J. Differential Equations 247 (2009) 3136–3162

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Multiplicity of solutions for a fourth order problem with exponential nonlinearity

Juan Dávila [a,*], Isabel Flores [b], Ignacio Guerra [c]

[a] Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático (UMI 2807, CNRS), Universidad de Chile, Casilla 170/3 Correo 3, Santiago, Chile
[b] Departamento de Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción, Casilla 160 C, Concepción, Chile
[c] Departamento de Matematica y C.C., Facultad de Ciencia, Universidad de Santiago de Chile, Casilla 307, Correo 2, Santiago, Chile

ARTICLE INFO

Article history:
Received 10 March 2009
Revised 14 July 2009
Available online 13 August 2009

ABSTRACT

Let B be the unit ball in $\mathbb{R}^N$, $N \geqslant 5$ and n be the exterior unit normal vector on the boundary. We consider radial solutions to

$$\Delta^2 u = \lambda e^u \quad \text{in } B, \qquad u = 0 \quad \text{and} \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial B,$$

where $\lambda \geqslant 0$. We show that there exists a unique $\lambda_S > 0$ such that if $\lambda = \lambda_S$ there is a radial singular solution. If $5 \leqslant N \leqslant 12$ then for $\lambda = \lambda_S$ there exist infinitely many regular radial solutions and as $\lambda \to \lambda_S$ the number of such solutions goes to infinity. If $N \geqslant 13$ we prove uniqueness of smooth radial solutions. We derive similar results for the same equation with Navier boundary conditions.

1. Summary

In this paper we study radial solutions to the fourth order problem

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } B, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial B, \end{cases} \tag{1}$$

where B is the unit ball in $\mathbb{R}^N$, $N \geqslant 5$, n is the exterior unit normal vector and $\lambda \geqslant 0$ is a parameter.

* Corresponding author.
E-mail addresses: jdavila@dim.uchile.cl (J. Dávila), iflores@udec.cl (I. Flores), ignacio.guerra@usach.cl (I. Guerra).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2009.07.023

Recently nonlinear higher order equations with a supercritical behavior have attracted much interest. See Arioli, Gazzola and Grunau [1], Arioli, Gazzola, Grunau and Mitidieri [2], Dávila, Dupaigne, Guerra and Montenegro [15] for the exponential nonlinearity, Berchio and Gazzola [7], Berchio, Gazzola and Mitidieri [8] for some properties for general nonlinearities, Ferrero and Grunau [18], Ferrero, Grunau and Karageorgis [19], Ferrero and Warnault [20] for power nonlinearities, Cassani, do O and Ghoussoub [10], Cowan, Esposito, Ghoussoub and Moradifam [14], Guo and Wei [29,30] and also [20] for singular nonlinearities. Some of these works treat the problem in a ball [2,10,14,15,18–20,29,30] and others in entire space [1,19,30,33].

The central result in this work is the existence of infinitely many radial smooth solutions of (1) for some $\lambda > 0$ if $5 \leqslant N \leqslant 12$. This value of λ is precisely the one for which a singular solution exists, so first we study singular solutions.

Theorem 1.1. *Assume $N \geqslant 5$. Let $\lambda > 0$ and suppose $u \in C^4((0,1))$, $u \geqslant 0$ satisfies*

$$\Delta^2 u = \lambda e^u \quad \text{in } B \setminus \{0\}. \tag{2}$$

Then either:

(a) *u can be extended as a $C^\infty(B)$ and (2) holds in B, or*
(b) *u is singular at $r = 0$ and satisfies*

$$\lim_{r\to 0} u(r) + 4\log(r) = \log\frac{8(N-2)(N-4)}{\lambda}, \tag{3}$$

$$\lim_{r\to 0} r u'(r) \quad \text{exists}. \tag{4}$$

We prove this in Section 3. In [2] the authors call u a weakly singular solution to (2) if it satisfies (4) and is singular. It turns out that this definition is natural for space phase analysis, after transforming the problem to a suitable first order autonomous system. From the PDE point of view, the following definition is also natural. We say that u is a weak solution of (1) if

$$\begin{cases} u \in H_0^2(B), \quad e^u \in L^1(B) \quad \text{and} \\ \displaystyle\int_B \Delta u \Delta\varphi = \lambda \int_B e^u \varphi \quad \forall \varphi \in C_0^\infty(B). \end{cases} \tag{5}$$

It is possible to show that weakly singular solutions are also weak solutions. Since weak solutions in the sense (5) are nonnegative (see [2]) as a consequence of Theorem 1.1 we are showing that both notions coincide for radial functions.

Combining the results of [2] and [15], some of which are obtained by a computer assisted proof, we know that for all $N \geqslant 5$, (1) has a singular solution for some $\lambda > 0$. We give a new proof of the existence, which is not computer assisted, and show its uniqueness.

Theorem 1.2. *Assume $N \geqslant 5$. There exists a unique $\lambda_S > 0$ such that (1) with $\lambda = \lambda_S$ admits a radial singular solution and this singular solution is unique in the class of radial solutions.*

Let λ^* denote the largest value of $\lambda \geqslant 0$ such that (1) has a radial classical solution. Then $\lambda^* > 0$ and finite, see [2]. Many authors have studied what happens to solutions when $\lambda = \lambda^*$ (for this see [2,15]). With respect to multiplicity of solutions we have the following:

Theorem 1.3. *Assume $5 \leqslant N \leqslant 12$. Then $\lambda_S < \lambda^*$ and (1) with $\lambda = \lambda_S$ admits infinitely many regular radial solutions. For $\lambda \neq \lambda_S$ then (1) has a finite number of regular radial solutions, and this number goes to infinity as $\lambda \to \lambda_S$.*

The fact that if $5 \leqslant N \leqslant 12$ then $\lambda_S < \lambda^*$ is also a consequence of the results in [2] and [15].

Theorem 1.4. *If $N \geqslant 13$ then $\lambda_S = \lambda^*$ and for all $0 < \lambda < \lambda^*$ (1) has a unique radial solution, which is regular. For $\lambda = \lambda^*$ there is a unique radial solution which is singular.*

Let

$$\mathcal{C} = \{(\lambda, u) \in (0, \infty) \times C^4(\bar{B}):\ u \text{ is radial and solves } (1)\}.$$

Following [30] we have:

Theorem 1.5. *Assume $N \geqslant 5$. The set $\mathcal{C}$ is homeomorphic to $(0, \infty)$ and the identification can be done through $(\lambda, u) \in \mathcal{C} \mapsto u(0)$.*

The inverse of the above identification can be extended continuously in a suitable topology to $[0, \infty]$ as $0 \mapsto (0, 0)$ and $\infty \mapsto (\lambda_S, u_S)$ where u_S is the unique singular solution of Theorem 1.2.

For the problem with Navier boundary conditions

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } B, \\ u = \Delta u = 0 & \text{on } \partial B \end{cases} \tag{6}$$

we have similar results.

Theorem 1.6. *Assume $N \geqslant 5$. There exists a unique $\lambda_S > 0$ such that (6) with $\lambda = \lambda_S$ admits a radial singular solution and this singular solution is unique in the class of radial solutions.*

Theorem 1.7. *Assume $5 \leqslant N \leqslant 12$. Then (6) with $\lambda = \lambda_S$ admits infinitely many regular radial solutions. For $\lambda \neq \lambda_S$ then (6) has a finite number of regular radial solutions and the number of radial regular solutions goes to infinity as $\lambda \to \lambda_S$.*

We prove the multiplicity results by phase space analysis, using ideas from the work of Bamón, Flores, del Pino [3] and which were subsequently applied also in [17,21,22]. By a change of variables we transform the ODE version of (1) into a reasonable first order 4-dimensional autonomous system, which has 2 stationary points P_1, P_2. Some properties of this system, such as the local character of P_1, P_2, were studied by Arioli, Gazzola, Grunau and Mitidieri [2]. We review this material in Section 2. It is important in our argument to know that there exists a heteroclinic connection from P_1 to P_2. This connection was found by Arioli, Gazzola and Grunau in [1], in the form of an entire solution of $\Delta^2 u = e^u$ with a special decay. We explain this in Section 4 and show that in dimensions $5 \leqslant N \leqslant 12$ this connection near P_2 is a spiral. In Section 5 we study some properties of the unstable manifold at P_2, which lead to the proof of Theorems 1.2 and 1.6. The proof of the multiplicity of solutions asserted in Theorems 1.3 and 1.6 is in Section 6. Finally Section 7 is dedicated to the study of some properties of the solution set $\mathcal{C}$. In particular we prove there Theorems 1.4 and 1.5.

It is natural to ask whether the uniqueness result of Theorem 1.4 is true for problem (6). Using the techniques in this work it is possible to show that if the extremal solution u^* of (6) is singular, then for all $\lambda \in (0, \lambda^*)$ there is a unique radial solution. We conjecture that this is the case for all $N \geqslant 13$, and the proof could be done using similar ideas as in [15].

The counterpart of the results in this paper for the classical problem

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } B, \\ u = 0 & \text{on } \partial B \end{cases} \tag{7}$$

are well known. In dimension 1 this problem was first studied by Liouville [35], then by Bratu [9], Chandrasekhar [11] and Frank-Kamenetskii [23]. Barenblatt [25] proved that in dimension 3 for $\lambda = 2$ there are infinitely many solutions, and Joseph and Lundgren [32] completed the description of the classical solutions to (7) in all dimensions. The literature on second order problems like (7), including other nonlinearities and general domains, is very extensive, see [4,34].

In a forthcoming work [16] we will address similar multiplicity results for the bilaplacian with power-type nonlinearities.

2. Preliminaries

With the change of variables $v(t) = u(r)$, $r = e^t$ Eq. (1) is equivalent to

$$(\partial_t + N - 4)(\partial_t - 2)(\partial_t + N - 2)\partial_t v(t) = \lambda e^{v+4t} \quad \text{for all } t < 0 \tag{8}$$

with the boundary conditions

$$v(0) = 0, \qquad v'(0) = 0 \tag{9}$$

and the behavior at $-\infty$ given by

$$\lim_{t\to-\infty} v(t) \in \mathbb{R}, \qquad \lim_{t\to-\infty} e^{-t} v'(t) = 0.$$

Let

$$\begin{cases} v_1(t) = \dfrac{\lambda}{8(N-2)(N-4)} e^{v(t)+4t} = \dfrac{\lambda}{8(N-2)(N-4)} e^{u(e^t)+4t}, \\ v_2(t) = \partial_t v(t) = e^t u'\big(e^t\big), \\ v_3(t) = (\partial_t - 2 + N) v_2(t) = e^{2t} \Delta u\big(e^t\big), \\ v_4(t) = (\partial_t - 2) v_3(t) = e^{3t} (\Delta u)'\big(e^t\big). \end{cases} \tag{10}$$

Then (8) becomes

$$\begin{cases} v_1' = v_1(v_2 + 4), \\ v_2' = -(N-2)v_2 + v_3, \\ v_3' = 2v_3 + v_4, \\ v_4' = -(N-4)v_4 + 8(N-2)(N-4)v_1 \end{cases} \tag{11}$$

while (9) is equivalent to

$$v_2(0) = 0, \qquad v_1(0) = \frac{\lambda}{8(N-2)(N-4)}. \tag{12}$$

The only stationary points of the system (11) are

$$\begin{cases} P_1 = (0,0,0,0) \\ P_2 = \big(1, -4, -4(N-2), 8(N-2)\big). \end{cases} \tag{13}$$

Let $V = (v_1, \dots, v_4)$. From Theorem 6 in [2] we learn that u is a regular solution of (1) if and only if

$$\lim_{t\to-\infty} V(t) = P_1$$

while u is a weakly singular solution if and only if

$$\lim_{t\to-\infty} V(t) = P_2.$$

The linearization of (11) around the point P_1 is given by $Z' = M_1 Z$ where

$$M_1 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & -(N-2) & 1 & 0 \\ 0 & 0 & 2 & 1 \\ A & 0 & 0 & -(N-4) \end{bmatrix}$$

and $A = 8(N-4)(N-2)$. The eigenvalues of this matrix are 2, 4, $-N+4$, $-N+2$. Thus, if $N \geqslant 5$ then P_1 is a hyperbolic point with a 2-dimensional unstable manifold $W^u(P_1)$ and a 2-dimensional stable manifold $W^s(P_1)$.

