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New Nonlinear Equations with Soliton-Like Solutions

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Abstract. We introduce new nonlinear equations which provide soliton-like solutions. The existence of such solutions is obtained by a minimization argument and by using a variant of the concentration-compactness principle.

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1. Introduction

The aim of this Letter is to present new nonlinear equations which provide soliton-like solutions and to study their dynamics. In the last few years, several new researches have been devoted to the study of existence and multiplicity of soliton-like solutions for nonlinear problems (see [1–7, 9, 11, 12]).

A soliton is a solution of a field equation whose energy travels as a localized packet and which preserves its form under perturbations. In this respect, solitons have a particle-like behavior and they occur in many questions of mathematical physics, such as classical and quantum field theory, nonlinear optics, fluid mechanics, plasma physics (see [11, 14, 16–18]).

To ensure some form of stability of solutions under perturbations to certain nonlinear equations, it is natural to look for solutions which are local minima of the energy functional related to the problem. For instance, the classical Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V_0\psi - |\psi|^{p-1}\psi, \quad \psi = \psi(x,t), \quad x \in \mathbb{R}^N, \quad t \in \mathbb{R} \tag{1.1}$$

does not provide soliton-like solutions: the energy functional is not bounded from below, it has no local minima, and solutions correspond to saddle points. On the other hand, if a nondispersive term is added in Equation (1.1), the problem gains stability and, in particular, soliton-like solutions (see [1]).

A common pattern of problems which have soliton-like solutions is the fact that the existence of such solutions is guaranteed by some suitable topological constraint:

they naturally appear from the discussion of all admissible configurations on which the related functional is finite (see, for instance, [12, 13, 16]) .

The sine–Gordon equation is the simplest example of equation with soliton-like solutions. A first generalization of the sine–Gordon equation is the case of the 3-space dimension model

$$\Box\phi + V'(\phi) = 0 \quad \phi = \phi(x,t), \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R} \tag{1.2}$$

where $\Box\phi = -c^2\Delta\phi + \partial^2\phi/\partial t^2$ (Δ is the three-dimensional Laplace operator) and V' is the gradient of a C^1 nonnegative real function.

For stability reasons, one is interested in static solutions of (1.2), that is functions $u = u(x)$ satisfying

$$-\Delta u + V'(u) = 0 \text{ in } \mathbb{R}^3, \tag{1.3}$$

which correspond to local minima of the energy functional

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} V(u)\,\mathrm{d}x. \tag{1.4}$$

In [10], the author proves that $\mathcal{E}$ does not admit local minima: in fact, assume u is a local minimum for $\mathcal{E}$, then $u_\lambda(x) = u(\lambda x)$ with $\lambda > 0$ is such that $\mathcal{E}(u_\lambda) < \mathcal{E}(u)$ for certain $\lambda > 1$.

In order to overcome this difficulty, Derrick proposed some models which are the Euler–Lagrange equations of the action functional

$$\mathcal{S} = \int\int \mathcal{L}\,\mathrm{d}x\,\mathrm{d}t \tag{1.5}$$

with special Lagrange density $\mathcal{L}$.

In [2], the authors carry out a wide existence analysis of the finite energy static solutions for a class of Lagrangian density $\mathcal{L}$ of the form

$$\mathcal{L}(\phi, \varrho) = -\tfrac{1}{2}\alpha(\varrho) - V(\phi) \tag{1.6}$$

where $\phi\colon \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$, $\varrho = |\nabla\phi|^2 - |\phi_t|^2$ ($\nabla\phi$ denotes the Jacobian with respect to $x \in \mathbb{R}^N$ and ϕ_t the derivative with respect to $t \in \mathbb{R}$), V is a real nonnegative function defined on an open subset $\Omega \subset \mathbb{R}^{N+1}$ and $\alpha(\varrho) = a\varrho + b|\varrho|^{p/2}$ with $p \geqslant N$, $a \geqslant 0$ and $b > 0$ (see also [6, 7, 9]).

The Euler–Lagrange equations related to $\mathcal{S}$ (see (1.5)) with $\mathcal{L}$ of the form (1.6) are

$$\frac{\partial}{\partial t}\big(\alpha'(\varrho)\phi_t\big) - \nabla\big(\alpha'(\varrho)\nabla\phi\big) + V'(\phi) = 0. \tag{1.7}$$

A static solution of (1.7) is a map $u\colon \mathbb{R}^N \to \mathbb{R}^{N+1}$ which satisfies the following system:

$$-a\Delta u - b\frac{p}{2}\Delta_p u + V'(u) = 0, \tag{1.8}$$

where $\Delta_p u = \nabla\big(|\nabla u|^{p-2}\nabla u\big)$; the vector field

$$\phi_v(x, t) = u\left(\frac{x_1 - vt}{\sqrt{1 - v^2}}, x_2, \dots, x_N\right), \quad |v| < 1,$$

is then a solution to (1.7).

