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Expanding the asymptotic explosive boundary behavior of large solutions to a semilinear elliptic equation

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ABSTRACT

The main goal of this paper is to study the asymptotic expansion near the boundary of the large solutions of the equation

$$-\Delta u+\lambda u^m=f \quad in\ \Omega,$$

where $\lambda>0, m>1, f\in\mathcal{C}(\Omega), f\geq 0$, and Ω is an open bounded set of $\mathbb{R}^N$, $N>1$, with boundary smooth enough. Roughly speaking, we show that the number of explosive terms in the asymptotic boundary expansion of the solution is finite, but it goes to infinity as m goes to 1. We prove that the expansion consists in two eventual geometrical and non-geometrical parts separated by a term independent on the geometry of $\partial\Omega$, but dependent on the diffusion. For low explosive sources the non-geometrical part does not exist; all coefficients depend on the diffusion and the geometry of the domain by means of well-known properties of the distance function $\operatorname{dist}(x,\partial\Omega)$. For high explosive sources the preliminary coefficients, relative to the non-geometrical part, are independent on Ω and the diffusion. Finally, the geometrical part does not exist for very high explosive sources.

1. Introduction

In this paper we are interested in the solutions of the equation

$$-\Delta u+g(u)=f \quad \text{in } \Omega, \tag{1}$$

with an *explosive* behavior on the boundary

$$u(x)\to\infty \quad \text{as } x\to\partial\Omega. \tag{2}$$

In general, the solutions of (1) and (2) are called *large solutions* if a Comparison Principle holds. This is because the inequality

$$u(x)\geq v(x),\quad x\in\overline{\Omega},$$

is satisfied for any other solution v of (1) with bounded boundary values.

Singular boundary value problems as (1)–(2) have been extensively studied in the literature starting with the results of L. Bieberbach and H. Rademacher for precise choices of the function g (see for instance [1–4]). From our point of view, the pioneer works in the topic are due to Keller [5] and Osserman [6] on 1957 who proved the existence of large solutions of (1)

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provided that $f \equiv 0$, g is a nondecreasing function and Ω is a bounded open set of $\mathbb{R}^N$, $N > 1$. They also establish necessary and sufficient conditions to guarantee that the large solutions exist under the so called *Keller–Osserman* condition

$$\int^{\infty} \frac{ds}{\sqrt{\int_0^s g(\tau)d\tau}} < +\infty. \tag{3}$$

From that time forward an extensive literature has been produced (see again [1–4,7] and the references therein). In sight of results in [3] or [7] about the existence and uniqueness of the classical large solutions of (1), we focus our attention on their asymptotic behavior on the boundary $\partial\Omega$.

As it is usual in studying properties near the boundary, the distance function $\text{dist}(x, \partial\Omega)$, here denoted by $d(x)$, plays an important role. As it is well known, if the boundary is bounded with $\partial\Omega \in \mathcal{C}^k$, $k \geq 1$, one proves $d(\cdot) \in \mathcal{C}^k$ in the parallel strip near the boundary

$$\Omega_{\delta_0} = \{x \in \Omega : 0 \leq d(x) < \delta_0\}. \tag{4}$$

Obviously, the positive constant δ_0 only depends on $\partial\Omega$ (see [2] or [8]). In particular, as it was proved in [3] if $\partial\Omega \in \mathcal{C}^2$ then the first term of the boundary explosive expansion is uniform and independent on Ω for the large solution of

$$-\text{div}(|\nabla u|^{p-2}\nabla u) + \lambda u^m = f \quad \text{in } \Omega \ (1 < p < \infty)$$

provided the condition $m > p - 1$ which is the extended version of (3). Other sharp properties on the uniform first term of the expansion of the large solution of (1), for $f \equiv 0$, have been obtained by C. Bandle, G. Díaz, J. García Melián, A. Greco, A. Lazer, S. Kim, N. Kondrat'ev, R. Letelier, J. López-Gómez, M. Marcus, J. Matero, P. McKenna, V. Nikishkin, M. del Pino, G. Porru, J. Sabina and L. Véron among many other authors. We remit to [1] and [2] for some illustrations.

Certainly the geometric properties of the domain can appear in the asymptotic expansion near the boundary. Indeed this influence occurs in secondary terms under more regularity assumptions on the boundary. It is obtained by considering terms containing $\Delta d(x)$ neglected in the leading coefficient of the expansion. We note the important property

$$\Delta d(x) = -(N-1)\mathbf{H}(x),$$

where $\mathbf{H}(x)$ denotes the mean curvature of $\partial\{y \in \Omega : d(y) < d(x)\}$ at x (see again [2] or [8]). The simplest geometry is derived on balls, as $\Omega = \mathbf{B}_R(0)$, for which

$$\Delta d(x) = -\frac{N-1}{|x|}, \quad |x| < R.$$

The first contribution on this geometrical influence is due to M. del Pino and R. Letelier who proved in [9] that the large solution of (1), for $g(r) = r^m$, $1 < m < 3$, $\partial\Omega \in \mathcal{C}^4$, $N > 1$ and $f \equiv 0$, admits the expansion

$$u(x) = \left(\frac{2(m+1)}{\lambda(m-1)^2}\right)^{\frac{1}{m-1}} \big(d(x)\big)^{-\frac{2}{m-1}} \left(1 - \left(\frac{(N-1)\mathbf{H}(x_0)}{m+3} + o(1)\right) d(x)\right), \tag{5}$$

where $\mathbf{H}(x_0)$ is the mean curvature of the boundary at the point $x_0 \in \partial\Omega$, given by $d(x) = |x - x_0|$, and $o(1) \to 0$ as $d(x) \to 0$. More recently, C. Bandle and M. Marcus have extended the results of [9] by obtaining the dependence on the mean curvature of $\partial\Omega$ in the second order term of the asymptotic behavior of the large solution of (1), again if $f \equiv 0$ (see [2]).

As it was pointed out in the Abstract, the main goal of this paper is to study the whole asymptotic explosive expansion near the boundary of the large solution of (1), here viewed as the source equation

$$-\Delta u + \lambda u^m = f \quad \text{in } \Omega \ (m > 1, f \geq 0). \tag{6}$$

As in [3], we will use a simple scheme characterized by means of the behavior

$$f(x) \approx f_0\big(d(x)\big)^{-q_\tau} \quad \text{as } d(x) \to 0$$

with

$$\alpha_\tau = \frac{2+\tau}{m-1} \quad \text{and} \quad q_\tau = m\alpha_\tau, \quad (\tau \text{ is a non-negative integer}),$$

for which the *low explosive sources* are given by $\tau = 0$ and $f_0 \geq 0$ and the *high explosive sources* by $\tau > 0$ and $f_0 > 0$. We note that large solutions for low explosive sources have been considered in the literature, mainly for null sources $f \equiv 0$ (see the above references). On the other hand, to the best of our knowledge only in [3, Theorem 3.8] large solutions for high explosive sources have been studied.

So that, our main contribution is sketched as follows (see Theorem 1). Let us assume $\partial\Omega$ smooth enough and $f \in \mathcal{C}(\Omega)$, $f \geq 0$, verifying

$$f(x) = \big(d(x)\big)^{-q_\tau}\left(f_0 + \sum_{n=1}^{M_\tau} f_n\big(d(x)\big)^n\right), \quad x \in \Omega_{\delta_0},$$

where f_n, $0 \le n \le \mathrm{M}_\tau$, are known constants, with $f_0 \ge 0$, and M_τ to be defined later (see (8)). Then we prove that the large solution of (6) admits an explosive expansion given by

$$u(x) = \mathrm{C}_0(\mathrm{d}(x))^{-\alpha_\tau}\Bigg(1 + \overbrace{\sum_{n=1}^{\min\{\tau,\mathrm{M}_\tau\}-1} \mathrm{C}_n(\mathrm{d}(x))^n}^{\text{the non-geometrical and non-diffused part}} + \overbrace{\mathrm{C}_{\min\{\tau,\mathrm{M}_\tau\}}(\mathrm{d}(x))^{\min\{\tau,\mathrm{M}_\tau\}}}^{\text{it does not appear if } \min\{\tau,\mathrm{M}_\tau\}=0}$$
$$+ \underbrace{\sum_{n=\min\{\tau,\mathrm{M}_\tau\}+1}^{\mathrm{M}_\tau} \mathrm{C}_n(x)(\mathrm{d}(x))^n}_{\text{the geometrical part}}\Bigg) + \mathrm{o}\left(\left(\mathrm{d}(x)\right)^{-\alpha_\tau+\mathrm{M}_\tau}\right),$$

where $\mathrm{M}_\tau + 1$ is the number of all explosive terms. As it will be proved later, if $3 + \tau \le m$ the expansion is very simple, it consists of a unique explosive term (see Remarks 1 and 6). Furthermore, one has

$$\lim_{m\to 1} \mathrm{M}_\tau = \infty$$

(see (9)). We prove that the main explosive rate C_0 is a precise positive constant independent on Ω, even independent on the diffusion whenever $\tau > 0$. Moreover, C_n, $1 \le n \le \min\{\tau, \mathrm{M}_\tau\} - 1$ are precise constants independent on Ω and the diffusion and $\mathrm{C}_{\min\{\tau,\mathrm{M}_\tau\}}$ is a constant independent on Ω but dependent on the diffusion. The other explosive coefficients $\mathrm{C}_n(x)$, $\min\{\tau, \mathrm{M}_\tau\} + 1 \le n \le \mathrm{M}_\tau$, are explicit functions depending on the geometry of Ω and the diffusion. Equality $\min\{\tau, \mathrm{M}_\tau\} = 0$ corresponds with the *low explosive* source case for which only the first term is uniform and independent on Ω; otherwise one has the *high explosive* source case. Certainly, if $\min\{\tau, \mathrm{M}_\tau\} = \mathrm{M}_\tau$ the sources can be called *very high explosive* because all $\mathrm{M}_\tau + 1$ explosive coefficients in the expansion are uniform and independent on the geometry and the diffusion.

For the simple case $\Omega = \mathbf{B}_{\mathrm{R}}(0)$ the geometrical part is uniform on $\partial\Omega$, consequently the expansion is uniform on $\partial\Omega$. In general, we may illustrate the results by noting that for two boundary points $x_0, y_0 \in \partial\Omega$ if

$$\left|\mathrm{C}_n\left(x_0 - s\overrightarrow{\mathbf{n}}_{x_0}\right) - \mathrm{C}_n\left(y_0 - s\overrightarrow{\mathbf{n}}_{y_0}\right)\right| \to 0 \quad \text{as } s \to 0$$

is satisfied for $\min\{\tau, \mathrm{M}_\tau\} + 1 \le n \le \mathrm{M}_\tau$, then we deduce

$$\left|u\left(x_0 - s\overrightarrow{\mathbf{n}}_{x_0}\right) - u\left(y_0 - s\overrightarrow{\mathbf{n}}_{y_0}\right)\right| \to 0 \quad \text{as } s \to 0;$$

otherwise

$$\left|u\left(x_0 - s\overrightarrow{\mathbf{n}}_{x_0}\right) - u\left(y_0 - s\overrightarrow{\mathbf{n}}_{y_0}\right)\right| \to \infty \quad \text{as } s \to 0,$$

here $\overrightarrow{\mathbf{n}}_{x_0}$ and $\overrightarrow{\mathbf{n}}_{y_0}$ denote the relative unit outward vector.

