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Large number of fast decay ground states to Matukuma-type equations

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ABSTRACT

In this article we consider the Matukuma type equation

$$\Delta u + K\big(|x|\big)u^p = 0 \quad \text{in } \mathbb{R}^N, \tag{$*$}$$

for positive radially symmetric solutions. When K satisfies some suitable monotonicity assumption, there exists a unique ground state of $(*)$. In this work we find a large class of K functions for which this monotonicity assumption fails and a large number of bubble-tower ground states exist.

1. Introduction

This paper concerns to the study of positive radial solutions for equations of the type

$$\Delta u + K(r)u^p = 0 \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N > 2$, $p > 1$, $|x| = r$ and $K(r) \geqslant 0$, which was proposed by Matukuma [18] as a model in Celestial Mechanics for the dynamics of a cluster of stars, where u is the gravitational potential and $K(r)u^p$ is the density of stars, see Li [13] for more details.

This type of equations has been studied in last decades by many authors under certain monotonicity conditions related with K. Under this assumption the solution set is very simple and there is only one fast decay ground state.

Recently, Felmer and Quaas [7] found examples of K functions such that the solution set becomes very complex and a large number or infinitely many fast decay ground states exist.

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In this article we prove the existence of a large number of bubble-tower fast decay ground states to (1.1) for a large class of K functions. This establishes two interesting features: i) the complexity found in [7] some how persists for a large class of K functions, and ii) this type of equations are closely connected with other type of semilinear elliptic equations without weight function, as we will see below. Moreover, this paper is a first step in relating these types of problems since analogous results hold.

We first start reviewing some known results.

If $K \equiv 1$, then (1.1) is known as Emden–Fowler equation, and there exists only one fast decay ground state, up to scaling, for $p = \frac{N+2}{N-2}$ the critical number. Here we understand as a fast decay ground states a positive solution satisfying $\lim_{r\to\infty} r^{N-2}u(r) = c$ for certain $c > 0$.

When K is given by a pure power function $K = r^{\ell}$, then there is a new shifted critical value which is

$$\frac{N+2+2\ell}{N-2},$$

as was proved by Ni and Nussbaum [20].

To continue with the known results, let us define now the growth rate function of K as

$$P(r) = \frac{rK'(r)}{K(r)}.$$

If this function is not constant, then the critical exponent $\frac{N+2+2P(r)}{N-2}$ will vary with r and the structure will be more complex. Under the condition on P

$$(H) \qquad P(r) \text{ is non-increasing and non-constant over } (0, \infty).$$

Writing

$$\sigma = \lim_{r\to 0} P(r) \quad \text{and} \quad \ell = \lim_{r\to\infty} P(r),$$

then the Sobolev critical number is shifted to an interval (p_∞, p_0) where

$$p_0 = \frac{N+2+2\sigma}{N-2} \quad \text{and} \quad p_\infty = \max\left\{1, \frac{N+2+2\ell}{N-2}\right\}.$$

In this situation Yanagida and Yotsutani [26] proved, under the additional condition $p_0 > 1$, that there is a unique ξ such that the initial value problem

$$u'' + \frac{N-1}{r}u' + K(r)u^p = 0, \quad r > 0, \qquad u(0) = \xi > 0, \qquad u'(0) = 0,$$

has a positive fast decay solution if $p \in (p_\infty, p_0)$.

In another paper García-Huidobro, Manásevich and Yarur [11] studied Eq. (1.1) for the operator p-Laplacian and formulated the growth rate of K by means of a different function

$$m(r) = \frac{2r^N K(r)}{(N-2)\int_0^r s^{N-1}K(s)\,ds}.$$

Moreover, assuming that

$$(\tilde{H}) \qquad m(r) \text{ is non-increasing and non-constant over } (0, \infty),$$

the authors in [11] defined the critical numbers

$$p_0 = \lim_{r \to 0} m(r) - 1 \quad \text{and} \quad p_\infty = \max\Big\{1, \lim_{r \to \infty} m(r) - 1\Big\}$$

and also proved that, for $p \in (p_\infty, p_0)$, a unique fast decay solution exists. Asymptotically, p_∞ and p_0 are equivalent for m and P, but $(\tilde{H})$ and (H) are not. In fact, the authors in [11] gave an example for which condition $(\tilde{H})$ holds while (H) is not true. The existence of a fast decaying solution can be done by a topological argument, however the proof of the uniqueness of the fast decaying solution is highly non-trivial. For the Matukuma equation, that is $K(r) = \frac{1}{1+r^2}$, the uniqueness was first proved by Yanagida in [23]. Since then many authors contributed to the study of this type of equations. For instance, we mention here the work by García-Huidobro, Kufner, Manásevich and Yarur [10], Kawano, Yanagida and Yotsutani [12], Li and Ni [14–16], Ni and Yotsutani [21], and Yanagida and Yotsutani [24, 25].

In the present paper we want to study the case $0 \leqslant \sigma < \ell$ or equivalently $p_0 < p_\infty$ for a general class of K functions, not only for an example as in [7]. Roughly speaking, under this condition the equation behaves like supercritical for small values of r and subcritical for large values of r and the structure of the solution set that appears will be the same as in other equations mixing supercritical and subcritical non-linearities, as we will see next. We start with the problem

$$\Delta u + u^p + u^q = 0 \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

$p < \frac{N+2}{N-2} < q$, first considered by Lin and Ni [17] and further investigated by Bamón, Flores and del Pino [1], Flores [9] through a dynamical system approach and recently Campos [2] which is closely connected with our results.

Another type of equation with this phenomena is

$$\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

with f given by $f(u) = u^p$ if $0 \leqslant u < 1$ and $f(u) = u^q$ if $u \geqslant 1$, where $1 < p < \frac{N+2}{N-2} < q$. When the role of p and q are reversed, the structure of positive solutions has been completely described by Erbe and Tang in [6], see also [22] and [5], where there is a unique fast decay solution.

Now let us state our results. We start with the precise assumption on K. The function $K : [0, +\infty) \mapsto \mathbb{R}$ is non-negative and continuous such that

$$\lim_{r \to 0} \frac{r^{-\sigma} K(r) - C_0}{r^\gamma} = C_1 > 0 \quad \text{and} \quad \lim_{r \to \infty} \frac{K(r)}{r^\ell} = C_\infty > 0, \tag{1.4}$$

where $0 \leqslant \sigma < \ell < 2(\sigma + 2)$ and $\frac{N+\sigma}{2} < \gamma$.

A model case is $K(r) = C_0 r^\sigma + B r^\mu + C_\infty r^\ell$ for $\sigma < \mu < \ell$, with the above condition on the parameters and $B \geqslant 0$.

Our first result gives the existence of a large number of fast decay ground states to (1.1) with an exact asymptotic formula.

Theorem 1.1. *Let $N > 2$. For any $k \in \mathbb{N}$, $k > 2$, there exists $\varepsilon_0 > 0$, such that, for all $0 < \varepsilon < \varepsilon_0$, the problem* (1.1) *with $p = p_0 + \varepsilon$ has a solution u_ε of the form*

$$u_\varepsilon(x) = \xi_\sigma \sum_{i=1}^{k} \frac{\alpha_i^* \varepsilon^{-(i-1-\frac{1}{p_0+1})}}{(1 + (\alpha_i^*)^{p_0-1} \varepsilon^{-(i-1-\frac{1}{p_0+1})(p_0-1)} |x|^{2+\sigma})^{\frac{2}{p_0-1}}} \big(1 + o(1)\big),$$

with $o(1) \to 0$ uniformly in $[0, +\infty)$ as $\varepsilon \to 0$. Here, $\xi_\sigma = (\frac{(N+\sigma)(N-2)}{4C_0})^{\frac{1}{p_0-1}}$ and the α_i^*'s are explicit constants depending only of N and K.

Remark 1.1. a) This bubble-tower type of solution were first found by Chen and Lin [3] in the case of Eq. (1.1) with a function K which is a perturbation from a constant and p is the critical exponent. This result is obtained through an ODE approach. Note that Eq. (1.1) arises also from problems in conformal geometry (see [3] and the references therein).

b) We believe that our result has an analogous for Eq. (1.2) with $1 < p < \frac{N+2}{N-2} < q$.

c) In [26] and [19] the authors show, through some numerical example, that the uniqueness of the fast decay solutions fails.

Finally, we give some how the dual of our main theorem, which corresponds to flat bubbles.

Theorem 1.2. *Let $N > 2$. For any $k \in \mathbb{N}$, $k > 2$, there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$, problem* (1.1) *with $p = p_\infty - \varepsilon$ has a solution u_ε of the form*

$$u_\varepsilon(x) = \xi_\ell \sum_{i=1}^{k} \frac{\beta_i^* \varepsilon^{(i-1-\frac{1}{p_\infty+1})}}{(1 + (\beta_i^*)^{p_\infty-1} \varepsilon^{(i-1-\frac{1}{p_\infty+1})(p_\infty-1)} |x|^{2+\sigma})^{\frac{2}{p_\infty-1}}} \big(1 + o(1)\big),$$

with $o(1) \to 0$ uniformly in $[0, +\infty)$ as $\varepsilon \to 0$. Here, $\xi_\ell = (\frac{(N+\sigma)(N-2)}{4C_\infty})^{\frac{1}{p_\infty-1}}$ and the β_i^*'s are explicit constants depending only of N and K.

The method used in the proof of our two theorems is a variation of Lyapunov–Schmidt reduction, that has become now very classical in singular perturbed problems. This reduction was first used by Floer and Weinstein [8] in the context of partial differential equations. This method was adapted to find bubble-tower solution in the Brezis–Nirenberg problem by Del Pino, Dolbeault and Musso [4] and after that by many other authors in similar problems. We mention here the paper by Campos [2] for Eq. (1.2) which is close to our work. Here we do not give the proof of our second theorem, since the arguments are similar to those used in the proof of Theorem 1.1.

In all equations before mentioned the existence of a large number of fast decay solutions can be seen in a three-dimensional dynamical systems (by Emden–Fowler transformation) as a large number of intersection points between a two-dimensional stable manifold with a two-dimensional unstable. Our results and the results in [2] and [1], can be seen as a perturbation argument from a homoclinic orbit in some plane of this dynamical system. Moreover, if there exists a slow decay solution (which is always unique, because it corresponds to a one-dimensional manifold) it implies that these intersection points are infinitely many, see Flores [9] for Eq. (1.2). Notice that Eq. (1.1) with the model case $K(r) = 1 + r^2$ admits a slow decay solution of type $u(r) = A(B + r^2)^s$, $s = -\frac{2}{p-1}$ for suitable exponents p and constants A and B. In this case using the same argument as in [9] it can be proven that there exist infinitely many fast decay solutions.

