Concentrating solutions in a two-dimensional elliptic problem with exponential Neumann data

Juan Dávila\textsuperscript{a,}\textsuperscript{*}, Manuel del Pino\textsuperscript{a}, Monica Musso\textsuperscript{b,}\textsuperscript{c}

\textsuperscript{a}Departamento de Ingeniería Matemática and CMM, Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile
\textsuperscript{b}Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy
\textsuperscript{c}Departamento de Matemática, Pontificia Universidad Católica de Chile, Avda. Vicuña Mackenna 4860, Macul, Chile

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Abstract

We consider the elliptic equation $-\Delta u + u = 0$ in a bounded, smooth domain $\Omega$ in $\mathbb{R}^2$ subject to the nonlinear Neumann boundary condition $\frac{\partial u}{\partial n} = \varepsilon e^u$. Here $\varepsilon > 0$ is a small parameter. We prove that any family of solutions $u_\varepsilon$ for which $\varepsilon \int_\Omega e^u$ is bounded, develops up to subsequences a finite number $m$ of peaks $\zeta_i \in \partial \Omega$, in the sense that $\varepsilon e^u \rightharpoonup 2\pi \sum_{k=1}^m \delta_{\zeta_i}$ as $\varepsilon \to 0$. Reciprocally, we establish that at least two such families indeed exist for any given $m \geq 1$.

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1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial \Omega$. This paper deals with the analysis of solutions of the boundary value problem

$$
\begin{aligned}
-\Delta u + u &= 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= \varepsilon e^u \quad \text{on } \partial \Omega,
\end{aligned}
$$

(1)

where $\nu$ denotes outer unit normal vector to $\partial \Omega$ and $\varepsilon > 0$ is a small parameter.

Elliptic equations with this type of nonlinear Neumann boundary condition arise in conformal geometry (prescribing Gaussian curvature of the domain and curvature of the boundary), see for instance [8] and references therein, and in corrosion modelling, see [3,6,9]. The Trudinger–Moser and trace inequalities imply the validity of the Sobolev–Orlicz (compact) trace embedding

$$
\sup_{u \in H^1(\Omega) \setminus \{0\}} \int_{\partial \Omega} \exp \left( \frac{u}{\|u\|_{H^1}} \right) < +\infty,
$$

(2)

see [6] for a proof. Note that an extremal of this inequality with $\|u\|_{H^1(\Omega)} = 1$ solves (1) for certain $\varepsilon > 0$. Given a fixed value of the parameter $\varepsilon$, solutions of (1) correspond precisely to critical points in $H^1(\Omega)$ of the free energy functional

$$
J_\varepsilon(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + u^2 - \varepsilon \int_{\partial \Omega} e^u.
$$

(3)

The maximum principle implies that solutions of (1) are automatically positive. Smallness of $\varepsilon$ is necessary for existence of a solution as integration against a suitable test function shows. On the other hand, inequality (2) implies that a (unique) local minimizer exists near zero, provided that $\varepsilon > 0$ is sufficiently small. This minimizer represents a “small” solution of Problem (1). The functional is not bounded below, thus suggesting the presence of a second, large solution for $\varepsilon > 0$ small. Compactness of the trace embedding yields the sufficient PS condition for this second solution to exist thanks to the standard mountain pass theorem. In [3,6,9], the following related problem was analyzed:

$$
\begin{aligned}
\Delta u &= 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= \varepsilon \sinh u \quad \text{on } \partial \Omega,
\end{aligned}
$$

(4)

Evenness of the associated energy functional

$$
I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \varepsilon \int_{\partial \Omega} \cosh u
$$
and the above-mentioned compactness makes Ljusternik–Schnirelmann theory applicable to find that actually infinitely many solutions exist associated to critical values

\[ c_1^\varepsilon \leq c_2^\varepsilon \leq \cdots \leq c_k^\varepsilon \leq \cdots. \]

For each fixed \( k \) it turns out that \( c_k^\varepsilon \) is bounded above by \( C_k \log \frac{1}{\varepsilon} \). This is shown to imply that \( \varepsilon \int_{\partial \Omega} \cosh u \) is uniformly bounded for the associated solutions as \( \varepsilon \to 0 \). Applying similar arguments as those in [6] one can show that the mountain pass solution of (1) has a similar property, namely that \( \varepsilon \int_{\partial \Omega} e^{\varepsilon u} \) remains bounded. Our first result characterizes the asymptotic behavior of families of solutions \( u_\varepsilon \) with \( \varepsilon \int_{\partial \Omega} e^{\varepsilon u} \) bounded.

It turns out that, up to subsequences, there is an integer \( m \), such that \( \varepsilon \int_{\partial \Omega} e^{\varepsilon u} \to 2m\pi \). More precisely, \( \varepsilon e^{\varepsilon u} \) approaches the sum of \( m \) Dirac masses at the boundary. The location of these possible points of concentration may be further characterized as critical points of a functional of \( m \) points of the boundary which we introduce next: let us consider the Green’s function for the Neumann problem

\[
\begin{aligned}
-\Delta_x G(x, y) + G(x, y) &= 0, \quad x \in \Omega, \\
\frac{\partial G}{\partial \nu_x}(x, y) &= 2\pi \delta_y(x), \quad x \in \partial \Omega
\end{aligned}
\]

and its regular part

\[ H(x, y) = G(x, y) - \log \frac{1}{|x - y|^2}. \]

We define \( \varphi_m \) on \((\partial \Omega)^m\) by

\[ \varphi_m(\xi_1, \ldots, \xi_m) = -\left[ \sum_{j=1}^{m} H(\xi_j, \xi_j) + \sum_{i \neq j} G(\xi_i, \xi_j) \right]. \]

**Theorem 1.1.** Let \( u_\varepsilon \) be a family of solutions to (1) with \( \varepsilon \to 0 \). If \( \varepsilon \int_{\partial \Omega} e^{\varepsilon u} \leq C \) for some constant \( C \) independent of \( \varepsilon \) there exists a subsequence (denoted the same way) and a finite collection of distinct points \( \xi_i \in \partial \Omega \) \( i = 1, \ldots, m \) such that

\[ u_\varepsilon \to u^*, \]

where \( u^* \) is the solution to

\[
\begin{aligned}
-\Delta u^* + u^* &= 0 \quad \text{in } \Omega, \\
\frac{\partial u^*}{\partial \nu} &= 2\pi \sum_{i=1}^{m} \delta_{\xi_i} \quad \text{on } \partial \Omega.
\end{aligned}
\]
Moreover

\[ \varepsilon \int_{\partial \Omega} e^{u_\varepsilon} \to 2\pi m \quad \text{and} \quad \varepsilon e^{u_\varepsilon} \to 2\pi \sum_{i=1}^{m} \delta_{\xi_i}, \]

weakly in the sense of Radon measures in \( \partial \Omega \), \( u_\varepsilon \to u^* \) in \( L^p(\partial \Omega) \) and \( L^p(\Omega) \) for all \( 1 \leq p < \infty \) and in \( C^1_{\text{loc}}(\overline{\Omega} - \{\xi_1, \ldots, \xi_m\}) \). Additionally \( (\xi_1, \ldots, \xi_m) \) is a critical point of \( \varphi_m \), that is for all \( k = 1, \ldots, m \)

\[ \nabla_{\tau(\xi_k)} H(\xi_k, \xi_k) + \sum_{i \neq k} \nabla_{\tau(\xi_k)} G(\xi_k, \xi_i) = 0, \quad (7) \]

where \( \tau(\xi_k) \) is a tangent vector to \( \partial \Omega \) at \( \xi_k \).

We remark that if \( \varepsilon \int_{\partial \Omega} e^{u_\varepsilon} \) is unbounded after extracting a subsequence for which \( \varepsilon \int_{\partial \Omega} e^{u_\varepsilon} \to \infty \) we have \( u_\varepsilon \rightharpoonup \infty \) uniformly in \( \overline{\Omega} \).

A natural question is whether families of solutions such as those described in the previous theorem do indeed exist. It can be shown that the mountain pass large solution does correspond to one exhibiting a single spike. However, it is not clear how to set up a Ljusternik–Schnirelmann scheme that predicts the existence of higher-energy solutions, in particular since the functional (3) does not seem to exhibit any useful symmetries. In this paper, we develop a completely different approach to this question which allows us to prove an existence result, which we suspect optimal: given any integer \( m \geq 1 \), there are at least two distinct families of solutions \( u_\varepsilon \) for which \( \varepsilon \int_{\partial \Omega} e^{u_\varepsilon} \to 2m\pi \).

**Theorem 1.2.** Let \( m \geq 1 \). Then for \( \varepsilon > 0 \) sufficiently small there exist two solutions \( u_\varepsilon \) to (1) satisfying

\[ \lim_{\varepsilon \to 0} \varepsilon \int_{\partial \Omega} e^{u_\varepsilon} = 2\pi m. \]

The peaks of these two solutions are located near points \( \xi_1, \ldots, \xi_m \in \partial \Omega \) corresponding to two distinct critical points of \( \varphi_m \).

We can actually show stronger versions of this result. For instance, if \( \partial \Omega \) has more than one component, then pairs of families of \( m \)-peak solutions on each component happen to exist. In reality, associated to each topologically nontrivial critical point situation associated to \( \varphi_m \) (for instance local maxima or saddle points possibly degenerate), a solution with concentration peaks at a corresponding critical point exists. We elaborate further on these issues at the end of Section 8.
It is important to remark the interesting analogy between these results and those known for the Liouville-type equation

\[
\begin{cases}
\Delta u + \varepsilon e^u = 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]  

(8)

Asymptotic behavior of families \(u_\varepsilon\) of solutions of (8) for which \(\varepsilon \int_{\Omega} e^u\) remains uniformly bounded is well understood after the works [2,7,10]. It is known that up to subsequences,

\[
\lim_{\varepsilon \to 0} \varepsilon \int_{\Omega} e^{u_\varepsilon} = 8m\pi
\]  

(9)

for some integer \(m \geq 1\). More precisely, \(\varepsilon e^{u_\varepsilon}\) peaks up as \(m\) Dirac masses at points of the domain which correspond to a critical point of a functional similar to \(\varphi_m\) defined from Green’s function of \(-\Delta\) under Dirichlet boundary condition. The reciprocal question of existence has been addressed among other works in [1,4,5]. In particular in [4], a result is established which may be thought of as an analog of Theorem 2: if the domain is not simply connected, then at least one solution with \(m\) peaks exists.

In the rest of this paper we will prove Theorems 1 and 2. Sections 2–8 are devoted to the proof of Theorem 2. Scaling out properly \(\varepsilon\) around a single point of the boundary leads us formally to the limiting problem

\[
\begin{cases}
\Delta v = 0 & \text{in } \mathbb{R}^2_+,
\\
\frac{\partial v}{\partial \nu} = e^v & \text{on } \partial \mathbb{R}^2_+,
\end{cases}
\]  

(10)

where \(\mathbb{R}^2_+\) denotes the upper half-plane \(\{(x_1, x_2) : x_2 > 0\}\) and \(\nu\) the unit exterior normal to \(\partial \mathbb{R}^2_+\).

A family of solutions to (10) is given by

\[
w_{\mu,t}(x_1, x_2) = \log \left( \frac{2\mu}{(x_1 - t)^2 + (x_2 + \mu)^2} \right),
\]  

(11)

where \(t \in \mathbb{R}\) and \(\mu > 0\) are parameters. It is interesting to point out that the results in [8,11,12] imply that any solution \(v\) of (10) which satisfies additionally

\[
\int_{\mathbb{R}^2_+} e^v < +\infty
\]

must be of the form (11). The solutions predicted in Theorem 2 are constructed using as building blocks these solutions, suitably scaled and projected to make it up to a good order for the boundary condition. Solutions are found as a small additive perturbation
of these initial approximations. A linearization procedure leads to a finite-dimensional reduction, where the reduced problem corresponds to that of adjusting variationally the location of the concentration points.

Theorem 1 is established in Section 9. The point concentration behavior of the family is established first, then Pohozaev-type identities in balls around the singularities lead to the desired result.

2. An equation in the upper half-plane

The family of solutions (11) is invariant under translations in the $x_1$-direction and under the scaling $s \mapsto w(sx) + 2 \log s$, $s > 0$. An important property that we will need is the nondegeneracy of these solutions (11) except for the above natural invariances of Eq. (10). Let us define

$$z_0 = \frac{1}{\mu} - 2 \frac{x_2 + \mu}{x_1^2 + (x_2 + \mu)^2}$$

and

$$z_1 = -2 \frac{x_1}{x_1^2 + (x_2 + \mu)^2}.$$ 

We have the following:

**Proposition 2.1.** Any bounded solution of

$$\begin{cases} \Delta \phi = 0 & \text{in } \mathbb{R}^2_+, \\ \frac{\partial \phi}{\partial v} - \frac{2\mu}{x_1^2 + \mu^2} \phi = 0 & \text{on } \partial \mathbb{R}^2_+,
\end{cases}$$

is a linear combination of $z_0$ and $z_1$.

**Proof.** Let $\phi$ be a solution to (14) and set

$$w(y) = \phi \left( \frac{y}{|y|^2} - (0, \mu) \right).$$

The function $w$ is just the Kelvin transform of $\phi$ about the point $(0, -\mu)$. The domain of $w$ is the disk $D = B((0, \frac{1}{2\mu}), \frac{1}{2\mu})$ and $w$ is a bounded function that satisfies $\Delta w = 0$ in $D$,

$$\frac{\partial w}{\partial v'} = 2\mu w \text{ on } \partial D \setminus \{0\},$$

(15)
where \( \nu' \) is the exterior unit normal to \( D \). To see this observe that the map \( y \mapsto K(y) = \frac{y}{|y|^2} - (0, \mu) \) is anti-conformal (preserves angles and reverses orientation) and maps the normal vector to \( D \) to a normal vector to \( \partial \mathbb{R}^2_+ \). More precisely, if \( \nu' \) is the exterior unit normal vector to \( D \) then

\[
\frac{\partial w}{\partial \nu'} = \frac{1}{|y|^2} \frac{\partial \phi}{\partial \nu}.
\]

Thus on \( \partial D \)

\[
\frac{\partial w}{\partial \nu'} = \frac{1}{|y|^2} e^{w_{\mu,0}(K(y))} w
\]

and a calculation shows that

\[
\frac{1}{|y|^2} e^{w_{\mu,0}(K(y))} = \frac{1}{|y|^2} \frac{2\mu}{\frac{\mu^2}{|y|^2} + \mu^2} = 2\mu.
\]

Since \( w \) is bounded, by elliptic regularity (15) holds in all \( \partial D \).

By translating in the \( y_2 \) direction we can assume that \( D \) is the disk centered at the origin with radius \( \frac{1}{2\mu} \). We think of \( w \) as the real part of an analytic function \( \tilde{w} \) and write

\[
\tilde{w}(y) = \sum_{k=0}^{\infty} a_k k^k e^{ik\theta}
\]

with \( y = r e^{i\theta} \). Condition (15) is equivalent to

\[
\text{Re} \left( \sum_{k=0}^{\infty} a_k (k-1) e^{ik\theta} \right) = 0 \quad \forall \theta
\]

and hence \( a_0 = 0 \), \( a_k = 0 \) for all \( k > 1 \). Looking at the real part \( w \) of \( \tilde{w} \), and recalling that we shifted in the \( y_2 \) direction we see that it is a linear combination of

\[
y_1 = \frac{x_1}{x_1^2 + (x_2 + \mu)^2} \quad \text{and} \quad y_2 = \frac{1}{2\mu} \left( \frac{x_2 + \mu}{x_1^2 + (x_2 + \mu)^2} - \frac{1}{2\mu} \right)
\]

\( \square \)
3. Ansatz for the solution

We can produce a solution to
\[
\begin{cases}
\Delta u = 0 & \text{in } \mathbb{R}^2_+,
\
\frac{\partial u}{\partial v} = \varepsilon e^u & \text{on } \partial \mathbb{R}^2_+.
\end{cases}
\]

by taking
\[
u(x) = w(\mu x / \varepsilon) - 2 \log \varepsilon = \log \frac{2\mu}{x_1^2 + (x_2 + \varepsilon \mu)^2}.
\]

Based on this, given \( \xi_j \in \partial \Omega, \mu_j > 0 \) we define
\[
u_j(x) = \log \frac{2\mu_j}{|x - \xi_j - \varepsilon \mu_j v(\xi_j)|^2}.
\]

The choice of \( \xi_j \) and \( \mu_j \) will be made later on.

The ansatz is
\[
U(x) = \sum_{j=1}^{m} u_j(x) + H_j^\varepsilon(x),
\]

where \( H_j^\varepsilon \) is a correction term defined as the solution of
\[
\begin{cases}
-\Delta H_j^\varepsilon + H_j^\varepsilon = -u_j & \text{in } \Omega,
\
\frac{\partial H_j^\varepsilon}{\partial v} = \varepsilon e^{u_j} - \frac{\partial u_j}{\partial v} & \text{on } \partial \Omega.
\end{cases}
\]

Lemma 3.1. For any \( 0 < \varepsilon < 1 \)
\[
H_j^\varepsilon(x) = H(x, \xi_j) - \log 2\mu_j + O(\varepsilon^2)
\]
uniformly in \( \overline{\Omega} \), where \( H \) is the regular part of Green’s function defined (6).

We will give the proof of this lemma at the end of the section.

It will be convenient to work with the scaling of \( u \) given by
\[
v(y) = u(\varepsilon y) + 2 \log \varepsilon.
\]
If \( u \) is a solution of (1) then \( v \) satisfies

\[
\begin{cases}
-\Delta v + \varepsilon^2 v = 2\varepsilon^2 \log \varepsilon & \text{in } \Omega_\varepsilon, \\
\frac{\partial v}{\partial v} = e^v & \text{on } \partial \Omega_\varepsilon,
\end{cases}
\]

(19)

where \( \Omega_\varepsilon = \Omega / \varepsilon \). With this scaling \( u_j \) becomes

\[
v_j(y) = \log \frac{2\mu_j}{|y - \zeta_j - \mu_j v(\zeta_j)|^2},
\]

where \( \zeta_j = \zeta_j / \varepsilon \) and where we will write \( v \) for the exterior normal unit vector to \( \partial \Omega \) and \( \partial \Omega_\varepsilon \).