The paper is organized as follows. The influence of the geometric properties of the domain requires several awful straightforward computations in constructing a formal boundary explosive expansion. It is studied in Section 2. In Section 3 we apply the formal expansions to obtain the boundary explosive expansion of the large solution of (6). Examples 1 and 2 can illustrate the contribution. The paper ends with some technicalities. So, in Appendix A we expand the power of polynomials by means of an explicit expression which extends the old formula of Federico Villarreal (1850–1923). It is applied in Appendix B where we obtain representations of the power of auxiliar sub and supersolutions used in the paper.

We finish this Introduction by noting that the partial differential equation (6) appears in several contexts: equilibrium of a charged gas in a container, invariance under conformal or projective transformations (see [3] and the references therein). We also note that for the particular case $m = 2$, problem (6)–(2) is of interest in the study of the subsonic motion of a gas (see [10]) and when $1 < m \le 2$ it is related to a problem involving superdiffusion (see [11], [12]). Also the singular value boundary problem (6)–(2) can be viewed as the Dynamic Programming approach of a Stochastic Optimal Control problem (state constraints). Here, at least in a heuristic way, the nonlinear term $\left(u(x)\right)^{m-1}$ denotes a kind of optimal feedback control.

2. Constructing the boundary explosive expansion of the large solutions

As in Theorem 3.8 of [3] we study the boundary behavior by two different ways to proving that $\left(\mathrm{d}(x)\right)^{-\alpha}$ satisfies

$$-\Delta u + \lambda u^m = f \quad \text{near } \partial\Omega.$$

The first one is based on the scheme

$$\overbrace{\left(\mathrm{d}(x)\right)^{-\alpha_0-2}}^{\Delta u} \approx \overbrace{\left(\mathrm{d}(x)\right)^{-m\alpha_0}}^{\lambda u^m} - \overbrace{\left(\mathrm{d}(x)\right)^{-q}}^{f} \quad \text{near } \partial\Omega \quad \Rightarrow \quad q \le \alpha_0 m$$

for which $\alpha_0 + 2 = \alpha_0 m \Leftrightarrow \alpha_0 = \frac{2}{m-1}$ is the explosive exponent. The second scheme is

$$\overbrace{\big(\mathrm{d}(x)\big)^{-\alpha-2}}^{\Delta u} \ll \overbrace{\big(\mathrm{d}(x)\big)^{-m\alpha}}^{\lambda u^m} \approx \overbrace{\big(\mathrm{d}(x)\big)^{-q}}^{f} \quad \text{near } \partial\Omega \quad \Rightarrow \quad q > \alpha_0 m.$$

Now $\alpha m = q \Leftrightarrow \alpha = \frac{q}{m} > \alpha_0$ is the explosive exponent. Both cases can be represented by

$$\alpha_\tau = \frac{2+\tau}{m-1} \quad \text{and} \quad q_\tau = m\alpha_\tau, \tag{7}$$

where τ is a non-negative integer number.

Therefore, the main boundary behavior can be written as

$$\mathrm{C}_0\big(\mathrm{d}(x)\big)^{-\alpha_\tau} + o\Big(\big(\mathrm{d}(x)\big)^{-\alpha_\tau}\Big) \quad \text{as } \mathrm{d}(x) \to 0.$$

Next we expand this behavior by means of formal expansions near the boundary

$$\mathrm{C}_0\big(\mathrm{d}(x)\big)^{-\alpha_\tau}\left(1 + \sum_{n\geq 1} \mathrm{C}_n(x)\big(\mathrm{d}(x)\big)^n\right).$$

Here C_0 is a positive constant and $\mathrm{C}_n(x)$, $n \geq 1$, are real functions. Certainly we are interested in to obtain the explosive terms, thus, governed by $n < \alpha_\tau$. So the maximum numbers of explosive terms $\mathrm{M}_\tau + 1$ is given by $\alpha_\tau - 1 \leq \mathrm{M}_\tau < \alpha_\tau$, whence

$$\mathrm{M}_\tau = \begin{cases} \alpha_\tau - 1, & \text{if } \alpha_\tau \text{ is an integer number,} \\ [\alpha_\tau], & \text{otherwise,} \end{cases} \tag{8}$$

where $[\alpha_\tau]$ denotes the integer part of α_τ.

Remark 1. Consequently, a maximum number of explosive terms $\mathrm{M}_\tau + 1$ is available if

$$\frac{2+\tau}{m-1} - 1 \leq \mathrm{M}_\tau < \frac{2+\tau}{m-1}$$

whence

$$m \in \mathrm{I}_{\mathrm{M}_\tau} \doteq \left[\frac{\mathrm{M}_\tau + 3 + \tau}{\mathrm{M}_\tau + 1}, \frac{\mathrm{M}_\tau + 2 + \tau}{\mathrm{M}_\tau}\right[\quad \Leftrightarrow \quad \alpha_\tau \in \left]\mathrm{M}_\tau, \mathrm{M}_\tau + 1\right]. \tag{9}$$

Since $\mathrm{I}_0 = [3+\tau, \infty[$, one proves

$$]1, \infty[= \bigcup_{\mathrm{M}_\tau \geq 0} \mathrm{I}_{\mathrm{M}_\tau}.$$

For the purpose of the paper we focus our attention in the case $\mathrm{M}_\tau \geq 1$ or, equivalently, $1 < m < 3 + \tau$. □

We will assume that $\Omega \subset \mathbb{R}^{\mathrm{N}}$, $\mathrm{N} > 1$, is a bounded open set with $\partial\Omega$ smooth enough. Then, we consider the functions

$$\mathrm{V}_\delta^\pm(x) = \mathrm{C}_0 \sum_{n=0}^{\mathrm{M}_\tau} \mathrm{V}_{\delta,n}^\pm(x)$$

with

$$\mathrm{V}_{\delta,0}^\pm(x) = \big(\mathrm{d}(x) \mp \delta\big)^{-\alpha_\tau} \quad \text{and} \quad \mathrm{V}_{\delta,n}^\pm(x) = \mathrm{C}_n(x)\big(\mathrm{d}(x) \mp \delta\big)^{-\alpha_\tau + n}, \quad 1 \leq n \leq \mathrm{M}_\tau,$$

defined for $x \in \Omega$ such that $\mathrm{d}(x) \mp \delta > 0$ and $\delta > 0$ small enough. Straightforward computations yield

$$\begin{aligned} \Delta\mathrm{V}_{\delta,0}^\pm(x) &= \alpha_\tau(\alpha_\tau + 1)\big(\mathrm{d}(x) \mp \delta\big)^{-\alpha_\tau - 2}|\nabla \mathrm{d}(x)|^2 - \alpha_\tau \Delta\mathrm{d}(x)\big(\mathrm{d}(x) \mp \delta\big)^{-\alpha_\tau - 1} \\ \Delta\mathrm{V}_{\delta,n}^\pm(x) &= (-\alpha_\tau + n)(-\alpha_\tau + n - 1)\mathrm{C}_n(x)\big(\mathrm{d}(x) \mp \delta\big)^{-\alpha_\tau + (n-2)} \\ &\quad + (-\alpha_\tau + n)\big[2\langle \nabla \mathrm{C}_n(x), \nabla \mathrm{d}(x)\rangle + \mathrm{C}_n(x)\Delta\mathrm{d}(x)\big]\big(\mathrm{d}(x) \mp \delta\big)^{-\alpha_\tau + (n-1)} \\ &\quad + \Delta\mathrm{C}_n(x)\big(\mathrm{d}(x) \mp \delta\big)^{-\alpha_\tau + n}, \quad 1 \leq n \leq \mathrm{M}_\tau. \end{aligned}$$

So that we derive

$$\Delta\mathrm{V}_\delta^\pm(x) = \mathrm{C}_0\big(\mathrm{d}(x) \mp \delta\big)^{-\alpha_\tau - 2}\left(\mathrm{A}_0|\nabla \mathrm{d}(x)|^2 + \sum_{n=1}^{\mathrm{M}_\tau + 2} \mathrm{A}_n(x)\big(\mathrm{d}(x) \mp \delta\big)^n\right),$$

with

$$\begin{cases} A_0 = \alpha_\tau(\alpha_\tau+1), \\ A_1(x) = \alpha_\tau(\alpha_\tau-1)C_1(x) - \alpha_\tau \Delta d(x), \\ A_2(x) = (-\alpha_\tau+2)(-\alpha_\tau+1)C_2(x) + (-\alpha_\tau+1)\big[2\langle\nabla C_1(x), \nabla d(x)\rangle + C_1(x)\Delta d(x)\big], \\ A_n(x) = (-\alpha_\tau+n)(-\alpha_\tau+n-1)C_n(x) \\ \quad + (-\alpha_\tau+n-1)\big[2\langle\nabla C_{n-1}(x), \nabla d(x)\rangle + C_{n-1}(x)\Delta d(x)\big] + \Delta C_{n-2}(x), \quad 3 \le n \le M_\tau, \\ A_{M_\tau+1}(x) = (-\alpha_\tau+M_\tau)\big[2\langle\nabla C_{M_\tau}(x), \nabla d(x)\rangle + C_{M_\tau}(x)\Delta d(x)\big] + \Delta C_{M_\tau-1}(x), \\ A_{M_\tau+2}(x) = \Delta C_{M_\tau}(x). \end{cases} \tag{10}$$

Remark 2. We note that all functions $A_n(x)$, $1 \le n \le M_\tau + 2$, depend on the geometry of Ω through the distance function $d(x)$. More precisely, $A_1(x)$ depends only on the mean curvature. On the other hand, since $|\nabla d(x)| = 1, x \in \Omega_{\delta_0}$ (see (4) and [8]), in these parallel strip near the boundary one has

$$\Delta V_\delta^{\pm}(x) = C_0\big(d(x) \mp \delta\big)^{-\alpha_\tau-2}\left(A_0 + \sum_{n=1}^{M_\tau+2} A_n(x)\big(d(x)\mp\delta\big)^n\right). \quad \square \tag{11}$$

In order to construct the semilinear differential operator on $V_\delta^{\pm}$, we need a representation as

$$\big(V_\delta^{\pm}(x)\big)^m = C_0^m\big(d(x)\mp\delta\big)^{-\alpha_\tau m}\left(1 + \sum_{n=1}^{M_\tau} D_n(x)\big(d(x)\mp\delta\big)^n + \sum_{n=M_\tau+1}^{\infty} D_n(x)\big(d(x)\mp\delta\big)^n\right) \tag{12}$$

that will be obtained in (43) later. Certainly it requires straightforward and tedious computations that, in order to simplify the exposition, we have collected in Appendix B. So, we prove in (45)

$$D_n(x) = mC_n(x) + \sum_{i=2}^{n}\binom{m}{i} B_{n-i,i}(x), \quad 1 \le n \le M_\tau, \tag{13}$$

where

$$B_{i,n}(x) = \sum_{j=1}^{i}\binom{n}{j}\big(C_1(x)\big)^{n-j} \sum_{\substack{\ell_1\cdot\gamma_{\ell_1}+\ell_2\cdot\gamma_{\ell_2}+\cdots+\ell_j\cdot\gamma_{\ell_j}=i+j \\ \gamma_{\ell_1}+\gamma_{\ell_2}+\cdots+\gamma_{\ell_j}=j \\ 2\le\ell_1<\cdots<\ell_j\le i-j+2 \\ \{\gamma_{\ell_k}\}_{k=1}^{j}\subset\{0,1,\ldots,j\}}} \frac{j!}{\gamma_{\ell_1}!\gamma_{\ell_2}!\cdots\gamma_{\ell_j}!}\big(C_{\ell_1}(x)\big)^{\gamma_{\ell_1}}\cdots\big(C_{\ell_j}(x)\big)^{\gamma_{\ell_j}}$$

for $i = 1, 2, \ldots, n$ (see (42)). Moreover one proves that, in (13), each $C_n(x)$, $1 \le n \le M_\tau$, does not appear in $B_{n-i,i}(x)$, $i \ne 1$. On the other hand, all coefficients $C_n(x)$, $1 \le n \le M_\tau$, are involved in $D_n(x)$, $M_\tau + 1 \le n$.