In [2], it is proved that (1.8) has a weak solution, which is a minimizer of the energy functional in the class of maps which are topologically nontrivial (in a certain sense). Moreover, under certain assumptions of symmetry on V, a result of existence of infinitely many solutions is provided.

In this Letter we are concerned with existence of soliton-like solutions for the following system of equations

$$\Box\phi_1 + \phi_1 + V'(\phi_2) = 0, \qquad \Box\phi_2 = \phi_1, \tag{1.9}$$

where ϕ_1, ϕ_2 are maps defined in $\mathbb{R}^{3+1}$ with values in $\mathbb{R}^4$ and V' is the gradient of a real nonnegative function defined on an open subset of $\mathbb{R}^4$.

This model corresponds to coupling two wave equations which actually corresponds to considering a nonlinear nonlocal wave equation for one of the waves, or a fourth-order equation for the other one.

In fact, plugging the second equation in the first one, it is easy to see that (ϕ_1, ϕ_2) is a solution to (1.9) if $\phi_1 = \Box\phi_2$, where ϕ_2 solves

$$\Box^2\phi_2 + \Box\phi_2 + V'(\phi_2) = 0 \quad \text{in } \mathbb{R}^3. \tag{1.10}$$

Static solutions to (1.10) are maps which satisfy

$$-\Delta\varphi^j + \Delta^2\varphi^j + \frac{\partial V}{\partial\xi_j}(\varphi) = 0 \quad \text{in } \mathbb{R}^3 \tag{1.11}$$

for $j = 1, \dots, 4$, where $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^4$ is a map whose components in $\mathbb{R}^4$ are $(\varphi^1, \dots, \varphi^4)$.

It is standard to verify that, if $\varphi = \varphi(x)$ is a solution of (1.11) and $\tilde\varphi(x) = -\Delta\varphi(x)$, then

$$\phi_1^{\mathbf{v}}(x, t) = \tilde\varphi\left(\frac{x_1 - vt}{\sqrt{1 - v^2}}, x_2, x_3\right)$$

and

$$\phi_2^{\mathbf{v}}(x, t) = \varphi\left(\frac{x_1 - vt}{\sqrt{1 - v^2}}, x_2, x_3\right)$$

are solutions of system (1.9), for any vector field $\mathbf{v} = (v, 0, 0)$ with $|v| < 1$.

Weak solutions of (1.11) are maps $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^4$ which satisfy

$$\int_{\mathbb{R}^3}\left(D\varphi^j D\psi^j + \Delta\varphi^j\Delta\psi^j + \frac{\partial V}{\partial\xi_j}(\varphi)\psi^j\right)\mathrm{d}x = 0,$$
$$\forall j = 1, \dots, 4, \quad \forall\psi \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^4).$$

Hence, weak solutions of (1.11) correspond to critical points of the energy functional

$$E(\varphi) = \int_{\mathbb{R}^3} \left[\tfrac{1}{2}(|\nabla\varphi|^2 + |\Delta\varphi|^2) + V(\varphi)\right] \mathrm{d}x; \tag{1.12}$$

in particular we are interested in those critical points of E which are minimum points of E in a suitable class of maps.

Since Equations (1.11) involve maps which are defined on $\mathbb{R}^3$, our variational problem has a lack of compactness, that is the Palais–Smale condition fails. Hence, in this case the direct methods of the calculus of variations are useless.

In order to overcome this obstruction, we use the Splitting Lemma (Lemma 3.1), which is a direct consequence of the concentration-compactness principle (see [6, 15]) and it gives a precise description of the behavior of the minimizing sequences.

Moreover, we point out that the maps on which we minimize the functional E are classified by means of a topological invariant, the charge, which also gives a qualitative characterization of the solutions of (1.11).

Let $\bar{\xi}$ be a fixed point in $\mathbb{R}^4$. The potential V is a real function defined on $\Omega = \mathbb{R}^4 \setminus \{\bar{\xi}\}$ such that

(V1) $V \in C^1(\Omega; \mathbb{R})$;
(V2) $V(x) \geqslant V(0) = 0$ for every $x \in \Omega$;
(V3) V is twice differentiable in 0 and the Hessian matrix $V''(0)$ is nondegenerate;
(V4) there exist $c, r > 0$ such that

$$\text{if } |\xi| < r, \text{ then } V(\bar{\xi} + \xi) \geqslant c|\xi|^{-6}; \tag{1.13}$$

for sake of simplicity we consider $|\bar{\xi}| = 1$.

