EXISTENCE OF POSITIVE RADIAL SOLUTIONS FOR A WEAKLY COUPLED SYSTEM VIA BLOW UP

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ABSTRACT. The existence of positive solutions to certain systems of ordinary differential equations is studied. Particular forms of these systems are satisfied by radial solutions of associated partial differential equations.

1. INTRODUCTION

In this paper we will study existence of positive solutions to a system of the form

$$(D)\qquad \begin{cases} -(r^{N-1}\phi_i(u_i'(r)))' = r^{N-1} f_i(u_{i+1}(r)), \\ \qquad\qquad\qquad\qquad\qquad i = 1,\dots,n \\ u_i'(0) = 0 = u_i(R), \end{cases}$$

where it is understood that $u_{n+1} = u_1$. Here, for $i = 1, \dots, n$, the functions ϕ_i are odd increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$ and the $f_i : \mathbb{R} \mapsto \mathbb{R}$ are odd continuous functions such that $sf_i(s) > 0$ for $s \neq 0$. Also $' = \frac{d}{dr}$.

System (D) is particularly important when the homeomorphisms ϕ_i take the form $\phi_i(s) = sa_i(|s|)$, $s \in \mathbb{R}$ since it is satisfied by the radial solutions of the system

$$(P)\qquad \begin{cases} \operatorname{div}(a_i(|\nabla u_i|)\nabla u_i) + f_i(u_{i+1}(|x|)) = 0, \quad x \in \Omega, \\ \qquad\qquad\qquad\qquad\qquad i = 1,\dots,n \\ u_i(|x|) = 0, \quad x \in \partial\Omega, \end{cases}$$

where Ω denotes the ball in $\mathbb{R}^N$ centered at zero and with radius $R > 0$.

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Furthermore, concerning the functions $\phi_i,\ f_i$, $i = 1, \dots, n$, we will assume that they belong to the class of asymptotically homogeneous functions (AH for short). We say that $h : \mathbb{R} \mapsto \mathbb{R}$ is AH at $+\infty$ of exponent $\delta > 0$ if for any $\sigma > 0$

$$\lim_{s\to+\infty} \frac{h(\sigma s)}{h(s)} = \sigma^{\delta}. \tag{1.1}$$

By replacing $+\infty$ by 0 in (1.1) we obtain a similar equivalent definition for a function h to be AH of exponent δ at zero. AH functions have been recently used in [GMU] and [GMS] in connection with quasilinear problems. They form an important class of non homogeneous functions which without being necessarily asymptotic to any power have the suitable homogeneous asymptotic behavior given by (1.1). In a very different context they have been used in applied probability and statistics where they are known as regularly varying functions, see for example [R], [S].

By a solution to (D) we understand a vector function $\mathbf{u} = (u_1, \dots, u_n)$ such that $\mathbf{u} \in C^1([0,T], \mathbb{R}^n)$ and $\phi_i(u_i') \in C^1([0,T], \mathbb{R})$, $i = 1, \dots, n$, which satisfies (D).

In [CMM], the existence of solutions with positive components for a system of the form (D) with $n = 2$ and with the functions ϕ_i and f_i having the particular form $\phi_i(s) = |s|^{p_i-2}s$, $\phi_i(0) = 0$, $p_i > 1$, $f_i(s) = |s|^{\delta_i-1}s$, $f_i(0) = 0$, $\delta_i > 0$, $i = 1, 2$, was done. In [GMU], within the scope of the AH functions, the case of a single equation was considered. In both situations the central idea to obtain a-priori bounds was the blow-up method of Gidas and Spruck, see [GS]. As a consequence of our results in this paper, those in [CMM] and [GMU] are greatly generalized.

Next we develop some preliminaries in order to state our main theorem. For $i = 1, \dots, n$, let $\delta_i, \bar{\delta}_i$ be positive real numbers and $p_i, \bar{p}_i$ real numbers greater than one, and assume that the functions $\phi_i,\ f_i$, $i = 1, \dots, n$ satisfy

$$\lim_{s\to+\infty} \frac{\phi_i(\sigma s)}{\phi_i(s)} = \sigma^{p_i-1}, \quad \lim_{s\to+\infty} \frac{f_i(\sigma s)}{f_i(s)} = \sigma^{\delta_i}, \tag{H_1}$$

for all $\sigma > 0$,

$$\prod_{i=1}^{n} \frac{\delta_i}{(p_i - 1)} > 1. \tag{H_2}$$

To the exponents p_i, δ_i, let us associate the system

$$\begin{cases} (p_i - 1)E_i - \delta_i E_{i+1} = -p_i, \quad i = 1, \dots, n, \\ E_{n+1} = E_1. \end{cases} \tag{AS}$$

From (H_2), it turns out that (AS) has a unique solution $(E_1, \dots, E_N)$, such that $E_i > 0$ for each $i = 1, \dots, n$. An explicit form for these solutions is given in the Appendix at the end of the paper.

Now we can establish our main existence theorem.

Theorem 1.1. *For $i = 1, \dots, n$, let ϕ_i be odd increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$ and $f_i : \mathbb{R} \mapsto \mathbb{R}$ odd continuous functions with $xf_i(x) > 0$ for*

$x \neq 0$, which satisfy (H_1), (H_2), and

$$(H_3) \qquad \lim_{s\to 0} \frac{\phi_i(\sigma s)}{\phi_i(s)} = \sigma^{\bar{p}_i - 1}, \quad \lim_{s\to 0} \frac{f_i(\sigma s)}{f_i(s)} = \sigma^{\bar{\delta}_i},$$

for any $\sigma > 0$. Additionally, for $i = 1, \dots, n$, let us assume that

$$(H_4) \qquad \prod_{i=1}^{n} \frac{\bar{\delta}_i}{(\bar{p}_i - 1)} > 1,$$

$$(H_5) \qquad p_i < N, \ i = 1, \dots, n, \qquad \max_{i=1,\dots,n} \{E_i - \theta_i\} \geq 0,$$

where $\theta_i = \frac{N-p_i}{p_i-1}$ and the $E_i{}'s$ are the solutions to (AS). Then problem (D) has a solution $(u_1, \dots, u_n)$ such that $u_i(r) > 0$, $r \in [0, R)$, for each $i = 1, \dots, n$.

The plan of this paper is as follows. We begin section 2 by discussing some properties of the AH functions that will be used throughout the paper. Then we provide an abstract functional analysis setting for problem (D) so that finding solutions to that problem is equivalent to solving a fixed point problem. Section 3 is first devoted to the study of a-priori bounds for positive solutions to problem (D) and then to prove our main theorem by using Leray Schauder degree arguments. To show the a-priori bounds we argue by contradiction and thus by using some suitable *rescaling functions* we find that there must exist a vector solution $\mathbf{v} = (v_1, \dots, v_n)$ defined on $[0, +\infty)$ (vector ground state) to a system of the form

$$(D_p) \qquad \begin{cases} -(r^{N-1}|v_i'(r)|^{p_i-2}v_i'(r))' \\ = C_i r^{N-1}|v_{i+1}(r)|^{\delta_i - 1}v_{i+1}(r), \ r \in [0, +\infty), \\ \qquad\qquad\qquad\qquad i = 1, \dots, n, \\ \quad v_i'(0) = 0, \quad v_i(r) \geq 0, \ r \in [0, +\infty), \end{cases}$$

where $v_{n+1} = v_1$ and C_i are positive constants, $i = 1, \dots, n$. We observe here the interesting fact that in this asymptotic system only properties of ϕ_i, f_i at $+\infty$ appear. We reach then a contradiction, and hence the existence of a-priori bounds, by using hypothesis (H_5) which prevents the existence of such a vector ground state.

In all of our previous argument the existence of suitable rescaling functions is crucial. The lemma for their existence (as well as some of their key properties) is stated without proof at the beginning of section 3 and its proof (which is delicate and rather lengthy and technical) is postponed to section 4. In section 5 we give some applications that illustrate our existence result. In particular, in example 5.2 we apply our existence results to a system that contains operators of the form $(-\Delta_p)^n$, $(-\Delta_q)^m$, where for $t > 1$ $\Delta_t u := \mathrm{div}(|\nabla u|^{t-2}\nabla u)$. We end the paper with an Appendix which contains some technical results.

We introduce now some notation. Throughout the paper vectors in $\mathbb{R}^n$ will be written in boldface. $C_\#$ will denote the closed linear subspace of $C[0, R]$ defined by $C_\# = \{u \in C[0, R] \mid u(R) = 0\}$. We have that $C_\#$ is a

Banach space with respect to the norm $\|\cdot\| := \|\cdot\|_\infty$. Also we will denote by $C^n_\#$, the Banach space of the $n-$tuples of elements of $C_\#$ endowed with the norm $||\mathbf{u}||_n := \sum_{i=1}^n ||u_i||$, where $\mathbf{u} = (u_1, \dots, u_n) \in C^n_\#$.

Finally we adopt the following conventions. By $\mathbb{R}_+$ and $\mathbb{R}^+$ we mean $[0, +\infty)$ and $(0, +\infty)$ respectively. For a function $H : \mathbb{R} \mapsto \mathbb{R}$ (with $\lim_{s\to 0} \frac{H(s)}{s} = 0$) we define $\hat{H}(s) := \frac{H(s)}{s}$, $s \neq 0$, $\hat{H}(0) = 0$, and we note that if H is AH of exponent p (at $+\infty$ or zero) then $\hat{H}$ is AH of exponent $p-1$. Also if γ_i, $i = 1, \dots, n$, are real numbers or functions, we define $\gamma_{n+i} = \gamma_i$ for all $i = 1, \dots, n$.

2. Preliminaries and Abstract Formulation

We begin this section with a proposition.

Proposition 2.1. *Let $h : \mathbb{R} \mapsto \mathbb{R}$ be a continuous function with $h(0) = 0$, $th(t) > 0$ for $t \neq 0$, and let $H(t) := \int_0^t h(s)ds$ and $\hat{H} : \mathbb{R} \mapsto \mathbb{R}$ as defined in the Introduction.*

(i) *If h is AH of exponent $\rho > 0$ at $+\infty$, then there exists $t_0 > 0$ and positive constants d_1 and d_2 with $1 < d_1 \leq d_2$ such that*

$$d_1 \leq \frac{th(t)}{H(t)} \leq d_2, \quad \textit{for all} \quad t \geq t_0, \tag{2.1}$$

$h(s) \to +\infty$ as $s \to +\infty$, $\hat{H}(t)$ is increasing for $t \geq t_0$ and

$$d_1 h(s) \leq d_2 h(t) \quad \textit{for all } s,\ t \textit{ such that} \quad t_0 \leq s \leq t. \tag{2.2}$$

(ii) *If h is AH of exponent $\rho > 0$ at 0 then there exists $t_0 > 0$ and positive constants d_1 and d_2, with $1 < d_1 \leq d_2$ such that*

$$d_1 \leq \frac{th(t)}{H(t)} \leq d_2, \quad \textit{for all } |t| \leq t_0,$$

$\hat{H}(t)$ is increasing in $[-t_0, t_0]$, and

$$d_1|h(s)| \leq d_2|h(t)|$$

for all $s,\ t$ with $|s| \leq |t| \leq t_0$.

Proof. We only prove (i), since (ii) is similar. From Karamata's theorem (see [R], page 17, Theorem 0.6), it follows that for any $\sigma > 0$

$$\lim_{t\to+\infty} \frac{h(\sigma t)}{h(t)} = \sigma^\rho \qquad \text{if and only if} \qquad \lim_{t\to+\infty} \frac{H(t)}{th(t)} = \frac{1}{\rho+1}, \tag{2.3}$$

and thus, if h is AH of exponent $\rho > 0$, for $\varepsilon \geq 0$ (less than $\min\{\rho, 1\}$) there is a $t_0 > 0$, such that for all $t \geq t_0$,

$$\frac{\rho + 1 - \varepsilon}{t} \leq \frac{h(t)}{H(t)} \leq \frac{\rho + 1 + \varepsilon}{t}. \tag{2.4}$$

Setting $d_1 := \rho + 1 - \varepsilon > 1$ and $d_2 := \rho + 1 + \varepsilon$ we have that (2.1) holds. Now since $h(t) = H'(t)$, from (2.4) we obtain that

$$C_1 t^{d_1 - 1} \leq h(t) \leq C_2 t^{d_2 - 1} \quad \text{for all} \quad t \geq t_0, \tag{2.5}$$

for some positive constants C_1, C_2 and thus $h(t) \to +\infty$ as $s + \infty$.