Finally, observe that the complete understanding of the dynamical systems or all solution set is wide open in these three types of equations, with exception of the particular case found in [7]. So, many basic and challenging questions still remain open for all these equations. Moreover, we strongly believe that they are closely connected and that the complexity found in [7] is present in all of them.

This paper is organized as follows. In Section 2 we compute the energy for our approximate solution of the transformed problem, through Emden–Fowler change of variable. In Section 3 we discuss the finite-dimensional reduction scheme that we will use to establish our main result, which is proved in Section 4 by means of degree theory.

2. Preliminaries and the reduced energy

We start introducing the change of variable

$$v(t) = e^{-t} u\left(e^{-\frac{2}{N-2}t}\right), \quad \forall t \in \mathbb{R},$$

which is a slight variation of the Emden–Fowler transformation so fast decay solution of (1.1) satisfies

$$\begin{cases} v''(t) = v(t) - \beta^2 e^{-\beta\sigma t} e^{\varepsilon t} K\left(e^{\beta t}\right) v^{p_0+\varepsilon}(t), \quad t \in \mathbb{R}, \\ \lim\limits_{t \to +\infty} v'(t) = 0, \\ \lim\limits_{t \to -\infty} v(t) e^t = 0, \end{cases} \tag{2.1}$$

where $\beta = -\frac{2}{N-2}$. Note that if $\varepsilon \to 0$, the equation above is carried out to the following limit equation

$$v''(t) = v(t) - \beta^2 e^{-\beta\sigma t} K\left(e^{\beta t}\right) v^{\frac{2+N+2\sigma}{N-2}}(t). \tag{2.2}$$

On the other hand, note that conditions over K in (1.4) are equivalent to

$$\lim_{t \to +\infty} \frac{e^{-\beta\sigma t} K(e^{\beta t}) - C_0}{e^{\beta\gamma t}} = C_1 > 0 \quad \text{and} \quad \lim_{t \to -\infty} \frac{K(e^{\beta t})}{e^{\beta \ell t}} = C_\infty > 0. \tag{2.3}$$

In particular, the condition above on the left-hand side implies that

$$\lim_{t \to +\infty} e^{-\beta\sigma t} K\left(e^{\beta t}\right) = C_0 > 0. \tag{2.4}$$

Then, one can choose constants $\bar{t}_1, \bar{t}_2$, with $\bar{t}_1 \leqslant 0 \leqslant \bar{t}_2$ and $\bar{t}_2, |\bar{t}_1|$ sufficiently large, such that

$$\begin{cases} \left(C_\infty - \dfrac{1}{2n_0}\right) e^{\beta \ell t} \leqslant K\left(e^{\beta t}\right) \leqslant \left(C_\infty + \dfrac{1}{2n_0}\right) e^{\beta \ell t}, & \text{if } t < \bar{t}_1, \\ 0 \leqslant K\left(e^{\beta t}\right) \leqslant \sup\limits_{t \in [\bar{t}_1, \bar{t}_2]} K\left(e^{\beta t}\right), & \text{if } \bar{t}_1 \leqslant t \leqslant \bar{t}_2, \\ \left(C_0 - \dfrac{1}{2n_0}\right) e^{\beta\sigma t} \leqslant K\left(e^{\beta t}\right) \leqslant \left(C_0 + \dfrac{1}{2n_0}\right) e^{\beta\sigma t}, & \text{if } t > \bar{t}_2, \end{cases} \tag{2.5}$$

for some $n_0 \in \mathbb{N}$ fixed large enough verifying $C_0 - \frac{1}{2n_0} > 0$ and $C_\infty - \frac{1}{2n_0} > 0$. In this way, for $\rho > 0$ fixed but arbitrary, it is suitable to consider the following equation

$$U'' - U + \rho \left(\frac{2}{N-2}\right)^2 U^{\frac{2+N+2\sigma}{N-2}} = 0 \quad \text{in } \mathbb{R},$$

and its explicit solution is

$$U(\rho; t) = \left(\frac{(N+\sigma)(N-2)}{\rho}\right)^{\frac{N-2}{2(2+\sigma)}} e^{-t} \left(1 + e^{-\frac{2(2+\sigma)}{N-2}t}\right)^{-\frac{N-2}{(2+\sigma)}}.$$

Now, we define $U := U(C_0; \cdot)$ and introduce the functions

$$U_i(t) = U(t - \tau_i) \quad \text{and} \quad V(t) = \sum_{i=1}^{k} U_i(t), \tag{2.6}$$

where $\tau_i \in \mathbb{R}$ and $k \in \mathbb{N}$, $k \geqslant 2$. Roughly speaking, we are looking for solutions v of (2.1) which are approximately of the form

$$v(t) = V(t) + \phi(t),$$

which for suitable points $\bar{t}_2 < \tau_1 < \tau_2 < \cdots < \tau_k$, with $\bar{t}_2$ given by (2.5), we will have the remainder term ϕ of small order all over $\mathbb{R}$.

Here we do the following choice of the points τ_i

$$\begin{cases} \tau_1 = -\dfrac{1}{p_0+1}\log\varepsilon - \log\lambda_1, \\ \tau_{i+1} - \tau_i = -\log\varepsilon - \log\lambda_{i+1}, \quad \forall i = 1, 2, \ldots, k-1, \end{cases} \tag{2.7}$$

and by simplicity we put $\vec{\tau} = (\tau_1, \tau_2, \ldots, \tau_k) \in \mathbb{R}^k$ and $\vec{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{R}^k$. Since solutions of (2.1) correspond to stationary points of its associated energy functional E_ε defined by

$$E_\varepsilon(v) = J_\varepsilon(v) - \frac{\beta^2}{p_0+1+\varepsilon} \int_{-\infty}^{+\infty} e^{\varepsilon s}\left(e^{-\beta\sigma s} K\left(e^{\beta s}\right) - C_0\right) v^{p_0+1+\varepsilon}\, ds, \tag{2.8}$$

where

$$J_\varepsilon(v) = \frac{1}{2}\int_{-\infty}^{+\infty} \left(v'\right)^2 + \frac{1}{2}\int_{-\infty}^{+\infty} v^2 - \frac{C_0\beta^2}{p_0+1+\varepsilon}\int_{-\infty}^{+\infty} e^{\varepsilon s} v^{p_0+1+\varepsilon}\, ds, \tag{2.9}$$

our first goal is to estimate $E_\varepsilon(V)$.

Lemma 2.1. *Let $N > 2$, $\sigma < \ell < 2\sigma + N$, $k \in \mathbb{N}$, $k \geqslant 2$ and let $\delta > 0$ be fixed. Moreover, assume that*

$$\delta < \lambda_i < \delta^{-1}, \quad \forall i = 1, 2, \ldots, k. \tag{2.10}$$

Then, for V defined by (2.6) *and points τ_i as in* (2.7)*, there are positive numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 depending only on N, K, such that*

$$E_\varepsilon(V) = k\alpha_1 + \varepsilon\Psi_k(\vec{\lambda}) + \varepsilon k\left(\frac{2+(k-1)(p_0+1)}{2(p_0+1)}\log\varepsilon\right)\alpha_5 + \varepsilon\theta_\varepsilon(\vec{\lambda}), \tag{2.11}$$

where

$$\Psi_k(\vec{\lambda}) = -\sum_{i=2}^{k} \lambda_i\alpha_2 - \lambda_1^{p_0+1}\alpha_3 + k\alpha_4 + \left(\sum_{i=1}^{k}(k-i+1)\log\lambda_i\right)\alpha_5 \tag{2.12}$$

and $\theta_\varepsilon(\vec{\lambda}) \to 0$ as $\varepsilon \to 0$, uniformly in the C^1-sense with respect to the values λ_i satisfying (2.10).

Here and in the rest of this paper, we denote by C a generic positive constant which is independent of ε and of the particular τ_i's chosen satisfying (2.7).

Proof of Lemma 3.1. Firstly, we estimate $J_\varepsilon(V)$. Note that

$$J_\varepsilon(V) = J_0(V) + A_{1,\varepsilon} + A_{2,\varepsilon} + A_{3,\varepsilon},$$

where

$$J_0(V) = \frac{1}{2}\int_{-\infty}^{+\infty} (V')^2 + \frac{1}{2}\int_{-\infty}^{+\infty} V^2 - \frac{C_0\beta^2}{p_0+1}\int_{-\infty}^{+\infty} V^{p_0+1},$$

$$A_{1,\varepsilon} = \left(\frac{C_0\beta^2}{p_0+1} - \frac{C_0\beta^2}{p_0+1+\varepsilon}\right)\int_{-\infty}^{+\infty} V^{p_0+1},$$

$$A_{2,\varepsilon} = \left(\frac{C_0\beta^2}{p_0+1+\varepsilon}\right)\int_{-\infty}^{+\infty} \left(V^{p_0+1} - V^{p_0+1+\varepsilon}\right)$$

and

$$A_{3,\varepsilon} = \left(\frac{C_0\beta^2}{p_0+1+\varepsilon}\right)\int_{-\infty}^{+\infty} \left(1 - e^{\varepsilon s}\right) V^{p_0+1+\varepsilon}\, ds.$$

Using a Taylor expansion, is not difficult to check that

$$A_{1,\varepsilon} = \frac{k\varepsilon C_0\beta^2}{(p_0+1)^2}\int_{-\infty}^{+\infty} U^{p_0+1} + o(\varepsilon), \tag{2.13}$$

$$A_{2,\varepsilon} = -\frac{k\varepsilon C_0\beta^2}{p_0+1}\int_{-\infty}^{+\infty} U^{p_0+1}\ln U + o(\varepsilon) \tag{2.14}$$

and if we consider from now on

$$\mu_0 = -\infty, \qquad \mu_i = \frac{\tau_i + \tau_{i+1}}{2}, \quad \text{for } i = 1, 2 \ldots, k-1, \quad \text{and} \quad \mu_k = +\infty, \tag{2.15}$$

we obtain

$$A_{3,\varepsilon} = -\sum_{i=1}^{k} \varepsilon C_0 \int_{\mu_{i-1}}^{\mu_i} s V^{p_0+1}\, ds + o(\varepsilon) = -\varepsilon C_0 \sum_{i=1}^{k} \tau_i \int_{-\infty}^{+\infty} U^{p_0+1} + o(\varepsilon).$$

