1)\rho^2\Big)-2.$$

A straightforward computations show that $T'(\rho)=-p(p-1)(2-p)\,\frac{(\rho+1)^{p-3}}{(1-t)^3}\leqslant 0$ since $1<p<2$. Hence $T(\rho)\leqslant T(0)=0$ and the claim follows.

Now, using Taylor-Maclaurin Formula

$$h_\alpha(t)-h_\alpha(0)=th'_\alpha(0)+\int_0^t(t-s)h''_\alpha(s)ds,$$

and observing that $h_\alpha(0)=|\alpha|^p$ and $h'_\alpha(0)=-(p-1)|\alpha|^p+p|\alpha|^{p-2}\alpha-1$, we have that

$$\begin{aligned}|\alpha|^p-\frac{|\alpha(1-t)+t|^p-t}{1-t} &= h_\alpha(0)-h_\alpha(t)\\ &= t((p-1)|\alpha|^p-p|\alpha|^{p-2}\alpha+1)+\int_0^t(s-t)h''_\alpha(s)ds,\\ &\geqslant t((p-1)|\alpha|^p-p|\alpha|^{p-2}\alpha+1)\end{aligned}$$

since $h''_\alpha\leqslant 0$ by the previous claim. Therefore we conclude that

$$\frac{1}{t}\left\{|\alpha|^p-\frac{|\alpha(1-t)+t|^p-t}{1-t}\right\}\geqslant((p-1)|\alpha|^p-p|\alpha|^{p-2}\alpha+1).$$

It is clear that

$$((p-1)|\alpha|^p-p|\alpha|^{p-2}\alpha+1)(|\alpha|+1)^{2-p}\geqslant C_p(\alpha-1)^2$$

Hence the result follows in this case.

The second case: $\alpha<0$**.** Setting $\tilde\alpha=-\alpha$, then we need to prove that

$$\frac{1}{t}\left\{|\tilde\alpha|^p-\frac{|\tilde\alpha(1-t)-t|^p-t}{1-t}\right\}\geqslant C_p\frac{(\tilde\alpha+1)^2}{(|\tilde\alpha|+1)^{2-p}}=(1+\tilde\alpha)^p\qquad\forall\,\tilde\alpha>0,\ t\in(0,1).\tag{3.26}$$

It is clear that

$$\frac{1}{t}\left\{|\tilde\alpha|^p-\frac{|\tilde\alpha(1-t)-t|^p-t}{1-t}\right\}=$$
$$\frac{1}{t}\left\{|\tilde\alpha|^p-\frac{|\tilde\alpha(1-t)+t|^p-t}{1-t}\right\}+\frac{1}{t}\left\{\frac{|\tilde\alpha(1-t)+t|^p}{1-t}-\frac{|\tilde\alpha(1-t)-t|^p}{1-t}\right\}$$

We set

$$R_1(\tilde\alpha,t)=\frac{1}{t}\left\{|\tilde\alpha|^p-\frac{|\tilde\alpha(1-t)+t|^p-t}{1-t}\right\}$$

and

$$R_2(\tilde\alpha,t)=\frac{1}{t}\left\{\frac{|\tilde\alpha(1-t)+t|^p}{1-t}-\frac{|\tilde\alpha(1-t)-t|^p}{1-t}\right\}$$

Since $\tilde\alpha>0$, then $R_2(\tilde a,t)\geqslant 0$. Now, using the first step we reach that

$$R_1(\tilde\alpha,t)(|\tilde\alpha|+1)^{2-p})\geqslant C(p)(\tilde\alpha-1)^2.$$

It is clear that, independently of the values of R_2 and $t\in(0,1)$,

$$(\tilde\alpha-1)^2\geqslant\bar C(\tilde\alpha+1)^2\text{ for all }\tilde\alpha\in I\!R^+\setminus(\frac{2}{3},\frac{3}{2}).$$

Hence we will consider the case $\tilde{\alpha} \in [\frac{2}{3}, \frac{3}{2}]$. To get the desired result it suffices to show that

$$R_2(\tilde{\alpha}, t) \geqslant C_2 > 0 \text{ for all } (\tilde{\alpha}, t) \in [\frac{2}{3}, \frac{3}{2}] \times (0,1). \tag{3.27}$$

By a direct computation we get

$$\frac{\partial}{\partial \tilde{\alpha}} R_2(\tilde{\alpha}, t) = \frac{p}{t}\Big(|\tilde{\alpha}(1-t)+t|^{p-1} - |\tilde{\alpha}(1-t)-t|^{p-2}(\tilde{\alpha}(1-t)-t) \Big)$$