We observe now that the function $\hat{H}$ is a C^1 function for $t > 0$, and that $\hat{H}'(t) = \frac{th(t) - H(t)}{t^2}$. Then from (2.4) and since $d_1 > 1$, we find that $\hat{H}'(t) > 0$ for $t \geq t_0$, i.e., $\hat{H}$ is ultimately increasing. Finally, and again from (2.4) for $t_0 \leq s \leq t$, we have that $d_1 h(s) \leq d_1 d_2 \hat{H}(s) \leq d_1 d_2 \hat{H}(t) \leq d_2 h(t)$, ending the proof of the proposition. ∎

As a consequence of this proposition we have the following result, which will be used to prove our main result.

Proposition 2.2. *Let $h : \mathbb{R} \mapsto \mathbb{R}$ be continuous and asymptotically homogeneous at $+\infty$ (at 0) of exponent $\rho > 0$ satisfying $th(t) > 0$ for $t \neq 0$. Let $\{w_n\}$ and $\{t_n\} \subseteq \mathbb{R}^+$ be sequences such that $w_n \to w$ and $t_n \to +\infty$ ($t_n \to 0$) as $n \to \infty$. Then,*

$$\lim_{n\to\infty} \frac{h(t_n w_n)}{h(t_n)} = w^\rho. \tag{2.6}$$

Proof. We only prove the case when h is AH at $+\infty$, the other case being similar. Let $H(s) := \int_0^s h(t)dt$ and assume first $w \neq 0$. Then $t_n w_n \to +\infty$ and by writing

$$\frac{h(t_n w_n)}{h(t_n)} = \frac{t_n w_n h(t_n w_n)}{H(t_n w_n)} \frac{\hat{H}(t_n w_n)}{\hat{H}(t_n)} \frac{H(t_n)}{t_n h(t_n)} \tag{2.7}$$

we see from (2.3) that to obtain (2.6) it suffices to prove that

$$\lim_{n\to\infty} \frac{\hat{H}(t_n w_n)}{\hat{H}(t_n)} = w^\rho. \tag{2.8}$$

Since by proposition 2.1, $\hat{H}$ is ultimately increasing, given $\varepsilon > 0$ sufficiently small, there exists $n_0 > 0$ such that for all $n \geq n_0$

$$\frac{\hat{H}(t_n(w - \varepsilon))}{\hat{H}(t_n)} \leq \frac{\hat{H}(t_n w_n)}{\hat{H}(t_n)} \leq \frac{\hat{H}(t_n(w + \varepsilon))}{\hat{H}(t_n)}$$

and thus (2.8) follows by using the fact that $\hat{H}$ is AH of exponent ρ and $\varepsilon > 0$ is arbitrarily small. Assume now that $w = 0$. We claim then that

$$\lim_{n\to\infty} \frac{h(t_n w_n)}{h(t_n)} = 0.$$

If not,

$$\frac{h(t_{n_k} w_{n_k})}{h(t_{n_k})} \geq \mu,$$

for some subsequences $\{t_{n_k}\}$, $\{w_{n_k}\}$, which implies that $t_{n_k} w_{n_k}$ must tend to $+\infty$. Let now $\varepsilon > 0$ be such that $\varepsilon < \mu^{1/\rho}$. Since $w_{n_k} \to 0$, there exists

$k_0 > 0$ such that $w_{n_k} < \varepsilon$ and as $t_{n_k} w_{n_k} \to +\infty$, both $t_{n_k} w_{n_k}$ and $t_{n_k}\varepsilon$ belong to the range where $\hat{H}$ is increasing for $k \geq k_0$.. Hence,

$$0 \leq \frac{\hat{H}(t_{n_k} w_{n_k})}{\hat{H}(t_{n_k})} \leq \frac{\hat{H}(t_{n_k}\varepsilon)}{\hat{H}(t_{n_k})}.$$

Using now that $\hat{H}$ is AH of exponent ρ, by letting $k \to \infty$ we find that $\limsup_{k\to\infty} \frac{\hat{H}(t_{n_k} w_{n_k})}{\hat{H}(t_{n_k})} \leq \varepsilon^\rho$ and hence, by (2.7), $\mu \leq \limsup_{k\to\infty} \frac{h(t_{n_k} w_{n_k})}{h(t_{n_k})} \leq \varepsilon^\rho < \mu$, a contradiction. ∎

Finally, regarding properties of AH (at ∞ or 0) that we will need later on, it is simple to see that if $\chi,\ \psi : \mathbb{R} \mapsto \mathbb{R}$ are AH functions of exponent p and q respectively, then $\chi \circ \psi$ is AH of exponent $r = pq$. Also, if ϕ is an increasing odd homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$ which is AH of exponent $p-1$, then its inverse ϕ^{-1} is AH of exponent p^*-1, where $p^* = \frac{p}{p-1}$.

We now find a functional analysis setting for problem (D). A simple calculation shows that finding non trivial solutions with positive components to problem (D) is equivalent to finding non trivial solutions to the problem

$$(A) \qquad \begin{cases} -(r^{N-1}\phi_i(u_i'(r)))' = r^{N-1} f_i(|u_{i+1}(r)|), \\ \\ u_i'(0) = 0 = u_i(R). \end{cases} \qquad i = 1, \dots, n,$$

Let $(u_1(r), \dots, u_n(r))$ be a non trivial solution of (A). Then for each $i = 1, \dots, n$, we have that $u_i(r) \geq 0$ and is non increasing on $[0, R]$. By integrating the equations in (A), it follows that $u_i(r)$ satisfies

$$u_i = M_i(u_{i+1})$$

where $M_i : C_\# \mapsto C_\#$ is given by

$$M_i(v)(r) = \int_r^R \phi_i^{-1}[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1} f_i(|v(\xi)|)d\xi]ds,$$

for each $i = 1, \dots, n$. Let us define $T_0 : C_\#^n \mapsto C_\#^n$ by

$$T_0(\boldsymbol{u}) := (M_1(u_2), ..., M_i(u_{i+1}), ..., M_n(u_1)),$$

where $\boldsymbol{u} = (u_1, \dots, u_n)$. Clearly T_0 is well defined and fixed points of T_0 will provide solutions of (A), and hence componentwise positive solutions of (D).

Define now the operator $T_h : C_\#^n \times [0,1] \mapsto C_\#^n$ by

$$T_h(\mathbf{u}, \lambda) := \left(\tilde{M}_1(u_2, \lambda), ..., M_i(u_{i+1}), ..., M_n(u_1)\right)$$

where $\tilde{M}_1 : C_\# \times [0,1] \mapsto C_\#$ is the operator defined by

$$\tilde{M}_1(v, \lambda)(r) : \int_r^R \phi_1^{-1}[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1}(f_1(|v(\xi)|) + \lambda h)d\xi]ds$$

with $h > 0$ a constant to be fixed later. Define also $S : C^n_\# \times [0,1] \mapsto C^n_\#$ by

$$S(\mathbf{u}, \lambda) = (N_1(u_2, \lambda), ..., N_i(u_{i+1}, \lambda), ..., N_n(u_1, \lambda)) \tag{2.9}$$

where $N_i : C_\# \times [0,1] \mapsto C_\#$ is the operator defined by

$$N_i(v, \lambda)(r) = \int_r^R \phi_i^{-1}[\frac{\lambda}{s^{N-1}} \int_0^s \xi^{N-1} f_i(|v(\xi)|) d\xi] ds, \ i = 1, \dots, n. \tag{2.10}$$

It follows from Proposition 2.2 of [GMU] that all the operators $\tilde{M}_1, M_i, N_i$, $i = 1, \dots, n$, are completely continuous, hence the operators T_0, T_h and S are also completely continuous. We note that $T_h(\cdot, 0) = T_0 = S(\cdot, 1)$.

To prove existence of a fixed point of T_0 we use suitable a-priori estimates and degree theory. Indeed, we will show that there exists $R_1 > 0$, such that $deg_{LS}(I - T_0, B(0, R_1), 0) = 0$, and also that the index $i(T_0, 0, 0)$ is defined and it satisfies $i(T_0, 0, 0) = 1$, from where the existence of a fixed point of T_0 follows by the excision property of the degree.

Finally in this section, in our next lemma we will select the constant h that appears in the definition of the operator T_h, and hence fix this operator once for all.

Lemma 2.1. *For $i = 1, \dots, n$ let the homeomorphisms ϕ_i, and the functions f_i satisfy (H_1) and (H_2). Then there exists $h_0 > 0$ such that the problem*

$$\mathbf{u} = T_h(\mathbf{u}, 1) \tag{2.11}$$

has no solutions for $h \geq h_0$.

Proof. We argue by contradiction and thus we assume that there exists a sequence $\{h_k\}_{k \in \mathbb{N}}$, with $h_k \to +\infty$ as $k \to \infty$, such that the problem

$$\mathbf{u} = T_{h_k}(\mathbf{u}, 1)$$

has a solution $\mathbf{u}_k = (u_{1,k}, \dots, u_{n,k})$, for each $k \in \mathbb{N}$. Then $\boldsymbol{u}_k$ satisfies

$$u_{1,k}(r) = \int_r^R \phi_1^{-1}[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1}(f_1(|u_{2,k}(\xi)|) + h_k) d\xi] ds \tag{2.12}$$

$$u_{i,k}(r) = \int_r^R \phi_i^{-1}[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1} f_i(|u_{i+1,k}(\xi)|) d\xi] ds, \ i = 2, \dots, n, \tag{2.13}$$

for each $k \in \mathbb{N}$. Clearly $u_{i,k}(r) > 0$, $r \in [0, R)$, and is non increasing for $r \in [0, R]$, for all $k \in \mathbb{N}$, and all $i = 1, \dots, n$. From (2.12)

$$u_{1,k}(r) \geq (R - r)\phi_1^{-1}(\frac{r h_k}{N}), \quad \text{for all } r \in [0, R]$$

and thus for $r \in [0, \frac{7R}{8}]$, (we choose this interval for convenience, but any other interval of the form $[0, T] \subset [0, R)$ will work as well) we find that

$$u_{1,k}(r) \geq \frac{R}{8}\phi_1^{-1}(\frac{R h_k}{4N}) \tag{2.14}$$

where we have used that $u_{1,k}(r) \geq u_{1,k}(R/4)$ for all $r \in [0, R/4]$. Then, using that $f_i(x) \to +\infty$ as $x \to +\infty$, from (2.13) and (2.14), by iteration, we conclude that for any $A > 0$, there exists $k_A > 0$ such that for all $r \in [0, \frac{3R}{4}]$

$$u_{i,k}(r) \geq A \quad \text{for all} \quad k \geq k_A \quad \text{and all} \quad i = 1, \dots, n. \tag{2.15}$$

Now, from the second of (H_1) and (i) of proposition 2.1 there exist $t_0 > 0$, $1 < d_1 \leq d_2$ such that

$$d_1 f_i(\tau) \leq d_2 f_i(t) \tag{2.16}$$

for all $t \geq \tau \geq t_0$ and all $i = 1, \dots, n$. Hence, by (2.11), and by increasing A if necessary,

$$d_1 f_i(|u_{i+1,k}(r)|) \leq d_2 f_i(|u_{i+1,k}(\xi)|) \tag{2.17}$$

for all $\xi \in [0, r]$ with $r \in [0, 3R/4]$. Since from (2.12) and (2.13) we also have that

$$u_{i,k}(r) \geq \int_r^{\frac{3R}{4}} \phi_i^{-1}[\frac{1}{s^{N-1}} \int_0^r \xi^{N-1} f_i(|u_{i+1,k}(\xi)|) d\xi] ds, \ i = 1, \dots, n, \tag{2.18}$$

then, for $k \geq k_A$, from (2.17) and the monotonicity of ϕ_i^{-1} we have that

$$u_{i,k}(r) \geq \int_r^{\frac{3R}{4}} \phi_i^{-1}[d \ f_i(u_{i+1,k}(r)) d\xi] ds, \quad r \in [\frac{R}{4}, \frac{3R}{4}],$$

where $d = \frac{d_1 R}{4N d_2 3^{N-1}}$. Thus for all $r \in [\frac{R}{4}, \frac{R}{2}]$, we find that

$$u_{i,k}(r) \geq \frac{R}{4} \phi_i^{-1}(d \ f_i(u_{i+1,k}(r))), \tag{2.19}$$

for all k large enough and for all $i = 1, \dots, n$. Next, setting

$$b_{i,k}(r) := \frac{R}{4} \frac{\phi_i^{-1}(d \ f_i(u_{i+1,k}(r)))}{\phi_i^{-1}(f_i(u_{i+1,k}(r)))}, \tag{2.20}$$