Remark 3. In order to illustrate we give some examples in Remark 12 (see Appendix B). $\square$

A last comment on the power $\big(V_\delta^{\pm}(x)\big)^m$. From (12) we may write

$$\big(V_\delta^{\pm}(x)\big)^m = C_0^m\big(d(x)\mp\delta\big)^{-\alpha_\tau m}\left(1 + \sum_{n=1}^{M_\tau} D_n(x)\big(d(x)\mp\delta\big)^n + \Psi\big(x; d(x)\mp\delta\big)\right) \tag{14}$$

for the continuous function

$$\Psi(x; r) \doteq \sum_{n=M_\tau+1}^{\infty} D_n(x) r^n.$$

In fact, since Ψ is continuous uniformly on the set

$$\{x \in \Omega : 0 \le 2d(x) \le \delta_0\},$$

we may prove an inequality as

$$\Psi^-(r) \le \Psi(x; r) \le \Psi^+(r) \quad (r \text{ small enough}) \tag{15}$$

for some functions

$$\Psi^-(r) \le 0 \le \Psi^+(r)$$

with

$$\lim_{r\to 0}\Psi^{\pm}(r)=0.$$

Remark 4. In Remark 13 also it is proved that if m is an integer for which $\mathrm{M}_\tau \geq 1$, then we have

$$\Psi\big(x;\mathrm{d}(x)\mp\delta\big)=\sum_{n=\mathrm{M}_\tau+1}^{m\mathrm{M}_\tau}\mathrm{D}_n(x)\big(\mathrm{d}(x)\mp\delta\big)^n$$

□

So that, we assume on the source function $f\in\mathcal{C}(\Omega)$ the explosive expansion near the boundary

$$f(x)=\big(\mathrm{d}(x)\big)^{-q_\tau}\left(f_0+\sum_{n=1}^{\mathrm{M}_\tau}f_n\big(\mathrm{d}(x)\big)^n\right),\quad x\in\Omega_{\delta_0},\tag{16}$$

where f_n, $0\leq n\leq \mathrm{M}_\tau$, are real constants with $f_0\geq 0$. With the above notation, an explosive expansion of the equation near the boundary is

$$-\Delta\mathrm{V}_\delta^{\pm}(x)+\lambda\big(\mathrm{V}_\delta^{\pm}(x)\big)^m-f(x)=-\mathrm{C}_0\big(\mathrm{d}(x)\mp\delta\big)^{-\alpha_\tau-2}\left(\mathrm{A}_0+\sum_{n=1}^{\mathrm{M}_\tau+2}\mathrm{A}_n(x)\big(\mathrm{d}(x)\mp\delta\big)^n\right)$$
$$+\lambda\mathrm{C}_0^m\big(\mathrm{d}(x)\mp\delta\big)^{-\alpha_\tau m}\left(1+\sum_{n=1}^{\mathrm{M}_\tau}\mathrm{D}_n(x)\big(\mathrm{d}(x)\mp\delta\big)^n+\Psi\big(x;\mathrm{d}(x)\mp\delta\big)\right)-\big(\mathrm{d}(x)\big)^{-q_\tau}\left(f_0+\sum_{n=1}^{\mathrm{M}_\tau}f_n\big(\mathrm{d}(x)\big)^n\right)$$

(see (11), (14) and (16)).

3. Proving the boundary asymptotic expansion of the solution

From the schemes of Section 2 we consider the parametrization

$$(\alpha_\tau+2)+\tau=q_\tau=\alpha_\tau m$$

(see (7)), for which

$$-\Delta\mathrm{V}_\delta^{\pm}(x)+\lambda\big(\mathrm{V}_\delta^{\pm}(x)\big)^m-f(x)=\big(\mathrm{d}(x)\mp\delta\big)^{-q_\tau}\Bigg[-\mathrm{C}_0\left(\mathrm{A}_0\big(\mathrm{d}(x)\mp\delta\big)^\tau+\sum_{n=1}^{\max\{\mathrm{M}_\tau-\tau,0\}}\mathrm{A}_n(x)\big(\mathrm{d}(x)\mp\delta\big)^{n+\tau}\right)$$
$$+\big(\lambda\mathrm{C}_0^m-f_0\big)+\sum_{n=1}^{\mathrm{M}_\tau}\big(\lambda\mathrm{C}_0^m\mathrm{D}_n(x)-f_n\big)\big(\mathrm{d}(x)\mp\delta\big)^n+\Phi\big(x;\mathrm{d}(x)\mp\delta\big)\Bigg],\tag{17}$$

for

$$\Phi(x;r)=-\mathrm{C}_0\sum_{\ell=\max\{\mathrm{M}_\tau-\tau,0\}+1}^{\mathrm{M}_\tau+2}\mathrm{A}_\ell(x)r^{\ell+\tau}+\lambda\mathrm{C}_0^m\Psi(x;r).\tag{18}$$

As it was pointed out in the Introduction there are several class of coefficients in the boundary asymptotic expansion of the solutions.

(a) *Coefficients independent on the geometry and the diffusion*. If $\tau>0$ we choose C_0 and $\mathrm{C}_1,\ldots,\mathrm{C}_{\min\{\tau,\mathrm{M}_\tau\}-1}$ from the equalities

$$-\mathrm{C}_0\cdot 0+\lambda\mathrm{C}_0^m\mathrm{D}_n(x)=f_n,\quad 0\leq n\leq\min\{\tau,\mathrm{M}_\tau\}-1.$$

Since $n=0$ implies $\lambda\mathrm{C}_0^m=f_0$, one has

$$\mathrm{C}_n=\frac{1}{mf_0}\left(f_n-f_0\sum_{i=2}^{n}\binom{m}{i}\mathrm{B}_{n-i,i}\right),\quad 0\leq n\leq\min\{\tau,\mathrm{M}_\tau\}-1.\tag{19}$$

From the properties of D_n (see (13)), the coefficients C_n, $1\leq n\leq\min\{\tau,\mathrm{M}_\tau\}-1$, are constants independent on Ω. Obviously, they are independent on the diffusion too. Certainly, we will assume

$$f_0>0\quad\text{whenever }\tau>0.\tag{20}$$

Remark 5. The examples of Remark 12 lead to

$$\begin{cases} C_0 = \left(\frac{f_0}{\lambda}\right)^{\frac{1}{m}}, \\ C_1 = \frac{1}{mf_0} f_1, \\ C_2 = \frac{1}{mf_0}\left(f_2 - \frac{m-1}{2}\frac{1}{mf_0} f_1^2\right), \end{cases}$$

provided $2 \le \min\{\tau, M_\tau\} - 1$. □

(b) *The coefficient* $C_{\min\{\tau,M_\tau\}}$ *independent on the geometry but dependent on the diffusion*. It is obtained by

$$-C_0 A_0 + \lambda C_0^m D_{\min\{\tau,M_\tau\}} = f_{\min\{\tau,M_\tau\}},$$

i.e.

$$C_{\min\{\tau,M_\tau\}} = \frac{1}{m\lambda C_0^m}\left(f_{\min\{\tau,M_\tau\}} + C_0\alpha_\tau(\alpha_\tau + 1) - \lambda C_0^m \sum_{i=2}^{\min\{\tau,M_\tau\}} \binom{m}{i} B_{\min\{\tau,M_\tau\}-i,i}\right). \tag{21}$$

Clearly, here there are two limit cases.

(b.1) If $C_{\min\{\tau,M_\tau\}}$ is the last coefficient of the eventual explosive expansion of the solution, thus if

$$\min\{\tau, M_\tau\} = M_\tau \ge 0 \tag{22}$$

holds, one has

$$-\alpha_\tau(\alpha_\tau + 1)C_0 + \lambda C_0^m D_{M_\tau} = f_{M_\tau}.$$

Therefore, if $\tau > 0$ one has

$$C_{M_\tau} = \frac{1}{mf_0}\left[f_{M_\tau} + \frac{(2+\tau)(m+\tau+1)}{(m-1)^2}\left(\frac{f_0}{\lambda}\right)^{\frac{1}{m}} - f_0 \sum_{i=2}^{M_\tau} \binom{m}{i} B_{M_\tau - i,i}\right]. \tag{23}$$

Then the relative high explosive sources involved, called *very high explosive sources*, induce that all coefficients on the expansion are independent on the geometry. Also they are independent on the diffusion, unless this last coefficient C_{M_τ}.

Remark 6. 1. Remark 1 implies

$$M_\tau = 0 \quad \Leftrightarrow \quad 3 + \tau \le m,$$

for which the expansion has a unique explosive term uniform and independent on Ω.

2. In general, condition (22) implies

$$\begin{cases} (m-2)\tau \ge 3 - m, & \text{if } \alpha_\tau \text{ is an integer number,} \\ (m-2)\tau > 3 - m, & \text{otherwise} \end{cases}$$

(see Remark 1 again). □

(b.2) If $0 = \tau < M_\tau$, the coefficient $C_{\min\{\tau,M_\tau\}} = C_0$ is obtained from

$$-C_0 A_0 + \lambda C_0^m = f_0 \quad \Leftrightarrow \quad \lambda C_0^m - \alpha_0(\alpha_0 + 1)C_0 = f_0. \tag{24}$$

We note that C_0 is independent on the geometry but dependent on the diffusion and it coincides with

$$C_0 = \left(\frac{2(m+1)}{\lambda(m-1)^2}\right)^{\frac{1}{m-1}},$$

whenever $f_0 = 0$. This case corresponds with low explosive sources for which only the geometrical part of the expansion is available.

(c) *Coefficients dependent on the geometry and the diffusion*. We choose $C_{\min\{\tau,M_\tau\}+1}(x), \dots, C_{M_\tau}(x)$ from the equalities

$$-C_0 A_{n-\min\{\tau,M_\tau\}}(x) + \lambda C_0^m D_n(x) = f_n, \quad \min\{\tau, M_\tau\} + 1 \le n \le M_\tau. \tag{25}$$

By means of $A_n(x)$, $\min\{\tau, M_\tau\} + 1 \le n \le M_\tau$, these coefficients depend on the geometry of Ω. In particular, $C_{\min\{\tau,M_\tau\}+1}(x)$ depends only on the mean curvature (see Remark 2).