Observe that, since 0 is a nondegenerate minimum for V, we can choose a base in $\mathbb{R}^4$ which diagonalizes $V''(0)$ so that

$$V''(0) = \begin{pmatrix} m_1^2 & & 0 \\ & \ddots & \\ 0 & & m_4^2 \end{pmatrix},$$

where $m_j \neq 0$ for any $j = 1, \ldots, 4$.

We obtain the following result.

THEOREM 1.1. *There exists at least one weak solution of system* (1.11) *which is a minimum of the energy functional* (1.12) *in the class of maps which are not homotopic to the null map (defined in a suitable sense, see Definition* 2.2*).*

Remark. If the target space has a more complicated topology, if say $\Omega = \mathbb{R}^4 \setminus \{\xi_1, \ldots, \xi_k\}$, then a proper topological invariant can be associated to each map φ and a similar existence result can be easily obtained (see [7] for a related problem).

This Letter is organized as follow: in Section 2 we first choose the space Λ of maps on which the functional E is defined, we introduce a topological invariant of such maps, the charge, and we state some useful properties of it. Then we study the behaviour of the functional E. In Section 3 we give the proof of Theorem 1.1.

2. Variational Framework

Let H denote the space obtained as the completion of $C_0^\infty(\mathbb{R}^3; \mathbb{R}^4)$ with respect to the norm

$$||\varphi|| = ||\varphi||_{L^2} + ||\nabla\varphi||_{(L^2)^3} + ||\Delta\varphi||_{L^2}, \tag{2.1}$$

where $||\cdot||_{L^2}$ denotes the norm in $L^2(\mathbb{R}^3; \mathbb{R}^4)$. H is a Banach space continuously embedded in $W^{2,2}(\mathbb{R}^3; \mathbb{R}^4)$.

From well-known Sobolev embeddings (see [8]), the space H has the following useful properties.

(I) There exist two constants C_1, C_2 such that, for every $\varphi \in H$,

$$||\varphi||_\infty \leqslant C_1||\varphi|| \tag{2.2}$$

and

$$|\varphi(x) - \varphi(y)| \leqslant C_2||\varphi|||x - y|^{\frac{1}{2}}. \tag{2.3}$$

(II) For every $\varphi \in H$

$$\lim_{|x|\to\infty} \varphi(x) = 0. \tag{2.4}$$

(III) If $(\varphi_n)_n \subset H$ (weakly) converges in H to φ, then it converges uniformly on every compact set contained in $\mathbb{R}^3$.

Because of (2.3) the set

$$\Lambda = \{\varphi \in H : \forall x \in \mathbb{R}^3 \ \ \varphi(x) \neq \overline{\xi}\}$$

is well-defined; it is open in H since (2.2) and (2.3) hold.

Moreover, the boundary of Λ is given by

$$\partial\Lambda = \{\varphi \in H : \exists \overline{x} \in \mathbb{R}^3 \text{ such that } \varphi(\overline{x}) = \overline{\xi}\}.$$

The set of maps on which we define E is precisely Λ.

For any $\varphi \in \Lambda$ we get $E(\varphi) < +\infty$. Indeed, $\int_{\mathbb{R}^3} V(\varphi)\,\mathrm{d}x < +\infty$ since $(V1)$ and $(V2)$ hold, $\varphi \in \Lambda$ and $\int_{\mathbb{R}^3} |\varphi|^2\,\mathrm{d}x < +\infty$. Moreover $E \in C^1(\Lambda; \mathbb{R})$.

Let us now introduce the topological charge defined for maps in Λ.

Let us consider the projection $P: \Omega \to \Sigma$ defined by

$$P(\xi) = \bar{\xi} + \frac{\xi - \bar{\xi}}{|\xi - \bar{\xi}|}, \quad \forall \xi \in \Omega,$$

where $\Sigma = \{\xi \in \mathbb{R}^4 : |\xi - \bar{\xi}| = 1\}$ and $\bar{\xi}$ is the point which appears in (V.4).