(2.19) becomes

$$u_{i,k}(r) \geq b_{i,k}(r) \phi_i^{-1}(f_i(u_{i+1,k}(r))), \quad r \in [\frac{R}{4}, \frac{R}{2}]. \tag{2.21}$$

Observing that by (2.15) and (H_1), $b_{i,k}(r) \to c_i$ as $k \to \infty$, uniformly in $[\frac{R}{4}, \frac{R}{2}]$, where c_i is a positive constant, we have that $b_{i,k}(r) \geq \tilde{C}$ for all $r \in [\frac{R}{4}, \frac{R}{2}]$, for all $i = 1, \dots, n$ and for all k sufficiently large and where $\tilde{C}$ is a positive constant. Hence, by (2.5) in the proof of proposition 2.1, for $\varepsilon > 0$ small there is a $k_0 \in \mathbb{N}$ such that

$$u_{i,k}(r) \geq C u_{i+1,k}^{\frac{\delta_i}{p_i - 1} - \varepsilon}(r), \quad r \in [\frac{R}{4}, \frac{R}{2}], \tag{2.22}$$

for all $k \geq k_0$ and all $i = 1, \dots, n$, and where C is a positive constant. Now, by iterating in (2.22), we find that

$$u_{1,k}(r) \geq C (u_{1,k}(r))^{\prod_{i=1}^{n} \left(\frac{\delta_i}{p_i - 1} - \varepsilon \right)}, \tag{2.23}$$

where C is a new positive constant. Since by (H_2) we may choose $0<\varepsilon<\min\{\frac{\delta_i}{p_i-1},\ i=1,\dots,n\}$ so that $\prod_{i=1}^{n}\left(\frac{\delta_i}{p_i-1}-\varepsilon\right)>1$, from (2.23), we have

$$(u_{1,k}(r))^{\prod_{i=1}^{n}\left(\frac{\delta_i}{p_i-1}-\varepsilon\right)-1}\le\frac{1}{C},\ \text{for any fixed } r\in[\frac{R}{4},\frac{R}{2}],$$

which by (2.15) gives a contradiction for large k. This ends the proof of the lemma. ∎

3. A-priori bounds and proof of the main result

In this section we will use the blow up method to find a priori bounds for the positive solutions of problem (D_h) and then prove Theorem 1.1. Let ϕ_i, f_i, $i=1,\dots,n$ be as in Theorem 1.1 and set

$$\Phi_i(s)=\int_0^s\phi_i(t)dt,\quad F_i(s)=\int_0^s f_i(t)dt,\ i=1,\dots,n. \tag{3.1}$$

In extending the blow up method to our situation it turns out that a key step is to find a solution $(x_1,\dots,x_n)$ in terms of s (for s near $+\infty$) to the system

$$F_i(x_{i+1})x_i=x_{i+1}\Phi_i(x_is),\qquad i=1,\dots,n. \tag{3.2}$$

In this respect we can prove the following.

Lemma 3.1. *Assume that the homeomorphisms ϕ_i, and the functions f_i, $i=1,\dots,n$ satisfy (H_1), (H_2), (H_3), and (H_4). Then*

(i) *there exist positive numbers s_0, x_i^0, and increasing diffeomorphisms α_i defined from $[s_0,+\infty)$ onto $[x_i^0,+\infty)$, $i=1,\dots,n$, which satisfy*

$$F_i(\alpha_{i+1}(s))\alpha_i(s)=\alpha_{i+1}(s)\Phi_i(\alpha_i(s)s), \tag{3.3}$$

for all $s\in[s_0,+\infty)$.

(ii) *The functions α_i satisfy*

$$\lim_{s\to\infty}\frac{f_i(\alpha_{i+1}(s))}{s\phi_i(\alpha_i(s)s)}=\frac{\delta_i+1}{p_i},\quad i=1,\dots,n. \tag{3.4}$$

(iii) *The functions α_i satisfy*

$$\lim_{s\to+\infty}\frac{\alpha_i(\sigma s)}{\alpha_i(s)}=\sigma^{E_i}\quad \textit{for all}\quad \sigma\in(0,+\infty)\quad i=1,\dots,n,$$

where the E_i's are the solutions to (AS).

We call these α_i's functions *rescaling* variables for system (D).

The proof of this lemma is rather lengthy and delicate and thus in order not to deviate the attention of the reader we postpone it until section 4.

We next find a-priori bounds for positive solutions. To this end let h satisfy the conditions of Lemma 2.1 and consider the family of problems

$$(D_\lambda)\quad \begin{cases}[r^{N-1}\phi_1(u_1')]'+r^{N-1}(f_1(|u_2(r)|)+\lambda h)=0,\\ [r^{N-1}\phi_i(u_i')]'+r^{N-1}f_i(|u_{i+1}(r)|)=0,\quad \lambda\in[0,1],\ i=2,\dots,n,\end{cases}$$

$$u_i'(0) = 0 = u_i(R) \quad \text{for } i = 1, \dots, n.$$

Clearly, a solution to (D_λ) is a fixed point of $T_h(\cdot, \lambda)$.

Theorem 3.1. *Under the conditions of Theorem 1.1, solutions to problem (D_λ) are a-priori bounded.*

Proof. We argue by contradiction and thus we assume that there exists a sequence $\{(\mathbf{u}_k, \lambda_k)\} \in C^n_\# \times [0,1]$, with $\mathbf{u}_k = (u_{1,k}, \dots, u_{n,k})$, such that $(\mathbf{u}_k, \lambda_k)$ satisfies (D_{λ_k}) and $\|\mathbf{u}_k\| = \sum_{i=1}^n \|u_{i,k}\| \to \infty$ as $k \to \infty$. It is not difficult to check by using the equations in (D_{λ_k}) that $\sum_{i=1}^n \|u_{i,k}\| \to \infty$ as $k \to \infty$ if and only if $\|u_{i,k}\| \to \infty$ as $k \to \infty$ for each $i = 1, \dots, n$. Hence, by redefining the sequence $(\mathbf{u}_k, \lambda_k)$ if necessary, we can assume that $\|u_{i,k}\| \geq s_0$ (s_0 as in lemma 3.1) for all $i = 1, \dots, n$ and for all $k > 0$. Let us set

$$\gamma_k = \sum_{i=1}^n \alpha_i^{-1}(\|u_{i,k}\|) \text{ and } t_{i,k} = \alpha_i(\gamma_k). \tag{3.5}$$

Then, $\gamma_k \to \infty$ as $k \to \infty$, and $\|u_{i,k}\| \leq t_{i,k}$, for each $i = 1, \dots, n$. Also by (3.4)

$$\lim_{k \to \infty} \frac{f_i(t_{i+1,k})}{\gamma_k \phi_i(t_{i,k}\gamma_k)} = \frac{\delta_i + 1}{p_i}. \tag{3.6}$$

Next we define the change of variables $y = \gamma_k r$, $w_{i,k}(y) = \frac{u_{i,k}(r)}{t_{i,k}}$ and set $\mathbf{w}_k := (w_{1,k}, \dots, w_{n,k})$. Clearly we have $|w_{i,k}(y)| \leq 1$ for all $y \in [0, \gamma_k R]$. In terms of these new variables and since $(\mathbf{u}_k, \lambda_k)$ satisfies (D_{λ_k}), we obtain that $(\mathbf{w}_k, \lambda_k)$ satisfies

$$-(y^{N-1}\phi_1(t_{1,k}\gamma_k w_{1,k}'(y)))' = y^{N-1}[\frac{f_1(t_{2,k}|w_{2,k}(y)|)}{\gamma_k} + \frac{\lambda_k h}{\gamma_k}], \tag{3.7}$$

$$-(y^{N-1}\phi_i(t_{i,k}\gamma_k w_{i,k}'(y)))' = y^{N-1}\frac{f_i(t_{i+1,k}|w_{i+1,k}(y)|)}{\gamma_k}, \ i = 2, \dots, n, \tag{3.8}$$

$$w_{i,k}'(0) = 0 = w_{i,k}(\gamma_k R) \quad \text{for} \quad i = 1, \dots, n, \tag{3.9}$$

where now $' = \frac{d}{dy}$. Let now $T > 0$ be fixed and assume, by passing to a subsequence if necessary, that $\gamma_k R > T$ for all $k \in \mathbb{N}$. We observe that by the usual argument, $w_{i,k}'(y) \leq 0$ and $w_{i,k}(y) \geq 0$ for all $i = 1, \dots, n$, for all $k \in \mathbb{N}$, and for all $y \in [0, T]$.

Claim. The sequences $\{w_{i,k}'\}_k$, $i = 1, \dots, n$, are bounded in $C[0,T]$. Indeed, assume by contradiction that for some $i = 1, \dots, n$, $\{w_{i,k}'\}$ contains a subsequence, renamed the same, with $||w_{i,k}'||_{C[0,T]} \to \infty$ as $k \to \infty$. Then there exists a sequence $\{y_k\}$, $y_k \in [0,T]$, such that for any $A > 0$ there is n_0 such that $|w_{i,k}'(y_k)| > A$ for all $k > n_0$. Integrating (3.7) (resp. (3.8) from 0 to y_k, we obtain

$$\phi_i(t_{i,k}\gamma_k|w_{i,k}'(y_k)|) = {y_k}^{1-N}\int_0^{y_k} \frac{s^{N-1}f_i(t_{i+1,k}w_{i+1,k}(s))}{\gamma_k} ds + \frac{\lambda_k h y_k}{N\gamma_k}. \tag{3.10}$$

Now let t_0, d_1, d_2 be as in Proposition 2.1 and set $M = \max_{i\in\{1,\dots,n\}} \sup_{x\in[0,t_0]} f_i(x)$. Since $t_{i+1,k} \to +\infty$ as $k \to \infty$, by redefining the sequence if necessary, we may assume that $\frac{M}{f_i(t_{i+1,k})} \leq \frac{d_2}{d_1}$ for all $i = 1,\dots,n$ and all $k \in \mathbb{N}$. Also, since $w_{i+1,k}(s) \leq 1$, if $t_{i+1,k}w_{i+1,k}(s) \geq t_0$, then by Proposition 2.1 we have that

$$\frac{f_i(t_{i+1,k}w_{i+1,k}(s))}{f_i(t_{i+1,k})} \leq \frac{d_2}{d_1}. \tag{3.11}$$

Since if $t_{i+1,k}w_{i+1,k}(s) \leq t_0$ (3.11) holds by the definition of M, we have that indeed (3.11) holds for all $i = 1,\dots,n$, all $k \in \mathbb{N}$ and all $s \in [0,T]$. Hence from (3.10) and the monotonicity of ϕ_i we find that

$$\frac{\phi_i(t_{i,k}\gamma_k A)}{\phi_i(t_{i,k}\gamma_k)} \leq \frac{d_2}{d_1}\frac{f_i(t_{i+1,k})T}{\phi_i(t_{i,k}\gamma_k)\gamma_k N} + \frac{hT}{N\phi_i(t_{i,k}\gamma_k)\gamma_k}.$$

Thus, by (H_1) and (3.6), and by letting $k \to \infty$ in this last inequality we find that

$$A^{p_i-1} \leq \frac{d_2}{d_1}\frac{(\delta_i+1)T}{p_i N},$$

which is a contradiction since A can be taken arbitrarily large and hence the claim follows.

From this claim and Arzela Ascoli Theorem, by passing to a subsequence if necessary, we have that $\mathbf{w}_k \to \mathbf{w} := (w_1,\dots,w_n)$ in $C^n[0,T]$. Also, by (3.5),

$$1 = \sum_{i=1}^n \frac{\alpha_i^{-1}(t_{i,k}w_{i,k}(0))}{\gamma_k} = \sum_{i=1}^n \frac{\alpha_i^{-1}(t_{i,k}w_{i,k}(0))}{\alpha_i^{-1}(t_{i,k})},$$

and hence, by letting $k \to \infty$ and using (iii) of lemma 3.1, we obtain

$$1 = \sum_{i=1}^n w_i^{\frac{1}{E_i}}(0),$$

which implies that w is not identically zero.

