

MULTI-BUBBLING FOR THE EXPONENTIAL NONLINEARITY IN THE SLIGHTLY SUPERCRITICAL CASE

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ABSTRACT

We consider radial solutions of an equation involving a p -Laplacian type operator and an exponential nonlinearity in dimension n , which turns out to be critical for $p = n$. For such a nonlinearity, the equation can be reduced to an autonomous ODE, thus allowing a very precise study of the *multi-bubbling* phenomenon as the solutions in the critical case are approached by solutions corresponding to the supercritical case $p < n$.

Keywords. Emden-Fowler equation, Gelfand problem, supercritical case, bifurcation diagram, Emden-Fowler transform, p -Laplacian, branches of solutions, critical and super-critical problems, bubbles, spikes, multi-peaks, dynamical systems, phase plane analysis

1. Introduction and main result

Since the pioneering work of Joseph and Lundgreen [26], it is well known that the solutions of

$$-\Delta u = \lambda e^u \tag{1.1}$$

with zero Dirichlet boundary conditions in the unit ball Ω of \mathbb{R}^n have different behaviours depending on the dimension: see [30, 2] for reviews of related results. This equation is used in stellar dynamics, combustion and chemotaxis models. It is often called the Emden-Fowler equation [10, 23, 31] or Gelfand's problem. Let us summarize the main properties of the bifurcation diagrams in $L^\infty(\Omega)$, in terms of the parameter λ .

- (1) If $n = 2$, the branch of bounded solutions has an asymptote at $\lambda = \lambda^* = 0$, the equation has exactly two solutions for any $\lambda \in (0, \lambda_1^+)$ and no solution if $\lambda > \lambda_1^+$. These solutions are moreover explicit [3]. See Fig. 5.
- (2) If $2 < n < 10$, the branch of bounded solutions has an asymptote at $\lambda = \lambda^* > 0$, the equation has at least one solution for any $\lambda \in (0, \lambda_1^+)$, $\lambda_1^+ > \lambda^*$, and no solution if $\lambda > \lambda_1^+$. The branch of solutions oscillates around $\lambda = \lambda^*$. See Fig. 3.

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- (3) If $n \geq 10$, the branch of bounded solutions has an asymptote at $\lambda = \lambda^* > 0$, the equation has exactly one solution for any $\lambda \in (0, \lambda^*)$ and no solution if $\lambda > \lambda^*$.

In any case, $\lambda^* = 2(n-2)$, and it is known that for $\lambda = \lambda^*$, $n > 2$, there exists a unique radial singular solution u^* such that

$$e^{u^*(x)} = \frac{1}{|x|^2},$$

see [7, 8, 9, 30]. As long as we consider bounded solutions, these solutions are smooth and therefore radial according to [24]. Equation (1.1) is then equivalent to

$$\begin{cases} u'' + \frac{n-1}{r} u' + \lambda e^u = 0, & r \in (0, 1) \\ u'(0) = 0, \quad u(1) = 0 \end{cases} \quad (1.2)$$

and it is very natural to consider now n as a real parameter. Note that looking for radial solutions of the equation

$$|x|^{N-2-\varepsilon} \operatorname{div}(|x|^{-(N-2-\varepsilon)} \nabla u) + \lambda e^u = 0$$

in the unit ball of \mathbb{R}^N , $N \in \mathbb{N}$, with zero Dirichlet boundary conditions, is exactly equivalent to solving Equation (1.2) with $n = 2 + \varepsilon$. The above discussion still applies and we will actually recover it as a consequence of our results. If we look at the bifurcation diagram, then $n = 2$ appears as the *critical* case, while $n = 2 + \varepsilon$, $\varepsilon > 0$ is *supercritical*, in analogy with the situation observed in the Brezis-Nirenberg problem [5].

Let us be a little bit more specific about criticality in the Brezis-Nirenberg problem. Consider

$$-\Delta u = u^p + \lambda u$$

in Ω , with zero Dirichlet boundary conditions on $\partial\Omega$, for which, for $n \geq 3$, the *critical* exponent is $(n+2)/(n-2)$. In terms of the parameter λ , the first branch is monotone decreasing for $p = (n+2)/(n-2)$ and oscillating in the *supercritical* regime $p > (n+2)/(n-2)$, around an asymptotic value $\lambda = \lambda^*$. By first branch, we mean the branch of positive radial bounded solutions which bifurcates from the trivial solution at the first eigenvalue of $-\Delta$. In the supercritical regime, there exists a radial singular solution if and only if $\lambda = \lambda^*$ [29]. How the slightly supercritical regime approaches the critical one has recently been studied in a series of papers [14, 15, 16] (also see [17, 18, 19]) and the analogy with the case of the exponential nonlinearity has been underlined in [15].

Rather than solutions of (1.2), we shall consider radial solutions corresponding to the more general supercritical equation

$$\begin{cases} \Delta_p u + \lambda e^u = 0 & \text{in } \Omega \\ u > 0, \quad u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.3)$$

with $p < n$, where Ω is the unit ball in \mathbb{R}^n . Here we use the standard notation $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$. Written in radial coordinates, the equation is

$$\begin{cases} \Delta_{p,n} u + \lambda e^u = 0, & r \in (0, 1) \\ u(0) > 0, \quad \frac{du}{dr}(0) = 0, \quad u(1) = 0 \end{cases} \quad (1.4)$$

where

$$\Delta_{p,n}u := \frac{1}{r^{n-1}} \frac{d}{dr} \left(r^{n-1} \left| \frac{du}{dr} \right|^{p-2} \frac{du}{dr} \right)$$

is the radial version of Δ_p in dimension n . Here we abusively use the same notations for the solutions of Equation (1.3) and Equation (1.4).

In the rest of this paper, we shall assume that $p > 1$. Unless it is explicitly specified, by solutions we mean bounded solutions. As long as we consider solutions of (1.4), it is possible to regard p and n as two independent real parameters. The small parameter in the slightly supercritical regime is now $\varepsilon = n - p > 0$. The properties of the bifurcation diagram for $p > 1$ are very similar to the ones of the special case $p = 2$ [25] (see Fig. 1):

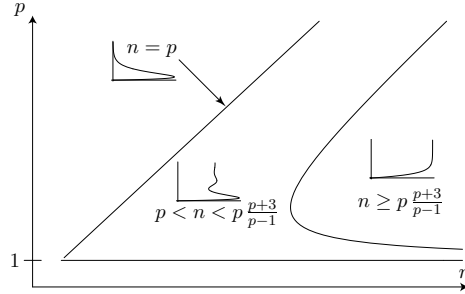


FIGURE 1. Types of bifurcation diagrams in terms of n and p .

- (1) If $n = p$, the branch of bounded solutions has an asymptote at $\lambda = 0$ (see Fig. 5) and the equation has exactly two solutions for any $\lambda \in (0, \lambda_1^+)$.
- (2) If $p < n < p(p+3)/(p-1)$ which means

$$1 < p < n \quad \text{if } 1 < n < 9$$

$$1 < p < p_-(n) \quad \text{or} \quad p_+(n) < p < n \quad \text{if } n \geq 9$$

for $p_{\pm}(n) = \frac{1}{2}[n - 3 \pm \sqrt{(n-1)(n-9)}]$, the branch of bounded solutions has an asymptote at $\lambda = \lambda^* := p^{p-1}(n-p) > 0$ (see Fig. 3) and the equation has at least one solution for any $\lambda \in (0, \lambda_1^+)$, $\lambda_1^+ > \lambda^*$ if $3 \leq n < 10$. The branch oscillates around $\lambda = \lambda^*$. There is a unique radial singular solution $u^* := -p \log r$, for $\lambda = \lambda^*$.

