EXISTENCE AND UNIQUENESS OF SOLUTIONS OF NONLINEAR ELLIPTIC EQUATIONS WITHOUT GROWTH CONDITIONS AT INFINITY

By

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Abstract. In this paper, we consider the nonlinear elliptic problem

$$-\Delta u + |u|^{p-1}u + |\nabla u|^q = f$$

in $\mathbb{R}^N$, where $p > 1$ and $q > 0$. We show that if $f \in L^r_{\rm loc}(\mathbb{R}^N)$ for suitable $r \geq 1$, then there exists a distributional solution of the equation, independently of the behavior of f at infinity. We also analyze the uniqueness of this solution in some cases.

1 Introduction

In [6], the following somewhat surprising result was obtained: if $p > 1$ and $f \in L^1_{\rm loc}(\mathbb{R}^N)$, then there exists a unique distributional solution of

$$-\Delta u + |u|^{p-1}u = f \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

The surprising fact is that neither the existence nor the uniqueness of solutions of (1.1) depends on the behavior of f at infinity, but strongly relies on the fact that $p > 1$.

The question of existence and uniqueness of solutions without prescribing a growth on f at infinity has subsequently been considered for equations more general than (1.1). Such equations are obtained when the Laplacian is replaced with a divergence operator of p-Laplacian type (see [4] or [12]) or with a fully nonlinear operator (cf. [7]). The extension to parabolic equations has also been studied in [5] and [13].

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Our intention in the present paper is to analyze whether the existence and uniqueness features in (1.1) still hold when we introduce a term that depends on the gradient into the equation. More precisely, we are interested in the problem

$$-\Delta u + |u|^{p-1}u + |\nabla u|^q = f \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $p > 1$, $q > 0$, and $f \in L^1_{\text{loc}}(\mathbb{R}^N)$. It can be deduced from our proofs below that more general problems, for instance the m-Laplacian version of (1.2), can be considered,. However, optimal conditions on the parameters m, p, q, and r when $f \in L^r_{\text{loc}}(\mathbb{R}^N)$ are far from clear. For simplicity, we restrict our attention to (1.2).

By a solution u of (1.2) we mean $u \in L^p_{\text{loc}}(\mathbb{R}^N)$, $|\nabla u|^q \in L^1_{\text{loc}}(\mathbb{R}^N)$, and u satisfies (1.2) in the distributional sense, i.e.,

$$-\int u\Delta\phi + \int (|u|^{p-1}u + |\nabla u|^q)\phi = \int f\phi$$

for every $\phi \in C_0^\infty(\mathbb{R}^N)$. It is known that since $\Delta u \in L^1_{\text{loc}}(\mathbb{R}^N)$, $u \in W^{1,s}_{\text{loc}}(\mathbb{R}^N)$ for every s, $1 \le s < N/(N-1)$. As usual, a slight increase of regularity in f is reflected in a gain of regularity for u.

We begin with the case $0 < q < 2p/(p+1)$, where the regularity $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ is enough to obtain a solution. In this case, the gradient term does not "interfere too much" with the structure of (1.1).

Theorem 1. *Let $p > 1$ and $0 < q < 2p/(p+1)$. Then for every $f \in L^1_{\text{loc}}(\mathbb{R}^N)$, there exists a distributional solution u to the problem*

$$-\Delta u + |u|^{p-1}u + |\nabla u|^q = f \quad \text{in } \mathbb{R}^N.$$

This solution satisfies $u \in L^p_{\text{loc}}(\mathbb{R}^N) \cap W^{1,s}_{\text{loc}}(\mathbb{R}^N)$ for every s in the interval $(1, \max\{2p/(p+1), N/(N-1)\})$. Moreover, if $f \ge 0$, then $u \ge 0$.

When $q \ge 2p/(p+1)$, the L^1_{loc} regularity for f seems insufficient to ensure existence. Also, let us remark that the sign of f is an important issue here (mainly when $q > 2$). This is due to the fact that the equation (1.2) is not invariant under the changes of u to $-u$ and f to $-f$, as happens with (1.1). Thus, in the main results of present paper, Theorems 1–4, we restrict ourselves to the case $f \ge 0$, delaying the study of negative f to future work. The differences in the problem between a positive and negative f can be seen even when the problem is posed in a bounded domain in $\mathbb{R}^N$. Our next result is valid for all $q > 1$.

Theorem 2. *Let $p, q > 1$ and $f \in L^r_{\text{loc}}(\mathbb{R}^N)$ for some $r > N$ with $f \ge 0$. Then there exists a strong solution $u \in C^1(\mathbb{R}^N) \cap W^{2,r}_{\text{loc}}(\mathbb{R}^N)$ of the equation*

$$-\Delta u + |u|^{p-1}u + |\nabla u|^q = f \quad \text{in } \mathbb{R}^N.$$

Moreover, the solution is positive.

We next analyze how the condition $f \in L^r_{\mathrm{loc}}(\mathbb{R}^N)$ with $r > N$ can be weakened. For this purpose, we consider a radially symmetric, nonnegative function f and try to construct radially symmetric, nonnegative solutions. It turns out that in this framework, $r > 1$ suffices, provided that $1 < q < N/(N-1)$. We mention in passing that the next theorem is valid for all $p > 1$, but it only gives better results than Theorem 1 when $p < N/(N-2)$, $q \geq 2p/(p+1)$.

Theorem 3. *Let* $1 < p < N/(N-2)$ *and* $2p/(p+1) \leq q < N/(N-1)$. *For every radially symmetric, nonnegative function* $f \in L^r_{\mathrm{loc}}(\mathbb{R}^N)$, $r > 1$, *there exists a radially symmetric, nonnegative distributional solution u of the equation*

$$-\Delta u + u^p + |\nabla u|^q = f \quad \text{in } \mathbb{R}^N.$$

The proof of existence of solutions in all of our theorems is achieved by first considering the problem in a smooth bounded domain of $\mathbb{R}^N$, complemented with a Dirichlet boundary condition. The essential step is then to obtain good estimates for the solutions and their gradients.

Finally, we analyze the question of uniqueness of the constructed solutions. We are able to prove uniqueness only if the regularity of f is improved slightly.

Theorem 4. *Assume* $f \in L^r_{\mathrm{loc}}(\mathbb{R}^N)$ *for some* $r > N$ *and that* f *is nonnegative. Then if* $0 < q < p$, *problem* (1.2) *admits a unique solution* $u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N)$. *Moreover,* $u \in C^1(\mathbb{R}^N) \cap W^{2,r}_{\mathrm{loc}}(\mathbb{R}^N)$ *and u is nonnegative.*

It is worthy of mention that the condition $q < p$ is optimal in the uniqueness assertion since if $q \geq p$, infinitely many smooth solutions can be constructed when $f = 0$; see Remark 4.

For the proof of Theorem 4, we follow a device used in [6] with a significant variation: we use the minimal solution U_R of the boundary blow-up problem

$$\begin{aligned} -\Delta U + c\,U^p - d|\nabla U|^q &= 0 \quad \text{in } B_R, \\ U &= \infty \quad \text{on } \partial B_R, \end{aligned}$$

where $c, d > 0$; this was shown to exist in [1]. A property worth noticing is that $U_R \to 0$ uniformly in compacts of $\mathbb{R}^N$ as $R \to \infty$.

The rest of the paper is organized as follows. In Section 2, we consider problem (1.2) in smooth bounded domains of $\mathbb{R}^N$ with a Dirichlet boundary condition, while Section 3 is dedicated to obtaining local estimates for these approximate solutions and their gradients. Finally, the proofs of Theorems 1, 2, 3, and 4 are presented in Section 4.

After acceptance of this paper, the authors learned of references [14] and [17], which are closely related to the results contained here.