For every $\xi \in \Omega$

$$P(\xi) = 2\bar{\xi} \Leftrightarrow \xi = (1 + |\xi - \bar{\xi}|)\bar{\xi}$$

and therefore

$$P(\xi) = 2\bar{\xi} \Rightarrow |\xi| > 1. \tag{2.5}$$

DEFINITION 2.1. For $\varphi \in \Lambda$, we call support of φ the compact set

$$K(\varphi) = \overline{\{x \in \mathbb{R}^3 : 1 < |\varphi(x)|\}}$$

(here the value 1 depends on the norm of the singularity $\bar{\xi}$). Then the topological charge of φ is the Brower degree of $P \circ \varphi$ in the support of φ with respect to $2\bar{\xi}$, namely

$$\operatorname{car}(\varphi) = \deg(P \circ \varphi, \operatorname{int}(K(\varphi)), 2\bar{\xi}).$$

In [6] some properties of the topological charge, introduced in a different setting of maps, are showed; the same properties hold in our case and the proofs are quite similar to those given in [6]. Hence, we limit ourselves in stating them.

PROPOSITION 2.2. *The topological charge satisfies the following properties.*

(i) *For every $\varphi \in \Lambda$ and for every $R > 0$ such that $K(\varphi) \subset B(0, R)$,*

$$\operatorname{car}(\varphi) = \deg(P \circ \varphi, B(0, R), 2\bar{\xi});$$

moreover, if $K(\varphi)$ consists of m connected components $K_1, \dots, K_m$, we can define also

$$\operatorname{car}(\varphi, K_j) = \deg(P \circ \varphi, K_j, 2\bar{\xi});$$

then, by additivity of the degree,

$$\operatorname{car}(\varphi) = \sum_{j=1}^{m} \operatorname{car}(\varphi, K_j).$$

(ii) *If a sequence $(\varphi_n)_n \subset \Lambda$ converges to $\varphi \in \Lambda$ uniformly on $A \subset \mathbb{R}^3$, then also $P \circ \varphi_n$ converges to $P \circ \varphi$ uniformly on A.*

(iii) *The topological charge is locally constant in Λ.*

Now for every $q \in \mathbb{Z}$, we set

$$\mathcal{A}_q = \{\varphi \in \Lambda : \operatorname{car}(\varphi) = q\}.$$

From (iii), it follows that $\mathcal{A}_q$ is open in H; moreover $\Lambda = \bigcup_{q\in\mathbb{Z}} \mathcal{A}_q$, with $\mathcal{A}_q \cap \mathcal{A}_p = \emptyset$ if $p \neq q$. Hence, $\mathcal{A}_q$ is a connected component of Λ.

If we set $\Lambda^* = \bigcup_{q\neq 0} \mathcal{A}_q$, from (2.5) it follows that

$$\forall\varphi \in \Lambda^* \Rightarrow ||\varphi||_\infty > 1 \tag{2.6}$$

and, hence, $K(\varphi) \neq \emptyset$.

Let us now proceed with the properties of the functional E we want to minimize.

PROPOSITION 2.4. *The functional E is coercive in H. Moreover, there exists a positive constant Δ^* such that, for every $\varphi \in \Lambda$,*

$$||\varphi||_\infty \geqslant 1 \Rightarrow E(\varphi) \geqslant \Delta^*. \tag{2.7}$$

Proof. Let $(\varphi_n)_{n\geqslant 1}$ be a sequence of maps in H such that $||\varphi_n|| \to +\infty$. We need to show that $E(\varphi_n) \to +\infty$. If $||\nabla\varphi_n||_{(L^2)^3} + ||\Delta\varphi_n||_{L^2} \to +\infty$ we are done.

On the other hand, if

$$||\nabla\varphi_n||_{(L^2)^3} + ||\Delta\varphi_n||_{L^2} \leqslant C_1, \quad ||\varphi_n||_{L^2} \to +\infty,$$

then $\int_{\mathbb{R}^3} V(\varphi_n)\,\mathrm{d}x \to +\infty$.

Indeed, since 0 is a nondegenerate minimum of V, there exist $\varrho > 0$ and $\lambda > 0$ such that

$$|\xi| < \varrho \quad \Rightarrow \quad V(\xi) \geqslant \lambda|\xi|^2.$$

For any $n \geqslant 1$, let

$$A_n := \{x \in \mathbb{R}^3 : |\varphi_n(x)| \leqslant \varrho\}.$$

By a well known Sobolev inequality and the boundedness of $||D\varphi_n||_{(L^2)^3}$, we infer that $||\varphi_n||_{L^6} \leqslant C_2$; hence $\operatorname{meas}(CA_n) \leqslant C_3$, where CA_n denotes $\mathbb{R}^3 \setminus A_n$.