Now by integrating (3.7) (respectively (3.8)) from 0 to $y \in [0,T]$ and using (3.9), we obtain

$$-\phi_i(t_{i,k}\gamma_k w'_{i,k}(y)) = \tilde{f}_{i,k}(y)\frac{f_i(t_{i+1,k})}{\gamma_k}, \tag{3.12}$$

for $i = 1,\dots,n$ and all $k \in \mathbb{N}$, where

$$\tilde{f}_{1,k}(y) = y^{1-N}\int_0^y s^{N-1}\frac{f_1(t_{2,k}w_{2,k}(s))}{f_1(t_{2,k})}ds + \frac{\lambda_k h y}{N f_1(t_{2,k})}, \tag{3.13}$$

and

$$\tilde{f}_{i,k}(y) = y^{1-N}\int_0^y s^{N-1}\frac{f_i(t_{i+1,k}w_{i+1,k}(s))}{f_i(t_{i+1,k})}ds, \quad i = 2,\dots,n. \tag{3.14}$$

Using now Proposition 2.2, we have that $\frac{f_i(t_{i+1,k}w_{i+1,k}(s))}{f_i(t_{i+1,k})} \to (w_{i+1}(s))^{\delta_i}$ for each $s \in [0,T]$ and $i = 1,\dots,n$, and thus by (3.11) we may use the

Lebesgue's dominated convergence theorem to conclude that

$$\lim_{k\to\infty} \tilde{f}_{i,k}(y) = y^{1-N}\int_0^y s^{N-1} w_{i+1}^{\delta_i}(s)ds := \tilde{f}_i(y) \tag{3.15}$$

for each $y \in (0, T]$. From (3.12),

$$-w'_{i,k}(y) = \frac{\phi_i^{-1}(\tilde{g}_{i,k}(y)\mu_k)}{\phi_i^{-1}(\mu_k)}, \tag{3.16}$$

where

$$\tilde{g}_{i,k}(y) = \frac{\tilde{f}_{i,k}(y) f_i(\alpha_{i+1}(\gamma_k))}{\gamma_k \phi_i(\gamma_k \alpha_i(\gamma_k))} \quad \text{and} \quad \mu_k = \phi_i(\gamma_k t_{i,k}).$$

Then $\mu_k \to +\infty$ as $k \to \infty$ and by (3.15) and (ii) of lemma 3.1,

$$\tilde{g}_{i,k}(y) \to \frac{\delta_i + 1}{p_i}\tilde{f}_i(y) \quad \text{as} \quad k \to \infty \quad \text{for each } y \in [0, T]. \tag{3.17}$$

Integrating (3.16) over $[0, y]$, we obtain

$$w_{i,k}(0) - w_{i,k}(y) = \int_0^y \frac{\phi_i^{-1}(\tilde{g}_{i,k}(s)\mu_k)}{\phi_i^{-1}(\mu_k)}\, ds. \tag{3.18}$$

Then, since by (3.15) there exists $A > 0$ such that $|\tilde{f}_i(y)| \leq A$ for all $i = 1, \dots, n$ and all $y \in [0, T]$, using (3.17) and the monotonicity of ϕ^{-1}, by another application of the Lebesgue's dominated convergence theorem, we find that

$$w_i(0) - w_i(y) = (\frac{\delta_i + 1}{p_i})^{\frac{1}{p_i - 1}} \int_0^y \Big(s^{1-N}\int_0^s t^{N-1} w_{i+1}^{\delta_i}(t)dt\Big)^{\frac{1}{p_i-1}}\, ds,$$

and hence that w_i satisfies

$$(D_p)_T \qquad \begin{cases} -(y^{N-1}|w_i'(y)|^{p_i-2}w_i'(y))' = (\frac{\delta_i+1}{p_i})y^{N-1}w_{i+1}^{\delta_i}(y), & y \in (0, T], \\ w_i'(0) = 0, \quad w_i(y) \geq 0 \text{ for all } y \in [0, T]. \end{cases}$$

We observe next that each component $w_i(y)$ is decreasing on $[0, T]$. Thus if for some i, $w_i(0) = 0$, then necessarily $w_i(y) = 0$ for all $y \in [0, T]$. But from $(D_p)_T$ it follows that $w_{i+1}(y) = 0$ for all $y \in [0, T]$ and hence by iterating, that $\mathbf{w} \equiv \mathbf{0}$ on $[0, T]$, which cannot be. Now, for the purpose of our next argument let us call $\{\mathbf{w}_k^T\}$ the final subsequence, solution to (3.7), (3.8) and (3.9), which by the limiting process provided us with the non trivial solution $\mathbf{w}$ to $(D_p)_T$ defined in $[0, T]$. We also set $\mathbf{w}^T \equiv \mathbf{w}$. Let us choose next $T_1 > T$. By repeating the limiting process following (3.9), this time starting from the sequence $\{\mathbf{w}_k^T\}$, we will find a subsequence $\{\mathbf{w}_k^{T_1}\}$, which as $k \to \infty$ will provide us with a non trivial solution $\mathbf{w}^{T_1}$ to $(D_p)_{T_1}$. Clearly $\mathbf{w}^{T_1}$ is an extension of $\mathbf{w}^T$ to the interval $[0, T_1]$, which satisfies $w_i^{T_1}(y) \geq 0$, $i = 1, \dots, n$. It is then clear that by this argument we can obtain a non trivial solution (called again $\mathbf{w}$) to (D_p), i.e. $\mathbf{w}$ satisfies

$$(D_p) \qquad \begin{cases} -(y^{N-1}|w_i'(y)|^{p_i-2}w_i'(y))' = (\frac{\delta_i+1}{p_i})y^{N-1}w_{i+1}^{\delta_i}(y), & y \in (0, +\infty), \\ w_i'(0) = 0, \quad w_i(y) \geq 0 \text{ for all } y \in [0, +\infty). \end{cases}$$

We claim now that under the hypotheses of Theorem 1.1 such a non trivial solution cannot exist. The proof of this claim is entirely similar to lemma 2.1 in [CMM] so we just sketch it. An integration of the equations of (D_p) over $[0,r]$, $r \in (0,+\infty)$, shows that $w_i'(r) \le 0$, for all $r>0$, and that

$$-r^{N-1}|w_i'(r)|^{p_i-2}w_i'(r) \ge \left(\frac{\delta_i+1}{p_i}\right)^{\frac{1}{p_i-1}} \frac{r^N}{N} w_{i+1}^{\delta_i}(r), \quad \text{for all } r>0, \tag{3.19}$$

Also it must be that $w_i(r)>0$ for all $r>0$ and all $i=1,\dots,n$. Now by Proposition 2.1 and Lemma 2.1 in [CMM], see also [MI] for related results, we have that for all $i=1,\dots,n$, $w_i \in C^2(0,+\infty)$ and that

$$rw_i'(r) + \theta_i w_i(r) \ge 0, \qquad \text{for all } r>0. \tag{3.20}$$

Hence, from (3.19)

$$\frac{\theta_i w_i(r)}{r} \ge -w_i'(r) \ge C r^{\frac{1}{p_i-1}} w_{i+1}^{\frac{\delta_i}{p_i-1}}(r) \quad \text{for all } r>0,$$

where C is a positive constant. (In the rest of this argument C will denote a positive constant that may change from one step to the other). Multiplying this inequality by r^{E_i+1}, using (3.20) and system (AS), we obtain

$$r^{E_i} w_i(r) \ge C(r^{E_{i+1}} w_{i+1}(r))^{\frac{\delta_i}{p_i-1}}, \quad \text{for all } r>0 \text{ and } i=1,\dots,n. \tag{3.21}$$

Iterating this expression $n-1$ times, we find first that

$$r^{E_i} w_i(r) \ge C(r^{E_i} w_i(r))^{\prod_{j=1}^{n} \frac{\delta_j}{p_j-1}} \qquad \text{for each} \quad i=1,\dots,n, \tag{3.22}$$

and thus by hypothesis (H_2),

$$w_i(r) \le C r^{-E_i} \quad \text{for each} \quad i=1,\dots,n. \tag{3.23}$$

By (3.20), $r^{\theta_i} w_i(r)$ is non decreasing, and thus combining with (3.22),

$$C_i \equiv w_i(r_0) r_0^{\theta_i} \le w_i(r) r^{\theta_i} \le C r^{-E_i} r^{\theta_i} = C r^{-(E_i-\theta_i)}, \tag{3.24}$$

for all $r > r_0 > 0$, for all $i=1,\dots,n$ and where the C_i's are positive constants. If the strict inequality holds in hypothesis (H_5), we obtain a contradiction by letting $r \to +\infty$ in (3.24) and the claim follows in this case.

Next, let us assume that for some $j \in \{1,\dots,n\}$ we have that $E_j = \theta_j$. Integrating the j-th equation of (D_p) on (r_0,r), $r_0>0$, using (3.21) and iterating $n-2$ times, we obtain

$$r^{N-1}|w_j'(r)|^{p_j-1} \ge C \int_{r_0}^{r} s^{N-1-E_{j+1}\delta_j} (s^{E_j} w_j(s))^{P_j'} ds$$

where $P_j' := \prod_{i=1, i\neq j}^{n} \frac{\delta_i}{(p_i-1)}$. Hence, since $w_j(r) r^{\theta_j}$ is non decreasing, and using that $E_j = \theta_j$,

$$r^{N-1}|w_j'(r)|^{p_j-1} \ge C \int_{r_0}^{r} s^{-1} ds \quad \text{for all } r>r_0,$$

where C is a positive constant. Hence by (3.20), we find

$$r^{\theta_1} w_j(r) \geq C(\log(\frac{r}{r_0}))^{\frac{1}{p_j-1}} \quad \text{for all } r > r_0, \tag{3.25}$$

which combined with (3.23) ($i=j$) and using that $E_j = \theta_j$, yields again a contradiction and thus the claim follows.

In this form we have concluded the proof that solutions to (D_λ) are a priori bounded. ∎

To prove Theorem 1.1 we need a last lemma. Let S be as defined by (2.9) and $B(0,\rho)$ denote the open ball centered at 0 and having radius ρ in $C^n_\#$.

Lemma 3.2. *Under the assumptions of Theorem 1.1, there exists $\rho_0 > 0$ such that the equation*

$$\mathbf{u} = S(\mathbf{u}, \lambda) \tag{3.26}$$

has no solutions $(\mathbf{u}, \lambda) \in \Big(\overline{B(0,\rho)} \setminus \{0\}\Big) \times [0,1]$ for all $0 \leq \rho \leq \rho_0$. In particular, the index $i(S(\cdot,1),0,0) \equiv i(T_0,0,0)$ is defined and $i(T_0,0,0) = 1$.

Proof. We argue by contradiction and thus we assume that there exist sequences $\{\mathbf{u}_k\}$ in $C^n_\#$, $\{\rho_k\}$, $\rho_k > 0$ such that $||\boldsymbol{u}_k|| = \rho_k \to 0$, and a sequence $\{\lambda_k\}$, $\lambda_k \in [0,1]$, such that

$$\mathbf{u}_k = S(\mathbf{u}_k, \lambda_k). \tag{3.27}$$

Let $\mathbf{u}_k = (u_{1,k}, \dots, u_{n,k})$. Since $u_{i,k}(s) \leq ||\boldsymbol{u}_k||$ for all $s \in [0,R]$, by (ii) of Proposition 2.1 we find that there exist $k_0 > 0$, $d_1 > 1$, $d_2 \geq d_1$, such that $f_i(u_{i,k}(s)) \leq \frac{d_2}{d_1} f_i(||\boldsymbol{u}_k||)$, for all $s \in [0,R]$, for all $k \geq k_0$, and for all $i = 1, \dots, n$.

