AN IMPROVED HARDY INEQUALITY FOR A NONLOCAL OPERATOR

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Abstract. Let \(0 < s < 1\) and \(1 < p < 2\) be such that \(ps < N\) and let \(Ω\) be a bounded domain containing the origin. In this paper we prove the following improved Hardy inequality:

given \(1 \le q < p\), there exists a positive constant \(C ≡ C(Ω,q,N,s)\) such that

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy - \Lambda_{N,p,s}^N \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} \, dx \ge C \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+qs}} \, dx \, dy
\]

for all \(u \in C^\infty_0(Ω)\). Here \(\Lambda_{N,p,s}^N\) is the optimal constant in the Hardy inequality (1.1).

1. Introduction and statement of main results. In [17], Frank and Seiringer proved the following Hardy inequality: for \(p > 1\) with \(sp < N\) and for all \(\phi \in C^\infty_0(\mathbb{R}^N)\),

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\phi(x) - \phi(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \ge \Lambda_{N,p,s}^N \int_{\mathbb{R}^N} \frac{|\phi(x)|^p}{|x|^{ps}} \, dx \quad (1.1)
\]

where the constant \(\Lambda_{N,p,s}^N\) is given by

\[
\Lambda_{N,p,s}^N = 2 \int_0^\infty |1 - \sigma^{-\gamma}|^{p-2} (1 - \sigma^{-\gamma}) \sigma^{N-\beta-1} K(\sigma) \, d\sigma \quad (1.2)
\]

with

\[
K(\sigma) = \int_{|y'|=1} \frac{dH^{n-1}(y')}{|x' - \sigma y'|^{N+ps}}.
\]

Moreover they proved that if \(p \ge 2\), then if we set

\[
G_{s,p}(u) ≡ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy - \Lambda_{N,p,s}^N \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} \, dx,
\]

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Theorem 1.1. (Authors in [1] obtained an improved Hardy inequality stated in the next theorem.

\[ G_{s,p}(u) \geq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+qs}} w^\frac{2}{q}(x) w^\frac{2}{q}(y) \, dx \, dy \]  

(1.3)

where \( w(x) = |x|^{-\frac{N+qs}{p}} \) and \( v(x) = \frac{u(x)}{w(x)} \). The above inequality turns out to be equality for \( p = 2 \) with \( C = 1 \).

For \( p = 2 \), the author in [15], proves the following improved Hardy inequality,

\[ G_{s,2}(u) \geq C(\Omega, q, N, s) \|u\|^2_{W^{s,2}_0(\Omega)} \]  

for all \( u \in C_0^\infty(\Omega) \) and all \( s/2 < \tau < s \). (1.4)

See also [2] and [14] for alternative proofs without using the Fourier transform.

Lemma 1.4. (1.7) Assume that \(\Omega \subset \mathbb{R}^N\) is a bounded domain, then for all \(1 < q < p\), there exists a positive constant \(C = C(\Omega, q, N, s)\) such that for all \(u \in C_0^\infty(\Omega)\),

\[ G_{s,p}(u) \geq C \int_{\Omega} \int_{\Omega} \frac{|v(x) - v(y)|^p}{|x-y|^{N+qs}} v(x)^\tau v(y)^\tau \, dx \, dy. \]  

(1.5)

The first main aim of this note is to extend inequality (1.3) to the range \( p < 2 \). Precisely we prove the following ground state inequality.

Theorem 1.2. Let \( p < 2 \), \(0 < s < 1\) and \(N > ps\). Assume that \(\Omega \subset \mathbb{R}^N\) is a bounded domain, then for all \(1 < q < p\), there exists a positive constant \(C = C(\Omega, q, N, s)\) such that for all \(u \in C_0^\infty(\Omega)\),

\[ G_{s,p}(u) \geq C \int_{\Omega} \int_{\Omega} \frac{|v(x) - v(y)|^p}{|x-y|^{N+qs}} v(x) w(x)^\tau w(y)^\tau \, dx \, dy. \]  

(1.6)

As a consequence, we get the next improved Hardy inequality.

Theorem 1.3. Suppose that the hypotheses of Theorem 1.2 hold, then for all \(1 < q < p\), there exists a positive constant \(C = C(\Omega, q, N, s)\) such that for all \(u \in C_0^\infty(\Omega)\),

\[ G_{s,p}(u) \geq C \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{N+qs}} \, dx \, dy. \]  

(1.7)

Before closing the introduction we recall the following algebraic inequalities which will be useful throughout the paper. The first result is proved in [17].

Lemma 1.4. ([17]) Assume that \(p > 1\), then for all \(0 \leq t \leq 1\) and \(a \in \mathbb{C}\), we have

\[ |a - t|^p \geq (1 - t)^p - (|a|^p - t). \]  

(1.8)

The second one is proved in [24].

Lemma 1.5. ([24]) For any \(1 < p < 2\) there exists a constant \(c \equiv c(p) > 0\) such that for all \(a, b \in \mathbb{R}^N\) we have :

\[ |a|^p - |b|^p - p|b|^{p-2}(a - b) \geq c \frac{|a - b|^2}{(|a| + |b|)^2 - p} \]  

(1.9)

and

\[ |a + b|^p - |b|^p - p|b|^{p-2}(b, a) \leq c|a|^p. \]  

(1.10)
The paper is organized as follows. In Section 2 we give some nonlinear tools in order to get the improved Hardy inequality. In particular we define the weighted fractional Sobolev space, we prove a Picone’s type inequality in that’s space and the weighted Hardy inequality.

In Section 3 we prove the main results of the paper. We begin by proving Lemma 3.1 that can be seen as a ground state representation. As a consequence we get the proof of Theorems 1.2 and 1.3.