Furthermore, by Holder inequality, we have

$$\int_{CA_n} |\varphi_n|^2\,\mathrm{d}x \leqslant \left(\int_{CA_n} |\varphi_n|^6\,\mathrm{d}x\right)^{\frac{1}{3}} (\operatorname{meas}(CA_n))^{\frac{2}{3}} \leqslant C_4.$$

Finally, we get

$$\begin{aligned}\int_{\mathbb{R}^3} V(\varphi_n)\,\mathrm{d}x &\geqslant \int_{A_n} V(\varphi_n)\,\mathrm{d}x \geqslant \lambda \int_{A_n} \varphi_n^2\,\mathrm{d}x\\ &= \lambda\left(||\varphi_n||_{L^2}^2 - \int_{CA_n} \varphi_n^2\,\mathrm{d}x\right)\\ &\geqslant \lambda\big(||\varphi_n||_{L^2}^2 - C_4\big)\end{aligned}$$

that is the coercivity of E on H.

Since (2.2) holds one easily gets (2.7). □

PROPOSITION 2.4. *Let $(\varphi_n)_n \subset \Lambda$ be bounded in the H-norm and weakly convergent to $\varphi \in \partial\Lambda$, then $\int_{\mathbb{R}^3} V(\varphi_n)\,\mathrm{d}x \to +\infty$. That is, if $(\varphi_n)_n \subset \Lambda$ is weakly convergent to φ and $E(\varphi_n)$ is bounded, then $\varphi \in \Lambda$.*

Proof. Since $\varphi \in \partial\Lambda$ there exists $\overline{x} \in \mathbb{R}^3$ such that $\varphi(\overline{x}) = \overline{\xi}$. Since V is positive it suffices to show that

$$\int_{B(\overline{x},\varrho)} V(\varphi_n)\,\mathrm{d}x \to +\infty \tag{2.8}$$

for a suitable $\varrho > 0$.

By uniform convergence on compact sets, we have

$$\lim_{n\to\infty} \varphi_n(\overline{x}) = \overline{\xi}. \tag{2.9}$$

Since $(||\varphi_n||)_n$ is bounded, (2.3) implies that

$$|\varphi_n(x) - \varphi_n(\overline{x})| \leqslant C|x - \overline{x}|^{\frac{1}{2}}, \quad \forall\, x \in \mathbb{R}^3 \tag{2.10}$$

for a suitable constant C.

From (2.9) and (2.10) it follows that

$$|\varphi_n(x) - \overline{\xi}| \leqslant C|x - \overline{x}|^{\frac{1}{2}} + \mathrm{o}(1) \tag{2.11}$$

for n sufficiently large; hence we can fix $\varrho > 0$ such that, for all $x \in B(\overline{x}, \varrho)$, $|\varphi_n(x) - \overline{\xi}| \leqslant r$ where r is the number which appears in (1.3). Hence, using (1.3) and (2.11), for every $x \in B(\overline{x}, \varrho)$ we have

$$V(\varphi_n(x)) \geqslant \frac{C'}{|\varphi_n(x) - \overline{\xi}|^{\frac{1}{6}}} \geqslant \frac{C''}{|x - \overline{x}|^3 + \mathrm{o}(1)},$$

from which (2.8) follows. □

PROPOSITION 2.5. *For every $a > 0$, there exists $d > 0$ such that, for every $\varphi \in \Lambda$,*

$$E(\varphi) \leqslant a \Rightarrow \min_{x\in\mathbb{R}^3} |\varphi(x) - \overline{\xi}| \geqslant d.$$

PROPOSITION 2.6. *For every* $\varphi \in \Lambda$ *and for every sequence* $(\varphi_n)_n \subset \Lambda$, *if* $(\varphi_n)_n$ *weakly converges to* φ, *then* $E(\varphi) \leqslant \liminf_{n\to\infty} E(\varphi_n)$. *For the proof of Propositions* 2.6, *one can argue like in [6].*

PROPOSITION 2.7. *The minimum points* $\varphi \in \Lambda$ *for the functional E are weak solutions of the system* (1.1).

Proof. Let $\varphi \in \Lambda$ be a minimum point of E and fix ψ arbitrarily in $C_0^\infty(\mathbb{R}^3; \mathbb{R})$. Let $\bar{\mathbf{e}}_j$ denote the jth vector of the canonical base in $\mathbb{R}^4$.