Then, from (3.27) we obtain that $\boldsymbol{u}_k$, $k \geq k_0$, satisfies

$$\|u_{i,k}\| \leq \phi_i^{-1}[\frac{\lambda_k R d_2 f_i(\|u_{i+1,k}\|)}{N d_1}]R, \quad 1 = 1, \dots, n,$$

and hence

$$\phi_i\Big(\frac{\|u_{i,k}\|}{R}\Big) \leq \frac{R d_2}{N d_1} f_i(\|u_{i+1,k}\|). \tag{3.28}$$

Using the fact that the functions ϕ_i and f_i, $i = 1, \dots, n$, are AH near zero, we have that given $\varepsilon > 0$ small, there are $s_0 > 0$ and positive constants C, $\tilde{C}$, such that

$$C s^{\overline{\delta}_i + \varepsilon} \leq f_i(s) \leq \tilde{C} s^{\overline{\delta}_i - \varepsilon} \text{ for all } 0 \leq s \leq s_0, \tag{3.29}$$

and

$$C s^{\overline{p}_i - 1 + \varepsilon} \leq \phi_i(s) \leq \tilde{C} s^{\overline{p}_i - 1 - \varepsilon} \text{ for all } 0 \leq s \leq s_0. \tag{3.30}$$

Hence by combining (3.28), (3.29), and (3.30), we obtain that $||u_{i,k}|| \leq C||u_{i+1,k}||^{\frac{\overline{\delta}_i - \varepsilon}{\overline{p}_i - 1 + \varepsilon}}$ for all $i = 1, \dots, n$ and $k \in \mathbb{N}$ sufficiently large. Then by iteration, there is a positive constant C such that $||u_{i,k}||^{1 - \prod_{j=1}^{n} \frac{\overline{\delta}_j - \varepsilon}{\overline{p}_j - 1 + \varepsilon}} \leq C$ for each $i = 1, \dots, n$. But this is not possible for $\varepsilon > 0$ small since $||u_{i,k}|| \to 0$ as $k \to \infty$, and since by (H_4) $1 - \prod_{j=1}^{n} \frac{\overline{\delta}_j - \varepsilon}{\overline{p}_j - 1 + \varepsilon} < 0$ for $\varepsilon > 0$ small enough. That the index $i(S(\cdot, 1), 0, 0) = i(T_0, 0, 0)$ is defined and that $i(T_0, 0, 0) = 1$ is elementary. ∎

Proof of Theorem 1.1. It follows from Theorem 3.1 that if $(\mathbf{u}, \lambda)$ is a solution to the equation

$$\mathbf{u} = T_h(\mathbf{u}, \lambda), \quad \lambda \in [0, 1],$$

$\mathbf{u} = (u_1, \dots, u_n)$, then there is a positive constant R_1 such that $\sum_{i=1}^{n} \|u_i\| \le R_1$ for all $\lambda \in [0, 1]$ and where we may assume $R_1 > \rho_0$. Thus if $B(0, R_1)$ denotes the ball centered at 0 in $C^n_{\#}$ with radius $R_1 > C$, we have that the Leray-Schauder degree of the operator

$$I - T_h(\cdot, \lambda) : \overline{B(0, R_1)} \mapsto C^n_{\#}$$

is well defined and constant with $\lambda \in [0, 1]$. Then, by Lemma 2.1

$$\begin{aligned} deg_{LS}(I - T_0, B(0, R_1), 0) &= deg_{LS}(I - T_h(\cdot, 0), B(0, R_1), 0) \\ &= deg_{LS}(I - T_h(\cdot, 1), B(0, R_1), 0) \\ &= 0. \end{aligned} \tag{3.31}$$

Thus by Lemma 3.2, the excision property of the Leray Schauder degree, and (3.31), we conclude that there must be a solution of the equation

$$\mathbf{u} = T_0(\mathbf{u})$$

with $\mathbf{u} \in B(0, R_1) \setminus \overline{B(0, \varepsilon_0)}$, for $\varepsilon_0 > 0$ small enough. ∎

Remark 3.1. We point out that condition (H_5) in our main Theorem 1.1 is only used to conclude that problem (D_p) has no non trivial solutions on $[0, +\infty)$. Thus it can be replaced by any other condition which ensures this property and enlarges the set of parameters $\{\delta_i, p_i\}$ $i = 1, \dots, n$, for which the conclusion of Theorem 1.1 remains true. This remark will be illustrated in Example 5.3.

4. Proof of Lemma 3.1

Throughout this section we will use freely the definition $\hat{H}$ for a function H that we gave in the Introduction.

To prove Lemma 3.1 we need some preliminary propositions. We begin by noting that the functions $\hat{\Phi}_i$, $\hat{F}_i$, $i = 1, \cdots, n$ defined in (3.1) are C^1 functions from $\mathbb{R}^+$ onto $\mathbb{R}^+$. Also $\hat{\Phi}_i$ is AH of exponent $p_i - 1 > 0$ at $+\infty$ and of exponent $\overline{p}_i - 1 > 0$ at zero and $\hat{F}_i$ is AH of exponent $\delta_i > 0$ at $+\infty$ and of exponent $\overline{\delta}_i > 0$ at zero. Furthermore $\hat{F}_i$ is strictly increasing in some interval of the form $(-t_1, t_1)$, $t_1 > 0$, and in some interval of the form $(t_2, +\infty)$, $t_2 \ge t_1$.

For $\boldsymbol{x} = (x_1, \dots, x_n) \in (\mathbb{R}^+)^n$, we have that solving (3.2) is equivalent to solving

$$\hat{F}_i(x_{i+1}) - \hat{\Phi}_i(x_i s)s = 0, \quad i = 1, \dots, n. \tag{4.1}$$

Proposition 4.1. *For each fixed $s > 0$ there exists a solution $\boldsymbol{x} \in (\mathbb{R}^+)^n$ of the system (4.1).*

Proof. Let us fix $s > 0$ and suppose that $\boldsymbol{x} \in (\mathbb{R}^+)^n$ is a solution to (4.1). We have that (4.1) is in turn equivalent to the system

$$x_i = \Psi_i(x_{i+1}) \quad i = 1, \dots, n, \tag{4.2}$$

where $\Psi_i(t) := \frac{1}{s}\hat{\Phi}_i^{-1}(\frac{\hat{F}_i(t)}{s})$ for $t \geq 0$, and $i = 1, \dots, n$. Here and in what follows, for simplicity of the notation we will not show the dependence on s. Hence the component x_1 of the solution satisfies $l(x_1) = 0$, where

$$l(t) = t - (\Psi_1 \circ \Psi_2 \circ \cdots \circ \Psi_n)(t), \tag{4.3}$$

Conversely, if x_1 satisfies $l(x_1) = 0$,, then by recursively defining $x_i = \Psi_i(x_{i+1})$, $i = 2, \dots, n$, we find that $\boldsymbol{x} = (x_1, \dots, x_n)$ satisfies (4.1). Thus we are led to study the zeros of the function l. Since the function Ψ_i is AH of exponent $\frac{\delta_i}{p_i - 1}$ at $+\infty$ and of exponent $\frac{\overline{\delta}_i}{\overline{p}_i - 1}$ at zero, $i = 1, \dots, n$, we obtain that the function $\Psi_1 \circ \Psi_2 \ldots \circ \Psi_n$ is AH at $+\infty$ of exponent $\prod_{i=1}^{n} \frac{\delta_i}{p_i - 1}$ and at zero of exponent $\prod_{i=1}^{n} \frac{\overline{\delta}_i}{\overline{p}_i - 1}$. Thus for a given $\varepsilon > 0$ there are $t_2 > t_1 > 0$ and two positive constants $C_1 \equiv C_1(s)$ and $C_2 \equiv C_2(s)$ such that

$$\frac{l(t)}{t} \leq 1 - C_2 t^{\prod_{i=1}^{n}(\frac{\delta_i}{p_i - 1} - \varepsilon) - 1} \quad \text{for all } t \geq t_2, \tag{4.4}$$

and

$$\frac{l(t)}{t} \geq 1 - C_1 t^{\prod_{i=1}^{n}(\frac{\overline{\delta}_i}{\overline{p}_i - 1} - \varepsilon) - 1} \quad \text{for all } 0 < t \leq t_1. \tag{4.5}$$

Since by (H_2) and (H_4) we may choose $\varepsilon > 0$ such that

$$\prod_{i=1}^{n}\Big(\frac{\delta_i}{p_i - 1} - \varepsilon\Big) - 1 > 0 \text{ and } \prod_{i=1}^{n}\Big(\frac{\overline{\delta}_i}{\overline{p}_i - 1} - \varepsilon\Big) - 1 > 0,$$

we have by (4.4) that $l(t) < 0$ for all large t and by (4.5) that $l(t) > 0$ for all small positive t. Thus the equation $l(t) = 0$ has at least one solution, which is what we wanted to prove. ∎

We note that for each $s > 0$ the set of solutions of $l(t) = 0$ is bounded (the bound depending on s) but may not be a singleton. For $s > 0$ let us define $\beta_1(s) := \min\{t \mid l(t) = 0\}$ and $\gamma_1(s) := \max\{t \mid l(t) = 0\}$ (these max and min are reached), and define recursively $\beta_2(s)$ to $\beta_n(s)$ by $\beta_i(s) = \Psi_i(\beta_{i+1}(s))$, $\gamma_2(s)$ to $\gamma_n(s)$ by $\gamma_i(s) = \Psi_i(\gamma_{i+1}(s))$. Then, $\boldsymbol{\beta} := (\beta_1, \dots, \beta_n) : \mathbb{R}^+ \mapsto (\mathbb{R}^+)^n$, $\boldsymbol{\gamma} := (\gamma_1, \dots, \gamma_n) : \mathbb{R}^+ \mapsto (\mathbb{R}^+)^n$ and $\boldsymbol{\beta}(s)$, $\boldsymbol{\gamma}(s)$ are solutions to (4.1) for each $s > 0$.

Proposition 4.2. *We have that*

(i) $\beta_i(s) \to +\infty$ *as* $s \to +\infty$ *for each* $i = 1, \dots, n$.

(ii) *For each* $m > 0$ *there is a constant* $M = M(m)$ *such that for all* $0 < s \le m$, *it follows that* $||\boldsymbol{\gamma}(s)|| \le M$.

Proof. (i) We will show first that

$$\liminf_{s \to +\infty} \beta_i(s) > C \quad \text{for all} \quad i = 1, \dots, n, \tag{4.6}$$

where C is a positive constant. Suppose by contradiction that for some $j \in \{1, \dots, n\}$ there is a sequence $\{s_k\} \to +\infty$ such that $\beta_j(s_k) \to 0$. By (4.1),

$$\hat{\Phi}_i^{-1}(\hat{F}_i(\beta_{i+1}(s_k))) \ge \beta_i(s_k) \quad i = 1, \dots, n, \tag{4.7}$$

for $s_k \ge 1$ and thus by an iteration process starting with $i = j - 1$ we conclude that $\beta_i(s_k) \to 0$ for all $i = 1, \dots, n$. Using now that the function $\hat{\Phi}_i^{-1} \circ \hat{F}_i$ is AH at 0 of exponent $\frac{\overline{\delta}_i}{\overline{p}_i - 1}$, we obtain from (4.7) that given $\varepsilon > 0$ small enough, there is a positive constant C such that

$$\Big(\beta_{i+1}(s_k)\Big)^{\frac{\overline{\delta}_i}{\overline{p}_i - 1} - \varepsilon} \ge C\beta_i(s_k) \quad i = 1, \dots, n,$$

and hence

$$\Big(\beta_j(s_k)\Big)^{\prod_{i=1}^{n} (\frac{\overline{\delta}_i}{\overline{p}_i - 1} - \varepsilon) - 1} \ge C$$

for some other positive constant C. Since by (H_4) we may choose $\varepsilon > 0$ so that $\prod_{i=1}^{n} (\frac{\overline{\delta}_i}{\overline{p}_i - 1} - \varepsilon) > 1$, this is a contradiction and thus (4.6) holds. We conclude then that there are a positive constant C_1 and $s_0 > 0$ such that $\beta_i(s) \ge C_1$ for all $s \ge s_0$ and all $i = 1, \dots, n$. Hence, $\hat{F}_i(\beta_{i+1}(s)) \ge s\hat{\Phi}_i(sC_1)$ for all $s \ge s_0$, which implies that $\lim_{s \to +\infty} \beta_{i+1}(s) = +\infty$ and (i) is proved.