Then, from the choice of the τ_i's in (2.7), we yield

$$A_{3,\varepsilon} = \varepsilon C_0 \left(\left(\frac{k}{p_0+1} + \frac{k(k-1)}{2} \right) \log\varepsilon + \sum_{i=1}^{k} (k-i+1) \log\lambda_i \right) \int_{-\infty}^{+\infty} U^{p_0+1} + o(\varepsilon). \quad (2.16)$$

On the other hand,

$$J_0(V) - \sum_{i=1}^{k} J_0(U_i) = \frac{C_0\beta^2}{p_0+1} \int_{-\infty}^{+\infty} \left(\sum_{i=1}^{k} U_i^{p_0+1} - \left(\sum_{i=1}^{k} U_i \right)^{p_0+1} \right) + C_0\beta^2 \int_{-\infty}^{\infty} \sum_{j=2}^{k} \sum_{\substack{i=1 \\ i<j}}^{k-1} U_i^{p_0} U_j$$

and so

$$J_0(V) = \sum_{i=1}^{k} J_0(U_i) + \sum_{i=1}^{k} (B_{1,i} + B_{2,i} + B_{3,i}),$$

where

$$B_{1,l} = \frac{C_0\beta^2}{p_0+1} \int_{\mu_{l-1}}^{\mu_l} \left(U_l^{p_0+1} - \left(\sum_{i=1}^{k} U_i \right)^{p_0+1} + (p_0+1) U_l^{p_0} \sum_{\substack{j=1 \\ j\neq l}}^{k} U_j \right),$$

$$B_{2,l} = -C_0\beta^2 \int_{\mu_{l-1}}^{\mu_l} U_l^{p_0} \sum_{\substack{j=1 \\ j<l}}^{k} U_j$$

and

$$B_{3,l} = \frac{C_0\beta^2}{p_0+1} \int_{\mu_{l-1}}^{\mu_l} \left(\sum_{\substack{i=1 \\ i\neq l}}^{k} U_i^{p_0+1} + (p_0+1) \sum_{\substack{j=1 \\ j\neq l}}^{k} \sum_{\substack{i=1 \\ i<j}}^{k} U_i^{p_0} U_j \right).$$

From the Mean Value Theorem we have that

$$|B_{1,l}| \leqslant C \int_{\mu_{l-1}}^{\mu_l} \left(\sum_{\substack{i=1 \\ i\neq l}}^{k} U_i \right)^2 \sum_{i=1}^{k} U_i^{p_0-1}.$$

Hence, for $l \in \{2, 3, \ldots, k-1\}$, putting $\varrho = |\log\varepsilon|$ and using the fact that $U(t) = O(e^{-|t|})$, we obtain

$$|B_{1,l}| \leqslant C \int_{0}^{\frac{\varrho}{2}+M} e^{-2(\varrho-s)} e^{-(p_0-1)s} \, ds \leqslant C e^{-2\varrho} \int_{0}^{\frac{\varrho}{2}+M} e^{-(p_0-3)s} \, ds = o(\varepsilon),$$

where M is a constant that depends only on δ, and if $l = \{1, k\}$, we easily get

$$|B_{1,l}| = o(\varepsilon).$$

Now, if $\varepsilon > 0$ is small enough, then from the choice of the τ_l's in (2.7) and of the μ_l's in (2.15) we get

$$\begin{aligned} B_{2,l} &= -C_0\beta^2 \int_{\mu_{l-1}}^{\mu_l} U^{p_0} U_{l-1} + o(\varepsilon) \\ &= -C_0\beta^2 \tilde{\gamma}_N e^{-(\tau_l-\tau_{l-1})} \int_{-\infty}^{+\infty} U^{p_0} e^{-|s|}\, ds + o(\varepsilon) \\ &= -\varepsilon C_0\beta^2 \tilde{\gamma}_N \lambda_l \int_{-\infty}^{+\infty} U^{p_0} e^{-|s|}\, ds + o(\varepsilon), \end{aligned}$$

where $\tilde{\gamma}_N = 2^{-\frac{2+\sigma}{N-2}} U(0)$ and $B_{2,1} = 0$. To estimate $B_{3,l}$ we note that

$$|B_{3,l}| \leqslant C \int_{\mu_{l-1}}^{\mu_l} \sum_{\substack{j=1\\ j\neq l}}^{k} \sum_{\substack{i=1\\ i<j}}^{k} U_i^{p_0} U_j \leqslant C \int_{\mu_{l-1}}^{\mu_l} U_l^{p_0} \sum_{\substack{i=1\\ i>l}}^{k} U_i,$$

where μ_l's are given by (2.15). Hence, setting again $\varrho = |\log\varepsilon|$ and since $U(t) = O(e^{-|t|})$, we obtain

$$|B_{3,l}| \leqslant C e^{\varrho} \int_0^{\frac{\varrho}{2}+M} e^{-(p_0-1)s}\, ds = o(\varepsilon),$$

where M is a constant that depends only on δ. So, we get

$$J_0(V) = k J_0(U) - \varepsilon\beta^2 \tilde{\gamma}_N \sum_{i=2}^{k} \lambda_i \int_{-\infty}^{+\infty} U^{p_0} e^{-|s|}\, ds + o(\varepsilon). \tag{2.17}$$

Finally, we have that

$$\int_{-\infty}^{+\infty} e^{\varepsilon s}\big(e^{-\beta\sigma s} K\big(e^{\beta s}\big) - C_0\big)\Bigg(\sum_{i=1}^{k} U_i\Bigg)^{p_0+1+\varepsilon} ds = C_{1,\varepsilon} + C_{2,\varepsilon} + C_{3,\varepsilon},$$

where

$$\begin{aligned} C_{1,\varepsilon} &= \int_{-\infty}^{+\infty} e^{-\beta\sigma s} K\big(e^{\beta s}\big)\big(e^{\varepsilon s} - 1\big)\Bigg(\sum_{i=1}^{k} U_i\Bigg)^{p_0+1+\varepsilon} ds, \\ C_{2,\varepsilon} &= \int_{-\infty}^{+\infty} \big(e^{-\beta\sigma s} K\big(e^{\beta s}\big) - C_0\big)\Bigg(\sum_{i=1}^{k} U_i\Bigg)^{p_0+1+\varepsilon} ds \end{aligned}$$

and

$$C_{3,\varepsilon} = C_0 \int_{-\infty}^{+\infty} \left(1 - e^{\varepsilon s}\right) \left(\sum_{i=1}^{k} U_i \right)^{p_0+1+\varepsilon} ds.$$

Bearing in mind the constraints (2.10) over λ_i's and the choices of μ_i's in (2.15), and considering $\sigma < \ell < N + \sigma$ and constants $\tilde{C}_\infty$ and $\tilde{C}_0$ such that

$$\max\left\{|\tilde{C}_0 - C_0|, |\tilde{C}_\infty - C_\infty|\right\} < \frac{1}{n_0},$$

we obtain by means of straightforward calculations

$$C_{1,\varepsilon} = \varepsilon \tilde{C}_0 \sum_{i=1}^{k} \tau_i \int_{-\infty}^{+\infty} U^{p_0+1} + o(\varepsilon)$$

and

$$C_{2,\varepsilon} = e^{-(p_0+1)\tau_1} C \left(\tilde{C}_\infty \int_{-\infty}^{\bar{t}_1} e^{-(p_\infty - 2p_0 - 1)s}\, ds + \int_{\bar{t}_1}^{\bar{t}_2} e^{-\beta(2\sigma+N)s} K\left(e^{\beta s}\right) ds \right)$$
$$+ \beta^2 \tilde{\gamma}_N (\tilde{C}_0 - C_0) \sum_{i=2}^{k} e^{-(\tau_i - \tau_{i-1})} \int_{-\infty}^{+\infty} U^{p_0} e^{-|s|}\, ds + k(\tilde{C}_0 - C_0) \int_{-\infty}^{+\infty} U^{p_0+1} + o(\varepsilon).$$

Also we obtain

$$C_{3,\varepsilon} = -\varepsilon C_0 \sum_{i=1}^{k} \tau_i \int_{-\infty}^{+\infty} U^{p_0+1} + o(\varepsilon).$$

Hence, from previous estimates for $C_{1,\varepsilon}$, $C_{2,\varepsilon}$ and $C_{3,\varepsilon}$, we get

$$\int_{-\infty}^{+\infty} e^{\varepsilon s}\left(e^{-\beta\sigma s} K\left(e^{\beta s}\right) - 1\right) \left(\sum_{i=1}^{k} U_i \right)^{p_0+1+\varepsilon} ds$$
$$= \varepsilon (C_0 - \tilde{C}_0) \left(\left(\frac{k}{p_0+1} + \frac{k(k-1)}{2} \right) \log \varepsilon + \sum_{i=1}^{k} (k-i+1) \log \lambda_i \right) \int_{-\infty}^{+\infty} U^{p_0+1}$$
$$+ \varepsilon \lambda_1^{p_0+1} C \left(\tilde{C}_\infty \int_{-\infty}^{\bar{t}_1} e^{-(p_\infty - 2p_0 - 1)s}\, ds + \int_{\bar{t}_1}^{\bar{t}_2} e^{-\beta(2\sigma+N)s} K\left(e^{\beta s}\right) ds \right)$$
$$+ \varepsilon (\tilde{C}_0 - C_0) \beta^2 \tilde{\gamma}_N \sum_{i=2}^{k} \lambda_i \int_{-\infty}^{+\infty} U^{p_0} e^{-|s|}\, ds + k(\tilde{C}_0 - C_0) \int_{-\infty}^{+\infty} U^{p_0+1} + o(\varepsilon). \tag{2.18}$$