2 A problem in bounded domains

As mentioned in the Introduction, the construction of solutions to (1.2) relies on the solvability of a related Dirichlet problem in a smooth bounded domain Ω of $\mathbb{R}^N$. The purpose of this brief section is to analyze such a problem. Thus, we consider

$$\begin{aligned} -\Delta u + |u|^{p-1}u + |\nabla u|^q &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $p > 1$ and $q > 0$. In the present context, we may always take f to be sufficiently smooth; hence we assume $f \in C^\infty(\overline{\Omega})$. The result we need is the following.

Theorem 5. *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and $f \in C^\infty(\overline{\Omega})$, $p > 1$, $q > 0$. When $q > 2$, assume, additionally, that $f \geq 0$. Then problem* (2.1) *admits a unique classical solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Moreover, if $f \geq 0$, then $u > 0$ in Ω.*

Proof. It is clear that the unique solution $\bar{u}$ to the problem

$$\begin{aligned} -\Delta u + |u|^{p-1}u &= |f|_\infty \quad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

is a supersolution of (2.1) which, according to the strong maximum principle, satisfies $\bar{u} > 0$ in Ω. Next, assume $0 < q \leq 2$. We claim that the problem

$$\begin{aligned} -\Delta u + |u|^{p-1}u + |\nabla u|^q &= -|f|_\infty \quad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{2.2}$$

has a unique (negative) solution. Indeed, setting $\theta = |f|_\infty^{\frac{1}{p}}$ and $v = u + \theta$, we see that (2.2) is equivalent to

$$\begin{aligned} \Delta v - |\nabla v|^q &= h(v) \quad \text{in } \Omega, \\ v &= \theta \qquad \text{on } \partial\Omega, \end{aligned} \tag{2.3}$$

where $h(v) = \theta^p + |v - \theta|^{p-1}(v - \theta)$ is increasing and satisfies $h(0) = 0$. From [1, Proposition 5] and the strong maximum principle, we deduce that problem (2.3) has a unique solution $\underline{v}$ which satisfies $\underline{v} < \theta$ in Ω. Letting $\underline{u} = \underline{v} - \theta$, we obtain the unique solution of (2.2), which is a subsolution of (2.1).

Since $0 < q \leq 2$, we obtain by standard results (see, for example, [2] or [18]) the existence of a weak solution $u \in C^1(\overline{\Omega})$ of (2.1) which satisfies $\underline{u} < u < \bar{u}$ in Ω. By classical regularity, we also have $u \in C^2(\Omega)$, although this regularity can be improved, depending on the values of p and q.

When $q > 2$ and $f \geq 0$, we may still take $\bar{u}$ as a supersolution and $\underline{u} = 0$ as a subsolution, and we obtain the existence of a solution u verifying $0 < u < \bar{u}$ in Ω thanks to [15, Theorem III.1]. □

3 Interior estimates for solutions and their gradients

In order to construct solutions to (1.2), we need local bounds for solutions and their gradients. When $q < 2p/(p+1)$, these bounds can be obtained merely under the assumption that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, but the case $q \geq 2p/(p+1)$ is not so straightforward and calls for a different strategy.

We begin by considering the bounds for weak solutions when $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $0 < q < 2p/(p+1)$. Recall that $u \in H^1(B_{2R})$ is a weak solution of

$$-\Delta u + |u|^{p-1}u + |\nabla u|^q = f \quad \text{in } B_{2R}$$

if

$$\int \nabla u \nabla \phi + (|u|^{p-1}u + |\nabla u|^q)\phi = \int f\phi$$

for every $\phi \in C_0^\infty(B_{2R})$. The proof of the next result is inspired by [4].

Theorem 6. *Let $p > 1$ and $0 < q < 2p/(p+1)$. Then for every $R > 0$, there exists a constant $C = C(R) > 0$ such that for every weak solution $u \in H^1(B_{2R})$ of*

$$-\Delta u + |u|^{p-1}u + |\nabla u|^q = f \quad \text{in } B_{2R} \tag{3.1}$$

with $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$,

$$\int_{B_R} |u|^p \leq C\left(\int_{B_{2R}} |f| + 1\right). \tag{3.2}$$

Also, for every $s \in \big(0, 2p/(p+1)\big)$, there exists $C = C(s,R)$ such that

$$\int_{B_R} |\nabla u|^s \leq C\left(\int_{B_{2R}} |f| + 1\right). \tag{3.3}$$

Proof. For $m > 0$, we introduce the function

$$\phi_m(\sigma) = m \int_0^\sigma \frac{dt}{(1+|t|)^{m+1}}, \quad \sigma \in \mathbb{R},$$

which is odd and satisfies $\left|\phi_m(\sigma)\right| \le 1$. Choose $\theta \in C_0^\infty(B_{2R})$ satisfying $0 \le \theta \le 1$ and $\theta \equiv 1$ in B_R. Next, take $\phi_m(u)\theta^\alpha$ as a test function in the weak formulation of (3.1) (where $\alpha > 0$ is to be chosen later on) to obtain

$$m \int \frac{|\nabla u|^2}{(1+|u|)^{m+1}}\theta^\alpha + \int |u|^{p-1}u\phi_m(u)\theta^\alpha \le \int |f|\theta^\alpha + C\int \theta^{\alpha-1}|\nabla u| + \int \theta^\alpha |\nabla u|^q,$$

thanks to the definition of ϕ_m. We adopt the usual convention that the letter C denotes different constants, not depending on u or f. Observe now that, on the one hand, from Young's inequality, we have

$$\theta^{\alpha-1}|\nabla u| \le \varepsilon \frac{|\nabla u|^2\theta^\alpha}{(1+|u|)^{m+1}} + C(\varepsilon)\,\theta^{\alpha-2}(1+|u|)^{m+1}$$

for every $\varepsilon > 0$, where $C(\varepsilon)$ depends only on ε. On the other hand, we can take q_0 such that $\max\{1, q\} < q_0 < 2p/p+1)$, and since $|\nabla u|^q \le 1 + |\nabla u|^{q_0}$, we obtain, again by Young's inequality,

$$|\nabla u|^q \le 1 + \varepsilon \frac{|\nabla u|^2}{(1+|u|)^{m+1}} + C(\varepsilon)(1+|u|)^{(m+1)q_0/(2-q_0)}.$$

Hence, if we fix $\varepsilon > 0$ sufficiently small, on the one hand, we arrive at

$$\begin{aligned}\int \frac{|\nabla u|^2}{(1+|u|)^{m+1}}\theta^\alpha &+ \int |u|^{p-1}u\phi_m(u)\theta^\alpha \\ &\le C\int_{B_{2R}} |f| + C\int (1+|u|)^{m+1}\theta^{\alpha-2} + C\int (1+|u|)^{(m+1)q_0/(2-q_0)}\theta^\alpha + C \\ &\le C\int_{B_{2R}} |f| + C\int (1+|u|)^{(m+1)q_0(2-q_0)}\theta^{\alpha-2} + C,\end{aligned}$$

since $q_0/(2-q_0) > 1$. On the other hand, it is easily seen that $|u|^{p-1}u\phi_m(u) \ge C|u|^p - 1$ for $u \in \mathbb{R}$. Hence

$$\int \frac{|\nabla u|^2}{(1+|u|)^{m+1}}\theta^\alpha + \int |u|^p\theta^\alpha \le C\int_{B_{2R}} |f| + C\int (1+|u|)^{(m+1)q_0/(2-q_0)}\theta^{\alpha-2} + C.$$

A further application of Young's inequality gives

$$\int (1+|u|)^{(m+1)q_0/(2-q_0)}\theta^{\alpha-2} \le \varepsilon \int (1+|u|)^p\theta^\alpha + C(\varepsilon)\int \theta^{\alpha-2p/(p-\mu)},$$

where $\mu = (m+1)q_0/(2-q_0)$. We note that we can achieve $\mu < p$ if we choose m small enough. Taking $\alpha \geq 2p/(p-\mu)$ and recalling that $\theta \equiv 1$ in B_R, we obtain

$$\int_{B_R} \frac{|\nabla u|^2}{(1+|u|)^{m+1}} + \int_{B_R} |u|^p \leq C\left(\int_{B_{2R}} |f| + 1\right). \tag{3.4}$$

Finally, observe that (3.4) holds for all $m > 0$, since it holds for small m and the left-hand side is decreasing in m.

