Proceedings of the Royal Society of Edinburgh, **144A**, 225–239, *2014*

Existence and non-existence of solutions to elliptic equations with a general convection term

Salomón Alarcón
Departamento de Matemática,
Universidad Técnica Federico Santa María,
Casilla V-110, Avenida España 1680, Valparaíso, Chile
(salomon.alarcon@usm.cl)

Jorge García-Melián
Departamento de Análisis Matemático, Universidad de La Laguna,
Campus de Anchieta, Avenida Astrofísico Francisco Sánchez s/n,
38271 La Laguna, Spain and
Instituto Universitario de Estudios Avanzados en Física Atómica,
Molecular y Fotónica, Facultad de Física, Universidad de La Laguna,
Campus de Anchieta, Avenida Astrofísico Francisco Sánchez s/n,
38203 La Laguna, Spain (jjgarmel@ull.es)

Alexander Quaas
Departamento de Matemática,
Universidad Técnica Federico Santa María,
Casilla V-110, Avenida España 1680, Valparaíso, Chile
(alexander.quaas@usm.cl)

(MS received 28 May 2012; accepted 26 February 2013)

In this paper we consider the nonlinear elliptic problem $-\Delta u + \alpha u = g(|\nabla u|) + \lambda h(x)$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain of $\mathbb{R}^N$, $\alpha \geqslant 0$, g is an arbitrary C^1 increasing function and $h \in C^1(\bar{\Omega})$ is non-negative. We completely analyse the existence and non-existence of (positive) classical solutions in terms of the parameter λ. We show that there exist solutions for every λ when $\alpha = 0$ and the integral $\int_1^\infty 1/g(s)\,\mathrm{d}s = \infty$, or when $\alpha > 0$ and the integral $\int_1^\infty s/g(s)\,\mathrm{d}s = \infty$. Conversely, when the respective integrals converge and h is non-trivial on $\partial\Omega$, existence depends on the size of λ. Moreover, non-existence holds for large λ. Our proofs mainly rely on comparison arguments, and on the construction of suitable supersolutions in annuli. Our results include some cases where the function g is superquadratic and existence still holds without assuming any smallness condition on λ.

1. Introduction

The concern of the present paper is the existence and non-existence of solutions to the following nonlinear elliptic problem:

$$\left.\begin{aligned} -\Delta u + \alpha u &= g(|\nabla u|) + \lambda h(x) && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega,\end{aligned}\right\} \tag{1.1}$$

where Ω is a bounded domain of class $C^{2,\eta}$ of $\mathbb{R}^N$ for some $\eta \in (0,1)$, $\alpha \geqslant 0$, and $g \in C^1(\mathbb{R})$ is increasing with $g(0) = 0$. We assume that the function $h \in C^1(\bar{\Omega})$ is non-negative, while λ is regarded as a positive parameter.

We focus our attention on general functions g, obtaining sharp conditions that imply that either (a) (1.1) has a unique solution for every λ or (b) there exists a critical size of λ that divides existence from non-existence for (1.1) when $h \not\equiv 0$ on $\partial\Omega$.

This type of problem has been extensively studied. We give a quick review of the topic here; other references can be found in the papers quoted below. The pioneering work on the subject seems to be due to Serrin [28], Amann and Crandall [5] and Lions [23]. The case $\alpha > 0$ is considered in [10, 11], where existence holds when g has at most quadratic growth (see also [13]). The case $\alpha = 0$ and $g(t) = t^2$ was studied, for example, in [16, 17] (see also [2, 19]). For related results see [1, 12].

More recently, related fully nonlinear equations were also considered in [29] (see also the case $\alpha < 0$ in [21], where multiplicity results were obtained).

We finally mention that a starting point of our work can be found in [3]; actually we solve a problem given in that paper (see [3, remark, p. 29]). More precise information on our contribution with respect to the known results is given in the remarks after our main theorems.

By a solution to (1.1) we mean a function $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ verifying the equation in the classical sense. Remark that, on the one hand, standard bootstrapping gives $u \in C^{2,\eta}(\bar{\Omega})$, while, on the other hand, solutions are strictly positive in Ω by the maximum principle, since $-\Delta u + \alpha u \geqslant 0$ in Ω. An important remark with regard to (1.1) is that uniqueness of solutions holds by the comparison principle (see, for example, [18, theorem 10.1] or the results in [15, 26]). Thus, we need only show existence and non-existence of solutions. For other uniqueness results see, for example, [7–9]. Observe also that non-uniqueness holds with less regularity on the solution; see, for example, [2].

We now state our main results. We begin with the case $\alpha = 0$. It turns out that the existence of solutions depends on the condition

$$\int_1^\infty \frac{\mathrm{d}s}{g(s)} = \infty. \tag{1.2}$$

More precisely, we have the following.

THEOREM 1.1. *Assume that $g \in C^1(\mathbb{R})$ is increasing with $g(0) = 0$, while $h \in C^1(\bar{\Omega})$ is such that $h \geqslant 0$ in Ω. If $\alpha = 0$, then*

(i) *if (1.2) holds, there exists a unique solution to (1.1) for every $\lambda > 0$;*

(ii) *if (1.2) does not hold and $h \not\equiv 0$ on $\partial\Omega$, then there exists $\Lambda > 0$ such that, for $\lambda \in (0, \Lambda)$, (1.1) has a unique solution, while there are no solutions when $\lambda > \Lambda$.*

REMARK 1.2. (a) The non-existence part in (ii) is already proved in the case where g is convex (see [3, theorem 2.1]). In the particular case, $g(t) = t^p$ with $p > 2$; see [20] for existence when h is a measure and λ is small.

(b) Part (ii) of theorem 1.1 extends the case $g(t) = t^2$ of the above-mentioned papers.

We now turn to the case $\alpha > 0$. In this case, the condition for the existence of solutions is

$$\int_1^\infty \frac{s}{g(s)}\,\mathrm{d}s = \infty. \tag{1.3}$$

Observe that condition (1.3) is implied by (1.2). We have the following.