Now, we choose

$$\begin{cases} \alpha_1 = J_0(U) - \dfrac{(\tilde{C}_0 - C_0)\beta^2}{p_0+1} \displaystyle\int_{-\infty}^{+\infty} U^{p_0+1}, \\ \alpha_2 = \tilde{C}_0 \beta^2 \tilde{\gamma}_N \displaystyle\int_{-\infty}^{+\infty} U^{p_0} e^{-|s|}\, ds, \\ \alpha_3 = \dfrac{C\beta^2}{p_0+1}\left(\tilde{C}_\infty \displaystyle\int_{-\infty}^{\bar{t}_1} e^{-(p_\infty - p_0 - 1)s}\, ds + \int_{\bar{t}_1}^{\bar{t}_2} e^{-\beta(2\sigma+N)s} K\left(e^{\beta s}\right) ds\right), \\ \alpha_4 = \dfrac{C_0\beta^2}{(p_0+1)^2} \displaystyle\int_{-\infty}^{+\infty} U^{p_0+1} - \dfrac{C_0\beta^2}{p_0+1}\int_{-\infty}^{+\infty} U^{p_0+1} \ln U, \\ \alpha_5 = \dfrac{\tilde{C}_0\beta^2}{p_0+1} \displaystyle\int_{-\infty}^{+\infty} U^{p_0+1}, \end{cases} \tag{2.19}$$

and since $\sigma < \ell < 2\sigma + N$, we have that $\alpha_3 \in \mathbb{R}^+$. From estimates (2.13), (2.14), (2.16), (2.17), (2.18) and the choice of the constants α_i's in (2.19), it follows that

$$E_\varepsilon(V) = k\alpha_1 + \varepsilon \Psi_k(\vec{\lambda}) + \varepsilon k\left(\frac{2 + (k-1)(p_0+1)}{2(p_0+1)} \log \varepsilon\right)\alpha_5 + o(\varepsilon), \tag{2.20}$$

where Ψ_k is given by (2.12). Moreover, in all previous estimates the quantity $o(\varepsilon)$ is actually of this size in the C^1-norm as function of the values λ_i's satisfying (2.10). Therefore, (2.11) is obtained from (2.20). $\square$

Remark 2.1. Note that Ψ_k has a unique critical point which is non-degenerate and it is given by:

$$\vec{\lambda}^* = \left(\left(\frac{k\alpha_5}{\alpha_3(p_0+1)}\right)^{\frac{1}{p_0+1}}, \frac{(k-1)\alpha_5}{\alpha_2}, \frac{(k-2)\alpha_5}{\alpha_2}, \ldots, \frac{2\alpha_5}{\alpha_2}, \frac{\alpha_5}{\alpha_2}\right).$$

3. The finite-dimensional reduction

Let us consider points τ_i such that $\bar{t}_2 < \tau_1 < \tau_2 < \cdots < \tau_k$, with $\bar{t}_2$ given by (2.5) and functions U_i, V, defined in (2.6). Now, for each $i = 1, 2, \ldots, k$, we define the following functions

$$Z_i(t) = U_i'(t) = \left(\frac{e^{\beta(\sigma+2)(t-\tau_i)} - 1}{e^{\beta(\sigma+2)(t-\tau_i)} + 1}\right) U_i(t). \tag{3.1}$$

Here we are interesting in the problem of finding a function ϕ such that

$$\begin{cases} -(V+\phi)'' + (V+\phi) - \beta^2 e^{\varepsilon t} e^{-\beta\sigma t} K\big(e^{\beta t}\big)(V+\phi)^{p_0+\varepsilon} = \displaystyle\sum_{i=1}^{k} c_i Z_i \quad \text{in } \mathbb{R}, \\ \displaystyle\int_{-\infty}^{+\infty} Z_i \phi = 0, \quad \forall i = 1, 2, \ldots, k, \\ \displaystyle\lim_{t\to\pm\infty} \phi(t) = 0, \end{cases} \tag{3.2}$$

for certain scalars c_i. Note that $V+\phi$ is a solution of (2.1) if the scalars c_i in (3.2) are all zero. Also, we note that the differential equation in (3.2) is equivalent to

$$L_\varepsilon(\phi) = N_\varepsilon(\phi) + R_\varepsilon + \sum_{i=1}^{k} c_i Z_i \quad \text{in } \mathbb{R}, \tag{3.3}$$

where

$$L_\varepsilon(\phi) = -\phi'' + \phi - \beta^2 e^{\varepsilon t} e^{-\beta\sigma t} K\big(e^{\beta t}\big)(p_0+\varepsilon) V^{p_0+\varepsilon-1}\phi, \tag{3.4}$$

$$N_\varepsilon(\phi) = \beta^2 e^{\varepsilon t} e^{-\beta\sigma t} K\big(e^{\beta t}\big)\big((V+\phi)_+^{p_0+\varepsilon} - V^{p_0+\varepsilon} - (p_0+\varepsilon) V^{p_0+\varepsilon-1}\phi\big) \tag{3.5}$$

and

$$R_\varepsilon = \beta^2 e^{\varepsilon t} e^{-\beta\sigma t} K\big(e^{\beta t}\big) - \sum_{i=1}^{k} U_i^{p_0}. \tag{3.6}$$

A first step is to study the following linear problem: given $h \in C(\mathbb{R})$, find ϕ such that

$$\begin{cases} L_\varepsilon(\phi) = h + \displaystyle\sum_{i=1}^{k} c_i Z_i \quad \text{in } \mathbb{R}, \\ \displaystyle\int_{-\infty}^{+\infty} Z_i \phi = 0, \quad \forall i = 1, 2, \ldots, k, \\ \displaystyle\lim_{t\to\pm\infty} \phi(t) = 0, \end{cases} \tag{3.7}$$

for certain constants c_i. We prove the next lemma.

Lemma 3.1. *Assume that $\sigma < \ell < 2(\sigma+2)$ and that there exist a sequence $\varepsilon_n \to 0$ and points $0 < \tau_1^n < \tau_2^n < \cdots < \tau_k^n$ depending on ε_n which verify*

$$\tau_1^n \to +\infty, \qquad \min_{i=1,\ldots,k}\big(\tau_{i+1}^n - \tau_i^n\big) \to +\infty, \quad \textit{and} \quad \tau_k^n = o\big(\varepsilon_n^{-1}\big), \tag{3.8}$$

such that for certain scalars c_i^n, and functions ϕ_n and h_n, with $\|h_n\|_* \to 0$, one has

$$\begin{cases} L_{\varepsilon_n}(\phi_n) = h_n + \sum_{i=1}^{k} c_i^n Z_i^n \quad \text{in } \mathbb{R}, \\ \int_{-\infty}^{+\infty} Z_i^n \phi_n = 0, \quad \forall i = 1, 2, \ldots, k, \\ \lim_{t \to \pm\infty} \phi_n(t) = 0, \end{cases} \tag{3.9}$$

where $Z_i^n(t) = U'(t - \tau_i^n)$. Then

$$\lim_{n \to \infty} \|\phi_n\|_* = 0.$$

Here

$$\|\psi\|_* = \sup_{t \in \mathbb{R}} \left| \left(\sum_{i=1}^{k} e^{-\bar{\eta}|t-\tau_i|} \right)^{-1} \psi(t) \right|,$$

where $\bar{\eta} > 0$ is a number to be fixed.

Proof. Firstly we prove that

$$\lim_{n \to \infty} \|\phi_n\|_\infty = 0.$$

Arguing by contradiction, we can assume that $\|\phi_n\|_\infty = 1$. Testing the differential equation in (3.9) with Z_i^n and integrating twice by parts, we obtain

$$\int_{-\infty}^{+\infty} L_\varepsilon\big(Z_i^n\big)\phi_n - \int_{-\infty}^{+\infty} h_n Z_i^n = \sum_{l=1}^{k} c_l^n \int_{-\infty}^{+\infty} Z_l^n Z_i^n.$$

The previous equality defines an almost diagonal system on the c_l^n's as $n \to +\infty$ because if $i \neq l$, then by the Dominated Convergence Theorem we have that

$$\int_{-\infty}^{+\infty} Z_l^n Z_i^n \to 0$$

and if $i = l$, then directly we obtain

$$\int_{-\infty}^{+\infty} Z_l^n Z_i^n = \int_{-\infty}^{+\infty} \big|U_l'\big|^2.$$

On the other hand, $\|h_n\|_* \to 0$ implies that

$$\big|h_n(t)\big| \leqslant \theta_n(t) \sum_{i=1}^{k} e^{-\bar{\eta}|t-\tau_i^n|},$$

for some $\theta_n \to 0$ uniformly, and bearing in mind that $Z_i^n(t) = O(e^{-|t-\tau_i^n|})$, we get

$$\left|\int_{-\infty}^{+\infty} h_n Z_i^n\right| \leqslant C\|\theta_n\|_\infty \left|\int_{-\infty}^{+\infty} e^{-|s|}\,ds\right| \to 0$$

as $n \to +\infty$. Also, since

$$-Z_i^{n\prime\prime} + Z_i^n - p_0\beta^2 C_0 U_i^{p_0-1} Z_i^n = 0 \quad \text{in } \mathbb{R},$$

$\sigma < \ell < 2(\sigma+2)$ and from (2.5) one has $|K(e^{\beta t}) - C_0| \leqslant \frac{1}{n_0}$ for all $t > \bar{t}_2$, it follows that

$$\begin{aligned}\left|\int_{-\infty}^{+\infty} L_\varepsilon\big(Z_i^n\big)\phi_n\right| &= \left|\int_{-\infty}^{+\infty} \big(-Z_i^{n\prime\prime} + Z_i^n - (p_0+\varepsilon)\beta^2 e^{\varepsilon t} e^{-\beta\sigma t} K\big(e^{\beta t}\big) V^{p_0+\varepsilon-1} Z_i^n\big)\phi_n\right| \\ &\leqslant C\left(\int_{-\infty}^{\bar{t}_1} e^{-\beta(\sigma-\ell)t} V^{p_0-1} Z_i^n + \int_{\bar{t}_1}^{\bar{t}_2} V^{p_0-1} Z_i^n + \int_{\bar{t}_2}^{+\infty} U_i^{p_0-1} Z_i^n\right) \\ &\to 0\end{aligned}$$

as $n \to +\infty$. Therefore $c_i^n \to 0$ as $n \to +\infty$.

Now we choose $t_n \in \mathbb{R}$ such that $\phi_n(t_n) = 1$. By theory of elliptic regularity, we can assume that $\exists i \in \{1, 2, \dots, k\}$ such that for n large enough one has

$$\exists R > 0 \quad \text{such that} \quad \big|t_n - \tau_i^n\big| < R. \tag{3.10}$$

Let us fix an index i such that (3.10) holds and put $\hat{\phi}_n(t) = \phi_n(t + \tau_i^n)$. From (3.9), (3.10) and elliptic estimates, choosing a suitable subsequence, $\hat{\phi}_n(t)$ converges uniformly on compacts to a non-trivial solution $\bar{\phi}$ of

$$-\bar{\phi}'' + \bar{\phi} - \beta^2 p_0 C_0 U^{p_0-1}\bar{\phi} = 0 \quad \text{in } \mathbb{R}^N.$$

Hence $\bar{\phi} = CU'$ for some positive constant C. Nevertheless

$$0 = \int_{-\infty}^{+\infty} Z_l^n \hat{\phi}_n \to C\int_{-\infty}^{+\infty} \big|U'\big|^2 > 0,$$

which is a contradiction. Then $\|\phi_n\|_\infty \to 0$.