The linearization of (11) around P_2 is given by $Z' = MZ$ where

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -(N-2) & 1 & 0 \\ 0 & 0 & 2 & 1 \\ A & 0 & 0 & -(N-4) \end{bmatrix} \tag{14}$$

and $A = 8(N-4)(N-2)$. The eigenvalues of M are given by

$$\begin{cases} \nu_1 = \frac{1}{2}\left(4 - N + \sqrt{M_1(N) + M_2(N)}\right), \\ \nu_2 = \frac{1}{2}\left(4 - N - \sqrt{M_1(N) + M_2(N)}\right), \\ \nu_3 = \frac{1}{2}\left(4 - N + \sqrt{M_1(N) - M_2(N)}\right), \\ \nu_4 = \frac{1}{2}\left(4 - N - \sqrt{M_1(N) - M_2(N)}\right) \end{cases} \tag{15}$$

where

$$M_1(N) = (N-2)^2 + 4, \qquad M_2(N) = 4\sqrt{(N-2)^2 + A}.$$

Then

$$\nu_2 < 0 < \nu_1.$$

If $5 \leqslant N \leqslant 12$ then $M_1(N) - M_2(N) < 0$ and ν_3, ν_4 are complex conjugate with nonzero imaginary part and negative real part. If $N \geqslant 13$ all eigenvalues are real and ν_3, ν_4 are negative. For all $N \geqslant 5$, P_2 is a hyperbolic stationary point with a 1-dimensional unstable manifold $W^u(P_2)$ and a 3-dimensional stable manifold $W^s(P_2)$.

Concerning the eigenvectors of M we have:

Lemma 2.1. *Let $v^{(1)}, \dots, v^{(4)}$ be the eigenvectors of M associated to $\nu_1, \dots, \nu_4$. Then*

$$v^{(k)} = \bigl[1, \nu_k, \nu_k(\nu_k + N - 2), \nu_k(\nu_k + N - 2)(\nu_k - 2)\bigr]. \tag{16}$$

We have that $v^{(1)}, v^{(2)}$ are always real, and $v^{(3)}, v^{(4)}$ are complex conjugate if $5 \leqslant N \leqslant 12$. Let us write $v^{(i)} = (v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, v_4^{(i)})$, $i = 1, \dots, 4$. If $N \geqslant 3$ then

$$v_1^{(1)} > 0, \qquad v_2^{(1)} > 0, \qquad v_3^{(1)} > 0, \qquad v_4^{(1)} > 0, \tag{17}$$

and

$$v_1^{(2)} > 0, \qquad v_2^{(2)} < 0, \qquad v_3^{(2)} > 0, \qquad v_4^{(2)} < 0. \tag{18}$$

Proof. That the vectors defined by (16) are eigenvector of M follows from a direct calculation. Let $v^{(1)} = (t_1, t_2, t_3, t_4)$ an eigenvector for M with eigenvalue ν_1. Then

$$t_1 = 1 > 0, \qquad t_2 = \nu_1 > 0,$$
$$t_3 = (\nu_1 + N - 2)\nu_1 > 0, \qquad t_4 = (\nu_1 - 2)(\nu_1 + N - 2)\nu_1 > 0.$$

In fact, since $\nu_1 > 0$, it is sufficient to prove that $\nu_1 - 2 > 0$. This holds if $\sqrt{M_1(N) + M_2(N)} > N$ and this is equivalent to $A = 8(N - 2)(N - 4) > 0$.

The proof of (18) is similar. □

It will be convenient to have the following result:

Lemma 2.2. *The system* (11) *is C^1-conjugate to its linearization around the point P_2.*

Proof. We use a result of Belickiĭ [5,6], see also the book [39, p. 25]. To apply it we need to verify that no relation of the form $\mathrm{Re}(\nu_i) = \mathrm{Re}(\nu_j) + \mathrm{Re}(\nu_k)$ holds for different indices i, j, k in $\{1, \dots, 4\}$ such that $\mathrm{Re}(\nu_j) < 0$ and $\mathrm{Re}(\nu_k) > 0$, where $\nu_1, \dots, \nu_4$ are the eigenvalues of M defined in (14). This can be verified by calculation. □

3. Behavior of singular solutions

The purpose of this section is to prove Theorem 1.1. In what follows we assume that $u \in C^4(0, 1)$, $u \geqslant 0$ satisfies

$$\Delta^2 u = 8(N - 2)(N - 4)e^u \quad \text{in } (0, 1), \tag{19}$$

where we have assumed, by using a scaling, that $\lambda = 8(N - 2)(N - 4)$. That the interval is $(0, 1)$ is not relevant for the next arguments. The arguments in this section are based on the work of Ferrero and Grunau [18] where they show that radial singular solutions to a problem with power-type nonlinearity are weakly singular for that problem.

Define $v(t) = u(e^t)$, $w(t) = v(t) + 4t$ for $t \leqslant 0$. We also let $\nu_1, \dots, \nu_4$ be defined by (10). We note that w satisfies

$$\begin{aligned} &w^{(4)} + 2(N - 4)w''' + \bigl(N^2 - 10N + 20\bigr)w'' - 2(N - 2)(N - 4)w' \\ &\quad = 8(N - 2)(N - 4)\bigl(e^w - 1\bigr) \quad \text{for all } t < 0. \end{aligned} \tag{20}$$

Lemma 3.1. *If $\delta > 0$ there exists a constant C depending only on δ such that if $[a,b] \subset (-\infty, 0]$ is such that $w(t) \geqslant \delta$ for all $t \in [a,b]$ then $b - a \leqslant C$. A consequence is that*

$$\liminf_{t \to -\infty} w(t) \leqslant 0. \tag{21}$$

Proof. We follow an idea of Mitidieri and Pokhozhaev [37], which has also been used in [1,18,24], called the test-function method in these references.

We proceed by contradiction. We may suppose, by shifting time, that $w(t) \geqslant \delta$ for all $t \in [-L, 0]$ for arbitrary large $L > 0$. Let $\phi \in C^\infty(\mathbb{R})$ be such that, $0 \leqslant \phi \leqslant 1$, $\phi(t) = 0$ for $t \leqslant -3$, $\phi(t) > 0$ for $t \in (-3, 0)$, $\phi(t) = 0$ for $t \geqslant 0$, $\phi(t) = 1$ for $t \in [-2, -1]$, and for $i = 1, 2, 3, 4$

$$\int_{-3}^{0} \frac{(\phi^{(i)})^2}{\phi}\, dt < +\infty.$$

Let $\tau > 1$ and $\phi_\tau(t) = \phi(t/\tau)$ and assume that $3\tau \leqslant L$. Let us rewrite Eq. (20) in the form

$$\sum_{i=1}^{4} a_i w^{(i)}(t) = A\big(e^w - 1\big) \quad \text{for } t < 0 \tag{22}$$

where $A = 8(N-2)(N-4)$ and $a_i \in \mathbb{R}$. Multiplying Eq. (22) by ϕ_τ and integrating we find

$$\sum_{i=1}^{4} a_i (-1)^i \int_{-3\tau}^{0} \phi_\tau^{(i)} w\, dt = A \int_{-3\tau}^{0} \big(e^w - 1\big)\phi_\tau\, dt. \tag{23}$$

Let $\varepsilon > 0$ to be fixed later on. For all $t > -3\tau$

$$\big|w\phi_\tau^{(i)}\big| \leqslant \varepsilon w^2 \phi_\tau + C_\varepsilon \frac{(\phi_\tau^{(i)})^2}{\phi_\tau}$$

so that from (23) we know that

$$A \int_{-3\tau}^{0} \big(e^w - 1\big)\phi_\tau\, dt \leqslant \varepsilon K \int_{-3\tau}^{0} w^2 \phi_\tau\, dt + C_\varepsilon K \max_{i=1,\dots,4} \int_{-3\tau}^{0} \frac{(\phi_\tau^{(i)})^2}{\phi_\tau}\, dt$$

where $K = \sum_{i=1}^4 |a_i|$. Since $w(t) \geqslant \delta$, we can select $\varepsilon > 0$ sufficiently small so that $A(e^w - 1) - \varepsilon K w^2 \geqslant \delta/4$ for all $t \in [-3\tau, 0]$. It follows that

$$\frac{\delta}{4}\tau \leqslant C_\varepsilon K \max_{i=1,\dots,4} \int_{-3\tau}^{0} \frac{(\phi_\tau^{(i)})^2}{\phi_\tau}\, dt.$$

But

$$\int_{-3\tau}^{0} \frac{(\phi_\tau^{(i)})^2}{\phi_\tau}\, dt = \tau^{1-2i} \int_{-3}^{0} \frac{(\phi^{(i)})^2}{\phi}\, dt \leqslant C_i \tau^{1-2i}.$$

It follows that

$$\frac{\delta}{4}\tau \leqslant C_\varepsilon K \max_{i=1,\dots,4} C_i \tau^{1-2i} \quad \text{for all } \tau > 1,$$

which is not possible. □

Lemma 3.2.

$$\limsup_{t\to-\infty} w(t) < +\infty.$$

Proof. We follow the idea of Lemma 1 in [18]. Assume by contradiction that $\limsup_{t\to-\infty} w(t) = +\infty$. Since (21) also holds there is a sequence $t_k \to -\infty$ such that $w(t_k) \to +\infty$, and for all $k \geqslant 1$ we have $t_{k+1} + \log 2 < t_k$, $w(t_{k+1}) \geqslant w(t_k)$, $w'(t_k) = 0$, and $w''(t_k) \leqslant 0$. It suffices to take as t_k a sequence of well separated local maxima of w along which it goes to $+\infty$.

Let $M_k = w(t_k)$, $r_k = e^{t_k}$ and $\rho_k = \frac{r_{k+1}}{r_k}$. Note that $0 < \rho_k \leqslant 1/2$. Define

$$u_k(r) = u(rr_k) - M_k + 4\log(r_k),$$

where u is the original solution to (19). Then

$$\begin{aligned} &\Delta^2 u_k = Ae^{M_k}e^{u_k} \quad \text{for } r \in \left(0, r_k^{-1}\right), \\ &u_k(1) = 0, \\ &u_k(\rho_k) = w(t_{k+1}) - M_k + 4(t_k - t_{k+1}) \geqslant 0. \end{aligned} \tag{24}$$

Moreover, since

$$\Delta u_k(r) = \frac{1}{r^2}\left[w''\left(\log(rr_k)\right) + (N-2)w'\left(\log(rr_k)\right) - 4(N-2)\right]$$

and $w'(t_k) = 0$ and $w''(t_k) \leqslant 0$ we have

$$\begin{aligned} \Delta u_k(1) &= w''(t_k) - 4(N-2) \leqslant 0, \\ \Delta u_k(\rho_k) &= \frac{1}{\rho_k^2}\left[w''(t_{k+1}) - 4(N-2)\right] \leqslant 0. \end{aligned}$$

Let λ_k be the first eigenvalue for $-\Delta$ with Dirichlet boundary condition in the annulus $B \setminus B_{\rho_k}$ and $\phi_k > 0$ be an associated eigenfunction, that is

$$\begin{cases} -\Delta\phi_k = \lambda_k\phi_k & \text{in } B \setminus B_{\rho_k}, \\ \phi_k = 0 & \text{on } \partial(B \setminus B_{\rho_k}). \end{cases}$$

Then $\Delta^2\phi_k = \lambda_k^2\phi_k$. Multiplying (24) by ϕ_k and integrating by parts we obtain

$$
\begin{aligned}
Ae^{M_k} \int\limits_{B\setminus B_{\rho_k}} e^{u_k}\phi_k\,dx &= \int\limits_{B\setminus B_{\rho_k}} \Delta^2 u_k\phi_k\,dx \\
&= \int\limits_{\partial(B\setminus B_{\rho_k})} \left[\frac{\partial\Delta u_k}{\partial n}\phi_k - \Delta u_k\frac{\partial\phi_k}{\partial n} + \frac{\partial u_k}{\partial n}\Delta\phi_k - u_k\frac{\partial\Delta\phi_k}{\partial n}\right] \\
&\quad + \int\limits_{B\setminus B_{\rho_k}} u_k\Delta^2\phi_k\,dx.
\end{aligned}
$$