- (3) The branch of bounded solutions has an asymptote at $\lambda = \lambda^* > 0$ and the equation has exactly one solution for any $\lambda \in (0, \lambda^*)$ if $n \geq p(p+3)/(p-1)$, or, in terms of p , if $p \in [p_-(n), p_+(n)]$.

We are interested in understanding how the *supercritical regime* approaches the *critical regime*, i.e., in what happens when n approaches 2 from above for $p = 2$ fixed, or what happens when $p \rightarrow n$ from below, $n > 1$ fixed. Our main result is that the solutions exhibit a *multi-bubbling* phenomenon.

THEOREM 1. *Let k be a positive integer. There exists a positive constant $\bar{\lambda}_k^+$ such that for any $\lambda \in (0, \bar{\lambda}_k^+)$, the following property holds: if $(\varepsilon_i)_{i \in \mathbb{N}}$ is a sequence of positive numbers with $\lim_{i \rightarrow +\infty} \varepsilon_i = 0$, then up to the extraction of a subsequence,*

there exist k functions w_j^k and, for any $i \in \mathbb{N}$, k parameters $\mu_{i,j}$, $j = 1, 2, \dots, k$, for which

$$\lim_{i \rightarrow +\infty} (\mu_{i,j+1} - \mu_{i,j}) = +\infty \quad \forall j = 1, 2, \dots, k-1,$$

such that, with $\varepsilon = \varepsilon_i$, Equation (1.3) has a solution u^{ε_i} which can be written as:

$$\lambda |x|^p e^{u^{\varepsilon_i}(x)} = \left[\sum_{j=1}^k w_j^k \left(\log |x| + \mu_{i,j}(\varepsilon_i) \right) \right] (1 + o(1)) \quad \text{as } i \rightarrow +\infty$$

uniformly on Ω . Here the functions w_j^k are smooth, even, positive and such that $w_j^k(s) \searrow 0$ as $s \rightarrow \pm\infty$. Besides, $\bar{\lambda}_j^+ \leq \max w_j^k$ and $(\bar{\lambda}_k^+)_{k \in \mathbb{N}}$ is a strictly decreasing sequence.

Numerical evidence suggests the following conjecture: w_j^k depends neither on k nor on the particular sequence $(\varepsilon_i)_{i \in \mathbb{N}}$. Moreover, and this is the main difference with the Brezis-Nirenberg problem, it turns out that these functions differ from each other and for instance $(m_j^k := \max w_j^k)_j$ is strictly decreasing in $j = 1, 2, \dots, k$. Although the $w_j^k = -y'$, where y solves the ODE:

$$y' + m_j^k = \frac{|y|^{p^*}}{p^*} + p y + p^{p-1}, \quad y(0) = -p^{p-1}$$

where p^* is the Hölder conjugate of p . The special case $p = 2$ is simpler:

$$w_j^k(s) = m_j^k \left[1 - \left(\tanh \left(\sqrt{m_j^k/2} s \right) \right)^2 \right].$$

Multi-bubbling at a single point has already been observed in relation with critical exponent, see for instance [11]. In the Brezis-Nirenberg problem, it has been studied using a Lyapunov-Schmidt reduction in [14, 16]. Here, since the problem is reduced to an ODE, we use more direct methods based on phase plane analysis. By means of a generalized Emden-Fowler change of variables, the study of radial solutions of (1.3) reduces to the analysis of an autonomous ODE system whose qualitative behaviour is completely understood. This allows us to describe in a fairly precise way how the supercritical regime $p < n$ approaches the critical case $p = n$.

The interest of the Emden-Fowler transformation [21] is that it decouples the scales at which concentrations occur. The superposition of bubbles is then reduced to a more standard multi-peak problem. In terms of dynamical systems, the transformed equation describes a trajectory corresponding to an heteroclinic orbit, which is very degenerate in the critical limit.

A main feature of the exponential nonlinearity compared to power laws is that the bubbles do not have all the same shape. The extension to non radial geometries is perhaps possible through variational methods, but as detailed estimates as those given in Sections 3 and 3.2 seem for the moment out of reach by such approaches.

In the case of a star-shaped domain, the existence of a critical value $\bar{\lambda}_1^+$ which is also the first turning point of the bifurcation diagram in a ball has been extensively studied after [32, 33, 23]. For completeness, let us mention that for the nonradial case in dimension 2, the investigation of possible behaviours of the solutions when $\lambda \rightarrow 0$ has been the subject of [34, 35, 1] and extended to the case of an external potential or weight in [4, 27]. For the fact that endpoints of the branch are still weak solutions, we shall refer to [4, 8]. The case of the n -Laplacian has not been

studied as much, but one can for instance refer to [12, 22, 28, 25]. Many other papers deal with related evolution problems, see for instance [20], or with other geometrical situations like annuli and tori, which are out of the scope of our study.

This paper is organized as follows. We first perform a generalized Emden-Fowler change of variables which is then used to parametrize the solutions both in the supercritical and in the critical case. Multi-bubbling (proof of Theorem 1) easily follows up to two essential properties whose proofs are rejected in Section 3: one can consider as many bubbles as desired, and these bubbles have different heights.

The results of this paper have been announced without proofs (to be precise, without the estimates of Section 3) in [15], where the emphasis was put on the analogy with the Brezis-Nirenberg problem in the slightly supercritical case.

2. Preliminary results

We are going to state a series of very elementary results, which provides the proof of Theorem 1, up to the two properties which are summarized in Lemma 7 and whose proof is the subject of Section 3.

2.1. The generalized Emden-Fowler change of variables

Since (1.3) is invariant under rotations, for bounded solutions it makes sense to restrict the study to the case of radial solutions. See [13] and [6] for some recent result on the symmetry properties of the solutions. Let u be a solution of Equation (1.4). For $r = e^s$, $s \in (-\infty, 0]$, define $v(s) := u(r)$. Then (1.4) is equivalent to

$$\begin{cases} (p-1)|v'|^{p-2}v'' + (n-p)|v'|^{p-2}v' + \lambda e^{v+ps} = 0, & s \in (-\infty, 0) \\ \lim_{s \rightarrow -\infty} v(s) > 0, & \lim_{s \rightarrow -\infty} e^{-s}v'(s) = 0, & v(0) = 0 \end{cases}$$

where $v' = \frac{dv}{ds}$. Note that the change of variables means that

$$\lim_{s \rightarrow -\infty} v(s) = u(0).$$

The equation for v can be reduced to an *autonomous* ODE system as follows. Let

$$x(s) = \lambda e^{v(s)+ps} \quad \text{and} \quad y(s) = |v'(s)|^{p-2}v'(s).$$

Then

$$\begin{cases} x' = x(v' + p) \\ y' = (p-1)|v'|^{p-2}v'' \end{cases}$$

and (1.4) is finally equivalent to the system

$$\begin{cases} x' = x(|y|^{p^*-2}y + p), & x(0) = \lambda \\ y' = (p-n)y - x, & \lim_{s \rightarrow -\infty} e^{-s}|y(s)|^{p^*-2}y(s) = 0 \end{cases} \quad (2.1)$$

where $p^* = (1 - 1/p)^{-1}$ is the Hölder conjugate of p , so that $y = |v'|^{p-2}v' \iff v' = |y|^{p^*-2}y$. The change of coordinates is somewhat classical, see for instance [3, 21, 10, 23, 26, 30, 2, 25], at least for $p = 2$. The novelty of our approach is to use it in order to understand the limit $n - p = \varepsilon \rightarrow 0$, $\varepsilon > 0$.