THEOREM 1.3. *Assume that $g \in C^1(\mathbb{R})$ is increasing with $g(0) = 0$, while $h \in C^1(\bar{\Omega})$ is such that $h \geqslant 0$ in Ω. If $\alpha > 0$, then*

(i) *if (1.3) holds, there exists a unique solution to (1.1) for every $\lambda > 0$;*

(ii) *if (1.3) does not hold and $h \not\equiv 0$ on $\partial\Omega$, then there exists $\Lambda > 0$ such that, for $\lambda \in (0, \Lambda)$, (1.1) admits a unique solution, while no solutions exist when $\lambda > \Lambda$.*

REMARK 1.4. (a) The non-existence part in (ii) is already proved in the particular case $g(t) = t^p$ with $p > 2$ (see [3, proposition 2.3]).

(b) Part (i) in theorem 1.3 applies, for instance, to $g(t) = t^2 \ln(1 + t)$, which is superquadratic, and so existence holds without the smallness restriction on the right-hand side. This means that, in the setting of classical solutions, and with smooth data, most of the previous existence theorems are not optimal with respect to the growth condition in g, since at most quadratic behaviour is required in the case $\alpha > 0$.

(c) Part (i) in our two theorems answers an open question raised in [3] (see [3, p. 29, remark]). We are, indeed, a little more precise here, since our optimal conditions are different for the cases $\alpha = 0$ and $\alpha > 0$. When $g(t) = t$, part (i) holds in both theorems, this particular case already being covered in [3] (see [3, theorem 3.1]).

(d) The case $g(t) = O(t)$ has usually been the reference case for a general solvability result (part (i) in theorem 1.1 or theorem 1.3); see, for example, [4, 14] (note that the symmetrization approach reduces the problem to a radial one, which is related to our approach). The superlinear model case $g(t) = |t|^q$, $q > 1$, is studied in depth in [20], in particular, as far as necessary conditions for the existence are concerned. In [25], the case $g(t) = t^q$ with the limit $\alpha \to 0$ is described, and the maximal constant Λ is characterized in terms of stochastic state constraint ergodic problems. For the superquadratic case, also see [6], where the existence of a generalized viscosity solution is proved when $\alpha > 0$, though this solution is not classical and, in particular, may not attain the boundary datum.

We mention in passing that the positivity condition on h is only imposed in order to simplify the presentation. In particular, it is relevant for the non-existence results only. For a function h that takes both signs, we may still assert the existence of solutions for every $\lambda > 0$ in case (i) in both theorems, and for small λ in case (ii), although in this last situation the results are not expected to be optimal. Note also that, when h is negative, the change of u to $-u$ in (1.1) amounts to replacing g

by $-g$. In most of the previous work, no distinction is made between the two cases, but the results are far from optimal. Here, we have decided to restrict our attention to non-negative h (hence, positive g) for definiteness. Also, the restriction that h is non-trivial on $\partial\Omega$ is probably not necessary for the non-existence of solutions. The same result is expected to hold for functions h that are non-trivial on $\bar{\Omega}$, although the method of proof has to be modified. We are not pursuing this matter further in the present paper.

On the other hand, we believe that the proofs can be adapted to deal with some more general operators than the Laplacian, for instance, the p-Laplacian or even some fully nonlinear operators that depend on the second derivatives of the solution.

The basic idea for proving the existence of solutions to (1.1) comes from [23] (see also [22, 24]). It consists in truncating the term $g(|\nabla u|)$ in order to obtain a problem in a classical setting, i.e. with subquadratic growth in the gradient. The standard method of sub- and supersolutions can then be used to get a solution to the truncated problem, and the final step is to show that the solution to the truncated problem is indeed a solution to the original one. This can be achieved by obtaining appropriate estimates for the gradient ∇u of the solution u in $\bar{\Omega}$. By an adaptation of the classical method of Bernstein (see [23, 28]), these estimates are a consequence of a kind of maximum principle for $|\nabla u|^2 + u^2$, so everything is reduced to estimating $|\nabla u|$ on $\partial\Omega$. This in turn can be done by comparing with a suitable supersolution.

It is important to note that our approach does not rely on obtaining a supersolution $\bar{u}$ to (1.1) that vanishes on the whole $\partial\Omega$, something which is required to apply [23, theorem III.1]. Rather, we construct the supersolution by analysing (1.1) in an annulus that, after a suitable translation, is tangent to $\partial\Omega$ at every fixed $x_0 \in \partial\Omega$. This enables us to deal with a radial problem that is in some sense integrable, so we are able to find conditions that are both necessary and sufficient for existence.

The paper has the following structure. In § 2 we construct supersolutions to (1.1) in the particular case where Ω is an annulus. Section 3 is dedicated to showing the non-existence of solutions to (1.1) when Ω is a ball. Finally, in § 4 we deal with the proofs of theorems 1.1 and 1.3.

2. Supersolutions for problems in annuli

We prove in § 4 that the existence of a radial supersolution to (1.1) posed in an annulus when h is constant suffices to ensure the existence of a solution to (1.1). Thus, this section is dedicated to constructing a positive radial function u verifying

$$\left.\begin{aligned} -(r^{N-1}u')' \geqslant r^{N-1}(-\alpha u + g(|u'|) + c), \quad R_1 < r < R_2,\\ u(R_1) = 0, \qquad u(R_2) \geqslant 0, \end{aligned}\right\} \tag{2.1}$$

for suitable values of c, depending on whether $\alpha = 0$ or $\alpha > 0$, and also on the integrability conditions on g at ∞ considered in § 1. In what follows, $R_2 > R_1 > 0$ are fixed.

LEMMA 2.1. *Assume that $g \in C^1(\mathbb{R})$ is increasing with $g(0) = 0$ and $\alpha = 0$. Then, if*

$$\int_1^\infty \frac{\mathrm{d}s}{g(s)} = \infty, \tag{2.2}$$

for every $c > 0$ there exists a positive radial function u verifying (2.1). If (2.2) does not hold, the existence of such a function also follows provided that c is small enough.