But on $\partial(B\setminus B_{\rho_k})$, $\phi_k=\Delta\phi_k=0$, $\frac{\partial\phi_k}{\partial n}\leqslant 0$ and $\frac{\partial\Delta\phi_k}{\partial n}\geqslant 0$. Hence

$$
\Delta u_k\frac{\partial\phi_k}{\partial n}\geqslant 0 \quad\text{and}\quad u_k\frac{\partial\Delta\phi_k}{\partial n}\geqslant 0 \quad \text{on } \partial(B\setminus B_{\rho_k}).
$$

Using also the inequality $e^u\geqslant u$ it follows that

$$
Ae^{M_k}\leqslant\lambda_k^2.
$$

But since the annulus $B\setminus B_{\rho_k}$ has a width that does not converge to zero, λ_k remains uniformly bounded, even if $\rho_k\to 0$. It follows that M_k remains bounded as $k\to\infty$, which is a contradiction. □

Lemma 3.3. *For all $i=0,1,2,3,4$*

$$
\left|w^{(i)}(t)\right|\leqslant C\left(1+|t|\right)\quad\forall t\leqslant 0,
$$

and for $i=1,2,3,4$

$$
\left|v_i(t)\right|\leqslant C\left(1+|t|\right)\quad\forall t\leqslant 0. \tag{25}
$$

Proof. The fact that $|w(t)|\leqslant C(1+|t|)$ follows from $u\geqslant 0$ and that w is bounded above. We regard (20) as an elliptic equation, or use interpolation inequalities such as in Chapter 6 of [26] to obtain the following assertion. For any $t\leqslant -1$ and $i=1,2,3,4$

$$
\left|w^{(i)}(t)\right|\leqslant C\sup_{[t-1,t+1]}\left(|w|+A\left|e^w-1\right|\right).
$$

Since w is bounded above the second term in the supremum is bounded and the conclusion follows from the bound for w. □

Lemma 3.4. *For $i=1,2,3,4$*

$$
\left|v_i(t)\right|\leqslant C\quad\forall t\leqslant 0,
$$

and for $i=1,2,3$

$$
\left|w^{(i)}(t)\right|\leqslant C\quad\forall t\leqslant 0.
$$

Proof. Integrating the equation

$$\left(v_4(t)e^{(N-4)t}\right)' = Ae^{(N-4)t}v_1(t)$$

in $[t,t_0]$ with $t \leqslant t_0 \leqslant 0$ we find

$$v_4(t) = e^{-(N-4)t}\left(v_4(t_0)e^{(N-4)t_0} - A\int_t^{t_0} e^{(N-4)s}v_1(s)\,ds\right). \tag{26}$$

Since $v_1(t) = \frac{\lambda}{A}e^{w(t)}$ and w is bounded above we have that $v_1(t)$ is bounded as $t \to -\infty$. Hence the integral $\int_{-\infty}^{t_0} e^{(N-4)s}v_1(s)\,ds$ exists. If

$$A\int_{-\infty}^{t_0} e^{(N-4)s}v_1(s)\,ds \neq v_4(t_0)e^{(N-4)t_0}$$

we deduce from (26) that $|v_4(t)|$ grows exponentially as $t \to -\infty$, which contradicts (25). It follows that

$$v_4(t_0) = Ae^{-(N-4)t_0}\int_{-\infty}^{t_0} e^{(N-4)s}v_1(s)\,ds \quad \forall t_0 \leqslant 0. \tag{27}$$

Since v_1 is bounded we see from this formula that

$$\left|v_4(t)\right| \leqslant C \quad \text{for all } t \leqslant 0.$$

This in turn implies that v_3 is bounded as well. In fact from (11) we have

$$\left(v_3(t)e^{-2t}\right)' = e^{-2t}v_4(t)$$

and integrating on $[t,0]$, $t \leqslant 0$ yields

$$v_3(t) = e^{2t}v_3(0) - \int_t^0 e^{2(t-s)}v_4(s)\,ds. \tag{28}$$

Using that v_4 is bounded it follows that v_3 is bounded as well as $t \to -\infty$.

Let us prove that v_2 remains bounded as $t \to -\infty$. Arguing as for v_4 we integrate the equation

$$\left(v_2(t)e^{(N-2)t}\right)' = e^{(N-2)t}v_3(t)$$

in $[t,t_0]$ with $t \leqslant t_0 \leqslant 0$:

$$v_2(t) = e^{-(N-2)t}\left(v_2(t_0)e^{(N-2)t_0} - \int_t^{t_0} e^{(N-2)s}v_3(s)\,ds\right). \tag{29}$$

The integral $\int_t^{t_0} e^{(N-2)s} v_3(s)\,ds$ converges because v_3 is bounded. If

$$\int_{-\infty}^{t_0} e^{(N-4)s} v_3(s)\,ds \neq v_2(t_0) e^{(N-2)t_0}$$

we deduce from (29) that $|v_2(t)|$ grows exponentially as $t \to -\infty$, which contradicts (25). It follows that

$$v_2(t_0) = e^{-(N-2)t_0} \int_{-\infty}^{t_0} e^{(N-4)s} v_3(s)\,ds \quad \forall t_0 \leqslant 0. \tag{30}$$

Since v_3 is bounded we see from this formula that v_2 is also bounded.

The fact that $w^{(i)}$ are bounded as $t \to -\infty$ follows from the formulas

$$\begin{aligned} w' &= v_2 + 4, \qquad w'' = -(N-2)v_2 + v_3, \\ w''' &= (N-2)^2 v_2 - (N-4)v_3 + v_4, \\ w^{(4)} &= Av_1 + (N-2)^3 v_2 - \big((N-2)^2 + 2(N-4)\big)v_3 - 2(N-4)v_4. \qquad \square \end{aligned}$$

As in [1] we consider the energy

$$E(t) = \frac{1}{2} w''(t)^2 - \frac{1}{2}\big(N^2 - 10N + 20\big) w'(t)^2 + A\big(e^w - w\big).$$

A computation reveals that if $t_1 \leqslant t_2$ then

$$\begin{aligned} E(t_2) - E(t_1) = {} & w'w'''\big|_{t_1}^{t_2} + 2(N-4)w'w''\big|_{t_1}^{t_2} - 2(N-4)\int_{t_1}^{t_2} w''(s)^2\,ds \\ & - 2(N-2)\int_{t_1}^{t_2} w'(s)^2\,ds. \end{aligned} \tag{31}$$

Lemma 3.5. *If*

$$\liminf_{t \to -\infty} w(t) = -\infty$$

then $w(t) \to -\infty$, $v_i(t) \to 0$ as $t \to -\infty$ for $i = 1, 2, 3, 4$ and u is a regular solution.

Proof. We first show that $w(t) \to -\infty$ as $t \to -\infty$ by contradiction. Suppose that $w(t)$ does not approach $-\infty$. Then one can find sequences $t_k, s_k \to -\infty$ such that $s_k > t_k$,

$$\begin{gathered} w(t_k) \text{ remains bounded}, \qquad w'(t_k) = 0, \\ w(s_k) \to -\infty, \qquad w'(s_k) = 0. \end{gathered}$$

Then by (31) $E(s_k) \leqslant E(t_k)$. But $E(t_k)$ remains bounded while $E(s_k) \to \infty$, which is a contradiction.

Now that we know that $w(t) \to -\infty$ as $t \to -\infty$, we deduce immediately that $v_1(t) \to 0$ as $t \to -\infty$. Then using formulas (27), (28) and (30) we also obtain $v_i(t) \to 0$ as $t \to -\infty$ for $i = 2, 3, 4$. By Theorem 6 in [2] we deduce that u is a regular solution. □

Lemma 3.6. *If*

$$\liminf_{t \to -\infty} w(t) > -\infty$$

then $w(t) \to 0$, $(v_1(t), \dots, v_4(t)) \to P_2$ *as* $t \to -\infty$ *and* u *is a weakly singular solution.*

Proof. In this case, since w is also bounded above by Lemma 3.2, we have that w is bounded. By Lemma 3.4 the derivatives of w are bounded as well and we deduce that $E(t)$ remains bounded as $t \to -\infty$. The boundedness of E together with the boundedness of the derivatives of w and formula (31) imply that

$$\int_{-\infty}^{0} (w')^2 \, dt < +\infty, \qquad \int_{-\infty}^{0} (w'')^2 \, dt < +\infty. \tag{32}$$

Then we can select a strictly decreasing sequence $t_k \to -\infty$ such that

$$\lim_{k \to \infty} (t_k - t_{k+1}) = 0$$

and

$$w'(t_k) \to 0 \quad \text{as } k \to \infty.$$

If $t \leqslant s \leqslant 0$ we have by (32)

$$\left| w'(t) - w'(s) \right| \leqslant C |t - s|^{1/2}.$$

Hence for $t \in [t_k, t_{k+1}]$

$$\left| w'(t) \right| \leqslant \left| w'(t_k) \right| + C (t_{k+1} - t_k)^{1/2}.$$

This shows that $w'(t) \to 0$ as $t \to \infty$. Using then elliptic estimates we deduce

$$w^{(i)}(t) \to 0 \quad \text{as } t \to -\infty$$

for $i = 1, 2, 3, 4$. Using the equation we also deduce that $w(t) \to 0$ as $t \to -\infty$. We hence obtain that $(v_1(t), \dots, v_4(t)) \to P_2$ as $t \to -\infty$. Then by Theorem 6 in [2] we have u is a weakly singular solution. □

Proof of Theorem 1.1. It is a consequence of Lemmas 3.5 and 3.6. □

4. Heteroclinic connection from P_1 to P_2

Proposition 4.1. *For $N \geqslant 5$, system* (11) *has an heteroclinic orbit from P_1 to P_2.*

Proof. We use one of the main results in [1] on the initial value problem

$$\begin{gathered}\Delta^2 u = 8(N-2)(N-4)e^u, \quad r \in \big(0, R(\beta)\big),\\ u(0)=0, \qquad u'(0)=0, \qquad \Delta u(0)=\beta, \qquad (\Delta u)'(0)=0\end{gathered} \tag{33}$$

where $\beta \in \mathbb{R}$ is a parameter and $R(\beta) > 0$ is the maximal time of existence of the solution. Here the constant in front of e^u is taken, without loss of generality, to be $8(N-2)(N-4)$. In Theorem 2 of [1] the authors show that there exists a unique β such that $R(\beta) = +\infty$ and

$$\lim_{r\to\infty} u(r) + 4\log(r) = 0. \tag{34}$$

In what follows we fix β is such that (34) holds. Let $v(t) = u(r)$ where $r = e^t$ and $V = (v_1, \dots, v_4)$ be defined by (10). Then, since u is smooth at the origin

$$\lim_{t\to-\infty} V(t) = P_1$$

and (34) tells us that

$$\lim_{t\to\infty} v_1(t) = 1.$$

It remains only to show that

$$\lim_{t\to\infty} V(t) = P_2. \tag{35}$$

Set $w(t) = v(t) + 4t$. Then w satisfies the following equation

$$(\partial_t + N - 4)(\partial_t - 2)(\partial_t + N - 2)\partial_t w = 8(N-2)(N-4)\big(e^w - 1\big) \quad \text{in } \mathbb{R}.$$

Note that (34) is equivalent to $\lim_{t\to\infty} w(t) = 0$. To prove (35) it suffices to show that

$$\lim_{t\to\infty} w^{(i)}(t) = 0 \quad \text{for } i = 1, 2, 3. \tag{36}$$

This follows from the lemma below. $\square$

Lemma 4.2. *Assume that $z : [T_0, \infty) \to \mathbb{R}$ exists for some T_0 and solves*

$$z^{(4)}(t) + K_3 z'''(t) + K_2 z''(t) + K_1 z'(t) = f\big(z(t)\big) \quad \forall t > T_0$$

where $f \in C^1(\mathbb{R})$ and $K_i \in \mathbb{R}$. Let z_0 be such that $f(z_0) = 0$ and assume that $\lim_{t\to\infty} z(t) = z_0$. Then for $k = 1, \dots, 4$

$$\lim_{t\to\infty} z^{(k)}(t) = 0.$$

For the proof see [18, Proposition 1].