Now (3.2) follows immediately from (3.4). With regard to (3.3), we can use Hölder's inequality for $s \in (0, 2p/(p+1))$ to obtain

$$\begin{aligned} \int_{B_R} |\nabla u|^s &= \int_{B_R} \frac{|\nabla u|^s}{(1+|u|)^\nu}(1+|u|)^\nu \\ &\leq \left(\int_{B_R} \frac{|\nabla u|^2}{(1+|u|)^{\frac{2\nu}{s}}}\right)^{s/2}\left(\int_{B_R}(1+|u|)^{2\nu/(2-s)}\right)^{(2-s)/2} \end{aligned} \tag{3.5}$$

for every $\nu > 0$. Note that we can choose ν so that $2\nu/s > 1$, $2\nu/(2-s) \leq p$, since this is equivalent to $s/2 < \nu \leq p(2-s)/2$. This choice is possible since $s < 2p/(p+1)$. Thus (3.3) follows at once from (3.5) and (3.4). □

Remark 1. When $f \in L^r_{\text{loc}}(\mathbb{R}^N)$, $r > 1$, we can obtain better estimates for weak nonnegative, bounded solutions (these last two assumptions are made only for the sake of simplicity in the present proof; they do not seem necessary). Indeed, if u is such a solution, we have

$$\int_{B_R} u^{pr} \leq C\left(\int_{B_{2R}} |f|^r + 1\right). \tag{3.6}$$

To see this, take as a test function in (3.1) $(u+\varepsilon)^m\theta^\alpha$, where θ is as in the previous proof, $m = p(r-1)$, $\varepsilon > 0$ is small, and $\alpha > 0$ is to be chosen. This leads to

$$\begin{aligned} m\int \theta^\alpha(u+\varepsilon)^{m-1}|\nabla u|^2 + \int \theta^\alpha u^p(u+\varepsilon)^m \\ \leq \int |f|\theta^\alpha(u+\varepsilon)^m + \alpha\int \theta^{\alpha-1}(u+\varepsilon)^m|\nabla u||\nabla\theta|. \end{aligned}$$

From Young's inequality applied to the last integral, we obtain

$$\alpha\int \theta^{\alpha-1}(u+\varepsilon)^m|\nabla u||\nabla\theta| \leq \frac{m}{2}\int \theta^\alpha(u+\varepsilon)^{m-1}|\nabla u|^2 + C\int \theta^{\alpha-2}(u+\varepsilon)^{m+1},$$

where C is a positive constant. Hence

$$\int \theta^\alpha u^p(u+\varepsilon)^m \leq \int |f|\theta^\alpha(u+\varepsilon)^m + C\int \theta^{\alpha-2}(u+\varepsilon)^{m+1}.$$

We can first let ε tend to 0 and then apply Young's inequality to obtain

$$|f|u^m \le \frac{1}{2}u^{pr} + C|f|^r,$$

so that

$$\frac{1}{2}\int \theta^\alpha u^{pr} \le C\int |f|^r\theta^\alpha + C\int \theta^{\alpha-2}u^{m+1}. \tag{3.7}$$

Next, observe that applying Young's inequality once again to the last integrand, we obtain $\theta^{\alpha-2}u^{m+1} \le \theta^\alpha u^{pr}/4 + C\theta^{\alpha-2p'r}$, where from now on $p' = p/(p-1)$, so that $1/p + 1/p' = 1$. Choosing now $\alpha > 2p'r$ in (3.7), we get

$$\int \theta^\alpha u^{pr} \le C\left(\int |f|^r\theta^\alpha + 1\right).$$

We obtain (3.6), since $0 \le \theta \le 1$ and $\theta = 1$ in B_R.

In the complementary case $q \ge 2p/(p+1)$, we impose an extra regularity condition on f. Namely, we assume that $f \in L^r_{\mathrm{loc}}(\mathbb{R}^N)$ for some $r > N$. Although the estimates for the solutions can be obtained by arguing as in [6], it is not so clear how to get appropriate bounds for the gradients needed to pass to the limit. Hence our approach is completely different from that in the previous case: it uses a mixture of methods from [11] and [16] and is valid for all $q > 1$.

Theorem 7. *Let $p, q > 1$, $f \in C^1(\mathbb{R}^N)$ and $r > N$. If $u \in C^3(B_{4R})$ is a nonnegative classical solution of*

$$-\Delta u + u^p + |\nabla u|^q = f \quad \text{in } B_{4R},$$

then there exists a positive constant C, depending on R and $|f|_{L^r(B_{4R})}$, such that

$$\sup_{B_{R/2}} (u + |\nabla u|) \le C. \tag{3.8}$$

Proof. We first claim that there exists a positive constant C, not depending on f or on u, such that

$$\int_{B_{2R}} u^p \le C\left(\int_{B_{4R}} |f| + 1\right). \tag{3.9}$$

To prove (3.9) we can argue exactly as in [6]. Notice that $-\Delta u + u^p \le f$ in $\mathbb{R}^N$. Now take a cut-off function $\xi \in C^\infty(B_{4R})$ satisfying $0 \le \xi \le 1$ and $\xi \equiv 1$ in B_{2R}. Testing the equation with ξ^α, where $\alpha > 0$ is to be chosen later, and using Hölder's inequality, we have

$$\int u^p \xi^\alpha \leq \int_{B_{4R}} |f| + \int u \Delta \xi^\alpha \leq \int_{B_{4R}} |f| + C \int u \xi^{\alpha-2}$$
$$\leq \int_{B_{4R}} |f| + C \left(\int u^p \, \xi^{(\alpha-2)p} \right)^{1/p},$$

where C is a positive constant (depending only on α and ξ). Setting $\alpha = 2p/(p-1)$, we obtain

$$\int u^p \xi^\alpha \leq \int_{B_{2R}} |f| + C \left(\int u^p \xi^\alpha \right)^{1/p},$$

so that (3.9) follows by recalling that $\xi \equiv 1$ in B_{2R}. Now observe that $-\Delta u \leq f$ in B_{2R}, so that we may apply Theorem 8.17 in [9] to arrive at $u \in L^\infty(B_R)$, with the bound $\sup_{B_R} u \leq C$, where C depends on R, $|f|_{L^r(B_{4R})}$, and p.