Proof. Introducing the change of variables

$$s = \begin{cases} \log r, & N = 2, \\ -\dfrac{1}{N-2}\dfrac{1}{r^{N-2}}, & N \geqslant 3, \end{cases} \tag{2.3}$$

and defining $u(r) = v(s)$, (2.1) is transformed into

$$-v'' \geqslant r^{2(N-1)}\left(g\left(\frac{1}{r^{N-1}}|v'|\right) + c\right),$$
$$v(a) = 0, \quad v(b) \geqslant 0,$$

where $a = \log R_1$, $b = \log R_2$ when $N = 2$, while

$$a = -\frac{1}{N-2}\frac{1}{R_1^{N-2}}, \quad b = -\frac{1}{N-2}\frac{1}{R_2^{N-2}} \quad \text{if } N \geqslant 3.$$

Since g is increasing and positive, it suffices to have that

$$-v'' \geqslant R_2^{2(N-1)}\left(g\left(\frac{1}{R_1^{N-1}}|v'|\right) + c\right),$$
$$v(a) = 0, \qquad v(b) \geqslant 0.$$

Setting $w(s) = v(s+a)$, this suggests that we consider the one-dimensional autonomous initial-value problem

$$\left.\begin{aligned} -w'' = R_2^{2(N-1)}\left(g\left(\frac{1}{R_1^{N-1}}|w'|\right) + c\right), \\ w(0) = 0, \qquad w'(0) = \gamma > 0, \end{aligned}\right\} \tag{2.4}$$

which has a unique solution for every $\gamma > 0$, and find a positive solution in $(0, b-a)$. Observe that the solutions to (2.4) verify, on one hand, that $w'' \leqslant -cR_2^{2(N-1)}$, so an integration gives that $w(s) \leqslant s(\gamma - cR_2^{2(N-1)}s/2)$. On the other hand, since w' is decreasing, we have that $w'(s_0) = 0$ for some $s_0 > 0$, and it follows, by the symmetry of the problem, that w is symmetric with respect to s_0 and $w(2s_0) = 0$. Letting $2s_0$ be the first zero of w and integrating the equation in $(0, s_0)$, we obtain that

$$s_0 = \left(\frac{R_1}{R_2^2}\right)^{N-1} \int_0^{\gamma/R_1^{N-1}} \frac{\mathrm{d}t}{g(t) + c}.$$

We conclude that s_0 is an increasing function of γ that verifies

$$s_0 \to \left(\frac{R_1}{R_2^2}\right)^{N-1} \int_0^\infty \frac{\mathrm{d}t}{g(t) + c} \tag{2.5}$$

as $\gamma \to +\infty$. Therefore, in the case where (2.2) holds, since g is increasing, the integral in (2.5) diverges. So, we can always choose γ large enough such that $s_0 > (b-a)/2$, and this provides us with a positive solution of (2.1). When the integral converges, we can also obtain that $s_0 > (b-a)/2$ if we select c small enough, since the integral

$$\int_0^\infty \frac{\mathrm{d}t}{g(t)} \tag{2.6}$$

diverges at 0, due to $g(0) = 0$ and $g \in C^1(\mathbb{R})$. This concludes the proof. □

Lemma 2.2. *Assume that $g \in C^1(\mathbb{R})$ is increasing with $g(0) = 0$ and $\alpha > 0$. Then, if*

$$\int_1^\infty \frac{s}{g(s)}\,\mathrm{d}s = \infty, \tag{2.7}$$

for every $c > 0$ there exists a positive radial function u verifying (2.1). If (2.7) does not hold and c is small enough, such a function also exists.

Proof. Setting $z = c/\alpha - u$, we look for a function verifying

$$(r^{N-1}z')' \geqslant r^{N-1}(\alpha z + g(|z'|)),$$
$$z(R_1) = \frac{c}{\alpha}, \qquad z(R_2) \leqslant \frac{c}{\alpha}.$$

We look for a positive solution z to this inequality. With the change of variables (2.3), and letting $v(s) = z(r)$, we find, as before, that v is a supersolution, provided, for instance, that

$$v'' \geqslant R_2^{2(N-1)}\biggl(\alpha v + g\biggl(\frac{1}{R_1^{N-1}}|v'|\biggr)\biggr),$$
$$v(a) = \frac{c}{\alpha}, \qquad v(b) = 0.$$

Setting $w(s) = v(b-s)$, it is thus natural to consider the initial-value problem

$$\left.\begin{aligned} w'' &= R_2^{2(N-1)}\biggl(\alpha w + g\biggl(\frac{1}{R_1^{N-1}}|w'|\biggr)\biggr), \\ w(0) &= 0, \qquad w'(0) = \gamma > 0, \end{aligned}\right\} \tag{2.8}$$

which has a unique solution for every $\gamma > 0$, and see if we can select γ such that $w(b-a)$ is as large as we please.

Note that $w'' \geqslant 0$ as long as $w \geqslant 0$, so it is not hard to see that solutions are positive, increasing and convex for $s > 0$. For every $\gamma > 0$, the solution is defined in an interval $[0, T(\gamma))$, and when $T(\gamma) < \infty$ we have that

$$\lim_{s\to T(\gamma)} w(s) = +\infty \quad \text{or} \quad \lim_{s\to T(\gamma)} w'(s) = +\infty. \tag{2.9}$$

We shall see that, when the integral condition (2.7) is satisfied, we always have both conditions in (2.9). Indeed, the first one implies the second, and if we had that $w(T(\gamma)) < +\infty$, then

$$w'' \leqslant R_2^{2(N-1)}\biggl(\alpha w(T(\gamma)) + g\biggl(\frac{1}{R_1^{N-1}}w'\biggr)\biggr).$$

Multiplying by w' and integrating, we arrive at

$$\int_0^T \frac{w'w''}{\alpha w(T(\gamma)) + g(w'/R_1^{N-1})} \leqslant R_2^{2(N-1)} w(T(\gamma)),$$

which yields that

$$\int_{\gamma/R_1^{N-1}}^{\infty} \frac{s}{\alpha w(T(\gamma)) + g(s)}\, \mathrm{d}s \leqslant \frac{R_2^{2(N-1)}}{R_1^{N-1}} w(T(\gamma)),$$

contradicting condition (2.7). Thus, $w, w' \to \infty$ as $s \to T(\gamma)$.