We will seek a solution \( v \) of (19) of the form

\[
v = V + \phi,
\]

where

\[
V(y) = U(\varepsilon y) + 2 \log \varepsilon
\]

(20)

and \( U \) is defined by (16). Problem (19) can be stated as to find \( \phi \) a solution to

\[
\begin{cases}
-\Delta \phi + \varepsilon^2 \phi = 0 & \text{in } \Omega_\varepsilon, \\
\frac{\partial \phi}{\partial v} = e^V \phi + N(\phi) + R & \text{on } \partial \Omega_\varepsilon,
\end{cases}
\]

(21)

where the “nonlinear term” is

\[
N(\phi) = e^V (e^\phi - 1 - \phi)
\]

(22)

and the “error term” is given by

\[
R = e^V - \frac{\partial V}{\partial v}.
\]

(23)

At this point it is convenient to make a choice of the parameters \( \mu_j \), the objective being to make the error term small. We claim that if

\[
\log 2\mu_j = H(\zeta_j, \zeta_j) + \sum_{i \neq j} G(\zeta_i, \zeta_j),
\]

(24)
then we achieve the following behavior for $R$: for any $0 < \varepsilon < 1$ there exists $C$ independent of $\varepsilon$, such that

$$|R(y)| \leq C \varepsilon^2 \sum_{j=1}^{m} \frac{1}{1 + |y - \xi_j'|} \quad \forall y \in \Omega_e$$

and for $W = e^V$

$$W(y) = \sum_{j=1}^{m} \frac{2\mu_j}{|y - \xi_j' - \mu_j v(\xi_j')|^2} (1 + \theta(\varepsilon y) \quad \forall y \in \Omega_e$$

with $\theta(\varepsilon)$ satisfying the following estimate:

$$|\theta(\varepsilon y)| \leq C \varepsilon^2 + C \sum_{j=1}^{m} |y - \xi_j'| \quad \forall y \in \Omega_e.$$

**Proof of (26).**

$$W(y) = \varepsilon^2 \exp \left( \sum_{i=1}^{m} u_i(\varepsilon y) + H_i^{\varepsilon}(\varepsilon y) \right)$$

$$= \varepsilon^2 \exp \left( \sum_{i=1}^{m} \left( \log \frac{2\mu_i}{\varepsilon^2 |y - \xi_i' - \mu_j v(\xi_j')|^2} + H_i^{\varepsilon}(\varepsilon y) \right) \right).$$

Let us fix a small constant $\delta > 0$ and consider this expression for $|y - \xi_j'| < \delta$.

$$W(y) = \frac{2\mu_j}{|y - \xi_j' - \mu_j v(\xi_j')|^2} \exp \left( H_j^{\varepsilon}(\varepsilon y) + \sum_{i \neq j} \left[ \log \frac{2\mu_i}{\varepsilon^2 |y - \xi_i' - \mu_i v(\xi_i')|^2} + H_i^{\varepsilon}(\varepsilon y) \right] \right).$$

Using (18) and the fact that $H$ is $C^1(\tilde{c}\Omega^2)$ we have

$$H_i^{\varepsilon}(\varepsilon y) = H(\varepsilon y, \xi_i) - \log(2\mu_i) + O(\varepsilon^2) \quad \forall y \in \Omega_e$$

$$= H(\xi_i, \xi_j) - \log(2\mu_i) + O(\varepsilon^2) + O(\varepsilon |y - \xi_j'|) \quad \forall y \in \Omega_e.$$
Hence for \(|y - \zeta'_j| < \frac{\delta}{\varepsilon}\)

\[
H_j^\varepsilon(\varepsilon y) + \sum_{i \neq j}^m \left( \log \frac{2\mu_i}{\varepsilon^2 |y - \zeta'_i - \mu_i v(\zeta'_i)|^2} + H_i^\varepsilon(\varepsilon y) \right)
\]

\[
= H(\zeta_j, \zeta_j) - \log(2\mu_j) + \sum_{i \neq j}^m \left( \log \frac{2\mu_i}{|\zeta_j - \zeta_i|^2} + H(\zeta_j, \zeta_i) - \log(2\mu_i) \right)
\]

\[
+ O(\varepsilon^2) + O(\varepsilon |y - \zeta'_j|),
\]

\[
= H(\zeta_j, \zeta_j) - \log(2\mu_j) + \sum_{i \neq j}^m G(\zeta_j, \zeta_i) + O(\varepsilon^2) + O(\varepsilon |y - \zeta'_j|)
\]

\[
= O(\varepsilon^2) + O(\varepsilon |y - \zeta'_j|)
\]

by the choice of \(\mu_j\), cf. (24). Therefore

\[
W(y) = \frac{2\mu_j}{|y - \zeta'_j - \mu_j v(\zeta'_j)|^2} (1 + O(\varepsilon^2) + O(\varepsilon |y - \zeta'_j|)) \quad \forall |y - \zeta'_j| < \frac{\delta}{\varepsilon}. \quad (27)
\]

If \(|y - \zeta'_j| > \frac{\delta}{\varepsilon}\) for all \(j = 1, \ldots, m\) we have \(W = O(\varepsilon^2)\), and this together with (27) implies (26). \(\square\)

**Proof of (25).** We defined \(R = e^V - \frac{\partial V}{\partial y} \frac{\partial V}{\partial y} \) with \(V\) given by (20). We need to compute \(\frac{\partial V}{\partial y} = e \frac{\partial U}{\partial y}\). But

\[
\frac{\partial U}{\partial y} = \sum_{i=1}^m \frac{\partial u_i}{\partial y} + \frac{\partial H_i^\varepsilon}{\partial y} = \varepsilon \sum_{i=1}^m e^{u_i} = \varepsilon \sum_{i=1}^m \frac{2\mu_i}{|x - \xi_i - \varepsilon \mu_i v(\xi_i)|^2}.
\]

Hence

\[
\frac{\partial V}{\partial y} (y) = \varepsilon \frac{\partial U}{\partial y} (\varepsilon y) = \sum_{i=1}^m \frac{2\mu_i}{|y - \zeta'_i - \mu_i v(\zeta'_i)|^2}.
\]

Thus, near \(\zeta'_j\) by the above computation and (27) we obtain

\[
R(y) = e^V - \frac{\partial V}{\partial y} = \frac{2\mu_j}{|y - \zeta'_j - \mu_j v(\zeta'_j)|^2} (O(\varepsilon^2) + O(\varepsilon |y - \zeta'_j|)), \quad |y - \zeta'_j| < \frac{\delta}{\varepsilon}.
\]
If $|y - \xi_j| > \frac{\delta}{\varepsilon}$ for all $j = 1, \ldots, m$ then $e^V = O(\varepsilon^2)$ and $\frac{\partial V}{\partial v} = O(\varepsilon^2)$ and (25) follows. □

**Proof of Lemma 3.1.** The boundary condition satisfied by $H^\varepsilon_j$ is

$$\frac{\partial H^\varepsilon_j}{\partial v} = \varepsilon \mu_j - \frac{\partial u_j}{\partial v} = 2 \varepsilon \mu_j + \frac{(x - \xi_j - \varepsilon \mu_j v(\xi_j)) \cdot v(x)}{|x - \xi_j - \varepsilon \mu_j v(\xi_j)|^2}$$

$$= 2 \varepsilon \mu_j \frac{1 - v(\xi_j) \cdot v(x)}{|x - \xi_j - \varepsilon \mu_j v(\xi_j)|^2} + 2 \frac{(x - \xi_j) \cdot v(x)}{|x - \xi_j - \varepsilon \mu_j v(\xi_j)|^2}.$$ 

Thus

$$\lim_\varepsilon \to 0 \frac{\partial H^\varepsilon_j}{\partial v} (x) = \frac{2(x - \xi_j) \cdot v(x)}{|x - \xi_j|^2} \quad \forall x \neq \xi_j. \quad (28)$$

The regular part of Green’s function satisfies

$$\begin{cases}
-\Delta_x H(x, y) + H(x, y) = -\log \frac{1}{|x - y|^2}, & x \in \Omega, \\
\frac{\partial H}{\partial v_x} (x, y) = 2 \frac{(x - y) \cdot v(x)}{|x - y|^2}, & x \in \partial \Omega.
\end{cases}$$

For the difference $z^\varepsilon(x) = H^\varepsilon_j(x) + \log 2 \mu_j - H(x, \xi_j)$ we have

$$\begin{cases}
-\Delta_z z^\varepsilon + z^\varepsilon = -\log \frac{1}{|x - \xi_j - \varepsilon \mu_j v(\xi_j)|^2} + \log \frac{1}{|x - \xi_j|^2} & \text{in } \Omega, \\
\frac{\partial z^\varepsilon_x}{\partial v} = \frac{\partial H^\varepsilon_j}{\partial v} - 2 \frac{(x - y) \cdot v(x)}{|x - y|^2} & \text{on } \partial \Omega.
\end{cases}$$

We claim that for any $p > 1$ there exists $C > 0$ such that

$$\left\| \frac{\partial H^\varepsilon_j}{\partial v} - 2 \frac{(x - \xi_j) \cdot v(x)}{|x - \xi_j|^2} \right\|_{L^p(\partial \Omega)} \leq C \varepsilon^{1/p}. \quad (29)$$

For this it will be convenient to observe first that

$$|1 - v(\xi_j) \cdot v(x)| \leq C |x - \xi_j|^2, \quad |(x - \xi_j) \cdot v(x)| \leq C |x - \xi_j|^2 \quad \forall x \in \partial \Omega, \quad (30)$$
which can be proved, for example, assuming that \( \zeta_j = 0 \) and that near the origin \( \partial\Omega \) is the graph of a function \( G : (-a, a) \to \mathbb{R} \) with \( G(0) = G'(0) = 0 \). Now

\[
\frac{\partial H^\varepsilon_j}{\partial v} - 2 \frac{(x - \zeta_j) \cdot v(x)}{|x - \zeta_j|^2} = 2\varepsilon \mu_j \frac{1 - v(\zeta_j) \cdot v(x)}{|x - \zeta_j - \varepsilon \mu_j v(\zeta_j)|^2} + 2\varepsilon \mu_j \frac{(x - \zeta_j) \cdot v(x)(2(x - \zeta_j) \cdot v(\zeta_j) - \varepsilon \mu_j)}{|x - \zeta_j|^2|x - \zeta_j - \varepsilon \mu_j v(\zeta_j)|^2}.
\]

By (30)

\[
\left| \frac{\partial H^\varepsilon_j}{\partial v} - 2 \frac{(x - \zeta_j) \cdot v(x)}{|x - \zeta_j|^2} \right| \leq C\varepsilon + C\varepsilon \frac{|2(x - \zeta_j) \cdot v(\zeta_j) - \varepsilon \mu_j|}{|x - \zeta_j - \varepsilon \mu_j v(\zeta_j)|^2}.
\]

(31)

Fix \( \rho > 0 \) small. Then

\[
\left| \frac{\partial H^\varepsilon_j}{\partial v} - 2 \frac{(x - \zeta_j) \cdot v(x)}{|x - \zeta_j|^2} \right| \leq C\varepsilon \quad \forall |x - \zeta_j| \geq \rho, \ x \in \partial\Omega.
\]

(32)

Now let \( p > 1 \). Changing variables \( x - \zeta_j = \varepsilon y \) we have

\[
\int_{B_p(\zeta_j) \cap \partial\Omega} \left| \frac{\varepsilon|2(x - \zeta_j) \cdot v(\zeta_j) - \varepsilon \mu_j|}{|x - \zeta_j - \varepsilon \mu_j v(\zeta_j)|^2} \right|^p \, dx = C\varepsilon \int_{B_p(0) \cap \partial\Omega} \left| \frac{2y \cdot v(0) - \mu_j}{|y - \mu_j v(0)|^2} \right|^p \, dy
\]

\[
\leq C\varepsilon \int_0^{p/e} \frac{1}{(1 + s)^p} \, ds
\]

\[
\leq C\varepsilon.
\]

Combining this with (31) and (32) we conclude that (29) holds.

For \( p > 1 \) let us estimate now

\[
\left\| \log \frac{1}{|x - \zeta_j|^2} - \log \frac{1}{|x - \zeta_j - \varepsilon \mu_j v(\zeta_j)|^2} \right\|^p_{L^p(\Omega)} = \int_{B_{10\varepsilon \mu_j}(\zeta_j) \cap \Omega} \cdots + \int_{\Omega \setminus B_{10\varepsilon \mu_j}(\zeta_j)} \cdots = I_1 + I_2.
\]

For \( I_1 \) observe that

\[
\int_{B_{10\varepsilon \mu_j}(\zeta_j) \cap \Omega} \left| \log \frac{1}{|x - \zeta_j|^2} \right|^p \, dx \leq C \int_0^{10\varepsilon} |\log r|^p r \, dr \leq C\varepsilon^2 \left( \log \frac{1}{\varepsilon} \right)^p.
\]
The same bound is true for the integral of \( |\log \frac{1}{|x - \xi_j|^2} - \log \frac{1}{|x - \xi_j - \epsilon \mu_j \nu(\xi_j)|^2}| \) in \( B_{10 \epsilon \mu_j}(\xi_j) \cap \Omega \). Hence

\[
|I_1| \leq C \epsilon^2 \left( \log \frac{1}{\epsilon} \right)^p.
\]

Let us estimate \( I_2 \) as follows:

\[
\left| \log \frac{1}{|x - \xi_j|^2} - \log \frac{1}{|x - \xi_j - \epsilon \mu_j \nu(\xi_j)|^2} \right| \leq \sup_{0 \leq t \leq 1} \frac{C \epsilon}{|x - \xi_j - t \epsilon \mu_j \nu(\xi_j)|}.
\]

But if \( |x - \xi_j| \geq 10 \epsilon \mu_j \) then \( |x - \xi_j| \leq C |x - \xi_j - t \epsilon \mu_j \nu(\xi_j)| \) for any \( t \in [0, 1] \) as can be seen from \( |x - \xi_j| \leq |x - \xi_j - t \epsilon \mu_j \nu(\xi_j)| + \mu_j \epsilon \leq |x - \xi_j - t \epsilon \mu_j \nu(\xi_j)| + \frac{1}{10} |x - \xi_j| \). Thus

\[
\left| \log \frac{1}{|x - \xi_j|^2} - \log \frac{1}{|x - \xi_j - \epsilon \mu_j \nu(\xi_j)|^2} \right| \leq \frac{C \epsilon}{|x - \xi_j|}.
\]

Take \( 1 < p < 2 \) and integrate

\[
|I_2| \leq C \epsilon^p \int_{10 \epsilon \mu_j}^D r^{1-p} \, dr \leq C \epsilon^p,
\]

where \( D \) is the diameter of \( \Omega \). In conclusion, for any \( 1 < p < 2 \) we have

\[
\left\| \log \frac{1}{|x - \xi_j|^2} - \log \frac{1}{|x - \xi_j - \epsilon \mu_j \nu(\xi_j)|^2} \right\|_{L^p(\Omega)} \leq C \epsilon.
\]

By \( L^p \) theory

\[
\| z_\epsilon \|_{W^{1+s, p}(\Omega)} \leq C \left( \| \frac{\partial z_\epsilon}{\partial y} \|_{L^p(\hat{\Omega})} + \| \Delta z_\epsilon \|_{L^p(\Omega)} \right) \leq C \epsilon^{1/p}
\]

for any \( 0 < s < \frac{1}{p} \). By the Morrey embedding we obtain

\[
\| z_\epsilon \|_{C^{\gamma}(\Omega)} \leq C \epsilon^{1/p}
\]

for any \( 0 < \gamma < \frac{1}{2} + \frac{1}{p} \). This proves the result (with \( z = \frac{1}{p} \)). □
Remark. The convergence (28) is not uniform in general because \( \frac{\partial H^\varepsilon}{\partial v}(\xi_j) = 0 \) while the function \( x \mapsto 2 \frac{(x-\xi_j) \cdot v(x)}{|x-\xi_j|^2} \) can be extended continuously to \( \xi_j \) with a value equal to the curvature of \( \partial \Omega \) at \( \xi_j \).

4. Solvability of a linear equation

The main result of this section is the solvability of the following linear problem: given \( h \) find \( \phi, c_1, \ldots, c_m \), such that

\[
\begin{cases}
-\Delta \phi + \varepsilon^2 \phi = 0 & \text{in } \Omega_\varepsilon, \\
\frac{\partial \phi}{\partial v} - W \phi = h + \sum_{j=1}^m c_j \chi_j Z_{1j} & \text{on } \partial \Omega_\varepsilon, \\
\int_{\Omega_\varepsilon} \chi_j Z_{1j} \phi = 0 & \forall j = 1, \ldots, m,
\end{cases}
\]

where \( W \) is a function on \( \partial \Omega_\varepsilon \) that satisfies (26), \( h \in L^\infty(\partial \Omega_\varepsilon) \) and \( Z_{1j}, \chi_j \) are defined as follows: let \( z_{ij} \) denote the functions \( z_0, z_1 \) defined in (12) and (13) with parameter \( \mu = \mu_j \) \((i = 0, 1; j = 1, \ldots, m)\)

\[
z_{0j} = \frac{1}{\mu_j} - \frac{2}{x_1^2 + (x_2 + \mu_j)^2}, \quad z_{1j} = -\frac{2}{x_1^2 + (x_2 + \mu_j)^2}.
\]

Around each point \( \xi_j' \in \partial \Omega_\varepsilon \) we consider a smooth change of variables

\[
F^\varepsilon_j(y) = \frac{1}{\varepsilon} F_j(\varepsilon y),
\]

where \( F_j : B_\rho(\xi_j) \to M \) is a diffeomorphism and \( M \) an open neighborhood of the origin such that \( F(\Omega \cap B_\rho(\xi_j)) = \mathbb{R}^2_+ \cap M \), \( F(\partial \Omega \cap B_\rho(\xi_j)) = \partial \mathbb{R}^2_+ \cap M \). We can select \( F_j \) so that it preserves area. Define

\[
Z_{ij}(y) = z_{ij}(F^\varepsilon_j(y)), \quad i = 0, 1; j = 1, \ldots, m.
\]

Next, we choose a large but fixed number \( R_0 \) and nonnegative smooth function \( \chi : \mathbb{R} \to \mathbb{R} \) so that \( \chi(r) = 1 \) for \( r \leq R_0 \) and \( \chi(r) = 0 \) for \( r \geq R_0 + 1 \), \( 0 \leq \chi \leq 1 \). Then set

\[
\chi_j(y) = \chi(|F^\varepsilon_j(y)|).
\]

All functions above depend on \( \varepsilon \) but we omit this dependence in the notation.
Eq. (33) will be solved for $h \in L^\infty(\partial \Omega_\varepsilon)$, but we will be able to estimate the size of the solution in terms of the following norm:

$$\|h\|_{*,\partial \Omega_\varepsilon} = \sup_{y \in \partial \Omega_\varepsilon} \frac{|h(y)|}{\varepsilon + \sum_{j=1}^{m} (1 + |y - \xi_j'|)^{-1-\sigma}},$$

where we fix $0 < \sigma < 1$ although the precise choice will be made later on.

**Proposition 4.1.** Let $d > 0$ and $m$ a positive integer. Then there exist $\varepsilon_0 > 0$, $C$ such that for any $0 < \varepsilon < \varepsilon_0$, any family of points $\xi_1, \ldots, \xi_m \in \partial \Omega$ with

$$|\xi_i - \xi_j| \geq d \quad \forall i \neq j$$

and any $h \in L^\infty(\partial \Omega_\varepsilon)$ there is a unique solution $\phi \in L^\infty(\Omega_\varepsilon)$, $c_1, \ldots, c_m \in \mathbb{R}$ to (33). Moreover

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C \log \frac{1}{\varepsilon} \|h\|_{*,\partial \Omega_\varepsilon}.$$

To prove this result we shall study first the linear equation

$$\begin{cases}
-\Delta \phi + \varepsilon^2 \phi = f & \text{in } \Omega_\varepsilon, \\
\frac{\partial \phi}{\partial v} - W \phi = h & \text{on } \partial \Omega_\varepsilon,
\end{cases}$$

where $W$ satisfies (26) and $f, h$ are in suitable weighted spaces: we consider for $h$ the norm defined in (35) and for $f$

$$\|f\|_{**,\Omega_\varepsilon} = \sup_{y \in \Omega_\varepsilon} \frac{|f(y)|}{\varepsilon^2 + \sum_{j=1}^{m} (1 + |y - \xi_j'|)^{-2-\sigma}}.$$

We begin by stating an a priori estimate for solutions of (37) satisfying orthogonality conditions with respect to $Z_{0j}$ and $Z_{1j}$.

**Lemma 4.2.** There are $R_0 > 0$ and $\varepsilon_0 > 0$ so that for $0 < \varepsilon < \varepsilon_0$ and any solution $\phi$ of (37) with the orthogonality conditions

$$\int_{\Omega_\varepsilon} Z_{ij} \phi = 0 \quad \forall i = 0, 1 \quad \forall j = 1, \ldots, m,$$

we have

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C(\|h\|_{*,\partial \Omega_\varepsilon} + \|f\|_{**,\Omega_\varepsilon}),$$

where $C$ is independent of $\varepsilon$. 

The idea behind this estimate comes from looking at (37) with \( f \equiv 0, h \equiv 0 \) as \( \varepsilon \to 0 \) at a fixed distance from one of the points, say \( \zeta_j' \). After a translation and a rotation so that \( \Omega_\varepsilon \) converges to the upper half-plane \( \mathbb{R}^2_+ \) and \( \zeta_j' \) is located at the origin this equation approaches precisely (14).

For the proof of this lemma we need to construct a suitable barrier.

**Lemma 4.3.** For \( \varepsilon > 0 \) small enough there exist \( R_1 > 0 \), and

\[
\psi : \Omega_\varepsilon \left\{ \bigcup_{j=1}^{m} B_{R_1}(\zeta_j') \to \mathbb{R} \right.
\]

smooth and positive so that

\[
-\Delta \psi + \varepsilon^2 \psi \geq \sum_{j=1}^{m} \frac{1}{|y - \zeta_j'|^{2+\sigma}} + \varepsilon^2 \quad \text{in} \quad \Omega_\varepsilon \left\{ \bigcup_{j=1}^{m} B_{R_1}(\zeta_j') \right.
\]

\[
\frac{\partial \psi}{\partial v} - W \psi \geq \sum_{j=1}^{m} \frac{1}{|y - \zeta_j'|^{1+\sigma}} + \varepsilon \quad \text{on} \quad \partial \Omega_\varepsilon \left\{ \bigcup_{j=1}^{m} B_{R_1}(\zeta_j') \right.
\]

\[
\psi > 0 \quad \text{in} \quad \Omega_\varepsilon \left\{ \bigcup_{j=1}^{m} B_{R_1}(\zeta_j') \right.
\]

\[
\psi \geq 1 \quad \text{on} \quad \Omega_\varepsilon \cap \left( \bigcup_{j=1}^{m} \partial B_{R_1}(\zeta_j') \right).
\]

The constants \( R_1 > 0, c > 0 \) can be chosen independently of \( \varepsilon \) and \( \psi \) is bounded uniformly

\[
0 < \psi \leq C \quad \text{in} \quad \Omega_\varepsilon \left\{ \bigcup_{j=1}^{m} B_{R_1}(\zeta_j') \right.
\]

**Proof of Lemma 4.2.** We take \( R_0 = 2R_1 \), \( R_1 \) being the constant of Lemma 4.3. Thanks to the barrier \( \psi \) of that lemma we deduce that the following maximum principle holds
in $\Omega_\varepsilon \setminus \bigcup_{j=1}^{m} B_{R_1}(\xi'_j)$: if $\phi \in H^1(\Omega_\varepsilon \setminus \bigcup_{j=1}^{m} B_{R_1}(\xi'_j))$ satisfies:

$$
\begin{cases}
-\Delta \phi + \varepsilon^2 \phi \geq 0 & \text{in } \Omega_\varepsilon \\
\frac{\partial \phi}{\partial v} - W \phi \geq 0 & \text{on } \partial \Omega_\varepsilon \\
\phi \geq 0 & \text{on } \Omega_\varepsilon \cap \left( \bigcup_{j=1}^{m} \partial B_{R_1}(\xi'_j) \right)
\end{cases}
$$

then $\phi \geq 0$ in $\Omega_\varepsilon \setminus \bigcup_{j=1}^{m} B_{R_1}(\xi'_j)$.