If $s > 0$ is sufficiently small, then $\varphi + s\psi\bar{\mathbf{e}}_j \in \Lambda$ and $E(\varphi + s\psi\bar{\mathbf{e}}_j) < +\infty$. Differentiating with respect to s in 0, taking into account that φ is a minimum point, one gets

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}s} E(\varphi + s\psi\bar{\mathbf{e}}_j)_{|s=0} \\ &= \int_{\mathbb{R}^3} \left(\nabla\varphi^j \nabla\psi + \Delta\varphi^j \Delta\psi + \frac{\partial V}{\partial \xi_j}(\varphi)\psi \right) dx. \end{aligned} \qquad \square$$

3. Existence Theorem

We recall the splitting lemma, whose proof derives straightforwardly from the concentration compactness method (see [6, 15]).

LEMMA 3.1 (sSplitting lemma). *Let* $(\varphi_n)_n \subset \Lambda^*$ *be such that*

$$E(\varphi_n) \leqslant a. \tag{3.1}$$

Then there exists $l \in \mathbb{N}$, *with*

$$1 \leqslant l \leqslant \frac{a}{\Delta^*} \tag{3.2}$$

(Δ^* *has been introduced in* (2.7)) *and there exist* $\overline{\varphi}_1, \ldots, \overline{\varphi}_l \in \Lambda, (x_n^1)_n, \ldots, (x_n^l)_n \subset \mathbb{R}^3$, $R_1, \ldots, R_l > 0$ *such that, up to subsequences,*

$$\varphi_n(\cdot + x_n^i) \rightharpoonup \overline{\varphi}_i \quad in \quad \Lambda \quad as \quad n \to \infty, \tag{3.3}$$

$$||\overline{\varphi}_i||_\infty \geqslant 1, \tag{3.4}$$

$$|x_n^i - x_n^j| \to +\infty \quad if \quad i \neq j, \quad as \quad n \to \infty, \tag{3.5}$$

$$\sum_{i=1}^{l} E(\overline{\varphi}_i) \leqslant \liminf_{n\to\infty} E(\varphi_n), \tag{3.6}$$

$$\forall\, x \in \mathbb{R}^3 \setminus \left(\bigcup_{i=1}^{l} B(x_n^i, R_i) \right) \quad |\varphi_n(x)| \leqslant 1 \quad \textit{for } n \textit{ sufficiently large.} \tag{3.7}$$

Then we have also

$$\mathrm{car}(\varphi_n) = \sum_{j=1}^{l} \mathrm{car}(\overline{\varphi}_j) \quad \textit{for } n \textit{ large enough,} \tag{3.8}$$

$$\limsup_{n\to\infty} ||\varphi_n - \sum_{j=1}^{l} \overline{\varphi}_j(\cdot - x_n^j)||_\infty \leqslant 1. \tag{3.9}$$

Remark 3.2. From (3.4) it follows that

$$E(\overline{\varphi}_i) \geqslant \Delta^* \quad \forall i = 1, \dots, l. \tag{3.10}$$

Remark 3.3. Using (2.4) and (3.5), it is not difficult to see that, for n large enough,

$$\sum_{i=1}^{l} \overline{\varphi}_i(\cdot - x_n^i) \in \Lambda,$$

and

$$\mathrm{car}\left(\sum_{i=1}^{l} \overline{\varphi}_i(\cdot - x_n^i) \right) = \sum_{i=1}^{l} \mathrm{car}(\overline{\varphi}_i).$$

THEOREM 3.4. *There exists* $\overline{\varphi} \in \Lambda^*$ *such that* $E(\overline{\varphi}) = \inf_{\Lambda^*} E$.

Proof. Let us set $E^* = \inf_{\Lambda^*} E$. From (2.6) it follows that $\Delta^* \leqslant E^*$.

Consider now a minimizing sequence $(\varphi_n)_n \subset \Lambda^*$. Let us apply the splitting lemma: there exists $l \in \mathbb{N}$ and $\overline{\varphi}_1, \dots, \overline{\varphi}_l \in \Lambda$ such that, up to a subsequence,

$$\sum_{i=1}^{l} E(\overline{\varphi}_i) \leqslant \liminf_{n\to\infty} E(\varphi_n) = E^* \tag{3.11}$$

and

$$\mathrm{car}(\varphi_n) = \sum_{i=1}^{l} \mathrm{car}(\overline{\varphi}_i). \tag{3.12}$$

Since $\mathrm{car}(\varphi_n) \neq 0$, from (3.12) we deduce that there exists $i \in \{1, \dots, l\}$, for sake of simplicity $i = 1$, such that $\mathrm{car}(\overline{\varphi}_1) \neq 0$.