Now, we note that

$$-\phi_n'' + \phi_n = g_n \quad \text{in } \mathbb{R}, \tag{3.11}$$

with

$$g_n(t) = h_n(t) + \sum_{i=1}^{k} c_i^n Z_i^n(t) + \beta^2 e^{\varepsilon t} e^{-\beta\sigma t} K\big(e^{\beta t}\big)(p_0+\varepsilon) V^{p_0+\varepsilon-1}(t)\phi_n(t).$$

Since $\|h_n\|_* \to 0$, $c_i^n \to 0$, $Z_i^n(t) = O(e^{-|t-\tau_i|})$,

$$\begin{aligned}&\big|\beta^2 e^{\varepsilon t} e^{-\beta\sigma t} K\big(e^{\beta t}\big)(p_0+\varepsilon) V^{p_0+\varepsilon-1}(t)\phi_n(t)\big| \\ &\quad\leqslant \begin{cases} C\|\phi_n\|_\infty \sum_{i=1}^k e^{-(2p_0-p_\infty-1)|t-\tau_i^n|}, & \text{if } t<\bar{t}_1, \\ C\|\phi_n\|_\infty \sum_{i=1}^k e^{-(p_0-1)|t-\tau_i^n|}, & \text{if } \bar{t}_1<t<\bar{t}_2, \\ C\|\phi_n\|_\infty \sum_{i=1}^k e^{-|t-\tau_i^n|}, & \text{if } t>\bar{t}_2, \end{cases}\end{aligned}$$

with $\|\phi_n\|_\infty \to 0$ as $n \to +\infty$, and $\sigma < \ell < 2(2+\sigma)$, it follows that if $0 < \bar{\eta} < \min\{1, p_0 - 1, 2p_0 - 1 - p_\infty\}$, then

$$\big|g_n(t)\big| \leqslant \theta_n(t) \sum_{i=1}^k e^{-\bar{\eta}|t-\tau_i^n|},$$

with $\theta_n \to 0$ uniformly. Choosing $\bar{C} > 0$ large enough, we have that

$$\varphi_n(t) = \bar{C}\theta_n(t) \sum_{i=1}^k e^{-\bar{\eta}|t-\tau_i^n|}$$

is a super-solution of (3.11), and $-\varphi_n(t)$ is a sub-solution of (3.11). Therefore

$$|\phi_n| \leqslant \theta_n(t) \sum_{i=1}^k e^{-\bar{\eta}|t-\tau_i^n|},$$

for some $\theta_n \to 0$ uniformly. The proof is finished. □

Proposition 3.1. *There exist positive numbers ε_0, δ_0 and R_0 such that if $\vec{\tau} \in \mathbb{R}^k$ satisfies*

$$R_0 < \tau_1, \qquad R_0 < \min_{i=1,\dots,k} (\tau_{i+1} - \tau_i) \quad \textit{and} \quad \tau_k < \frac{\delta_0}{\varepsilon},$$

then for all $\varepsilon \in (0, \varepsilon_0)$ and all $h \in C(\mathbb{R})$, with $\|h\|_* < \infty$, problem (3.7) *admits a unique solution $\phi := T_\varepsilon(h)$. Besides, there exists a constant $C > 0$ such that*

$$\big\|T_\varepsilon(h)\big\|_* \leqslant C\|h\|_* \quad \textit{and} \quad |c_i| \leqslant C\|h\|_*.$$

Proof. Let us consider the space

$$H_\varepsilon = \left\{\phi \in H^1(\mathbb{R}): \int_{-\infty}^{+\infty} Z_i\phi = 0,\ i = 1,\dots,k\right\}$$

endowed with the usual inner product of $H^1(\mathbb{R})$ that here we denote by $[\cdot,\cdot]$. Then problem (3.7) written in sense weak with respect to H_ε is equivalent to find $\phi \in H_\varepsilon$ such that

$$[\phi, \psi] = \beta^2 \int_{-\infty}^{+\infty} \big(e^{\varepsilon t} e^{-\beta\sigma t} K\big(e^{\beta t}\big)(p_0+\varepsilon) V^{p_0+\varepsilon-1}(t)\big)\phi\psi + \int_{-\infty}^{+\infty} h\psi, \quad \forall \psi \in H_\varepsilon.$$

Moreover, H_ε is Hilbert, then from the Riesz Representation Theorem we deduce that there exists a linear isomorphism $\mathcal{I}_\varepsilon \in \mathcal{L}(H_\varepsilon^*, H_\varepsilon)$ such that to each $\phi^* \in H_\varepsilon^*$ corresponds a unique $\phi \in H_\varepsilon$ which verifies

$$\mathcal{I}_\varepsilon\big(\phi^*\big)[\psi] = [\phi, \psi], \quad \forall \psi \in H_\varepsilon.$$

Hence, we can identify ϕ with $\mathcal{I}_\varepsilon(\phi^*)$. Also, note that the operator $M_\varepsilon : H_\varepsilon \to H_\varepsilon^*$ defined, for each $\phi \in H_\varepsilon$, by the functional

$$\psi \mapsto M_\varepsilon(\phi)[\psi] = \beta^2 \int_{-\infty}^{+\infty} \big(e^{\varepsilon t} e^{-\beta\sigma t} K\big(e^{\beta t}\big)(p_0 + \varepsilon) V^{p_0+\varepsilon-1}(t)\big)\phi\psi$$

is compact, and the functional

$$\psi \mapsto \tilde{h}(\psi) = \int_{-\infty}^{+\infty} h\psi$$

belongs to H_ε^*, and clearly depends linearly of h. Then, (3.7) can be interpreted by way operational in H_ε as: find $\phi \in H_\varepsilon$ such that

$$\phi := T_\varepsilon(h) = M_\varepsilon(\phi) + \tilde{h}.$$

The Fredholm Alternative Theorem guarantees that this problem possesses a unique solution for any $h \in H_\varepsilon$ under the supposition that the homogeneous equation

$$\phi = M_\varepsilon(\phi)$$

has by solution only to the null solution in H_ε. Observe now that in sense weak in H_ε this last equation is equivalent to problem

$$\begin{cases} L_\varepsilon(\phi) = \sum_{i=1}^{k} c_i Z_i \quad \text{in } \mathbb{R}, \\ \int_{-\infty}^{+\infty} Z_i\phi = 0, \quad \forall i = 1, 2, \ldots, k, \\ \lim_{t\to\pm\infty} \phi(t) = 0, \end{cases} \tag{3.12}$$

for certain constants c_i. For proving that (3.12) has only by solution the null solution in H_ε we argue by contradiction. Let ϕ be a non-null solution of (3.12). Without loss of generality we can assume that $\|\phi\|_* = 1$. Hence, if we put $\phi = \phi_n$, $h_n = 0$ and we consider some sequence $\varepsilon_n \to$ as $n \to +\infty$ and τ_i^n's as in (3.8), then we have all conditions for applying Lemma 3.1 and conclude that $\|\phi_n\|_* \to 0$; but this is a contradiction. Therefore, for suitable ε_0, δ_0 and R_0, we have that for $0 < \varepsilon < \varepsilon_0$ and $h \in C(\mathbb{R})$, with $\|h\|_* < \infty$, problem (3.7) admits only one solution in H_ε.

Now we check that $\phi = T_\varepsilon(h)$ verifies $\|\phi\|_* \leqslant C\|h\|_*$ for some constant $C > 0$. Again, we argue by contradiction. Let ϕ be a non-null solution of (3.12). Without loss of generality we can assume that $\|\phi\|_* = 1$. If we put $h = h_n$ and $\phi = \phi_n$ with $\|h_n\|_* < n^{-1}\|\phi_n\|_*$, then all conditions for applying Lemma 3.1 are given and we can conclude that $\|\phi_n\|_* \to 0$; which is a contradiction.

Finally, we recall that from the proof of Lemma 3.1 we have that

$$|c_i|\left(\int_{-\infty}^{+\infty} Z_i^2 + o(1)\right) \leqslant \left|\int_{-\infty}^{+\infty} L_\varepsilon(Z_i)\phi\right| + \left|\int_{-\infty}^{+\infty} hZ_i\right| \leqslant C\big(\|\phi\|_* + \|h\|_*\big).$$

Therefore, $|c_i| \leqslant C\|h\|_*$. $\square$

Now, we are interested in study properties of differentiability of T_ε in the variables τ_i, which will be very important in future purposes. By simplicity, from now on we will consider the Banach space

$$C_* = \big\{f \in C(\mathbb{R}):\ \|f\|_* < \infty\big\},$$

endowed of the $\|\cdot\|_*$-norm, and the space $\mathcal{L}(C_*)$ of the linear operators in C_*. Also we consider numbers ε_0, δ_0 and R_0, given by Proposition 3.1, and the set

$$\mathcal{M}_\varepsilon = \left\{\vec{\tau} \in \mathbb{R}^k:\ R_0 < \tau_1,\ R_0 < \min_{i=1,\dots,k}(\tau_{i+1} - \tau_i) \text{ and } \tau_k < \frac{\delta_0}{\varepsilon}\right\},$$

for $0 < \varepsilon < \varepsilon_0$. We define the map

$$\begin{aligned} S_\varepsilon : \mathcal{M}_\varepsilon \times C_* &\to \mathcal{L}(C_*) \\ (\vec{\tau}, h) &\to S_\varepsilon(\vec{\tau}, h) = T_\varepsilon(h). \end{aligned}$$

Proposition 3.2. *For each $h \in C_*$ the map $\vec{\tau} \mapsto S_\varepsilon(\vec{\tau}, h)$ is of class C^1. Besides, there exists a constant $C > 0$ such that*

$$\big\|D_{\vec{\tau}} T_\varepsilon(h)\big\|_* \leqslant C\|h\|_*$$

uniformly on vectors $\vec{\tau} \in \mathcal{M}_\varepsilon$.