Our next task is to obtain estimates for the gradient of u. Let $w = |\nabla u|^2$, and observe that w is smooth. It is not hard to see that

$$-\Delta w + q |\nabla u|^{q-2} \nabla u \nabla w = -2|D^2 u|^2 + 2 \nabla u \nabla f - 2p u^{p-1} w$$

in $\mathbb{R}^N$. Next take a smooth cut-off function φ such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in B_R, $\varphi \equiv 0$ in $\mathbb{R}^N \backslash B_{2R}$, and $|\Delta \varphi| \leq C \varphi^\theta$, $|\nabla \varphi|^2 \leq C \varphi^{1+\theta}$ for some positive constant C and certain $\theta \in (0, 1)$ to be chosen later. We have

$$-\Delta(\varphi w) + q w^{(q-2)/2} \nabla u \nabla(\varphi w) + 2 \frac{\nabla \varphi}{\varphi} \nabla(\varphi w) + 2 |D^2 u|^2 \varphi$$
$$= q w^{(q-2)/2} \nabla u \nabla \varphi w - \Delta \varphi w + 2 \nabla u \nabla f \varphi - 2p u^{p-1} w + 2 \frac{|\nabla \varphi|^2}{\varphi} w$$

in $\mathbb{R}^N$. Taking now $m > 0$ and using $(\varphi w)^m$ as a test function, we obtain

$$m \int |\nabla(\varphi w)|^2 (\varphi w)^{m-1} + q \int w^{\frac{q-2}{2}} \nabla u \nabla(\varphi w)(\varphi w)^m$$
$$+ 2 \int \frac{\nabla \varphi}{\varphi} \nabla(\varphi w)(\varphi w)^m + 2 \int |D^2 u|^2 \varphi (\varphi w)^m$$
$$= q \int w^{\frac{q-2}{2}} \nabla u \nabla \varphi (\varphi w)^m w - \int \Delta \varphi w (\varphi w)^m$$
$$+ 2 \int \nabla u \nabla f \varphi (\varphi w)^m - 2p \int u^{p-1} w (\varphi w)^m + 2 \int \frac{|\nabla \varphi|^2}{\varphi} w (\varphi w)^m.$$

We next observe that thanks to Cauchy-Schwarz inequality,

$$|D^2 u|^2 \geq \frac{1}{N} (\Delta u)^2 \geq \frac{1}{N} (|\nabla u|^q + u^p - f)^2$$
$$\geq \frac{1}{2N} \left(|\nabla u|^q + u^p \right)^2 - \frac{2}{N} |f|^2 \geq \frac{1}{2N} |\nabla u|^{2q} - \frac{2}{N} |f|^2.$$

It follows that

$$
\begin{aligned}
m \int |\nabla(\varphi w)|^2 (\varphi w)^{m-1} &+ q \int w^{\frac{q-2}{2}} \nabla u \nabla(\varphi w)(\varphi w)^m \\
&+ 2 \int \frac{\nabla\varphi}{\varphi} \nabla(\varphi w)(\varphi w)^m + \frac{1}{2N} \int w^q \varphi(\varphi w)^m + \int |D^2 u|^2 \varphi(\varphi w)^m \\
&\leq q \int w^{\frac{q-2}{2}} |\nabla\varphi| w(\varphi w)^m + \int |\Delta\varphi| w(\varphi w)^m + \frac{4}{N} \int |f|^2 (\varphi w)^m \\
&+ 2 \int \frac{|\nabla\varphi|^2}{\varphi} w(\varphi w)^m + 2 \int \nabla u \nabla f \varphi(\varphi w)^m + C,
\end{aligned}
$$

where here and in the rest of the proof, C represents a constant independent of u, f, and m and can take different values at different places. Now by the choice we have made for φ,

$$
\begin{aligned}
q \int w^{(q-1)/2} |\nabla\varphi| w(\varphi w)^m &\leq C \int \varphi^{m+(\theta+1)/2} w^{m+(q+1)/2} \leq C \int \varphi^{m+\theta} w^{m+(q+1)/2} \\
\int |\Delta\varphi| w(\varphi w)^m &\leq C \int \varphi^{m+\theta} w^{m+1} \\
2 \int \frac{|\nabla\varphi|^2}{\varphi} w(\varphi w)^m &\leq C \int \varphi^{m+\theta} w^{m+1}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
m \int |\nabla(\varphi w)|^2 (\varphi w)^{m-1} &+ q \int w^{(q-2)/2} \nabla u \nabla(\varphi w)(\varphi w)^m \\
&+ 2 \int \frac{\nabla\varphi}{\varphi} \nabla(\varphi w)(\varphi w)^m + \frac{1}{2N} \int \varphi^{m+1} w^{m+q} + \int |D^2 u|^2 \varphi(\varphi w)^m \\
&\leq C \int \varphi^{m+\theta} w^{m+(q+1)/2} + C \int \varphi^{m+\theta} w^{m+1} \\
&+ \frac{4}{N} \int |f|^2 (\varphi w)^m + 2 \int \nabla u \nabla f \varphi(\varphi w)^m + C \\
&\leq C \int \varphi^{m+\theta} w^{m+\frac{q+1}{2}} + \frac{4}{N} \int |f|^2 (\varphi w)^m + 2 \int \nabla u \nabla f \varphi(\varphi w)^m + C.
\end{aligned}
\tag{3.10}
$$

The next step is to estimate the integrals in the right-hand side of (3.10) with the aid of Young's inequality in the form $ab \leq \varepsilon a^2 + b^2/(4\varepsilon)$ for $a, b > 0$ and arbitrary $\varepsilon > 0$. With regard to the integrals that do not contain f, we have

$$
\begin{aligned}
-q \int w^{(q-2)/2} \nabla u \nabla(\varphi w)(\varphi w)^m &\leq q \int w^{(q-1)/2} |\nabla(\varphi w)| (\varphi w)^m \\
&\leq \varepsilon \int |\nabla(\varphi w)|^2 (\varphi w)^{m-1} + \frac{q}{4\varepsilon} \int w^{q-1} (\varphi w)^{m+1} \\
&= \varepsilon \int |\nabla(\varphi w)|^2 (\varphi w)^{m-1} + \frac{q}{4\varepsilon} \int \varphi^{m+1} w^{m+q}
\end{aligned}
$$

and

$$\begin{aligned}
-2\int \frac{\nabla\varphi}{\varphi}\nabla(\varphi w)(\varphi w)^m &\le 2\int \frac{|\nabla\varphi|}{\varphi}|\nabla(\varphi w)|(\varphi w)^m \\
&\le C\int \varphi^{(\theta-1)/2}|\nabla(\varphi w)|(\varphi w)^m \\
&\le \varepsilon\int |\nabla(\varphi w)|^2(\varphi w)^{m-1} + \frac{C}{\varepsilon}\int \varphi^{\theta-1}(\varphi w)^{m+1} \\
&\le \varepsilon\int |\nabla(\varphi w)|^2(\varphi w)^{m-1} + \frac{C}{\varepsilon}\int \varphi^{m+\theta} w^{m+1}.
\end{aligned}$$

Next, we consider the integral involving f in the right-hand side of (3.10). Integrating by parts and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
2\int \nabla u\nabla f\varphi(\varphi w)^m &= -2\int f\,\mathrm{div}\big(\nabla u\varphi(\varphi w)^m\big) \\
&= -2\int f\,\Delta u\varphi(\varphi w)^m - 2\int f\nabla u\nabla\varphi(\varphi w)^m - 2\int f\nabla u\varphi\nabla(\varphi w)^m \\
&\le 2\int |f||\Delta u|\varphi(\varphi w)^m + 2\int |f||\nabla u||\nabla\varphi|(\varphi w)^m \\
&\quad + 2m\int |f||\nabla u|\varphi(\varphi w)^{m-1}|\nabla(\varphi w)| \\
&\le \int |D^2u|^2\varphi(\varphi w)^m + C\int |f|^2\varphi(\varphi w)^m + \int |f|^2(\varphi w)^m \\
&\quad + \int |\nabla u|^2|\nabla\varphi|^2(\varphi w)^m + \frac{m}{2}\int |\nabla(\varphi w)|^2(\varphi w)^{m-1} \\
&\quad + Cm\int |f|^2\varphi|\nabla u|^2(\varphi w)^{m-1} \\
&\le \int |D^2u|^2\varphi(\varphi w)^m + Cm\int |f|^2(\varphi w)^m + \frac{m}{2}\int |\nabla(\varphi w)|^2(\varphi w)^{m-1} \\
&\quad + C\int \varphi^{m+1+\theta} w^{m+1}.
\end{aligned}$$