Certainly, when $\tau > 0$, from the properties of $\mathrm{D}_n(x)$ (see (13)), one has

$$\mathrm{C}_n(x) = \frac{1}{mf_0}\left(f_n + \mathrm{C}_0\mathrm{A}_{n-\min\{\tau,\mathrm{M}_\tau\}}(x) - f_0\sum_{i=2}^{n}\binom{m}{i}\mathrm{B}_{n-i,i}(x)\right), \quad \min\{\tau,\mathrm{M}_\tau\}+1 \le n \le \mathrm{M}_\tau,$$

that is a simple explicit formula. Whenever $\tau = 0$ the condition (25) becomes

$$-\mathrm{C}_0\mathrm{A}_n(x) + \lambda\mathrm{C}_0^m\mathrm{D}_n(x) = f_n, \quad 1 \le n \le \mathrm{M}_0. \tag{26}$$

Then the relative coefficients $\mathrm{C}_n(x)$, $1 \le n \le \mathrm{M}_0$, chosen in (26), also admit an explicit expression as

$$\mathcal{A}_n\mathrm{C}_n(x) = \mathcal{F}\big(m, \lambda, f_0, \ldots, f_n, \mathrm{C}_0, \mathrm{C}_1(x), \ldots, \mathrm{C}_{n-1}(x)\big),$$

where

$$\begin{aligned}\mathcal{A}_n &\doteq \lambda m C_0^m - (-\alpha_0+n)(-\alpha_0+n-1)\mathrm{C}_0 \\ &= \mathrm{C}_0\big((2+n)(\alpha_0+1) + n(\alpha_0-n)\big) + mf_0\end{aligned}$$

is a positive constant due to the definition of C_0 and $-\alpha_0 + n \le -\alpha_0 + \mathrm{M}_0 < 0$.

Remark 7. The obtainment of functions $\mathrm{C}_n(x)$ requires tedious computations. For example, for $\tau > 0$ one obtains

$$\begin{aligned}\mathrm{C}_{\min\{\tau,\mathrm{M}_\tau\}+1}(x) = \frac{1}{mf_0}\Bigg(&f_{\min\{\tau,\mathrm{M}_\tau\}+1} + \left(\frac{f_0}{\lambda}\right)^{\frac{1}{m}}\big[\alpha_\tau(\alpha_\tau-1)\mathrm{C}_1(x) - \alpha_\tau\Delta\mathrm{d}(x)\big] \\ &- f_0\sum_{i=2}^{\min\{\tau,\mathrm{M}_\tau\}+1}\binom{m}{i}\mathrm{B}_{\min\{\tau,\mathrm{M}_\tau\}+1-i,i}(x)\Bigg).\end{aligned}$$

When $\tau = f_0 = 0$ the computations are easier. So, one proves

$$\begin{cases}\mathrm{C}_0 = \left(\dfrac{2(m+1)}{\lambda(m-1)^2}\right)^{\frac{1}{m-1}}, \\ \mathrm{C}_1(x) = \dfrac{1}{m+3}\Big[\gamma(m)f_1 - \Delta\mathrm{d}(x)\Big], \\ \mathrm{C}_2(x) = \dfrac{m-1}{12\mathrm{C}_0}f_2 - \dfrac{1}{12(m+3)}\Big[(m-3)\left\{2\langle\nabla(\Delta\mathrm{d}(x)), \nabla\mathrm{d}(x)\rangle - \gamma(m)f_1\Delta\mathrm{d}(x) + (\Delta\mathrm{d}(x))^2\right\} \\ \quad + \dfrac{m(m+1)}{m+3}\big(\gamma(m)f_1 - \Delta\mathrm{d}(x)\big)^2\Big],\end{cases}$$

for

$$\gamma(m) = \left(\frac{\lambda(m-1)^{m+1}}{2^m(m+1)}\right)^{\frac{1}{m-1}}. \quad \square$$

The above choices lead to

$$-\Delta\mathrm{V}_\delta^\pm(x) + \lambda\big(\mathrm{V}_\delta^\pm(x)\big)^m - f(x) = \big(\mathrm{d}(x) \mp \delta\big)^{-q_\tau}\Phi\big(x; \mathrm{d}(x) \mp \delta\big) \tag{27}$$

(see (17)). Then the relative properties of $\Phi(x; r)$ (see (18)) prove

Proposition 1. *Let us consider $f \in \mathcal{C}(\Omega)$ verifying (16) and (20), as well as $\partial\Omega \in \mathcal{C}^{2(\mathrm{M}_\tau+1)}$. Then the function*

$$\mathrm{V}(x) = \mathrm{C}_0\big(\mathrm{d}(x)\big)^{-\alpha_\tau}\left(1 + \sum_{n=1}^{\mathrm{M}_\tau}\mathrm{C}_n(x)(\mathrm{d}(x))^n\right), \tag{28}$$

where the coefficients C_n, $0 \le n \le \min\{\tau,\mathrm{M}_\tau\} - 1$ are given by (19), $\mathrm{C}_{\min\{\tau,\mathrm{M}_\tau\}}$ is given by (21) and $\mathrm{C}_n \in \mathcal{C}^{2(\mathrm{M}_\tau-n)}(\Omega_{\delta_0})$, $\min\{\tau,\mathrm{M}_\tau\}+1 \le n \le \mathrm{M}_\tau$, are given by (25), is a well defined $\mathcal{C}^2$ function near $\partial\Omega$. Moreover, one has

$$\big(\mathrm{d}(x)\big)^{q_\tau}\left(-\Delta\mathrm{V}(x) + \lambda\big(\mathrm{V}(x)\big)^m - f(x)\right) = \mathrm{O}(\mathrm{d}(x)). \quad \square$$

Clearly, the function V is the candidate to govern the boundary asymptotic behavior of the large solution. In order to prove it, sending $\delta \to 0$ in (27) we may obtain

$$-\Delta \mathrm{V}(x) + \lambda\big(\mathrm{V}(x)\big)^m - f(x) = \big(\mathrm{d}(x)\big)^{-q_\tau}\Big(\mathcal{P}_\tau(\mathrm{C}_0) + \Phi\big(x;\mathrm{d}(x)\big)\Big),$$

where C_0 is the positive root of

$$\mathcal{P}_\tau(\mu) = \begin{cases}\lambda\mu^m - \alpha_0(\alpha_0+1)\mu - f_0, & \text{if } \tau = 0,\\ \lambda\mu^m - f_0, & \text{if } \tau > 0,\end{cases}$$

obtained in (19), if $\tau > 0$, or in (24), whenever $\tau = 0$. So that the main contribution is

Theorem 1. *Under the assumption of Proposition 1, the explosive boundary expansion of the large solution of (6) has the property*

$$u(x) = \mathrm{C}_0\big(\mathrm{d}(x)\big)^{-\alpha_\tau}\Bigg(1 + \overbrace{\sum_{n=1}^{\min\{\tau,\mathrm{M}_\tau\}-1}\mathrm{C}_n\big(\mathrm{d}(x)\big)^n}^{\text{the non-geometrical and non-diffused part}} + \overbrace{\mathrm{C}_{\min\{\tau,\mathrm{M}_\tau\}}\big(\mathrm{d}(x)\big)^{\min\{\tau,\mathrm{M}_\tau\}}}^{\text{it does not appear if } \min\{\tau,\mathrm{M}_\tau\}=0}$$
$$+ \underbrace{\sum_{n=\min\{\tau,\mathrm{M}_\tau\}+1}^{\mathrm{M}_\tau}\mathrm{C}_n(x)(\mathrm{d}(x))^n}_{\text{the geometrical part}}\Bigg) + o\Big(\big(\mathrm{d}(x)\big)^{-\alpha_\tau+\mathrm{M}_\tau}\Big).$$

Proof. In order to apply a comparison argument, we consider the modifications

$$\mathrm{W}_\delta^{\pm\varepsilon}(x) = \mathrm{C}_0\big(\mathrm{d}(x)\mp\delta\big)^{-\alpha_\tau}\left(1 \pm \varepsilon + \sum_{n=1}^{\mathrm{M}_\tau}\mathrm{C}_n(x)\big(\mathrm{d}(x)\mp\delta\big)^n\right),$$

where $\varepsilon > 0$ will be sent to 0. So, we construct the perturbed polynomials

$$\mathcal{P}_\tau^{\pm\varepsilon}(\mu) = \begin{cases}\lambda\big((1\pm\varepsilon)\mu\big)^m - \alpha_0(\alpha_0+1)(1\pm\varepsilon)\mu - f_0, & \text{if } \tau = 0,\\ \lambda\big((1\pm\varepsilon)\mu\big)^m - f_0, & \text{if } \tau > 0,\end{cases}$$

for which

$$\mathcal{P}_\tau^{+\varepsilon}(\mathrm{C}_0) > 0 \quad \text{and} \quad \mathcal{P}_\tau^{-\varepsilon}(\mathrm{C}_0) < 0.$$

The reasoning is based on to prove that $\mathrm{W}_\delta^{\pm\varepsilon}(x)$ are upper and lower solutions in a thin strip near the boundary. Then, arguing as in Proposition 1, we have

$$-\Delta \mathrm{W}_\delta^{+\varepsilon}(x) + \lambda(\mathrm{W}_\delta^{+\varepsilon}(x))^m - f(x) = \big(\mathrm{d}(x)-\delta\big)^{-q_\tau}\Big(\mathcal{P}_\tau^{+\varepsilon}(\mathrm{C}_0) + \Phi\big(x;\mathrm{d}(x)-\delta\big)\Big),$$

thus

$$-\Delta \mathrm{W}_\delta^{+\varepsilon}(x) + \lambda(\mathrm{W}_\delta^{+\varepsilon}(x))^m > f(x)$$

in a parallel strip $\delta < \mathrm{d}(x) < \delta_1$, provided $2\delta_1 < \delta_0$ small enough (see (4), (15) and (18)). So that Comparison Principle leads to

$$u(x) - \mathrm{W}_\delta^{+\varepsilon}(x) \le \sup_{\mathrm{d}(y)=\delta_1}\big(u(y) - \mathrm{W}_\delta^{+\varepsilon}(y)\big), \quad \delta < \mathrm{d}(x) < \delta_1,$$

or

$$\frac{u(x)}{\mathrm{W}_\delta^{+\varepsilon}(x)} - 1 \le \frac{\sup_{\mathrm{d}(y)=\delta_1}\big(u(y) - \mathrm{W}_\delta^{+\varepsilon}(y)\big)}{\mathrm{W}_\delta^{+\varepsilon}(x)}, \quad \delta < \mathrm{d}(x) < \delta_1.$$

Now, sending $\delta_1 \to 0$ and then $\varepsilon \to 0$ we derive

$$\limsup_{\mathrm{d}(x)\to 0}\frac{u(x)}{\mathrm{V}(x)} \le 1,$$

where $\mathrm{V}(x)$ is the expansion function (see (28)). Analogously, one proves

$$-\Delta \mathrm{W}_\delta^{-\varepsilon}(x) + \lambda(\mathrm{W}_\delta^{-\varepsilon}(x))^m - f(x) = \big(\mathrm{d}(x)+\delta\big)^{-q_\tau}\Big(\mathcal{P}_\tau^{-\varepsilon}(\mathrm{C}_0) + \Phi\big(x;\mathrm{d}(x)+\delta\big)\Big),$$

thus

$$-\Delta W_\delta^{-\varepsilon}(x)+\lambda(W_\delta^{-\varepsilon}(x))^m<f(x)$$

in a parallel strip $0<d(x)<\delta_1$, provided $2\delta_1<\delta_0$ whence

$$1-\frac{u(x)}{W_\delta^{-\varepsilon}(x)}\le\frac{\sup_{d(y)=\delta_1}\left(W_\delta^{-\varepsilon}(y)-u(y)\right)}{W_\delta^{-\varepsilon}(x)},\quad 0<d(x)<\delta_1.$$

As above, sending $\delta\to 0$ and then $\varepsilon\to 0$ we conclude

$$\limsup_{d(x)\to 0}\frac{u(x)}{V(x)}\le 1\le\liminf_{d(x)\to 0}\frac{u(x)}{V(x)}.\quad\square$$

Remark 8. Certainly Theorem 1 extends Theorem 3.8 of [3] as well as the results obtained in [1,2] or [9] where only the second explosive term was considered for $f\equiv 0$. $\square$