2.2. Parametrization of the solutions

The behaviour of the solutions easily follows from the study of the vector field and a linearization around the two fixed points: $P^- = (0, 0)$ and $P^+ = p^{p-1}(n-p, -1)$. The linearization of (2.1) at P^- is

$$\begin{pmatrix} X \\ Y \end{pmatrix}' = \begin{pmatrix} p & 0 \\ -1 & -(n-p) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

with eigenvalues p and $-(n-p)$, and, at P^+ ,

$$\begin{pmatrix} X \\ Y \end{pmatrix}' = \begin{pmatrix} 0 & p(n-p)/(p-1) \\ -1 & -(n-p) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

with eigenvalues

$$\frac{1}{2} \sqrt{n-p} \left(-\sqrt{n-p} \pm i \sqrt{p(p+3)/(p-1) - n} \right),$$

as long as $n < p(p+3)/(p-1)$.

LEMMA 2. *Assume that $p < n < p(p+3)/(p-1)$. Then the following properties hold:*

- (i) P^- is hyperbolic and P^+ is elliptic.
- (ii) Any trajectory of (2.1) is such that $x(s)$ does not change sign. Any trajectory with $x > 0$ enters the lower quadrant corresponding to $x > 0, y < 0$.
- (iii) P^+ (resp. P^-) is attracting all trajectories with $x > 0$ as $s \rightarrow +\infty$ (resp. all bounded trajectories with $x > 0$ as $s \rightarrow -\infty$).
- (iv) There exists a bounded trajectory $s \mapsto (x(s), y(s))$ such that

$$\lim_{s \rightarrow \pm\infty} (x(s), y(s)) = P^\pm.$$

This heterocline trajectory is unique, up to any translation in s .

Note that for $n \geq p(p+3)/(p-1)$, to the linearization of (2.1) at P^+ corresponds two negative eigenvalues, so that the trajectory connecting P^- to P^+ is unique, up to any translation in s , and monotone in y . As a consequence, we recover for instance that for $p = 2, n \geq 10$, the branch of the solutions of (1.3) in $L^\infty(\Omega)$ is monotone. From now on we assume that $p \leq n < p(p+3)/(p-1)$. Let (\bar{x}, \bar{y}) be the unique trajectory such that $\lim_{s \rightarrow -\infty} (\bar{x}(s), \bar{y}(s)) = P^-$ and $\bar{x}(0) = \max_{s \in \mathbb{R}} \bar{x}(s)$. In order to emphasize the dependence on ε , we shall write $(\bar{x}^\varepsilon, \bar{y}^\varepsilon)$ whenever needed.

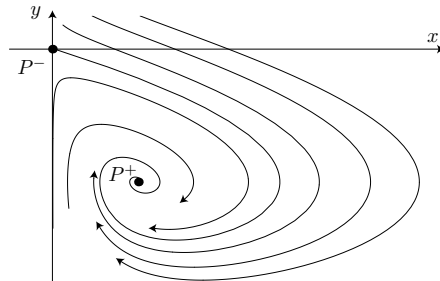


FIGURE 2. Phase portrait in the supercritical case $p < n < p \frac{p+3}{p-1}$ (here $n = 2, p = 1.5$).

The proofs of Lemma 2 and of the next result are standard (see Fig. 3) and therefore left to the reader.

LEMMA 3. *Assume that $p \leq n < p(p+3)/(p-1)$. For a given λ , to any solution v of (1.4) corresponds a unique $s_0 \in \mathbb{R}$ such that*

$$\lambda e^{v(s)+ps} = \bar{x}(s+s_0)$$

for any $s \leq 0$. Reciprocally, for any $\lambda \in (0, \lambda_1^+]$, where $\lambda_1^+ := \max_{s \in \mathbb{R}} \bar{x}(s) = \bar{x}(0)$, the equation $\bar{x}(s_0) = \lambda$ has at least one solution and

$$v(s) = \log \left(\frac{1}{\lambda} \bar{x}(s+s_0) \right) - ps$$

is a solution of (1.4).

Note that with the change of variables $s = t - s_0$,

$$v(t-s_0) = \log \left(\frac{\bar{x}(t)}{\bar{x}(s_0)} \right) - pt + ps_0 \quad \forall t \in (-\infty, s_0).$$

The corresponding solution u of (1.3) is fully determined by $\lambda = \bar{x}(s_0)$, $u'(0) = 0$ and

$$u(0) = \lim_{t \rightarrow -\infty} v(t-s_0) = \lim_{t \rightarrow -\infty} \log \left(\frac{\bar{x}(t) e^{-pt}}{\bar{x}(s_0) e^{-ps_0}} \right).$$

2.3. The supercritical case: $p < n$

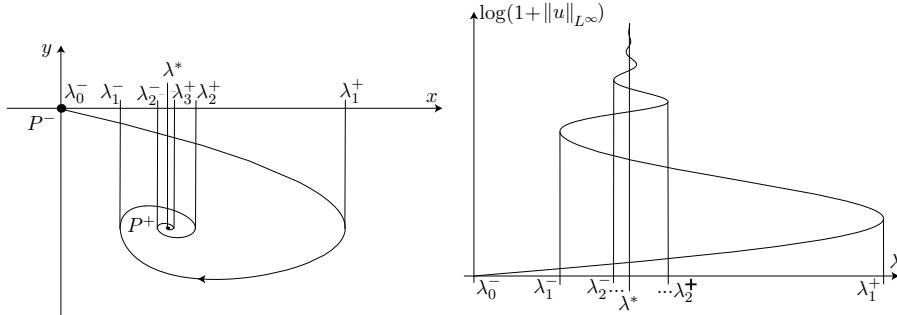


FIGURE 3. Parametrization of the solutions of (1.3) in the supercritical case ($n = 2$, $p = 1.5$). Left: (\bar{x}, \bar{y}) in the phase space. Right: the bifurcation diagram for (1.3).

The parametrization in Lemma 3 is a straightforward consequence of the Emden-Fowler change of coordinates. The next result only involves an elementary phase plane analysis which is described in Fig. 3. Details of the proof are left to the reader.