Hence, plugging everything into (3.10) yields

$$\begin{aligned}
\left(\frac{m}{2} - 2\varepsilon\right)\int |\nabla(\varphi w)|^2(\varphi w)^{m-1} + \left(\frac{1}{2N} - \frac{C}{\varepsilon}\right)\int \varphi^{m+1} w^{m+q} \\
\le C\int \varphi^{m+\theta} w^{m+1} + C\int \varphi^{m+\theta} w^{m+\frac{q+1}{2}} + Cm\int |f|^2(\varphi w)^m + C.
\end{aligned}$$

Choosing and fixing large enough ε and then large m, we obtain

$$\begin{aligned}
\frac{m}{3}\int |\nabla(\varphi w)|^2(\varphi w)^{m-1} + \frac{1}{4N}\int \varphi^{m+1} w^{m+q} \\
\le C\int \varphi^{m+\theta} w^{m+(q+1)/2} + C\int |f|^2(\varphi w)^m + C.
\end{aligned} \tag{3.11}$$

Now we observe that it is possible to choose $\theta \in (0, 1)$, independent of m, such that

$$\frac{m+\theta}{m+(q+1)/2} > \frac{m+1}{m+q}.$$

With this choice of θ, we obtain

$$\varphi^{m+\theta} w^{m+(q+1)/2} \le \frac{1}{4N}\varphi^{m+1} w^{m+q} + C,$$

and hence from (3.11),

$$\frac{m}{3}\int |\nabla(\varphi w)|^2(\varphi w)^{m-1} \le C\int |f|^2(\varphi w)^m + C. \tag{3.12}$$

On the other hand, Sobolev's inequality gives

$$\begin{aligned}\frac{m}{3}\int |\nabla(\varphi w)|^2(\varphi w)^{m-1} &= \frac{4m}{3(m+1)^2}\int |\nabla(\varphi w)^{(m+1)/2}|^2 \\ &\ge C\left(\int (\varphi w)^{(m+1)N/N-2}\right)^{(N-2)/N},\end{aligned}$$

where the constant C depends also on m. Applying Hölder's inequality to the integral in (3.12) containing f yields

$$\int |f|^2(\varphi w)^m \le \left(\int (\varphi w)^{(m+1)N/(N-2)}\right)^{1/\beta}\left(\int_{B_{2R}} |f|^{2\beta'}\right)^{1/\beta'}, \tag{3.13}$$

where $\beta = \big((m+1)/m\big)\big(N/(N-2)\big)$. Since $\beta' \to N/2 < r/2$ as $m \to \infty$, we may choose m so large that $2\beta' < r$, so that the last integral in (3.13) is controlled by $|f|_{L^r(B_{2R})}$. Hence, from (3.12), we obtain

$$\left(\int (\varphi w)^{(m+1)N/(N-2)}\right)^{(N-2)/N} \le C\left(\int (\varphi w)^{(m+1)N/N-2}\right)^{1/\beta} + C,$$

which immediately yields $\int(\varphi w)^{(m+1)N/(N-2)} \le C$. Taking into account that $\varphi \equiv 1$ in B_R and the definition $w = |\nabla u|^2$, we obtain $\int_{B_R} |\nabla u|^{2(m+1)N/(N-2)} \le C$. Since m can be taken to be arbitrarily large, we obtain local bounds for $|\nabla u|$ in L^s for every $s > 1$. Finally, since $u \in C^3(B_{4R})$ is a nonnegative classical solution of $-\Delta u + u^p + |\nabla u|^q = f$ in B_{4R}, $\Delta u = h$ in B_R, where $h \in L^r_{\text{loc}}(B_{R/2})$ for some $r > N$. Inequality (3.8) then follows thanks to [9, Remark, p. 70].

Remark 2. Estimates for $|\nabla u|$ in L^q can be obtained as in [6], when $q > 1$. We take a cut-off function $\xi \in C^\infty(B_{2R})$ with $0 \le \xi \le 1$, $\xi \equiv 1$ in B_R. For $\alpha > 0$

to be chosen, we test the equation with ξ^α. Since u is nonnegative,

$$\begin{aligned}\int |\nabla u|^q \xi^\alpha &\le \int_{B_{2R}} |f| - \alpha \int \xi^{\alpha-1} \nabla u \nabla \xi \\ &\le \int_{B_{2R}} |f| + C \int \xi^{\alpha-1} |\nabla u| \\ &\le \int_{B_{2R}} |f| + C \Big(\int \xi^{(\alpha-1)q} |\nabla u|^q \Big)^{1/q},\end{aligned}$$

by Hölder's inequality. Taking $\alpha = (\alpha - 1)q$, i.e., $\alpha = q/(q-1)$, we obtain

$$\int |\nabla u|^q \xi^\alpha \le \int_{B_{2R}} |f| + C \Big(\int \xi^\alpha |\nabla u|^q \Big)^{1/q},$$

where C depends on R; in particular, this gives bounds for $|\nabla u|^q$ in $L^1(B_R)$. However, these estimates are not sharp enough to pass to the limit in the proof of Theorem 2.

Finally, we consider a particular case in which appropriate estimates for the gradient of the solutions can be obtained with only a slight improvement to the regularity of f. We restrict ourselves to the radially symmetric situation.

Theorem 8. *Assume $p > 1$ and $1 < q < N/(N-1)$. Let $f \in L^r_{\text{loc}}(\mathbb{R}^N)$ for some $r > 1$ be radially symmetric. For every $R > 0$, there exists $\delta > 0$ and a positive constant $C = C(\delta, R, |f|_{L^r(B_{2R})})$ such that*

$$\int_{B_R} |\nabla u|^{q+\delta} \le C. \tag{3.14}$$

for every radially symmetric smooth nonnegative solution of

$$-\Delta u + u^p + |\nabla u|^q = f \quad \text{in } B_{2R}.$$

Proof. Since u is radially symmetric and nonnegative,

$$-u'' - \frac{N-1}{s} u' + |u'|^q \le |f|,$$

where $s = |x|$ and throughout this proof, $'$ stands for the derivative with respect to s. Multiply this by $|u'|^\varepsilon$, where $\varepsilon > 0$ is small and to be chosen. Observe that the resulting inequality can be written as

$$-\frac{1}{1+\varepsilon} s^{-\tilde{N}+1} \big(s^{\tilde{N}-1} |u'|^\varepsilon u' \big)' + |u'|^{q+\varepsilon} \le |f| |u'|^\varepsilon, \tag{3.15}$$

where $\tilde{N} = 1 + (N-1)(1+\varepsilon) > N$. We now proceed as in Remark 2. Select a radially symmetric cut-off function $\xi \in C^\infty(B_{2R})$ such that $0 \le \xi \le 1$ and $\xi = 1$

in B_R. Multiplying (3.15) by ξ^α, where $\alpha > 0$ is to be chosen, and using Hölder's inequality, we find that

$$\begin{aligned}\frac{\alpha}{1+\varepsilon}\int_0^{2R} s^{\tilde N-1}|u'|^\varepsilon u'\xi^{\alpha-1}\xi' + \int_0^{2R} s^{\tilde N-1}|u'|^{q+\varepsilon}\xi^\alpha \\ \le \int_0^{2R} s^{\tilde N-1}|f||u'|^\varepsilon\xi^\alpha \le \int_0^{2R} s^{\tilde N-1}|f||u'|^\varepsilon \\ \le \left(\int_0^{2R} s^{\tilde N-1}|f|^{\theta'}\right)^{1/\theta'}\left(\int_0^{2R} s^{\tilde N-1}|u'|^{\varepsilon\theta}\right)^{1/\theta},\end{aligned}$$

where $\theta > 1$. Taking $\theta = q/\varepsilon$ and using the estimates obtained in Remark 2, we obtain

$$\int_0^{2R} s^{\tilde N-1}|u'|^{q+\varepsilon}\xi^\alpha \le C|f|_{L^{q/(q-\varepsilon)}(B_{2R})} + C\int_0^{2R} s^{\tilde N-1}|u'|^{\varepsilon+1}\xi^{\alpha-1}. \tag{3.16}$$