Theorem 1 can be illustrated as follows

Example 1 (*Low Explosive Sources*). As it was pointed out, the influence of the geometry was obtained in [9] (see also [2]) where one proves that the large solution verifies (5) assumed $\partial\Omega\in\mathcal{C}^4$, $1<m<3$ and $f\equiv 0$. It can be improved by Theorem 1 whenever the values of m are more accurate. For instance, let us suppose $\frac{5}{3}\le m<2$ (or equivalently $2<\alpha_0\le 3$), for which $M_0=2$, and

$$f(x)=\left(d(x)\right)^{-q_0}\left(f_1 d(x)+f_2\left(d(x)\right)^2\right),\quad f_1\ge 0,$$

if $\partial\Omega\in\mathcal{C}^6$, then Remark 7 enables us to obtain

$$\begin{aligned}u(x)=&\left(\frac{2(m+1)}{\lambda(m-1)^2}\right)^{\frac{1}{m-1}}\left(d(x)\right)^{-\frac{2}{m-1}}\Bigg\{1+\frac{1}{m+3}\Big[\gamma(m)f_1-\Delta d(x)\Big]d(x)\\&+\frac{1}{12}\Bigg(\left(\frac{\lambda(m-1)^{m+1}}{2(m+1)}\right)^{\frac{1}{m-1}}f_2-\frac{1}{m+3}\Bigg[(m-3)\big\{2\langle\nabla(\Delta d(x)),\nabla d(x)\rangle\\&-\gamma(m)f_1\Delta d(x)+(\Delta d(x))^2\big\}+\frac{m(m+1)}{m+3}\left(\gamma(m)f_1-\Delta d(x)\right)^2\Bigg]\Bigg)\left(d(x)\right)^2\Bigg\}+o\left(\left(d(x)\right)^{-\frac{2(2-m)}{m-1}}\right),\end{aligned}$$

where $\gamma(m)$ was given in Remark 7. $\square$

Example 2 (*High Explosive Sources*). 1. In order to simplify, we start by constructing an example without geometrical part in the expansion. So, for instance an inequality as $\tau\ge M_\tau=1$ requires

$$\begin{cases}M_\tau=1\Leftrightarrow\dfrac{4+\tau}{2}\le m<3+\tau & \text{(see Remark 1)}\\ \tau\ge M_\tau\Leftrightarrow(m-2)\tau>3-m\Leftrightarrow\dfrac{3+2\tau}{\tau+1}<m & \text{(see Remark 6).}\end{cases}$$

Since $\frac{3+2\tau}{\tau+1}\le\frac{4+\tau}{2}$ for $\tau\ge 1$, both conditions hold when $\frac{4+\tau}{2}<m<3+\tau$, for which

$$f(x)=\left(d(x)\right)^{-q_\tau}\left(f_0+f_1 d(x)\right),\quad f_0>0.$$

Theorem 1 proves that the expansion of all explosive terms of the large solution is

$$u(x)=\left(\frac{f_0}{\lambda}\right)^{\frac{1}{m}}\left(d(x)\right)^{-\alpha_\tau}\left\{1+\frac{1}{mf_0}\left(f_1+\frac{(2+\tau)(m+\tau+1)}{(m-1)^2}\left(\frac{f_0}{\lambda}\right)^{\frac{1}{m}}\right)d(x)\right\}+o\left(\left(d(x)\right)^{-\alpha_\tau+1}\right),$$

provided $\partial\Omega\in\mathcal{C}^4$ (see Remarks 5 and 6 and (23)). Clearly, the first coefficient is independent on the geometry of Ω and the diffusion, however the second one depends on the diffusion. Here τ is an arbitrary positive integer number.

2. Finally, we construct an example where the expansion has two coefficients uniform and independent on Ω plus three coefficients dependent on Ω; it implies $\tau=1$ and $M_1+1=5$. So, Remark 1 enables us to consider $\frac{8}{5}\le m<\frac{7}{4}$ (or equivalently $4<\alpha_1\le 5$) and, for simplicity, we suppose

$$f(x)=f_0\left(d(x)\right)^{-\frac{3m}{m-1}},\quad f_0>0.$$

Then the expansion of all explosive terms of the large solution is

$$u(x)=\mathrm{C}_0\big(\mathrm{d}(x)\big)^{-\frac{3}{m-1}}\left(1+\mathrm{C}_1\mathrm{d}(x)+\mathrm{C}_2(x)\big(\mathrm{d}(x)\big)^2+\mathrm{C}_3(x)\big(\mathrm{d}(x)\big)^3+\mathrm{C}_4(x)\big(\mathrm{d}(x)\big)^4\right)+\mathrm{o}\left(\big(\mathrm{d}(x)\big)^{-\frac{7-4m}{m-1}}\right),$$

for the coefficients

$$\begin{cases}\mathrm{C}_0=\left(\dfrac{f_0}{\lambda}\right)^{\frac{1}{m}}, \quad \text{(independent on the diffusion)}\\ \mathrm{C}_1=\dfrac{\alpha_1(\alpha_1+1)}{mf_0}\mathrm{C}_0, \quad \text{(dependent on the diffusion)}\\ \mathrm{C}_2(x)=\dfrac{\alpha_1\mathrm{C}_0}{mf_0}\Big[(\alpha_1-1)\mathrm{C}_1-\Delta\mathrm{d}(x)\Big]-\dfrac{m-1}{2}\mathrm{C}_1^2,\\ \mathrm{C}_3(x)=\dfrac{(1-\alpha_1)\mathrm{C}_0}{mf_0}\Big[(2-\alpha_1)\mathrm{C}_2(x)+\mathrm{C}_1\Delta\mathrm{d}(x)\Big]-\dfrac{m-1}{6}\mathrm{C}_1\Big[(m-2)\mathrm{C}_1^2+6\mathrm{C}_2(x)\Big],\\ \mathrm{C}_4(x)=\dfrac{(2-\alpha_1)\mathrm{C}_0}{mf_0}\Big[(3-\alpha_1)\mathrm{C}_3(x)+2\langle\nabla\mathrm{C}_2(x),\nabla\mathrm{d}(x)\rangle+\mathrm{C}_2(x)\Delta\mathrm{d}(x)\Big]\\ \quad-\dfrac{m-1}{2}\Big[\dfrac{(m-2)(m-3)}{12}\mathrm{C}_1^4+(m-2)\mathrm{C}_1^2\mathrm{C}_2(x)+2\mathrm{C}_1\mathrm{C}_3(x)+\big(\mathrm{C}_2(x)\big)^2\Big],\end{cases}$$

where $\alpha_1=\frac{3}{m-1}$ and provided $\partial\Omega\in\mathcal{C}^{10}$ (see Remarks 3 and 7). □

We end this Section with a careful glance on the proof of Theorem 1 for which we note that the above boundary behavior holds for the interior and the exterior boundaries of open sets with holes. It enables us to extend the result for more general domains. So that, we derive

Theorem 2. *Let $z\in\partial\Omega$ be a regular boundary point in the sense of an interior and exterior ball condition are satisfied. If $f\in\mathbb{L}^\infty(\mathbb{R}^N)$, $f\geq 0$, then the behavior*

$$\lim_{s\to 0}u(z-s\overrightarrow{\mathbf{n}}_z)s^{\frac{2}{m-1}}=\left(\frac{2(m+1)}{\lambda(m-1)^2}\right)^{\frac{1}{m-1}}$$

holds for the large solution of (6). Here $\overrightarrow{\mathbf{n}}_z$ stands for the unit outward normal vector to $\partial\Omega$ at z.

Proof. Let $\mathbf{B}_{\mathrm{R}_z^+}(x_0^z)\subset\Omega$ such that $\overline{\mathbf{B}}_{\mathrm{R}_z^+}(x_0^z)\cap(\mathbb{R}^N\setminus\Omega)=\{z\}$ and $\overline{u}_\varepsilon$ the radially symmetric large solution of

$$-\Delta\overline{u}_\varepsilon+\lambda\overline{u}_\varepsilon^m=f \quad \text{in } \mathbf{B}_{(1-\varepsilon)\mathrm{R}_z^+}(x_0^z)$$

for $0<\varepsilon\ll 1$. Comparison Principle implies inequality

$$u(x)\leq\overline{u}_\varepsilon(x), \quad x\in\mathbf{B}_{(1-\varepsilon)\mathrm{R}_z^+}(x_0^z).$$

Since all coefficients of the expansion (16) of f near $\partial\mathbf{B}_{(1-\varepsilon)\mathrm{R}_z^+}(x_0^z)$ are all nulls, applying Theorem 1 to $\overline{u}_\varepsilon$ we deduce

$$\limsup_{s\to 0}u(z-s\overrightarrow{\mathbf{n}}_z)s^{\frac{2}{m-1}}\leq\left(\frac{2(m+1)}{\lambda(m-1)^2}\right)^{\frac{1}{m-1}}$$

by sending $\varepsilon\to 0$. On the other hand, let $\mathbf{B}_{\mathrm{R}_z^-}(y_0^z)\subset\mathbb{R}^N\setminus\Omega$ such that $\overline{\mathbf{B}}_{\mathrm{R}_z^-}(y_0^z)\cap\overline{\Omega}=\{z\}$, with R_z^- small enough, and $\underline{u}_\varepsilon$ the radially symmetric solution of

$$\begin{cases}-\Delta\underline{u}_\varepsilon+\lambda\underline{u}_\varepsilon^m=f \quad \text{in } \mathbf{B}_{2\mathrm{R}_z^-}(x_0)\setminus\overline{\mathbf{B}}_{(1+\varepsilon)\mathrm{R}_z^-}(x_0),\\ \underline{u}_\varepsilon(x)\to\infty \quad \text{as } |x-z|\to(1+\varepsilon)\mathrm{R}_z^-,\\ \underline{u}_\varepsilon(x)\to 0 \quad \text{as } |x-z|\to 2\mathrm{R}_z^-.\end{cases}$$

Since function u is non-negative, Comparison Principle implies

$$u(x)\geq\underline{u}_\varepsilon(x), \quad x\in\Omega, \qquad (1+\varepsilon)\mathrm{R}_z^-\leq|x-z|\leq 2\mathrm{R}_z^-.$$

On the other hand the relative coefficients of the expansion (16) of f near $\partial\mathbf{B}_{(1+\varepsilon)\mathrm{R}_z^-}(x_0^z)$ are all nulls too. Now Theorem 1 applied to $\underline{u}_\varepsilon$ leads to

$$\liminf_{s\to 0}u(z-s\overrightarrow{\mathbf{n}}_z)s^{\frac{2}{m-1}}\geq\left(\frac{2(m+1)}{\lambda(m-1)^2}\right)^{\frac{1}{m-1}}$$

by sending $\varepsilon\to 0$. □

Remark 9. For $f \equiv 0$ Theorem 2 was first proved in [7] by using the asymptotic explosive behavior on interior boundaries of annulus and exterior boundaries of balls. □

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Appendix A. Expanding the power of polynomials