This quantity is clearly positive, then R_2 is increasing in $\tilde{\alpha}$ and we obtain that

$$R_2(\tilde{\alpha}, t) \geqslant R_2(\frac{2}{3}, t) = \frac{(\frac{2}{3})^p}{t(1-t)}\Big(|1+\frac{1}{2}t|^p - |1-\frac{5}{2}t|^p \Big) \geqslant C_p \qquad \text{for all } (\tilde{\alpha}, t) \in [\frac{2}{3}, \frac{3}{2}] \times (0,1)$$

Indeed, we use Formula (1.9) with $a = 1 + \frac{1}{2}t$ and $b = |1 - \frac{5}{2}t|$. For $t < \frac{2}{5}$ Formula (1.9) yields

$$|1+\frac{1}{2}t|^p - |1-\frac{5}{2}t|^p \geqslant c\frac{9t^2}{(|1+\frac{1}{2}t| + |1-\frac{5}{2}t|)^{2-p}} + 3pt|1-\frac{5}{2}t|^{p-1}$$

and for $t > \frac{2}{5}$ Formula (1.9) yields

$$|1+\frac{1}{2}t|^p - |1-\frac{5}{2}t|^p \geqslant c\frac{4(1-t)^2}{(|1+\frac{1}{2}t| + |1-\frac{5}{2}t|)^{2-p}} + 2p(1-t)|1-\frac{5}{2}t|^{p-1}.$$

Clearly in the two cases we have that

$$\frac{1}{t(1-t)}\Big(|1+\frac{1}{2}t|^p - |1-\frac{5}{2}t|^p \Big) \geqslant C_p.$$

This proves (3.27). Thus (3.26) follows and then we conclude. □

We have now all the ingredients to prove Theorem 1.2. This is the aim of the next subsection.

3.1. **Proof of Theorem 1.2.** Recalling the definitions given in the proof of Lemma 3.1

$$\frac{|v(x)-v(y)|^p}{|x-y|^{N+qs}}\, w(x)^{\frac{p}{2}} w(y)^{\frac{p}{2}} \quad = \quad f_1(x,y) = f_2(x,y)$$

where

$$f_1(x,y) = \frac{\Big|(u(x)-u(y)) - \dfrac{u(y)}{w(y)}(w(x)-w(y))\Big|^p}{|x-y|^{N+qs}} \left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}}$$

and

$$f_2(x,y) = \frac{\Big|(u(y)-u(x)) - \dfrac{u(x)}{w(x)}(w(y)-w(x))\Big|^p}{|x-y|^{N+qs}} \left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}}.$$

We define the subsets of $\Omega \times \Omega$,

$$D_1 = \{(x,y) \in \Omega \times \Omega : w(y) \leqslant w(x)\} \quad \text{and} \quad D_2 = \{(x,y) \in \Omega \times \Omega : w(x) \leqslant w(y)\}.$$

Then

$$\begin{aligned} \int_\Omega \int_\Omega \frac{|v(x)-v(y)|^p}{|x-y|^{N+qs}}\, w(x)^{\frac{p}{2}} w(y)^{\frac{p}{2}}\, dx\, dy &= \iint_{D_1} f_1(x,y)\, dx\, dy + \iint_{D_2} f_2(x,y)\, dx\, dy \\ &= J_1 + J_2. \end{aligned}$$

We first estimate J_1. To do so, we set

$$I_1(x,y) = \big|(u(x)-u(y)) - \frac{u(y)}{w(y)}(w(x)-w(y))\big|^p \left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}}.$$

We rewrite I_1 as

$$\begin{aligned} I_1(x,y) &= \frac{\big|(u(x)-u(y)) - \frac{u(y)}{w(x)}(w(x)-w(y))\big|^p}{\big(|u(x)-u(y)| + |\frac{u(y)}{w(x)}(w(x)-w(y))|\big)^{(2-p)\frac{p}{2}}} \left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}} \\ &\times \ \big((|u(x)-u(y)| + |\frac{u(x)}{w(x)}(w(x)-w(y))|\big)^{(2-p)\frac{p}{2}} \end{aligned}$$

Using Hölder's inequality, it follows that

$$\begin{aligned} \iint_{D_1} f_1(x,y)\,dx\,dy &\leqslant \left(\iint_{D_1} \frac{\big|(u(x)-u(y)) - \frac{u(y)}{w(y)}(w(x)-w(y))\big|^2}{|x-y|^{N+qs}(|u(x)-u(y)| + |\frac{u(x)}{w(x)}(w(x)-w(y))\big|)^{(2-p)}} \frac{w(y)}{w(x)}\right)^{\frac{p}{2}} \\ &\times \left(\iint_{D_1} \frac{(|(u(x)-u(y))| + |\frac{u(y)}{w(y)}(w(x)-w(y))|)^p}{|x-y|^{N+qs}}\right)^{\frac{2-p}{2}}. \end{aligned} \tag{3.28}$$