Then, by (3.11), we obtain

$$E^* \geqslant \sum_{i=1}^{l} E(\overline{\varphi}_i) \geqslant E(\overline{\varphi}_1) \geqslant E^*;$$

so we get $E(\overline{\varphi}_1) = E^*$. □

For every $q \in \mathbb{N}$, we set $E_q = \inf_{\mathcal{A}_q} E = \inf_{\mathcal{A}_{-q}} E$. It is obvious that the functional E takes its absolute minimum 0 in the class $\mathcal{A}_0$. Let us observe that, under a further condition on V, one can prove the existence of a nontrivial weak solution for (1.11) with charge 0. On the other hand, by Theorem 3.4, there exists at least one $\overline{q} = \mathrm{car}(\overline{\varphi}) \neq 0$ such that $E_{\overline{q}}$ is attained.

PROPOSITION 3.5. *For every $q \neq 0$, a sufficient condition to guarantee that E_q is attained is that there exists a minimizing sequence in $\mathcal{A}_q$ which satisfies the properties of the splitting lemma with $l = 1$. So, if E_q is not attained, then every minimizing sequence in $\mathcal{A}_q$ satisfies the properties of the splitting lemma with $l \geqslant 2$.*

Proof. Let $(\varphi_n)_n$ be a minimizing sequence in $\mathcal{A}_q$ which satisfies the properties of Lemma 3.1 with $l = 1$. From (3.8) we have $\mathrm{car}(\overline{\varphi}_1) = \mathrm{car}(\varphi_n) = q$, which implies $E(\overline{\varphi}_1) \geqslant E_q$. On the other hand, by (3.6),

$$E_q = \liminf_{n\to\infty} E(\varphi_n) \geqslant E(\overline{\varphi}_1).$$

So we conclude that $E(\overline{\varphi}_1) = E_q$. □

PROPOSITION 3.6. *For every $q \neq 0$, if $E_q < 2E^*$, then the value E_q is attained in $\mathcal{A}_q$.*

Proof. Let $(\varphi_n)_n$ be a minimizing sequence in $\mathcal{A}_q$. For n sufficiently large we have

$$E(\varphi_n) < 2E^*. \tag{3.13}$$

Using the splitting lemma we see that there exist $l \in \mathbb{N}$ and $\overline{\varphi}_1, \ldots, \overline{\varphi}_l \in \Lambda$ such that, up to a subsequence,

$$\sum_{i=1}^{l} E(\overline{\varphi}_i) \leqslant \liminf_{n\to\infty} E(\varphi_n) = E_q \tag{3.14}$$

and

$$0 \neq q = \mathrm{car}(\varphi_n) = \sum_{i=1}^{l} \mathrm{car}(\overline{\varphi}_i). \tag{3.15}$$

By (3.15) there exist $i \in \{1, \ldots, l\}$ such that $\mathrm{car}(\overline{\varphi}_i) \neq 0$; by (3.13) and (3.14) such index is unique; say $i = 1$. Then we have $\mathrm{car}(\overline{\varphi}_1) = \mathrm{car}(\varphi_n) = q$ and, substituting in (3.14),

$$E_q \geqslant \sum_{i=1}^{l} E(\overline{\varphi}_i) \geqslant E_q + \sum_{i=2}^{l} E(\overline{\varphi}_i).$$

So we obtain

$$\sum_{i=2}^{l} E(\overline{\varphi}_i) = 0$$

and we necessarily have $l = 1$. Then we can apply Proposition 3.5. □

Now let us define

$$S = \{q \in \mathbb{N}^* : E_q \text{ is attained}\}.$$

S is not empty, by Theorem 3.4. Moreover, the following characterization holds.

THEOREM 3.7. *The ideal spanned by S coincides with $\mathbb{Z}$.*

Remark 3.8. Theorem 3.7 means that either the set S contains 1, that is E attains its minimum in the classes $\mathcal{A}_{\pm 1}$; or the set S contains at least two different elements, that is E has two (pairs of) minima, with different charge.

Proof. Arguing by contradiction, assume that $A = \mathbb{Z} \backslash \text{span} S \neq \emptyset$. Then we set $\mathcal{A} = \bigcup_{q \in A} \mathcal{A}_q$ and $E_A = \inf_{\mathcal{A}} E$.