In 1879 the mathematician peruvian Federico Villarreal (1850–1923) obtained a simple algorithm in order to expand the power of polynomials (see *La Gaceta Científica*, **2**, Mars 1886, (Perú)). Here we show a short presentation by using the expression

$$(G(x))^n = F(x) \tag{29}$$

where

$$\mathrm{G}(x) = \sum_{j=0}^{q} a_j x^j \quad \text{and} \quad \mathrm{F}(x) = \sum_{j=0}^{qn} b_j x^j$$

and the coefficients $a_j, b_j \in \mathbb{R}$ with $a_0 \neq 0$ and $q, n \in \mathbb{N}$. Since differentiating the expression (29) one obtains

$$\mathrm{F}'(x) = n(\mathrm{G}(x))^{n-1}\mathrm{G}'(x) = n\,\frac{\mathrm{F}(x)}{\mathrm{G}(x)}\,\mathrm{G}'(x),$$

it must verify the equality

$$n\mathrm{F}(x)\mathrm{G}'(x) = \mathrm{F}'(x)\mathrm{G}(x). \tag{30}$$

where

$$\begin{cases} \mathrm{G}'(x) = \displaystyle\sum_{j=1}^{q} j a_j x^{j-1} = \sum_{j=0}^{q-1} (j+1) a_{j+1} x^j \\ \mathrm{F}'(x) = \displaystyle\sum_{j=1}^{qn} j b_j x^{j-1} = \sum_{j=0}^{qn-1} (j+1) b_{j+1} x^j. \end{cases}$$

Our introduction of the Villarreal formula is based on the general equality

$$\begin{aligned}\left(\sum_{j=0}^{\mu} \alpha_j x^j\right)\left(\sum_{j=0}^{\nu} \beta_j x^j\right) &= \sum_{k=0}^{\nu}\left(\sum_{j=0}^{k} \beta_j \alpha_{k-j}\right)x^k + \sum_{k=\nu+1}^{\mu}\left(\sum_{j=0}^{\nu} \beta_j \alpha_{k-j}\right)x^k + \sum_{k=\mu+1}^{\mu+\nu}\left(\sum_{j=k-\mu}^{\nu} \beta_j \alpha_{k-j}\right)x^k \\ &= \sum_{k=0}^{\mu+\nu}\left(\sum_{j=\max\{0,k-\mu\}}^{\min\{k,\nu\}} \beta_j \alpha_{k-j}\right)x^k,\end{aligned}$$

obtained by straightforward computations, provided $\mu, \nu \in \mathbb{N}$ with $\mu \geq \nu$. Next several choices are considered. So

- $\mu = qn$, $\alpha_j = b_j$, $\nu = q-1$, $\beta_j = (j+1)a_{j+1}$ lead

$$\begin{aligned}\mathrm{F}(x)\mathrm{G}'(x) &= \sum_{k=0}^{q-1}\left(\sum_{j=0}^{k} (j+1)a_{j+1}b_{k-j}\right)x^k + \sum_{k=q}^{qn}\left(\sum_{j=0}^{q-1} (j+1)a_{j+1}b_{k-j}\right)x^k + \sum_{k=qn+1}^{qn+q-1}\left(\sum_{j=k-qn}^{q-1} (j+1)a_{j+1}b_{k-j}\right)x^k \\ &= \sum_{k=0}^{q-1}\left(\sum_{j=1}^{k+1} ja_jb_{k-j+1}\right)x^k + \sum_{k=q}^{qn}\left(\sum_{j=1}^{q} ja_jb_{k-j+1}\right)x^k + \sum_{k=qn+1}^{qn+q-1}\left(\sum_{j=k-qn+1}^{q} ja_jb_{k-j+1}\right)x^k.\end{aligned} \tag{31}$$

- $\mu = qn-1$, $\alpha_j = (j+1)b_{j+1}$, $\nu = q$, $\beta_j = a_j$ lead

$$\begin{aligned}\mathrm{F}'(x)\mathrm{G}(x) &= \sum_{k=0}^{q}\left(\sum_{j=0}^{k} (k-j+1)a_jb_{k-j+1}\right)x^k + \sum_{k=q+1}^{qn-1}\left(\sum_{j=0}^{q} (k-j+1)a_jb_{k-j+1}\right)x^k \\ &\quad + \sum_{k=qn}^{qn+q-1}\left(\sum_{j=k-qn+1}^{q} (k-j+1)a_jb_{k-j+1}\right)x^k.\end{aligned} \tag{32}$$

By substituting (31) and (32) in equality (30) and identifying the relative powers of k, one obtains the following relations.

- If $k = 0, 1, \ldots, q-1$ then

$$n \sum_{j=1}^{k+1} j a_j b_{k-j+1} = \sum_{j=0}^{k} (k-j+1) a_j b_{k-j+1}$$

and therefore

$$n \sum_{j=1}^{k+1} j a_j b_{k-j+1} = \sum_{j=1}^{k} (k-j+1) a_j b_{k-j+1} + (k+1) a_0 b_{k+1}.$$

Therefore, as $a_0 \neq 0$, one has

$$\begin{aligned} b_{k+1} &= \frac{1}{(k+1)a_0} \left(n \sum_{j=1}^{k+1} j a_j b_{k-j+1} - \sum_{j=1}^{k} (k-j+1) a_j b_{k-j+1} \right) \\ &= \frac{1}{(k+1)a_0} \left(n(k+1) a_{k+1} b_0 + \sum_{j=1}^{k} (nj - k - 1 + j) a_j b_{k-j+1} \right) \\ &= \frac{1}{(k+1)a_0} \sum_{j=1}^{k+1} ((n+1)j - k - 1) a_j b_{k-j+1} \\ &= \frac{1}{(k+1)a_0} \sum_{j=0}^{k} ((n+1)(k-j+1) - (k+1)) a_{k-j+1} b_j. \end{aligned}$$

In this way, we obtain the coefficients $\{b_i\}_{i=0}^{q}$ given by

$$\begin{cases} b_0 = a_0^n \\ b_i = \dfrac{1}{i a_0} \displaystyle\sum_{j=0}^{i-1} ((n+1)(i-j) - i) a_{i-j} b_j, \quad i = 1, 2, \ldots, q. \end{cases} \tag{33}$$

- If $k = q$ then

$$n \sum_{j=1}^{q} j a_j b_{q+1-j} = \sum_{j=0}^{q} (q+1-j) a_j b_{q+1-j}$$

whence

$$n \sum_{j=1}^{q} j a_j b_{q+1-j} = \sum_{j=1}^{q} (q+1-j) a_j b_{q+1-j} + (q+1) a_0 b_{q+1}.$$

Again, as $a_0 \neq 0$, one has

$$\begin{aligned} b_{q+1} &= \frac{1}{(q+1)a_0} \left(n \sum_{j=1}^{q} j a_j b_{q-j+1} - \sum_{j=1}^{q} (q-j+1) a_j b_{q-j+1} \right) \\ &= \frac{1}{(q+1)a_0} \sum_{j=1}^{q} ((n+1)j - q - 1) a_j b_{q-j+1} \\ &= \frac{1}{(q+1)a_0} \sum_{j=1}^{q} ((n+1)(q-j+1) - (q+1)) a_{q-j+1} b_j. \end{aligned} \tag{34}$$

- If $k = q+1, q+2, \ldots, qn-1$ then

$$n \sum_{j=1}^{q} j a_j b_{k-j+1} = \sum_{j=0}^{q} (k-j+1) a_j b_{k-j+1}$$

hence

$$n \sum_{j=1}^{q} j a_j b_{k-j+1} = \sum_{j=1}^{k} (k-j+1) a_j b_{k-j+1} + (k+1) a_0 b_{k+1}.$$

As $a_0 \neq 0$, one has

$$\begin{aligned} b_{k+1} &= \frac{1}{(k+1)a_0}\left(n\sum_{j=1}^{q} j a_j b_{k-j+1} - \sum_{j=1}^{q}(k-j+1)a_j b_{k-j+1}\right) \\ &= \frac{1}{(k+1)a_0}\sum_{j=1}^{q}((n+1)j-k-1)a_j b_{k-j+1} \\ &= \frac{1}{(k+1)a_0}\sum_{j=k-q+1}^{k}((n+1)(k-j+1)-(k+1))a_{k-j+1}b_j. \end{aligned} \tag{35}$$

Now, we obtain the coefficients $\{b_i\}_{i=q+2}^{qn}$ given by

$$b_i = \frac{1}{ia_0}\sum_{j=i-q}^{i-1}((n+1)(i-j)-i)a_{i-j}b_j, \quad i = q+2, q+3, \dots, qn.$$

Finally, from (33), (34) and (35) we conclude

Theorem 3 (*Extended Villarreal Formula*). *For all $q, n \in \mathbb{N}$ the coefficients of the expansion*

$$\left(\sum_{j=0}^{q} a_j x^j\right)^n = \sum_{j=0}^{qn} b_j x^j \quad (a_j \in \mathbb{R}, a_0 \neq 0) \tag{36}$$

satisfy the extended Villarreal formula

$$\begin{aligned} b_i &= a_0^n, \quad \text{if } i = 0, \\ &\frac{1}{ia_0}\sum_{j=0}^{i-1}\Big((n+1)(i-j)-i\Big)a_{i-j}b_j, \quad \text{if } i = 1, 2, \dots, q, \\ &\frac{1}{ia_0}\sum_{j=i-q}^{i-1}\Big((n+1)(i-j)-i\Big)a_{i-j}b_j, \quad \text{if } i = q+1, q+2, \dots, qn. \quad \square \end{aligned} \tag{37}$$

Remark 10. Straightforward computations lead to

$$\begin{cases} b_0 = a_0^n \binom{n}{0}, \quad q, n \in \mathbb{N} \\ b_1 = a_0^{n-1}\binom{n}{1} a_1, \quad q, n \in \mathbb{N} \\ b_2 = a_0^{n-2}\left[\binom{n}{1} a_0 a_2 + \binom{n}{2} a_1^2\right], \quad \text{if } \min\{q,n\} \geq 2, \\ b_3 = a_0^{n-3}\left[\binom{n}{1} a_0^2 a_3 + \binom{n}{2} 2a_0 a_1 a_2 + \binom{n}{3} a_1^3\right], \quad \text{if } \min\{q,n\} \geq 3, \\ b_4 = a_0^{n-4}\left[\binom{n}{1} a_0^3 a_4 + \binom{n}{2} a_0^2(2a_1 a_3 + a_2^2) + \binom{n}{3} 3a_0 a_1^2 a_2 + \binom{n}{4} a_1^4\right], \quad \text{if } \min\{q,n\} \geq 4, \\ b_5 = a_0^{n-5}\left[\binom{n}{1} a_0^4 a_5 + \binom{n}{2} 2a_0^3(a_1 a_4 + a_2 a_3) + \binom{n}{3} 3a_0^2(a_1 a_2^2 + a_1^2 a_3) + \binom{n}{4} 4a_0 a_1^3 a_2 + \binom{n}{5} a_1^5\right], \end{cases}$$

provided $\min\{q, n\} \geq 5$. $\square$

The next contribution here is devoted with the explicit version of (37). More precisely, we note that each summand in the brackets of the coefficients in Remark 10 can be written as

$$\binom{n}{j} a_0^{i-j} \sum_{\substack{\ell_1\cdot\gamma_{\ell_1}+\ell_2\cdot\gamma_{\ell_2}+\cdots+\ell_j\cdot\gamma_{\ell_j}=i \\ \gamma_{\ell_1}+\gamma_{\ell_2}+\cdots+\gamma_{\ell_j}=j \\ 1\leq\ell_1<\ell_2<\cdots<\ell_j\leq i-j+1 \\ \{\gamma_{\ell_k}\}_{k=1}^{j}\in\{0,1,2,\dots,j\}}} \binom{j}{\gamma_{\ell_1}\gamma_{\ell_2}\cdots\gamma_{\ell_j}} a_{\ell_1}^{\gamma_{\ell_1}} a_{\ell_2}^{\gamma_{\ell_2}}\cdots a_{\ell_j}^{\gamma_{\ell_j}}, \quad 0 \leq j \leq i \leq \min\{q,n\},$$

where

$$\binom{j}{\gamma_{\ell_1}\gamma_{\ell_2}\cdots\gamma_{\ell_j}} = \frac{j!}{\gamma_{\ell_1}!\gamma_{\ell_2}!\cdots\gamma_{\ell_j}!}$$

denotes the permutations of j objects of which γ_{ℓ_1} are of one kind, γ_{ℓ_2} are of a second kind, …, γ_{ℓ_j} are of a jth kind.