LEMMA 4. *Let $\lambda^* = p^{p-1}(n-p)$. Assume that $p < n < p(p+3)/(p-1)$. There exists two sequences $(\lambda_k^-)_{k \geq 1}$ and $(\lambda_k^+)_{k \geq 1}$ such that:*

- (i) $(\lambda_k^-)_{k \geq 1}$ is increasing and $\lim_{k \rightarrow +\infty} \lambda_k^- = \lambda^*$.
- (ii) $(\lambda_k^+)_{k \geq 1}$ is decreasing and $\lim_{k \rightarrow +\infty} \lambda_k^+ = \lambda^*$.
- (iii) Equation (1.4) has no solutions if $\lambda > \lambda_1^+$, $2k-1$ solutions if $\lambda = \lambda_k^+$ or $\lambda \in (\lambda_{k-1}^-, \lambda_k^-)$ with the convention $\lambda_0^- = 0$, and $2k$ solutions if $\lambda = \lambda_k^-$ or $\lambda \in$

$(\lambda_{k+1}^+, \lambda_k^+)$, $k \geq 1$.

(iv) Equation (1.4) has infinitely many solutions if and only if $\lambda = \lambda^*$.

2.4. *The critical case: $p = n$*

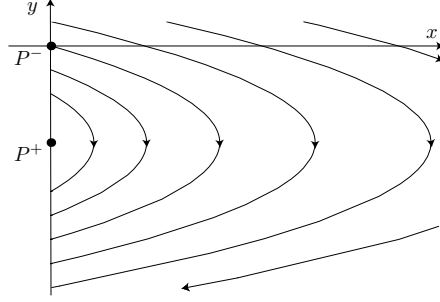


FIGURE 4. Phase portrait in the critical case $n = p$ (here $n = 2$).

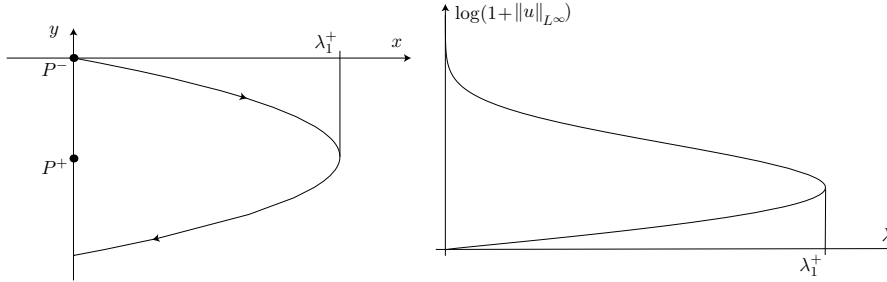


FIGURE 5. Parametrization of the solutions of (1.3) in the critical case ($n = p = 2$).
Left: (\bar{x}, \bar{y}) in the phase space. Right: the bifurcation diagram for (1.3).

In the limit case $p = n$, (2.1) becomes an Hamiltonian system:

$$x' = x(|y|^{p^*-2}y + p), \quad y' = -x, \quad (2.2)$$

which is explicitly solvable in the case $p = 2$ [3]: $u(r) = 2 \log(a^2 + 1) - 2 \log(a^2 + r^2)$ is a solution of (1.3) for any $a > 0$ such that $\lambda = 8a^2(a^2 + 1)^{-2}$. See Fig. 4. The counterpart of Lemma 4 in the critical case is the

LEMMA 5. *Assume that $p = n$ and let $\lambda_1^+ := \sup_{s \in \mathbb{R}} \bar{x}(s)$. Then Equation (1.4) has no solutions if $\lambda > \lambda_1^+$, one and only one solution if $\lambda = \lambda_1^+$ and two and only two solutions if $\lambda \in (0, \lambda_1^+)$.*

2.5. *Description of the critical limit*

This regime corresponds to the limit $\varepsilon = n - p \rightarrow 0$, $\varepsilon \geq 0$. For any $\varepsilon > 0$ (resp. $\varepsilon = 0$), define by $s_k(\varepsilon)$ (resp. $s_1(0)$) the sequence of the points of local maximum of \bar{x}^ε (resp. the unique point of maximum of \bar{x}^0), where $(\bar{x}^\varepsilon, \bar{y}^\varepsilon)$ is the unique

trajectory such that

$$\lim_{s \rightarrow \pm\infty} (\bar{x}^\varepsilon(s), \bar{y}^\varepsilon(s)) = P^\pm$$

and $\bar{x}^\varepsilon(0) = \max_{s \in \mathbb{R}} \bar{x}^\varepsilon(s) =: \lambda_1^+(\varepsilon)$. Note that as a consequence, $\bar{y}^\varepsilon(0) = -p^{p-1}$. By definition of $(\bar{x}^\varepsilon, \bar{y}^\varepsilon)$,

$$s_1(\varepsilon) = 0 \quad \forall \varepsilon \in [0, p \frac{p+3}{p-1} - n) .$$

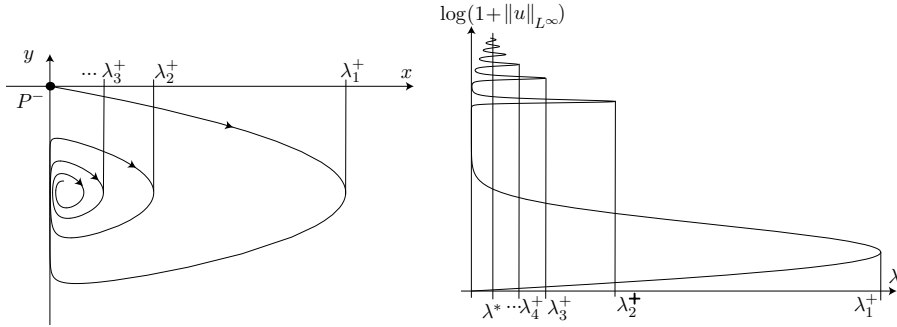


FIGURE 6. *Left: the solution $(\bar{x}^\varepsilon, \bar{y}^\varepsilon)$ in the slightly supercritical regime $\varepsilon > 0$, $\varepsilon \rightarrow 0$. Right: the corresponding bifurcation diagram for (1.3). Here $n = 2$, $\varepsilon = 0.05$.*

LEMMA 6. *For any $k \geq 1$, $\lim_{\varepsilon \rightarrow 0} [s_{k+1}(\varepsilon) - s_k(\varepsilon)] = +\infty$.*

The proof easily follows from the properties of the phase plane (see Fig. 6). To study the critical limit, we emphasize the dependence on ε . Let $\lambda_k^{\varepsilon,+} = \bar{x}^\varepsilon(s_k(\varepsilon))$. According to Lemma 4, $(\lambda_k^{\varepsilon,+})_{k \geq 1}$ is a positive decreasing sequence. Define $\bar{\lambda}_k^+ := \lim_{\varepsilon \rightarrow 0} \lambda_k^{\varepsilon,+}$. It is not clear that for any sequence $(\varepsilon_i)_{i \in \mathbb{N}}$ with $\varepsilon_i > 0$, $\lim_{i \rightarrow \infty} \varepsilon_i = 0$, the limit of $\lambda_k^{\varepsilon_i,+}$ is unique and well defined so that one should consider a special sequence $(\varepsilon_i)_{i \in \mathbb{N}}$ and potentially extract subsequences. For the sake of simplicity, we will speak of “the limit $\varepsilon \rightarrow 0$ ” in the rest of this section.