2. Some functional setting. In this section we will introduce some notations, definitions and prove some intermediate lemmas that will be useful later to prove the main results of this paper.

Let $1 \leq q \leq p$, $0 < \beta < \frac{N-2q}{2}$ and $\Omega \subset \mathbb{R}^N$ a smooth bounded domain with $0 \in \Omega$, the weighted Sobolev space $X^{s,p,q,\beta}(\Omega)$ is defined by

$$X^{s,p,q,\beta}(\Omega) := \{ \phi \in L^p(\Omega, \frac{dx}{|x|^{2\beta}}) : \int_{\Omega} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^p}{|x-y|^{N+qs}|x|^{\beta}|y|^{\beta}} dxdy < +\infty \}.$$ 

It is not difficult to prove that $X^{s,p,q,\beta}(\Omega)$ is a Banach space endowed with the norm

$$||\phi||_{X^{s,p,q,\beta}(\Omega)} := \left( \int_{\Omega} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^p}{|x-y|^{N+qs}|x|^{\beta}|y|^{\beta}} dxdy \right)^{\frac{1}{p}}.$$ \hfill (2.11)

Now, we define the weighted Sobolev space $X^{s,p,q,\beta}_0(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the previous norm (2.11).

Following the arguments in [3], see also [12], we can prove the next extension result.

**Lemma 2.1.** Assume that $\Omega \subset \mathbb{R}^N$ is a smooth domain, then for all $w \in X^{s,p,q,\beta}(\Omega)$, there exists $\tilde{w} \in X^{s,p,q,\beta}(\mathbb{R}^N)$ such that $\tilde{w}|_{\Omega} = w$ and

$$||\tilde{w}||_{X^{s,p,q,\beta}(\mathbb{R}^N)} \leq C ||w||_{X^{s,p,q,\beta}(\Omega)}$$

where $C \equiv C(N, s, p, \Omega) > 0$.

**Remark 2.2.** As in the case $\beta = 0$ and $q = p$, if $\Omega$ is a smooth bounded domain, we can endow $X^{s,p,q,\beta}_0(\Omega)$ with the equivalent norm

$$|||\phi|||_{X^{s,p,q,\beta}_0(\Omega)} := \left( \int_{\Omega} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^p}{|x-y|^{N+qs}|x|^{\beta}|y|^{\beta}} dxdy \right)^{\frac{1}{p}}.$$ 

Now, for $w \in X^{s,p,q,\beta}(\Omega)$, we set

$$\tilde{L}(w)(x) \equiv \tilde{L}_{s,p,\beta}(w)(x) = \text{P.V.} \int_{\mathbb{R}^N} \frac{|w(x) - w(y)|^{p-2}(w(x) - w(y))}{|x-y|^{N+qs}|x|^{\beta}|y|^{\beta}} dy.$$ 

It is clear that for all $w, v \in X^{s,p,q,\beta}(\mathbb{R}^N)$, we have

$$\langle \tilde{L}(w), v \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(x) - w(y)|^{p-2}(w(x) - w(y))(v(x) - v(y))}{|x-y|^{N+qs}|x|^{\beta}|y|^{\beta}} dxdy.$$ 

The next result that we will be needed later is the following Picone’s type inequality.
Lemma 2.3. (Picone’s inequality) Let $w \in X_{0}^{s,p,q,\beta}(\Omega)$ be such that $w > 0$ in $\Omega$. Assume that $\tilde{L}(w)(x) \geq 0$, then for all $u \in C_{0}^{\infty}(\Omega)$, we have
\[
\frac{1}{2} \int_{Q} \frac{|u(x) - u(y)|^{p}}{|x - y|^{N+qs}|x^{\beta}|^{\gamma}}
\geq \langle \tilde{L}(w), \frac{|u|^{p}}{w^{p-1}} \rangle
\]
where $Q = \mathbb{R}^{N} \times \mathbb{R}^{N} \setminus (\mathcal{C}_{\Omega} \times C_{\Omega})$.

Proof. This result has been proven for particular cases of $p, q$ and $\beta$. For example, Leonori et al. in [22] treated the case $q = p = 2$ and $\beta = 0$ and then Abdellaoui et al. in [1] proved the result for $q = p \neq 2$. Here, we will give some details for the case $\beta \neq 0$ and $1 < q \leq p$.

We set $v(x) = \frac{|u(x)|^{p}}{|w(x)|^{p-1}}$ and $k(x,y) = \frac{1}{|x-y|^{N+qs}|x^{\beta}|^{\gamma}}$, then
\[
\langle \tilde{L}(w)(w(x)), v(x) \rangle = \int_{\Omega} v(x) \int_{\mathbb{R}^{N}} |w(x) - w(y)|^{p-2}(w(x) - w(y))k(x,y) dy dx
\]
\[= \int_{\Omega} \frac{|u(y)|^{p}}{|w(y)|^{p-1}} \int_{\mathbb{R}^{N}} |w(x) - w(y)|^{p-2}(w(x) - w(y))k(x,y) dy dx.
\]
Since $k$ is symmetric, then we clearly get
\[
\langle \tilde{L}(w)(w(x)), v(x) \rangle = \frac{1}{2} \int_{Q} \left( \frac{|u(x)|^{p}}{|w(x)|^{p-1}} - \frac{|u(y)|^{p}}{|w(y)|^{p-1}} \right) |w(x) - w(y)|^{p-2}(w(x) - w(y))k(x,y) dy dx.
\]
Now, we define $v_{1} := \frac{u}{w}$ so that the above quantity can be rewritten as
\[
\langle \tilde{L}(w)(w(x)), v(x) \rangle = \frac{1}{2} \int_{Q} \left( |v_{1}(x)|^{p}w(x) - |v_{1}(y)|^{p}w(y) \right) |w(x) - w(y)|^{p-2}(w(x) - w(y))k(x,y) dy dx.
\]
We next define
\[
\Phi(x,y) := |u(x)-u(y)|^{p}-(|v_{1}(x)|^{p}w(x) - |v_{1}(y)|^{p}w(y)) |w(x) - w(y)|^{p-2}(w(x) - w(y)).
\]
It follows that
\[\langle \tilde{L}(w)(w(x)), v(x) \rangle + \frac{1}{2} \int_{Q} \Phi(x,y) k(x,y) dy dx
\]
\[= \frac{1}{2} \int_{Q} |u(x) - u(y)|^{p} k(x,y) dy dx.
\]
We claim that $\Phi \geq 0$. Indeed, it is clear, by a symmetry argument, that we can assume $w(x) \geq w(y)$. Then, letting $t = w(y)/w(x)$ and $a = u(x)/u(y)$ and using inequality (1.8), we easily obtain the claim: $\Phi(x,y) \geq 0$. This proves the lemma. 