The next lemma shows that a connection from P_1 to P_2 necessarily reaches P_2 in an oscillatory way if $5 \leqslant N \leqslant 12$, but the statement below holds for all $N \geqslant 5$.

Lemma 4.3. *Let $N \geqslant 5$. Assume $V : (T, \infty) \to \mathbb{R}^4$, $V = (v_1, v_2, v_4, v_4)$ is a solution to* (11) *such that $\lim_{\to\infty} V(t) = P_2$ and either*

$$\frac{V'(t)}{|V'(t)|} + \frac{v^{(2)}}{|v^{(2)}|} \to 0 \quad \text{as } t \to +\infty \tag{37}$$

or

$$\frac{V'(t)}{|V'(t)|} - \frac{v^{(2)}}{|v^{(2)}|} \to 0 \quad \text{as } t \to +\infty. \tag{38}$$

Then V cannot be extended to a connection from P_1 to P_2.

Proof. Assume first that (37) holds. Then by (18) we have

$$v_1'(t) < 0, \qquad v_2'(t) > 0, \qquad v_3'(t) < 0, \qquad v_4'(t) > 0$$

for all t near $+\infty$ and

$$v_1(t) > 1, \qquad v_2(t) < -4, \qquad v_3(t) > -4(N-2), \qquad v_4(t) < 8(N-2) \tag{39}$$

for all t near $+\infty$. We claim that

$$v_2(t) < -4 \quad \forall t > T. \tag{40}$$

Assume by contradiction that this fails. Then from (39) we can define $t_1 > T$ to be the last time such that $v_2(t_1) = -4$. Then $v_2'(t_1) \leqslant 0$. Using Eqs. (11) we deduce that

$$v_3(t_1) \leqslant -4(N-2).$$

Then thanks to (39) we can define $t_2 \geqslant t_1$ to be the last time such that $v_3(t_2) = -4(N-2)$. This implies that $v_3'(t_2) \geqslant 0$ and by the system (11)

$$v_4(t_2) \geqslant -8(N-4).$$

Let $t_3 \geqslant t_2$ be the last time such that $v_4(t_3) = -8(N-4)$. Then $v_4'(t_3) \leqslant 0$. We deduce from (11) that

$$v_1(t_3) \leqslant 1.$$

Let $t_4 \geqslant t_3$ be the last time such that $v_1(t_4) = 1$. Then $v_1'(t_4) \geqslant 0$ and by (11)

$$v_2(t_4) \geqslant -4.$$

But $v_2(t) < -4$ for all $t \in (t_1, \infty)$, which is a contradiction. This proves the claim (40) and shows that the trajectory defined by V cannot come from P_1.

Assume now that (38) holds. We claim that in this case

$$v_3(t) < -4(N-2) \quad \text{for all } t > T. \tag{41}$$

The proof is similar as before. Note that under the assumption (38) we have the opposite inequalities in (39). If the statement (41) fails we can define the last time t_1 such that $v_3(t_1) = -4(N-2)$. Then define successively $t_2 \geqslant t_1$ such that $v_4(t_2) = 8(N-2)$, $v_4'(t_2) \geqslant 0$, $t_3 \geqslant t_2$ such that $v_1(t_3) = 1$, $v_1'(t_3) \leqslant 0$, $t_4 \geqslant t_3$ such that $v_2(t_4) = -4$ and $v_2'(t_4) \geqslant 0$, which leads to $v_3(t_4) \geqslant -4(N-2)$ which yields a contradiction. This shows that the trajectory cannot come from P_1. $\square$

5. The unstable manifold at P_2

In this section we study $W^u(P_2)$ and as a consequence we obtain Theorems 1.2 and 1.6. Let $v^{(j)}$ denote the eigenvectors of the linearization of (11) at P_2 with corresponding eigenvalue ν_j. Then $W^u(P_2)$ is 1-dimensional and tangent to $v^{(1)}$ at P_2. Hence, if $V = (v_1, \dots, v_4) : (-\infty, T) \to \mathbb{R}^4$ is any trajectory in $W^u(P_2)$ there are 2 cases:

$$\begin{aligned} &\left\langle V'(t), v^{(1)}\right\rangle < 0 \quad \text{for } t \text{ near } -\infty, \\ &\left\langle V'(t), v^{(1)}\right\rangle > 0 \quad \text{for } t \text{ near } -\infty. \end{aligned}$$

The main results in this section are

Proposition 5.1. *Suppose that $V = (v_1, \dots, v_4) : (-\infty, T) \to \mathbb{R}^4$ is the trajectory in $W^u(P_2)$ such that $\langle V'(t), v^{(1)}\rangle < 0$ for t near $-\infty$. Then*

(a) *$v_2(t) < -4$ for all $t \in (-\infty, T)$, and*
(b) *$v_3(t) < -4(N-2)$ for all $t \in (-\infty, T)$.*

Proposition 5.2. *Let $V = (v_1, \dots, v_4) : (-\infty, T) \to \mathbb{R}^4$ be the trajectory in $W^u(P_2)$ such that $\langle V'(t), v^{(1)}\rangle > 0$ for t near $-\infty$, where T is the maximal time of existence. Then*

(a) *$v_1(t) > 1$ for all $t < T$.*
(b) *There exists a unique t_0 such that $v_2(t_0) = 0$. Moreover $v_2'(t) > 0$ for all $t < T$. In particular the trajectory of V intersects the hyperplane $\{v_2 = 0\}$ transversally.*
(c) *There exists a unique t_1 such that $v_3(t_1) = 0$. Moreover $v_3'(t) > 0$ for all $t < T$. In particular the trajectory of V intersects the hyperplane $\{v_3 = 0\}$ transversally.*

Proof of Proposition 5.1. (a) The relations (17) and the hypothesis $\langle V'(t), v^{(1)}\rangle < 0$ for $t \to -\infty$ imply that for t near $-\infty$

$$\begin{cases} v_1(t) < 1, & v_2(t) < -4, \\ v_3(t) < -4(N-2), & v_4(t) < 8(N-2). \end{cases} \tag{42}$$

Assume by contradiction that $v_2(t) \geqslant -4$ for some $t < T$. Thus we may define $t_0 < T$ the first time such that $v_2(t) = -4$. Then $v_2'(t_0) \geqslant 0$. Then by (11) $0 \leqslant v_2'(t_0) = v_3(t_0) + 4(N-2)$, that is,

$$v_3(t_0) \geqslant -4(N-2).$$

By (42) we can define $t_1 \leqslant t_0$ as the first time such that $v_3(t) = -4(N-2)$. Then $v_3'(t_1) \geqslant 0$ and (11) implies

$$v_4(t_1) \geqslant 8(N-2). \tag{43}$$

Again using (42), let $t_2 \leqslant t_1$ be the first time that $v_4(t) = 8(N-2)$. Then $v_4'(t_2) \geqslant 0$ and by (11)

$$v_1(t_2) \geqslant 1.$$

Thanks to (42) we must have a first time $t_3 \leqslant t_2$ such that $v_1(t) = 1$. But then $v_1'(t_3) \geqslant 0$ which by (11) implies

$$v_2(t_3) + 4 \geqslant 0.$$

Thus $v_2(t_3) \geqslant -4$. This cannot happen if $t_3 < t_0$ because $v_2(t) < -4$ for all $t < t_0$. If $t_3 = t_2 = t_1 = t_0$ then $v_1'(t_0) = v_2'(t_0) = v_3'(t_0) = v_4'(t_0)$, which means $V \equiv P_2$, a contradiction. This proves that $v_2(t) < -4$ for all $t < T$.

(b) Let us show now that $v_3(t) < -4(N-2)$ for all $t < T$. If not, we can define $t_1 < T$ as the first time such that $v_3(t) = -4(N-2)$. Then $v_3'(t_1) \geqslant 0$ and we may repeat the same argument starting at (43) to find $t_3 \leqslant t_1$ such that $v_2(t_3) \geqslant -4$. This is impossible and proves the result. $\square$

Proof of Proposition 5.2. By (17) and the hypothesis $\langle V'(t), v^{(1)} \rangle > 0$ for $t \to -\infty$ we have

$$v_1'(t) > 0, \qquad v_2'(t) > 0, \qquad v_3'(t) > 0, \qquad v_4'(t) > 0 \tag{44}$$

for t near $-\infty$.

Let us prove first that

$$v_1(t) > 0 \quad \forall t < T. \tag{45}$$

This is valid for t near $-\infty$ by (44). If $v_1(t) = 0$ for some t then v_1 would be constant by the equation, which is not possible.

Before proving (b) and (c) we will claim that (44) is valid for all $t < T$.

First we establish that

$$v_3'(t) > 0 \quad \forall t < T. \tag{46}$$

To prove (46) suppose it fails. Let $s_0 < T$ be the first time such that $v_3'(s_0) = 0$. Using (11) we see that

$$0 = v_3'(s_0) = 2v_3(s_0) + v_4(s_0).$$

But $v_3(s_0) > -4(N-2)$ and we deduce $v_4(s_0) < 8(N-2)$. Let $s_1 \leqslant s_0$ be the first time such that $v_4(t) = 8(N-2)$. Then $v_4'(s_1) \leqslant 0$ and hence

$$v_1(s_1) \leqslant 1.$$

Let $s_2 \leqslant s_1$ be the first time such that $v_1(s_2) = 1$. Then $v_1'(s_2) \leqslant 0$ and we conclude

$$v_2(s_2) \leqslant -4. \tag{47}$$

Let $s_3 \leqslant s_2$ be the first time such that $v_2(s_3) = -4$. Then $v_2'(s_3) \leqslant 0$ and we conclude

$$v_3(s_2) \leqslant -4(N-2). \tag{48}$$

Now since $s_2 < s_0$, we have $v_3(s_2) > -4(N-2)$, a contradiction. This establishes our claim (46).

Since (46) holds we have then $v_3(t) > -4(N-2)$ for all $t < T$. From the second equation in (11), we have

$$v_2'' = -(N-2)v_2' + v_3'.$$

We claim that $v_2' > 0$. By contradiction, if s_0 is the first time such that $v_2'(s_0) = 0$ then using (46), we have that $v_2''(s_0) > 0$ so v_2 has a local minimum at s_0 which is not possible, since v_2 is increasing near $t = -\infty$. We conclude that

$$v_2'(t) > 0 \quad \forall t < T. \tag{49}$$

Similarly differentiating the first equation in (11), and using (45), and (49), we obtain that

$$v_1'(t) > 0 \quad \forall t < T \tag{50}$$

and again using now the fourth equation in (11), and (50), we have

$$v_4'(t) > 0 \quad \forall t < T, \tag{51}$$

this proves that (44) is valid for all $-\infty < t < T$.

Now since $v_1'(t) > 0$ for all $t < T$ and $\lim_{t\to-\infty} v_1(t) = 1$, part (a) of the proposition follows.