To show (ii) we assume there is an $m_1 > 0$ and a sequence $\{s_k\} \subset [0, m_1]$ such that $\gamma_j(s_k) \to +\infty$ as $k \to \infty$ for some component γ_j of $\boldsymbol{\gamma}(s) = (\gamma_1(s), \dots, \gamma_n(s))$. Since by (4.1)

$$\frac{1}{m_1}\hat{\Phi}_i^{-1}\Big(\frac{1}{m_1}\Big(\hat{F}_i(\gamma_{i+1}(s_k))\Big)\Big) \le \gamma_i(s_k) \quad i = 1, \dots, n, \tag{4.8}$$

by iteration (starting with $i = j - 1$) we find that $\gamma_i(s_k) \to +\infty$ for all $i = 1, \dots, n$. Using now that $\hat{\Phi}_i^{-1}$ and $\hat{F}_i$ are AH at $+\infty$ of exponents $\frac{1}{p_i - 1}$ and δ_i respectively, for $\varepsilon > 0$ small enough we obtain from (4.8) that $(\gamma_{j+1}(s_k))^{\frac{\delta_j}{p_j - 1} - \varepsilon} \le \tilde{C}\gamma_j(s_k)$, and thus by iterating, we conclude that $\Big(\gamma_{j+1}(s_k)\Big)^{\prod_{j=1}^{n} (\frac{\delta_j}{p_j - 1} - \varepsilon) - 1} \le C$ for k large, where $\tilde{C}$ and C are positive constants. Since by (H_2) we may choose $\varepsilon > 0$ such that $\prod_{j=1}^{n} (\frac{\delta_j}{p_j - 1} - \varepsilon) > 1$, we have reached a contradiction. Hence (ii) is proved and the proposition follows. ∎

We begin now the proof of Lemma 3.1.

Proof of (i) of Lemma 3.1. Suppose first that there is a function $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) : \mathbb{R}^+ \mapsto (\mathbb{R}^+)^n$ such that $\boldsymbol{\alpha}(s)$ is a solution of class C^1 to (4.1) for s in some subinterval I of $\mathbb{R}^+$. Then, for $s \in I$, $\boldsymbol{\alpha}(s)$ satisfies

$$\hat{F}_i(\alpha_{i+1}(s)) - \hat{\Phi}_i(\alpha_i(s)s)s = 0, \quad i = 1, \dots, n. \tag{4.9}$$

By differentiating with respect to s, we find that $\boldsymbol{\alpha}$ is a solution to the system of differential equations

$$a_i(s, \boldsymbol{\alpha}(s))\alpha_i'(s) - b_i(s, \boldsymbol{\alpha}(s))\alpha_{i+1}'(s) = -c_i(s, \boldsymbol{\alpha}(s)), \; i = 1, \dots, n, \tag{4.10}$$

where $' = \frac{d}{ds}$, and

$$a_i(s, \boldsymbol{\alpha}) = s^2 \hat{\Phi}_i'(s\alpha_i), \; b_i(s, \boldsymbol{\alpha}) = \hat{F}_i'(\alpha_{i+1}) \text{ and } c_i(s, \boldsymbol{\alpha}) = \phi(s\alpha_i). \tag{4.11}$$

Conversely if $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ is a solution to (4.10) in I, then $\boldsymbol{\alpha}(s)$ satisfies

$$\hat{F}_i(\alpha_{i+1}(s)) - \hat{\Phi}_i(\alpha_i(s)s)s = C_i, \; i = 1, \dots, n. \text{ for all } s \in I. \tag{4.12}$$

Hence if for some $s_0 \in I$ $(s_0, \boldsymbol{\alpha}(s_0))$ satisfies (4.1), then $(s, \boldsymbol{\alpha}(s))$ satisfies (4.1) for all $s \in I$. At this point the proof of (i) of Lemma 3.1 consists in showing that indeed the initial value problem

$$(IV) \qquad \begin{cases} a_i(s, \boldsymbol{x})x_i'(s) - b_i(s, \boldsymbol{x})x_{i+1}' = -c_i(s, \boldsymbol{x}), \quad i = 1, \dots, n, \\ x(s_0) = \boldsymbol{x}_0, \end{cases}$$

has a solution defined for all $s \geq s_0$, for some initial condition $(s_0, \boldsymbol{x}_0)$ which satisfies (4.1). Thanks to Proposition 4.1 we know that we can choose a pair $(s_0, \boldsymbol{x}_0)$ satisfying (4.1) for any $s_0 > 0$.

Observing that the system in (IV) has the form (6.1) in the Appendix, with $x_i'(s)$ in the place of X_i, we have that we can solve for the $x_i'(s)$ in any subset of $\mathbb{R}^+ \times (\mathbb{R}^+)^n$ where $\prod_{i=1}^n \frac{b_i(s,\boldsymbol{x})}{a_i(s,\boldsymbol{x})} \neq 1$ is satisfied. We will find next a point $(s_0, \boldsymbol{x}_0)$ and hence by continuity a neighborhood of this point where $\prod_{i=1}^n \frac{b_i(s,\boldsymbol{x})}{a_i(s,\boldsymbol{x})} \neq 1$ holds. To this end, let us define the lower and upper envelopes of $\hat{F}_i$ by

$$\hat{F}_i^-(x) = \inf_{s \in [0,x]} \hat{F}_i(s), \quad \hat{F}_i^+(x) = \sup_{s \in [0,x]} \hat{F}_i(s).$$

Then, v, $\hat{F}_i^-$, and $\hat{F}_i^+$ are nondecreasing and since $\hat{F}_i$ is ultimately increasing, there exists $\overline{m_1} > 0$ such that $\hat{F}_i^-(x) = \hat{F}_i^+(x) = \hat{F}_i(x)$ for all $x \geq \overline{m_1}$ and all $i = 1, \dots, n$.

After computing the derivatives in (4.11), we find that

$$\prod_{i=1}^n \frac{b_i(s, \boldsymbol{x})}{a_i(s, \boldsymbol{x})} = D(s, \boldsymbol{x}) \prod_{i=1}^n \frac{F_i(x_{i+1})}{\Phi_i(sx_i)}, \tag{4.13}$$

where $D(s, \boldsymbol{x}) = \prod_{i=1}^n (\frac{x_{i+1} f_i(x_{i+1})}{F_i(x_{i+1})} - 1)(\frac{sx_i \phi_i(sx_i)}{\Phi_i(sx_i)} - 1)^{-1}$. Now, since the functions f_i, ϕ_i are respectively AH at $+\infty$ of exponent δ_i and $p_i - 1$, from (2.3) in the proof of Proposition 2.1, we have that for $\varepsilon > 0$ small there is an $m_1 \geq \overline{m}_1$ such that for all $\boldsymbol{x} = (x_1, \dots, x_n)$ with $x_i \geq m_1$ and $s \geq 1$ it holds that

$$0 < \delta_i - \varepsilon \leq \frac{x_{i+1} f_i(x_{i+1})}{F_i(x_{i+1})} - 1 \leq \delta_i + \varepsilon$$

and

$$0 < p_i - 1 - \varepsilon \leq \frac{s x_i \phi_i(s x_i)}{\Phi_i(s x_i)} - 1 \leq p_i - 1 + \varepsilon,$$

for all $i = 1, \dots, n$. Thus $D(s, \boldsymbol{x}) \geq \prod_{i=1}^{n} \frac{\delta_i - \varepsilon}{p_i - 1 + \varepsilon}$ for all $(s, \boldsymbol{x})$ in the set

$$\mathcal{S} := \{(s, \boldsymbol{x}) \mid s \geq 1, x_i \geq m_1, \ i = 1, \dots, n\}.$$

Since by (H_2) we may choose ε small enough so that $\prod_{i=1}^{n} \frac{\delta_i - \varepsilon}{p_i - 1 + \varepsilon} > 1$, we have that $D(s, \boldsymbol{x}) > 1$ for all $(s, \boldsymbol{x}) \in \mathcal{S}$. Then from (4.13), we find that

$$\prod_{i=1}^{n} \frac{b_i(s, \boldsymbol{x})}{a_i(s, \boldsymbol{x})} > \prod_{i=1}^{n} \frac{F_i(x_{i+1})}{\Phi_i(s x_i)} = \prod_{i=1}^{n} \frac{\hat{F}_i(x_{i+1})}{s \hat{\Phi}_i(s x_i)},$$

for all $(s, \boldsymbol{x}) \in \mathcal{S}$.

By using (i) of Proposition 4.2 we choose now $s_0 > 1$ such that $\beta_i(s) > m_1$ for all $i = 1, \dots, n$ and all $s \geq s_0$ and set $\boldsymbol{x}_0 = \boldsymbol{\beta}(s_0)$. Then $(s_0, \boldsymbol{x}_0)$ satisfies (4.1) and $(s_0, \boldsymbol{x}_0) \in \mathrm{Int}\mathcal{S}$, implying that $\prod_{i=1}^{n} \frac{b_i(s_0, \boldsymbol{x}_0)}{a_i(s_0, \boldsymbol{x}_0)} > 1$. By continuity the same is true for $(s, \boldsymbol{x}) \in \Omega_0 := (s_0 - \mu_0, s_0 + \mu_0) \times B(\boldsymbol{x}_0, \varepsilon_0)$ for some small $\mu_0 > 0$ and $\varepsilon_0 > 0$ and where $B(\boldsymbol{x}_0, \varepsilon_0)$ is the ball in $\mathbb{R}^n$ centered at $\boldsymbol{x}_0$ and with radius ε_0. By using (6.1) in the Appendix we can solve for the derivatives x_i' in (IV) in terms of $(s, \boldsymbol{x}) \in \Omega_0$ to obtain the equivalent initial value problem

$$(IV_e) \qquad \begin{cases} x_i' = \dfrac{\dfrac{c_i(s, \boldsymbol{x})}{a_i(s, \boldsymbol{x})} + \displaystyle\sum_{k=1}^{n-1} \Big[\dfrac{c_{i+k}(s, \boldsymbol{x})}{a_{i+k}(s, \boldsymbol{x})} \prod_{\ell=0}^{k-1} \dfrac{b_{i+\ell}(s, \boldsymbol{x})}{a_{i+\ell}(s, \boldsymbol{x})} \Big]}{\displaystyle\prod_{j=1}^{n} \dfrac{b_j(s, \boldsymbol{x})}{a_j(s, \boldsymbol{x})} - 1}, \quad i = 1, \dots, n, \\[2ex] \boldsymbol{x}(s_0) = \boldsymbol{x}_0. \end{cases}$$

Since the right hand in the system in (IV_e) is continuous in Ω_0, by the theory of ordinary differential equation problem (IV_e) has a solution $\boldsymbol{\alpha} = (\alpha_1(s), \dots, \alpha_n(s))$ defined in an interval $(s_0 - \gamma_0, s_0 + \gamma_0)$, with $\gamma_0 \leq \mu_0$ which can be extended to the right as a solution of (IV_e) (this extension is also denoted by $\boldsymbol{\alpha}$) to a maximal interval of existence of the form $[s_0, w)$.
We claim that $w = +\infty$. We argue by contradiction and so we assume $w < +\infty$. Indeed, since $\boldsymbol{\alpha}(s)$ satisfies (4.1), by the definition of the vector function $\boldsymbol{\beta}$ we have that

$$\begin{aligned} \hat{F}_1^+(\alpha_2(s)) \geq \hat{F}_1(\alpha_2(s)) &= s \hat{\Phi}_1(s \alpha_1(s)) \\ &\geq s \hat{\Phi}_1(s \beta_1(s)) \\ &= \hat{F}_1(\beta_2(s)) = \hat{F}_1^+(\beta_2(s)), \end{aligned}$$

and thus $\alpha_2(s) \geq \beta_2(s) > m_1$ for all $s \in [s_0, w)$. By iteration we conclude that $\alpha_i(s) > \beta_i(s) > m_1$ for all $i = 1, \ldots, n$ and all $s \in [s_0, w)$. Hence, $(s, \boldsymbol{\alpha}(s)) \in \text{Int}\mathcal{S}$ for all $s \in [s_0, w)$. On the other hand by the choice of m_1 the function $\hat{F}_i(y)$ is strictly increasing for $y \in [m_1, +\infty)$, and thus it holds that $\hat{F}_i'(y) > 0$ for all $y \geq m_1$. Then from the definition of a_i, b_i, and c_i in (4.11) and the fact that $\hat{\Phi}_i'(x) > 0$ for all $x > 0$, we see that the numerator on the right hand side of the equations in (IV_e) is positive for all $(s, \boldsymbol{x}) \in \mathcal{S}$ and thus $\alpha_i'(s) > 0$ for all $s \in [s_0, w)$. Also, it can be easily verified that $\alpha_i(s) \leq \gamma_i(s)$ for all $s \in [s_0, w)$ and all $i = 1, \ldots, n$. Indeed, by the definition of $\hat{F}_i^-$,

$$\begin{aligned}
\hat{F}_1^-(\alpha_2(s)) \leq \hat{F}_1(\alpha_2(s)) &= s\hat{\Phi}_1(s\alpha_1(s)) \\
&\leq s\hat{\Phi}_1(s\gamma_1(s)) \\
&= \hat{F}_1(\gamma_2(s)) = \hat{F}_1^-(\gamma_2(s)),
\end{aligned}$$

and therefore by the monotonicity of $\hat{F}_1^-$, we have that $\alpha_2(s) \leq \gamma_2(s)$ for all $s \in [s_0, w)$, and thus iterating, we find that $\alpha_i(s) \leq \gamma_i(s)$ for all $s \in [s_0, w)$ and all $i = 1, \ldots, n$. Hence, by (ii) of Proposition 4.2, we obtain that $\alpha_i(s)$ is bounded in $[s_0, w)$ and then $\lim\limits_{s \to w^-} \boldsymbol{\alpha}(s) = \boldsymbol{d} = (d_1, \ldots, d_n)$ and $(w, \boldsymbol{d}) \in \mathcal{S}$. But from the continuity of the α_i's, and the fact that $\boldsymbol{\alpha}(s)$ satisfies (4.1), we obtain that $\prod\limits_{i=1}^{n} \frac{\hat{F}_i(d_{i+1})}{w\hat{\Phi}_i(wd_i)} = 1$ which implies $\prod\limits_{i=1}^{n} \frac{b_i(w,\boldsymbol{d})}{a_i(w,\boldsymbol{d})} > 1$. Hence we conclude that $\boldsymbol{\alpha}(s)$ can be extended to the right of w, a contradiction and our claim is proved.