We have also used

$$\int_0^{2R} s^{\tilde N-1}|f|^{q/(q-\varepsilon)} \le C\int_0^{2R} s^{N-1}|f|^{q/(q-\varepsilon)},$$

which holds since $\tilde N > N$. Next, choose $\varepsilon \le q(r-1)/r$ and again use Hölder's inequality in (3.16) to obtain

$$\int_0^{2R} s^{\tilde N-1}|u'|^{q+\varepsilon}\xi^\alpha \le C + C\left(\int_0^{2R} s^{\tilde N-1}|u'|^{q+\varepsilon}\xi^{(\alpha-1)(q+\varepsilon)(1+\varepsilon)}\right)^{(1+\varepsilon)(q+\varepsilon)},$$

where C depends additionally on $|f|_{L^r(B_{2R})}$. Choosing $\alpha = (q+\varepsilon)/(q-1)$ and recalling that $\xi \equiv 1$ in B_R, we obtain

$$\int_0^R s^{\tilde N-1}|u'|^{q+\varepsilon} \le C. \tag{3.17}$$

Our intention is to derive (3.14) from (3.17). We first observe that if $\delta > 0$, we have, by Hölder's inequality,

$$\begin{aligned}\int_0^R s^{N-1}|u'|^{q+\delta} &= \int_0^R s^{-\nu}s^{N-1+\nu}|u'|^{q+\delta} \\ &\le \left(\int_0^{2R} s^{-\nu\gamma/(\gamma-1)}\right)^{(\gamma-1)/\gamma}\left(\int_0^R s^{(N-1+\nu)\gamma}|u'|^{(q+\delta)\gamma}\right)^{1/\gamma},\end{aligned}$$

where $\nu > 0$ and $\gamma > 1$ are to be chosen presently. Observe that if ν and γ are taken to satisfy

$$\begin{aligned}&\nu\gamma/(\gamma-1) < 1 \\ &(N-1+\nu)\gamma \ge (N-1)(1+\varepsilon) \\ &(q+\delta)\gamma \le q+\varepsilon,\end{aligned} \tag{3.18}$$

then (3.14) follows, by (3.17). Hence, choose γ such that equality holds in the second line of (3.18), i.e., $\gamma = (N-1)(1+\varepsilon)/(N-1+\nu)$. In order that $\gamma > 1$, we need to restrict ν to $\nu < \varepsilon(N-1)$. The first line in (3.18) is then equivalent to $\nu < \varepsilon(N-1)/\big(1+(N-1)(1+\varepsilon)\big)$ (which implies, in particular, that $\nu < \varepsilon(N-1)$). Thus, choose $\nu = \tau\varepsilon$, where

$$\tau < \frac{N-1}{1+(N-1)(1+\varepsilon)}. \tag{3.19}$$

Finally, the third line in (3.18) can be satisfied with small δ if $q < \varepsilon/(\gamma-1)$, i.e., if

$$q < \frac{N-1+\tau\varepsilon}{N-1-\tau}. \tag{3.20}$$

Since $1 < q < N/(N-1)$, we can choose small ε and τ so that both (3.19) and (3.20) hold. Hence (3.18) holds, and this establishes (3.14), provided that δ is small enough. □

4 Proof of the main theorems

This section is devoted to the proofs of Theorems 1, 2, 3 and 4. We consider first the proofs of existence of solutions. The proofs are based on the corresponding theorems proved in the previous section.

Proof of Theorem 1. We follow the same procedure as in [3] or [4]. Choose a sequence $\{f_n\}_{n=1}^{\infty} \subset C^{\infty}(\mathbb{R}^N)$ such that $f_n \to f$ in $L^1_{\text{loc}}(\mathbb{R}^N)$. For each nonnegative integer n, consider the problem

$$\begin{aligned} -\Delta u + |u|^{p-1}u + |\nabla u|^q &= f_n \quad \text{in } B_n \\ u &= 0 \quad \text{on } \partial B_n. \end{aligned} \tag{4.1}$$

According to Theorem 5, there exists a unique solution $u_n \in C^2(B_n) \cap C^1(\bar{B}_n)$ of (4.1).

Choose s such that $q < s < 2p/(p+1)$. Theorem 6 implies that for every $R \in (0, n/2)$, there exists a constant $C > 0$, depending on R, such that

$$\int_{B_R} |u_n|^p + |\nabla u_n|^s \le C\left(\int_{B_{2R}} |f_n| + 1\right) \le C. \tag{4.2}$$

Since $2p/(p+1)) < p$, we obtain bounds in $W^{1,s}(B_R)$ so that (after passing to a subsequence and by means of a diagonal procedure) we obtain

$$u_n \to u \quad \text{weakly in } W^{1,s}_{\text{loc}}(\mathbb{R}^N).$$

In particular, we may assume

$$\begin{aligned} u_n &\to u \quad \text{in } L^s_{\mathrm{loc}}(\mathbb{R}^N),\\ u_n &\to u \quad \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

Our intention is to prove that u is a solution of (1.2). To this end, we first verify that $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ and $\nabla u_n \to \nabla u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)^N$.

Let $h_n = f_n - |u_n|^{p-1}u_n - |\nabla u_n|^q$. The sequence $\{h_n\}$ is bounded in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ by (4.2), and $-\Delta u_n = h_n$. Take $\varepsilon > 0$ and $\xi \in C^\infty(B_{2R})$ with $0 \le \xi \le 1$ and $\xi \equiv 1$ in B_R. Define

$$\psi(s) = \begin{cases} \inf\{s, \varepsilon\}, & s \ge 0,\\ -\psi(-s), & s \le 0. \end{cases}$$

Taking $\xi\psi(u_n - u_m)$ as a test function in the weak formulation of $-\Delta(u_n - u_m) = h_n - h_m$, we obtain

$$\int_{B_R \cap A_{n,m,\varepsilon}} |\nabla(u_n - u_m)|^2 \le \varepsilon \int_{B_{2R}} (|h_n| + |h_m|) + C\varepsilon \int_{B_{2R}} (|\nabla u_n| + |\nabla u_m|) \le C\varepsilon,$$

where $A_{n,m,\varepsilon} = \{x \in \mathbb{R}^N : \ |u_n(x) - u_m(x)| \le \varepsilon\}$. Moreover,

$$\begin{aligned} \int_{B_R} |\nabla(u_n - u_m)| &= \int_{B_R \cap A_{n,m,\varepsilon}} |\nabla(u_n - u_m)| + \int_{B_R \cap A^c_{n,m,\varepsilon}} |\nabla(u_n - u_m)|\\ &\le |B_R|^{1/2} \bigg(\int_{B_R \cap A_{n,m,\varepsilon}} |\nabla(u_n - u_m)|^2 \bigg)^{1/2}\\ &\qquad + |B_R \cap A^c_{n,m,\varepsilon}|^{1/q'} \bigg(\int |\nabla(u_n - u_m)|^q \bigg)^{1/q}\\ &\le C\varepsilon^{1/2} + C|B_R \cap A^c_{n,m,\varepsilon}|^{1/q'}, \end{aligned}$$

where $A^c_{n,m,\varepsilon} = \mathbb{R}^N \setminus A_{n,m,\varepsilon}$. Since $u_n \to u$ in measure, $|B_R \cap A^c_{n,m,\varepsilon}| \to 0$; hence ∇u_n is a Cauchy sequence in $L^1_{\mathrm{loc}}(\mathbb{R}^N)^N$, and $\nabla u_n \to w$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)^N$. Of course, this gives $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ with $w = \nabla u$.