Let $f$, $h$ be bounded and $\phi$ a solution to (37) satisfying (38). Following [4] we first claim that $\|\phi\|_{L^\infty(\Omega_\varepsilon)}$ can be controlled in terms of $\|f\|_{**,\Omega_\varepsilon}$, $\|h\|_{*,\partial \Omega_\varepsilon}$ and the following inner norm of $\phi$:

$$
\|\phi\|_i = \sup_{\Omega_\varepsilon \cap \left( \bigcup_{j=1}^{m} B_{R_1}(\xi'_j) \right)} |\phi|.
$$

Indeed, set

$$
\tilde{\phi} = C_1 \psi \left( \|\phi\|_i + \|f\|_{**,\Omega_\varepsilon} + \|h\|_{*,\partial \Omega_\varepsilon} \right)
$$

with $C_1$ a constant independent of $\varepsilon$. By the above maximum principle we have $\phi \leq \tilde{\phi}$ and $-\phi \leq \tilde{\phi}$ in $\Omega_\varepsilon \setminus \bigcup_{j=1}^{m} B_{R_1}(\xi'_j)$. Since $\psi$ is uniformly bounded we deduce

$$
\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C \left( \|\phi\|_i + \|f\|_{**,\Omega_\varepsilon} + \|h\|_{*,\partial \Omega_\varepsilon} \right)
$$

(39)

for some constant $C$ independent of $\phi$ and $\varepsilon$.

We prove the lemma by contradiction. Assume that there exist a sequence $\varepsilon_n \to 0$, points $\xi''_1, \ldots, \xi''_m$ on $\partial \Omega$ satisfying (36) and functions $\phi_n$, $f_n$ and $h_n$ with $\|\phi_n\|_{L^\infty(\Omega_{\varepsilon_n})} = 1$, $\|f_n\|_{**,\Omega_{\varepsilon_n}} \to 0$, $\|h_n\|_{*,\partial \Omega_{\varepsilon_n}} \to 0$ so that for each $n$ $\phi_n$ solves (37) and satisfies (38). By (39) we see that $\|\phi_n\|_i$ stays away from zero. For one of the indices, say $j$, we can assume that $\sup_{B_{R_1}(\xi'_j)} |\phi_n| \geq c > 0$ for all $n$. Consider $\hat{\phi}_n(z) = \phi_n(z - \xi'_j)$ and let us translate and rotate $\Omega_{\varepsilon_n}$ so that $\Omega_{\varepsilon_n}$ approaches the upper half-plane $\mathbb{R}^2_+$ and $\xi'_j = 0$. Then by elliptic estimates $\hat{\phi}_n$ converges uniformly on compact sets to a nontrivial solution of (14). By Proposition 2.1 $\hat{\phi}$ is a linear combination of $z_{0j}$ and $z_{1j}$. On the other hand, we can take the limit in the orthogonality relations (38), observing that limits of the functions $Z_{ij}$ are just rotations and translations of $z_{ij}$, and we find $\int_{\mathbb{R}^2_+} \hat{\phi} z_{ij} = 0$ for $i = 1, 2$. This contradicts the fact that $\hat{\phi} \neq 0$. □
Proof of Lemma 4.3. We take

$$\psi_{1j}(y) = \frac{(y - \xi_j') \cdot v(\xi_j')}{r^{1+\sigma}},$$

where $r = |y - \xi_j' - \mu_j v(\xi_j')|$. A computation shows that

$$\Delta \psi_{1j} = O(r^{-2-\sigma}) \quad \text{in } \Omega_\delta$$

(40)

and if $\delta > 0$ is small but fixed and $R_1 > 0$ is large and fixed then

$$\frac{\partial \psi_{1j}}{\partial v} \geq cr^{-1-\sigma} \quad \text{for } R_1 < r < \delta/\varepsilon,$$

where $c > 0$ is fixed. To prove the last assertion we may suppose that $\xi_j'$ is at the origin and assume that the normal vector at $\xi_j'$ is $(0, -1)$. Hence

$$\psi_{1j}(y) = -\frac{y_2}{r^{1+\sigma}}.$$

Let us write $\partial \Omega_\delta$ near $\xi_j'$ as the graph $\{(y_1, y_2) : y_2 = G_\delta(y_1)\}$ with $G_\delta(y_1) = \frac{1}{\varepsilon} G(\varepsilon y_1)$ and $G$ a smooth function such that $G(0) = 0$ and $G'(0) = 0$. Fix $\delta > 0$ small. Then for $R_1 < r < \delta/\varepsilon$ we have that $r$ is comparable with $y_1$, $G_\delta'(y_1) = O(\varepsilon r)$ and $G_\delta(y_1) = O(\varepsilon r^2)$. Then

$$\frac{\partial \psi_{1j}}{\partial v} = \frac{1}{\sqrt{G'(\varepsilon y_1)^2 + 1}} \left( -(1 + \sigma) \frac{G_\delta(y_1)G_\delta'(y_1)y_1}{r^{3+\sigma}} - \frac{1}{r^{1+\sigma}} ight)
\begin{align*}
&\quad + (1 + \sigma) \frac{G_\delta(y_1)^2}{r^{3+\sigma}} + \mu_j (1 + \sigma) \frac{G_\delta(y_1)}{r^{3+\sigma}} \\
&= \frac{1}{\sqrt{O(\varepsilon^2 r^4) + 1}} \left( -\frac{1}{r^{1+\sigma}} + \frac{O(\varepsilon^2 r^4)}{r^{3+\sigma}} + \frac{O(\delta)}{r^{2+\sigma}} \right) \quad \text{for } R_1 < r < \delta/\varepsilon
\end{align*}

$$= \frac{1}{\sqrt{O(\varepsilon^2) + 1}} \left( -\frac{1}{r^{1+\sigma}} + \frac{O(\delta^2)}{r^{1+\sigma}} + \frac{O(\delta)}{r^{2+\sigma}} \right) \quad \text{for } R_1 < r < \delta/\varepsilon,$$

from where the claim follows by taking $\delta$ small enough.

Consider also

$$\psi_{2j}(r) = 1 - \frac{1}{r^\sigma}.$$
Then
\[-\Delta \psi_{2j} = \sigma^2 \frac{1}{r^{2+\sigma}} \]  
(41)

and proceeding analogously as for \(\psi_{1j}\) we find
\[
\frac{\partial \psi_{2j}}{\partial v} = \frac{\sigma}{r^{2+\sigma}} \frac{1}{\sqrt{G'(\varepsilon y_1)^2 + 1}} \left( (-y_1 G'_e(y_1), G_e(y_1) + \mu_j) \right) \\
= \frac{\sigma}{r^{2+\sigma}} \frac{1}{\sqrt{O(\varepsilon r^2)}} O(\varepsilon r^2) \quad \forall R_1 < r < \delta / \varepsilon \\
= O \left( \frac{\varepsilon}{r^{\sigma}} \right) \quad \forall R_1 < r < \delta / \varepsilon.
\]

Now let

\[ \psi_{3j} = \psi_1 + C \psi_{2j}. \]

For \(C\) large enough (but independent of \(\varepsilon\)) using (40) and (41) we have

\[-\Delta \psi_{3j} + \varepsilon^2 \psi_{3j} \gtrless \sigma^2 \frac{1}{r^{2+\sigma}} \quad \forall R_1 < |y - \xi_j'| < \delta / \varepsilon. \]

(42)

Now recall that \(W\) satisfies (26) and therefore

\[ W(y) = O \left( \frac{1}{r^2} \right) \quad \forall R_1 < r < \frac{\delta}{\varepsilon}. \]

Thus

\[
\frac{\partial \psi_{3j}}{\partial v} - W \psi_{3j} \gtrless \frac{c}{r^{1+\sigma}} - C \frac{1}{r^2} \geq \frac{c'}{r^{1+\sigma}} \quad \text{for } R_1 < r < \delta / \varepsilon
\]

(43)

with a constant \(c' > 0\) if we choose \(R_1\) larger if necessary.

Let \(\eta_j \in C^\infty_0(\mathbb{R}^2)\) be such that \(0 \leq \eta_j \leq 1, \eta_j \equiv 1\) in \(\Omega_\varepsilon \cap B_{\delta/(2\varepsilon)}(\xi_j'), \eta_j \equiv 0\) in \(\Omega_\varepsilon \setminus B_{\delta/(\varepsilon)}(\xi_j'), |\nabla \eta_j| \leq C\varepsilon\) in \(\Omega_\varepsilon, |\Delta \eta_j| \leq C\varepsilon^2\) in \(\Omega_\varepsilon\). Let \(\psi_0(y) = \tilde{\psi}(\varepsilon y)\) where \(\tilde{\psi}\) is the solution to

\[
\begin{cases}
-\Delta \tilde{\psi} + \tilde{\psi} = 1 & \text{in } \Omega, \\
\frac{\partial \tilde{\psi}}{\partial v} = 1 & \text{on } \partial \Omega,
\end{cases}
\]

Let \(\eta_j \equiv 1\) in \(\Omega_\varepsilon \setminus B_{\delta/(\varepsilon)}(\xi_j'), |\nabla \eta_j| \leq C\varepsilon\) in \(\Omega_\varepsilon, |\Delta \eta_j| \leq C\varepsilon^2\) in \(\Omega_\varepsilon\). Let \(\psi_0(y) = \tilde{\psi}(\varepsilon y)\) where \(\tilde{\psi}\) is the solution to

\[
\begin{cases}
-\Delta \tilde{\psi} + \tilde{\psi} = 1 & \text{in } \Omega, \\
\frac{\partial \tilde{\psi}}{\partial v} = 1 & \text{on } \partial \Omega,
\end{cases}
\]

with a constant \(c' > 0\) if we choose \(R_1\) larger if necessary.
so that \(-\Delta \psi_0 + \varepsilon^2 \psi_0 = \varepsilon^2\) in \(\Omega_\varepsilon\) and \(\frac{\partial \psi_0}{\partial v} = \varepsilon\) on \(\partial \Omega_\varepsilon\). In particular, \(\psi_0\) is uniformly bounded in \(\Omega_\varepsilon\). The function

\[
\psi = \sum_{j=1}^{m} \eta_j \psi_{3j} + C \psi_0
\]

with \(C\) a sufficiently large constant meets the requirements. Indeed by (42)

\[
-\Delta \psi + \varepsilon^2 \psi \geq \sum_{j=1}^{m} \left( -\Delta \eta_j \psi_{3j} + 2\nabla \eta_j \nabla \psi_{3j} + \eta_j \frac{\sigma^2}{r_j^{2+\sigma}} \right) + C \varepsilon^2,
\]

where \(r_j = |y - \zeta_j'|\) and hence

\[
-\Delta \psi + \varepsilon^2 \psi \geq \frac{\sigma^2}{r_j^{2+\sigma}} + C \varepsilon^2, \quad R_1 < r_j < \frac{\delta}{2 \varepsilon}.
\]

By construction we have \(|\nabla \psi_{3j}| = O\left(\frac{1}{r_j^{1+\sigma}}\right)\) and hence, choosing \(C\) large we have

\[
-\Delta \psi + \varepsilon^2 \psi \geq O(\varepsilon^2) + O\left(\frac{1}{r_j^{1+\sigma}}\right) + C \varepsilon^2 \geq c \varepsilon^2, \quad \frac{\delta}{2 \varepsilon} < r_j < \frac{\delta}{\varepsilon}
\]

if \(\varepsilon\) is small enough, and also

\[
-\Delta \psi + \varepsilon^2 \psi \geq c \frac{1}{r_j^{2+\sigma}}, \quad \frac{\delta}{2 \varepsilon} < r_j < \frac{\delta}{\varepsilon}.
\]

Finally, a similar argument using (43) yields

\[
\frac{\partial \psi}{\partial v} - W \psi \geq c \frac{1}{r_j^{1+\sigma}} + c \varepsilon, \quad R_1 < r_j < \frac{\delta}{\varepsilon}
\]

for all \(j = 1, \ldots, m\). \(\square\)

We will establish next an a priori estimate for solutions to problem (37) that satisfy orthogonality conditions with respect to \(Z_{1j}\) only.

**Lemma 4.4.** For \(\varepsilon\) sufficiently small, if \(\phi\) solves

\[
\begin{cases}
-\Delta \phi + \varepsilon^2 \phi = f & \text{in } \Omega_\varepsilon, \\
\frac{\partial \phi}{\partial v} - W \phi = h & \text{on } \partial \Omega_\varepsilon
\end{cases}
\]

(44)
and satisfies

\[
\int_{\Omega_\varepsilon} Z_{1j} \chi_j \phi = 0 \quad \forall j = 1, \ldots, m, \tag{45}
\]

then

\[
\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C \log \frac{1}{\varepsilon} (\|h\|_{*,\Omega_\varepsilon} + \|f\|_{*,\Omega_\varepsilon}), \tag{46}
\]

where C is independent of \(\varepsilon\).

**Proof.** Let \(\phi\) satisfy (44) and (45). We will modify \(\phi\) to satisfy all orthogonality relations in (38) and for this purpose we consider modifications with compact support of the functions \(Z_{0j}\). Let \(R > R_0 + 1\) be large and fixed. Set

\[
\widehat{Z}_{0j}(y) = \psi Z_{0j}(y),
\]

where

\[
\psi(y) = \tilde{h}(|F^\varepsilon_j(y)|), \quad \tilde{h}(x) = \frac{\log(\delta/\varepsilon) - \log|x|}{\log(\delta/\varepsilon) - \log R}
\]

and \(F^\varepsilon_j\) is the change of variables defined in (34). Here \(\delta > 0\) is a small fixed constant. Note that \(\tilde{h}\) is just the solution to

\[
\begin{cases}
\Delta \tilde{h} = 0 & \text{in } B_{\delta/\varepsilon}(0) \setminus B_R(0), \\
\tilde{h} = 1 & |x| = R, \\
\tilde{h} = 0 & |x| = \delta/\varepsilon.
\end{cases}
\]

Let \(\tilde{\eta}_{1j}, \tilde{\eta}_{2j}\) be radial smooth cut-off functions on \(\mathbb{R}^2\) so that

\[
0 \leq \tilde{\eta}_{1j} \leq 1, \quad |\nabla \tilde{\eta}_{1j}| \leq C \text{ in } \mathbb{R}^2, \\
\tilde{\eta}_{1j} \equiv 1 \text{ in } B_R(0), \quad \tilde{\eta}_{1j} \equiv 0 \text{ in } \mathbb{R}^2 \setminus B_{R+1}(0)
\]

and

\[
\tilde{\eta}_{2j} \equiv 1 \text{ in } B_{\delta\varepsilon}(0), \quad \tilde{\eta}_{2j} \equiv 0 \text{ in } \mathbb{R}^2 \setminus B_{\delta\varepsilon}(0), \\
0 \leq \tilde{\eta}_{2j} \leq 1, \quad |\nabla \tilde{\eta}_{2j}| \leq C\varepsilon/\delta, \quad |\nabla^2 \tilde{\eta}_{2j}| \leq C\varepsilon^2/\delta^2 \text{ in } \mathbb{R}^2.
\]
Write
\[\eta_{1j}(y) = \bar{\eta}_{1j}(F_j^y(y)), \quad \eta_{2j}(x) = \bar{\eta}_{2j}(F_j^x(y)).\] (47)

Now define
\[\widetilde{Z}_{0j} = \eta_{1j}Z_{0j} + (1 - \eta_{1j})\eta_{2j}\widetilde{Z}_{0j}.\]

Given \(\phi\) satisfying (44) and (45) let
\[\widetilde{\phi} = \phi + \sum_{j=1}^{m} d_j\widetilde{Z}_{0j}, \quad \text{where} \quad d_j = -\frac{\int_{\Omega_\varepsilon} Z_{0j}\partial_j \phi}{\int_{\Omega_\varepsilon} \widetilde{Z}_{0j}^2 \partial_j \phi}.\]

Estimate (46) is a direct consequence of

**Claim.**

\[|d_j| \leq C \log \frac{1}{e} \left( \|h\|_{*,\partial\Omega_\varepsilon} + \|f\|_{**,\Omega_\varepsilon} \right) \quad \forall j = 1, \ldots, m.\] (48)

We start proving this by observing, using the notation \(L = -\Delta + \varepsilon^2\), that
\[L(\widetilde{\phi}) = f + \sum_{j=1}^{m} d_j L(\widetilde{Z}_{0j}) \quad \text{in} \quad \Omega_\varepsilon\] (49)

and
\[\left( \frac{\partial}{\partial \nu} - W \right) \widetilde{\phi} = h + \sum_{j=1}^{m} d_j \left( \frac{\partial}{\partial \nu} - W \right) \widetilde{Z}_{0j} \quad \text{on} \quad \partial\Omega_\varepsilon.\] (50)

Thus by Lemma 4.2 we have
\[\|\widetilde{\phi}\|_{L^\infty(\Omega_\varepsilon)} \leq C \sum_{j=1}^{m} |d_j| \left( \left\| \left( \frac{\partial}{\partial \nu} - W \right) \widetilde{Z}_{0j} \right\|_{**,\partial\Omega_\varepsilon} + \|L(\widetilde{Z}_{0j})\|_{**,\Omega_\varepsilon} \right) + C\|h\|_{*,\partial\Omega_\varepsilon} + C\|f\|_{**,\Omega_\varepsilon}.\] (51)
Multiplying Eq. (49) by $\tilde{Z}_{0k}$, integrating by parts and using (50) we find

$$
d_k \left[ \int_{\Omega_{\varepsilon}} L(\tilde{Z}_{0k}) \tilde{Z}_{0k} + \int_{\partial \Omega_{\varepsilon}} \tilde{Z}_{0k} \left( \frac{\partial}{\partial \nu} - W \right) \tilde{Z}_{0k} \right]
= - \int_{\partial \Omega_{\varepsilon}} h \tilde{Z}_{0k} - \int_{\Omega_{\varepsilon}} f \tilde{Z}_{0k} + \int_{\partial \Omega_{\varepsilon}} \tilde{\phi} \left( \frac{\partial}{\partial \nu} - W \right) \tilde{Z}_{0k} + \int_{\Omega_{\varepsilon}} \tilde{\phi} L(\tilde{Z}_{0k}).$$

This combined with (51) yields

$$
d_k \left[ \int_{\Omega_{\varepsilon}} L(\tilde{Z}_{0k}) \tilde{Z}_{0k} + \int_{\partial \Omega_{\varepsilon}} \tilde{Z}_{0k} \left( \frac{\partial}{\partial \nu} - W \right) \tilde{Z}_{0k} \right]
\leq C \|h\|_{*, \partial \Omega_{\varepsilon}} + C \|f\|_{**, \Omega_{\varepsilon}} + \|\tilde{\phi}\|_{L^\infty} \left\| \left( \frac{\partial}{\partial \nu} - W \right) \tilde{Z}_{0k} \right\|_{*, \partial \Omega_{\varepsilon}}
+ \|\tilde{\phi}\|_{L^\infty} \|L(\tilde{Z}_{0k})\|_{**, \Omega_{\varepsilon}}
\leq (C \|h\|_{*, \partial \Omega_{\varepsilon}} + \|f\|_{**, \Omega_{\varepsilon}}) \left( 1 + \left\| \left( \frac{\partial}{\partial \nu} - W \right) \tilde{Z}_{0k} \right\|_{*, \partial \Omega_{\varepsilon}} + \|L(\tilde{Z}_{0k})\|_{**, \Omega_{\varepsilon}} \right)
+ C \sum_{j=1}^{m} |d_j| \left( \left\| \left( \frac{\partial}{\partial \nu} - W \right) \tilde{Z}_{0j} \right\|_{*, \partial \Omega_{\varepsilon}}^2 + \|L(\tilde{Z}_{0j})\|_{**, \Omega_{\varepsilon}}^2 \right).
$$

We will achieve (48) proving the following estimates: for some constant $C > 0$ independent of $\varepsilon$ we have

$$
\int_{\Omega_{\varepsilon}} L(\tilde{Z}_{0j}) \tilde{Z}_{0j} + \int_{\partial \Omega_{\varepsilon}} \tilde{Z}_{0j} \left( \frac{\partial}{\partial \nu} - W \right) \tilde{Z}_{0j} \geq \frac{1}{C \log \frac{1}{\varepsilon}}, \quad (52)
$$

$$
\|L(\tilde{Z}_{0j})\|_{**, \Omega_{\varepsilon}} \leq \frac{C}{\log \frac{1}{\varepsilon}}, \quad (53)
$$

$$
\left\| \left( \frac{\partial}{\partial \nu} - W \right) \tilde{Z}_{0j} \right\|_{*, \partial \Omega_{\varepsilon}} \leq \frac{C}{\log \frac{1}{\varepsilon}}. \quad (54)
$$

Proof of (52). We write

$$
\int_{\Omega_{\varepsilon}} L(\tilde{Z}_{0j}) \tilde{Z}_{0j} = I_0 + I_1 + I_2 + I_3,
$$
where $I_l = \int_{R_l} L(\tilde{Z}_{0j}) \tilde{Z}_{0j}$ and the regions $R_0, \ldots, R_3$ are defined in terms of the change of variables $F^e_j$ defined in (34) as follows. Write $x = (F^e_j)^{-1}(y)$, $r = |x|$ and define the following subsets of $\Omega_\varepsilon$:

\begin{align*}
R_0 &= (F^e_j)^{-1}([r < R] \cap \mathbb{R}^2_+), \quad R_1 = (F^e_j)^{-1}([R < r < R + 1] \cap \mathbb{R}^2_+), \\
R_2 &= (F^e_j)^{-1}\left(\left\{ R + 1 < r < \frac{\delta}{4\varepsilon}\right\} \cap \mathbb{R}^2_+\right), \quad R_3 = (F^e_j)^{-1}\left(\left\{ \frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon}\right\} \cap \mathbb{R}^2_+\right).
\end{align*}

We will prove that

\[ I_1 \geq \frac{\bar{c}}{\log \frac{\delta}{\varepsilon}} \]

with $\bar{c} > 0$ independent of $\varepsilon$, $\delta$ and $R$ while the other terms $I_0$, $I_2$, $I_3$ and $\int_{\partial \Omega_\varepsilon} \tilde{Z}_{0j}(\frac{\partial}{\partial r} - W)\tilde{Z}_{0j}$ can be made small compared to $\frac{1}{\log \frac{\delta}{\varepsilon}}$ by choosing $\delta > 0$ small and $R > 0$ large, but fixed independently of $\varepsilon$.