Proof. Let us fix $h \in C_*$, and put $\phi = T_\varepsilon(h)$ for $0 < \varepsilon < \varepsilon_0$. We are interested in study the differentiability of ϕ respect to τ_j, for each $j = 1, 2, \dots, k$. Putting $\vartheta = \frac{\partial\phi}{\partial\tau_j}$, we obtain from (3.7) that ϑ verifies

$$\begin{cases} -\vartheta'' + \vartheta - \beta^2 e^{\varepsilon t} e^{-\beta\sigma t} K\big(e^{\beta t}\big)(p_0 + \varepsilon)\left(V^{p_0+\varepsilon-1}\vartheta + \dfrac{\partial}{\partial\tau_j}\big(V^{p_0+\varepsilon-1}\big)\phi\right) = \displaystyle\sum_{i=1}^k c_i \frac{\partial Z_i}{\partial\tau_j} \quad \text{in } \mathbb{R}, \\ \displaystyle\int_{-\infty}^{+\infty} \frac{\partial Z_l}{\partial\tau_j}\phi = -\int_{-\infty}^{+\infty} \vartheta Z_l = 0, \quad \forall l \neq j, \\ \displaystyle\int_{-\infty}^{+\infty} \frac{\partial Z_j}{\partial\tau_j}\phi = -\int_{-\infty}^{+\infty} \vartheta Z_j. \end{cases}$$

Consider now constants r_i such that

$$\int_{-\infty}^{+\infty}\left(\vartheta - \sum_{i=1}^k r_i Z_i\right) Z_l = 0, \quad \forall l = 1, 2, \dots, k.$$

These relations lead to

$$\sum_{i=1}^{k} r_i \int_{-\infty}^{+\infty} Z_i Z_l = -\int_{-\infty}^{+\infty} \frac{\partial Z_l}{\partial \tau_j}\phi, \quad \forall l = 1, 2, \ldots, k.$$

In other words, for each $i = 1, \ldots, k$, the constants r_i will be given by the following system

$$\sum_{i=1}^{k} r_i \int_{-\infty}^{+\infty} Z_i Z_l = 0, \quad \forall l \neq j, \quad \text{and} \quad \sum_{i=1}^{k} r_i \int_{-\infty}^{+\infty} Z_i Z_j = -\int_{-\infty}^{+\infty} \frac{\partial Z_j}{\partial \tau_j}\phi.$$

Clearly this system is almost diagonal. Hence, putting

$$\psi = \vartheta - \sum_{i=1}^{k} r_i Z_i,$$

it follows that

$$L_\varepsilon(\psi) = f \quad \text{in } \mathbb{R},$$

where

$$f(t) = \beta^2 e^{\varepsilon t} e^{-\beta\sigma t} K\left(e^{\beta t}\right)(p_0+\varepsilon)\frac{\partial}{\partial \tau_j}\left(V^{p_0+\varepsilon-1}\right)\phi + c_l \frac{\partial Z_j}{\partial \tau_j} - \sum_{i=1}^{k} r_i L_\varepsilon(Z_i) \tag{3.13}$$

and

$$\int_{-\infty}^{+\infty} Z_i \psi = 0, \quad \forall i = 1, 2, \ldots, k.$$

Moreover, it is easy to check that $\lim_{t\to\pm\infty} \psi(t) = 0$. Also, observe that from Proposition 3.1 and the definition of ϕ we have

$$\|\phi\|_* \leqslant C\|h\|_* \quad \text{and} \quad |c_l| < C\|h\|_*,$$

and, by definition of r_i, $|r_i| \leqslant C\|\phi\|_*$. Hence, from (3.13) we get $\|f\|_* < C\|h\|_*$, and so $f \in C_*$. Then, again from Proposition 3.1, we conclude that $\psi = T_\varepsilon(f)$, and in this way

$$\vartheta = T_\varepsilon(f) + \sum_{i=1}^{k} r_i Z_i \quad \text{in } \mathbb{R},$$

with ϑ verifying $\|\vartheta\|_* \leqslant C\|h\|_*$.

Finally, note that ϑ depends continuously on τ_j and h for the $\|\cdot\|_*$-norm, for each $j = 1, 2, \ldots, k$. □

For later purposes, from now on it is suitable to assume that, for $A > 0$ fixed and large enough, the following constraints hold

$$\begin{cases} \dfrac{1}{p_0+1}\log(A\varepsilon)^{-1} < \tau_1, \\ \log(A\tau_1)^{-1} < \min\limits_{i=2,3,\dots,k}\{\tau_i - \tau_{i-1}\}, \\ \tau_k < k\log(A\varepsilon)^{-1}. \end{cases} \tag{3.14}$$

Besides, we prove two technical results related with the size of $N_\varepsilon(\phi)$, R_ε and their derivative correspondents in the $\|\cdot\|_*$-norm.

Lemma 3.2. *There exists $C > 0$ such that if $\|\phi\|_* < \frac{1}{2}$, then*

$$\|N_\varepsilon(\phi)\|_* \leqslant C\big(\|\phi\|_*^{\min\{p_0,2\}} + \|\phi\|_*^{\min\{2p_0-p_\infty,2\}}\big) \tag{3.15}$$

and

$$\|D_\phi N_\varepsilon(\phi)\|_* \leqslant C\big(\|\phi\|_*^{\min\{p_0-1,1\}} + \|\phi\|_*^{\min\{2p_0-p_\infty-1,1\}}\big). \tag{3.16}$$

Proof. From the definition of $N_\varepsilon(\phi)$ in (3.5) and the Mean Value Theorem we have that

$$N_\varepsilon(\phi) = \tilde{t}\beta^2(p_0+\varepsilon)(p_0+\varepsilon-1)e^{\varepsilon t}e^{-\beta\sigma t}K\big(e^{\beta t}\big)|V+\bar{t}\phi|^{p_0+\varepsilon-2}\phi^2,$$

for some $\tilde{t}, \bar{t} \in (0,1)$. Then, if $|\phi| < C|V|$ we get

$$\begin{aligned} \big|N_\varepsilon(\phi)\big| &\leqslant Ce^{(\varepsilon-\beta\sigma)t}K\big(e^{\beta t}\big)|V|^{p_0+\varepsilon-2}|\phi|^2 \\ &\leqslant \begin{cases} C|V|^{p_0-2}|\phi|^2, & t > \bar{t}_2, \\ C(\sum_{i=1}^k e^{-(p_\infty-p_0)\tau_i})(\sum_{i=1}^k e^{-(p_0-p_\infty)|t-\tau_i|})|V|^{p_0-2}|\phi|^2, & t < \bar{t}_1, \end{cases} \\ &\leqslant \begin{cases} C|V|^{p_0-2}|\phi|^2, & t > \bar{t}_2, \\ C|V|^{2p_0-p_\infty-2}|\phi|^2, & t < \bar{t}_1, \end{cases} \end{aligned}$$

and, if $|\phi| \geqslant C|V|$ we get

$$\big|N_\varepsilon(\phi)\big| \leqslant \begin{cases} C|\phi|^{p_0}, & t > \bar{t}_2, \\ C|\phi|^{2p_0-p_\infty}, & t < \bar{t}_1. \end{cases}$$

Therefore (3.15) is obtained directly from estimates for $|N_\varepsilon(\phi)|$ above. On the other hand,

$$D_\phi N_\varepsilon(\phi) = \beta^2(p_0+\varepsilon)e^{\varepsilon t}e^{-\beta\sigma t}K\big(e^{\beta t}\big)\big((V+\phi)_+^{p_0+\varepsilon-1} - V^{p_0+\varepsilon-1}\big).$$

Hence, from the Mean Value Theorem, it follows that

$$D_\phi N_\varepsilon(\phi) = \bar{t}\beta^2(p_0+\varepsilon)(p_0+\varepsilon-1)e^{\varepsilon t}e^{-\beta\sigma t}K\big(e^{\beta t}\big)|V+\bar{t}\phi|^{p_0+\varepsilon-2}\phi,$$

for some $\bar{t} \in (0,1)$. In similar way that in the previous case we prove that (3.16) holds. □

Lemma 3.3. *Assume that constraints* (3.14) *hold. Then there exist* $C > 0$ *and* $0 < \bar{\rho} < 1$ *such that*

$$\|R_\varepsilon\|_* \leqslant C\varepsilon^{\bar{\rho}} \quad \textit{and} \quad \|D_{\bar{\tau}} R_\varepsilon\|_* \leqslant C\varepsilon^{\bar{\rho}}. \tag{3.17}$$

Proof. From the definition of R_ε in (3.6), we have that

$$\begin{aligned} R_\varepsilon &= \beta^2 e^{\varepsilon t}\left(e^{-\beta\sigma t}K\left(e^{\beta t}\right) - C_0\right)V^{p_0+\varepsilon} + C_0\beta^2\left(e^{\varepsilon t} - 1\right)V^{p_0+\varepsilon} \\ &\quad + C_0\beta^2\left(V^{p_0+\varepsilon} - V^{p_0}\right) + C_0\beta^2\left(V^{p_0} - \sum_{i=1}^{k} U_i^{p_0}\right) \\ &= R_{1,\varepsilon}(t) + R_{2,\varepsilon}(t) + R_{3,\varepsilon}(t) + R_{4,\varepsilon}(t). \end{aligned}$$

Note that by (2.3), after a redefining of $\bar{t}_2$ in (2.5) if is necessary, for all $t > \bar{t}_2$ one has

$$\left(C_1 - \frac{1}{2n_0}\right)e^{\beta\gamma t} < e^{-\beta\sigma t}K\left(e^{\beta t}\right) - C_0 < \left(C_1 + \frac{1}{2n_0}\right)e^{\beta\gamma t}.$$

Hence, for $t > \bar{t}_2$

$$\left|R_{1,\varepsilon}(t)\right| < \tilde{C}_1\beta^2\left|e^{\varepsilon t}e^{\beta\gamma t}V^{p_0+\varepsilon}\right| < C\left|e^{\beta\gamma t}\left(\sum_{i=1}^{k} e^{-|t-\tau_i|}\right)^{p_0}\right|.$$

Also we have for $t < \bar{t}_2$ that

$$|R_{1,\varepsilon}| < C\left|\left(\sum_{i=1}^{k} e^{\bar{t}_2-\tau_i}\right)^{p_0}\right|.$$

Therefore

$$\left\|R_{1,\varepsilon}(t)\right\|_* \leqslant C\varepsilon^{\rho_1},$$

for $\rho_1 = \min\{\frac{p_0}{p_0+1}, \frac{-\beta\gamma}{p_0+1}\}$. Similarly, the derivative $R_{1,\varepsilon}$ respect to τ_i satisfies

$$\left\|\frac{\partial R_{1,\varepsilon}}{\partial \tau_i}\right\|_* \leqslant C\varepsilon^{\rho_1}.$$

On the other hand, using a Taylor expansion, respectively, it is easy to check that

$$\left\|R_{j,\varepsilon}(t)\right\|_* \leqslant C\varepsilon \quad \text{and} \quad \left\|\frac{\partial R_{j,\varepsilon}}{\partial \tau_i}\right\|_* < C\varepsilon, \quad \text{for } j = 2, 3.$$