We have two cases to consider: either $T(\gamma_0)$ is infinite for some $\gamma_0 > 0$, or $T(\gamma)$ is finite for every $\gamma > 0$. In the first case, we see that this implies that $T(\gamma) = \infty$ for every $\gamma > 0$. Observe first that, when u, v are two solutions to the equation in (2.8) with $u(0) \geqslant v(0)$, $u'(0) = \gamma_1 > \gamma_2 = v'(0)$, then $u > v$ in the common interval of definition; hence, $T(\gamma_1) \leqslant T(\gamma_2)$.

In particular, $T(\gamma) = \infty$ for $\gamma < \gamma_0$. If $\gamma > \gamma_0$ and we temporarily denote by w_γ the unique solution to (2.8), there exists $\delta > 0$ such that $w'_{\gamma_0}(\delta) > \gamma$, since w'_{γ_0} is increasing and converges to ∞. Let

$$\bar{w}(x) = w_{\gamma_0}(x + \delta).$$

Then, $\bar{w}$ is a solution to the same equation with initial data $\bar{w}(0) = w_{\gamma_0}(\delta) > 0$, $\bar{w}'(0) = w'_{\gamma_0}(\delta) > \gamma$. It follows by the previous observation that $\bar{w} > w_\gamma$, and, in particular, $T(\gamma) = \infty$. Thus, all solutions are global in this case and it is easy to conclude that, since $w(x) \geqslant \gamma x$ by convexity, we can have $w(b-a)$ as large as we please, so a supersolution can be constructed with large values of c.

The second possibility is that all solutions blow up in finite time, i.e. $T(\gamma) < \infty$ for every $\gamma > 0$. We see that, in such a case, $T(\gamma)$ is a continuous function of γ. Take $\gamma_n \downarrow \gamma$. By comparison, we have that $T(\gamma_n) < T(\gamma)$. Moreover, we can choose $\delta_n \downarrow 0$ such that $w'_\gamma(\delta_n) > \gamma_n$. Arguing as before, $w_\gamma(x + \delta_n) > w_{\gamma_n}(x)$, so $T(\gamma) - \delta_n < T(\gamma_n)$ and we obtain that $T(\gamma_n) \to T(\gamma)$. When $\gamma_n \uparrow \gamma$ the proof is similar.

Next, we claim that $T(\gamma) \to \infty$ as $\gamma \to 0$. Indeed, assume that $T(\gamma) \leqslant T_0$ when $\gamma \to 0$. Since $w'' \geqslant 0$, we obtain that $w \leqslant T_0 w'$, so

$$w'' \leqslant R_2^{2(N-1)} \biggl(g\biggl(\frac{1}{R_1^{N-1}} w' \biggr) + \alpha T_0 w' \biggr),$$

and this leads, after an integration and a change of variables, to

$$\int_{\gamma/R_1^{N-1}}^{\infty} \frac{\mathrm{d}s}{g(s) + \alpha T_0 R_1^{N-1} s} \leqslant \biggl(\frac{R_2^2}{R_1} \biggr)^{N-1} T_0.$$

A contradiction is reached when we let $\gamma \to 0$, since the integral then diverges. Thus, $\lim_{\gamma \downarrow 0} T(\gamma) = \infty$.

We define $\bar{T} = \lim_{\gamma \to \infty} T(\gamma)$ (which is expected to be 0). If $\bar{T} \leqslant b - a$, we can use the continuity of T to obtain that $\gamma > 0$ such that $T(\gamma) = b - a + \varepsilon$ for small positive ε. Taking ε as small as we please, we obtain $w_\gamma(b-a)$ as large as we please, and this provides us with a supersolution for large values of c. If, on the contrary, $\bar{T} > b - a$, then all solutions would be defined at least in $[0, b-a]$, and since $w(b-a) \geqslant \gamma(b-a)$, we obtain that $w_\gamma(b-a)$ is as large as we please by taking large values of γ.

To conclude the proof, we now consider the case when

$$\int_1^\infty \frac{s}{g(s)}\,\mathrm{d}s < \infty$$

and c is small enough. Observe that, in this case, all solutions blow up in finite time. Indeed, let $T < T(\gamma)$. Since

$$w'' \geqslant R_2^{2(N-1)} g\left(\frac{1}{R_1^{N-1}} w'\right),$$

we can integrate in $(0,T)$ and let $T \to T(\gamma)$ to arrive at

$$R_2^{2(N-1)} T(\gamma) \leqslant R_1^{N-1} \int_\gamma^\infty \frac{\mathrm{d}s}{g(s)} < \infty, \tag{2.10}$$

since this last integral also converges.

It also follows from (2.10) that $T(\gamma) \to 0$ as $\gamma \to \infty$ (i.e. $\bar{T} = 0$ in the above proof). Since $T(\gamma)$ is continuous with $T(\gamma) \to \infty$ as $\gamma \to 0$, we can choose γ such that $T(\gamma) > b-a$ and obtain a supersolution for $c \leqslant \alpha w_\gamma(T(\gamma))$. It is worth mentioning that in the present case, where (2.7) does not hold, we cannot guarantee that the first equality in (2.9) holds, so the supersolution is not valid in principle for large values of c. □

3. Nonexistence of solutions in balls

In this section we tackle the question of non-existence of solutions to (1.1). We see in § 4 that it suffices to show non-existence of radial solutions when Ω is a ball of $\mathbb{R}^N$ and h is constant. Thus, under several hypotheses, we show that the problem

$$\left.\begin{aligned} -u'' - \frac{N-1}{r}u' = -\alpha u + g(|u'|) + c, \quad 0 < r < R,\\ u'(0) = 0, \qquad u(R) = 0, \end{aligned}\right\} \tag{3.1}$$

does not admit positive solutions for large values of c.