Let $(\varphi_n)_n \subset \mathcal{A}$ be such that $E(\varphi_n) \to E_A$. Since $\mathcal{A} \subset \Lambda^*$, we can apply the splitting lemma: there exist $l \in \mathbb{N}$, $\overline{\varphi}_1, \dots, \overline{\varphi}_l \in \Lambda$ such that, up to a subsequence,

$$E(\overline{\varphi}_i) \geqslant \Delta^* > 0; \tag{3.16}$$

$$\sum_{i=1}^{l} E(\overline{\varphi}_i) \leqslant \liminf_{n \to \infty} E(\varphi_n) = E_A; \tag{3.17}$$

$$\text{car}(\varphi_n) = \sum_{i=1}^{l} \text{car}(\overline{\varphi}_i) := Q. \tag{3.18}$$

Since $(\varphi_n)_n \subset \mathcal{A}$, we have

$$Q \in A. \tag{3.19}$$

Then, since $\mathcal{A}_Q \subset \mathcal{A}$ and (3.18) holds, we have $E_A \leqslant E_Q \leqslant E(\varphi_n)$. So $(\varphi_n)_n$ is a minimizing sequence in $\mathcal{A}_Q$.

From (3.18) and (3.19) we infer that there exists $i \in \{1, \dots, l\}$, say $i = 1$, such that $\text{car}(\overline{\varphi}_1) \in A$; indeed, otherwise, if $\text{car}(\overline{\varphi}_i) \notin A$ for all i, then $Q = \text{car}(\overline{\varphi}_i) \in S$, which contradicts (3.19).

On the other hand, E_Q is not attained, since $Q \in A$, then, by Proposition 3.6, we get $l \geqslant 2$. Therefore, using (3.16) and (3.17), we get a contradiction

$$E_A \geqslant E(\overline{\varphi}_1) + \sum_{i=1}^{l} E(\overline{\varphi}_i) \geqslant E_A + (l-1)\Delta^* > E_A. \qquad \square$$

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References

1. Badiale, M., Benci, V. and D'Aprile, T.: Semiclassical limit for a quasilinear elliptic field equation: one-peak and multi-peak solution, *Adv. Differential Equations* **6**(4) (2001), 385–418.
2. Benci, V., D'Avenia, P., Fortunato, D. and Pisani, L.: Solitons in several space dimensions: a Derrick's problem and infinitely many solutions, *Arch. Rational Mech. Anal.* **154**(4) (2000), 297–324.
3. Benci, V. and Fortunato, D.: On the existence of impossible pilot wave, *Proc. Conf.Calculus of Variations and Related Topics*, Haifa, Israel, March, 1998.
4. Benci, V. and Fortunato, D.: Existence of string-like solitons, *Ric. Mat.,* **48** *Suppl.* (1999), 249–271.
5. Benci, V., Fortunato, D., Masiello, A. and Pisani, L.: Solitons and electromagnetic field, *Math. Z.* **232**(1) (1999), 73–102.
6. Benci, V., Fortunato, D. and Pisani, L.: Soliton like solution of a Lorentz invariant equation in dimension 3, *Rev. Math. Phys.* **3** (1998), 315–344.
7. Benci, V., Fortunato, D. and Pisani, L.: Remarks on topological solitons, *Topol. Methods Nonlinear Anal.* **7** (1996), 349–367.
8. Brezis, H.: *Analyse functionnelle – théorie et applications*, Masson, Paris, 1983.
9. Cid, C. and Felmer, P.: A note on static solutions of a Lorentz invariant equation in dimension 3, *Lett. Math. Phys.* **53**(1) (2000), 1–10.
10. Derrick, C. H.: Comments on nonlinear wave equations as model for elementary particles, *J. Math. Phys.* **5** (1964), 1252–1254.
11. Dodd, R. K., Eilbeck, J. C., Gibbon, J. D. and Morris, H. C.: *Solitons and Nonlinear Wave Equations*, Academic Press, London, 1982.
12. Esteban, M. J. and Muller, S.: Sobolev maps with integer degree and applications to Skyrme's problem, *Proc. Royal Soc. London* **436** (1992), 197–201.
13. Esteban, M. J. and Lions, P. L.: Skyrmions and symmetry, *Analysis* **1** (1998), 187–192.
14. Kichenassamy, S.: *Nonlinear Wave Equations*, Marcel Dekker, New York, 1996.
15. Lions, P. L.: The concentration – compactness principle in the calculus of variations. The locally compact case -I, *Ann. Inst. H. Poincaré Anal. Nonlinéaire* **1** (1984), 109–145.
16. Rajaraman, R.: *Soliton and Instantons*, North-Holland, Amsterdam, 1998.
17. Skyrme, T. H.: A nonlinear field theory, *Proc. Royal Soc. A* **260** (1961), 127–138.
18. Whitam, G. B.: *Linear and Nonlinear Waves*, Wiley, New York, 1974.