Using Remark 3.2, we reach that

$$\iint_{D_1} \frac{(|(u(x)-u(y))| + |\frac{u(y)}{w(y)}(w(x)-w(y))|)^p dxdy}{|x-y|^{N+qs}} \leqslant$$

$$C_1 \int_\Omega\int_\Omega \frac{|(u(x)-u(y))|^p dxdy}{|x-y|^{N+qs}} + \frac{|\frac{u(y)}{w(y)}(w(x)-w(y))|)^p dxdy}{|x-y|^{N+qs}} \leqslant C(\Omega) H_\Omega(v).$$

We deal now with the first term in (3.28). Since $w(y) \leqslant w(x)$ in D_1, then by setting $t = \frac{w(y)}{w(x)}$ and $a = \frac{v(x)}{v(y)}$, we get

$$\begin{aligned} & \frac{\Big|(u(x)-u(y)) - \frac{u(y)}{w(y)}(w(x)-w(y))\Big|^2}{(\Big|u(x)-u(y)| + |\frac{u(x)}{w(x)}(w(x)-w(y))\Big|)^{(2-p)}} \frac{w(y)}{w(x)} \\ = \ & \frac{w^p(x)|v(y)|^p|a-1|^2 t}{(|a-t| + |1-t|)^{2-p}} \\ \leqslant \ & w^p(x)|v(y)|^p\Big(|a-t|^p - (1-t)^{p-1}(|a|^p - t)\Big) \\ = \ & w^p(x)|v(y)|^p\Big[|\frac{v(x)}{v(y)} - \frac{w(y)}{w(x)}|^p - (1-\frac{w(y)}{w(x)})^{p-1}(|\frac{v(x)}{v(y)}|^p - \frac{w(y)}{w(x)})\Big] \\ = \ & |u(x)-u(y)|^p - (w(x)-w(y))^{p-2}(w(x)-w(y))(\frac{|u(x)|^p}{w^{p-1}(x)} - \frac{|u(y)|^p}{w^{p-1}(y)}). \end{aligned}$$

Using the fact that, for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$,

$$|u(x)-u(y)|^p - (w(x)-w(y))^{p-2}(w(x)-w(y))(\frac{|u(x)|^p}{w^{p-1}(x)} - \frac{|u(y)|^p}{w^{p-1}(y)}) \geqslant 0,$$

it follows that

$$
\begin{aligned}
& C(\Omega)\int\int_{D_1} \frac{\left|(u(x)-u(y))-\dfrac{u(y)}{w(y)}(w(x)-w(y))\right|^2}{|x-y|^{N+qs}(|u(x)-u(y)|+|\dfrac{u(x)}{w(x)}(w(x)-w(y))|)^{2-p}}\frac{w(y)}{w(x)}\\
\leqslant \quad & \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|(u(x)-u(y))|^p dxdy}{|x-y|^{N+ps}}\\
- \quad & \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left(\frac{|u(x)|^p}{w^{p-1}(x)}-\frac{|u(y)|^p}{w^{p-1}(y)}\right)|w(x)-w(y)|^{p-2}(w(x)-w(y))dxdy}{|x-y|^{N+ps}}\\
= \quad & G_{s,p}(u).
\end{aligned}
$$

This implies that

$$\int\int_{D_1} f_1(x,y)dxdy \leqslant C(\Omega)G_{s,p}^{\frac{p}{2}}(u)H_{\Omega}^{\frac{2-p}{2}}(v). \tag{3.29}$$

Similarly, by symmetry arguments we obtain that

$$\int\int_{D_2} f_2(x,y)dxdy \leqslant C(\Omega)G_{s,p}^{\frac{p}{2}}(u)H_{\Omega}^{\frac{2-p}{2}}(v). \tag{3.30}$$

Combing (3.29) and (3.30), we get

$$H_{\Omega}(v) \leqslant C(\Omega)G_{s,p}^{\frac{p}{2}}(u)H_{\Omega}^{\frac{2-p}{2}}(v).$$

and then

$$H_{\Omega}(v) \leqslant C(\Omega)G_{s,p}(u)$$

which is the desired result. ■

Proof of Theorem 1.3 The proof follows combining the results of Lemma 3.1 and Theorem 1.2. ■

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