Next, recall that $|\nabla u_n|$ is bounded in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for every $s \in \big(0, 2p/(p+1)\big)$; thus, by Vitali's theorem, $\nabla u_n \to \nabla u$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)^N$ for $s \in \big(0, 2p/(p+1)\big)$, and, in particular, for $s = q$.

Finally, set $g_n = f_n - |\nabla u_n|^q$, and $g = f - |\nabla u|^q$, so that $g_n \to g$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Since $-\Delta(u_n - u_m) + |u_n|^{p-1}u_n - |u_m|^{p-1}u_m = g_n - g_m$ for arbitrary $n, m \in \mathbb{N}$, we may employ Kato's inequality (cf. [6, Appendix]) to arrive at

$$-\Delta|u_n - u_m| + \big||u_n|^{p-1}u_n - |u_m|^{p-1}u_m\big| \le |g_n - g_m| \quad \text{in } \mathbb{R}^N.$$

Multiplying this inequality by ξ, we obtain

$$\int_{B_R} \big||u_n|^{p-1}u_n - |u_m|^{p-1}u_m\big| \le \int_{B_{2R}} |g_n - g_m| + C\int_{B_{2R}} |u_n - u_m| \to 0.$$

In particular, $|u_n|^{p-1}u_n \to |u|^{p-1}u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, so that we may pass to the limit in the equation satisfied by u_n and conclude that u is a solution of (1.2). As a consequence, $u \in W^{1,s}_{\mathrm{loc}}(\mathbb{R}^N)$ for all $s \in \big(1, 2p/(p+1)\big)$ and $u \in L^p_{\mathrm{loc}}(\mathbb{R}^N)$. Finally, since $\Delta u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, we also have $u \in W^{1,s}_{\mathrm{loc}}(\mathbb{R}^N)$ for all $s \in (0, N/(N-1))$. □

Remark 3. If $f \ge 0$ and $f \in L^r_{\mathrm{loc}}(\mathbb{R}^N)$ for some $r > N$, a solution u can be constructed that also satisfies $u \in C^1(\mathbb{R}^N) \cap W^{2,r}_{\mathrm{loc}}(\mathbb{R}^N)$, provided that $0 < q \le 1$. Indeed, for a sequence $\{f_n\}$ which converges to f in $L^r_{\mathrm{loc}}(\mathbb{R}^N)$, and unique solution u_n of (4.1) (which is nonnegative), we have, by (3.6),

$$\int_{B_R} u_n^{pr} \le C\left(\int_{B_{2R}} |f_n|^r + 1\right) \le C.$$

Hence, passing to the limit, we find that $u^p \in L^r_{\mathrm{loc}}(\mathbb{R}^N)$. Thus $-\Delta u + |\nabla u|^q = h$ in $\mathbb{R}^N$ for some $h \in L^r_{\mathrm{loc}}(\mathbb{R}^N)$. We claim that this yields $u \in C^1(\mathbb{R}^N) \cap W^{2,r}_{\mathrm{loc}}(\mathbb{R}^N)$.

To verify the claim, observe that $u \in W^{1,s}_{\mathrm{loc}}(\mathbb{R}^N)$ for all $1 \le s < N/(N-1)$, since $\Delta u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Then $|\nabla u|^q \in L^{s/q}_{\mathrm{loc}}(\mathbb{R}^N)$, and so $\Delta u \in L^{\theta_1}_{\mathrm{loc}}(\mathbb{R}^N)$, where $\theta_1 = \min\{r, s/q\}$. We may assume that $\theta_1 = s/q$; otherwise $u \in W^{2,r}_{\mathrm{loc}}(\mathbb{R}^N)$ and we are done, since $W^{2,r}_{\mathrm{loc}}(\mathbb{R}^N) \subset C^1(\mathbb{R}^N)$ when $r > N$. The Sobolev Embedding Theorem implies that if $\theta_1 < N$, then $|\nabla u|^q \in L^{N\theta_1/(q(N-\theta_1))}_{\mathrm{loc}}(\mathbb{R}^N)$. Hence $\Delta u \in L^{\theta_2}_{\mathrm{loc}}(\mathbb{R}^N)$, where $\theta_2 = \min\{r, N\theta_1/(q(N-\theta_1))\}$. It is easily checked that $\theta_2 > \theta_1$. Continuing this way, we obtain an increasing sequence θ_k defined by

$$\theta_k = \min\left\{r, \frac{N\theta_{k-1}}{q(N-\theta_{k-1})}\right\},$$

provided that $\theta_{k-1} < N$, with the property that $u \in W^{2,\theta_k}_{\mathrm{loc}}(\mathbb{R}^N)$. It can be proved that there must exist k such that $\theta_k > N$. It follows that $u \in C^1(\mathbb{R}^N)$, by the Sobolev Embedding Theorem, and then that $u \in W^{2,r}_{\mathrm{loc}}(\mathbb{R}^N)$, by classical regularity.

Proof of Theorem 2. Take $f_n \in C^\infty(\mathbb{R}^N)$ such that $f_n \ge 0$ for every n and $f_n \to f$ in $L^r_{\mathrm{loc}}(\mathbb{R}^N)$. Consider again problem (4.1), which admits a unique solution $u_n \in C^2(B_n) \cap C^1(\bar{B}_n)$, by Theorem 5. Since $q > 1$, also $u_n \in C^3(B_n)$, by standard regularity.

The solution u_n is strictly positive; so we may use Theorem 7 to obtain

$$\sup_{B_R}\,(u_n + |\nabla u_n|) \le C, \tag{4.3}$$

where C depends on R and on $|f|_{L^r(B_{4R})}$. Arguing exactly as in the proof of Theorem 1, we obtain (passing to a subsequence) that $u_n \to u$ in $W^{1,s}_{\text{loc}}(\mathbb{R}^N)$ for every $s > 1$, where u is a solution of (1.2), which is nonnegative. Passing to the limit in (4.3), we also have $u \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^N)$. It then follows that $\Delta u \in L^r_{\text{loc}}(\mathbb{R}^N)$, and so $u \in W^{2,r}_{\text{loc}}(\mathbb{R}^N)$. The Sobolev Embedding Theorem then implies that $u \in C^1(\mathbb{R}^N)$. Finally, the strong maximum principle gives $u > 0$ in $\mathbb{R}^N$. □

Proof of Theorem 3. The proof is a minor variation of the existence proof in Theorem 1. We only have to choose radially symmetric functions f_n in (4.1) which forces the unique solution u_n of (4.1) to be radially symmetric. Hence Theorem 8 applies and, in particular, the application of Vitali's theorem as in Theorem 1 implies that $|\nabla u_n|^q \to |\nabla u|^q$ in $L^1_{\text{loc}}(\mathbb{R}^N)$. Hence u is a nonnegative solution of (1.2). □

Let us conclude by considering uniqueness. An important part in the proof of uniqueness for problem (1.2) is played by the minimal (classical) solution U_R of the boundary blow-up problem

$$(4.4)\qquad \begin{aligned} -\Delta U + c\,U^p - d|\nabla U|^q &= 0 \quad \text{in } B_R,\\ U &= \infty \quad \text{on } \partial B_R, \end{aligned}$$

where $c, d > 0$. The solution was shown to exist in [1, Corollary 13]. (It is unique when $0 < q \le 1$). An important property of this solution is the following.