Let us prove now that

$$\sup_{t<T} v_i(t) = +\infty, \quad \text{for all } i = 1 \ldots 4.$$

First we prove the statement for v_1. If we assume the contrary, i.e. that v_1 remains bounded, then (11) implies the estimate

$$\bigl|(v_1, \ldots, v_4)'(t)\bigr| \leqslant C\bigl|(v_1, \ldots, v_4)(t)\bigr| \quad \forall t < T,$$

for some $C > 0$ and from Gronwall's inequality we deduce that the solution is defined for all times, that is $T = +\infty$. Since v_1 is increasing, $v_1 \to L < +\infty$ as $t \to +\infty$ and $v_1'(t_k) \to 0$ along some sequence $t_k \to +\infty$. But v_1, v_2 are increasing and $v_2(t) > -4$, $v_1(t) > 1$ for all $t \in \mathbb{R}$. Then from the equation $v_1' = v_1(v_2 + 4)$ we obtain a contradiction. This proves that

$$v_1(t) \to \infty \quad \text{as } t \to T. \tag{52}$$

We prove similarly that $v_4(t) \to +\infty$ as $t \to T$. Arguing by contradiction we have $v_4 \to L < \infty$ as $t \to T$. If $T = +\infty$ the argument is the same as before: for some sequence $t_k \to +\infty$, $v_4'(t_k) \to 0$. Using the equation for v_4' we have a contradiction. If $T < +\infty$, the assumption that v_4 is bounded and the system (11) imply that v_3, v_2 and v_1 are bounded up to T, which is not possible by (52). Thus we have proved that

$$v_4(t) \to \infty \quad \text{as } t \to T. \tag{53}$$

Applying the same argument, now using (53) and the equation for v_3', we obtain

$$v_3(t) \to \infty \quad \text{as } t \to T. \tag{54}$$

For v_2, we use the same procedure now with the equation for v_2' and (54), and deduce that

$$v_2(t) \to \infty \quad \text{as } t \to T. \tag{55}$$

Finally the property (b) clearly follows from (55) and $v_2'(t) > 0$ for all $t < T$. Similarly, (c) is a consequence of (54) and that $v_3'(t) > 0$ for all $t < T$. $\square$

Proof of Theorems 1.2 and 1.6. Any weakly singular radial solution gives rise, through the changes of variable $v(t) = u(e^t)$, $t \leqslant 0$, and (10), to a solution $V : (-\infty, 0] \to \mathbb{R}^4$ of the system (11) such that the final conditions (12) hold. Since the solution is weakly singular, $\lim_{t\to-\infty} V(t) = P_2$. Hence $V((-\infty, 0])$ is contained in $W^u(P_2)$ and therefore there are 2 possibilities: either $\langle V'(t), v^{(1)}\rangle < 0$ for t near $-\infty$ or $\langle V'(t), v^{(1)}\rangle > 0$ for t near $-\infty$. The first case is not possible, because Proposition 5.1 shows that V cannot satisfy the end condition $v_2(0) = 0$. Thus we are in the second case and we can apply Proposition 5.2(b). Therefore there exists a unique $t_0 > -\infty$ such that $v_2(t_0) = 0$ since the system is autonomous by shifting time we can assume that $v_2(0) = 0$. This concludes the proof of Theorem 1.2. The proof of Theorem 1.6 is similar, we look at the component v_3 instead of v_2, since need to show that $v_3(0) = \Delta u(1) = 0$. Consequently, to conclude the proof we use Proposition 5.2(c). $\square$

6. Multiplicity results: Proofs of Theorems 1.3 and 1.7

By Propositions 5.1 and 5.2 we know that $W^u(P_2) \cap \{v_2 = 0\}$ is a single point, which we call $P^* = (P_1^*, P_2^*, P_3^*, P_4^*)$, with $P_1^* = \frac{\lambda_S}{8(N-2)(N-4)}$ and $P_2^* = 0$.

Let $\mathcal{E} = W^u(P_1) \cap \{v_2 = 0\}$. Each regular radial solution of (1) corresponds to exactly one point $v = (v_1, \dots, v_4) \in \mathcal{E}$ with $v_1 > 0$.

Throughout this section we assume that $5 \leqslant N \leqslant 12$. Let P_1, P_2 be the stationary points of the system (11) defined in (13). Then P_1 has a 2-dimensional unstable manifold $W^u(P_1)$ while P_2 has a 1-dimensional unstable manifold $W^u(P_2)$ and a 3-dimensional stable manifold $W^s(P_2)$.

Let $V_0 : \mathbb{R} \to \mathbb{R}^4$ be the heteroclinic connection from P_1 to P_2 of Proposition 4.1 and $\hat{V}_0 = V_0(-\infty, \infty)$. Then $\hat{V}_0$ is contained in both $W^u(P_1)$ and $W^s(P_2)$.

Lemma 6.1. *$W^u(P_1)$ and $W^s(P_2)$ intersect transversally on points of $\hat{V}_0$. More precisely for points $Q \in \hat{V}_0$ sufficiently close to P_2 there are directions in the tangent plane to $W^u(P_1)$ which are almost parallel to $v^{(1)}$, the tangent vector to $W^u(P_2)$ at P_2.*

Proof. Let $u(r, \beta)$ the solution to (33) defined in the maximal interval $[0, R(\beta))$. Let β_0 denote the unique value of β such that $R(\beta_0) = \infty$ and

$$\lim_{r\to\infty} u(r, \beta_0) + 4\log(r) \quad \text{exists,}$$

see [1]. From the proof of Lemma 8 of this reference it follows that for $\beta < \beta_0$ the following estimate holds:

$$\frac{\partial u}{\partial r}(r, \beta) \leqslant \frac{\partial u}{\partial r}(r, \beta_0) - \frac{\beta_0 - \beta}{N} r \quad \forall r \geqslant 0.$$

Then $\frac{\partial u}{\partial \beta}(r, \beta_0)$ satisfies the linearized equation at $u(\cdot, \beta_0)$ and

$$\frac{\partial}{\partial r}\frac{\partial u}{\partial \beta}(r, \beta_0) \geqslant \frac{r}{N} \quad \forall r \geqslant 0. \tag{56}$$

Let $v(t) = u(e^t, \beta_0)$, $t \in \mathbb{R}$ and $V = (v_1, \dots, v_4)$ be defined by (10). Define $Z = \frac{\partial V}{\partial \beta}$. Then Z satisfies

$$Z' = \big(M + R(t)\big)Z$$

where M is the matrix defined in (14) and

$$R(t) = \begin{bmatrix} v_2+4 & v_1-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Recall that $V(t) \to P_2$ as $t \to \infty$. Moreover the convergence is exponential, that is there are $C, \sigma > 0$ such that $|V(t) - P_2| \leqslant Ce^{-\sigma t}$ for all $t \geqslant 0$. This follows from Lemma 2.2 which shows that the system (11) is C^1-conjugate to its linearization near P_2 (it suffices here to show that (11) is C^0-conjugate to its linearization near P_2, which follows from the Hartman–Grobman theorem, see Theorem 7.1 in [31] or Theorem 1.1.3 in [28]). Recall that the eigenvalues of M are $\nu_1 > 0 > \nu_2$ and ν_3, ν_4 which have negative real part and nonzero imaginary part. Let $v^{(i)} \in \mathbb{C}^4$ denote an eigenvector associated to ν_i. By Theorem 8.1 in [13, Chapter 3] there are solutions φ_k to

$$\varphi_k' = \big(M + R(t)\big)\varphi_k, \quad t > 0$$

such that $\lim_{t\to\infty} \varphi_k(t)e^{-\nu_k t} = v^{(k)}$. Then

$$Z = \sum_{i=1}^{4} c_i \varphi_i \tag{57}$$

for some constants $c_1, \dots, c_4 \in \mathbb{C}$. The condition (56) and the definitions in (10) imply that for some $c > 0$

$$\left|\frac{\partial v_2}{\partial \beta}(t, \beta_0)\right| \geqslant ce^{2t} \quad \text{for all } t \geqslant 0. \tag{58}$$

If $c_1 = 0$ in (57), since ν_2, ν_3, ν_4 have negative real, we would obtain that $Z(t) \to 0$ as $t \to \infty$, contradicting (58). Hence $c_1 \neq 0$ and therefore

$$Z = c_1 v^{(1)} e^{\nu_1 t} + o\big(e^{\nu_1 t}\big) \quad \text{as } t \to \infty.$$

Since $v^{(1)}$ is the tangent vector to $W^u(P_2)$, we have that $\frac{\partial V}{\partial \beta}$ is not tangent to $W^s(P_2)$ for t large. On the other hand $\frac{\partial V}{\partial \beta}$ is tangent to $W^u(P_1)$ by construction. This shows that $W^s(P_2)$ and $W^u(P_1)$ intersect transversally on points of $\hat{V}_0$ close to P_2. By the invertibility of the flow away from the stationary points, $W^s(P_2)$ and $W^u(P_1)$ intersect transversally on all points of $\hat{V}_0$. $\square$

Proof of Theorem 1.3. We will write generic points in the phase space $\mathbb{R}^4$ as (v_1, v_2, v_3, v_4). Let $\{e_j\colon\ j = 1, \dots, 4\}$ denote the canonical basis of $\mathbb{R}^4$.

The multiplicity results asserted in Theorem 1.3 are consequences of the following claims:

(a) $\mathcal{E}$ contains a spiral $\mathcal{S}$ about the point P^*,
(b) $\mathcal{S}$ is contained in a 2-dimensional C^1 surface $\Sigma \subseteq \{v_2 = 0\}$, and
(c) the plane through P^* parallel to e_2, e_3, e_4 is transversal to the tangent plane to Σ at P^*.

More precisely, after a C^1 diffeomorphism of a neighborhood of P^* to a neighborhood of the origin in $\mathbb{R}^4$, which maps P^* to the origin, the curve $\mathcal{S}$ can be parametrized by a C^1 function of the form $(r(s)\cos(s), r(s)\sin(s), 0, 0)$, $s \in [0, \infty)$, such that $r(s) > 0$ for all $s \geqslant 0$ and $r(s) \to 0$ as $s \to \infty$. Moreover one can choose this diffeomorphism such that Σ corresponds to part of the surface $\{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4\colon\ x_3 = x_4 = 0\}$.

Assume (a), (b) and (c) have been proved and define hyperplane $H_\lambda = \{v_1 = \frac{\lambda}{8(N-2)(N-4)}\}$ where $\lambda > 0$. If $\lambda = \lambda_S$, the transversality condition (c) ensures that H_λ is transversal to Σ, and we will see that this implies that $H_\lambda \cap \mathcal{E}$ contains infinitely many points, which means that (1) has infinitely many radial regular solutions. Indeed, after the C^1 diffeomorphism described above we can assume that $\mathcal{S} = \{(r(s)\cos(s), r(s)\sin(s), 0, 0)\colon\ s \geqslant 0\}$ and $\Sigma = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4\colon\ x_3 = x_4 = 0\}$. The hyperplane H_λ is transformed into a C^1 hypersurface containing the origin, which is transversal to Σ. Then $H_\lambda \cap \Sigma$ is a C^1 curve through the origin. Using polar coordinates we then see that H_λ intersects the spiral $\mathcal{S}$ infinitely many times. If $\lambda \neq \lambda_S$ but λ is close to λ_S, by the transversality (c) we have that $H_\lambda \cap \mathcal{E}$ contains a large number of points, which yields a large number of radial regular solutions of (1).

In what follows we will prove (a), (b) and (c). Let X_t denote the flow generated by (11), that is, $X_t(\xi)$ is the solution to (11) at time t with initial condition $X_0(\xi) = \xi \in \mathbb{R}^4$. For fixed ξ, $X_t(\xi)$ is defined for t in a maximal open interval containing 0.

Let D be the 3-dimensional disk $D = \{v = (v_1, \ldots, v_4)\colon\ v_2 = 0,\ |v - P^*| < 1\}$, which by Proposition 5.2 is transversal to $W^u(P_2)$. Let $B^s \subseteq W^s(P_2) \cap N_{P_2}$ be an open neighborhood of P_2 relative to $W^s(P_2)$ diffeomorphic to a 3-dimensional disk. By choosing smaller neighborhoods if necessary, we may apply the λ-lemma of Palis [38]. Let D_t be the connected component of $X_t(D) \cap N_{P_2}$ that contains $X_t(P^*)$. Then, given $\varepsilon > 0$ there exists some $t_0 < 0$, $|t_0|$ large, such that D_{t_0} contains a 3-dimensional C^1 manifold $\mathcal{M}$ that is a ε C^1-close to B^s, which means that there is a diffeomorphism $\eta : \mathcal{M} \to B^s$ such that $\|i - \eta\|_{C^1(\mathcal{M})} \leqslant \varepsilon$ where $i : \mathcal{M} \to \mathbb{R}^4$ is the inclusion map.

Choose some point $Q \in \hat{V}_0$ such that $Q \in N_{P_2}$. By Lemma 6.1 we may choose a C^1 curve contained in $W^u(P_1)$, say $\Gamma = \{\gamma(s)\colon\ |s| < \delta\}$ with $\gamma : (-\delta, \delta) \to \mathbb{R}^4$ a C^1 function with $\gamma(0) = Q$, $\gamma'(0)$ not tangent to $W^s(P_2)$ at Q. We can assume also that this curve is contained in N_{P_2}. Choosing ε small we can assume that Γ intersects $\mathcal{M}$.

We have the following properties, which we prove after we complete the proof of Theorem 1.3.