Lemma 3.1. *Assume that $g \in C^1(\mathbb{R})$ is increasing with $g(0) = 0$ and $\alpha = 0$. Then, if*

$$\int_1^\infty \frac{\mathrm{d}s}{g(s)} < \infty,$$

there exists $c_0 > 0$ such that (3.1) does not admit positive solutions when $c \geqslant c_0$.

Proof. Assume that u is a solution to (3.1). We first claim that $u'(r) < 0$ for $r \in (0,R)$ and $u''(r) < 0$ in $[0,R)$. Observe that $u''(0) = -c/N < 0$, so $u'(r) < 0$ for $r > 0$ close enough to 0. If we had $u'(r_0) = 0$ for some $r_0 \in (0,R)$ with $u' < 0$ in $(0,r_0)$, then $u''(r_0) \geqslant 0$, so from the equation we obtain that $u''(r_0) = -c < 0$, which is impossible. Then, $u'(r) < 0$ if $0 < r < R$.

Assume now that for some $\tilde{r}_0 \in (0, R)$ we have that $u''(\tilde{r}_0) = 0$. Since $u'''(\tilde{r}_0) \geqslant 0$ in this case, we obtain, by differentiating the equation

$$-u''' - \frac{N-1}{r}u'' + \frac{N-1}{r^2}u' = -g'(-u')u''$$

such that $u'''(\tilde{r}_0) < 0$, a contradiction. Thus, $u''(r) < 0$ for $r \in (0, R)$ as well.

Next, if we rewrite the equation as $-(r^{N-1}u')' = r^{N-1}(g(-u') + c)$ and integrate in $(0, r)$, taking into account that $g(-u')$ is increasing, we obtain that

$$\begin{aligned} -r^{N-1}u'(r) &= \int_0^r s^{N-1}(g(-u'(s)) + c)\,\mathrm{d}s \\ &\leqslant (g(-u'(r)) + c)\int_0^r s^{N-1}\,\mathrm{d}s \\ &= \frac{r^N}{N}(g(-u'(r)) + c), \end{aligned}$$

so, substituting this into (3.1), we have that

$$-u'' \geqslant \frac{1}{N}(g(-u') + c) \quad \text{in } (0, R).$$

Integrating in $(0, R)$, we obtain that

$$\int_0^\infty \frac{\mathrm{d}t}{g(t) + c} > \int_0^{-u'(R)} \frac{\mathrm{d}t}{g(t) + c} \geqslant \frac{1}{N}R.$$

This implies that c cannot be too large in order to have a positive solution to (3.1). □

LEMMA 3.2. *Assume that $g \in C^1(\mathbb{R})$ is increasing with $g(0) = 0$ and $\alpha > 0$. Then, if*

$$\int_1^\infty \frac{s}{g(s)}\,\mathrm{d}s < \infty, \tag{3.2}$$

there exists $c_0 > 0$ such that (3.1) does not admit positive solutions when $c \geqslant c_0$.

Proof. Let u be a positive solution to (3.1). We first claim that $u < c/\alpha$. Indeed, if we had that $u(0) = c/\alpha$, then $u \equiv c/\alpha$ by uniqueness, which is not possible. If $u(0) > c/\alpha$, then $u''(0) > 0$ and u initially increases. According to the boundary condition $u(R) = 0$, there should be a point where u achieves its maximum, but this is in contradiction with the equation. We conclude that $u(0) < c/\alpha$ and, again by the equation in (3.1), u initially decreases and cannot reach a minimum, so u is always decreasing. It is seen, much as in the previous case, that $u'' < 0$ in $[0, R)$ also. Thus, arguing as in that proof, we obtain that

$$-u'' \geqslant \frac{1}{N}(-\alpha u + g(-u') + c). \tag{3.3}$$

Assume that there exists a sequence $c_n \to \infty$ such that a positive solution u_n to (3.1) exists with $c = c_n$ (with no loss of generality we may assume that c_n is increasing). Let $v_n = c_n/\alpha - u_n$. Then,

$$v_n'' + \frac{N-1}{r} v_n' = \alpha v_n + g(v_n'),$$
$$v_n'(0) = 0, \qquad v_n(R) = \frac{c_n}{\alpha},$$

with $v_n' > 0$, $v_n'' > 0$. We claim that $v_n(0)$ is bounded as $n \to \infty$. Indeed, since from (3.3) we have that

$$v_n'' \geqslant \frac{1}{N}(\alpha v_n + g(v_n')) \geqslant \frac{1}{N}(\alpha v_n(0) + g(v_n')),$$

we can integrate to arrive at

$$\frac{1}{N} R \leqslant \int_0^{v_n'(R)} \frac{\mathrm{d}s}{\alpha v_n(0) + g(s)} < \int_0^{\infty} \frac{\mathrm{d}s}{\alpha v_n(0) + g(s)}.$$

Therefore, if $v_n(0) \to \infty$, we arrive at a contradiction. Since solutions are increasing in c (due to uniqueness), we can guarantee that $v_n(0) \to \bar{v}$ for some $\bar{v} > 0$. It also follows that $v_n \to z$, the unique solution to

$$z'' + \frac{N-1}{r} z' = \alpha z + g(z'),$$
$$z(0) = \bar{v}, \qquad z'(0) = 0,$$

which is defined in a maximal interval $[0, T)$. When $T < \infty$, $\lim_{r \to T} z(r) = \infty$ or $\lim_{r \to T} z'(r) = \infty$. By comparison, we also have that $v_n \leqslant z$ in $[0, \min\{T, R\})$.