Now, we prove the following lemma.

Lemma 2.4. Fix $0 < \beta < \frac{N-qs}{2}$ and let $w(x) = |x|^{-\gamma}$ with $0 < \gamma < \frac{N-qs-2\beta}{p-1}$, then there exists a positive constant $\Lambda(\gamma) > 0$ such that
\[
\tilde{L}(w) = \Lambda(\gamma) \frac{w^{p-1}}{|x|^{qs+2\beta}} \quad a.e \quad \in \mathbb{R}^{N}\setminus\{0\}.
\]
Proof. Using polar coordinates: \( r = |x| \) and \( \rho = |y| \) and writing \( x = rx', y = \rho y' \)
with \( |x'| = |y'| = 1 \), we have that

\[
\tilde{L}(w) = \frac{1}{|x|^2} \int_0^{+\infty} |r^{-\gamma} - \rho^{-\gamma}|^{p-2} (r^{-\gamma} - \rho^{-\gamma}) \rho^{N-1} \left( \int_{|y'|=1} \frac{dH^{n-1}(y')}{|x' - \rho y'|^{N+qs}} \right) dp.
\]

Now, using the change of variables \( \sigma = \frac{\rho}{r} \), we clearly get

\[
\tilde{L}(w) = \frac{w^{p-1}(x)}{|x| q s + 2 \sigma} \int_0^{+\infty} |1 - \sigma^{-\gamma}|^{p-2} (1 - \sigma^{-\gamma}) \sigma^{N-\beta-1} \left( \int_{|y'|=1} \frac{dH^{n-1}(y')}{|x' - \sigma y'|^{N+qs}} \right) d\sigma.
\]

We set

\[ K(\sigma) = \int_{|y'|=1} \frac{dH^{n-1}(y')}{|x' - \sigma y'|^{N+qs}}, \]

since \( x' \in S^{N-1} \), then using a suitable change of variable, we reach that \( K \) does not depend on \( x' \), we refer to to the classical book of Grafakos [19] to see this fact.

Now as in [18], we obtain that

\[ K(\sigma) = 2 \pi \frac{N-1}{2} \int_0^{\pi} \sin^{N-2}(\theta) \frac{\sigma^{-N-qs}}{(1 - 2 \sigma \cos(\theta) + \sigma^2)^{\frac{N+qs}{2}}} d\theta. \tag{2.13} \]

It is clear that, for \( \sigma = 0 \), we have

\[ K(0) = 2 \pi \frac{N-1}{2} \int_0^{\pi} \sin^{N-2}(\theta) d\theta \]

and \( K(\sigma) \simeq \sigma^{-N-qs} \) as \( \sigma \to \infty \), moreover for \( \sigma \to 1 \), we can prove that \( K(\sigma) \leq C|\sigma - 1|^{-1-qs} \). Thus we conclude that

\[
\tilde{L}(w) = \frac{w^{p-1}(x)}{|x| q s + 2 \sigma} \int_0^{+\infty} \psi(\sigma) d\sigma,
\]

with

\[ \psi(\sigma) = |1 - \sigma^{-\gamma}|^{p-2} (1 - \sigma^{-\gamma}) \sigma^{N-\beta-1} K(\sigma). \tag{2.14} \]

Defining \( \Lambda(\gamma) := \int_0^{+\infty} \psi(\sigma) d\sigma \), to conclude we just need to show that \( 0 < \Lambda(\gamma) < \infty \). For this aim, it is convenient to write

\[
\Lambda(\gamma) = \int_0^{1} \psi(\sigma) d\sigma + \int_1^{+\infty} \psi(\sigma) d\sigma = I_1 + I_2.
\]

Notice that \( K(\frac{1}{\sigma}) = \xi^{N+qs} K(\xi) \) for any \( \xi > 0 \), then using the change of variable \( \xi = \frac{1}{\sigma} \) in \( I_1 \), it holds

\[
\Lambda(\gamma) = \int_1^{+\infty} K(\sigma)(\sigma^{-1})^{p-1} \left( \sigma^{N-1-\beta-\gamma(p-1)} - \sigma^{\beta+qs-1} \right) d\sigma. \tag{2.15}
\]

As \( \sigma \to \infty \), we have

\[
K(\sigma)(\sigma^{-1})^{p-1} \left( \sigma^{N-1-\beta-\gamma(p-1)} - \sigma^{\beta+qs-1} \right) \simeq \sigma^{-1-\beta-qs} \in L^1((2, \infty)).
\]
Now, as $\sigma \to 1$, we get

$$K(\sigma)(\sigma^\gamma - 1)^{p-1} \left(\sigma^{N-1-\beta-\gamma(p-1)} - \sigma^{\beta+qs-1}\right) \lesssim (\sigma - 1)^{p-1-q} \in L^1((1, 2)).$$

Therefore, combining the above estimates, we get $|\Lambda(\gamma)| < \infty$. On the other hand, since $0 < \gamma < \frac{N - qs - 2\beta}{p-1}$, then $\Lambda(\gamma) > 0$.