Lemma 6.2. *For large t, $X_t(\Gamma) \cap \mathcal{M}$ is a single point that we call P_t and the following properties hold*:

(1) *The collection of the points P_t for large t forms a spiral.*
(2) *There exists a 2-dimensional C^1 manifold $\tilde{\Sigma}$ that contains P_t for all t large.*
(3) *Let Q_{t_0} be the intersection of $\mathcal{M}$ with $W^u(P_2)$. Then the tangent plane to $\tilde{\Sigma}$ at Q_{t_0} becomes parallel to the one generated by $v^{(3)}$, $v^{(4)}$ (the eigenvectors corresponding to ν_3, ν_4) as $\varepsilon \to 0$.*
(4) *Moreover, for $s > 0$ suitably small the time t such that $X_t(\gamma(s)) \in \mathcal{M}$ satisfies*

$$s = ce^{-\nu_1 t} + o\big(e^{-\nu_1 t}\big) \tag{59}$$

where $c > 0$.

Let $\tilde{\mathcal{S}}$ denote the collection $\{P_t\colon\ t \geqslant t_1\}$ where t_1 is suitably large. Define $\mathcal{S} = X_{-t_0}(\tilde{\mathcal{S}})$ and $\Sigma = X_{-t_0}(\tilde{\Sigma})$. Since X_{-t_0} is a smooth diffeomorphism from M to a neighborhood of P^* inside the hyperplane $\{v_2 = 0\}$ we see that $\mathcal{S}$ is a spiral contained in a C^1 surface Σ. The points of $\mathcal{S}$ belong to $W^u(P_1)$ because they were obtained though the flow from points in $X_t(\Gamma)$.

This ends the proof of part (b).

We now prove statement (c). It is sufficient to show that inside the space $\{v_2 = 0\}$ the plane generated by e_3, e_4 is transversal to the tangent space to Σ at P^*. Let $V = (v_1, \ldots, v_4) : (-\infty, 0] \to \mathbb{R}^4$ denote the trajectory corresponding to the weakly singular solution, that is, $\lim_{t\to-\infty} V(t) = P_2$, $v_2(0) = 0$. To prove our claim we need to transport the plane generated by e_3 and e_4 back along V and this is accomplished by solving the linearized equation around V. More precisely, let $Z, \tilde{Z} : (-\infty, 0] \to \mathbb{R}^4$ be solutions to the linearization of (11) around V, that is, $Z = (z_1, z_2, z_3, z_4)$ satisfies for $t < 0$

$$\begin{cases} z_1' = z_1(v_2 + 4) + v_1 z_2, \\ z_2' = -(N-2)z_2 + z_3, \\ z_3' = 2z_3 + z_4, \\ z_4' = -(N-4)z_4 + 8(N-2)(N-4)z_1 \end{cases} \tag{60}$$

and similarly for $\tilde{Z} = (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4)$. As final conditions we take $Z(0) = e_3$, $\tilde{Z}(0) = e_4$.

By Theorem 8.1 in [13, Chapter 3] there are solutions $\varphi_k : (-\infty, 0] \to \mathbb{C}^4$ to (60) such that

$$\lim_{t \to -\infty} \varphi_k(t) e^{-\nu_k t} = v^{(k)} \tag{61}$$

where $v^{(1)}, \ldots, v^{(4)}$ are the eigenvectors of M. Recall that $v^{(1)}, v^{(2)}$ are real, and $v^{(3)}, v^{(4)}$ are complex conjugate. Thus one can assume that φ_1, φ_2 are real, and φ_3, φ_4 are complex conjugate. Then

$$Z(t) = \sum_{i=1}^{4} c_i \varphi_i(t), \quad \text{and} \quad \tilde{Z}(t) = \sum_{i=1}^{4} \tilde{c}_i \varphi_i(t)$$

for some constants $c_1, \ldots, c_4, \tilde{c}_1, \ldots, \tilde{c}_4 \in \mathbb{C}$. We note that $c_1, c_2, \tilde{c}_1, \tilde{c}_2$ are real and $c_3\varphi_3(t) + c_4\varphi_4(t) \in \mathbb{R}$, $\tilde{c}_3\varphi_3(t) + \tilde{c}_4\varphi_4(t) \in \mathbb{R}$ for all $t \leqslant 0$.

We claim that

$$c_2 \neq 0 \quad \text{or} \quad \tilde{c}_2 \neq 0. \tag{62}$$

Assume, by contradiction, that $c_2 = 0$ and $\tilde{c}_2 = 0$. Define

$$f(t) = e^{(N-4)t} \left(\frac{z_4(t)\tilde{z}_1(t)}{v_1(t)} - z_3(t)\tilde{z}_2(t) + z_2(t)\tilde{z}_3(t) - \frac{z_1(t)\tilde{z}_4(t)}{v_1(t)} \right) \quad \forall t \leqslant 0.$$

A calculation using (60) shows that f is constant. Using the final conditions for Z and $\tilde{Z}$ we see that $f(0) = 0$ and hence

$$f(t) = 0 \quad \forall t \leqslant 0.$$

Using (61), (16) and the assumption $c_2 = 0$, $\tilde{c}_2 = 0$ we can compute

$$\lim_{t \to -\infty} f(t) = (c_3\tilde{c}_4 - \tilde{c}_3 c_4) B$$

where

$$\begin{aligned} B &= \nu_3(\nu_3 + N - 2)(\nu_3 - 2) - \nu_3(\nu_3 + N - 2)\nu_4 + \nu_4(\nu_4 + N - 2)\nu_3 \\ &\quad - \nu_4(\nu_4 + N - 2)(\nu_4 - 2) \\ &= -\frac{1}{2} M_2(N) \sqrt{M_1(N) - M_2(N)}. \end{aligned}$$

Thus $B \in i\mathbb{R}$, $B \neq 0$ and we conclude that $(c_3\tilde{c}_4 - \tilde{c}_3 c_4) = 0$. This means that there exists a $\lambda \in \mathbb{C}$ such that $\tilde{c}_k = \lambda c_k$, $k = 3, 4$. Since $c_3\varphi_3(t) + c_4\varphi_4(t) \in \mathbb{R}$, $\tilde{c}_3\varphi_3(t) + \tilde{c}_4\varphi_4(t) \in \mathbb{R}$ for all $t \leqslant 0$, $\nu_1 > 0$ and we assume that $c_2 = \tilde{c}_2 = 0$, we must have $\lambda \in \mathbb{R}$. Using $Z(0) = e_3$ and $\tilde{Z}(0) = e_4$ we see that

$$(\tilde{c}_1 - \lambda c_1)\varphi_1(0) = e_4 - \lambda e_3.$$

But $\varphi_1 = cV'$, for some constant $c \in \mathbb{R}$, since both solve (60) and both tend to 0 as $t \to -\infty$. We know that $v_2'(0) > 0$ by Proposition 5.2 and this implies $\tilde{c}_1 - \lambda c_1 = 0$, a contradiction.

Finally, the condition (62) implies the assertion (c). Indeed, let us recall that $\Sigma = X_{-t_0}(\tilde{\Sigma})$ where $\tilde{\Sigma}$ is defined in Lemma 6.2 and $t_0 < 0$, with $|t_0|$ large. Using property 3 of that lemma and the condition (62) we see that for $|t_0|$ large at least one of the vectors $Z(t_0)$ or $\tilde{Z}(t_0)$ is transversal to the tangent plane to $\tilde{\Sigma}$ at Q_{t_0}.

To finish the proof of Theorem 1.3 we still need to verify one assertion: for $\lambda \neq \lambda_S$ (1) has a finite number of solutions. We will do this in Proposition 7.6 of Section 7. □

Proof of Lemma 6.2. By Lemma 2.2 there is a C^1 diffeomorphism $R : N_{P_2} \to N_0$ from an open neighborhood N_{P_2} of P_2 to an open neighborhood N_0 of 0 with $R(P_2) = 0$, $\det(R'(P_2)) > 0$, such that $RX_tR^{-1} = L_t$ where L_t is the flow generated by M, and the formula holds in some neighborhood of the origin. Note that $L_t = e^{Mt}$.

Thanks to the conjugation R, to prove the lemma we may assume that P_2 is at the origin and that near the origin the flow is given by $L_t = e^{Mt}$. Thus $W^s(P_2)$ in a neighborhood of the origin is $\{(y_1, \dots, y_4)\colon\ y_1 = 0\}$ and $B^s = \{(y_1, \dots, y_4)\colon\ y_1 = 0,\ |y| < \delta\}$ for some $\delta > 0$. We can also assume that the heteroclinic orbit V_0 near the origin in the new variables is given by

$$V_0(t) = \bigl(0, c_2 e^{\nu_2 t}, c_3 \operatorname{Re}\bigl(e^{\nu_3 t}\bigr), c_4 \operatorname{Im}\bigl(e^{\nu_3 t}\bigr)\bigr), \quad t \geqslant 0 \tag{63}$$

for some constants c_2, c_3, c_4. By Lemma 4.3 the curve V_0 cannot have a direction that becomes parallel to $e_2 = (0, 1, 0, 0)$ as $t \to \infty$. Since $|\nu_2| > |\operatorname{Re}(\nu_3)|$ by (15), $c_3 \neq 0$ or $c_4 \neq 0$. By choosing ε small, we can assume that the normal vector to $\mathcal{M}$ near P^* is almost parallel to $e_1 = (1, 0, 0, 0)$ after the change of variables. Thus by passing to a subset of $\mathcal{M}$ we may assume that $\mathcal{M}$ is a C^1 graph over the variables (y_2, y_3, y_4), that is, there exists a C^1 function $\psi : \{y' = (y_2, y_3, y_4) \in \mathbb{R}^3,\ |y'| < \delta\} \to \mathbb{R}$ with $\psi(0) > 0$ such that

$$\mathcal{M} = \bigl\{\bigl(\psi(y'), y'\bigr)\colon\ y' \in \mathbb{R}^3,\ |y'| < \delta\bigr\}.$$

By Lemma 6.1 the tangent plane to $W^u(P_1)$ at points close to the origin contains vectors almost parallel to $e_1 = (1, 0, 0, 0)$ and hence $\gamma_1'(0) \neq 0$. Using the implicit function theorem we see that for large t the intersection of $\mathcal{M}$ and $L_t(\Gamma)$ occurs at points of the form

$$P_t = \bigl(\gamma_1(s) e^{\nu_1 t}, \gamma_2(s) e^{\nu_2 t}, \gamma_3(s) \operatorname{Re}\bigl(e^{\nu_3 t}\bigr), \gamma_4(s) \operatorname{Im}\bigl(e^{\nu_3 t}\bigr)\bigr)$$

where $s = ce^{-\nu_1 t} + o(e^{-\nu_1 t})$ as $t \to \infty$ for some $c > 0$. Since $c_3 \neq 0$ or $c_4 \neq 0$ in (63) we can define a surface

$$\tilde{\Sigma} = \bigl\{y = (y_1, y_2, y_3, y_4)\colon\ |y| < \delta,\ y_1 = \psi(y_2, y_3, y_4),\ y_2 = g(y_3, y_4)\bigr\} \tag{64}$$

that contains the points P_t, where g is smooth away from the origin and has the property

$$g(y_3, y_4) = O\bigl(\bigl|(y_3, y_4)\bigr|^{\beta}\bigr)$$

with $\beta = \nu_2 / \operatorname{Re}(\nu_3)$. Thanks to (15) we see that $\beta > 1$. Therefore g is C^1 and $\tilde{\Sigma}$ is a C^1 surface. □

Proof of Theorem 1.7. By Propositions 5.1 and 5.2 we know that $W^u(P_2) \cap \{v_3 = 0\}$ is a single point, which we call $\bar{P}^* = (\bar{P}_1^*, \bar{P}_2^*, \bar{P}_3^*, \bar{P}_4^*)$, with $\bar{P}_1^* = \frac{\lambda_S}{8(N-2)(N-4)}$ and $\bar{P}_3^* = 0$.