**Estimate of $I_1$:** We change variables $x = F^e_j(y)$ and recall that this map preserves area, so

\[ I_1 = \int_{\{R < r < R + 1\} \cap \mathbb{R}^2_+} \tilde{L}(\tilde{z}_{0j}) \tilde{z}_{0j}, \]

\[ \tilde{z}_{0j}(x) = \tilde{Z}_{0j}((F^e_j)^{-1}(x)) = \tilde{\eta}_{1j}z_{0j} + (1 - \tilde{\eta}_{1j})\tilde{h}(x)z_{0j} \]  \hspace{1cm} (55)

and $\tilde{L}$ is a linear operator, which thanks to the definition $F^e_j(y) = \frac{1}{\varepsilon} F(\varepsilon y)$, has the expansion

\[ \tilde{L} = -\Delta + O(\varepsilon |x|) \nabla^2 + O(\varepsilon) \nabla + \varepsilon^2. \]  \hspace{1cm} (56)

Therefore

\[ I_1 = -\int_{\{R < r < R + 1\} \cap \mathbb{R}^2_+} \Delta(\tilde{z}_{0j}) \tilde{z}_{0j} + O(R\varepsilon). \]

Using that $z_{0j}$ is harmonic and that in the region $R < r < R + 1$ we have $\tilde{\eta}_{2j} \equiv 1$ we compute

\[ \Delta \tilde{z}_{0j} = \Delta(\tilde{\eta}_{1j}z_{0j} + (1 - \tilde{\eta}_{1j})\tilde{h}z_{0j}) = \Delta\tilde{\eta}_{1j}(1 - \tilde{h})z_{0j} + 2\nabla\tilde{\eta}_{1j}\nabla((1 - \tilde{h})z_{0j}) \]

\[ + (1 - \tilde{\eta}_{1j})\Delta(\tilde{h}z_{0j}), \quad R < r < R + 1. \]
Since $\Delta \tilde{h} = 0$, $\Lambda z_{0j} = 0$ for the last term in the expression above, we have

$$\Delta(\tilde{h}z_{0j}) = 2\nabla \tilde{h} \nabla z_{0j}, \quad R < r < R + 1.$$  

But

$$\frac{\partial z_{0j}}{\partial x_1}(x_1, x_2) = 4 \frac{x_1(x_2 + \mu_j)}{(x_1^2 + (x_2 + \mu_j)^2)^2}, \quad \frac{\partial z_{0j}}{\partial x_2}(x_1, x_2) = 2 \frac{(x_2 + \mu_j)^2 - x_1^2}{(x_1^2 + (x_2 + \mu_j)^2)^2}. $$

Thus

$$\nabla \tilde{h} \nabla z_{0j} = -\frac{2}{|x|^2(\log(\delta/\varepsilon) - \log R)} \frac{x_2(x_1^2 + (x_2 + \mu_j)^2) + 2x_1^2\mu_j}{(x_1^2 + (x_2 + \mu_j)^2)^2} \leq 0,$$

so that

$$\Delta(\tilde{\eta}_{1j}z_{0j} + (1 - \tilde{\eta}_{1j})\tilde{h}z_{0j}) \leq \Delta \tilde{\eta}_{1j}(1 - \tilde{h})z_{0j} + 2\nabla \tilde{\eta}_{1j} \nabla ((1 - \tilde{h})z_{0j}), \quad R < r < R + 1.$$  

It follows that

$$I_1 \geq -\int_{\{R < r < R + 1\} \cap \mathbb{R}_+^2} \Delta \tilde{\eta}_{1j}(1 - \tilde{h})z_{0j} \tilde{z}_{0j} + 2 \int_{\{R < r < R + 1\} \cap \mathbb{R}_+^2} \nabla \tilde{\eta}_{1j} \nabla \tilde{h}z_{0j} \tilde{z}_{0j}$$

$$-2 \int_{\{R < r < R + 1\} \cap \mathbb{R}_+^2} \nabla \tilde{\eta}_{1j} \nabla z_{0j} (1 - \tilde{h}) \tilde{z}_{0j} + O(R\varepsilon).$$

We integrate by parts the first term on the right-hand side above

$$I_1 \geq \int_{\{R < r < R + 1\} \cap \mathbb{R}_+^2} \nabla \tilde{\eta}_{1j} \nabla \tilde{h}z_{0j} \tilde{z}_{0j} - \int_{\{R < r < R + 1\} \cap \mathbb{R}_+^2} \nabla \tilde{\eta}_{1j} \nabla z_{0j} (1 - \tilde{h}) \tilde{z}_{0j}$$

$$+ \int_{\{R < r < R + 1\} \cap \mathbb{R}_+^2} \nabla \tilde{\eta}_{1j} \nabla \tilde{z}_{0j} (1 - \tilde{h}) z_{0j} + O(R\varepsilon), \quad (57)$$

observing that the boundary term on $\partial \mathbb{R}_+^2 \cap \{ R < r < R + 1 \}$ is zero because $\tilde{\eta}_{1j}$ is radial.

The second term on the right-hand side above is bounded by

$$\int_{\{R < r < R + 1\} \cap \mathbb{R}_+^2} |\nabla \tilde{\eta}_{1j} \nabla z_{0j} (1 - \tilde{h}) \tilde{z}_{0j}| \leq C \int_{\{R < r < R + 1\} \cap \mathbb{R}_+^2} |\tilde{h} - 1| |\nabla z_{0j}|.$$
But in the region \( R < r < R + 1 \) we have \(|\tilde{h} - 1| \leq \frac{C}{\log \frac{\delta}{\varepsilon}}\) and \(|\nabla z_0| \leq \frac{C}{R^2}\) which yields

\[
\int_{\{R < r < R + 1\} \cap \mathbb{R}^2_+} |\nabla \tilde{h}_{1j} \nabla z_{0j}(1 - \tilde{h})\tilde{z}_{0j}| \leq \frac{C}{R \log \frac{\delta}{\varepsilon}}. \tag{58}
\]

The third term on the right-hand side of (57) is similar since in the region \( R < r < R + 1 \) we have \(|\nabla \tilde{h}_{1j}| \leq \frac{C}{\log \frac{\delta}{\varepsilon}}\) and hence

\[
|\nabla \tilde{z}_{0j}| = |\nabla (\tilde{h}_{1j}(1 - \tilde{h})z_0) + \tilde{h}z_0| = |\nabla \tilde{h}_{1j}(1 - \tilde{h})z_0 - \tilde{h}_{1j}\nabla \tilde{h}z_0 + \tilde{h}_{1j}(1 - \tilde{h})\nabla z_0 + \nabla \tilde{h}z_0 + \tilde{h}\nabla z_0| \\
\leq \frac{C}{\log \frac{\delta}{\varepsilon}} + \frac{C}{R^2}, \quad R < r < R + 1.
\]

Integrating

\[
\int_{\{R < r < R + 1\} \cap \mathbb{R}^2_+} |\nabla \tilde{h}_{1j} \nabla \tilde{z}_{0j}(1 - \tilde{h})\tilde{z}_{0j}| \leq \frac{CR}{\log^2 \frac{\delta}{\varepsilon}} + \frac{C}{R \log \frac{\delta}{\varepsilon}}. \tag{59}
\]

Thus from (57)–(59) we obtain

\[
I_1 \geq \int_{\{R < r < R + 1\} \cap \mathbb{R}^2_+} \nabla \tilde{h}_{1j} \nabla \tilde{z}_{0j} \tilde{z}_{0j} + O(R\varepsilon) + O\left(\frac{R}{\log^2 \frac{\delta}{\varepsilon}}\right) + O\left(\frac{1}{R \log \frac{\delta}{\varepsilon}}\right).
\]

In the first integral above \( z_{0j} \) and \( \tilde{z}_{0j} \) have a lower bound independent of \( \varepsilon, \delta, R \) and

\[
|\nabla \tilde{h}_{1j}| = (|x|(\log(\delta/\varepsilon) - \log R))^{-1}.\]

Hence

\[
I_1 \geq \frac{\tilde{c}}{\log \frac{\delta}{\varepsilon}} + O(R\varepsilon) + O\left(\frac{R}{\log^2 \frac{\delta}{\varepsilon}}\right) + O\left(\frac{1}{R \log \frac{\delta}{\varepsilon}}\right) \tag{60}
\]

with \( \tilde{c} > 0 \) independent of \( \varepsilon, \delta, R \).

**Estimate of \( I_0 \):** By (56) and since \( \Delta \tilde{z}_{0j} = 0 \) we have

\[
\tilde{L}(\tilde{z}_{0j}) = O(\varepsilon), \quad r < R
\]

and this implies

\[
I_0 = O(R\varepsilon). \tag{62}
\]
Estimate of $I_2$: Changing variables as before

$$I_2 = \int_{\{R+1<r<\frac{\delta}{4\varepsilon}\} \cap \mathbb{R}_{+}^{2}} \tilde{L}(\tilde{z}_{0j})\tilde{z}_{0j}. $$

In the region $R + 1 < r < \frac{\delta}{4\varepsilon}$ we have $\tilde{z}_{0j} = \tilde{h}z_{0j}$ and therefore

$$|\Delta \tilde{z}_{0j}| = 2|\nabla \tilde{h} \nabla z_{0j}| \leq \frac{C}{r^3 \log \frac{\delta}{\varepsilon}}, \quad R + 1 < r < \frac{\delta}{4\varepsilon}. $$

For the other terms we find

$$|\nabla^2 \tilde{z}_{0j}| \leq |\nabla^2 \tilde{h}|z_{0j} + 2|\nabla \tilde{h} \nabla z_{0j}| + \tilde{h}|\nabla^2 z_{0j}|$$

$$= O \left( \frac{1}{r^2 \log \frac{\delta}{\varepsilon}} \right) + O \left( \frac{1}{r^3 \log \frac{\delta}{\varepsilon}} \right) + O \left( \frac{1}{r^3} \right), \quad R + 1 < r < \frac{\delta}{4\varepsilon},$$

so

$$O(\varepsilon|x|)|\nabla^2 \tilde{z}_{0j}| = O \left( \frac{\varepsilon}{r \log \frac{\delta}{\varepsilon}} \right) + O \left( \frac{\varepsilon}{r^2} \right), \quad R + 1 < r < \frac{\delta}{4\varepsilon}. $$

Also

$$|\nabla \tilde{z}_{0j}| \leq |\nabla \tilde{h}|z_{0j} + \tilde{h}|\nabla z_{0j}| = O \left( \frac{1}{r \log \frac{\delta}{\varepsilon}} \right) + O \left( \frac{1}{r^2} \right), \quad R + 1 < r < \frac{\delta}{4\varepsilon}. $$

Hence

$$\tilde{L}(\tilde{z}_{0j}) = O \left( \frac{1}{r^3 \log \frac{\delta}{\varepsilon}} \right) + O \left( \frac{\varepsilon}{r \log \frac{\delta}{\varepsilon}} \right) + O \left( \frac{\varepsilon^2}{r^2} \right) + \varepsilon^2 z_{0j}, \quad R + 1 < r < \frac{\delta}{4\varepsilon}. \quad (63)$$

This yields

$$\int_{\{R+1<r<\frac{\delta}{4\varepsilon}\} \cap \mathbb{R}_{+}^{2}} \tilde{L}(\tilde{z}_{0j})\tilde{z}_{0j} = O \left( \frac{1}{R \log \frac{\delta}{\varepsilon}} \right) + O \left( \frac{\delta}{\log \frac{\delta}{\varepsilon}} \right)$$

$$+ O(\varepsilon^2) \int_{\{R+1<r<\frac{\delta}{4\varepsilon}\} \cap \mathbb{R}_{+}^{2}} \tilde{z}_{0j}^2. $$
We estimate the last integral using the fact that in the region \( R + 1 < r < \frac{\delta}{4\varepsilon} \) \( \tilde{z}_{0j} = \tilde{h}z_{0j} \) and \( z_{0j} \) is bounded, thus

\[
\int_{\{R + 1 < r < \frac{\delta}{4\varepsilon}\}} \frac{\tilde{z}_{0j}^2}{r} \leq C \int_{R + 1}^{\frac{\delta}{4\varepsilon}} \left( \frac{\log \frac{\delta}{\varepsilon} - \log r}{\log \frac{\delta}{\varepsilon} - \log R} \right)^2 r \, dr \leq C \frac{\delta^2}{\varepsilon^2 \log^2 \frac{\delta}{\varepsilon}}.
\]

This and the previous estimate show that

\[
I_2 = O \left( \frac{1}{R \log \frac{\delta}{\varepsilon}} \right) + O \left( \frac{\delta}{\log \frac{\delta}{\varepsilon}} \right). \tag{64}
\]

**Estimate of \( I_3 \):** In the region \( \frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon} \) the definition of \( \tilde{z}_{0j} \) is \( \tilde{z}_{0j} = \tilde{h}_{2j} \tilde{h}z_{0j} \). We will estimate each term of (56) using the facts that \( \nabla \tilde{h}_{2j} = O(\varepsilon) \), \( |\nabla^2 \tilde{h}_{2j}| = O(\varepsilon^2) \) and that in the considered region \( \tilde{h} = O(\frac{1}{\log \frac{\delta}{\varepsilon}}) \) which implies also \( \tilde{z}_{0j} = O(\frac{1}{\log \frac{\delta}{\varepsilon}}) \). We obtain

\[
\Delta \tilde{z}_{0j} = \Delta \tilde{h}_{2j} \tilde{h}z_{0j} + 2 \nabla \tilde{h}_{2j} \nabla (\tilde{h}z_{0j}) + \tilde{h}_{2j} \Delta (\tilde{h}z_{0j})
\]

\[
= \Delta \tilde{h}_{2j} \tilde{h}z_{0j} + 2 \nabla \tilde{h}_{2j} \nabla \tilde{h}z_{0j} + 2 \nabla \tilde{h}_{2j} \nabla z_{0j} \tilde{h} + 2 \tilde{h}_{2j} \nabla \tilde{h} \nabla z_{0j}
\]

\[
= O \left( \frac{\varepsilon^2}{\delta^2 \log \frac{\delta}{\varepsilon}} \right) + O \left( \frac{\varepsilon}{r \delta \log \frac{\delta}{\varepsilon}} \right) + O \left( \frac{\varepsilon}{r^2 \delta \log \frac{\delta}{\varepsilon}} \right) + O \left( \frac{1}{r^3 \log \frac{\delta}{\varepsilon}} \right)
\]

\[
= O \left( \frac{\varepsilon^2}{\delta^2 \log \frac{\delta}{\varepsilon}} \right), \quad \frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon}.
\]

Next

\[
\nabla^2 \tilde{z}_{0j} = \nabla^2 \tilde{h}_{2j} \tilde{h}z_{0j} + 2 \nabla \tilde{h}_{2j} \nabla (\tilde{h}z_{0j}) + \tilde{h}_{2j} \nabla^2 (\tilde{h}z_{0j}), \quad \frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon}
\]

and by the above computations

\[
\nabla^2 \tilde{z}_{0j} = O \left( \frac{\varepsilon^2}{\delta^2 \log \frac{\delta}{\varepsilon}} \right) + \tilde{h}_{2j} (\nabla^2 \tilde{h}z_{0j} + 2 \nabla \tilde{h} \nabla z_{0j} + \tilde{h} \nabla^2 z_{0j})
\]

\[
= O \left( \frac{\varepsilon^2}{\delta^2 \log \frac{\delta}{\varepsilon}} \right), \quad \frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon}.
\]
Similarly
\[
\nabla \tilde{z}_0 = \nabla \tilde{\eta}_2 \tilde{h} z_0 + \tilde{\eta}_2 \hat{h} \nabla \tilde{z}_0 + \tilde{\eta}_2 \hat{h} \nabla z_0
\]
}\]
\[
= O \left( \frac{\varepsilon}{\delta \log \frac{\delta}{\varepsilon}} \right), \quad \frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon}.
\]

This shows that
\[
\tilde{L} (\tilde{z}_0) = O \left( \frac{\varepsilon^2}{\delta^2 \log \frac{\delta}{\varepsilon}} \right), \quad \frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon}.
\]
and integrating
\[
I_3 = O \left( \frac{1}{\log^2 \frac{\delta}{\varepsilon}} \right).
\]

**Estimate of** \( \int_{\partial \Omega} \tilde{Z}_0 (\frac{\partial}{\partial v} - W) \tilde{Z}_0 : \) **We change variables through the map** \( F^\varepsilon_j : \)
\[
\int_{\partial \Omega} \tilde{Z}_0 (\frac{\partial}{\partial v} - W) \tilde{Z}_0 = \int_{\partial \mathbb{R}^2_+} \tilde{z}_0 (B(\tilde{z}_0) - \tilde{W} \tilde{z}_0) b(x),
\]
where \( \tilde{z}_0 \) is defined in (55), \( \tilde{W}(x) = W((F^\varepsilon_j)^{-1}(x)) \) and \( b \) is a positive function arising from the change of variables bounded uniformly in \( \varepsilon \). \( B \) is a differential operator of order one on \( \partial \mathbb{R}^2_+ \). Rotating \( \Omega_\varepsilon \) so that \( \nabla F^\varepsilon_j (\xi^\varepsilon_j) = I \) we find the following expansion for \( B \)
\[
B = - \frac{\partial}{\partial x_2} + O(\varepsilon |x|) \nabla.
\]

Let us estimate first the integral in the region \( |x| < R \), where \( \tilde{z}_0 = z_0 \). Then
\[
B(\tilde{z}_0) = - \frac{\partial \tilde{z}_0}{\partial x_2} + O(\varepsilon), \quad |x| < R, x \in \partial \mathbb{R}^2_+.
\]