As a summary, we have proved that

$$\hat{L}(w) = \Lambda(\gamma) \frac{w^{p-1}}{|x|^{qs+2\beta}} \text{ a.e. in } \mathbb{R}^N \setminus \{0\},$$

where clearly the right hand-side $\frac{w^{p-1}}{|x|^{qs+2\beta}} \in L^1_{\text{loc}}(\mathbb{R}^N)$. This conclude the proof of the desired result.

As a consequence we have the following weighted Hardy inequality.

**Theorem 2.5.** Let $\beta < \frac{N-qs}{2}$, then for all $u \in C_0^\infty(\mathbb{R}^N)$, we have

$$2\Lambda(\gamma) \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{qs+2\beta}} dx \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+qs}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta}, \quad (2.16)$$

where $\Lambda(\gamma)$ is defined in (2.15).

**Proof.** Let $u \in C_0^\infty(\mathbb{R}^N)$ and $w(x) = |x|^{-\gamma}$ with $\gamma < \frac{N - ps - 2\beta}{p-1}$. By Lemma 2.4, we have

$$\hat{L}(w) = \Lambda(\gamma) \frac{w^{p-1}}{|x|^{qs+2\beta}}.$$ 

Since $\frac{w^{p-1}}{|x|^{qs+2\beta}} \in L^1_{\text{loc}}(\mathbb{R}^N)$, then using Picone’s inequality in Lemma 2.3, it follows that

$$\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+qs}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta} \geq (\hat{L}(w), \frac{w^{p-1}}{u^{p-1}}) = \Lambda(\gamma) \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{qs+2\beta}} dx.$$ 

Thus we conclude. \qed

Fix $q \leq p$ and define

$$\Lambda_{N,p,q,s,\beta} := \inf_{\{\phi \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}\}} \frac{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\phi(x) - \phi(y)|^p}{|x-y|^{N+qs}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta}}{\int_{\mathbb{R}^N} \frac{|\phi(x)|^p}{|x|^{qs+2\beta}} dx},$$

then, as in [17] (see also [1]), we can prove that $\Lambda_{N,p,s,\gamma} = 2\Lambda(\gamma_0)$ where $\gamma_0 = \frac{N-\beta-qs}{p}$.

In the case where we consider a smooth bounded domain $\Omega$ of $\mathbb{R}^N$ with $0 \in \Omega$, we can prove the following Hardy inequality.

**Lemma 2.6.** Let $\Omega$ be a smooth bounded domain such that $0 \in \Omega$, then there exists a constant $C \equiv C(\Omega, s, p, N) > 0$ such that for all $u \in C_0^\infty(\Omega)$, we have

$$C \int_{\Omega} \frac{|u(x)|^p}{|x|^{qs+2\beta}} dx \leq \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{N+qs}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta}. \quad (2.17)$$
3. **Proof of the main results.** Let us begin by proving the next Lemma.

**Lemma 3.1.** Let \( u \in C_0^\infty(\mathbb{R}^N) \) and define \( w(x) = |x|^{-\alpha} \) with \( \alpha = \frac{N - ps}{p} \). Consider \( v(x) = \frac{u(x)}{w(x)} \), then for all \( 1 \leq q < p \), we have

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N+qs}} w(x)^{\frac{2}{p}} w(y)^{\frac{2}{q}} dxdy \geq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+qs}} dxdy.
\]

**Proof.** The integrand in the left hand side can be written as

\[
\frac{|v(x) - v(y)|^p}{|x - y|^{N+qs}} w(x)^{\frac{2}{p}} w(y)^{\frac{2}{q}} = \frac{|w(y)u(x) - w(x)u(y)|^p}{|x - y|^{N+qs}} \frac{1}{w(x)^{\frac{2}{p}} w(y)^{\frac{2}{q}}} = \frac{|(u(x) - u(y)) - \frac{u(y)}{w(y)}(w(x) - w(y))|^p}{|x - y|^{N+qs}} \left( \frac{w(y)}{w(x)} \right)^{\frac{2}{q}} =: f_1(x,y).
\]

In the same way, thanks to the symmetry of \( f_1(x,y) \), it immediately follows that

\[
\frac{|v(x) - v(y)|^p}{|x - y|^{N+qs}} (w(x))^{\frac{2}{p}} (w(y))^{\frac{2}{q}} = \frac{|(u(y) - u(x)) - \frac{u(x)}{w(x)}(w(y) - w(x))|^p}{|x - y|^{N+qs}} \left( \frac{w(x)}{w(y)} \right)^{\frac{2}{q}} =: f_2(x,y).
\]

Hence, we can write the integral of the above term as

\[
H(v) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N+qs}} w(x)^{\frac{2}{p}} w(y)^{\frac{2}{q}} dxdy = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_1(x,y) dxdy + \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_2(x,y) dxdy.
\]

We define the quantities

\[
Q(x,y) := \left( \frac{w(x)w(y)}{w(x)^{p} + w(y)^{p}} \right)^{\frac{2}{q}},
\]

and

\[
D(x,y) \equiv \left( \frac{w(x)}{w(y)} \right)^{\frac{2}{q}} + \left( \frac{w(y)}{w(x)} \right)^{\frac{2}{q}} = \frac{w(x)^{p} + w(y)^{p}}{(w(x)w(y))^{\frac{2}{q}}}.
\]