We see that $T < R$. Indeed, if $T \geqslant R$, we have that $v_n(R) = c_n/\alpha \leqslant z(R)$, and then it follows that $T = R$ and $z(R) = \infty$. This is impossible, since $z'' \geqslant g(z')/N$, and multiplication by z' and another integration between $\frac{1}{2}R$ and $R - \varepsilon$ for some small positive ε yields that

$$\frac{1}{N}\left(z(R-\varepsilon) - z\left(\frac{R}{2}\right)\right) \leqslant \int_{z'(R/2)}^{z'(R-\varepsilon)} \frac{s}{g(s)}\,\mathrm{d}s < \int_{z'(R/2)}^{\infty} \frac{s}{g(s)}\,\mathrm{d}s.$$

Letting $\varepsilon \to 0$ we obtain a contradiction with (3.2). Thus, $T < R$.

Now choose a small $\varepsilon > 0$. Since $v_n' \to z'$ uniformly in $[0, T - \varepsilon]$, we have that $v_n'(T - \varepsilon) \geqslant z'(T - \varepsilon) - \varepsilon$ if n is large enough. Therefore,

$$\frac{1}{N}(R - T + \varepsilon) \leqslant \int_{v_n'(T-\varepsilon)}^{v'(R)} \frac{\mathrm{d}s}{g(s)} < \int_{z'(T-\varepsilon)-\varepsilon}^{\infty} \frac{\mathrm{d}s}{g(s)}.$$

Letting $\varepsilon \to 0$, we arrive at $T \geqslant R$, a contradiction, which shows that no solutions to (3.1) may exist if c is large enough. □

4. Proof of the theorems

This final section is dedicated to the proof of theorems 1.1 and 1.3. The idea of the proof of existence comes from [23], and it consists in truncating the function g in order to obtain a solution, and then estimating the gradient of this solution in $\bar{\Omega}$. The essential point is to obtain a suitable supersolution.

Since Ω verifies the uniform exterior sphere condition, there exists $R_1 > 0$ such that for every $x_0 \in \partial\Omega$ there exists $y_0 \in \mathbb{R}^N \setminus \Omega$ with $\overline{B_{R_1}(y_0)} \cap \bar{\Omega} = \{x_0\}$. Choose $R_2 > R_1$ large enough such that $\Omega \subset A := B_{R_2}(y_0) \setminus \overline{B_{R_1}}(y_0)$ for every $x_0 \in \partial\Omega$. Consider the radial problem

$$\left.\begin{aligned} -(r^{N-1}u')' = r^{N-1}(-\alpha u + g(|u'|) + c), \quad R_1 < r < R_2, \\ u(R_1) = 0, \qquad u(R_2) \geqslant 0, \end{aligned}\right\} \tag{4.1}$$

where $c > 0$. The first important existence result is the following.

LEMMA 4.1. *Assume that $g \in C^1(\mathbb{R})$ is increasing with $g(0) = 0$ and $h \in C^1(\bar{\Omega})$ is non-negative. If there exists a positive supersolution $\bar{u}$ to (4.1), then, for every $\lambda \in (0, c/|h|_\infty]$, (1.1) admits a positive solution.*

With regard to non-existence results, the reference situation is a radial problem in a ball. Observe that, since Ω verifies a uniform interior ball condition and $h \geqslant 0$, $h \not\equiv 0$ on $\partial\Omega$, we can find $x_0 \in \partial\Omega$, $y_0 \in \Omega$ and $R > 0$ such that $B_R(y_0) \subset \Omega$, $\overline{B_R(y_0)} \cap \partial\Omega = \{x_0\}$ and $h \geqslant h_0 > 0$ in $\overline{B_R(y_0)}$. Consider the problem

$$\left.\begin{aligned} -(r^{N-1}u')' = r^{N-1}(-\alpha u + g(|u'|) + c), \quad 0 < r < R, \\ u'(0) = 0, \qquad u(R) = 0, \end{aligned}\right\} \tag{4.2}$$

where $c > 0$. We then have the following.

LEMMA 4.2. *Assume that $g \in C^1(\mathbb{R})$ is increasing with $g(0) = 0$ and $h \in C^1(\bar{\Omega})$ is non-negative with $h \geqslant h_0 > 0$ in $B_R(y_0)$. If (4.2) does not admit a positive solution for some $c > 0$, then (1.1) does not have solutions for $\lambda \geqslant c/h_0$.*

As we have quoted in § 1, the method we follow for proving existence (and indeed also non-existence) relies on obtaining good estimates for the gradient of the solutions. This last part is achieved by means of a kind of maximum principle for the gradient of solutions to (1.1). The proof is inspired by the classical method of Bernstein (see, for example, [23, 28]).

LEMMA 4.3. *Let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a solution to (1.1). Assume that $g \in C^1(\mathbb{R})$ is increasing with $g(0) = 0$ and $h \in C^1(\bar{\Omega})$. There then exists a constant C that depends on $\sup_{\bar{\Omega}}|u|$, $\sup_{\partial\Omega}|\nabla u|$, $\sup_{\Omega}|\nabla h|$ and λ such that*

$$|\nabla u| \leqslant C \quad \text{in } \bar{\Omega}.$$

Proof. Let u be a solution to (1.1), and define $w = |\nabla u|^2 + u^2$. For simplicity, we define $g(|\xi|) = \tilde{g}(|\xi|^2)$. By standard regularity, it follows that $u \in C^3(\Omega_\rho)$, where $\Omega_\rho = \{x \in \Omega\colon |\nabla u|^2 > \rho\}$ for some $0 < \rho < \|u\|_\infty^2$, and, hence, $w \in C^2(\Omega_\rho) \cap C(\bar{\Omega}_\rho)$. It is then not hard to check that, in Ω_ρ, one has that

$$\Delta w = \frac{2}{N}|D^2u|^2 - 2\tilde{g}'(|\nabla u|^2)\nabla u \nabla(w - u^2) - 2\lambda \nabla h \nabla u + 2u\Delta u + (2 + 2\alpha)|\nabla u|^2.$$