Thus the domain of the solution $\boldsymbol{\alpha}$ to $(IV)_e$ is $[s_0, +\infty)$. Now for $i = 1, \ldots, n$, $\alpha_i'(s) > 0$, and $\alpha_i(s) \geq \beta_i(s)$, for all $s \geq s_0$. Hence by (i) of proposition 4.2 $\alpha_i(s) \to +\infty$ as $s \to +\infty$. Then $\alpha_i : [s_0, +\infty) \to [\beta_i(s_0), +\infty)$, is a diffeomorphism onto $[\beta_i(s_0), +\infty)$, for each $i = 1, \ldots, n$. Also $(s, \boldsymbol{\alpha}(s))$ satisfies (4.1) for each $s \in [s_0, +\infty)$. This concludes the proof of (i) of Lemma 3.1. ∎

Proof of (ii) of Lemma 3.1. By (4.1), for each $i = 1, \ldots, n$, we have that

$$\frac{F_i(\alpha_{i+1}(s))\alpha_i(s)}{\alpha_{i+1}(s)\Phi_i(\alpha_i(s)s)} = 1, \text{ for all } s > s_0.$$

Since we can write

$$\frac{f_i(\alpha_{i+1}(s))}{s\phi_i(\alpha_i(s)s)} = \frac{f_i(\alpha_{i+1}(s))\alpha_{i+1}(s)}{F_i(\alpha_{i+1}(s))} \frac{F_i(\alpha_{i+1}(s))\alpha_i(s)}{\Phi_i(\alpha_i(s)s)\alpha_{i+1}(s)} \frac{\Phi_i(\alpha_i(s)s)}{s\phi_i(\alpha_i(s)s)\alpha_i(s)},$$

and

$$\lim_{s \to \infty} \frac{f_i(\alpha_{i+1}(s))\alpha_{i+1}(s)}{F_i(\alpha_{i+1}(s))} = \delta_i + 1 , \qquad \lim_{s \to \infty} \frac{\Phi_i(\alpha_i(s)s)}{\phi(\alpha_i(s)s)\alpha_i(s)s} = \frac{1}{p_i},$$

we have that (3.4) follows immediately. ∎

Proof of (iii) of Lemma 3.1. We begin the proof by observing that if $h : \mathbb{R}^+ \to \mathbb{R}^+$ is continuously differentiable then by an obvious modification in

Karamata's theorem, (see [R], page 17, Theorem 0.6), we have that

(4.14)
$$\lim_{s\to+\infty}\frac{sh'(s)}{h(s)}=E>0,\ \text{if and only if}\ \lim_{s\to+\infty}\frac{h'(\sigma s)}{h'(s)}=\sigma^{E-1},$$

for all $\sigma>0$. Then, by L'Hôpital's rule, we find that

$$\lim_{s\to+\infty}\frac{h(\sigma s)}{h(s)}=\sigma^{E},\ \text{for all}\ \sigma>0. \tag{4.15}$$

From this observation the rest of the proof consists in showing that

$$\frac{s\alpha_i'(s)}{\alpha_i(s)}\to E_i,\ \text{as}\ s\to+\infty,\ \text{for each}\ i=1,\dots,n. \tag{4.16}$$

Since $\boldsymbol{\alpha}$ satisfies (4.10) and (4.11) for s large, by computing the derivatives of the coefficient functions in (4.10), we find that $\boldsymbol{\alpha}$ satisfies

$$A_i(s)\frac{s\alpha_i'(s)}{\alpha_i(s)}-B_i(s)\frac{s\alpha_{i+1}'(s)}{\alpha_{i+1}(s)}=-1,\ \text{for}\ i=1,\dots,n, \tag{4.17}$$

where

$$A_i(s)=\Big[1-\frac{\Phi_i(s\alpha_i(s))}{s\alpha_i(s)\phi_i(s\alpha_i(s))}\Big],\ B_i(s)=\frac{f_i(\alpha_{i+1}(s))}{s\phi_i(s\alpha_i(s))}\Big[1-\frac{F_i(\alpha_{i+1}(s))}{\alpha_{i+1}(s)f_i(\alpha_{i+1}(s))}\Big],$$

for $i=1,\dots,n$. We note that for each fixed s this system has the form (6.2) and thus it can be solved for $\frac{s\alpha_i'(s)}{\alpha_i(s)}$ if $\prod_1^n\frac{B_i(s)}{A_i(s)}\neq1$. Furthermore using the AH properties of the ϕ_i's and f_i's functions it can be seen that there exists $s_0>0$ such that $\prod_1^n\frac{B_i(s)}{A_i(s)}\neq1$, for all $s\geq s_0$ (we leave these calculations to the interested reader). Then, since $\lim_{s\to+\infty}A_i(s)=\frac{p_i-1}{p_i}$ and $\lim_{s\to+\infty}B_i(s)=\frac{\delta_i}{p_i}$, by letting $s\to+\infty$ in (4.17), it follows that (4.16) holds true, concluding the proof of (iii) of Lemma 3.1. This in turn ends the proof of Lemma 3.1. ∎

5. Applications

In this section we wish to show by means of simple examples the applicability of our main theorem. We will denote by Ω the open ball, centered at 0 with radius $R>0$ in $\mathbb{R}^N$.

Theorem 5.1. *Let ϕ, $\psi:\mathbb{R}\mapsto\mathbb{R}$ be defined by*

$$\phi(s)=|s|^{p_2-2}s+s\theta_1(s)+a|s|^{p_1-2}s,\quad p_2>p_1>1,$$
$$\psi(s)=|s|^{q_2-2}s+s\theta_2(s)+b|s|^{q_1-2}s,\quad q_2>q_1>1,$$

where a,b are positive constants, and for $i=1,2$, $\theta_i:\mathbb{R}\mapsto\mathbb{R}$, are even continuous functions with $0\leq\theta_i(s)$, $s\theta_i(s)$ non decreasing for all $s>0$, $\lim_{s\to0}s\theta_i(s)=0$, and such that

$$\lim_{s\to+\infty}\frac{s\theta_1(s)}{|s|^{p_2-2}s}=0\ \textit{and}\ \lim_{s\to0}\frac{s\theta_1(s)}{|s|^{p_1-2}s}=0,$$

$$\lim_{s\to+\infty}\frac{s\theta_2(s)}{|s|^{q_2-2}s}=0 \ \textit{and}\ \lim_{s\to 0}\frac{s\theta_2(s)}{|s|^{q_1-2}s}=0.$$

Let also $f,g:\mathbb{R}\mapsto\mathbb{R}$ *by odd continuous functions defined by*

$$f(s)=|s|^{\delta_2-1}s+\xi_1(s)+c|s|^{\delta_1-1}s,\quad \delta_2>\delta_1>0,$$
$$g(s)=|s|^{\mu_2-1}s+\xi_2(s)+d|s|^{\mu_1-1}s,\quad \mu_2>\mu_1>0,$$

where c,d *are positive constants and for* $i=1,2$, $\xi_i:\mathbb{R}\mapsto\mathbb{R}$, *are odd continuous (not necessarily increasing) functions such that* $0\le\xi_i(s)$, *for all* $s>0$, *and*

$$\lim_{s\to+\infty}\frac{\xi_1(s)}{|s|^{\delta_2-1}s}=0 \ \textit{and}\ \lim_{s\to 0}\frac{\xi_1(s)}{|s|^{\delta_1-1}s}=0,$$
$$\lim_{s\to+\infty}\frac{\xi_2(s)}{|s|^{\mu_2-1}s}=0 \ \textit{and}\ \lim_{s\to 0}\frac{\xi_2(s)}{|s|^{\mu_1-1}s}=0.$$

Then, if $\max\{p_2,\ q_2\}<N$, $\frac{\delta_2\mu_2}{(p_2-1)(q_2-1)}>1$, $\frac{\delta_1\mu_1}{(p_1-1)(q_1-1)}>1$, *and*

$$\max\Big\{\frac{p_2(q_2-1)+\delta_2q_2}{\delta_2\mu_2-(p_2-1)(q_2-1)}-\frac{N-p_2}{p_2-1},$$
$$\frac{q_2(p_2-1)+\mu_2p_2}{\delta_2\mu_2-(p_2-1)(q_2-1)}-\frac{N-q_2}{q_2-1}\Big\}\ge 0, \tag{5.1}$$

the problem

$$(P)\quad\begin{cases}-\mathrm{div}(|\nabla u|^{p_2-2}\nabla u)-\mathrm{div}(\theta_2(|\nabla u|)\nabla u)-a\mathrm{div}(|\nabla u|^{p_1-2}\nabla u)\\ \qquad =|v(x)|^{\delta_2-1}v(x)+\xi_1(v(x))+c|v(x)|^{\delta_1-1}v(x),\ x\ in\ \Omega\\ -\mathrm{div}(|\nabla v|^{q_2-2}\nabla v)-\mathrm{div}(\theta_2(|\nabla v|)\nabla v)-b\mathrm{div}(|\nabla u|^{q_1-2}\nabla v)\\ \qquad =|u(x)|^{\mu-1}u(x)+\xi_2(u(x))+d|u(x)|^{\mu_1-1}u(x),\ x\ in\ \Omega,\\ \qquad u(x)=v(x)=0,\ x\in\partial\Omega,\end{cases}$$

has a componentwise positive radial solution (u,v) *of class* C^1.

Proof. It can be easily shown that the function ϕ is AH of exponent p_2-1 at $+\infty$ and of exponent p_1-1 at zero, while ψ is AH of exponent q_2-1 at $+\infty$ and of exponent q_1-1 at zero. Also, the function f is AH of exponent δ_2 at $+\infty$ and of exponent δ_1 at zero, while g is AH of exponent μ_2 at $+\infty$ and of exponent μ_1 at zero. It only remains to show that condition (H_5) is fulfilled. Indeed, in this case system (AS) is given by

$$\begin{aligned}(p_2-1)E_1-\delta_2E_2&=-p_2\\ -\mu_2E_1+(q_2-1)E_2&=-q_2,\end{aligned}$$

and thus

$$E_1=\frac{p_2(q_2-1)+\delta_2q_2}{\delta_2\mu_2-(p_2-1)(q_2-1)}\quad\text{and}\quad E_2=\frac{q_2(p_2-1)+\mu_2p_2}{\delta_2\mu_2-(p_2-1)(q_2-1)}.$$

Also,

$$\theta_1=\frac{N-p_2}{p_2-1}\quad\text{and}\quad\frac{N-q_2}{q_2-1},$$

and thus (H_5) is given by (5.1) and the result follows directly from Theorem 1.1. ∎

Remark 5.1. A particular but illustrative case for the functions θ_i is given by

$$\theta_1(s) = \sum_{j=1}^{m_1} b_j|s|^{\alpha_j-2}, \qquad \theta_2(s) = \sum_{j=1}^{m_2} c_j|s|^{\beta_j-2},$$

where $b_j \geq 0,\ \alpha_j \in (p_1,p_2),\ j = 1,\dots,m_1$ and $c_j \geq 0,\ \beta_j \in (q_1,q_2),\ j = 1,\dots,m_2$. Thus

$$\phi(s) = |s|^{p_2-2}s + \sum_{j=1}^{m_1} b_j|s|^{\alpha_j-2}s + a|s|^{p_1-2}s$$

and

$$\psi(s) = |s|^{q_2-2}s + \sum_{j=1}^{m_2} c_j|s|^{\beta_j-2}s + b|s|^{q_1-2}s.$$

In the next example we show that our method allows us to find existence of positive solutions to some Δ_p, Δ_q systems.