Finally, using the Mean Value Theorem and analyzing $R_{4,\varepsilon}(t)$ for t and for σ in suitable ranges of $\mathbb{R}$, straightforward calculations lead to

$$\left\|R_{4,\varepsilon}(t)\right\|_* \leqslant C\varepsilon^{\rho_2} \quad \text{and} \quad \left\|\frac{\partial R_{4,\varepsilon}}{\partial \tau_i}\right\|_* < C\varepsilon^{\rho_2},$$

where $\rho_2 = \frac{1+\mu}{2}$ for $0 < \mu < \min\{p_0 - 1, 1\}$. Then, choosing $\bar{\rho} = \min\{\rho_1, \rho_2\}$ the proof of this lemma is completed. □

Proposition 3.3. *Assume that the constraints* (3.14) *hold. Then there exists $C > 0$ such that, for all $\varepsilon > 0$ small enough, there exists a unique solution $\phi = \phi(\vec{\tau})$ to problem* (3.2)*. Moreover, the map $\vec{\tau} \mapsto \phi(\vec{\tau})$ is of class C^1 for the $\|\cdot\|_*$-norm and satisfies*

$$\|\phi\|_* \leqslant C\varepsilon^{\bar{\rho}} \quad \text{and} \quad \|D_{\vec{\tau}}\phi\|_* \leqslant C\varepsilon^{\bar{\rho}},$$

with $\bar{\rho}$ as in (3.17)*.*

Proof. Let us consider the operator

$$\begin{aligned} F_\varepsilon : \mathcal{A}_r &\to H^1(\mathbb{R}) \\ \phi &\to F_\varepsilon(\phi) = -T_\varepsilon\big(N_\varepsilon(\phi) + R_\varepsilon\big), \end{aligned}$$

with T_ε given by Proposition 3.1 and

$$\mathcal{A}_r = \big\{\phi \in C_*:\ \|\phi\|_* \leqslant r\varepsilon^{\bar{\rho}}\big\},$$

for a suitable $r = r(N) > 0$ which will be chosen later. Note that if we show that F_ε is a contraction, then there is a fixed point in $\mathcal{A}_r$ for F_ε, which is equivalent to solving (3.2).

We have

$$\begin{aligned} \big\|F_\varepsilon(\phi)\big\|_* &\leqslant \big\|T_\varepsilon\big(N_\varepsilon(\phi) + R_\varepsilon\big)\big\|_* \\ &\leqslant C\big\|N_\varepsilon(\phi) + R_\varepsilon\big\|_* \\ &\leqslant \tilde{C}_1\big(\big(r\varepsilon^{\bar{\rho}}\big)^{\min\{p_0,2\}} + \big(r\varepsilon^{\bar{\rho}}\big)^{\min\{2p_0 - p_\infty,2\}} + \varepsilon^{\bar{\rho}}\big). \end{aligned}$$

Also we note that

$$\big\|F_\varepsilon(\phi_1) - F_\varepsilon(\phi_2)\big\|_* \leqslant C\big\|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)\big\|_*,$$

for $\phi_1, \phi_2 \in \mathcal{A}_r$. Hence, F_ε is a contraction if N_ε is.

We have

$$\big|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)\big| = \left|\frac{\partial N_\varepsilon}{\partial \phi}(\bar{\phi})\right| |\phi_1 - \phi_2|,$$

for some $\bar{\phi}$ on the line that join ϕ_1 with ϕ_2. It follows that

$$\big\|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)\big\|_* \leqslant C\big(\|\bar{\phi}\|_*^{\min\{p_0-1,1\}} + \|\bar{\phi}\|_*^{\min\{2p_\infty - p_0 - 1,1\}}\big)\|\phi_1 - \phi_2\|_*.$$

Hence

$$\big\|F_\varepsilon(\phi_1) - F_\varepsilon(\phi_2)\big\|_* \leqslant \tilde{C}_2\big(\big(r\varepsilon^{\bar{\rho}}\big)^{\min\{p_0-1,1\}} + \big(r\varepsilon^{\bar{\rho}}\big)^{\min\{2p_\infty - p_0 - 1,1\}}\big)\|\phi_1 - \phi_2\|_*.$$

Now, choosing $r > (3\tilde{C}_1 + 2\tilde{C}_2)$ one has

$$\left\|F_\varepsilon(\phi)\right\|_* \leqslant r\varepsilon^{\bar{\rho}}, \quad \forall \phi \in \mathcal{A}_r,$$

and

$$\left\|F_\varepsilon(\phi_1) - F_\varepsilon(\phi_2)\right\|_* < \|\phi_1 - \phi_2\|_*,$$

for $\varepsilon > 0$ sufficiently small.

Concerning to the differentiability properties, let us recall that ϕ is defined by the relation

$$B(\vec{\tau}, \phi) := \phi - T_\varepsilon\big(N_\varepsilon(\phi) + R_\varepsilon\big) = 0.$$

Hence, we see that

$$D_\phi B(\vec{\tau}, \phi)[\theta] = \theta - T_\varepsilon\big(\theta D_\psi N_\varepsilon(\phi)\big) := \theta + \tilde{M}(\theta),$$

where $\tilde{M}(\theta) = -T_\varepsilon(\theta D_\phi N_\varepsilon(\phi))$. Now, note that if we use the fact that $\phi \in \mathcal{A}_r$ one proves easily from (3.16) that

$$\left\|\tilde{M}(\theta)\right\|_* \leqslant C\varepsilon^{\bar{\rho}}\|\theta\|_*.$$

This implies that for ε small, the linear operator $D_\phi B(\vec{\tau}, \phi)$ is invertible in the space of the continuous functions in $\mathbb{R}$ with bounded $\|\cdot\|_*$-norm, with uniformly bounded inverse depending continuously on its parameters. Then, applying the Implicit Function Theorem we obtain that $\phi(\vec{\tau})$ is a C^1-function into C_*, with

$$D_{\vec{\tau}}\phi = -\big(D_\phi B(\vec{\tau}, \phi)\big)^{-1}\big(D_{\vec{\tau}} B(\vec{\tau}, \phi)\big).$$

Since

$$D_{\vec{\tau}} B(\vec{\tau}, \phi) = -D_{\vec{\tau}}\big(N_\varepsilon(\phi) + R_\varepsilon\big) - T_\varepsilon\big(D_{\vec{\tau}} N_\varepsilon(\phi) + D_{\vec{\tau}} R_\varepsilon\big),$$

where all these expressions depend continuously on their parameters, it follows that

$$\|D_{\vec{\tau}}\phi\|_* \leqslant C\big(\left\|N_\varepsilon(\phi) + R_\varepsilon\right\|_* + \left\|D_{\vec{\tau}} N_\varepsilon(\phi)\right\|_* + \|D_{\vec{\tau}} R_\varepsilon\|_*\big),$$

and using the first part of this proposition, the estimates in the previous lemmas, Proposition 3.1 and the constraints (3.14), we conclude that

$$\|D_{\vec{\tau}}\phi\|_* \leqslant C\varepsilon^{\bar{\rho}}. \qquad \square$$

4. The reduced functional and the proof of Theorem 1.1

Here we consider the constraints (3.14) and the function $\phi = \phi(\vec{\tau})$ given by Proposition 3.3. According to the previous sections, let us note that $c_i = 0$ in (3.2), for all $i = 1, 2, 3, \ldots, k$, is equivalent to say that $v = V + \phi(\vec{\tau})$ is a solution of problem (2.1), and therefore

$$u(r) = r^{\frac{N-2}{2}} v\big(\ln r^{-\frac{N-2}{2}}\big), \quad r \in (0, +\infty),$$

will be a solution of problem (1.1). A first result appearing to prove that problem (3.2) has solution, consists in proving that this problem is equivalent to a variational problem. For this, it is convenient to consider

$$I_\varepsilon(\vec{\tau}) = E_\varepsilon\big(V + \phi(\vec{\tau})\big).$$

Lemma 4.1. *The function $v = V + \phi(\vec{\tau})$ is a solution of* (2.1) *if and only if $\vec{\tau}$ is a critical point of I_ε.*

Proof. First we assume that $v = V + \phi(\vec{\tau})$ solves (2.1). Then directly one obtains

$$DE_\varepsilon\big(V + \phi(\vec{\tau})\big)\left[\frac{\partial(V + \phi(\vec{\tau}))}{\partial \tau_i}\right] = 0, \quad \forall i = 1, 2, 3, \dots, k.$$

In other words

$$\frac{\partial I_\varepsilon}{\partial \tau_i}(\vec{\tau}) = 0, \quad \forall i = 1, 2, 3, \dots, k,$$

so that $\vec{\tau}$ is a critical point of I_ε.

On the other hand, if $\vec{\tau}$ is a critical point of I_ε, then

$$DE_\varepsilon\big(V + \phi(\vec{\tau})\big)\left[\frac{\partial(V + \phi(\vec{\tau}))}{\partial \tau_j}\right] = 0, \quad \forall j = 1, 2, 3, \dots, k,$$

or equivalently, from (3.2),

$$\sum_{i=1}^{k} c_i\big(Z_i Z_j + o(1)\big) = 0, \quad \forall j = 1, 2, 3, \dots, k,$$

where $o(1) \to 0$ uniformly in the $\|\cdot\|_*$-norm, because $\frac{\partial(V+\phi(\vec{\tau}))}{\partial \tau_j} = Z_j + o(1)$. Now, noticing that the last system on c_i's is almost diagonal, one can conclude that $c_i = 0$ for all $i = 1, 2, 3, \dots, k$. Therefore $v = V + \phi(\vec{\tau})$ solves (2.1). □

The next step is to validate an expansion for I_ε which will be crucial to find its critical points.