On the other hand,

$$(\Delta u)^2 = \bigg(\sum_{i=1}^N \partial_{ii} u\bigg)^2 \leqslant N \sum_{i=1}^N (\partial_{ii} u)^2 \leqslant N|D^2u|^2,$$

and since $\tilde{g}$ is non-decreasing and $u \geqslant 0$, so $2\tilde{g}'(|\nabla u|^2)\nabla u \nabla(u^2) \geqslant 0$, we have that

$$\Delta w \geqslant \frac{2}{N}(\Delta u)^2 - 2\tilde{g}'(|\nabla u|^2)\nabla u \nabla w - 2\lambda \nabla h \nabla u + 2u\Delta u + (2+2\alpha)|\nabla u|^2$$

in Ω_ρ. An application of the Cauchy–Schwarz inequality leads to

$$\Delta w \geqslant \frac{1}{N}(\Delta u)^2 - 2\tilde{g}'(|\nabla u|^2)\nabla u \nabla w - \lambda^2 |\nabla h|^2 - N u^2 + |\nabla u|^2$$

in Ω_ρ. Fix

$$M > \sup_{\partial\Omega} |\nabla u|^2 + 2N\|u\|_\infty^2 + \lambda^2 \|\nabla h\|_\infty,$$

and assume that the open set $\Omega_0 = \{x \in \Omega \colon w > M\}$ is non-empty. It clearly follows that $\Omega_0 \subset\subset \Omega_\rho$, since

$$|\nabla u|^2 \geqslant \|u\|_\infty^2 > \rho \quad \text{in } \Omega_0.$$

Hence, $\mathcal{L}w \leqslant 0$ in Ω_0, where $\mathcal{L}w := -\Delta w - 2g'(|\nabla u|)\nabla u \cdot \nabla w$, and the strong maximum principle implies that $w < \sup_{\partial\Omega_0} w = M$ in Ω_0, which is a contradiction. Hence, $w \leqslant M$ in Ω. □

We now come to the proofs of lemmas 4.1 and 4.2.

Proof of lemma 4.1. For the moment, fix $y_0 \in \partial\Omega$ and define $\bar{v}(x) = \bar{u}(|x - y_0|)$. Take $K > 0$ and let $g_K \in C^1(\mathbb{R})$ be a bounded increasing function verifying that $g_K(t) = g(t)$ if $0 \leqslant t \leqslant K$.

We consider the truncated problem

$$\left.\begin{aligned} -\Delta u + \alpha u = g_K(|\nabla u|) + \lambda h(x) \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \partial\Omega. \end{aligned}\right\} \tag{4.3}$$

When $K > \sup|\bar{u}'|$, the function $\bar{v}$ is a supersolution to (4.3), and since $\underline{v} = 0$ is a subsolution, it follows that there exists a solution u to (4.3) (by using the results in [5, 27]), which verifies that $0 \leqslant u \leqslant \bar{v}$.

By the maximum principle, we have that $u > 0$ in Ω and that $\partial u/\partial\nu < 0$ on $\partial\Omega$. Moreover, since $\bar{v}(x_0) = 0$,

$$\frac{\partial u}{\partial \nu}(x_0) \geqslant \frac{\partial \bar{v}}{\partial \nu}(x_0) = -\bar{u}'(R_1).$$

We see that the same inequality holds for every $x_0' \in \partial\Omega$. Indeed, if we take such a x_0' and A' is the corresponding annulus, then, since $-\Delta u + \alpha u \leqslant g_K(|\nabla u|) + \lambda|h|_\infty$ in Ω, the function $\bar{u}(|x - y_0'|)$ is a supersolution to (4.3), considered in A'. By comparison, we obtain that $u(x) \leqslant \bar{u}(|x - y_0'|)$ in Ω. Thus,

$$\frac{\partial u}{\partial \nu}(x_0') \geqslant -\bar{u}'(R_1).$$

Hence,

$$-\bar{u}'(R_1) \leqslant \frac{\partial u}{\partial \nu} \leqslant 0 \quad \text{on } \partial\Omega.$$

We are in a position to apply lemma 4.3 to obtain a constant $M > 0$, which does not depend on K, such that $|\nabla u| < M$ in $\bar{\Omega}$. Taking $K > M$, we have that $g_K(|\nabla u|) = g(|\nabla u|)$ in Ω, and u is a solution to our original problem. This concludes the proof. □

Proof of lemma 4.2. Assume that (1.1) has a positive solution u for some $\lambda \geqslant c/h_0$. Then, u is a supersolution to the problem

$$\begin{aligned} -\Delta v + \alpha v &= g(|\nabla v|) + \lambda h_0 \quad \text{in } B_R(y_0), \\ v &= 0 \quad\quad\quad\quad\quad\quad\quad \text{on } \partial B_R(y_0). \end{aligned}$$

A similar procedure to that in the proof of lemma 4.1 yields the existence of a solution to this problem, which is unique, hence radial. This is in contradiction to the hypothesis. It is important to remark that this procedure works, since the supersolution vanishes at $x_0 \in \partial B_R(y_0) \cap \partial\Omega$, which allows us to estimate v' on $\partial B_R(y_0)$ in terms of $|\nabla u(x_0)|$. □

Finally, we proceed to prove our main theorems. We note that, once we have separately analysed the cases $\alpha = 0$ and $\alpha > 0$ in §§ 2 and 3, the rest of the proof is exactly the same in both cases.

Proof of theorems 1.1 and 1.3. (i) By lemmas 2.1 and 2.2, there exists a supersolution to (4.1) for every $c > 0$. The existence of a positive solution to (1.1) for every $\lambda > 0$ follows due to lemma 4.1.

(ii) Again by lemmas 2.1, 2.2 and 4.1, there exists a solution for small values of λ. On the other hand, using lemma 4.2 in conjunction with lemmas 3.1 and 3.2 we also have that no solutions to (1.1) exist for large values of λ. Hence, we can define

$$\Lambda = \sup\{\lambda > 0 \colon \text{there exists a solution to (1.1)}\},$$

and Λ is finite and positive. By its very definition, there exist no solutions to (1.1) for $\lambda > \Lambda$. Now choose an arbitrary $\lambda \in (0, \Lambda)$. There then exists $\mu \in (\lambda, \Lambda)$ such that (1.1) with λ substituted by μ admits a positive solution v. Since this solution is a supersolution to (1.1), the existence of a positive solution follows as in lemma 4.1. □

Acknowledgements

S.A. was partly supported by Universidad Técnica Federico Santa María (Grant no. 121002) and Fondecyt (Grant no. 11110482). J.G.-M. was partly supported by the Ministerio de Ciencia e Innovación (Grant no. MTM2011-27998) (Spain). A.Q. was partly supported by Fondecyt (Grant no. 1110210) and CAPDE (Anillo ACT-125). All three authors were partly supported by the Programa Basal CMM, Universidad de Chile.