On the other hand recall (26), that is
\[
W(y) = \frac{2\mu_j}{|y - \xi^\varepsilon_j - \mu_j y(\xi^\varepsilon_j)|^2} (1 + O(\varepsilon^x (1 + |y|))).
\]
Since we have the expansion \((F_j^\varepsilon)^{-1}(x) = \xi_j' + x + O(\varepsilon|x|)\) we find
\[
\tilde{W}(x) = W((F_j^\varepsilon)^{-1}(x)) = W(\xi_j' + x + O(\varepsilon|x|)) = \frac{2\mu_j}{x_1^2 + \mu_j^2} + O\left(\frac{\varepsilon x(1 + |x|)}{1 + |x|^2}\right), \quad x = (x_1, 0), |x| < \frac{\delta}{\varepsilon}. \quad (67)
\]

Thus
\[
B(\tilde{z}_0j) - \tilde{W}\tilde{z}_0j = O(\varepsilon^2), \quad x \in \partial \mathbb{R}^2_+, |x| < R \quad (68)
\]
and therefore
\[
\int_{\partial \mathbb{R}^2_+ \cap \{|x| < R\}} \tilde{z}_0j(B(\tilde{z}_0j) - \tilde{W}\tilde{z}_0j)b(x) = O(R\varepsilon^2). \quad (69)
\]

Next, in the region \(R < |x| < R + 1\) we have
\[
\nabla \tilde{z}_0j = \nabla(\tilde{\eta}_1j(1 - \tilde{h})z_0j + \tilde{h}z_0j) = \nabla \tilde{\eta}_1j(1 - \tilde{h})z_0j - \tilde{\eta}_1j \nabla \tilde{h}z_0j + \tilde{\eta}_1j(1 - \tilde{h})\nabla z_0j + \nabla \tilde{h}z_0j + \tilde{h}\nabla z_0j = O\left(\frac{1}{\log \frac{\delta}{\varepsilon}}\right) + \tilde{\eta}_1j(1 - \tilde{h})\nabla z_0j + \tilde{h}\nabla z_0j.
\]

Since \(\tilde{h}\) is radial this implies
\[
B(\tilde{z}_0j) = -\tilde{h} \frac{\partial z_0j}{\partial x_2} + O\left(\frac{1}{R^2 \log \frac{\delta}{\varepsilon}}\right) + O\left(\frac{R\varepsilon}{\log \frac{\delta}{\varepsilon}}\right), \quad R < |x| < R + 1, x \in \partial \mathbb{R}^2_+. \quad (67)
\]

Using (67) we see that
\[
B(\tilde{z}_0j) - \tilde{W}\tilde{z}_0j = O\left(\frac{1}{R^2 \log \frac{\delta}{\varepsilon}}\right) + O\left(\frac{R\varepsilon}{\log \frac{\delta}{\varepsilon}}\right), \quad R < |x| < R + 1, x \in \partial \mathbb{R}^2_+. \quad (69)
\]

It follows that
\[
\int_{\partial \mathbb{R}^2_+ \cap \{|x| < R + 1\}} \tilde{z}_0j(B(\tilde{z}_0j) - \tilde{W}\tilde{z}_0j)b(x) = O\left(\frac{1}{R^2 \log \frac{\delta}{\varepsilon}}\right) + O\left(\frac{R\varepsilon}{\log \frac{\delta}{\varepsilon}}\right). \quad (69)
\]
Using the fact that $\tilde{h}$ has zero normal derivative on $\partial \mathbb{R}^2_+$ we deduce

$$ B(\tilde{h}z_0j) = -\tilde{h} \frac{\partial z_0}{\partial x_2} + O(\varepsilon r)(\nabla \tilde{h}z_0j + \tilde{h}\nabla z_0j) $$

$$ = -\tilde{h} \frac{\partial z_0}{\partial x_2} + O\left(\frac{\varepsilon}{\log \frac{\delta}{\varepsilon}}\right) + O\left(\frac{\varepsilon}{r}\right), \quad R + 1 < r < \frac{\delta}{\varepsilon}. \quad (70) $$

On the other hand, using (67) we have

$$ B(\tilde{z}_0j) - \tilde{W}\tilde{z}_0j = O\left(\frac{\varepsilon}{\log \frac{\delta}{\varepsilon}}\right) + O\left(\varepsilon r\right) $$

and we conclude

$$ \int_{\partial \mathbb{R}^2_+ \cap \{R+1 < r < \frac{\delta}{\varepsilon}\}} \tilde{z}_0j(B(\tilde{z}_0j) - \tilde{W}\tilde{z}_0j)b(x) = O\left(\frac{\delta}{\log \frac{\delta}{\varepsilon}}\right). $$

Finally we consider $\frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon}$. Here we have $\tilde{z}_0j = \tilde{\eta}_2j\tilde{h}z_0j$ and $\tilde{h}, z_0j = O\left(\frac{1}{\log \frac{\delta}{\varepsilon}}\right)$,

$$ \nabla \tilde{\eta}_2j = O\left(\frac{\varepsilon}{\delta}\right). $$

Using these facts, estimate (70) and that $\tilde{\eta}_2j$ has zero normal derivative we find

$$ B(\tilde{z}_0j) = B(\tilde{\eta}_2j)\tilde{h}z_0j + \tilde{\eta}_2j B(\tilde{h}z_0j) $$

$$ = O\left(\frac{\varepsilon^2 r}{\delta \log \frac{\delta}{\varepsilon}}\right) + O\left(\frac{1}{r^2}\right) + O\left(\frac{\varepsilon}{\log \frac{\delta}{\varepsilon}}\right) + O\left(\frac{\varepsilon}{r}\right), \quad \frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon}. \quad (71) $$

Integrating we have

$$ \int_{\partial \mathbb{R}^2_+ \cap \{\frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon}\}} \tilde{z}_0j B(\tilde{z}_0j)b(x) = O\left(\frac{1}{\log^2 \frac{\delta}{\varepsilon}}\right). $$

From (67) we have

$$ \tilde{W} = O\left(\frac{\varepsilon^2}{r}\right), \quad \frac{\delta}{4\varepsilon} < r < \frac{\delta}{\varepsilon} \quad (72) $$

and this implies

$$ \int_{\partial \mathbb{R}^2_+ \cap \{\frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon}\}} \tilde{W}z_0j^2 b(x) = O\left(\varepsilon^2 \log \frac{\delta}{\varepsilon}\right). $$
Thus
\[
\int_{\Omega \cap \{ \frac{\delta}{4\epsilon} < r < \frac{\delta}{2\epsilon} \}} \tilde{z}_{0j} (B(\tilde{z}_{0j}) - \tilde{W}) b(x) = O \left( \frac{1}{\log \frac{\delta}{\epsilon}} \right) + O \left( \epsilon^2 \log \frac{\delta}{\epsilon} \right)
\]
and therefore
\[
\int_{\Omega} \tilde{z}_{0j} \left( \frac{\hat{\mathbf{c}}}{\mathbf{y}} - W \right) \tilde{z}_{0j} = O \left( \frac{1}{R^2 \log \frac{\delta}{\epsilon}} \right) + O \left( \frac{\delta}{\log \frac{\delta}{\epsilon}} \right). \tag{73}
\]
Combining (60), (62), (64), (66) and (73) we obtain
\[
\int_{\Omega} L(\tilde{Z}_{0j}) Z_{0j} \geq \frac{\tilde{c}}{\log \frac{\delta}{\epsilon}} + O \left( \frac{1}{R \log \frac{\delta}{\epsilon}} \right) + O \left( \frac{\delta}{\log \frac{\delta}{\epsilon}} \right). \tag{77}
\]
Choosing \( \delta > 0 \) small and \( R > 0 \) large (fixed independently of \( \epsilon \)) we conclude that (52) holds for \( \epsilon > 0 \) small enough.

**Proof of (53).** By (61) we deduce
\[
L(\tilde{Z}_{0}) = O(\epsilon), \quad r < R. \tag{74}
\]
Also (63) implies
\[
L(\tilde{Z}_{0j}) = O \left( \frac{1}{r^3 \log \frac{\delta}{\epsilon}} \right) + O \left( \frac{\epsilon}{r \log \frac{\delta}{\epsilon}} \right) + O \left( \frac{\epsilon}{r^2} \right) + \epsilon^2 \tilde{Z}_{0j}, \tag{75}
\]
\[
R + 1 < r < \frac{\delta}{4\epsilon}
\]
and from (65) we obtain
\[
L(\tilde{Z}_{0j}) = O \left( \frac{\epsilon^2}{\log \frac{\delta}{4\epsilon}} \right), \quad \frac{\delta}{4\epsilon} < r < \frac{\delta}{3\epsilon}. \tag{76}
\]
Thus, we only need to estimate the size of \( L(\tilde{Z}_{0j}) \) in the region \( R < r < R + 1 \). In this region we have \( \tilde{Z}_{0j} = \eta_{1j} Z_{0j} + (1 - \eta_{1j}) \psi Z_{0j} \) and hence
\[
\Delta \tilde{Z}_{0j} = \Delta \eta_{1j} (1 - \psi) Z_{0j} - 2 \nabla \eta_{1j} \nabla \psi Z_{0j} + 2 \nabla \eta_{1j} \nabla Z_{0j} (1 - \psi) + \eta_{1j} \Delta Z_{0j}
\]
\[ + (1 - \eta_{1j}) \Delta (\psi Z_{0j}) \]
\[ = O \left( \frac{1}{\log \frac{\delta}{\varepsilon}} \right) + \eta_{1j} \Delta Z_{0j} + (1 - \eta_{1j}) \Delta (\psi Z_{0j}), \quad R < r < R + 1. \]

Using the change of variables \( x = F^e_j(y) \) and recalling the definitions of \( Z_{0j} \) and \( \psi \) we have

\[ \Delta_y Z_{0j} = \Delta_x z_{0j} + O(\varepsilon) = O(\varepsilon), \quad R < r < R + 1 \]
and

\[ \Delta_y (\psi Z_{0j}) = \Delta_x (\tilde{h} z_{0j}) + O(\varepsilon) \]
\[ = 2 \nabla \tilde{h} \nabla z_{0j} + O(\varepsilon), \quad R < r < R + 1 \]
\[ = O \left( \frac{1}{\log \frac{\delta}{\varepsilon}} \right) + O(\varepsilon), \quad R < r < R + 1. \]

Thus

\[ L(\tilde{Z}_{0j}) = O \left( \frac{1}{\log \frac{\delta}{\varepsilon}} \right), \quad R < r < R + 1. \]

This bounds and (74)–(76) imply (53).

**Proof of (54).** By (68) we see that

\[ \frac{\partial \tilde{Z}_{0j}}{\partial v} - W \tilde{Z}_{0j} = O(\varepsilon), \quad y \in \partial \Omega_\varepsilon, |y| < R. \]

From (69) we also obtain

\[ \frac{\partial \tilde{Z}_{0j}}{\partial v} - W \tilde{Z}_{0j} = O \left( \frac{\varepsilon r}{\log \frac{\delta}{\varepsilon}} \right) + O(\varepsilon^2), \quad y \in \partial \Omega_\varepsilon, R < |y| < R + 1. \]

Finally using (70), (72) and (71) we also see that

\[ \frac{\partial \tilde{Z}_{0j}}{\partial v} - W \tilde{Z}_{0j} = O \left( \frac{\varepsilon^2 r}{\delta \log \frac{\delta}{\varepsilon}} \right) + O \left( \frac{1}{r^2} \right) + O \left( \frac{\varepsilon}{\log \frac{\delta}{\varepsilon}} \right) + O \left( \frac{\varepsilon}{r} \right), \]
\[ y \in \partial \Omega_\varepsilon, R + 1 < |y| < \frac{\delta}{3\varepsilon}. \]

These inequalities readily imply (54). \( \Box \)
Proof of Proposition 4.1. To prove the solvability of (33) we consider first a related problem: that of given \( h \in L^\infty (\partial \Omega_\varepsilon) \) find \( \phi \in L^\infty (\Omega_\varepsilon) \) and \( d_1, \ldots, d_m \in \mathbb{R} \), such that

\[
\begin{cases}
-\Delta \phi + \varepsilon^2 \phi = \sum_{j=1}^m d_j \chi_j Z_{1j} & \text{in } \Omega_\varepsilon, \\
\frac{\partial \phi}{\partial v} - W \phi = h & \text{on } \partial \Omega_\varepsilon, \\
\int_{\Omega_\varepsilon} \chi_j Z_{1j} \phi = 0 & \forall \ j = 1, \ldots, m.
\end{cases}
\]

(77)

First we prove that for any \( \phi, d_1, \ldots, d_m \) solution to (77) the bound

\[
\| \phi \|_{L^\infty (\Omega_\varepsilon)} \leq C \log \frac{1}{\varepsilon} \| h \|_{*, \partial \Omega_\varepsilon}
\]

(78)

holds. Indeed, by Lemma 4.4 we have

\[
\| \phi \|_{L^\infty (\Omega_\varepsilon)} \leq C \log \frac{1}{\varepsilon} \left( \| h \|_{*, \partial \Omega_\varepsilon} + \sum_{j=1}^m |d_j| \right)
\]

(79)

and therefore it is enough to prove that \( |d_j| \leq C \| h \|_{*, \partial \Omega_\varepsilon} \).

Let \( \eta_{2j} \) be the cut-off function defined in (47) and multiply Eq. (77) by \( \eta_{2k} Z_{1k} \). Integrating by parts we find

\[
d_k \int_{\Omega_\varepsilon} \chi_k Z_{1k}^2 = - \int_{\partial \Omega_\varepsilon} h \eta_{2k} Z_{1k} + \int_{\partial \Omega_\varepsilon} \phi \frac{\partial \eta_{2k}}{\partial v} Z_{1k} + \int_{\partial \Omega_\varepsilon} \phi \eta_{2k} \left( \frac{\partial Z_{1k}}{\partial v} - W Z_{1k} \right) \\
+ \int_{\Omega_\varepsilon} \phi (-\Delta (\eta_{2k} Z_{1k}) + \varepsilon^2 \eta_{2k} Z_{1k}).
\]

(80)

But \( Z_{1j} = O (\frac{1}{1+r}) \) and \( \nabla \eta_{2j} = O (\varepsilon) \) so \( \int_{\partial \Omega_\varepsilon} \phi \frac{\partial \eta_{2k}}{\partial v} Z_{1k} \leq C \varepsilon \log \frac{1}{\varepsilon} \). Also, using (67) and proceeding similarly as with (68) we obtain

\[
\frac{\partial Z_{1j}}{\partial v} - W Z_{1j} = O \left( \frac{\varepsilon}{1+r} \right) + O \left( \frac{\varepsilon^2}{1+r^2} \right), \quad |y| < \frac{\varepsilon}{r}, \ y \in \partial \Omega_\varepsilon
\]

and this implies

\[
\int_{\partial \Omega_\varepsilon} \left| \frac{\partial Z_{1j}}{\partial v} - W Z_{1j} \right| = O (\varepsilon^2).
\]

(81)
We also compute

\[ \Delta(\eta_j Z_{1,j}) = \Delta \eta_j Z_{1,j} + 2\nabla \eta_j \nabla Z_{1,j} + \eta_j \Delta Z_{1,j} \]

\[ = O \left( \frac{\varepsilon^2}{1 + r} \right) + O \left( \frac{\varepsilon}{1 + r^2} \right) + \eta_j \Delta Z_{1,j} . \]

But \(-\Delta Z_{1,j} + \varepsilon^2 Z_{1,j} = \tilde{L}(z_{1,j})\) where \(\tilde{L}\) is the linear operator (56). Thus

\[ -\Delta Z_{1,j} + \varepsilon^2 Z_{1,j} = O \left( \frac{\varepsilon}{1 + r^2} \right) + O \left( \frac{\varepsilon}{1 + r} \right) \]

and this readily implies

\[ \int_{\Omega_e} | - \Delta(\eta_j Z_{1,j}) + \varepsilon^2 \eta_j Z_{1,j} | = O \left( \varepsilon \log \frac{1}{\varepsilon} \right) . \quad (82) \]

Combining (80)–(82) we conclude that

\[ d_k \int_{\Omega_e} \chi_k Z^2_{1,k} \leq C \|h\|_{\ast, \partial \Omega_e} + C \varepsilon^2 \|\phi\|_{L^\infty(\Omega_e)} \]

and this combined with (79) yields

\[ |d_k| \leq C \left( \|h\|_{\ast, \partial \Omega_e} + C \varepsilon \log \frac{1}{\varepsilon} \sum_{j=1}^m |d_j| \right) . \]

This implies

\[ |d_k| \leq C \|h\|_{\ast, \partial \Omega_e} \quad (83) \]

which proves (78).

Now consider the Hilbert space

\[ H = \left\{ \phi \in H^1(\Omega_e) : \int_{\Omega_e} \chi_j Z_{1,j} \phi = 0 \quad \forall j = 1, \ldots, m \right\} \]

with the norm \(\|\phi\|_{H^1}^2 = \int_{\Omega_e} |\nabla \phi|^2 + \varepsilon^2 \phi^2\). Eq. (77) is equivalent to find \(\phi \in H\), such that

\[ \int_{\Omega_e} (\nabla \phi \nabla \psi + \varepsilon^2 \phi \psi) - \int_{\partial \Omega_e} W \phi \psi = \int_{\partial \Omega_e} h \psi \quad \forall \psi \in H. \]
By Fredholm’s alternative this is equivalent to the uniqueness of solutions to this problem, which is guaranteed by (78).

To show solvability of (33) let \( Y_i \in L^\infty(\Omega_\varepsilon) \), \( d_{ij} \in \mathbb{R} \) be the solution to (77) with \( h = \chi_i Z_1 \), that is

\[
\begin{cases}
-\Delta Y_i + \varepsilon^2 Y_i = \sum_{j=1}^{m} d_{ij} \chi_j Z_{1j} & \text{in } \Omega_\varepsilon, \\
\frac{\partial Y_i}{\partial \nu} - WY_i = \chi_i Z_1 & \text{on } \partial \Omega_\varepsilon, \\
\int_{\Omega_\varepsilon} \chi_j Z_{1j} Y_i = 0 & \forall j = 1, \ldots, m.
\end{cases}
\]  

(84)

By the previous argument there is a unique \( Y_i \in L^\infty(\Omega_\varepsilon) \) solution to this equation, and moreover we have the estimates

\[
\| Y_i \|_{L^\infty(\Omega_\varepsilon)} \leq C \log \frac{1}{\varepsilon}, \quad |d_{ij}| \leq C
\]  

(85)

for some constant \( C \) independent of \( \varepsilon \). We shall show that

\[
d_{ij} = A \delta_{ij} + O \left( \varepsilon^2 \log \frac{1}{\varepsilon} \right),
\]  

(86)

where \( A > 0 \) is independent of \( \varepsilon \) and \( \delta_{ii} = 1 \) and \( \delta_{ij} = 0 \) if \( i \neq j \) is Kronecker’s delta. Assuming this for a moment, we see that the matrix \( D \) with entries \( d_{ij} \) is invertible for small \( \varepsilon \) and \( \| D^{-1} \| \leq C \) uniformly in \( \varepsilon \). Then, given \( h \in L^\infty(\partial \Omega_\varepsilon) \) we find \( \phi_1, d_1, \ldots, d_m \) the solution to (77) and define

\[
\phi = \phi_1 + \sum_{i=1}^{m} c_i Y_i,
\]

where \( c_i \) is such that \( \sum_{i=1}^{m} c_i d_{ij} = -d_j \forall j = 1, \ldots, m \). Then \( \phi \) satisfies (33) and we have the estimate

\[
\| \phi \|_{L^\infty(\Omega_\varepsilon)} \leq \| \phi_1 \|_{L^\infty(\Omega_\varepsilon)} + \log \frac{1}{\varepsilon} \sum_{i=1}^{m} |c_i| \leq C \log \frac{1}{\varepsilon} \| h \|_{*,\partial \Omega_\varepsilon} + \log \frac{1}{\varepsilon} \sum_{i=1}^{m} |d_i| \leq C \log \frac{1}{\varepsilon} \| h \|_{*,\partial \Omega_\varepsilon},
\]

by (83).
To prove (86) we multiply (84) by \( \eta_{2j}Z_{1j} \) and integrate by parts

\[
d_{ij} \int_{\Omega_{\varepsilon}} \chi_j Z_{1j}^2 + \delta_{ij} \int_{\partial \Omega_{\varepsilon}} \chi_j Z_{1j}^2 = \int_{\Omega_{\varepsilon}} \left( \frac{\partial Z_{1j}}{\partial v} - W Z_{1j} \right) \eta_{2j} Y_i + \int_{\partial \Omega_{\varepsilon}} \frac{\partial \eta_{2j}}{\partial v} Z_{1j} Y_i
\]

\[
+ \int_{\Omega_{\varepsilon}} Y_i \left( -\Delta(\eta_{2j} Z_{1j}) + \varepsilon^2 \eta_{2j} Z_{1j} \right)
\]

\[
= O \left( \varepsilon^2 \log \frac{1}{\varepsilon} \right),
\]

using (81), (82) and (85). □

**Remark.** A slight modification of the proof above also shows that for any \( h \in L^\infty(\partial \Omega_{\varepsilon}) \) and \( f \in L^\infty(\Omega_{\varepsilon}) \) the equation

\[
\begin{cases}
-\Delta \phi + \varepsilon^2 \phi &= f \quad \text{in } \Omega_{\varepsilon}, \\
\frac{\partial \phi}{\partial v} - W \phi &= h + \sum_{j=1}^m c_j \chi_j Z_{1j} \quad \text{on } \partial \Omega_{\varepsilon}, \\
\int_{\Omega_{\varepsilon}} \chi_j Z_{1j} \phi &= 0 \quad \forall j = 1, \ldots, m
\end{cases}
\]

has a unique solution \( \phi, c_1, \ldots, c_m \) and that the estimates

\[
\| \phi \|_{L^\infty(\Omega_{\varepsilon})} \leq C \log \frac{1}{\varepsilon} (\| h \|_{\ast, \partial \Omega_{\varepsilon}} + \| f \|_{\ast \ast, \Omega_{\varepsilon}}),
\]

\[
|c_j| \leq C (\| h \|_{\ast, \partial \Omega_{\varepsilon}} + \| f \|_{\ast \ast, \Omega_{\varepsilon}}) \quad \forall j = 1, \ldots, m
\]

hold with \( C \) independent of \( \varepsilon \).