Proposition 4.1. *Under the assumptions of Lemma* 3.1*, and considering V as in* (2.6)*, $\phi = \phi(\vec{\tau})$ given by Proposition* 3.3 *and $\frac{N+\sigma}{2} < \gamma$, the following expansion holds*

$$I_\varepsilon(\vec{\tau}) = E_\varepsilon(V) + o(\varepsilon),$$

where $o(\varepsilon)$ is uniformly of this size in the C^1-sense on the vectors $\vec{\tau}$ satisfying (3.14)*, for given A.*

Proof. We note that

$$I_\varepsilon(\vec{\tau}) - E_\varepsilon(V) = E_\varepsilon(V + \phi) - E_\varepsilon(V),$$

where

$$E_\varepsilon(v) = \frac{1}{2}\int_{-\infty}^{+\infty} (v')^2 + \frac{1}{2}\int_{-\infty}^{+\infty} v^2 - \frac{1}{p_0+1+\varepsilon}\int_{-\infty}^{+\infty} e^{\varepsilon s} e^{-\beta\sigma s} K\big(e^{\beta s}\big) v^{p_0+1+\varepsilon}.$$

It is easy to check from the Fundamental Theorem of Calculus and one integrating by parts that

$$I_\varepsilon(\vec{\tau}) - E_\varepsilon(V) = -\int_0^1 t D^2 E_\varepsilon(V + t\phi)[\phi][\phi]\,dt.$$

Now, note that after a Taylor expansion and integrating by parts, one has for each $\phi \in H_\varepsilon$ that

$$DE_\varepsilon(V + t\phi)[\phi] = \int_{-\infty}^{+\infty} \big(-V'' + V + t\big(-\phi'' + \phi\big) - e^{\varepsilon s} e^{-\beta\sigma s} K\big(e^{\beta s}\big)(V + t\phi)^{p_0+\varepsilon}\big)\phi,$$

and since

$$-(V+\phi)'' + (V+\phi) - \beta^2 e^{\varepsilon t} e^{-\beta\sigma t} K\big(e^{\beta t}\big)(V+\phi)^{p_0+\varepsilon} = \sum_{i=1}^{k} c_i Z_i \quad \text{in } \mathbb{R},$$

and

$$\int_{-\infty}^{+\infty} Z_i \phi = 0, \quad \forall i = 1, 2, 3, \ldots, k,$$

it follows that

$$\begin{aligned}
&D^2 E_\varepsilon(V + t\phi)[\phi, \phi] \\
&\quad = (t-1)\left(\int_{-\infty}^{+\infty} \big(N_\varepsilon(\phi) + R_\varepsilon\big)\phi\right) + \int_{-\infty}^{+\infty} \big(e^{\varepsilon s} e^{-\beta\sigma s} K\big(e^{\beta s}\big)\big((V+\phi)^{p_0+\varepsilon} - (V+t\phi)^{p_0+\varepsilon}\big)\big)\phi.
\end{aligned}$$

Also, as $\frac{1}{2} < \bar{\rho} < 1$ and

$$\begin{aligned}
&\left|\int_0^1 t(t-1)\left(\int_{-\infty}^{+\infty} \big(N_\varepsilon(\phi) + R_\varepsilon\big)\phi\right) dt\right| \\
&\quad \leqslant \frac{1}{6}\big\|N_\varepsilon(\phi) + R_\varepsilon\big\|_* \|\phi\|_* \int_{-\infty}^{+\infty} \left(\sum_{i=1}^{k} e^{2(-\bar{\eta}|t-\tau_i|)}\right)^{-1} dt \\
&\quad \leqslant C\big(\big\|N_\varepsilon(\phi)\big\|_* + \|R_\varepsilon\|_*\big)\|\phi\|_* \\
&\quad \leqslant C\varepsilon^{2\bar{\rho}}
\end{aligned}$$

we have that

$$\int_0^1 t(t-1)\left(\int_{-\infty}^{+\infty}\big(N_\varepsilon(\phi)+R_\varepsilon\big)\phi\right)dt = o(\varepsilon).$$

On the other hand, after a Taylor expansion, we get

$$\begin{aligned}
&\int_0^1 t\left(\int_{-\infty}^{+\infty}\big(e^{\varepsilon s}e^{-\beta\sigma s}K\big(e^{\beta s}\big)\big((V+\phi)^{p_0+\varepsilon}-(V+t\phi)^{p_0+\varepsilon}\big)\big)\phi\right)dt\\
&\quad=(\varepsilon+p_0)\int_0^1 t(1-t)\left(\int_{-\infty}^{+\infty}e^{\varepsilon s}e^{-\beta\sigma s}K\big(e^{\beta s}\big)V^{\varepsilon+p_0-1}\phi^2\right)dt
\end{aligned}$$

and given that $\frac{1}{2}<\bar\rho<1$ and

$$\left|\int_{-\infty}^{t_1}e^{\varepsilon s}e^{-\beta\sigma s}K\big(e^{\beta s}\big)V^{\varepsilon+p_0-1}\phi^2\right|\leqslant Ce^{-(p_0+1)\tau_1}\|\phi\|_*^2\int_{-\infty}^{t_1}e^{(p_0-p_\infty)s}e^{(p_0-1)s}e^{2s}=O\big(\varepsilon^{1+2\bar\rho}\big),$$

$$\left|\int_{t_1}^{t_2}e^{\varepsilon s}e^{-\beta\sigma s}K\big(e^{\beta s}\big)V^{\varepsilon+p_0-1}\phi^2\right|\leqslant Ce^{-(p_0+1)\tau_1}\|\phi\|_*^2\int_{t_1}^{t_2}e^{-\beta\sigma s}e^{(p_0-1)s}e^{2s}=O\big(\varepsilon^{1+2\bar\rho}\big),$$

$$\left|\int_{t_2}^{+\infty}e^{\varepsilon s}e^{-\beta\sigma s}K\big(e^{\beta s}\big)V^{\varepsilon+p_0-1}\phi^2\right|\leqslant C\|\phi\|_*^2\int_{t_2}^{+\infty}V^{p_0+1}=O\big(\varepsilon^{2\bar\rho}\big),$$

it follows that

$$\left|-\int_0^1 t\left(\int_{-\infty}^{+\infty}\big(e^{\varepsilon s}e^{-\beta\sigma s}K\big(e^{\beta s}\big)\big((V+\phi)^{p_0+\varepsilon}-(V+t\phi)^{p_0+\varepsilon}\big)\big)\phi\right)dt\right|\leqslant o(\varepsilon).$$

Therefore

$$I_\varepsilon(\vec\tau)-E_\varepsilon(V)=o(\varepsilon).$$

For the differentiability only note that

$$\begin{aligned}
&\frac{\partial}{\partial\tau_i}\big(I_\varepsilon(\vec\tau)-E_\varepsilon(V)\big)\\
&\quad=-\int_0^1 t(t-1)\left(\int_{-\infty}^{+\infty}\frac{\partial}{\partial\tau_i}\big(\big(N_\varepsilon(\phi)+R_\varepsilon\big)\phi\big)\right)dt\\
&\qquad-(\varepsilon+p_0)\int_0^1 t(1-t)\left(\int_{-\infty}^{+\infty}e^{\varepsilon s}e^{-\beta\sigma s}K\big(e^{\beta s}\big)\frac{\partial}{\partial\tau_i}\big(V^{\varepsilon+p_0-1}\phi^2\big)\right)dt+o(\varepsilon)
\end{aligned}$$

and that

$$\left(\left\|\frac{\partial}{\partial\tau_i}N_\varepsilon(\phi)\right\|_* + \left\|\frac{\partial}{\partial\tau_i}R_\varepsilon\right\|_*\right)\|\phi\|_* + \left(\left\|N_\varepsilon(\phi)\right\|_* + \|R_\varepsilon\|_*\right)\left\|\frac{\partial}{\partial\tau_i}\phi\right\|_* = O\left(\varepsilon^{2\bar\rho}\right)$$

and

$$\|\phi\|_*^2 + \|\phi\|_*\left\|\frac{\partial}{\partial\tau_i}\phi\right\|_* = O\left(\varepsilon^{2\bar\rho}\right).$$

Therefore

$$\frac{\partial}{\partial\tau_i}\left(I_\varepsilon(\vec\tau) - E_\varepsilon(V)\right) = o(\varepsilon). \qquad \square$$

Proof of Theorem 1.1. Consider the change of variables

$$\tau_1 = -\frac{1}{p_0+1}\log\varepsilon - \log\lambda_1,$$
$$\tau_{i+1} - \tau_i = -\log\varepsilon - \log\lambda_i \quad \forall i = 2,3,\dots,k,$$

where the λ_i's are positive parameters. Hence, it is sufficient to find critical points of

$$\Phi_\varepsilon(\vec\lambda) = \varepsilon^{-1}I_\varepsilon\left(\vec\tau(\vec\lambda)\right).$$

From the previous lemma and the expansion given by Lemma 3.1, we obtain

$$\nabla\Phi_\varepsilon(\vec\lambda) = \nabla\Psi_k(\vec\lambda) + o(1),$$

where $o(1) \to 0$ uniformly on the vectors $\vec\lambda$ satisfying $M^{-1} < \lambda_i < M$ for any M fixed large sufficiently large. As we pointed in Remark 1, $\Psi_k(\vec\lambda)$ has an only one critical point $\vec\lambda^*$ which is non-degenerate. It follows from local theory degree that $\deg(\nabla\Phi_\varepsilon, \mathcal{U}, 0)$ is well defined and is non-zero, where $\mathcal{U}$ denotes an arbitrarily small neighborhood of $\vec\lambda^*$. Then, for $\varepsilon > 0$ small enough we have

$$\deg(I_\varepsilon, \mathcal{U}, 0) \neq 0.$$

We conclude that there exists a critical point $\vec\lambda^*$ of Φ_ε such that

$$\vec\lambda_\varepsilon^* = \vec\lambda^* + o(1).$$

Hence, for $\vec\tau^* = \vec\tau(\vec\lambda_\varepsilon^*)$ we get

$$v^*(t) = \sum_{i=1}^k U\left(t-\tau_i^*\right) + \phi\left(\tau_i^*\right) = \sum_{i=1}^k U\left(t-\tau_i^*\right)\left(1+o(1)\right)$$

is a solution of (2.1), and then we get

$$u(x) = \xi_\sigma\sum_{i=1}^k \varepsilon^{-(i-1)+\frac{1}{p_0+1}}\alpha_i^*\left(1+\left(\alpha_i^*\right)^{p_0-1}\varepsilon^{-((i-1)-\frac{1}{p_0+1})(p_0-1)}|x|^{2+\sigma}\right)^{-\frac{2}{p_0-1}}\left(1+o(1)\right)$$

is a solution of (1.1), where $\alpha_i^* = \prod_{j=1}^{i} (\lambda_j^*)^{-1}$ and

$$\vec{\lambda}^* = \left(\left(\frac{k\alpha_5}{\alpha_3(p_0+1)}\right)^{\frac{1}{p_0+1}}, \frac{(k-1)\alpha_5}{\alpha_2}, \frac{(k-2)\alpha_5}{\alpha_2}, \ldots, \frac{2\alpha_5}{\alpha_2}, \frac{\alpha_5}{\alpha_2}\right). \qquad \square$$

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