As in Theorem 1.3, the multiplicity results asserted in Theorem 1.7 are consequences of the following claims:

(a) $\mathcal{E} := W^u(P_1) \cap \{v_3 = 0\}$ contains a spiral $\mathcal{S}$ about the point $\bar{P}^*$,
(b) $\mathcal{S}$ is contained in a 2-dimensional C^1 surface $\Sigma \subseteq \{v_3 = 0\}$, and
(c) the plane through $\bar{P}^*$ parallel to e_2, e_3, e_4 is transversal to the tangent plane to Σ at $\bar{P}^*$.

The proofs are similar to the Dirichlet case, now changing $v_2 = 0$ for $v_3 = 0$. So to prove (c) it will be sufficient now to show that inside the space $\{v_3 = 0\}$ the plane generated by e_2, e_4 is transversal to the tangent space to Σ at $\bar{P}^*$. We define now Z satisfying (60) with the final condition $Z(0) = e_2$, and $\tilde{Z}$ remains unchanged. In the same form we claim that (62) holds. Indeed using the same argument as before with $Z(0) = e_2$ and $\tilde{Z}(0) = e_4$, we find

$$(\tilde{c}_1 - \lambda c_1)\varphi_1(0) = e_4 - \lambda e_2.$$

But we know by Proposition 5.2 that $v_3'(0) > 0$ and this implies $\tilde{c}_1 - \lambda c_1 = 0$, a contradiction. The rest of the proof is the same. $\square$

7. Structure of the solution set

In this section we study the properties of the solution set

$$\mathcal{C} = \big\{(\lambda, u) \in (0, \infty) \times C^4(\overline{B}):\ u \text{ is radial and solves (1)}\big\}.$$

We assume here that $N \geqslant 5$. We will see that all regular radial solutions u of (1) are characterized by $u(0)$ and that this value ranges from 0 to $+\infty$. To prove the first assertion we follow the strategy of Guo and Wei [30]. For this we recall a comparison result established by McKenna and Reichel [36, Lemma 3.2].

Lemma 7.1. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and increasing. Let $u, v \in C^4([0, R))$, $R > 0$ be such that*

$$\forall r \in [0, R) \quad \Delta^2 u(r) - f\big(u(r)\big) \geqslant \Delta^2 v(r) - f\big(v(r)\big),$$
$$u(0) \geqslant v(0), \qquad u'(0) \geqslant v'(0), \qquad \Delta u(0) \geqslant \Delta v(0), \qquad (\Delta u)'(0) \geqslant (\Delta v)'(0).$$

Then for all $r \in [0, R)$

$$u(r) \geqslant v(r), \qquad u'(r) \geqslant v'(r), \qquad \Delta u(r) \geqslant \Delta v(r), \qquad (\Delta u)'(r) \geqslant (\Delta v)'(r). \tag{65}$$

Moreover

(i) *the initial point 0 can be replaced by any initial point $\rho > 0$ if all four initial data are weakly ordered,*
(ii) *a strict inequality in one of the initial data at $\rho \geqslant 0$ or in the differential inequality on (ρ, R) implies a strict ordering of u, u', Δu, $(\Delta u)'$ and v, v', Δv, $(\Delta v)'$ in (65).*

Analogously to [30, Lemma 5.1] we have:

Lemma 7.2. *Suppose that u_1, u_2 are smooth radial solutions of (1) associated to parameters $\lambda_1 > 0$, $\lambda_2 > 0$ such that $u_1(0) = u_2(0)$. Then $\lambda_1 = \lambda_2$ and $u_1 \equiv u_2$.*

Proof. Suppose we have smooth radial solutions u_1, u_2 of (1) associated to parameters $\lambda_1 > \lambda_2$ such that $u_1(0) = u_2(0)$.

For $j = 1, 2$

$$v_j(r) = \frac{u(\lambda_j^{-1/4} r)}{u_1(0)} \quad \text{for } r \in \big[0, \lambda_j^{1/4}\big].$$

Then v_j satisfies

$$\Delta^2 v_j = f(v_j) \quad \text{for } r \in \left[0, \lambda_j^{1/4}\right]$$

where $f(t) = \frac{1}{u_1(0)} e^{u_1(0)t}$.

Assume that $\Delta v_1(0) < \Delta v_2(0)$. Then by Lemma 7.1 $v_1(r) < v_2(r)$ for all $r \in [0, \lambda_2^{1/4}]$. In particular $v_1(\lambda_2^{1/4}) < v_2(\lambda_2^{1/4}) = 0$ which is impossible because $v_1(r) > 0$ for all $r \in [0, \lambda_1^{1/4})$.

Assume now that $\Delta v_1(0) > \Delta v_2(0)$. Then by Lemma 7.1 $v_1(r) > v_2(r)$, $v_1'(r) > v_2'(r)$, $\Delta v_1(r) > \Delta v_2'(r)$, $(\Delta v_1)'(r) > (\Delta v_2)'(r)$ for all $r \in [0, \lambda_2^{1/4}]$. Since v_1 is defined up to $\lambda_1^{1/4}$, v_2 can be extended to $[0, \lambda_1^{1/4}]$ and the previous inequalities are valid in this interval. Evaluating at $\lambda_1^{1/4}$ we deduce that

$$0 = v_1'\left(\lambda_1^{1/4}\right) > v_2'\left(\lambda_1^{1/4}\right). \tag{66}$$

Since $w = \Delta v_2$ satisfies $\Delta w = f(v_2) > 0$ it is subharmonic and hence $w(r_1) \leqslant w(r_2)$ for all $0 \leqslant r_1 \leqslant r_2 \leqslant \lambda_1^{1/4}$. But the Green function for the bilaplacian in the ball of radius $R > 0$ with Dirichlet boundary conditions $G(x, y)$ satisfies $G(x, y) \geqslant c(R - |x|)^2(R - |y|)^2$ for some $c > 0$, see [27]. This implies that $\Delta v_2(\lambda_2^{1/4}) > 0$ and therefore $w(r) > 0$ for all $r \in [\lambda_2^{1/4}, \lambda_1^{1/4}]$. Thus

$$r^{N-1} v_2'(r) = \int_{\lambda_2^{1/4}}^{r} t^{N-1} \Delta v_2(t)\, dt > 0 \quad \text{for all } r \in \left(\lambda_2^{1/4}, \lambda_1^{1/4}\right].$$

In particular $v_2'(\lambda_1^{1/4}) > 0$ which contradicts (66).

It follows that $\Delta v_1(0) = \Delta v_2(0)$ and hence $v_1 \equiv v_2$. This implies that $\lambda_1 = \lambda_2$ and that $u_1 \equiv u_2$. $\square$

Proof of Theorem 1.4. By [2, Theorem 3] there exists λ^* such that if $0 \leqslant \lambda < \lambda^*$ then (8) has a minimal smooth solution u_λ and if $\lambda > \lambda^*$ then (8) has no weak solution. The limit $u^* = \lim_{\lambda \nearrow \lambda^*} u_\lambda$ exists pointwise, belongs to $H^2(B)$ and is a weak solution to (8) in the sense (5). The functions u_λ, $0 \leqslant \lambda < \lambda^*$ and u^* are radially symmetric and radially decreasing. Now, by [15, Theorem 1.4] we know that u^* is unbounded if $N \geqslant 13$.

Fix $\bar{\lambda} \in (0, \lambda^*)$ and let v be a smooth radial solution to (1) with parameter $\bar{\lambda}$. Since $\lambda \in (0, \lambda^*) \to u_\lambda(0)$ depends continuously on λ, and since $\lim_{\lambda \to \lambda^*} u_\lambda(0) \to \infty$ we see that there exists some $\lambda \in (0, \lambda^*)$ such that $v(0) = u_\lambda(0)$. By Lemma 7.2 we conclude that $\bar{\lambda} = \lambda$ and $v = u_\lambda$.

By [15, Proposition 1.8] we also know that u^* is a weakly singular solution. By Theorem 1.2 there is no weakly singular solution for any other value different than λ^*. Moreover, for $\lambda = \lambda^*$ by [15, Theorem 1.2], u^* is the unique weak solution of (1). $\square$

As in Section 6, we let $\mathcal{E} = W^u(P_1) \cap \{v_2 = 0\}$ and recall that each regular radial solution of (1) corresponds to exactly one point $v = (v_1, \dots, v_4) \in \mathcal{E}$ with $v_1 > 0$. It is therefore natural to define $\mathcal{E}_0 = W^u(P_1) \cap \{v_2 = 0, v_1 > 0\}$.

The curve of solutions $\mathcal{C}$ can also be parametrized by the shooting problem (33). Let u_β be the solution of (33) defined in the maximal interval of existence $[0, R(\beta))$. In Theorem 2 of [1], it is shown that for problem (33), given $\beta \in (\beta_0, 0)$ there exists a unique $R_0 \in (0, R(\beta))$ such that $u_\beta'(R_0) = 0$. Moreover, $u_\beta'(r) < 0$ in $(0, R_0)$ and $u_\beta'(r) > 0$ in $(R_0, R(\beta))$. It is not difficult to verify that $R_0(\beta)$ defines a C^1 function of $\beta \in (\beta_0, 0)$.

For $\beta \in (\beta_0, 0)$ we let $V_\beta = (v_{1,\beta}, \dots, v_{4,\beta}) : (-\infty, T(\beta)) \to \mathbb{R}^4$ be the function obtained from $v_\beta(t) = u_\beta(e^t)$ through the transformations (10), where $T(\beta) = \log(R(\beta))$. Define also $T_0(\beta) = \log(R_0(\beta))$ for $\beta \in (\beta_0, 0)$. Then V_β satisfies (11) and $v_{2,\beta}(T_0(\beta)) = 0$. Since $V_\beta(-\infty, T(\beta))$ lies in $W^u(P_1)$ we have $V_\beta(T_0(\beta)) \in \mathcal{E}$. Let us define $\phi : (\beta_0, 0) \to \mathbb{R}^4$ by

$$\phi(\beta) = V_\beta\big(T_0(\beta)\big) \quad \text{for all } \beta \in (\beta_0, 0).$$

It will also be convenient to introduce, for $\beta \in (\beta_0, 0)$ the function

$$U_\beta(r) = u_\beta\big(rR_0(\beta)\big) - u_\beta\big(R_0(\beta)\big), \quad 0 \leqslant r \leqslant 1. \tag{67}$$

Then U_β is a solution of (1) for the value of $\lambda = 8(N-2)(N-4)R_0(\beta)^4 e^{u_\beta(R_0(\beta))}$.

Lemma 7.3. *We have*

$$\lim_{\beta\to\beta_0} \phi(\beta) = P^*, \qquad \lim_{\beta\to\beta_0} T_0(\beta) = +\infty \quad \textit{and} \quad \lim_{\beta\to\beta_0} U_\beta(0) = +\infty.$$

Proof. Let $\mathcal{M}$, Q and $\Gamma = \{\gamma(s)\colon |s| < \delta\}$ with $\gamma : (-\delta, \delta) \to \mathbb{R}^4$ be as in the proof of Theorem 1.3. We choose $\gamma'(0)$ close to the direction $v^{(1)}$. Let $\Gamma_0 = \{\gamma(s)\colon 0 < s < \delta\}$. Then, taking δ sufficiently small, we can define the function $\tau : \Gamma_0 \to \mathbb{R}_+$ where $\tau(p)$ is such that $X_{\tau(p)}(p) \in \mathcal{M}$. Then τ is continuous, and by (59) $\tau(\gamma(s)) = \frac{1}{\nu_1}\log(1/s) + o(\log(1/s))$ as $s \to 0$, which shows that $\tau(p) \to +\infty$ as $p \to Q$. We note that for $\beta \in (\beta_0, 0)$ and β close to β_0 there is some time $t_1(\beta)$ such that $V_\beta(t_1(\beta)) \in \Gamma_0$. As $\beta \to \beta_0$, $V_\beta(t_1(\beta)) \to Q$ and then $T_0(\beta) \to \infty$.

As in the proof of Lemma 6.2 one can also show that as $p \to Q$, $p \in \Gamma_0$ the point $X_{\tau(p)}(p)$ approaches the intersection of $\mathcal{M}$ with $W^u(P_2)$. This shows that $\phi(\beta) \to P^*$ as $\beta \to \beta_0$.