The result of Proposition 4.1 implies that the unique solution \( \phi = T(h) \) of (33) defines a continuous linear map from the Banach space \( C_\ast \) of all functions \( h \) in \( L^\infty \) for which \( \| h \|_{\ast, \partial \Omega_{\varepsilon}} < \infty \), into \( L^\infty \).

It is important for later purposes to understand the differentiability of the operator \( T \) with respect to the variables \( \xi_l \). Fix \( h \in C_\ast \) and let \( \phi = T(h) \). We want to compute derivatives of \( \phi \) with respect to, say, \( \xi_k \). Formally, \( Z = \partial_{\xi_k} \phi \) should satisfy in \( \Omega_{\varepsilon} \) the equation

\[
-\Delta Z + \varepsilon^2 Z = 0 \quad \text{in } \Omega_{\varepsilon}
\]

and on \( \partial \Omega_{\varepsilon} \) the boundary condition

\[
\frac{\partial Z}{\partial v} - W Z = -\partial_{\xi_k} (W) \phi + c_k \partial_{\xi_k} (Z_{1k} \xi_k) + \sum_{j} d_j Z_j \xi_j,
\]
where (still formally) $d_j = \partial_v \phi(c_j)$. The orthogonality conditions now become

$$
\int_{\Omega_\varepsilon} Z_j \chi_j Z = 0 \quad \text{if } j \neq k,
$$
$$
\int_{\Omega_\varepsilon} Z_{1k} \chi_k Z = - \int_{\Omega_\varepsilon} \partial_v \phi(Z_{1k} \chi_k) \phi.
$$

Let us write $\tilde{Z} = Z + b_k \chi_k Z_{1k}$ where

$$
b_k \int_{\Omega_\varepsilon} \chi_k^2 |Z_{1k}|^2 \equiv \int_{\Omega_\varepsilon} \phi \partial_v \phi(\chi_k Z_{1k}).
$$

Hence $\int_{\Omega_\varepsilon} \tilde{Z} \chi_j Z_{1j} = 0$ for all $j$,

$$
-\Delta \tilde{Z} + \varepsilon^2 \tilde{Z} = a \quad \text{in } \Omega_\varepsilon
$$

and

$$
\frac{\nabla \tilde{Z}}{\nabla v} - W \tilde{Z} = b + \sum_j d_j Z_j \chi_j,
$$

where

$$
a = b_k (-\Delta(\chi_k Z_{1k}) + \varepsilon^2 \chi_k Z_{1k})
$$

and

$$
b = -\partial_v \phi(W) \phi + c_k \partial_v \phi(Z_{1k} \chi_k) + \frac{\partial(\chi_k Z_{1k})}{\partial v} - W \chi_k Z_{1k}
$$

with

$$
\|b\|_{*, \tilde{\Omega}_\varepsilon} \leq C \log \frac{1}{\varepsilon} \|h\|_{*, \tilde{\Omega}_\varepsilon}, \quad \|a\|_{**, \Omega_\varepsilon} \leq C \log \frac{1}{\varepsilon} \|h\|_{*, \tilde{\Omega}_\varepsilon}.
$$

The remark above gives

$$
\|\partial_v \phi\|_{L^\infty(\Omega_\varepsilon)} \leq C \left( \log \frac{1}{\varepsilon} \right)^2 \|h\|_{*, \tilde{\Omega}_\varepsilon}.
$$
5. The nonlinear problem

Consider the nonlinear equation

\[
\begin{align*}
-\Delta \phi + \varepsilon^2 \phi &= 0 & \text{in } \Omega_{\varepsilon}, \\
\frac{\partial \phi}{\partial n} - W \phi &= R + N(\phi) + \sum_{j=1}^{m} c_j Z_j Z_{1j} & \text{on } \partial \Omega_{\varepsilon}, \\
\int_{\Omega_{\varepsilon}} Z_j Z_{1j} \phi &= 0 & \forall j = 1, \ldots, m,
\end{align*}
\]  \tag{88}

where \( W \) is as in (26) and \( N, R \) are defined in (22) and (23), respectively.

**Lemma 5.1.** Let \( m > 0, \, d > 0 \). Then there exist \( \varepsilon_0 > 0, \, C > 0 \), such that for \( 0 < \varepsilon < \varepsilon_0 \) and any \( \xi_1, \ldots, \xi_m \in \partial \Omega \) satisfying

\[
|\xi_i - \xi_j| \geq d \quad \forall i \neq j,
\]

the problem (88) admits a unique solution \( \phi, c_1, \ldots, c_m \) such that

\[
\|\phi\|_{L^\infty(\Omega_{\varepsilon})} \leq C \varepsilon^2 |\log \varepsilon|, \tag{89}
\]

where \( \alpha \) is any number in the interval \((0, 1)\). Furthermore, the function \( \xi' \to \phi(\xi') \in C(\bar{\Omega}_{\varepsilon}) \) is \( C^1 \) and

\[
\|D_{\xi'} \phi\|_{L^\infty(\Omega_{\varepsilon})} \leq C \varepsilon^2 |\log \varepsilon|^2. \tag{90}
\]

**Proof.** In terms of the operator \( T \) defined in the previous section, problem (88) becomes

\[
\phi = T(N(\phi) + R) \equiv A(\phi). \tag{91}
\]

For a given number \( \gamma > 0 \), let us consider the region

\[
\mathcal{F}_{\gamma} \equiv \{ \phi \in C(\bar{\Omega}_{\varepsilon}) : \|\phi\|_{L^\infty(\Omega_{\varepsilon})} \leq \gamma \varepsilon^2 |\log \varepsilon| \}.
\]

From Proposition 4.1, we get

\[
\|A(\phi)\|_{L^\infty(\Omega_{\varepsilon})} \leq C \|\log \varepsilon\| \|N(\phi)\|_{s, \partial \Omega_{\varepsilon}} + \|R\|_{s, \partial \Omega_{\varepsilon}}.
\]

Estimate (25) implies that \( \|R\|_{s, \partial \Omega_{\varepsilon}} \leq C \varepsilon^{\alpha'} \), for any \( \alpha' \in (0, 1) \). Also, the definition of \( N \) in (22) immediately yields \( \|N(\phi)\|_{s, \partial \Omega_{\varepsilon}} \leq C \|\phi\|^2_{L^\infty(\Omega_{\varepsilon})} \). It is also immediate that \( N \)
satisfies, for $\phi_1, \phi_2 \in \mathcal{F}_\gamma$,
\[
\|N(\phi_1) - N(\phi_2)\|_{*} \leq C\gamma \varepsilon^2 |\log \varepsilon| \|\phi_1 - \phi_2\|_{L^\infty(\Omega_\varepsilon)},
\]
where $C$ is independent of $\gamma$. Hence we get
\[
\|A(\phi)\|_{L^\infty(\Omega_\varepsilon)} \leq C\varepsilon |\log \varepsilon| \left[\|\phi\|_{L^\infty(\Omega_\varepsilon)}\right]^{2} + 1,
\]
\[
\|A(\phi_1) - A(\phi_2)\|_{L^\infty(\Omega_\varepsilon)} \leq C\gamma \varepsilon^2 |\log \varepsilon| \|\phi_1 - \phi_2\|_{L^\infty(\Omega_\varepsilon)}.
\]
It follows that for all sufficiently small $\varepsilon$ we get that $A$ is a contraction mapping of $\mathcal{F}_\gamma$, and therefore a unique fixed point of $A$ exists in this region.

Let us now discuss the differentiability of $\phi$. Since $R$ depends continuously (in the $\ast$-norm) on the $m$-tuple $\xi' = (\xi'_1, \ldots, \xi'_m)$, the fixed point characterization obviously yields so for the map $\xi' \mapsto \phi$. Then, formally,
\[
-\partial_{\xi'_k} N(\phi) = \partial_{\xi'_k} W(e^\phi - \phi - 1) + W[e^\phi - 1] \partial_{\xi'_k} \phi.
\]
Since $\|\partial_{\xi'_k} W\|_{*,\xi_\varepsilon}$ is uniformly bounded, we conclude
\[
\|\partial_{\xi'_k} N(\phi)\|_{*,\xi_\varepsilon} \leq C \left[\|\phi\|_{L^\infty(\Omega_\varepsilon)} + \|\partial_{\xi'_k} \phi\|_{L^\infty(\Omega_\varepsilon)}\right] \|\phi\|_{L^\infty(\Omega_\varepsilon)}
\]
\[
\leq C \varepsilon^2 |\log \varepsilon| + \|\partial_{\xi'_k} \phi\|_{L^\infty(\Omega_\varepsilon)} \varepsilon^2 |\log \varepsilon|.
\]
Also observe that we have
\[
\partial_{\xi'_k} \phi = (\partial_{\xi'_k} T)(-(N(\phi) + R)) + T \left(-\partial_{\xi'_k} \left[N(\phi) + R\right]\right),
\]
so that, using (87),
\[
\|\partial_{\xi'_k} \phi\|_{L^\infty(\Omega_\varepsilon)} \leq C |\log \varepsilon| \left[|\log \varepsilon| (N(\phi) + R)\right]_{*,\xi_\varepsilon}
\]
\[
+ \|\partial_{\xi'_k} N(\phi)\|_{*,\xi_\varepsilon} + \|\partial_{\xi'_k} R\|_{*,\xi_\varepsilon}\right].
\]
Since it is also easily checked that $\|\partial_{\xi'_k} R\|_{*,\xi_\varepsilon} \leq C\varepsilon^{x'}$ for any $x' \in (0, 1)$, we conclude from the above computation that
\[
\|\partial_{\xi'_k} \phi\|_{L^\infty(\Omega_\varepsilon)} \leq C \varepsilon^{x'} |\log \varepsilon|^2 \text{ for all } k.
\]
The above computation can be made rigorous by using the implicit function theorem and the fixed point representation (91) which guarantees $C^1$ regularity in $\zeta'$. □

6. Variational reduction

In view of Lemma 5.1, given $\zeta = (\zeta_1, \ldots, \zeta_m) \in \partial \Omega^m$ satisfying $|\zeta_i - \zeta_j| \geq d \forall i \neq j$, we define $\phi(\zeta)$ and $c_j(\xi)$ to be the unique solution to (88) satisfying the bound (89).

Given $\zeta = (\zeta_1, \ldots, \zeta_m) \in \partial \Omega^m$ we write

$$U(\zeta) = \sum_{j=1}^{m} \left( u_j(x) + H_{j}(x) \right),$$

the ansatz defined in (16). Set

$$F_\epsilon(\zeta) = J_\epsilon(U(\zeta) + \tilde{\phi}(\zeta)), \quad (92)$$

where $J_\epsilon$ is the functional defined in (3) and

$$\tilde{\phi}(\zeta)(x) = \phi(\zeta) \left( \frac{x}{\epsilon} \right), \quad x \in \Omega. \quad (93)$$

**Lemma 6.1.** If $\zeta = (\zeta_1, \ldots, \zeta_m) \in (\partial \Omega)^m$ satisfying (36) is a critical point of $F_\epsilon$ then $u = U(\zeta) + \tilde{\phi}(\zeta)$ is a critical point of $J_\epsilon$, that is, a solution to (1).

**Proof.** Let

$$I_\epsilon(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 + \epsilon^2 v^2 - \int_{\partial \Omega} e^v.$$

Then $F_\epsilon(\zeta) = J_\epsilon(U(\zeta) + \tilde{\phi}(\zeta)) = I_\epsilon(V(\zeta') + \phi(\zeta'))$, where $\zeta' = \zeta/\epsilon$. Therefore

$$\frac{\partial F_\epsilon}{\partial \zeta_k} = \frac{1}{\epsilon} \frac{\partial I_\epsilon(V(\zeta') + \phi(\zeta'))}{\partial \zeta_k} = \frac{1}{\epsilon} D I_\epsilon(V(\zeta') + \phi(\zeta')) \left[ \frac{\partial V(\zeta')}{\partial \zeta_k} + \frac{\partial \phi(\zeta')}{\partial \zeta_k} \right].$$

Since $v = V(\zeta') + \phi(\zeta')$ solves (88)

$$\frac{\partial F_\epsilon}{\partial \zeta_k} = \frac{1}{\epsilon} \sum_{i=1}^{m} c_i \int_{\partial \Omega} Z_i \left[ \frac{\partial V(\zeta')}{\partial \zeta_k} + \frac{\partial \phi(\zeta')}{\partial \zeta_k} \right].$$
Let us assume that \( DF(\zeta) = 0 \). From the previous equation we conclude that

\[
\sum_{i=1}^{m} c_i \int_{\tilde{c}\Omega_\varepsilon} \chi_i Z_{1i} \left[ \frac{\partial V(\zeta')}{\partial \xi_k} + \frac{\partial \phi(\zeta')}{\partial \xi_k} \right] = 0 \quad \forall k = 1, \ldots, m.
\]

Since \( \| \frac{\partial \phi(\zeta')}{\partial \xi_k} \|_{L^\infty(\Omega_\varepsilon)} \leq C \varepsilon^2 |\log \varepsilon|^2 \) and \( \frac{\partial V(\zeta')}{\partial \xi_k} = \pm Z_{1k} + o(1) \) where \( o(1) \) is in the \( L^\infty \) norm, it follows that

\[
\sum_{i=1}^{m} c_i \int_{\tilde{c}\Omega_\varepsilon} \chi_i Z_{1i} (\pm Z_{1k} + o(1)) = 0 \quad \forall k = 1, \ldots, m,
\]

which is a strictly diagonal dominant system. This implies that \( c_i = 0 \) \( \forall i = 1, \ldots, m \). \( \square \)

In order to solve for critical points of the function \( F \), a key step is its expected closeness to the function \( J_\varepsilon(U) \), which we will analyze in the next section.

**Lemma 6.2.** The following expansion holds:

\[
F_\varepsilon(\zeta) = J_\varepsilon(U) + \theta_\varepsilon(\zeta),
\]

where

\[
|\theta_\varepsilon| + | \nabla \theta_\varepsilon | \to 0,
\]

uniformly on points satisfying the constraints (36).

**Proof.** Let \( \tilde{\theta}_\varepsilon(\zeta') = I_\varepsilon(V + \phi) - I_\varepsilon(V) \). In order to get the proof of this lemma, we need to show that

\[
|\tilde{\theta}_\varepsilon| + \varepsilon^{-1} | \nabla \tilde{\theta}_\varepsilon | = o(1).
\]

Taking into account \( DI_\varepsilon(V + \phi)[\phi] = 0 \), a Taylor expansion and an integration by parts give

\[
\begin{align*}
I_\varepsilon(V + \phi) - I_\varepsilon(V) & = \int_{0}^{1} D^2 I_\varepsilon(V + t\phi)[\phi]^2 (1 - t) \, dt \\
& = \int_{0}^{1} \left( \int_{\tilde{c}\Omega_\varepsilon} [N(\phi) + R] \phi + \int_{\tilde{c}\Omega_\varepsilon} e^V [1 - e^{t\phi}] \phi^2 \right) (1 - t) \, dt. \quad (94)
\end{align*}
\]
so we get

$$I_\varepsilon(V + \phi) - I_\varepsilon(V) = \tilde{\Theta}_\varepsilon = O(\varepsilon^{2z}|\log \varepsilon|^3).$$

taking into account that $\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C\varepsilon^z|\log \varepsilon|$. Let us differentiate with respect to $\xi_k$

$$\partial_{\xi_k}^z [I_\varepsilon(V + \phi) - I_\varepsilon(V)] = \int_0^1 \left( \int_{\partial \Omega_\varepsilon} \partial_{\xi_k}^z [(N(\phi) + R) \phi] + \int_{\partial \Omega_\varepsilon} \partial_{\xi_k}^z [e^V [1 - e^{t\phi} \phi]^2] \right)(1 - t) \, dt.$$

Using the fact that $\|\partial_{\xi_k}^z \phi\|_{*} \leq C \varepsilon^{2z}|\log \varepsilon|^2$ and the estimates of the previous sections we get

$$\partial_{\xi_k}^{zkl} [I_\varepsilon(V + \phi) - I_\varepsilon(V)] = \partial_{\xi_k}^{zkl} \tilde{\Theta}_\varepsilon = O(\varepsilon^{2z}|\log \varepsilon|^4).$$

The continuity in $\tilde{\xi}$ of all these expressions is inherited from that of $\phi$ and its derivatives in $\tilde{\xi}$ in the $L^\infty$ norm. The proof is complete. □

7. Expansion of the energy

**Lemma 7.1.** Let $\mu_j$ be given by (24). Then for any $0 < z < 1$

$$J_\varepsilon(U) = m(\beta - 2\pi + 2\pi \log 2) + 2\pi m \log \frac{1}{\varepsilon} - \pi \sum_{j=1}^{m} \left[ H(\tilde{\xi}_j, \tilde{\xi}_j) + \sum_{i \neq j} G(\tilde{\xi}_j, \tilde{\xi}_j) \right] + O(\varepsilon^{2z}),$$

where

$$\beta = \int_{-\infty}^{\infty} \frac{1}{1 + x^2} \log \frac{1}{1 + x^2} \, dx.$$

**Proof.** Define

$$U_j(x) = u_j(x) + H_j^\varepsilon(x),$$
so we may rewrite (16) in equivalent form \( U = \sum_{j=1}^{m} U_j \). Then

\[
J_{\varepsilon}(U) = \frac{1}{2} \int_{\Omega} \left( \sum_{j=1}^{m} \nabla U_j \right)^2 + \frac{1}{2} \int_{\Omega} \left( \sum_{j=1}^{m} U_j \right)^2 - \varepsilon \int_{\partial \Omega} \exp \left( \sum_{j=1}^{m} U_j \right)
\]

\[
= \sum_{j=1}^{m} \int_{\Omega} \left( |\nabla U_j|^2 + U_j^2 \right) + \sum_{i \neq j}^{m} \int_{\Omega} \left( \nabla U_i \nabla U_j + U_i U_j \right) - \varepsilon \int_{\partial \Omega} \exp \left( \sum_{j=1}^{m} U_j \right)
\]

\[= I_A + I_B + I_C.\]

Let us analyze the behavior of \( I_A \). We have

\[
\int_{\Omega} |\nabla U_j|^2 + U_j^2 = \int_{\Omega} |\nabla u_j|^2 + \int_{\Omega} u_j^2 + \int_{\Omega} |\nabla H_{\varepsilon j}|^2 + \int_{\Omega} (H_{\varepsilon j})^2 + 2 \int_{\Omega} \nabla u_j \nabla H_{\varepsilon j}
\]

\[= I_D + I_E + I_F.\]