Lemma 9. *$U_R \to 0$ uniformly on compact sets of $\mathbb{R}^N$ as $R \to \infty$.*

Proof. It is clear that for each $R > 0$, the solution U_R of (4.4) is radially symmetric and satisfies

$$\begin{aligned} -(r^{N-1}U_R')' + r^{N-1}(cU_R^p - d|U_R'|^q) &= 0 \quad \text{in } 0 < r < R.\\ U_R'(0) &= 0, \end{aligned}$$

where $' = d/dr$ and $U_R(0) = U_{0,R}$ for some $U_{0,R}$, with $0 < U_{0,R_2} \le U_{0,R_1}$ if $R_2 \le R_1$. It follows, by comparison arguments, that U_R is decreasing in R in the sense that given $z \in B_R \setminus \{0\}$, $U_{R_2}(z) < U_{R_1}(z)$ whenever $0 < R \le R_1 < R_2$. It can also be proved that U_R' is decreasing in R in an analogous sense (observe that $U_R' \ge 0$). This gives bounds on both U_R and U_R', and by standard arguments one gets $U_R \to U$ as $R \to \infty$, uniformly on compacts, where U is a (radial) solution of

$$-\Delta U + cU^p - d|\nabla U|^q = 0 \quad \text{in } \mathbb{R}^N.$$

Take $x_0 \in \mathbb{R}^N$, $R > 0$, and consider the function $V_R(x) = U_R(x - x_0)$. It is clear that V_R solves the problem

$$\begin{aligned} -\Delta V + cV^p - d|\nabla V|^q &= 0 \quad \text{in } B_R(x_0) \\ V &= \infty \quad \text{on } \partial B_R(x_0). \end{aligned}$$

Thus, by comparison, we have $U \leq V_R$ in $B_R(x_0)$ since $U < \infty$ on $\partial B_R(x_0)$. Letting R tend to ∞, we obtain $U(x) \leq U(x - x_0)$ in $\mathbb{R}^N$. Since x_0 is arbitrary, U is constant; hence $U \equiv 0$. □

Now we conclude with the proof of Theorem 4. Here, the cases $0 < q \leq 1$ and $1 < q < p$ are quite different.

Proof of Theorem 4. Assume first that $0 < q \leq 1$. Observe that by Remark 3, we have a nonnegative solution $u \in C^1(\mathbb{R}^N)$ of (1.2). Let $v \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^N)$ be another solution. The same argument as in the proof of Theorem 2 implies that $v \in C^1(\mathbb{R}^N)$.

Thus, $-\Delta(u - v) + |u|^{p-1}u - |v|^{p-1}v + |\nabla u|^q - |\nabla v|^q = 0$ in $\mathbb{R}^N$. Observe that

$$\big||\nabla u|^q - |\nabla v|^q\big| \leq \big|\nabla(u - v)\big|^q, \tag{4.5}$$

while

$$\big||u|^{p-1}u - |v|^{p-1}v\big| \geq c|u - v|^{p-1}(u - v) \tag{4.6}$$

for some positive constant c. Setting $z = u - v$, we have $-\Delta z + c|z|^{p-1}z - |\nabla z|^q \leq 0$ in $\mathbb{R}^N$.

Now, $z < U_R$ near ∂B_R, where U_R denotes the (unique) solution of (4.4) with $d = 1$. Using the comparison principle (cf., for example, [8, Lemma 2.1]) we obtain $z \leq U_R$ in B_R. Next let R tend to ∞ and use Lemma 9 to conclude that $z \leq 0$ in $\mathbb{R}^N$, i.e., $u \leq v$ in $\mathbb{R}^N$. The symmetric argument then gives $u = v$, and this shows uniqueness in the case $0 < q \leq 1$.

Now consider the case $1 < q < p$. Note that (4.5) is no longer valid. However, for fixed small $\delta > 0$, there exists a positive constant $C(\delta)$ such that

$$\big|(1 + \delta)a - b\big|^q \geq (1 + \delta)a^q - C(\delta)b^q, \quad a, b > 0.$$

Let U_R be the minimal solution of (4.4) with $d = C(\delta)$ and let c be as in (4.6).

Let $u \in C^1(\mathbb{R}^N)$ be the solution of (1.2) constructed in Theorem 2. We claim that $\bar{u} = (1+\delta)u + U_R$ is a supersolution of (1.2). Indeed,

$$\begin{aligned} -\Delta\bar{u} + \bar{u}^p + |\nabla\bar{u}|^q &= -(1+\delta)\Delta u - \Delta U_R + \big((1+\delta)u + U_R\big)^p \\ &\quad + \big|(1+\delta)\nabla u + \nabla U_R\big|^q \\ &= -(1+\delta)u^p - (1+\delta)|\nabla u|^q - cU_R^p + c(\delta)|\nabla U_R|^q + f \\ &\quad + \big((1+\delta)u + U_R\big)^p + \big|(1+\delta)\nabla u + \nabla U_R\big|^q. \end{aligned}$$

But

$$\big|(1+\delta)\nabla u + \nabla U_R\big|^q \geq \big|(1+\delta)|\nabla u| - |\nabla U_R|\big|^q \geq (1+\delta)|\nabla u|^q - C(\delta)|\nabla U_R|^q$$

and

$$\big((1+\delta)u + U_R\big)^p \geq (1+\delta)^p u^p + cU_R^p \geq (1+\delta)u^p + cU_R^p,$$

so $-\Delta\bar{u} + \bar{u}^p + |\nabla\bar{u}|^q \geq f$.

Next, observe that for every solution $v \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N)$, $\Delta v \in L^r_{\mathrm{loc}}(\mathbb{R}^N)$; hence $v \in W^{2,r}_{\mathrm{loc}}(\mathbb{R}^N)$ and, by the Sobolev Embedding Theorem, $v \in C^1(\mathbb{R}^N)$. In particular, $v < \bar{u}$ near ∂B_R, and it follows by comparison as before that

$$v \leq (1+\delta)u + U_R \quad \text{in } B_R$$

for every $R > 0$. Letting first $R \to \infty$, using Lemma 9, and then making $\delta \to 0$, we obtain $v \leq u$.

In a similar way, it can be proved that $(1-\delta)u - U_R$ is a subsolution to (1.2), and a comparison as before yields $(1-\delta)u - U_R \leq v$ in B_R. Letting first $R \to \infty$ and then $\delta \to 0$, we obtain $u = v$. □

Remark 4. If $q \geq p$, uniqueness of $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N)$ solutions does not hold. This can be seen by taking $f = 0$, where aside from the trivial solution, there are infinitely many negative radial (smooth) solutions. Indeed, set $v = -u$ and look for radial positive solutions of $-\Delta v + v^p - |\nabla v|^q = 0$ in $\mathbb{R}^N$ which solve the Cauchy problem

$$\begin{aligned} &(r^{N-1}v')' + r^{N-1}(v^p - |v'|^q) = 0, \\ &v(0) = v_0, \quad v'(0) = 0 \end{aligned}$$

for some $v_0 > 0$. Solutions of this problem are defined in an interval $[0, T)$, and from [1, Proposition 3], we know that $v' > 0$, $v'' \geq 0$; so, necessarily, $\lim_{r\to T-} v(r) = +\infty$. However, if $T < \infty$, then v is a solution of

$$\begin{aligned} \Delta v &= v^p - |\nabla v|^q && \text{in } B_T, \\ v &= \infty && \text{on } \partial B_T, \end{aligned}$$

which contradicts [1, Corollary 13], since $p \leq q$. Hence $T = \infty$, and this shows that $u = -v$ is a solution of $-\Delta u + |u|^{p-1}u + |\nabla u|^q = 0$ in $\mathbb{R}^N$.

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