LEMMA 7. *For any $k \geq 1$,*

$$\bar{\lambda}_k^+ > 0 \tag{P_k}$$

and $\bar{\lambda}_1^+ = \lambda_1^{0,+}$. Moreover $(\bar{\lambda}_k^+)_{k \in \mathbb{N}}$ is a strictly decreasing sequence.

As seen above, Property (P_1) is always satisfied: the property $\bar{\lambda}_1^+ = \lambda_1^{0,+}$ is easy. In Section 3 we will prove the rest of Lemma 7.

It is now possible to give a precise description of the asymptotic behaviour of the solutions of (1.3) as $\varepsilon \rightarrow 0$. Let (x_k, y_k) be the solution of (2.2) with

$$x_k(0) = \bar{\lambda}_k^+ \quad \text{and} \quad y_k(0) = -p^{p-1} ,$$

for any $k \geq 1$. With these notations, $(x_1, y_1) = (\bar{x}^0, \bar{y}^0)$ but $(x_k, y_k) \neq (\bar{x}^0, \bar{y}^0)$ for any $k \geq 2$.

COROLLARY 8. *For any $j \geq 1$, $\bar{x}^\varepsilon(\cdot + s_j(\varepsilon))$ converges to x_j uniformly on compact subset in \mathbb{R} .*

Corollary 8 can be rephrased into

COROLLARY 9. *For any $k \geq 1$, as $\varepsilon \rightarrow 0$,*

$$\bar{x}^\varepsilon(s) \rightarrow \sum_{j=1}^k x_j(s - s_j(\varepsilon))$$

uniformly on any interval $(-\infty, a(\varepsilon)) \in \mathbb{R}$ such that $s_k(\varepsilon) < a(\varepsilon) < s_{k+1}(\varepsilon)$ with $\liminf_{\varepsilon \rightarrow 0} (s_{k+1}(\varepsilon) - a(\varepsilon)) = \liminf_{\varepsilon \rightarrow 0} (a(\varepsilon) - s_k(\varepsilon)) = +\infty$.

Let $\lambda \in (0, \bar{\lambda}_k^+)$ and define $s_k^\pm(\lambda) \in \mathbb{R}$ as the two solutions of $x_k(s_k^\pm(\lambda)) = \lambda$, $\pm s_k^\pm(\lambda) > 0$. A careful rewriting of the Emden-Fowler change of variables then allows to see the solution of (1.4) as a superposition of bubbles.

LEMMA 10. *Let $\lambda \in (0, \bar{\lambda}_k^+]$ for some $k \geq 1$. Then there exist two solutions u^\pm of (1.4) which take the form*

$$\lambda r^p e^{u^\pm(r)} = \left[\sum_{j=1}^k x_j \left(\log r + s_k(\varepsilon) - s_j(\varepsilon) + s_k^\pm(\lambda) \right) \right] (1 + o(1)) \quad \forall r \in (0, 1)$$

as $\varepsilon \rightarrow 0$.

This actually amounts to saying that there is a *k-bubble solution*. Note that we have to assume that $\varepsilon > 0$ is small enough so that with the notations of Corollary 9,

$$a(\varepsilon) > s_k(\varepsilon) + s_k^\pm(\lambda).$$

Also note that the Property (P_k) is implicitly assumed in the statement of Lemma 10.

Proof. According to Lemma 5, for $\varepsilon > 0$ sufficiently small, we can define $s_{\varepsilon,k}^\pm(\lambda)$ as the two solutions of

$$\bar{x}^\varepsilon(s) = 0$$

which minimize $\pm (s_{\varepsilon,k}^\pm(\lambda) - s_k(\varepsilon)) > 0$. Then $\lim_{\varepsilon \rightarrow 0} (s_{\varepsilon,k}^\pm(\lambda) - s_k(\varepsilon)) = s_k^\pm(\lambda)$ and the statement is a consequence of Corollary 9. \square

Proof. The proof of Theorem 1 is now straightforward with $w_j^k = x_j$. Note that $\mu_{i,j}(\varepsilon) = s_{\varepsilon_i,k}^\pm(\lambda) - s_j(\varepsilon_i)$, where $(\varepsilon_i)_{i \in \mathbb{N}}$ is a sequence of positive numbers with $\lim_{i \rightarrow +\infty} \varepsilon_i = 0$. \square

3. Multi-Bubble solutions

This section is devoted to the proof of Lemma 7. We divide it in two steps. First, we prove that for any $k \geq 1$, $\bar{\lambda}_k^+$ is positive: *multi-bubbling* occurs, with an arbitrarily large number of bubbles provided $\lambda \in (0, \bar{\lambda}_k^+)$. Then we show that the bubbles do not have the same height, i.e., $(\bar{\lambda}_k^+)_{k \in \mathbb{N}}$ is strictly decreasing.

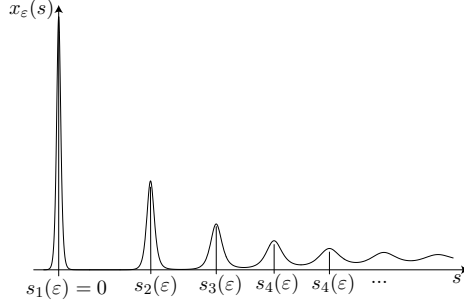


FIGURE 7. Bubbles in the logarithmic scale, after the Emden-Fowler transformation.

3.1. Multi-Bubbling

This section is devoted to the proof of Property (P_k) for $k \geq 2$. With the notations of Section 2.5, this means

$$\bar{\lambda}_k^+ > 0. \quad (P_k)$$

Before proving this result and actually more precise estimates, we start with some energy and angular velocity estimates in a new system of coordinates.

Consider

$$\begin{cases} x' = x(|y|^{p^*-2}y + p), & x(0) = \lambda_1^{\varepsilon,+} \\ y' = -\varepsilon y - x, & y(-\infty) = 0 \end{cases} \quad (2.1)$$

with $\varepsilon = n - p > 0$. To simplify the notations, we will omit the index ε . In the new coordinates

$$V = \log x, \quad U = -y,$$

System (2.1) becomes

$$\begin{cases} U' = e^V - \varepsilon U, & U(-\infty) = 0 \\ V' = U_*^{p^*-1} - U^{p^*-1}, & V(0) = \log \lambda_1^{\varepsilon,+} \end{cases} \quad (3.1)$$

where

$$U_* := p^{p-1} \iff U_*^{p^*-1} = p.$$

Note that to the trajectory (x, y) such that $\lim_{s \rightarrow -\infty} (x(s), y(s)) = (0, 0)$ now corresponds a trajectory such that $\lim_{s \rightarrow -\infty} (U(s), V(s)) = (0, -\infty)$ and such that $U(s) > 0$ for any $s \in \mathbb{R}$. With the notations of Section 2.1, this means

$$U(s) = -e^{(p-1)s} |u'(e^s)|^{p-2} u'(e^s),$$

$$V(s) = \log \lambda + u(e^s) + ps.$$

Let

$$(U_*, V_* = \log(\varepsilon U_*)) = \lim_{s \rightarrow +\infty} (U(s), V(s)).$$

The condition

$$x(0) = \max_{s \in \mathbb{R}} x(s) = \lambda_1^{\varepsilon,+}$$

now means

$$V(0) = \log \lambda_1^{\varepsilon,+} .$$

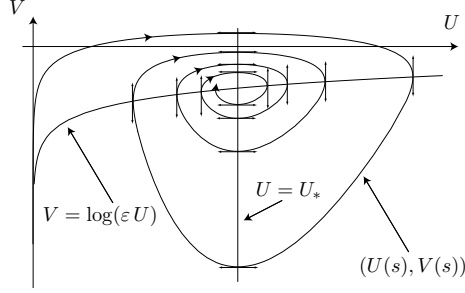


FIGURE 8. The trajectory in the (U, V) coordinates.