References

1 H. Abdel Hamid and M. F. Bidaut-Veron. Correlation between two quasilinear elliptic problems with a source term involving the function or its gradient. *C. R. Acad. Sci. Paris Sér. I* **346** (2008), 1251–1256.

2 B. Abdellaoui, A. Dall'Aglio and I. Peral. Some remarks on elliptic problems with critical growth in the gradient. *J. Diff. Eqns* **222** (2006), 21–62.

3 N. Alaa and M. Pierre. Weak solutions of some quasilinear elliptic equations with data measures. *SIAM J. Math. Analysis* **24** (1993), 23–35.

4 A. Alvino, G. Trombetti and P.-L. Lions. Comparison results for elliptic and parabolic equations via Schwarz symmetrization. *Annales Inst. H. Poincaré Analyse Non Linéaire* **7** (1990), 37–65.

5 H. Amann and M. G. Crandall. On some existence theorems for semi-linear elliptic equations. *Indiana Univ. Math. J.* **27** (1978), 779–790.

6 G. Barles. A short proof of the $C^{0,\alpha}$-regularity of viscosity subsolutions for superquadratic viscous Hamilton–Jacobi equations and applications. *Nonlin. Analysis* **73** (2010), 31–47.

7 G. Barles and F. Murat. Uniqueness and the maximum principle for quasilinear elliptic equations with quadratic growth conditions. *Arch. Ration. Mech. Analysis* **133** (1995), 77–101.

8 G. Barles and A. Porretta. Uniqueness for unbounded solutions to stationary viscous Hamilton–Jacobi equations. *Annali Scuola Norm. Sup. Pisa IV* **5** (2006), 107–136.

9 G. Barles, A. P. Blanc, C. Georgelin and M. Kobylanski. Remarks on the maximum principle for nonlinear elliptic PDE with quadratic growth conditions. *Annali Scuola Norm. Sup. Pisa* **28** (1999), 381–404.

10 L. Boccardo, F. Murat and J. P. Puel. Resultats d'existence pour certains problemes elliptiques quasilineaires. *Annali Scuola Norm. Sup. Pisa* **11** (1984), 213–235.

11 L. Boccardo, F. Murat and J. P. Puel. L^∞ estimate for some nonlinear elliptic partial differential equations and application to an existence result. *SIAM J. Math. Analysis* **23** (1992), 326–333.

12 L. Boccardo, T. Gallouët and F. Murat. A unified presentation of two existence results for problems with natural growth. In *Progress in partial differential equations: the Metz surveys*, vol. 2, pp. 127–137, Chemical Rubber Company Research Notes in Mathematics Series, vol. 296 (London: Taylor and Francis, 1993).

13 A. Dall'Aglio, D. Giachetti and J. P. Puel. Nonlinear elliptic equations with natural growth in general domains. *Annali Mat. Pura Appl.* **181** (2002), 407–426.

14 T. Del Vecchio and M. M. Porzio. Existence results for a class of non-coercive Dirichlet problems. *Ric. Mat.* **44** (1995), 421–438.

15 P. Felmer and A. Quaas. On the strong maximum principle for quasilinear elliptic equations and systems. *Adv. Diff. Eqns* **7** (2002), 25–46.

16 V. Ferone and F. Murat. Quasilinear problems having quadratic growth in the gradient: an existence result when the source term is small. In *Equations aux derivees partielles et applications*, pp. 497–515 (Elsevier, 1998).

17 V. Ferone and F. Murat. Nonlinear problems having quadratic growth in the gradient: an existence result when the source term is small. *Nonlin. Analysis TMA* **42** (2000), 1309–1326.

18 D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order* (Springer, 1983).

19 N. Grenon, F. Murat and A. Porretta. Existence and a priori estimate for elliptic problems with subquadratic gradient dependent terms. *C. R. Math.* **342** (2006), 23–28.

20 K. Hansson, V. Maz′ja and I. E. Verbitsky. Criteria of solvability for multidimensional Riccati equations. *Ark. Mat.* **37** (1999), 87–120.

21 L. Jeanjean and B. Sirakov. Existence and multiplicity for elliptic problems with quadratic growth in the gradient. *Commun. PDEs* **38** (2012), 244–264.

22 J. M. Lasry and P. L. Lions. Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints. I. The model problem. *Math. Ann.* **283** (1989), 583–630.

23 P. L. Lions. Résolution des problèmes elliptiques quasilinéaires. *Arch. Ration. Mech. Analysis* **74** (1980), 336–353.

24 P. L. Lions. Quelques remarques sur les problemes elliptiques quasilineaires du second ordre. *J. Analysis Math.* **45** (1985), 234–254.

25 A. Porretta. The 'ergodic limit' for a viscous Hamilton–Jacobi equation with Dirichlet conditions. *Rend. Lincei Mat. Appl.* **21** (2010), 59–78.

26 P. Pucci, J. Serrin and H. Zou. A strong maximum principle and a compact support principle for singular elliptic equations. *J. Math. Pures Appl.* **78** (1999), 769–789.

27 J. Schoenenberger-Deuel and P. Hess. A criterion for the existence of solutions of non-linear elliptic boundary value problems. *Proc. R. Soc. Edinb.* A **74** (1976), 49–54.

28 J. Serrin. The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables. *Phil. Trans. R. Soc. Lond.* A **264** (1969), 413–496.

29 B. Sirakov. Solvability of uniformly elliptic fully nonlinear PDE. *Arch. Ration. Mech. Analysis* **195** (2010), 579–607.

(*Issued 4 April 2014*)