It is clear that \( Q(x,y) \leq \frac{1}{2} \) and \( Q(x,y)D(x,y) = 1 \). Thus using (1.9), one easily obtain

\[
f_1(x,y) \geq C \left( Q(x,y) \left( \frac{w(y)}{w(x)} \right)^{\frac{2}{q}} \right) \times \left[ \frac{|u(x) - u(y)|^p}{|x - y|^{N+qs}} + p \frac{|u(x) - u(y)|^{p-2}}{|x - y|^{N+qs}} \left( u(x) - u(y), \frac{u(y)}{w(y)}(w(x) - w(y)) \right) \right].
\]
It then follows that

\[ f_1(x, y) \geq \left[ CQ(x, y) \left( \frac{w(y)}{w(x)} \right)^{\frac{p}{N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+qs}} \right] \]

\[ - \left[ pCQ(x, y) \left( \frac{w(y)}{w(x)} \right)^{\frac{p}{N}} \frac{|u(x) - u(y)|^{p-1}}{|x-y|^{N+qs}} \frac{u(y)}{w(y)} \right] \left[ (w(x) - w(y)) \right]. \]

The same argument applied to \( f_2 \) yields

\[ f_2(x, y) \geq \left[ CQ(x, y) \left( \frac{w(x)}{w(y)} \right)^{\frac{p}{N}} \frac{|u(y) - u(x)|^p}{|x-y|^{N+qs}} \right] \]

\[ - \left[ pCQ(x, y) \left( \frac{w(x)}{w(y)} \right)^{\frac{p}{N}} \frac{|u(y) - u(x)|^{p-1}}{|x-y|^{N+qs}} \frac{u(x)}{w(x)} \right] \left[ (w(x) - w(y)) \right]. \]

Combining the above two estimates we get

\[ H(v) \geq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} Q(x, y) \left( \left( \frac{w(y)}{w(x)} \right)^{\frac{p}{N}} + \left( \frac{w(x)}{w(y)} \right)^{\frac{p}{N}} \right) \frac{|u(x) - u(y)|^p}{|x-y|^{N+qs}} \, dx \, dy \]

\[ - C_1(p) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left( h_1(x, y) + h_2(x, y) \right) \, dx \, dy, \]

with

\[ h_1(x, y) = Q(x, y) \left( \frac{w(y)}{w(x)} \right)^{\frac{p}{N}} \frac{|u(x) - u(y)|^{p-1}}{|x-y|^{N+qs}} \frac{u(y)}{w(y)} \left[ (w(x) - w(y)) \right], \]

\[ h_2(x, y) = Q(x, y) \left( \frac{w(x)}{w(y)} \right)^{\frac{p}{N}} \frac{|u(x) - u(y)|^{p-1}}{|x-y|^{N+qs}} \frac{u(x)}{w(x)} \left[ (w(x) - w(y)) \right]. \]

Since \( h_1(x, y) \) and \( h_2(x, y) \) are symmetric functions, we just have to estimate the quantity

\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} h_2(x, y) \, dx \, dy. \]

To this aim we use Young’s inequality which yields

\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} h_2(x, y) \, dx \, dy \leq \epsilon \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+qs}} \, dx \, dy \]

\[ + C(\epsilon) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} G(x, y) \, dx \, dy, \]

with

\[ G(x, y) = (Q(x, y))^p \left( \frac{w(x)}{w(y)} \right)^{\frac{p}{N}} \frac{|u(x)|^p |(w(x) - w(y))|^p}{|x-y|^{N+qs}}. \]
We claim that
\[
I := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} G(x,y) \, dx \, dy \leq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x-y|^{N+qs}} \, w(x)^{\frac{q}{2}} \, w(y)^{\frac{q}{2}} \, dx \, dy.
\]
Indeed, notice that
\[
I = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x))^p}{|x-y|^{N+qs}} \frac{(w(x))^{p-2}(|w(x) - w(y)|)^p}{(w(x)^p + w(y)^p)^p} \, dx \, dy,
\]
where
\[
\rho = \left( \frac{1}{2} - \frac{1}{p} \right)^{-1}.
\]
To compute the above integral, we closely follow the arguments used in [18]. We set \( y = \rho y' \) and \( x = rx' \) with \( |x'| = |y'| = 1 \), then
\[
I = \int_{\mathbb{R}^N} u^p(x) \left( \int_{\mathbb{R}^N} \frac{||x|^\alpha - |y|^\alpha|^p}{(|x|^\alpha + |y|^\alpha)^p} \frac{|y|^{\alpha(p-1)}}{|x-y|^{N+qs}} \, dy \right) \, dx.
\]
We set \( \rho = r \sigma \), then
\[
I = \mu \int_{\mathbb{R}^N} \frac{u^p(x)}{|x|^{qs}} \, dx,
\]
where
\[
\mu = \int_0^\infty \frac{|x|^{-(1-\frac{N+qs}{p})} K(\sigma)}{(1+\sigma^p)^{\frac{N+qs}{p}}} \, d\sigma.
\]
As in the proof of Lemma 2.4, we can prove that \( 0 < \mu < \infty \). Since \( u(x) = v(x)|x|^{-\left(\frac{N+qs}{p}\right)} \), we get
\[
I = \mu \int_{\mathbb{R}^N} \frac{|v(x)|^p}{|x|^{N-s(p-q)}} \, dx.
\]
Let \( \beta_0 = \frac{N+qs}{2} + \frac{(q-p)s}{2} \), then \( \beta_0 < \frac{N+qs}{2} \). Applying Lemma 2.4, we obtain that
\[
I \leq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x-y|^{N+qs}} \, w(x)^{\frac{q}{2}} \, w(y)^{\frac{q}{2}} \, dx \, dy.
\]
and the claim follows.