Consider the two following quantities, which are functions of s :

(1) energy:

$$E = e^V - e^{V_*} - e^{V_*}(V - V_*) + \frac{1}{p^*} \left(U^{p^*} - U_*^{p^*} \right) - U_*^{p^*-1}(U - U_*) .$$

(2) angle: let $\theta = \theta(s)$ be such that

$$\cos \theta = \frac{U - U_*}{\sqrt{|U - U_*|^2 + |V - V_*|^2}} \quad \text{and} \quad \sin \theta = \frac{V - V_*}{\sqrt{|U - U_*|^2 + |V - V_*|^2}} .$$

We take the convention $\theta(0) = -\frac{\pi}{2}$ and assume that θ is continuous, which determines θ in a unique way.

LEMMA 11. *With the above notations, if (U, V) is a solution of (3.1), then U is uniformly bounded on $(0, +\infty)$ and there exists a constant $\nu > 0$ such that*

$$0 \geq \frac{dE}{ds} \geq -\varepsilon \nu E \quad \forall s \in \mathbb{R} . \quad (3.2)$$

As a consequence

$$E(s) \geq E(0) e^{-\varepsilon \nu s} \quad \forall s \geq 0 .$$

Note that the bound on U is also uniform in terms on $\varepsilon \in (0, 1)$, including in the limit $\varepsilon \rightarrow 0$.

Proof. A direct computation of $\frac{dE}{ds}$ gives

$$\begin{aligned} \frac{dE}{ds} &= (e^V - e^{V_*}) V' + \left(U^{p^*-1} - U_*^{p^*-1} \right) U' \\ &= (e^V - e^{V_*}) (U_*^{p^*-1} - U^{p^*-1}) + \left(U^{p^*-1} - U_*^{p^*-1} \right) (e^V - \varepsilon U) \\ \frac{dE}{ds} &= -\varepsilon (U - U_*) \left(U^{p^*-1} - U_*^{p^*-1} \right) \end{aligned} \quad (3.3)$$

using $e^{V^*} = \varepsilon U_*$. The function $V \mapsto e^V - e^{V^*} - e^{V^*}(V - V_*)$ is nonnegative for any $V \in \mathbb{R}^+$, which means that

$$\frac{1}{p^*} \left(U^{p^*}(s) - U_*^{p^*} \right) - U_*^{p^*-1}(U(s) - U_*) \leq E(s) \leq E(0) \quad (3.4)$$

for any $s \geq 0$. Note that at $s = 0$, $V'(0) = 0$ so that $U(0) = U_*$ and

$$E(0) = \lambda_1^{\varepsilon,+} - \varepsilon U_* \log \left(\frac{e \lambda_1^{\varepsilon,+}}{\varepsilon U_*} \right).$$

Since $\lambda_1^{\varepsilon,+} \rightarrow \bar{\lambda}_1^+$ as $\varepsilon \rightarrow 0$, $E(0)$ itself is uniformly bounded as $\varepsilon \rightarrow 0$. Combined with Inequality (3.4), this means that $U(s)$ is uniformly bounded in $s \in \mathbb{R}^+$, for $\varepsilon > 0$ fixed. Moreover, this bound is uniform as $\varepsilon \rightarrow 0$.

Independently of the uniform estimate on U , there exists a constant $\nu > 0$ such that

$$(U - U_*) \left(U^{p^*-1} - U_*^{p^*-1} \right) \leq \nu \left[\frac{1}{p^*} \left(U^{p^*} - U_*^{p^*} \right) - U_*^{p^*-1}(U - U_*) \right] \quad (3.5)$$

for any $s \in \mathbb{R}^+$. This, using again (3.4), ends the proof of Lemma 11. Let us prove (3.5). Define

$$F(t) := \frac{1}{p^*} (t^{p^*} - 1) - (t - 1) - \kappa (t - 1)(t^{p^*-1} - 1).$$

Then for $\kappa = 1/p^*$,

$$F'(t) = (1 - \kappa p^*) t^{p^*-1} + \kappa (p^* - 1) t^{p^*-2} + \kappa - 1$$

has the sign of $t - 1$ if $p^* \geq 2 \iff p \in (1, 2)$. If $p^* \in (1, 2)$, since

$$\frac{1}{p^* - 1} F''(t) = (1 - \kappa p^*) t^{p^*-2} + \kappa (p^* - 2) t^{p^*-3}$$

changes sign only once in $(0, +\infty) \ni t$, the condition $F(0) = 0$ together with $\kappa < 1/p^*$ means $\kappa = (p^* - 1)/p^* = 1/p$. Thus $F(t)$ is nonnegative for any $t \in (0, +\infty)$ if $\kappa = \min(1/p, 1/p^*)$, so that (3.5) holds with $\nu = \max(p, p^*)$. \square

Note that the exponential decay of E is not sufficient to assert that the multi-bubbling phenomenon occurs since the interval in s between two bubbles can be of a larger order than the scale $1/\varepsilon$. If this was the case, the height of the second bubble could converge to 0, i.e., $\bar{\lambda}_2^+ = 0$. It is the purpose of the rest of Section 3.1 to prove that this is not the case.

LEMMA 12. *Given $C > 0$, there exists a constant $\omega > 0$ such that, if*

$$V(s) \geq \log(\varepsilon U_*) - C \quad \forall s \in [s_1, s_2] \subset \mathbb{R}^+,$$

then for $\varepsilon > 0$ small enough and any $s \in [s_1, s_2]$,

$$\frac{d\theta}{ds} \geq \varepsilon \omega. \quad (3.6)$$

Proof. Let us remark that we can write

$$\tan \theta = -\frac{b}{a} \quad \text{where } a(s) := U(s) - U_* \text{ and } b(s) := V(s) - V_* = V(s) - \log(\varepsilon U_*)$$

for any $s \in \mathbb{R}$ such that $U(s) \neq U_*$. Differentiating with respect to s , we get

$$\begin{aligned} \frac{a^2 + b^2}{a^2} \frac{d\theta}{ds} &= (1 + \tan^2 \theta) \frac{d\theta}{ds} = \frac{a'b - a b'}{a^2}, \\ \frac{d\theta}{ds} &= \frac{a'b - a b'}{a^2 + b^2}. \end{aligned}$$