So that, one has

Theorem 4 (*Explicit Villarreal Formula*). *The first coefficients of the expansion (36) are given by*

$$\begin{aligned}
b_0 &= a_0^n \\
b_i &= a_0^{n-i}\Bigg[\binom{n}{1} a_0^{i-1}a_i + \binom{n}{2} a_0^{i-2} \sum_{\substack{\ell_1\cdot\gamma_{\ell_1}+\ell_2\cdot\gamma_{\ell_2}=i \\ \gamma_{\ell_1}+\gamma_{\ell_2}=2 \\ 1\le \ell_1<\ell_2\le i-1 \\ \{\gamma_{\ell_k}\}_{k=1}^{2}\in\{0,1,2\}}} \binom{2}{\gamma_{\ell_1}\gamma_{\ell_2}} a_{\ell_1}^{\gamma_{\ell_1}} a_{\ell_2}^{\gamma_{\ell_2}} \\
&\quad + \binom{n}{3} a_0^{i-3} \sum_{\substack{\ell_1\cdot\gamma_{\ell_1}+\ell_2\cdot\gamma_{\ell_2}+\ell_3\cdot\gamma_{\ell_3}=i \\ \gamma_{\ell_1}+\gamma_{\ell_2}+\gamma_{\ell_3}=3 \\ 1\le \ell_1<\ell_2<\ell_3\le i-2 \\ \{\gamma_{\ell_k}\}_{k=1}^{3}\in\{0,1,2,3\}}} \binom{3}{\gamma_{\ell_1}\gamma_{\ell_2}\gamma_{\ell_3}} a_{\ell_1}^{\gamma_{\ell_1}} a_{\ell_2}^{\gamma_{\ell_2}} a_{\ell_3}^{\gamma_{\ell_3}} + \dots \\
&\quad + \binom{n}{j} a_0^{i-j} \sum_{\substack{\ell_1\cdot\gamma_{\ell_1}+\ell_2\cdot\gamma_{\ell_2}+\cdots+\ell_j\cdot\gamma_{\ell_j}=i \\ \gamma_{\ell_1}+\gamma_{\ell_2}+\cdots+\gamma_{\ell_j}=j \\ 1\le \ell_1<\ell_2<\cdots<\ell_j\le i-j+1 \\ \{\gamma_{\ell_k}\}_{k=1}^{j}\in\{0,1,2,\dots,j\}}} \binom{j}{\gamma_{\ell_1}\gamma_{\ell_2}\cdots\gamma_{\ell_j}} a_{\ell_1}^{\gamma_{\ell_1}} a_{\ell_2}^{\gamma_{\ell_2}} \cdots a_{\ell_j}^{\gamma_{\ell_j}} + \dots \\
&\quad + \binom{n}{i-1} a_0 (i-1) a_1^{i-2} a_2 + \binom{n}{i} a_1^i \Bigg], \quad \text{if } i = 1, 2, \dots, \min\{q, n\}.
\end{aligned}$$

Thus

$$b_i = a_0^{n-i} \sum_{j=1}^{i} \binom{n}{j} a_0^{i-j} \sum_{\substack{\ell_1\cdot\gamma_{\ell_1}+\ell_2\cdot\gamma_{\ell_2}+\cdots+\ell_j\cdot\gamma_{\ell_j}=i \\ \gamma_{\ell_1}+\gamma_{\ell_2}+\cdots+\gamma_{\ell_j}=j \\ 1\le \ell_1<\ell_2<\cdots<\ell_j\le i-j+1 \\ \{\gamma_{\ell_k}\}_{k=1}^{j}\in\{0,1,2,\dots,j\}}} \binom{j}{\gamma_{\ell_1}\gamma_{\ell_2}\cdots\gamma_{\ell_j}} a_{\ell_1}^{\gamma_{\ell_1}} a_{\ell_2}^{\gamma_{\ell_2}} \cdots a_{\ell_j}^{\gamma_{\ell_j}} \tag{38}$$

for $i = 0, 1, 2, \dots, \min\{q, n\}$. *In particular, when* $n \ge q$ *the formula (38) provides the first* $q + 1$ *coefficients of (36).*

Sketch of the Proof. The obtainment of (38) requires awful computations based on transfinite induction arguments. In order to simplify, we only are going to obtain b_6 in terms of the coefficients $\{b_j\}_{j=0}^5$ given in Remark 10. By assuming $\min\{q, n\} \ge 6$, from definition, one has

$$\begin{aligned}
b_6 &= \frac{1}{6a_0}\Big[\Big((n+1)(6-0)-6\Big)a_6b_0 + \Big((n+1)(6-1)-6\Big)a_5b_1 + \Big((n+1)(6-2)-6\Big)a_4b_2 \\
&\quad + \Big((n+1)(6-3)-6\Big)a_3b_3 + \Big((n+1)(6-4)-6\Big)a_2b_4 + \Big((n+1)(6-5)-6\Big)a_1b_5\Big] \\
&= na_0^{n-1}a_6 + \frac{5n-1}{6}\, na_0^{n-2}a_1a_5 + \frac{2n-1}{3}\, a_0^{n-3}\Big(na_0a_2 + \frac{n(n-1)}{2}a_1^2\Big)a_4 \\
&\quad + \frac{n-1}{2}\, a_0^{n-4}\Big(na_0^2a_3 + \frac{n(n-1)}{2}\, 2a_0a_1a_2 + \frac{n(n-1)(n-2)}{6}\, a_1^3\Big)a_3 \\
&\quad + \frac{n-2}{3}\, a_0^{n-5}\Big(na_0^3a_4 + \frac{n(n-1)}{2}\, a_0^2(2a_1a_3 + a_2^2) + \frac{n(n-1)(n-2)}{6}\, 3a_0a_1^2a_2 + \frac{n(n-1)(n-2)(n-3)}{24}\, a_1^4\Big)a_2 \\
&\quad + \frac{n-5}{6}\, a_0^{n-6}\Big(na_0^4a_5 + \frac{n(n-1)}{2}\, 2a_0^3(a_1a_4 + a_2a_3) + \frac{n(n-1)(n-2)}{6}\, 3a_0^2(a_1^2a_3 + a_1a_2^2) \\
&\quad + \frac{n(n-1)(n-2)(n-3)}{24}\, 4a_0a_1^3a_2 + \frac{n(n-1)(n-2)(n-3)(n-4)}{120}\, a_1^5\Big)a_1.
\end{aligned}$$

Then by arrangement one proves

$$\begin{aligned}
b_6 &= a_0^{n-6}\Bigg[na_0^5 a_6 + \left(\frac{5n-1}{6} + \frac{n-5}{6}\right) na_0^4 a_1 a_5 + \left(\frac{2n-1}{3} + \frac{n-2}{3}\right) na_0^4 a_2 a_4 + \frac{n(n-1)}{2}\, a_0^4 a_3^2 \\
&\quad + \left(\frac{2n-1}{3} + \frac{n-5}{3}\right) \frac{n(n-1)}{2}\, a_0^3 a_1^2 a_4 + \left(\frac{n-1}{2} + \frac{n-2}{3} + \frac{n-5}{6}\right) n(n-1)\, a_0^3 a_1 a_2 a_3 \\
&\quad + \frac{n-2}{3}\,\frac{n(n-1)}{2}\, a_0^3 a_2^3 + \left(\frac{n-2}{3} + \frac{n-5}{6}\right) \frac{n(n-1)(n-2)}{2}\, a_0^2 a_1^2 a_2^2 \\
&\quad + \left(\frac{n-1}{6} + \frac{n-5}{6}\right) \frac{n(n-1)(n-2)}{2}\, a_0^2 a_1^3 a_3 + \left(\frac{n-2}{3} + \frac{2n-10}{3}\right) \frac{n(n-1)(n-2)(n-3)}{24}\, a_0 a_1^4 a_2 \\
&\quad + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{720}\, a_1^6 \Bigg] \\
&= a_0^{n-6}\Bigg[\binom{n}{1} a_0^5 a_6 + \binom{n}{2} a_0^4 (2a_1 a_5 + 2a_2 a_4 + a_3^2) + \binom{n}{3} a_0^3 (3a_1^2 a_4 + 6a_1 a_2 a_3 + a_2^3) \\
&\quad + \binom{n}{4} a_0^2 (6a_1^2 a_2^2 + 4a_1^3 a_3) + \binom{n}{5} 5a_0 a_1^4 a_2 + \binom{n}{6} a_1^6 \Bigg],
\end{aligned}$$

which corresponds with (38) whenever $i = 6$. □

Certainly without loss of generality we may assume $a_q \neq 0$. Then, multiplying by x^{-qn} we derive

$$\left(\sum_{j=0}^{q} a_j x^{-q+j}\right)^n = \sum_{j=0}^{qn} b_j x^{-qn+j},$$

whence $y = x^{-1}$ satisfies

$$\left(\sum_{j=0}^{q} \widehat{a}_j y^j\right)^n = \sum_{j=0}^{qn} \widehat{b}_j y^j \quad (\widehat{a}_0 = a_q \neq 0)$$

for $\widehat{a}_j = a_{q-j}, j = 0, 1, \ldots, n$, and $\widehat{b}_j = b_{qn-j}, j = 0, 1, \ldots, qn$. Therefore (38) enables us to conclude

Corollary 1. *When $a_q \neq 0$ the last coefficients of the expansion (36) are given*

$$\begin{aligned}
b_{qn} &= a_q^n, \\
b_{qn-i} &= a_q^{n-i} \sum_{j=1}^{i} \binom{n}{j} a_q^{i-j} \sum_{\substack{\ell_1\cdot\gamma_{\ell_1}+\ell_2\cdot\gamma_{\ell_2}+\cdots+\ell_j\cdot\gamma_{\ell_j}=i \\ \gamma_{\ell_1}+\gamma_{\ell_2}+\cdots+\gamma_{\ell_j}=j \\ 1\le\ell_1<\ell_2<\cdots<\ell_j\le i-j+1 \\ \{\gamma_{\ell_k}\}_{k=1}^{j}\in\{0,1,2,\ldots,j\}}} \binom{j}{\gamma_{\ell_1}\gamma_{\ell_2}\cdots\gamma_{\ell_j}} a_{q-\ell_1}^{\gamma_{\ell_1}} a_{q-\ell_2}^{\gamma_{\ell_2}} \cdots a_{q-\ell_j}^{\gamma_{\ell_j}}
\end{aligned} \tag{39}$$

for $i = 1, 2, \ldots, \min\{q, n\}$. In particular, when $n \geq q$ the formula (39) provides the last $q + 1$ coefficients of (36). □