As a direct consequence of the above estimates, we have proved that
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+qs}} \, dx \, dy \leq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x-y|^{N+qs}} \, w(x)^{\frac{q}{2}} \, w(y)^{\frac{q}{2}} \, dx \, dy
\]
which is the desired result. \( \square \)
Remark 3.2. Let Ω be a bounded domain such that 0 ∈ Ω and define
\[ H_Ω(v) = \int_Ω \int Ω \frac{|v(x) - v(y)|^p}{|x - y|^{N+qs}} |w(x)|^\frac{p}{2} w(y)^\frac{p}{2} dxdy \]
where \( u \in C_0^∞(Ω) \) and \( v(x) = \frac{u(x)}{u(x)} \), then using the same arguments as in the proof of the previous lemma and by the extension Lemma 2.1, we can prove that
\[ H_Ω(v) \geq C(Ω) \int_Ω \int Ω \frac{|u(x) - u(y)|^p}{|x - y|^{N+qs}} dxdy. \] (3.22)

Before proving the main results, we need the next algebraic inequality.

Lemma 3.3. Assume that 1 ≤ p ≤ 2, then for all 0 ≤ t ≤ 1 and \( a \in ℝ \), we have
\[ |a - t|^p - (1 - t)^{p-1} |a|^p - t \geq C_p \frac{|a - 1|^2}{t} \] (3.23)
for some positive constant \( C_p \) depending only on \( p \).

Proof. Clearly the above inequality is true for \( t = 0 \) and \( t = 1 \). In the same way we can prove that (3.23) holds if \( a = t \). Then we can assume that \( 0 < t < 1 \) and \( a \in ℝ\setminus\{t\} \).

We define \( \alpha = \frac{a-t}{t} \). We can rewrite (3.23) in the equivalent form
\[ \frac{1}{t} \left( |α|^p - \frac{|α(1-t) + t|^p - t}{1-t} \right) \geq C_p \frac{(α - 1)^2}{(\alpha + 1)^{2-p}}. \] (3.24)

We divide the proof in two cases:

1. **The first case**: \( α > 0 \). Notice that if (3.24) holds for \( α > 1 \) and for all \( t \in (0,1) \), then (3.24) holds in particular for \( z = 1 - t \). Thus
\[ \frac{1}{1-z} \left( |α|^p - \frac{|αz + (1-z)|^p - (1-z)}{z} \right) \geq C_p \frac{(α - 1)^2}{(\alpha + 1)^{2-p}}. \] (3.25)

Now, let \( ξ \in (0,1) \) and define \( α = \frac{1}{ξ} \), then \( α > 1 \). Substituting \( α \) in (3.25) we obtain that
\[ \frac{1}{z} \left( |ξ|^p - \frac{|ξ(1-z) + z|^p - z}{1-z} \right) \geq C_p \frac{(ξ - 1)^2}{(ξ + 1)^{2-p}}. \]

Thus, it suffices in this case to prove (3.24) for \( α \geq 1 \). For this aim, we define
\[ h_α(t) = \frac{|α(1-t) + t|^p - t}{1-t}. \]

By a direct computations we obtain that
\[ h'_α(t) = \frac{p(1-α)|α(1-t) + t|^{p-2}(α(1-t) + t) - 1}{1-t} + \frac{h_α(t)}{1-t} \]
and
\[ h''_α(t) = \left( \frac{1}{1-\frac{1}{2}} \right) \left( |α(1-t) + t|^{p-2} \left( 2(α(1-t) + t)^2 - 2p(α-1)(1-t)(α(1-t) + t) ight. \right. \] \[ \left. \left. + p(α-1)^2(1-t^2) \right) - 2 \right) \]

Using the fact that \( α(1-t) + t = (α - 1)(1-t) + 1 \), it follows that
\[ h''_α(t) = \left( \frac{1}{1-\frac{1}{2}} \right) \times \]
\[ \left\{ |α(1-t) + t|^{p-2} \left( - (2-p)(α-1)^2(1-t)^2 + 2(2-p)(α-1)(1-t) + 2 \right) - 2 \right\} \]
Since $\alpha(1 - t) + t \geq 1$, we claim that $h''(t) \leq 0$. Indeed, setting $\rho = (\alpha - 1)(1 - t)$, we clearly have that

$$h''(t) = \frac{1}{(1 - t)^3} T(\rho) = \frac{1}{(1 - t)^3} (\rho + 1)^{p-2} \left( 2 + 2(2 - p)\rho - (2 - p)(p - 1)\rho^2 \right) - 2.$$ 

A straightforward computations show that $T'(\rho) = -p(p - 1)(2 - p) \frac{(\rho + 1)^{p-3}}{(1 - t)^3} \leq 0$ since $1 < p < 2$. Hence $T(\rho) \leq T(0) = 0$ and the claim follows.

Now, using Taylor-Maclaurin Formula

$$h_\alpha(t) - h_\alpha(0) = th'_\alpha(0) + \int_0^t (t - s)h''_\alpha(s)ds,$$

and observing that $h_\alpha(0) = |\alpha|^p$ and $h'_\alpha(0) = -(p - 1)|\alpha|^p + p|\alpha|^{p-2}\alpha - 1$, we have that

$$|\alpha|^p - \frac{|\alpha(1 - t) + t|^p}{1 - t} = h_\alpha(0) - h_\alpha(t) = t((p - 1)|\alpha|^p - p|\alpha|^{p-2}\alpha + 1) + \int_0^t (s - t)h''_\alpha(s)ds,$$

$$\geq t((p - 1)|\alpha|^p - p|\alpha|^{p-2}\alpha + 1)$$

since $h''_\alpha \leq 0$ by the previous claim. Therefore we conclude that

$$\frac{1}{t} \left\{ |\alpha|^p - \frac{|\alpha(1 - t) + t|^p}{1 - t} \right\} \geq ((p - 1)|\alpha|^p - p|\alpha|^{p-2}\alpha + 1).$$

It is clear that

$$(p - 1)|\alpha|^p - p|\alpha|^{p-2}\alpha + 1)(|\alpha| + 1)^{2-p} \geq C_p(\alpha - 1)^2$$

Hence the result follows in this case.