On the one hand, $-a b' = (U - U_*) (U^{p^*-1} - U_*^{p^*-1}) \geq C_1 (U - U_*)^2$ for some positive constant C_1 . If $p^* < 2$, one has to use the fact that, according to Lemma 11, U is bounded. On the other hand

$$a'b = (e^V - \varepsilon U)(V - V_*) = (e^V - e^{V_*})(V - V_*) - \varepsilon(U - U_*)(V - V_*).$$

Thus

$$\frac{d\theta}{ds} \geq \frac{1}{a^2 + b^2} [C_1 a^2 - \varepsilon a b + C_2 \varepsilon b^2],$$

where

$$C_2 = \frac{1}{\varepsilon} \min_{s \in \mathbb{R}} \left(\frac{e^{V(s)} - e^{V_*}}{V(s) - V_*} \right).$$

By assumption,

$$\frac{e^{V(s)} - e^{V_*}}{V(s) - V_*} \geq e^{V_* - C} = \varepsilon U_* e^{-C},$$

which gives a lower estimate for C_2 which is independent of ε . It is now easy to prove that (3.6) holds for some $\omega > 0$. Namely we can estimate

$$C_1 a^2 - \varepsilon a b + C_2 \varepsilon b^2 = \left(\frac{1}{2} \sqrt{C_1} a - \frac{\varepsilon}{\sqrt{C_1}} b \right)^2 + \left(C_2 \varepsilon - \frac{\varepsilon^2}{C_1} \right) b^2 + \frac{3}{4} C_1 a^2$$

from below by

$$\frac{1}{2} C_2 (a^2 + b^2) \varepsilon$$

provided

$$\varepsilon \leq \frac{1}{2} C_1 \min(C_2, 3C_2^{-1}).$$

This ends the proof with $\omega = C_2/2$. \square

COROLLARY 13. *Let C be a positive constant and consider s_0, s_1 with $0 \leq s_0 < s_1$. Assume that*

- (i) *either $s_0 = 0$ or $s_0 > 0$ and $V(s_0) = \log(\varepsilon U_*) - C$*
- (ii) *$V(s) \geq \log(\varepsilon U_*) - C \quad \forall s \in [s_0, s_1]$.*

Then

$$E(s) \geq E(s_0) e^{-\frac{\nu}{\omega} [\theta(s) - \theta(s_0)]}$$

where ν is the constant of Lemma 11.

Proof. It is an easy consequence of (3.2) and (3.6):

$$\frac{1}{E} \frac{dE}{ds} \geq -\nu \varepsilon \geq -\frac{\nu}{\omega} \frac{d\theta}{ds}.$$

\square

LEMMA 14. *Let K be a positive constant and assume that*

$$V \leq \log(\varepsilon U) - K . \quad (3.7)$$

Then

$$U' \leq -(1 - e^{-K}) \varepsilon U . \quad (3.8)$$

Proof. Condition (3.7) can be rephrased into

$$e^V \leq e^{-K} \varepsilon U$$

and the result easily follows. \square

LEMMA 15. *Let C be a positive constant and consider $s_1, s_2 \in \mathbb{R}$, with $0 < s_1 < s_2$, such that*

$$V(s_i) = \log(\varepsilon U_*) - C \quad i = 1, 2 ,$$

$$V(s) < \log(\varepsilon U_*) - C \quad \forall s \in (s_1, s_2) .$$

Then there exists a constant $\kappa > 0$, which is independent of ε in the limit $\varepsilon \rightarrow 0$, such that

$$E(s_2) \geq \kappa E(s_1)$$

holds uniformly with respect to ε .

Proof. First of all, we may apply Lemma 14 with $K = C/2$. Since V' has the same sign as $U_* - U$, it is straightforward that

$$U(s_2) < U_* < U(s_1)$$

and that

$$s(K) = \inf\{s > s_1 : V(s) \geq \log(\varepsilon U(s)) - K\}$$

is such that

$$U(s(K)) < U_* .$$

Exactly as in Corollary 13, for any $s \in (s_1, s(K))$,

$$\frac{1}{E} \frac{dE}{ds} \geq -\nu \varepsilon \geq \frac{\nu}{(1 - e^{-K})} \frac{1}{U(s)} \frac{dU}{ds} ,$$

so that

$$E(s) \geq E(s_1) \left(\frac{U(s)}{U(s_1)} \right)^{\nu/(1-e^{-K})} \quad \forall s \in (s_1, s(K)) .$$

Let us argue by contradiction. If for $\varepsilon > 0$ small enough, $E(s_2)/E(s_1)$ can be taken arbitrarily small and if $s(K) < s_2$, then

$$U_* - U(s_2) \geq U_* - U(s) \geq U_* - U(s(K)) \geq 0 \quad \forall s \in (s(K), s_2)$$

can also be taken arbitrarily small, which contradicts the fact that $s(K) < s_2$. The condition

$$\log(\varepsilon U(s_2)) - \frac{C}{2} \leq \log(\varepsilon U(s(K))) - \frac{C}{2} = V(s(K)) < \log(\varepsilon U_*) - C$$

indeed means that

$$U(s_2) < e^{-C/2} U_* ,$$

which is impossible if $\frac{1}{p^*}(U^{p^*}(s_2) - U_*^{p^*}) - U_*^{p^*-1}(U(s_2) - U_*) \leq E(s_2)$ is taken arbitrarily small. Remind indeed that $E(s_1) \leq E(0)$ is uniformly bounded in terms of ε .

Thus if $E(s_2)/E(s_1)$ can be taken arbitrarily small, then $s(K) \geq s_2$. This means that $U_* - U(s)$ is either negative or positive but small on $(s_1, s_2) \ni s$:

$$U' = e^V - \varepsilon U \leq \varepsilon U e^{-K} - \varepsilon U \quad \forall s \in (s_1, s_2) \subset (s_1, s(K))$$

is therefore at most of the order of $-\varepsilon U_*(1 - e^{-K})$ and there exists a constant $\mu > 0$, uniform in ε such that

$$U' \leq -\mu \varepsilon \quad \forall s \in (s_1, s_2).$$

Combined with (3.2), this means that

$$\frac{1}{E} \frac{dE}{ds} \geq \frac{\nu}{\mu} U'$$

which by integration gives

$$E(s_2)/E(s_1) \geq e^{\frac{\nu}{\mu}(U(s_2) - U(s_1))}$$

and again provides a contradiction with the assumption that $E(s_2)/E(s_1)$ can be taken arbitrarily small. \square

For any $k \geq 1$, let $s_k(\varepsilon)$ be such that

$$\theta(s_k(\varepsilon)) = -\frac{\pi}{2} + (k-1)2\pi.$$

With the definition of $\bar{\lambda}_k^+$ given in Section 2.5, we get the following result.

PROPOSITION 16. *Consider a sequence $(\varepsilon_i)_{i \in \mathbb{N}}$ with $\lim_{i \rightarrow \infty} \varepsilon_i = 0$. Then up to the extraction of a subsequence,*

$$\lim_{i \rightarrow +\infty} E(s_k(\varepsilon_i)) = \bar{\lambda}_k^+$$

is positive. Moreover, there exists a constant $\kappa_0 \in (0, 1)$ such that

$$\forall k \geq 1, \quad \bar{\lambda}_k \geq \kappa_0^{k-1} \bar{\lambda}_1.$$

Proof. The fact that $\bar{\lambda}_k$ is positive is a consequence of Corollary 13 and Lemma 15. Looking more carefully into the proofs, it holds that

$$E(s_1) \geq e^{-2\pi\nu/\omega} E(s_0) =: \kappa_1 E(s_0)$$

in the case of Corollary 13 and $E(s_2) \geq \kappa_2 E(s_1)$ for some $\kappa_2 > 0$ in the case of Lemma 15, so that the Proposition holds with $\kappa_0 = \kappa_1 \cdot \kappa_2$. \square

REMARK.