For $p,\ q > 1,\ n,\ m \in \mathbb{N}$ and $\mu,\ \delta$ positive real numbers such that $\mu\delta \neq (p-1)^n(q-1)^m$, define

$$A := \frac{\delta q Q_m + p(q-1)^m P_n}{\mu\delta - (p-1)^n(q-1)^m}, \quad B := \frac{\mu p P_n + q(p-1)^n Q_m}{\mu\delta - (p-1)^n(q-1)^m},$$

where for any $k \in \mathbb{N}$, $P_k = \sum\limits_{i=1}^{k}(p-1)^{i-1}$ and $Q_k = \sum\limits_{i=1}^{k}(q-1)^{i-1}$. We have the following existence result.

Theorem 5.2. *Let $f,\ g : \mathbb{R} \mapsto \mathbb{R}$ be odd continuous functions such that f is AH at $+\infty$ of exponent $\delta > 0$ and AH at 0 of exponent $\overline{\delta} > 0$, g is AH at $+\infty$ of exponent $\mu > 0$ and AH at 0 of exponent $\overline{\mu} > 0$. Assume also that $\min\{\mu\delta,\ \overline{\mu}\overline{\delta}\} > (p-1)^n(q-1)^m$. Then, if $N > \max\{p,\ q\}$ and*

$$\begin{aligned} &\max\Big\{A - \frac{N-p}{p-1},\ (p-1)^{n-1}A + pP_{n-1} - \frac{N-p}{p-1}, \\ &\qquad B - \frac{N-q}{q-1},\ (q-1)^{m-1}B + qQ_{m-1} - \frac{N-q}{q-1}\Big\} \geq 0, \end{aligned} \tag{5.2}$$

the problem

$$(S) \qquad \begin{cases} (-\Delta_p)^n u = f(v); \quad (-\Delta_q)^m v = g(u), & \text{in } \Omega, \\ ((\Delta_p)^i u)(x) = ((\Delta_q)^j v)(x) = 0, & i = 0,1,\dots,n-1, \\ \qquad\qquad j = 0,1,\dots,m-1, & x \in \partial\Omega, \end{cases}$$

has a nontrivial radially symmetric solution $(u(x), v(x))$ such that $u(x) > 0$ and $v(x) > 0$ for all $x \in \Omega$.

Proof. We apply the result in Theorem 1.1 to the problem

$$(SS) \qquad \begin{cases} -\Delta_p u_i = u_{i+1},\ i = 1,\dots,n-1; \quad -\Delta_p u_n = f(u_{n+1}), & \text{in } \Omega \\ -\Delta_q u_{n+j} = u_{n+j+1},\ j = 1,\dots,m-1, \quad \text{in } \Omega \\ -\Delta_q u_{n+m} = g(u_1), \quad \text{in } \Omega \\ u_i(x) = 0,\ i = 1,\dots,n+m, \quad x \in \partial\Omega. \end{cases}$$

By a solution to (SS) we mean a vector function $(u_1(x),\dots,u_{n+m}(x))$, $x \in \overline{\Omega}$, that satisfies (SS). Indeed, in this case the functions ϕ_i, f_i defined by

$$\phi_i(t) = \begin{cases} |t|^{p-2}t & \text{for } i = 1,\dots,n, \\ |t|^{q-2}t & \text{for } i = n+1,\dots,n+m, \end{cases}$$

$$f_i(t) = \begin{cases} t & \text{for } i = 1.\dots,n-1, n+1,\dots,n+m-1, \\ f(t) & \text{for } i = n, \\ g(t) & \text{for } i = n+m, \end{cases}$$

satisfy the hypotheses of the theorem with $\delta_i = 1$ for $i = 1,\dots,n-1,n+1,\dots,n+m-1$, $\delta_n = \delta$, $\delta_{n+m} = \mu$, $\overline{\delta}_i = 1$ for $i = 1,\dots,n-1,n+1,\dots,n+m-1$, $\overline{\delta}_n = \overline{\delta}$, $\overline{\delta}_{n+m} = \overline{\mu}$, $\overline{p}_i = p_i = p$ for $i = 1,\dots,n$ and $\overline{q}_i = q_i = q$ for $i = n+1,\dots,n+m$. Furthermore, system (AS) for this problem is given by

$$(p-1)E_i - E_{i+1} = -p,\ i = 1,\dots,n-1,\ (p-1)E_n - \delta E_{n+1} = -p$$
$$(q-1)E_{n+i} - E_{n+i+1} = -q,\ i = 1,\dots,m-1,\ (q-1)E_{n+m} - \mu E_1 = -q$$

which has the unique solution $(E_1,\dots,E_{n+m})$ given by $E_1 = A$, $E_{n+1} = B$, $E_i = (p-1)^{i-1}A + pP_{i-1}$, $i = 2,\dots,n$, $E_{n+i} = (q-1)^{i-1}B + qQ_{i-1}$, $i = 2,\dots,m$. Also, $\theta_i = \frac{N-p}{p-1}$ for $i = 1,\dots,n$ and $\theta_i = \frac{N-q}{q-1}$ for $i = n+1,\dots,n+m$. Since, as it can be checked, either $E_1 \le \dots \le E_n$ or $E_1 \ge \dots \ge E_n$, and $E_{n+1} \le \dots \le E_{n+m}$ or $E_{n+1} \ge \dots \ge E_{n+m}$, we see that hypothesis (5.2) corresponds to hypothesis (H_5) in Theorem 1.1. Hence, according to that theorem, for $N > \max\{p,q\}$, (SS) will have a radial solution which is positive componentwise in Ω. The result follows now by setting $u(x) = u_1(x)$ and $v(x) = u_{n+1}(x)$. ∎

Remark 5.2. It is interesting to note that if

$$pP_{n-1} - \frac{N-p}{p-1} \ge 0 \quad \text{or} \quad qQ_{m-1} - \frac{N-q}{q-1} \ge 0$$

then (5.2) is automatically satisfied. Thus for instance, if $p = q = 2$, n, $m > 1$, then $P_{n-1} = n-1$ and $Q_{m-1} = m-1$, and we have that the problem

$$\begin{cases} (-\Delta)^n u = f(v); \quad (-\Delta)^m v = g(u), \quad \text{in } \Omega \\ u(x) = ((\Delta)^i u)(x) = v(x) = ((\Delta)^j v)(x) = 0, \quad x \in \partial\Omega, \\ i = 1,\dots,n-1, \quad j = 1,\dots,m-1, \end{cases}$$

has a non trivial radial positive componentwise solution (u,v) whenever

$$\max\{2n,\ 2m\} \ge N > 2$$

for any choice of $\mu, \overline{\mu}, \delta, \overline{\delta}$ satisfying $\mu\delta > 1$ and $\overline{\mu}\overline{\delta} > 1$.

Our last application illustrates the Remark 3.1 following the proof of Theorem 1.1. It is known from [SZ], Theorem 1.1, that the problem

$$(DD) \qquad \begin{cases} -\Delta u = |v|^{\delta-1}v; \quad -\Delta v = |u|^{\mu-1}u, \quad \text{in } \mathbb{R}^{\mathbb{N}}, \\ \qquad u(x) \ge 0, \quad v(x) \ge 0, \quad x \in \mathbb{R}^{\mathbb{N}}, \end{cases}$$

where $\delta > 0$, $\mu > 0$ does not possess non trivial positive radially symmetric solution if

$$\frac{N}{\delta+1} + \frac{N}{\mu+1} > N-2,$$

and thus we have the following existence result.

Theorem 5.3. *Let $f,\ g : \mathbb{R} \mapsto \mathbb{R}$ be odd continuous functions such that f is AH at $+\infty$ of exponent $\delta > 0$ and AH at 0 of exponent $\overline{\delta} > 0$, g is AH at $+\infty$ of exponent $\mu > 0$ and AH at 0 of exponent $\overline{\mu} > 0$ with $\mu\delta > 1$. Let also $\overline{p},\ \overline{q} > -1$ be such that $\overline{\delta}\overline{\mu} > (\overline{p}+1)(\overline{q}+1)$. Then, if*

$$\frac{N}{\delta+1} + \frac{N}{\mu+1} > N-2, \tag{5.3}$$

the problem

$$(DL) \qquad \begin{cases} -\mathrm{div}((\log(1+|\nabla u|))^{\overline{p}}\nabla u) = f(v), \ x \in \Omega, \\ -\mathrm{div}((\log(1+|\nabla u|))^{\overline{q}}\nabla v) = g(u), \ x \in \Omega, \\ u(x) = v(x) = 0, \quad x \in \partial\Omega, \end{cases}$$

has a non trivial radially symmetric solution (u, v) such that $u(x) > 0$ and $v(x) > 0$ for all $x \in \Omega$.

Proof. For this problem we have that $\phi_1(s) = (\log(1+|s|))^{\overline{p}}s$ and $\phi_2(s) = (\log(1+|s|))^{\overline{q}}s$ are AH at $+\infty$ of exponent 1 and AH at 0 of exponents $\overline{p}+1$ and $\overline{q}+1$ respectively. Moreover, the limiting problem at infinity is (DD) and thus the result follows. ∎

Remark 5.3. We observe that for (DL) condition (H_5) of Theorem 1.1 becomes

$$\max\Big\{\frac{\delta+1}{\delta\mu-1}, \frac{\mu+1}{\delta\mu-1}\Big\} \geq N-2.$$

Thus condition (5.3) above improves the set of δ, μ values for which existence of positive solutions is guaranteed by Theorem 1.1.

Finally, for related existence results of positive solutions for the case $\overline{p} = \overline{q} = 0$, in (DL), see [PvV] and [vV].

6. Appendix

Here we briefly consider the solutions to the system (AS), which for convenience of the reader we repeat here.

$$(AS) \quad (p_i - 1)E_i - \delta_{i+1}E_{i+1} = -p_i \quad \text{for} \quad i = 1, \dots, n.$$

This system is a particular case of the system

$$a_i X_i - b_i X_{i+1} = -c_i, \quad \text{for} \tag{6.1}$$

where $a_i,\ b_i,\ c_i$ are constants and which has played an important role in this paper.

It can be easily verified that if $a_i \neq 0,\ i = 1, \dots, n$ and $\prod_{i=1}^{n} \frac{b_i}{a_i} \neq 1$, then (6.1) has the unique solution $\mathbf{X} = (X_1, \dots, X_n)$

$$X_i = \frac{\dfrac{c_i}{a_i} + \displaystyle\sum_{k=1}^{n-1} \Big[\frac{c_{i+k}}{a_{i+k}} \prod_{\ell=0}^{k-1} \frac{b_{i+\ell}}{a_{i+\ell}} \Big]}{\displaystyle\prod_{j=1}^{n} \frac{b_j}{a_j} - 1}, \qquad i = 1, \dots, n, \tag{6.2}$$

with the usual convention that $a_{n+k} = a_k$, $b_{n+k} = b_k$ and $c_{n+k} = c_k$ for $k = 1, \dots, n$. Clearly if $a_i > 0, b_i > 0, c_i > 0$, $i = 1, \dots, n$, then $X_i > 0$ for all $i = 1, \dots, n$. Hence, if $p_i > 1$, $\delta_i > 0$ and $\prod_{i=1}^{n} \delta_i > \prod_{i=1}^{n} (p_i - 1)$ then (AS) has the unique solution $\mathbf{E} = (E_1, \dots, E_n)$

$$E_i = \frac{\dfrac{p_i}{p_i - 1} + \displaystyle\sum_{k=1}^{n-1} \Big[\frac{p_{i+k}}{p_{i+k} - 1} \prod_{\ell=0}^{k-1} \frac{\delta_{i+\ell}}{p_{i+\ell} - 1} \Big]}{\displaystyle\prod_{j=1}^{n} \frac{\delta_j}{p_j - 1} - 1}, \quad i = 1, \dots, n, \tag{6.3}$$

such that $E_i > 0$, $i = 1, \dots, n$.

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