Multiplying (17) by \( H_{\varepsilon j} \) yields

\[
\int_{\Omega} |\nabla H_{\varepsilon j}|^2 + (H_{\varepsilon j})^2 = - \int_{\Omega} u_j H_{\varepsilon j} + \int_{\partial \Omega} \frac{\partial H_{\varepsilon j}}{\partial \nu} H_{\varepsilon j}
\]

\[= - \int_{\Omega} u_j H_{\varepsilon j} + \varepsilon \int_{\partial \Omega} e^{u_j} H_{\varepsilon j} - \int_{\partial \Omega} \frac{\partial u_j}{\partial \nu} H_{\varepsilon j},\]

and replacing in (95) we obtain

\[
\int_{\Omega} |\nabla U_j|^2 + U_j^2 = \int_{\Omega} |\nabla u_j|^2 + \int_{\Omega} u_j^2 + 2 \int_{\Omega} \nabla u_j \nabla H_{\varepsilon j} + \int_{\Omega} u_j H_{\varepsilon j}
\]

\[= I_D + I_E + I_F + \varepsilon \int_{\partial \Omega} e^{u_j} H_{\varepsilon j} - \int_{\partial \Omega} \frac{\partial u_j}{\partial \nu} H_{\varepsilon j}.\]

Multiplying (17) by \( u_j \) and integrating we find

\[
\int_{\Omega} u_j^2 + \int_{\Omega} H_{\varepsilon j} u_j = - \int_{\Omega} \nabla H_{\varepsilon j} \nabla u_j + \varepsilon \int_{\partial \Omega} e^{u_j} u_j - \int_{\partial \Omega} \frac{\partial u_j}{\partial \nu} u_j.
\]

Combining this and (97) we arrive at

\[
\int_{\Omega} |\nabla U_j|^2 + U_j^2 = \varepsilon \int_{\partial \Omega} e^{u_j} (u_j + H_{\varepsilon j}).
\]
where we have used

\[
\int_{\Omega} \nabla u_1 \nabla u_j = \int_{\partial \Omega} \frac{\partial u_1}{\partial \nu} u_j, \quad \int_{\Omega} \nabla u_1 \nabla H^i_j = \int_{\partial \Omega} \frac{\partial u_1}{\partial \nu} H^i_j
\]

(99)

with \(i = j\). Let us find the asymptotic behavior of the expression:

\[
\int_{\Omega} |\nabla U_j|^2 + U_j^2 = \varepsilon \int_{\partial \Omega_{\varepsilon \mu j}} \frac{2\mu_j}{|x - \xi_j - \varepsilon \mu_j v(\xi_j)|^2} \left( \log \frac{1}{|x - \xi_j - \varepsilon \mu_j v(\xi_j)|^2} + H(x, \xi_j) + O(\varepsilon^2) \right).
\]

Changing variables \(\varepsilon \mu_j y = x - \xi_j\)

\[
\int_{\Omega} |\nabla U_j|^2 + U_j^2 = \int_{\partial \Omega_{\varepsilon \mu j}} \frac{2}{|y - v(0)|^2} \left( \log \frac{1}{|y - v(0)|^2} + H(\xi_j + \varepsilon \mu_j y, \xi_j) - 2 \log(\varepsilon \mu_j) \right) + O(\varepsilon^2).
\]

But

\[
\int_{\partial \Omega_{\varepsilon \mu j}} \frac{2}{|y - v(0)|^2} = \pi + O(\varepsilon), \quad \int_{\partial \Omega_{\varepsilon \mu j}} \frac{2}{|y - v(0)|^2} \log \frac{1}{|y - v(0)|^2} = \beta + O(\varepsilon^2).
\]

and for \(0 < \varepsilon < 1\)

\[
\int_{\partial \Omega_{\varepsilon \mu j}} \frac{2}{|y - v(0)|^2} (H(\varepsilon \mu_j y, \xi_j) - H(\xi_j, \xi_j)) = \int_{\partial \Omega_{\varepsilon \mu j}} \frac{2}{|y - v(0)|^2} O(\varepsilon^2 |y|^2) = O(\varepsilon^2).
\]

Therefore

\[
\int_{\Omega} |\nabla U_j|^2 + U_j^2 = 2\beta + 2\pi H(\xi_j, \xi_j) - 4\pi \log(\varepsilon \mu_j)
\]

\[
+ \int_{\partial \Omega_{\varepsilon \mu j}} \frac{2}{|y - v(0)|^2} (H(\varepsilon \mu_j y, \xi_j) - H(\xi_j, \xi_j)) + O(\varepsilon^2)
\]

\[
= 2\beta + 2\pi H(\xi_j, \xi_j) - 4\pi \log(\varepsilon \mu_j) + O(\varepsilon^2).
\]
Thus

\[ I_A = m\beta + 2\pi m \log \frac{1}{\varepsilon} + \pi \sum_{j=1}^{m} \left[ H(\zeta_j, \zeta_j) - 2 \log(\varepsilon \mu_j) \right] + O(\varepsilon^2). \]  

(100)

We consider now

\[ I_B = \sum_{i \neq j}^{m} \int_{\Omega} \nabla U_i \nabla U_j + U_i U_j \]

\[ = \sum_{i \neq j}^{m} \int_{\Omega} \nabla u_i \nabla u_j + 2 \int_{\Omega} \nabla u_i \nabla H_{\varepsilon}^j + \int_{\Omega} \nabla H_{\varepsilon}^i \nabla H_{\varepsilon}^j + \int_{\Omega} u_i u_j + 2 \int_{\Omega} u_i H_{\varepsilon}^j \]

\[ + \int_{\Omega} H_{\varepsilon}^i \nabla H_{\varepsilon}^j. \]

Multiplying Eq. (17) by \( H_{\varepsilon}^i \) and integrating we find

\[ \int_{\Omega} \nabla H_{\varepsilon}^j \nabla H_{\varepsilon}^i + \int_{\Omega} H_{\varepsilon}^i H_{\varepsilon}^j = - \int_{\Omega} u_j H_{\varepsilon}^i + \varepsilon \int_{\Omega} \frac{\partial u_j}{\partial v} H_{\varepsilon}^i. \]

Hence

\[ I_B = \sum_{i \neq j}^{m} \int_{\Omega} \nabla u_i \nabla u_j + 2 \int_{\Omega} \nabla u_i \nabla H_{\varepsilon}^j + \int_{\Omega} u_i u_j + \int_{\Omega} u_i H_{\varepsilon}^j + \varepsilon \int_{\Omega} \frac{\partial u_j}{\partial v} H_{\varepsilon}^i \]

\[ - \int_{\Omega} \frac{\partial u_j}{\partial v} H_{\varepsilon}^i. \]

Multiplication of (17) by \( u_i \) and integration by parts yields

\[ \int_{\Omega} u_j u_i + \int_{\Omega} H_{\varepsilon}^j u_i = - \int_{\Omega} \nabla H_{\varepsilon}^j \nabla u_i + \varepsilon \int_{\Omega} \frac{\partial u_j}{\partial v} u_i - \int_{\Omega} \frac{\partial u_j}{\partial v} u_i. \]

Replacing in the expression above and using (99) we find

\[ I_B = \frac{\varepsilon}{2} \sum_{i \neq j}^{m} \int_{\Omega} e^{u_i} (u_j + H_{\varepsilon}^j). \]
A similar argument as for $I_A$ shows that
\[
I_B = \pi \sum_{i \neq j}^m G(\xi_i, \xi_j) + O(\varepsilon^2). \tag{101}
\]

Regarding the expression $I_C$ we have
\[
I_C = -\varepsilon \int_{\partial \Omega} e^{\sum_{j=1}^m U_j} \quad = -\varepsilon \sum_{k=1}^m \int_{\partial \Omega} e^{\sum_{j=1}^m u_j + H_j^r}.
\]

Using the definition of $u_j$ and (18) for each term we have
\[
\varepsilon \int_{\partial \Omega} e^{\sum_{j=1}^m u_j + H_j^r} = \varepsilon \int_{\partial \Omega} \frac{e^{H(x, \xi_j) + O(\varepsilon^2)}}{|x - \xi_j - \varepsilon \mu_j v(\xi_j)|^2} E_j(x),
\]
where
\[
E_j(x) = \exp \left( \sum_{i \neq j} \log \frac{1}{|x - \xi_i - \varepsilon \mu_i v(\xi_i)|} + H(x, \xi_i) + O(\varepsilon^2) \right).
\]

Changing variables $\varepsilon \mu_j y = x - \xi_j$ we have
\[
e^{H(\xi_j + \varepsilon \mu_j y, \xi_j) + O(\varepsilon^2)} = e^{H(\xi_j, \xi_j) + O(\varepsilon^2 |y|^2)}
\]
and
\[
E_j(\xi_j + \varepsilon \mu_j y, \xi_j) = \exp \left( \sum_{i \neq j} \log \frac{1}{|\xi_j - \xi_i + \varepsilon \mu_j y - \varepsilon \mu_i v(\xi_i)|^2} \right.
\]
\[
+ H(\xi_j + \varepsilon \mu_j y, \xi_i) + O(\varepsilon^2) \right)
\]
\[
= \exp \left( \sum_{i \neq j} \log \frac{1}{|\xi_i - \xi_j|^2 + H(\xi_j, \xi_i)} \right) + O(\varepsilon^2 |y|^2)
\]
\[
= \exp \left( \sum_{i \neq j} G(\xi_j, \xi_i) \right) + O(\varepsilon^2 |y|^2).
\]
Therefore, by the definition of $\mu_j$ in (24)
\[ \varepsilon \int_{\tilde{\Omega}} e^{\sum_{j=1}^{m} a_j + H_j} = \frac{1}{\mu_j} \pi e^{H(\xi_j, \bar{\xi}_j) + \sum_{i \neq j} G(\xi_j, \bar{\xi}_i)} + O(\varepsilon^2) \]
\[ = 2\pi + O(\varepsilon^2). \]
Thus
\[ I_C = -2\pi m + O(\varepsilon^2). \quad (102) \]

Thanks to (100)–(102) we have
\[ J_\varepsilon(U) = m(\beta - 2\pi) + 2\pi m \log \frac{1}{\varepsilon} + \pi \sum_{j=1}^{m} \left[ -2 \log(\mu_j) + H(\bar{\xi}_j, \bar{\xi}_j) \right. \]
\[ + \left. \sum_{i \neq j} G(\bar{\xi}_i, \bar{\xi}_j) \right] + O(\varepsilon^2). \]

Employing again (24) we have
\[ J_\varepsilon(U) = m(\beta - 2\pi) + 2\pi m \log \frac{1}{\varepsilon} + 2\pi m \log 2 - \pi \sum_{j=1}^{m} \left[ H(\bar{\xi}_j, \bar{\xi}_j) + \sum_{i \neq j} G(\bar{\xi}_i, \bar{\xi}_j) \right] \]
\[ + O(\varepsilon^2). \quad \square \]

8. Proof of Theorem 1.2

Let $\hat{\Omega}_m = (\tilde{\Omega})^m \setminus D$, where $D$ denotes the diagonal. Namely,
\[ \hat{\Omega}_m = \{(\xi_1, \ldots, \xi_m) \in (\tilde{\Omega})^m : \xi_i \neq \xi_j \text{ if } i \neq j \}. \]

Proof. According to Lemma 6.2, the function $U(\xi) + \hat{\phi}(\xi)$, where $U$ and $\hat{\phi}$ are defined, respectively, by (16) and (93), is a solution of Problem (1) if we adjust $\xi$ so that it is a critical point of $F_\varepsilon(\hat{\xi}) = J_\varepsilon(U(\hat{\xi}) + \hat{\phi}(\hat{\xi}))$ defined by (92). This is obviously equivalent to finding a critical point of
\[ \hat{F}_\varepsilon(\hat{\xi}) = \frac{1}{\pi} \left( F_\varepsilon(\hat{\xi}) - m\beta + 2\pi m(1 - \log 2) + 2\pi m \log \varepsilon \right). \]
On the other hand, from Lemmas 6.2 and 7.1, we have that for \( \xi \in \tilde{\Omega}_m \), such that its components satisfy \( |\xi_i - \xi_j| \geq d \),

\[
\tilde{F}_\varepsilon(\tilde{\xi}) = \varphi_m(\tilde{\xi}) + \varepsilon^2 \Theta_\varepsilon(\tilde{\xi}),
\]

(103)

where \( \Theta_\varepsilon \) and \( \nabla_\xi \Theta_\varepsilon \) are uniformly bounded in the considered region as \( \varepsilon \to 0 \).

Given one component \( C_0 \) of \( \tilde{\Omega} \), let \( \Lambda : S^1 \to C_0 \) be a continuous bijective function that parametrizes \( C_0 \). We call \( \tilde{\Omega}_m \) the region in \( C_m^0 \setminus D \), where \( |\xi_i - \xi_j| > d \) and we show that \( \varphi_m \) has at least two distinct critical points in \( \tilde{\Omega}_m \).

The function \( \varphi_m \) is \( C^1 \), bounded from above in \( \hat{\Omega}_m \) (and hence in \( \tilde{\Omega}_m \)) and such that

\[
\varphi_m(\xi_1, \ldots, \xi_m) \to -\infty \text{ as } |\xi_i - \xi_j| \to 0 \text{ for some } i \neq j.
\]

Hence, since \( d \) is arbitrarily small, \( \varphi_m \) has an absolute maximum \( M \) in \( \tilde{\Omega}_m \).

On the other hand, the Ljusternik–Schnirelmann theory is applicable in our setting so that the number of critical points for \( \varphi_m \) can be estimate from below by \( \text{cat}(\tilde{\Omega}_m) \), the Ljusternik–Schnirelmann category of \( \tilde{\Omega}_m \) relative to \( \tilde{\Omega}_m \). Let us recall that \( \text{cat}(\tilde{\Omega}_m) \) is the minimal number of closed and contractible in \( \tilde{\Omega}_m \) sets whose union covers \( \tilde{\Omega}_m \).

Observe that \( \text{cat}(\tilde{\Omega}_m) \geq 1 \). Indeed, by contradiction, assume that \( \text{cat}(\tilde{\Omega}_m) = 1 \). This means that \( \tilde{\Omega}_m \) is contractible in itself, namely there exist a point \( \bar{\xi} \in \tilde{\Omega}_m \) and a continuous function \( \Gamma : [0, 1] \times \tilde{\Omega}_m \to \tilde{\Omega}_m \), such that, for all \( \xi \in \tilde{\Omega}_m \),

\[
\Gamma(0, \xi) = \bar{\xi}, \quad \Gamma(1, \xi) = \bar{\xi}^0.
\]

Define \( f : S^1 \to \tilde{\Omega}_m \) to be the continuous function given by

\[
f(\bar{\xi}) = (\Lambda(\bar{\xi}), \Lambda(e^{2\pi i \frac{1}{m}} \bar{\xi}), \ldots, \Lambda(e^{2\pi i \frac{m-1}{m}} \bar{\xi})).
\]

Let \( \eta : [0, 1] \times S^1 \to S^1 \) be the well defined continuous map given by

\[
\eta(t, \xi) = \Lambda^{-1} \circ \pi_1 \circ \Gamma(t, f(\bar{\xi})),
\]

where \( \pi_1 \) denotes the projection on the first component. The function \( \eta \) is a contraction of \( S^1 \) to a point and this gives a contradiction.

Thus we conclude that \( \text{cat}(\tilde{\Omega}_m) \geq 2 \), for any \( m \geq 1 \). Hence, if we define

\[
\Xi = \{ C \subset \tilde{\Omega}_m : C \text{ closed and } \text{cat}(C) \geq 2 \}.
\]
Ljusternik–Schnirelmann theory gives that $c$ is a critical level.

If $c \neq M$, we conclude that there are at least two distinct critical points for $\varphi_m$ in $\tilde{\Omega}_m$. If $c = M$, hence (104) implies that there is at least one set $C$, with $\text{cat}(C) \geq 2$, where the function $\varphi_m$ reaches its absolute maximum. In this case we conclude that there are infinitely many critical points for $\varphi_m$ in $\tilde{\Omega}_m$.

These critical points persist under small $C^0$-perturbation of the function. For this reason, from (103) we can conclude that also the function $\tilde{F}_\varepsilon$, which is $C^0$ close to $\varphi_m$ in $\tilde{\Omega}_m$, has at least two distinct critical points in $\tilde{\Omega}_m$. Since $d$ is arbitrarily small, $\tilde{F}_\varepsilon$ has at least two critical points in $\tilde{\Omega}_m$ and hence problem (1) has at least two distinct solutions. □

Remark 8.1. As mentioned in the introduction, one can get a stronger result than Theorem 1.2 under the assumption that the function $\varphi_m$ has, in addition to the ones described in the proof of Theorem 1.2, some other critical points in $\tilde{\Omega}_m$ with the property of being \textit{topologically nontrivial}, for instance (possibly degenerate) local minima or maxima, or saddle points.

Let us define what we mean by \textit{topologically nontrivial} critical point for $\varphi_m$.

Let $\Sigma$ be an open set compactly contained in $\tilde{\Omega}_m$ with smooth boundary. We recall that $\varphi_m$ \textit{links in} $\Sigma$ at critical level $C$ relative to $B$ and $B_0$ if $B$ and $B_0$ are closed subsets of $\Sigma$ with $B$ connected and $B_0 \subset B$ such that the following conditions hold: let us set $\Psi \in C([0, 1] \times B, \Sigma)$, such that

$$
\Psi(0, \cdot) = \text{Id}_B, \quad \Psi(1, \cdot) = \Phi, \quad \Psi(t, \cdot)|_{B_0} = \text{Id}_{B_0} \text{ for all } t \in [0, 1].
$$

We assume

$$
\sup_{y \in B_0} \varphi_m(y) < C \equiv \inf_{\Phi \in \Gamma} \sup_{y \in B} \varphi_m(\Phi(y)) \quad (105)
$$

and for all $y \in \partial \Sigma$, such that $\varphi_m(y) = C$, there exists a vector $\tau_y$ tangent to $\partial \Sigma$ at $y$ such that

$$
\nabla \varphi_m(y) \cdot \tau_y \neq 0. \quad (106)
$$

Under these conditions a critical point $\tilde{y} \in \Sigma$ of $\varphi_m$ with $\varphi_m(\tilde{y}) = C$ exists. Not only this: any function $C^1$ close to $\varphi_m$ inherits such critical point.
Going back to our problem, Lemma 6.2 and 7.1 yield that, if $\varphi_m$ has a topologically nontrivial critical point $\bar{\xi} = (\xi_1, \ldots, \xi_m)$ in $\hat{\Omega}_m$ which satisfies (36), then $F_\varepsilon$ itself has a critical point $\xi_\varepsilon = (\xi_1^\varepsilon, \ldots, \xi_m^\varepsilon)$, close to $\bar{\xi}$ for $\varepsilon$ small, such that

$$\nabla \varphi_m(\xi_1^\varepsilon, \ldots, \xi_m^\varepsilon) \to 0, \quad \varphi_m(\xi_1^\varepsilon, \ldots, \xi_m^\varepsilon) \to C.$$

Hence Lemma 6.1 guarantees the existence of a solution $u_\varepsilon$ for (1). Furthermore, from the ansatz (16), we get that, as $\varepsilon \to 0$, $u_\varepsilon$ remains uniformly bounded on $\Omega \setminus \bigcup_{j=1}^m B_\delta(\xi_j^\varepsilon)$, and

$$\sup_{B_\delta(\xi_j^\varepsilon)} u_\varepsilon \to +\infty$$

for any $\delta > 0$.

9. Blow up behavior as $\varepsilon \to 0$

In this section we give a proof of Theorem 1.1, but before we need a couple of preliminaries.

Consider the linear equation

$$\begin{cases}
-\Delta u + u = 0 \quad &\text{in } \Omega, \\
\frac{\partial u}{\partial v} = h \quad &\text{on } \partial \Omega
\end{cases} \quad (107)$$

with $h \in L^1(\partial \Omega)$.

The next result is a variant of an estimate of Brezis and Merle [2].