Appendix B. Expanding the power of the auxiliar sub and supersolutions

As it was pointed out, the proof of Theorem 1 uses the power of suitable polynomials relative to certain auxiliar sub and supersolutions. So we get back to the formal expansion

$$\mathrm{V}_\delta^{\pm}(x) = \mathrm{C}_0\big(\mathrm{d}(x) \mp \delta\big)^{-\alpha_\tau} \left(1 + \sum_{n=1}^{\mathrm{M}_\tau} \mathrm{C}_n(x)\big(\mathrm{d}(x) \mp \delta\big)^n\right)$$

for which

$$\begin{aligned}
(\mathrm{V}_\delta^{\pm}(x))^m &= \mathrm{C}_0^m\big(\mathrm{d}(x) \mp \delta\big)^{-\alpha_\tau m} \left(1 + \sum_{n=1}^{\mathrm{M}_\tau} \mathrm{C}_n(x)\big(\mathrm{d}(x) \mp \delta\big)^n\right)^m \\
&= \mathrm{C}_0^m\big(\mathrm{d}(x) \mp \delta\big)^{-\alpha_\tau m}\, \Phi \left(\sum_{n=1}^{\mathrm{M}_\tau} \mathrm{C}_n(x)\big(\mathrm{d}(x) \mp \delta\big)^{n-1}\right),
\end{aligned}$$

where

$$\Phi(s) = \big(1 + s(\mathrm{d}(x) \mp \delta)\big)^m.$$

Since the Taylor expansion of function Φ gives

$$\Phi(s) = \sum_{n\geq 0}\left(\frac{1}{n!}\frac{\mathrm{d}^n\Phi}{\mathrm{d}s^n}(0)\right)s^n,$$

we get

$$(\mathrm{V}_\delta^\pm(x))^m = \mathrm{C}_0^m\big(\mathrm{d}(x)\mp\delta\big)^{-\alpha_\tau m}\sum_{n\geq 0}\binom{m}{n}\left(\sum_{k=1}^{\mathrm{M}\tau}\mathrm{C}_k(x)\big(\mathrm{d}(x)\mp\delta\big)^{k-1}\right)^n\big(\mathrm{d}(x)\mp\delta\big)^n,$$

due to

$$\frac{1}{n!}\frac{\mathrm{d}^n\Phi}{\mathrm{d}s^n}(0) = \binom{m}{n}\big(\mathrm{d}(x)\mp\delta\big)^n = \frac{m(m-1)\cdots(m-n+1)}{n!}\big(\mathrm{d}(x)\mp\delta\big)^n, \quad m\in\mathbb{R}.$$

On the other hand, we may write

$$\left(\sum_{k=1}^{\mathrm{M}\tau}\mathrm{C}_k(x)\big(\mathrm{d}(x)\mp\delta\big)^{k-1}\right)^n = \left(\sum_{k=0}^{\mathrm{M}\tau-1}\mathrm{C}_{k+1}(x)\big(\mathrm{d}(x)\mp\delta\big)^{k}\right)^n = \sum_{i=0}^{(\mathrm{M}\tau-1)n}\mathrm{B}_{i,n}(x)\big(\mathrm{d}(x)\mp\delta\big)^i \tag{40}$$

where

$$\mathrm{B}_{i,n}(x) = \begin{cases}\big(\mathrm{C}_1(x)\big)^n, & \text{if } i=0,\\ \dfrac{1}{i\mathrm{C}_1(x)}\displaystyle\sum_{\ell=0}^{i-1}\big((i-\ell)(n+1)-i\big)\mathrm{C}_{i-\ell+1}(x)\mathrm{B}_{\ell,n}(x), & \text{if } i=1,2,\ldots,\mathrm{M}_\tau-1,\\ \dfrac{1}{i\mathrm{C}_1(x)}\displaystyle\sum_{\ell=i-\mathrm{M}_\tau+1}^{i-1}\big((i-\ell)(n+1)-i\big)\mathrm{C}_{i-\ell+1}(x)\mathrm{B}_{\ell,n}(x), & \text{if } i=\mathrm{M}_\tau,\ldots,(\mathrm{M}_\tau-1)n\end{cases} \tag{41}$$

(see (37) in the Appendix A).

Remark 11. The coefficients $\mathrm{B}_{i,n}(x)$, for $i = 0,1,\ldots,n$ and $n\in\mathbb{N}$ are obtained by straightforward computations. For instance

$$\begin{cases}\mathrm{B}_{0,n}(x) = \big(\mathrm{C}_1(x)\big)^n\binom{n}{0},\\ \mathrm{B}_{1,n}(x) = \big(\mathrm{C}_1(x)\big)^{n-1}\binom{n}{1}\mathrm{C}_2(x),\\ \mathrm{B}_{2,n}(x) = \big(\mathrm{C}_1(x)\big)^{n-2}\left[\binom{n}{1}\mathrm{C}_1(x)\mathrm{C}_3(x) + \binom{n}{2}\big(\mathrm{C}_2(x)\big)^2\right],\\ \mathrm{B}_{3,n}(x) = \big(\mathrm{C}_1(x)\big)^{n-3}\left[\binom{n}{1}\big(\mathrm{C}_1(x)\big)^2\mathrm{C}_4(x) + \binom{n}{2}2\mathrm{C}_1(x)\mathrm{C}_3(x)\mathrm{C}_2(x) + \binom{n}{3}\big(\mathrm{C}_2(x)\big)^3\right],\\ \mathrm{B}_{4,n}(x) = \big(\mathrm{C}_1(x)\big)^{n-4}\left[\binom{n}{1}\big(\mathrm{C}_1(x)\big)^3\mathrm{C}_5(x) + \binom{n}{2}\big(\mathrm{C}_1(x)\big)^2\Big(2\mathrm{C}_4(x)\mathrm{C}_2(x) + \big(\mathrm{C}_3(x)\big)^2\Big)\right.\\ \quad\left. + \binom{n}{3}3\mathrm{C}_1(x)\mathrm{C}_3(x)\big(\mathrm{C}_2(x)\big)^2 + \binom{n}{4}\big(\mathrm{C}_2(x)\big)^4\right].\end{cases}$$

Adjusting the formula (38) (see again the Appendix A), by means of a transfinite induction argument, we obtain the explicit expression of (41)

$$\mathrm{B}_{i,n}(x) = \sum_{j=1}^{i}\binom{n}{j}\big(\mathrm{C}_1(x)\big)^{n-j}\sum_{\substack{\ell_1\cdot\gamma_{\ell_1}+\ell_2\cdot\gamma_{\ell_2}+\cdots+\ell_j\cdot\gamma_{\ell_j}=i+j\\ \gamma_{\ell_1}+\gamma_{\ell_2}+\cdots+\gamma_{\ell_j}=j\\ 2\leq\ell_1<\cdots<\ell_j\leq i-j+2\\ \{\gamma_{\ell_k}\}_{k=1}^{j}\subset\{0,1,\ldots,j\}}}\frac{j!}{\gamma_{\ell_1}!\gamma_{\ell_2}!\cdots\gamma_{\ell_j}!}\big(\mathrm{C}_{\ell_1}(x)\big)^{\gamma_{\ell_1}}\cdots\big(\mathrm{C}_{\ell_j}(x)\big)^{\gamma_{\ell_j}} \tag{42}$$

for $i = 1,2,\ldots,n$. □

Then one has

$$\begin{aligned}\left(\mathrm{V}_\delta^{\pm}(x)\right)^m &= \mathrm{C}_0^m\left(\mathrm{d}(x)\mp\delta\right)^{-\alpha_\tau m}\sum_{n\geq 0}\binom{m}{n}\sum_{i=0}^{(\mathrm{M}_\tau-1)n}\mathrm{B}_{i,n}(x)\left(\mathrm{d}(x)\mp\delta\right)^{i+n}\\ &= \mathrm{C}_0^m\left(\mathrm{d}(x)\mp\delta\right)^{-\alpha_\tau m}\left(\mathrm{D}_0(x)+\sum_{n=1}^{\mathrm{M}_\tau}\mathrm{D}_n(x)\left(\mathrm{d}(x)\mp\delta\right)^n+\sum_{n=\mathrm{M}_\tau+1}^{\infty}\mathrm{D}_n(x)\left(\mathrm{d}(x)\mp\delta\right)^n\right)\end{aligned} \tag{43}$$

where

$$\mathrm{D}_n(x)=\sum_{i=1}^{n}\binom{m}{i}\mathrm{B}_{n-i,i}(x),\quad n=1,2,\ldots. \tag{44}$$

Remark 12. In order to illustrate we note that the first five coefficients $\mathrm{D}_n(x)$ are

$$\begin{cases}\mathrm{D}_0(x)=\binom{m}{0}\mathrm{B}_{0,0}(x)=\left(\mathrm{C}_1(x)\right)^0=1,\\ \mathrm{D}_1(x)=\binom{m}{1}\mathrm{B}_{0,1}(x)=\binom{m}{1}\mathrm{C}_1(x),\\ \mathrm{D}_2(x)=\binom{m}{2}\mathrm{B}_{0,2}(x)+\binom{m}{1}\mathrm{B}_{1,1}(x)=\binom{m}{2}\left(\mathrm{C}_1(x)\right)^2+\binom{m}{1}\mathrm{C}_2(x),\\ \mathrm{D}_3(x)=\binom{m}{3}\mathrm{B}_{0,3}(x)+\binom{m}{2}\mathrm{B}_{1,2}(x)+\binom{m}{1}\mathrm{B}_{2,1}(x)\\ \quad=\binom{m}{3}\left(\mathrm{C}_1(x)\right)^3+\binom{m}{2}2\mathrm{C}_1(x)\mathrm{C}_2(x)+\binom{m}{1}\mathrm{C}_3(x),\\ \mathrm{D}_4(x)=\binom{m}{4}\mathrm{B}_{0,4}(x)+\binom{m}{3}\mathrm{B}_{1,3}(x)+\binom{m}{2}\mathrm{B}_{2,2}(x)+\binom{m}{1}\mathrm{B}_{3,1}(x)\\ \quad=\binom{m}{4}\left(\mathrm{C}_1(x)\right)^4+\binom{m}{3}3\left(\mathrm{C}_1(x)\right)^2\mathrm{C}_2(x)+\binom{m}{2}\left[2\mathrm{C}_1(x)\mathrm{C}_3(x)+\left(\mathrm{C}_2(x)\right)^2\right]+\binom{m}{1}\mathrm{C}_4(x),\end{cases}$$

provided $\mathrm{M}_\tau\geq 4$. □

Choosing $n=1$ in (40) we deduce

$$\mathrm{B}_{i,1}(x)=\mathrm{C}_{i+1}(x),\quad i=0,1,2,\ldots,\mathrm{M}_{\tau-1},$$

so that, from (44), we obtain

$$\mathrm{D}_n(x)=\binom{m}{1}\mathrm{C}_n(x)+\sum_{i=2}^{n}\binom{m}{i}\mathrm{B}_{n-i,i}(x),\quad 1\leq n\leq\mathrm{M}_\tau, \tag{45}$$

whence, in (45), each $\mathrm{C}_n(x)$, $1\leq n\leq\mathrm{M}_\tau$, does not appear in $\mathrm{B}_{n-i,i}(x)$, $i\neq 1$. Certainly all coefficients $\mathrm{C}_n(x)$, $1\leq n\leq\mathrm{M}_\tau$, are involved in the other $\mathrm{D}_n(x)$, $\mathrm{M}_\tau+1\leq n$.

Remark 13. When m is an integer number one has

$$\frac{1}{n!}\frac{\mathrm{d}^n\Phi}{\mathrm{d}s^n}(0)=0,\quad n>m,$$

therefore the Taylor expansion is finite. So, from (44) we deduce that $\mathrm{D}_n(x)=0$ if $n>(M_\tau-1)m$. □

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