**The second case:** $\alpha < 0$. Setting $\tilde{\alpha} = -\alpha$, then we need to prove that

$$\frac{1}{t} \left\{ \tilde{\alpha}^{|\tilde{\alpha}|} - \frac{\tilde{\alpha}(1 - t) - t|^{|\tilde{\alpha}|}}{1 - t} \right\} \geq C_p(\frac{\tilde{\alpha} + 1)^{2-p}}{|\tilde{\alpha}| + 1)^{2-p}} = (1 + \tilde{\alpha})^p \quad \forall \tilde{\alpha} > 0, t \in (0, 1).$$

It is clear that

$$\frac{1}{t} \left\{ \tilde{\alpha}^{|\tilde{\alpha}|} - \frac{\tilde{\alpha}(1 - t) - t|^{|\tilde{\alpha}|}}{1 - t} \right\} = \frac{1}{t} \left\{ \tilde{\alpha}^{|\tilde{\alpha}|} - \frac{\tilde{\alpha}(1 - t) - t|^{|\tilde{\alpha}|}}{1 - t} \right\} + \frac{1}{t} \left\{ \tilde{\alpha}(1 - t) - t|^{|\tilde{\alpha}|}}{1 - t} - \frac{\tilde{\alpha}(1 - t) - t|^{|\tilde{\alpha}|}}{1 - t} \right\}$$

We set

$$R_1(\tilde{\alpha}, t) = \frac{1}{t} \left\{ |\tilde{\alpha}|^p - \frac{|\tilde{\alpha}(1 - t) + t|^p}{1 - t} \right\}$$

and

$$R_2(\tilde{\alpha}, t) = \frac{1}{t} \left\{ \frac{|\tilde{\alpha}(1 - t) + t|^p - |\tilde{\alpha}(1 - t) - t|^p}{1 - t} \right\}$$

Since $\tilde{\alpha} > 0$, then $R_2(\tilde{\alpha}, t) \geq 0$. Now, using the first step we reach that

$$R_1(\tilde{\alpha}, t)(|\tilde{\alpha}| + 1)^{2-p} \geq C(p)(\tilde{\alpha} - 1)^2.$$ 

It is clear that, independently of the values of $R_2$ and $t \in (0, 1)$,

$$(\tilde{\alpha} - 1)^2 \geq C(\tilde{\alpha} + 1)^2$$

for all $\tilde{\alpha} \in \mathbb{R}^+ \setminus \left( \frac{2}{3}, \frac{3}{2} \right)$.  

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Hence we will consider the case \( \tilde{\alpha} \in [\frac{2}{3}, \frac{3}{2}] \). To get the desired result it suffices to show that

\[
R_2(\tilde{\alpha}, t) \geq C_2 > 0 \quad \text{for all } (\tilde{\alpha}, t) \in \left[\frac{2}{3}, \frac{3}{2}\right] \times (0, 1).
\]  

(3.27)

By a direct computation we get

\[
\frac{\partial}{\partial \tilde{\alpha}} R_2(\tilde{\alpha}, t) = \frac{p}{t} \left( |\tilde{\alpha}(1 - t) + t|^{p-1} - |\tilde{\alpha}(1 - t) - t|^{p-2}(\tilde{\alpha}(1 - t) - t) \right)
\]

This quantity is clearly positive, then \( R_2 \) is increasing in \( \tilde{\alpha} \) and we obtain that

\[
R_2(\tilde{\alpha}, t) \geq R_2 \left( \frac{2}{3}, t \right) = \frac{(\frac{3}{2})^p}{t(1 - t)} \left( |1 + \frac{1}{2} t|^{p} - |1 - \frac{5}{2} t|^{p} \right) \geq C_p \quad \text{for all } (\tilde{\alpha}, t) \in \left[\frac{2}{3}, \frac{3}{2}\right] \times (0, 1)
\]

Indeed, we use Formula (1.9) with \( a = 1 + \frac{1}{2} t \) and \( b = |1 - \frac{5}{2} t| \). For \( t < \frac{2}{5} \) Formula (1.9) yields

\[
|1 + \frac{1}{2} t|^p - |1 - \frac{5}{2} t|^p \geq c \frac{9t^2}{(1 + \frac{1}{2} t) + (1 - \frac{5}{2} t)^2 - p} + 3pt|1 - \frac{5}{2} t|^{p-1}
\]

and for \( t > \frac{2}{5} \) Formula (1.9) yields

\[
|1 + \frac{1}{2} t|^p - |1 - \frac{5}{2} t|^p \geq c \frac{4(1 - t)^2}{(1 + \frac{1}{2} t) + (1 - \frac{5}{2} t)^2 - p} + 2p(1 - t)|1 - \frac{5}{2} t|^{p-1}.
\]

Clearly in the two cases we have that

\[
\frac{1}{t(1 - t)} \left( |1 + \frac{1}{2} t|^p - |1 - \frac{5}{2} t|^p \right) \geq C_p.
\]

This proves (3.27). Thus (3.26) follows and then we conclude. \( \square \)

We have now all the ingredients to prove Theorem 1.2. This is the aim of the next subsection.