(i) Note that $\bar{\lambda}_k^+$ may depend on the sequence $(\varepsilon_i)_{i \in \mathbb{N}}$. It is an open question to prove that for each $k \in \mathbb{N}$, $k \geq 2$, the limit as $i \rightarrow +\infty$ is actually unique, and to identify the value of $\bar{\lambda}_k^+$.

(ii) We will see in the next Section that $(\bar{\lambda}_k^+)_{k \in \mathbb{N}}$ is decreasing and converges to 0. This means that a different phenomenon occurs, compared to multi-bubbling in the slightly supercritical Brezis-Nirenberg problem, where all bubbles are identical up to a scaling factor.

3.2. Bubbles have different heights

LEMMA 17. *The sequence $(\bar{\lambda}_k^+)_{k \geq 1}$ is a strictly decreasing sequence of positive numbers.*

Proof. Assume by contradiction that

$$\bar{\lambda}_{k+1}^+ = \bar{\lambda}_k^+ =: \bar{\lambda} \quad (3.9)$$

for some $k \geq 1$. On the one hand, according to (3.3),

$$\frac{dE_\varepsilon}{ds} = -\varepsilon (|\bar{y}^\varepsilon(s)| - |y_*|) \left(|\bar{y}^\varepsilon(s)|^{p^*-1} - |y_*|^{p^*-1} \right)$$

for some positive constant $\nu > 0$, where $y_* = -p^{p-1}$ and

$$E_\varepsilon(s) := \bar{x}^\varepsilon(s) - \varepsilon p^{p-1} \left[1 - \log \left(\frac{\bar{x}^\varepsilon(s)}{\varepsilon p^{p-1}} \right) \right] + \frac{1}{p^*} \left(|\bar{y}^\varepsilon(s)|^{p^*} - |y_*|^{p^*} \right) + p(\bar{y}^\varepsilon(s) - y_*).$$

On the other hand, (3.9) means that there exists sequences $(\varepsilon_i)_{i \in \mathbb{N}}$, $(s_i^1)_{i \in \mathbb{N}}$ and $(s_i^2)_{i \in \mathbb{N}}$ such that:

- (i) For any $i \in \mathbb{N}$, $\varepsilon_i > 0$, and $\lim_{i \rightarrow +\infty} \varepsilon_i = 0$.
- (ii) For any $i \in \mathbb{N}$, $s_i^1 < s_i^2$, and

$$\frac{dy_{\varepsilon_i}}{ds}(s_i^j) = 0 \quad \text{and} \quad \lim_{i \rightarrow +\infty} (\bar{x}_{\varepsilon_i}(s_i^j), \bar{y}_{\varepsilon_i}(s_i^j)) = (0, \bar{y}_j), \quad j = 1, 2,$$

where $\bar{y} = \bar{y}_1, \bar{y}_2$ are the two solutions of

$$\frac{1}{p^*} |\bar{y}|^{p^*} + p \bar{y} = \bar{\lambda} - p^{p-1}$$

such that $\bar{y}_1 < -p^{p-1} < \bar{y}_2 \leq 0$. Here we use the conservation of the energy along the limiting trajectory corresponding to $\varepsilon = 0$: if $\frac{dx}{ds} = x(|y|^{p^*-2}y + p)$, $\frac{dy}{ds} = -x$, then $\frac{d}{ds}(x + \frac{1}{p^*}|y|^{p^*} + py) = 0$.

- (iii) Asymptotically, the energy does not decay on (s_i^1, s_i^2) :

$$\lim_{i \rightarrow +\infty} [E_{\varepsilon_i}(s_i^2) - E_{\varepsilon_i}(s_i^1)] = 0. \quad (3.10)$$

Let $\delta := \bar{y}_2 - \bar{y}_1 > 0$. Since

$$\bar{y}'_{\varepsilon_i} = -\varepsilon_i \bar{y}_{\varepsilon_i} - \bar{x}_{\varepsilon_i} \leq -\varepsilon_i \bar{y}_{\varepsilon_i},$$

it is straightforward to see that

$$\bar{y}_{\varepsilon_i}(s) \leq \bar{y}_{\varepsilon_i}(s_i^1) e^{\varepsilon_i (s_i^1 - s)} \quad \forall s \geq s_i^1,$$

which implies that

$$\bar{y}_{\varepsilon_i}(s_i^2) - \bar{y}_{\varepsilon_i}(s_i^1) \leq \bar{y}_{\varepsilon_i}(s_i^1) \left(e^{\varepsilon_i (s_i^1 - s_i^2)} - 1 \right).$$

Since $\lim_{i \rightarrow +\infty} \bar{y}_{\varepsilon_i}(s_i^1) = -|\bar{y}_1|$ and $\lim_{i \rightarrow +\infty} (\bar{y}_{\varepsilon_i}(s_i^2) - \bar{y}_{\varepsilon_i}(s_i^1)) = \delta$, this means that asymptotically as $i \rightarrow +\infty$,

$$\delta \leq |\bar{y}_1| \left(1 - e^{\varepsilon_i (s_i^1 - s_i^2)} \right) (1 + o(1)),$$

$$\varepsilon_i (s_i^2 - s_i^1) \geq \kappa (1 + o(1)),$$

where $\kappa := -\log \left(1 - \frac{\delta}{|\bar{y}_1|} \right) > 0$.

On $(s_i^2 - s_i^1) \ni s$, if $|y_{\varepsilon_i}(s) - y_*| > \delta/4$, then $E_{\varepsilon_i}(s)$ compares with $E_{\varepsilon_i}(s_i^j)$ which is itself of the same order as $\frac{1}{\varepsilon_i} \frac{d}{ds} E_{\varepsilon_i}(s)$ since

$$\bar{x}^\varepsilon(s_i^j) - \varepsilon p^{p-1} \left[1 - \log \left(\frac{\bar{x}^\varepsilon(s_i^j)}{\varepsilon p^{p-1}} \right) \right] = -\varepsilon \bar{y}^\varepsilon(s_i^j) - \varepsilon p^{p-1} \left[1 - \log \left(\frac{-\bar{y}^\varepsilon(s_i^j)}{p^{p-1}} \right) \right] \rightarrow 0$$

as $\varepsilon \rightarrow 0$.

Summarizing these estimates, this means that

$$E_{\varepsilon_i}(s_i^2) \leq E_{\varepsilon_i}(s_i^1) e^{-\mu} \quad \text{as } i \rightarrow +\infty$$

for some $\mu > 0$, a contradiction with (3.10), since $\bar{\lambda} > 0$ implies $\liminf_{i \rightarrow +\infty} E_{\varepsilon_i}(s_i^1) > 0$. \square

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