Finally, since $T_0(\beta) \to \infty$ as $\beta \to \beta_0$ we see from formula (67) that $U_\beta(0) \to \infty$ as $\beta \to \beta_0$. $\square$

Lemma 7.4. *We have*

$$\lim_{\beta\to 0} \phi(\beta) = 0 \quad \textit{and} \quad \lim_{\beta\to 0} U_\beta(0) = 0.$$

Proof. Using the implicit function theorem there is $\delta > 0$ such that for $\lambda > 0$ small there is a unique small solution u_λ of (1). The map $\lambda \mapsto u_\lambda$ is C^1 into $C^4(\bar{B})$. Set $\tilde{u}_\lambda(r) = u_\lambda(A_\lambda r) - u_\lambda(0)$ where $A_\lambda = (\frac{8(N-2)(N-4)}{\lambda e^{u_\lambda(0)}})^{1/4}$. Then $\tilde{u}_\lambda$ is the solution of (33) with $\beta = \beta(\lambda) = A_\lambda^2 \Delta u_\lambda(0)$ by uniqueness of that initial value problem. In particular $U_\beta = u_\lambda$ if $\beta = A_\lambda^2 \Delta u_\lambda(0)$.

Using Theorem 4 of [2] we know that $u_\lambda/\lambda \to \frac{1}{8N(N+2)}(1-r^2)^2$ uniformly in B as $\lambda \to 0$. By elliptic estimates the convergence is also in $C^4(\bar{B})$. It follows that $\beta(\lambda) = O(\lambda^{1/2})$ as $\lambda \to 0$. Thus for small $\beta < 0$ the solution of the shooting problem (33) is $\tilde{u}_\lambda$ with $\lambda > 0$ such that $A_\lambda^2 \Delta u_\lambda(0) = \beta$, and this $\lambda > 0$ is uniquely determined. Then as $\beta \to 0$, $\lambda \to 0$ and $U_\beta(0) = u_\lambda(0) \to 0$. Also $R_0(\beta) = 1/A_\lambda \to 0$ and $\phi(\beta) \to 0$ as $\beta \to 0$ (since $\phi(\beta)$ is expressed in terms of derivatives of u_λ). $\square$

Lemma 7.5. *We have that*

$$\mathcal{E}_0 = \big\{\phi(\beta)\colon \beta \in (\beta_0, 0)\big\}$$

is a real analytic curve.

By $\mathcal{E}_0$ being real analytic we mean that each point of this set as a neighborhood in $\mathcal{E}_0$ which can be parametrized by a real analytic function.

Proof. By construction $\phi(\beta) \in \mathcal{E}_0$ for each $\beta \in (\beta_0, 0)$. To prove $\mathcal{E}_0 \subseteq \{\phi(\beta)\colon \beta \in (\beta_0, 0)\}$ we need to show that given any radial regular solution u of (1) there exists $\beta \in (\beta_0, 0)$ such that $u = U_\beta$. Using Lemma 7.2 it is sufficient to find β such that $u(0) = U_\beta(0)$. We have by Lemma 7.3 that $U_\beta(0) \to +\infty$

as $\beta \to \beta_0$, while by Lemma 7.4 that $U_\beta(0) \to 0$ as $\beta \to 0$. Since $U_\beta(0)$ varies continuously with β there is $\beta \in (\beta_0, 0)$ such that $u(0) = U_\beta(0)$.

The unstable manifold of P_1 is a real analytic surface, since the vector field is real analytic, in fact a polynomial, see for instance [12, p. 104]. By the implicit function theorem, each point in $W^u(P_1) \cap \{v_2 = 0\}$ where the intersection is transversal has a neighborhood in this set which can be parametrized by a real analytic function. At points in the intersection of the sets $W^u(P_1)$ and $\{v_2 = 0,\ v_1 > 0\}$ the transversality condition holds. Indeed, the points in this set are given bu $\phi(\beta)$ with $\beta \in (\beta_0, 0)$. Let U_β be defined by (67) and recall that it is a positive solution of (1). We recall also that Green function in the ball with Dirichlet boundary conditions $G(x, y)$ satisfies $G(x, y) \geqslant c(1 - |x|)^2(1 - |y|)^2$ for some $c > 0$, see [27]. So we actually have $U_\beta''(1) > 0$. This implies that at $t = T_0(\beta)$ we have $v_2'(t) = e^{2t}\Delta U_\beta(1) > 0$ which shows that the intersection is transversal. It follows that the intersection of the sets $W^u(P_1)$ and $\{v_2 = 0, v_1 > 0\}$ is a real analytic curve. □

Proof of Theorem 1.5. It is a consequence of Lemmas 7.2 and 7.5. □

Proposition 7.6. *Assume* $5 \leqslant N \leqslant 12$. *If* $\lambda \neq \lambda_S$, *then there exists a finite number of regular radial solutions of* (1).

Proof. By Lemmas 7.3 and 7.4 we can consider P_1 and P^* as the endpoints of $\mathcal{E}_0$. If $\lambda = 0$ then $u = 0$ is the only solution of (1). Let $\lambda \neq 0$, $\lambda \neq \lambda^*$. By analyticity $\mathcal{E}_0 \cap \{v_1 = \lambda\}$ can only accumulate at either P_1 or P^*. Since P^* is not included in $\{v_1 = \lambda\}$ accumulation in P^* is not possible. Similarly, since $P_1 \notin \{v_1 = \lambda\}$ the set $\mathcal{E}_0 \cap \{v_1 = \lambda\}$ cannot accumulate at P_1. Thus $\mathcal{E}_0 \cap \{v_1 = \lambda\}$ consists of a finite number of points, which correspond to regular radial solutions of (1). □

Acknowledgments

J.D. was partially supported by Fondecyt 1090167, Fondap and Basal-CMM grants, I.F. was partially supported by Fondecyt 1090518 and I.G. was partially supported by Fondecyt 1090470.

References

[1] G. Arioli, F. Gazzola, H.-C. Grunau, Entire solutions for a semilinear fourth order elliptic problem with exponential nonlinearity, J. Differential Equations 230 (2) (2006) 743–770.
[2] G. Arioli, F. Gazzola, H.-C. Grunau, E. Mitidieri, A semilinear fourth order elliptic problem with exponential nonlinearity, SIAM J. Math. Anal. 36 (4) (2005) 1226–1258.
[3] R. Bamón, I. Flores, M. del Pino, Ground states of semilinear elliptic equations: A geometric approach, Ann. Inst. H. Poincaré Anal. Non Linéaire 17 (5) (2000) 551–581.
[4] J. Bebernes, D. Eberly, Mathematical Problems from Combustion Theory, Applied Mathematical Sciences, vol. 83, Springer-Verlag, New York, 1989.
[5] G.R. Belickiĭ, Equivalence and normal forms of germs of smooth mappings, Uspekhi Mat. Nauk 33 (1 (199)) (1978) 95–155, 263.
[6] G.R. Belickiĭ, Normal'nye Formy, Invarianty i Lokal'nye Otobrazheniya (Normal Forms, Invariant and Local Mappings), Naukova Dumka, Kiev, 1979, 173 pp. (in Russian).
[7] E. Berchio, F. Gazzola, Some remarks on biharmonic elliptic problems with positive, increasing and convex nonlinearities, Electron. J. Differential Equations (2005), No. 34, 20 pp.
[8] E. Berchio, F. Gazzola, E. Mitidieri, Positivity preserving property for a class of biharmonic elliptic problems, J. Differential Equations 229 (1) (2006) 1–23.
[9] G. Bratu, Sur les équations intégrales non linéaires, Bull. Soc. Math. France 42 (1914) 113–142.
[10] D. Cassani, J.M. do O, N. Ghoussoub, On a fourth order elliptic problem with a singular nonlinearity, J. Adv. Nonlinear Stud. 9 (2009) 177–197.
[11] S. Chandrasekhar, An Introduction to the Study of Stellar Structures, Dover, New York, 1957.
[12] S.N. Chow, J.K. Hale, Methods of Bifurcation Theory, Grundlehren der Mathematischen Wissenschaften, vol. 251, Springer-Verlag, New York–Berlin, 1982.
[13] E.A. Coddington, N. Levinson, Theory of Ordinary Differential Equations, McGraw–Hill Book Company, Inc., New York–Toronto–London, 1955.
[14] C. Cowan, P. Esposito, N. Ghoussoub, A. Moradifam, The critical dimension for a fourth order elliptic problem with singular nonlinearity, Arch. Ration. Mech. Anal., in press.

[15] J. Dávila, L. Dupaigne, I. Guerra, M. Montenegro, Stable solutions for the bilaplacian with exponential nonlinearity, SIAM J. Math. Anal. 39 (2) (2007) 565–592.

[16] J. Dávila, I. Flores, I. Guerra, Multiplicity of solutions for a fourth order equation with power-type nonlinearity, preprint.

[17] J. Dolbeault, I. Flores, Geometry of phase space and solutions of semilinear elliptic equations in a ball, Trans. Amer. Math. Soc. 359 (9) (2007) 4073–4087.

[18] A. Ferrero, H.-C. Grunau, The Dirichlet problem for supercritical biharmonic equations with power-type nonlinearity, J. Differential Equations 234 (2) (2007) 582–606.

[19] A. Ferrero, H.-C. Grunau, P. Karageorgis, Supercritical biharmonic equations with power-type nonlinearity, Ann. Mat. Pura Appl. 188 (2009) 171–185.

[20] A. Ferrero, G. Warnault, On solutions of second and fourth order elliptic equations with power-type nonlinearities, Nonlinear Anal. 70 (8) (2009) 2889–2902.

[21] I. Flores, A resonance phenomenon for ground states of an elliptic equation of Emden–Fowler type, J. Differential Equations 198 (1) (2004) 1–15.

[22] I. Flores, Singular solutions of the Brezis–Nirenberg problem in a ball, Comm. Pure Appl. Anal. 8 (2) (2009) 673–682.

[23] D.A. Frank-Kamenetskii, Diffusion and Heat Exchange in Chemical Kinetics, Princeton Univ. Press, Princeton, NJ, 1955.

[24] F. Gazzola, H. Grunau, Radial entire solutions for supercritical biharmonic equations, Math. Ann. 334 (4) (2006) 905–936.

[25] I.M. Gelfand, Some problems in the theory of quasilinear equations, Section 15, due to G.I. Barenblatt, Amer. Math. Soc. Transl. Ser. 2 29 (1963) 295–381.

[26] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[27] H. Grunau, G. Sweers, Positivity for equations involving polyharmonic operators with Dirichlet boundary conditions, Math. Ann. 307 (4) (1997) 589–626.

[28] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Applied Mathematical Sciences, vol. 42, Springer-Verlag, New York, 1990. Revised and corrected reprint of the 1983 original.

[29] Z. Guo, J. Wei, On a fourth order nonlinear elliptic equation with negative exponent, SIAM J. Math. Anal. 40 (5) (2009) 2034–2054.

[30] Z. Guo, J. Wei, Entire solutions and global bifurcations for a biharmonic equation with singular non-linearity in $\mathbb{R}^3$, Adv. Differential Equations 13 (7–8) (2008) 753–780.

[31] P. Hartman, Ordinary Differential Equations, Classics in Applied Mathematics, vol. 38, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002.

[32] D.D. Joseph, T.S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, Arch. Ration. Mech. Anal. 49 (1972) 241–269.

[33] P. Karageorgis, Stability and intersection properties of solutions to the nonlinear biharmonic equation, preprint.

[34] P.-L. Lions, On the existence of positive solutions of semilinear elliptic equations, SIAM Rev. 24 (4) (1982) 441–467.

[35] J. Liouville, Sur l'equation aux différences partielles $\frac{d^2 \log \lambda}{du\, dv} \pm \frac{\lambda}{2a^2} = 0$, J. Math. Pure Appl. 36 (1853) 71–72.

[36] P.J. McKenna, W. Reichel, Radial solutions of singular nonlinear biharmonic equations and applications to conformal geometry, Electron. J. Differential Equations (2003), No. 37, 13 pp.

[37] È. Mitidieri, S.I. Pokhozhaev, A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities, Proc. Steklov Inst. Math. 234 (3) (2001) 1–362.

[38] J. Palis, On Morse–Smale dynamical systems, Topology 8 (1968) 385–404.

[39] D. Ruelle, Elements of Differentiable Dynamics and Bifurcation Theory, Academic Press, Inc., Boston, MA, 1989.