3.1. Proof of Theorem 1.2. Recalling the definitions given in the proof of Lemma 3.1

\[
\frac{|v(x) - v(y)|^p}{|x - y|^{N+qs}} w(x)^\frac{p}{q} w(y)^\frac{p}{q} = f_1(x, y) = f_2(x, y)
\]

where

\[
f_1(x, y) = \frac{|(u(x) - u(y)) - \frac{u(y)}{w(y)}(w(x) - w(y))|}{|x - y|^{N+qs}} \left( \frac{w(y)}{w(x)} \right)^\frac{p}{q}
\]

and

\[
f_2(x, y) = \frac{|(u(y) - u(x)) - \frac{u(x)}{w(x)}(w(y) - w(x))|}{|x - y|^{N+qs}} \left( \frac{w(x)}{w(y)} \right)^\frac{p}{q}.
\]

We define the subsets of \( \Omega \times \Omega \),

\[
D_1 = \{(x, y) \in \Omega \times \Omega : w(y) \leq w(x)\} \quad \text{and} \quad D_2 = \{(x, y) \in \Omega \times \Omega : w(x) \leq w(y)\}.
\]

Then

\[
\int_{\Omega} \int_{\Omega} \frac{|v(x) - v(y)|^p}{|x - y|^{N+qs}} w(x)^\frac{p}{q} w(y)^\frac{p}{q} dx dy = \int_{D_1} f_1(x, y) dx dy + \int_{D_2} f_2(x, y) dx dy = J_1 + J_2.
\]
We first estimate $J_1$. To do so, we set

$$I_1(x, y) = |(u(x) - u(y)) - \frac{u(y)}{w(y)}(w(x) - w(y))|^p \left( \frac{w(y)}{w(x)} \right)^{\frac{p^2}{2}}.$$  

We rewrite $I_1$ as

$$I_1(x, y) = \frac{|(u(x) - u(y)) - \frac{u(y)}{w(x)}(w(x) - w(y))|^p}{(u(x) - u(y)) + \frac{u(y)}{w(x)}(w(x) - w(y))} \left( \frac{w(y)}{w(x)} \right)^{\frac{p^2}{2}} \times \left( \frac{|(u(x) - u(y)) + \frac{u(x)}{w(x)}(w(x) - w(y))|^p}{w(x)} \right)^{\frac{p^2}{2}}.$$  

Using Hölder’s inequality, it follows that

$$\int\int_{D_1} f_1(x, y) \, dx \, dy \leq \left( \int\int_{D_1} \frac{|(u(x) - u(y)) - \frac{u(y)}{w(x)}(w(x) - w(y))|^2}{|x - y|^{N+qs}} \, dx \, dy \right)^{\frac{p}{2}} \times \left( \int\int_{D_1} \frac{|(u(x) - u(y)) + \frac{u(x)}{w(x)}(w(x) - w(y))|^p}{|x - y|^{N+qs}} \, dx \, dy \right)^{\frac{2-p}{2}}. \quad (3.28)$$

Using Remark 3.2, we reach that

$$\int\int_{D_1} \frac{|(u(x) - u(y)) + \frac{u(x)}{w(x)}(w(x) - w(y))|^p \, dx \, dy}{|x - y|^{N+qs}} \leq C(\Omega) H_\Omega(v).$$

We deal now with the first term in (3.28). Since $w(y) \leq w(x)$ in $D_1$, then by setting $t = \frac{w(y)}{w(x)}$ and $a = \frac{w(x)}{w(y)}$, we get

$$\frac{|(u(x) - u(y)) - \frac{u(y)}{w(y)}(w(x) - w(y))|^2}{w(y)} \left( \frac{|u(x) - u(y)| + \frac{w(x)}{w(y)}(w(x) - w(y))}{w(x)} \right)^{2-p} \leq w^p(x)|v(y)|^p|a - t|^2 \left( |a - t| + |1 - t| \right)^{2-p} \leq w^p(x)|v(y)|^p \left( |a - t|^p - (1 - t)^p - 1 \right).$$

Using the fact that, for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$,

$$|u(x) - u(y)|^p - (w(x) - w(y))^{p-2}(w(x) - w(y))(\frac{|u(x)|^p}{w^{p-1}(x)} - \frac{|u(y)|^p}{w^{p-1}(y)}) \geq 0,$$
it follows that
\[
C(\Omega) \int_{D_1} \frac{|(u(x) - u(y)) - u(y)|}{|x-y|^{N+\eta s}}\left(|u(x) - u(y)| + \left|\frac{u(x)}{w(x)}(w(x) - w(y))\right|^2\right) w(y) \, dxdy 
\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|(u(x) - u(y))|^p}{|x-y|^{N+ps}} dxdy 
\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\left(\frac{|u(x)|^p}{w^{p-1}(x)} - \frac{|u(y)|^p}{w^{p-1}(y)}\right)|w(x) - w(y)|^{p-2}(w(x) - w(y))}{|x-y|^{N+ps}} dxdy 
= G_{s,p}(u).
\]
This implies that
\[
\int_{D_1} f_1(x,y) dxdy \leq C(\Omega) G_{s,p}^p(u) H^{2-p}_{\Omega} (v). \tag{3.29}
\]
Similarly, by symmetry arguments we obtain that
\[
\int_{D_2} f_2(x,y) dxdy \leq C(\Omega) G_{s,p}^p(u) H^{2-p}_{\Omega} (v). \tag{3.30}
\]
Combining (3.29) and (3.30), we get
\[
H_{\Omega}(v) \leq C(\Omega) G_{s,p}^p(u) H^{2-p}_{\Omega} (v).
\]
and then
\[
H_{\Omega}(v) \leq C(\Omega) G_{s,p}(u)
\]
which is the desired result. 

**Proof of Theorem 1.3** The proof follows combining the results of Lemma 3.1 and Theorem 1.2.

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