Lemma 9.1. For any $0 < k < \pi$ there is a constant $C$ depending on $k$ and $\Omega$ such that for any $h \in L^1(\partial \Omega)$ and $u$ the solution of (107) we have

$$\int_{\partial \Omega} \exp \left( \frac{k |u(x)|}{\|h\|_{L^1(\partial \Omega)}} \right) \, dx \leq C.$$

Proof. We have the representation formula

$$u(x) = \int_{\partial \Omega} \bar{G}(x, y) h(y) \, dy, \quad (108)$$
where $\overline{G} = \frac{1}{2\pi} G$ and $G$ is Green’s function defined in (5). Hence

$$\int_{\partial \Omega} \exp \left( \frac{k|u(x)|}{\|h\|_{L^1(\partial \Omega)}} \right) \, dx \leq \int_{\partial \Omega} \exp \left( k \int_{\partial \Omega} \frac{|\overline{G}(x, y)| |h(y)|}{\|h\|_{L^1(\partial \Omega)}} \, dy \right) \, dx.$$ 

Using Jensen’s inequality we find

$$\int_{\partial \Omega} \exp \left( \frac{k|u(x)|}{\|h\|_{L^1(\partial \Omega)}} \right) \, dx \leq \int_{\partial \Omega} \int_{\partial \Omega} \exp \left( k|\overline{G}(x, y)| \frac{|h(y)|}{\|h\|_{L^1(\partial \Omega)}} \right) \, dy \, dx.$$ 

But $|\overline{G}(x, y)| \leq |\log |x - y||/\pi + C$ so

$$\exp(k|\overline{G}(x, y)|) \leq \frac{C'}{|x - y|^{k/\pi}}$$

for all $x, y$ since we are in a bounded domain. Therefore

$$\int_{\partial \Omega} \exp \left( \frac{k|u(x)|}{\|h\|_{L^1(\partial \Omega)}} \right) \, dx \leq C' \int_{\partial \Omega} \int_{\partial \Omega} \frac{1}{|x - y|^{k/\pi}} \, ds(x) \frac{|h(y)|}{\|h\|_{L^1(\partial \Omega)}} \, dy$$

$$\leq C'' \frac{1}{1 - k/\pi}. \quad \Box$$

We also need the following “strong maximum principle”.

**Lemma 9.2.** There exists a constant $c > 0$ such that for all $h \in L^1(\partial \Omega)$ with $h \geq 0$, the solution $u$ of (107) satisfies

$$u(x) \geq c \int_{\partial \Omega} h \, ds \quad \text{a.e. } \Omega$$

**Proof.** First note that $G \geq 0$ and by the classical strong maximum principle, for each $y \in \partial \Omega$ $G(\cdot, y)$ cannot attain its minimum in $\Omega$. Also, by the Hopf lemma if $G(x, y) = 0$ for some $x, y \in \partial \Omega, x \neq y$ then the normal derivative $\frac{\partial G}{\partial \nu_x}(x, y)$ is negative, which is impossible. Therefore, for each $y \in \partial \Omega$ we have $G(\cdot, y) > 0$ in $\overline{\Omega}$.

By a compactness argument we can find a constant $c > 0$ such that

$$G(x, y) \geq c$$
for all \( y \in \partial \Omega \) and all \( x \in \overline{\Omega} \). If \( h \in L^1(\partial \Omega) \), \( h \geq 0 \), from the representation formula (108) we see that the conclusion holds. \( \square \)

Let \( u_\varepsilon \) be family of solutions to (1). If \( \varepsilon e^{u_\varepsilon} \) is unbounded in \( L^1(\partial \Omega) \) then by Lemma 9.2 we see that for a subsequence \( u_\varepsilon \not\rightarrow \infty \) uniformly in \( \overline{\Omega} \).

**Proof of Theorem 1.1.** The first part of this proof is an adaptation of the argument used in [2]. Since we assume that \( \varepsilon e^{u_\varepsilon} \) is bounded in \( L^1(\partial \Omega) \) we can select a sequence \( \varepsilon_j \rightarrow 0 \) and a Radon measure \( \gamma \geq 0 \) in \( \partial \Omega \) such that \( \varepsilon_j e^{u_j} \rightharpoonup \gamma \) weakly in the sense of Radon measures in \( \partial \Omega \) where

\[ u_j = u_{\varepsilon_j}. \]

We keep this notation throughout the rest of this section.

**Claim.** There is a constant \( \beta_0 > 0 \), such that if for some \( x \in \partial \Omega \) we have

\[ \gamma(\{x\}) \leq \beta_0 \]

then there exists \( R > 0 \) so that

\[ \limsup_j \|u_j\|_{L^\infty(\Omega \cap B_R(x))} < \infty. \] (109)

Indeed, fix some \( p > 1 \) and choose \( \beta_0 = \frac{\pi}{4p} \). Let \( B_R(x) \) denote the open ball with center at \( x \) and radius \( R > 0 \). Note that \( \gamma(B_R(x)) \rightarrow \gamma(\{x\}) \) as \( R \rightarrow 0^+ \) so we can select \( R > 0 \) so that

\[ \gamma(B_{2R}(x)) \leq 2\beta_0 \]

and from now on we fix this \( R > 0 \) depending only on \( x \).

By standard properties of the weak convergence of Radon measures

\[ \limsup_j \varepsilon_j \int_{\partial \Omega \cap B_R(x)} e^{u_j} ds \leq \gamma(B_R(x)) \leq 2\beta_0. \] (110)

Let \( a_j = \varepsilon_j e^{u_j} \chi_{B_R(x)} \) and \( v_j \) be the solution of

\[
\begin{cases}
-\Delta v_j + v_j = 0 & \text{in } \Omega, \\
\frac{\partial v_j}{\partial v} = a_j & \text{on } \partial \Omega.
\end{cases}
\]
Let also $b_j = \varepsilon e^{aj} - a_j$ and $w_j = u_j - v_j$. Note that $b_j \equiv 0$ in $B_R(x)$ therefore by elliptic estimates

$$\|w_j\|_{L^\infty(B_{R/2}(x))} \leq C \|w_j\|_{L^1(B_R(x))} \leq C.$$ 

Therefore

$$\int_{\partial \Omega \cap B_{R/2}(x)} (\varepsilon e^{aj})^p ds = \varepsilon^p \int_{\partial \Omega \cap B_{R/2}(x)} e^{pw_j} e^{pv_j} \leq C \varepsilon^p \int_{\partial \Omega \cap B_{R/2}(x)} e^{pv_j} \leq C \varepsilon^p \int_{\partial \Omega} \exp \left( k_j \frac{v_j}{\|a_j\|_{L^1(\partial \Omega)}} \right),$$

(111)

where $k_j = p \|a_j\|_{L^1(\partial \Omega)}$. But observe that by (110) and the definition of $a_j$ we have

$$\limsup_{j} k_j = \limsup_{j} p \|a_j\|_{L^1(\partial \Omega)} \leq 2p \beta_0 < \pi.$$ 

Hence from (111) and Lemma 9.1 we find

$$\int_{\partial \Omega \cap B_{R/2}(x)} (\varepsilon e^{aj})^p ds \leq C \varepsilon^p \to 0.$$ 

This inequality and elliptic estimates imply that

$$\limsup_{j} \|u_j\|_{L^\infty(\Omega \cap B_{R/4}(x))} < \infty$$

which is the desired conclusion.

Let $S$ denote the set

$$S = \{ x \in \partial \Omega \mid \gamma([x]) > \beta_0 \}.$$ 

Then $S$ is finite and for every $x \in \partial \Omega \setminus S$ we have that $u_j$ is bounded in a neighborhood of $x$. Therefore $u_j$ is bounded in compact subsets of $\partial \Omega \setminus S$ and so $\varepsilon e^{uj} \to 0$ uniformly on compact subsets of $\partial \Omega \setminus S$. This shows that the support of $m$ is contained in $S$ and therefore we can write

$$\gamma = \sum_{j=1}^{m} a_j \delta_{\xi_j},$$
where \( a_j > 0 \) and \( \xi_j \in \partial \Omega \). From the preceding remarks we see that \( u_j \to u^* \) and \( u^* \) satisfies

\[
\begin{cases}
-\Delta u^* + u^* = 0 & \text{in } \Omega, \\
\frac{\partial u^*}{\partial v} = \sum_{j=1}^{m} a_j \delta_{\xi_j} & \text{on } \partial \Omega.
\end{cases}
\]

From this it follows that

\[
 u^*(x) = \frac{1}{2\pi} \sum_{j=1}^{m} a_j G(x, \xi_j).
\]  

We shall now prove (7) through Pohozaev-type identities in balls around the singularities. Let us concentrate on \( \xi_1 \) and assume that it is located in the origin. For the computations we will make a change of variables to flatten the boundary of \( \Omega \) around 0. Pick some radius \( R_0 \) small enough and consider a map \( \Psi : H \cap B_{R_0} \to \Omega \cap B_r \), where \( H = \{(y_1, y_2) \mid y_2 > 0\} \). We can choose \( \Psi \) to be a conformal diffeomorphism, \( C^3 \) up \( \partial H \cap B_{R_0} \), and such that \( \Psi(0) = 0 \) and \( D\Psi(0) = I \) (after rotation of \( \Omega \)). Define

\[
\tilde{u}_j(y) = u_j(\Psi(y)), \quad y \in H \cap B_{R_0}.
\]

Then \( \tilde{u}_j \) satisfies

\[
\begin{cases}
-\Delta \tilde{u}_j + b(y)\tilde{u}_j = 0 & \text{in } H \cap B_{R_0}, \\
\frac{\partial \tilde{u}_j}{\partial v} = \varepsilon_j h(y)e \tilde{u}_j & \text{on } \partial H \cap B_{R_0},
\end{cases}
\]  

(113)

where \( b \) and \( h \) are smooth functions, given by

\[
 b(y) = |\det D\Psi(y)|, \\
 h(y) = |D\Psi(y)e_1|
\]

and \( e_1 = (1, 0) \). Note, since we assume \( D\Psi(0) = I \) we can drop the absolute values in \( b \) and \( h \).

For simplicity we will drop the index \( j \) in \( \tilde{u}_j \) and we write the partial derivative \( \frac{\partial}{\partial y_k} \) with a subscript \((\cdot)_y \), e.g. \( \frac{\partial \tilde{u}}{\partial y_1} = \tilde{u}_y_1 \). We use the convention of summation over repeated indices, and denote by \( v \) the exterior normal vector to \( \partial (H \cap B_R) \), \( v_1 \) and \( v_2 \) are the components of \( v \) and we write a partial derivative with respect to \( v \) as \( \frac{\partial \tilde{u}}{\partial v} = \tilde{u}_v \).
Take now $0 < R < R_0$ and multiply the equation in (113) by $\tilde{u}_{y_1}$ and integrate on $H \cap B_R$ to find

$$\int_{H \cap B_R} -\tilde{u}_{y_1}^2 \tilde{u}_{y_1} + b(y) \tilde{u}_{y_1} = 0.$$ 

Integrating by parts, and using the boundary condition in (113) we get

$$\varepsilon \int_{\partial H \cap B_R} h e^{\tilde{u}_{y_1}} + \int_{\partial B_R \cap H} \tilde{u}_{y_1} \tilde{u}_{y_1} = \int_{H \cap B_R} \frac{1}{2} (\tilde{u}_{y_1}^2)_{y_1} + \frac{1}{2} b (\tilde{u}^2)_{y_1} = \frac{1}{2} \int_{\partial B_R \cap H} b_{y_1} \tilde{u}_{y_1}^2.$$ (114)

Integrating by parts the first term in (114) we find

$$\int_{\partial H \cap B_R} h e^{\tilde{u}_{y_1}} = h e^{\tilde{u}} \bigg|_{-R}^{R} - \int_{\partial H \cap B_R} h' e^{\tilde{u}}$$

and substituting in (114) we obtain

$$\varepsilon h e^{\tilde{u}} \bigg|_{-R}^{R} - \varepsilon \int_{\partial H \cap B_R} h' e^{\tilde{u}} + \int_{\partial B_R \cap H} \tilde{u}_{y_1} \tilde{u}_{y_1} = \frac{1}{2} \int_{\partial B_R \cap H} (|\nabla \tilde{u}|^2 + b \tilde{u}^2)_{y_1} - \frac{1}{2} \int_{H \cap B_R} b_{y_1} \tilde{u}_{y_1}^2.$$ (115)

Before we take the limit as $j \to \infty$ we recall that $\Psi : H \cap B_{R_0} \to \Omega$ is a conformal map, smooth up to $\partial H \cap B_{R_0}$, and that we assumed $\Psi(0) = 0$ and $D \Psi(0) = I$. Using complex variables $z = y_1 + i y_2$, and expanding $\Psi$ in its Taylor series, we have

$$\Psi(z) = z + \frac{c}{2} z^2 + O(z^3),$$ (116)

where $O(z^3)$ denotes a quantity which is bounded by $|z|^3$ for $z$ in a fixed neighborhood of the origin. Let $\alpha, \beta \in \mathbb{R}$ denote the real and imaginary parts of $c = \Psi''(0)$ that is $c = \alpha + i \beta$. Then $\beta$ is the curvature of the $\partial \Omega$ at 0 and $\alpha$ is curvature at 0 of $t \mapsto \Psi(0, t)$ which is a curve transverse to $\partial \Omega$. We can modify $\Psi$ to prescribe this number. Indeed, consider a change of variables

$$z = \frac{1}{\lambda} (e^{\lambda w} - 1),$$
where \( w \) is in a neighborhood of the origin. For \( \lambda \in \mathbb{R} \) this map restricted to a neighborhood of the origin sends the upper half-plane in itself, the real line into the real line, and the lower half-plane into itself again. A computation shows that the expansion of \( \Psi \) in the variable \( w \) is

\[
\Psi(w) = w + \frac{1}{2}(c + \lambda)w^2 + O(w^3).
\]

Let \( \tilde{u}^*(y) = u^*(\Psi(y)) \) denote the limit function in the \( y \) coordinates, and observe that

\[\tilde{u}_j \to \tilde{u}^* \quad \text{in } C^1_{\text{loc}}(\overline{H} \cap B_R - \{0\}).\]

Taking the limit in (115) we get

\[
-\alpha a_1 + \int_{\partial H \cap \Omega} \tilde{u}_j^* \tilde{u}_j^* = \frac{1}{2} \int_{\partial H \cap \Omega} |\nabla \tilde{u}^*|^2 v_1 + \frac{1}{2} \int_{\partial \Omega} b \tilde{u}^* v_1 \quad \text{in } C^1_{\text{loc}}(H \cap B_R - \{0\}).
\]

We rewrite now (112) into a singular and a regular part near the origin

\[ u^* = s + w, \]

where

\[
s(x) = \frac{a_1}{\pi} \log \frac{1}{|x|} \quad \text{and} \quad w(x) = \frac{a_1}{2\pi} H(x, 0) + \frac{1}{2\pi} \sum_{j=2}^m a_j G(x, \xi_j).
\]

We define then the corresponding functions in the new coordinates

\[ \tilde{s}(y) = s(\Psi(y)), \quad \tilde{w}(y) = w(\Psi(y)), \quad y \in H \cap B_{R_0}. \]

Using this decomposition (117) takes the form

\[
-\alpha a_1 + \int_{\partial H \cap \Omega} \tilde{s}_v \tilde{s}_{y_1} + \tilde{s}_v \tilde{w}_{y_1} + \tilde{s}_{y_1} \tilde{w}_v + \tilde{w}_v \tilde{w}_{y_1} = \int_{\partial H \cap \Omega} \left( \frac{1}{2} |\nabla \tilde{s}|^2 + \nabla \tilde{s} \nabla \tilde{w} + \frac{1}{2} |\nabla \tilde{w}|^2 \right) v_1 + \int_{\partial H \cap \Omega} b \left( \frac{1}{2} \tilde{s}^2 + \tilde{s} \tilde{w} + \frac{1}{2} \tilde{w}^2 \right) v_1 - \int_{H \cap B_R} b_{y_1} \left( \frac{1}{2} \tilde{s}^2 + \tilde{s} \tilde{w} + \frac{1}{2} \tilde{w}^2 \right). \quad (118)
\]
Since \( w \) satisfies \( -\Delta w + w = -\frac{a_1}{\pi} \log \frac{1}{|x|} \) in \( \Omega \) we find for \( \tilde{w} \)
\[
-\Delta \tilde{w} + b(y)\tilde{w} = -\frac{a_1}{\pi} m(y) \quad \text{in \( H \cap B_R \)},
\]
where
\[
m(y) = b(y) \log \frac{1}{|\Psi(y)|}.
\]
Multiplying this equation by \( \frac{\partial \tilde{w}}{\partial y_1} = \tilde{w}_y \) and integrating on \( H \cap B_R \) (similarly as for \( \tilde{u} \))
\[
\int_{\partial B_R \cap H} \tilde{w}_y \tilde{w}_1 = \int_{\partial H \cap B_R} \tilde{w}_2 \tilde{w}_1 - \frac{1}{2} \int_{\partial B_R \cap H} (|\nabla \tilde{w}|^2 + b \tilde{w}^2) v_1 - \frac{1}{2} \int_{H \cap B_R} b_{y_1} \tilde{w}^2
\]
\[
- \frac{a_1}{\pi} \int_{B_R \cap H} m(y) \tilde{w}_1.
\]
Solving for \( \int_{B_R \cap H} \tilde{w}_y \tilde{w}_1 \) in this equation and replacing in (118) we obtain
\[
-za_1 + \int_{\partial B_R \cap H} \tilde{s}_y \tilde{s}_1 + \tilde{s}_y \tilde{w}_1 + \tilde{s}_1 \tilde{w}_y
\]
\[
= \int_{\partial B_R \cap H} \left( \frac{1}{2} |\nabla \tilde{s}|^2 + \nabla \tilde{s} \nabla \tilde{w} \right) v_1 + \int_{\partial B_R \cap H} b \left( \frac{1}{2} \tilde{s}_y^2 + \tilde{s} \tilde{w} \right) v_1
\]
\[
- \int_{H \cap B_R} b_{y_1} \left( \frac{1}{2} \tilde{s}_y^2 + \tilde{s} \tilde{w} \right) - \int_{H \cap B_R} \tilde{w}_{y_2} \tilde{w}_1
\]
\[
+ \frac{a_1}{\pi} \int_{B_R \cap H} m(y) \tilde{w}_1.
\] (119)

Now we take the limit in this relation as \( R \to 0 \).

**Lemma 9.3.** Recall that \( c = \psi''(0) = \alpha + i\beta \). We have
\[
\lim_{R \to 0} \int_{\partial B_R \cap H} \tilde{s}_y \tilde{s}_1 = \frac{3\alpha}{4\pi} a_1^2,
\]
\[
\lim_{R \to 0} \int_{\partial B_R \cap H} \tilde{s}_y \tilde{w}_1 = -a_1 \tilde{w}_{y_1}(0),
\]
\[
\lim_{R \to 0} \frac{1}{2} \int_{\partial B_R \cap H} |\nabla \tilde{s}|^2 v_1 = \frac{\alpha}{4\pi} a_1^2,
\]
\[
\lim_{R \to 0} \int_{\partial B_R \cap H} \nabla \tilde{s} \nabla \tilde{w} v_1 = -\frac{a_1}{2} \tilde{w}_{y_1}(0)
\] (120)

and all other terms in (119) have limit zero as \( R \to 0 \).
We prove this lemma later on.

**Proof of (7) completed.** Using this lemma together with (119) we obtain

\[-\varpi a_1 + \frac{3\varpi}{4\pi} a_1^2 - a_1 \tilde{w}_{y_1}(0) = \frac{\varpi}{4\pi} a_1^2 - \frac{a_1}{2} \tilde{w}_{y_1}(0)\]

that is

\[\varpi a_1 \left( \frac{a_1}{2\pi} - 1 \right) = \frac{1}{2} a_1 \tilde{w}_{y_1}(0). \tag{121}\]

But \(a_1 \neq 0\) and \(\varpi\) can be taken to be any real number, so \(a_1 = 2\pi\) and \(\tilde{w}_{y_1}(0) = 0\), which is equivalent to \(\nabla_\tau w(0) = 0\), where \(\tau\) is tangent to \(\partial \Omega\) at 0. \(\square\)

**Proof of Lemma 9.3.** We present a proof of (120) only, the others being analogous. Recall that

\[\tilde{s}(y) = s(\Psi(y)), \quad s(x) = \frac{a_1}{\pi} \log \frac{1}{|x|}.\]

Using the expansion (116) for \(\Psi\) we have

\[\nabla \tilde{s}(y) = -\frac{a_1}{\pi} \left( \frac{y}{|y|^2} + \frac{1}{2}(\varpi, -\beta) \right) + O(|y|), \tag{122}\]

where we recall that \(c = \varpi + i\beta = \Psi''(0)\). One way to see this is to consider the complex valued function \(\log \Psi\), and express the expansion of its derivative in terms of \(z = y_1 + iy_2\). Using (122) we have

\[\int_{\partial B_R \cap H} \tilde{s}_v \tilde{s}_{y_1} = \frac{a_1^2}{\pi^2} \int_{\partial B_R \cap H} \left( \frac{1}{R} + \frac{1}{2} (\varpi v_1 - \beta v_2) + O(R) \right) \left( \frac{v_1}{R} + \frac{1}{2} \varpi + O(R) \right) ds\]

\[= \frac{a_1^2}{\pi^2} \int_{\partial B_R \cap H} \frac{v_1}{\sqrt{2}} + \frac{1}{2R} (\varpi (1 + v_1^2) - \beta v_1 v_2) + O(1) ds\]

\[= \frac{a_1^2}{\pi^2} \frac{3}{4} \pi \varpi + O(R). \quad \square\]

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