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Calculus of Variations

Existence and multiplicity results for Pucci's operators involving nonlinearities with zeros

Salomón Alarcón · Leonelo Iturriaga · Alexander Quaas

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Abstract We study the problem

$$\begin{cases} -\varepsilon^2 \mathcal{M}^+_{\lambda,\Lambda}(D^2 u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N > 2$, and show it possesses nontrivial solutions for small values of ε provided f is a nonnegative continuous function which has a positive zero. The multiplicity result is based on degree theory together with a new Liouville type theorem for $-\mathcal{M}^+_{\lambda,\Lambda}(D^2 u) = f(u)$ in $\mathbb{R}^N$ for nonnegative nonlinearities with zeros.

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1 Introduction

In last years an increasing interest in the study of non-proper fully nonlinear elliptic equations has arisen. This has been motivated by the current classical viscosity solution framework that started in [11], see also [12], and combined with a very rich knowledge of more simple equations that, for example, involving the Laplacian. In particular, existence of viscosity solutions to fully nonlinear elliptic equations has been extensively investigated in last 20 years

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S. Alarcón (✉) · L. Iturriaga · A. Quaas
Departamento de Matemática, Universidad Técnica Federico Santa María, Avenida España 1680, Casilla 110-V, Valparaíso, Chile
e-mail: salomon.alarcon@usm.cl

L. Iturriaga
e-mail: leonelo.iturriaga@usm.cl

A. Quaas
e-mail: alexander.quaas@usm.cl

for proper equations, most of them through adaptations of Perron's method, see for instance [8,12].

In the context of non-proper fully nonlinear equations, most results have been obtained through Leray-Schauder degree theory, see [14,5,30,10,9,17,1]. One of the crucial points to use degree theory is to establish a priori bounds. The most classical method for obtaining a priori bounds is to use the blow-up method introduced by Gidas and Spruck, see [21], and then a Liouville type theorem in all space or in a half space. In this study, we use this kind of arguments together with a truncation procedure in order to obtain an unstable solution whereas a stable solution is obtained by the sub- and supersolutions method. Moreover, here we prove that the stable solution is isolated, from where we deduce a multiplicity result.

Before continuing, we mention that to the best of our knowledge, this article is a first effort towards the study of singular perturbed equations in a fully nonlinear setting, and in the obtaining some asymptotic behavior of the solution. We recall that early works on singular perturbed problems are due to Ni and Takagi [27,28], and since then this topic has become one of the most active research fields in partial differential equations. We also emphasize that most of the results in singular perturbations use strongly the variational structure of the equation, which is not present in our context.

Specifically, this article deals with positive viscosity solutions to the problem

$$(P_\varepsilon)\begin{cases} -\varepsilon^2\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = f(x,u) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $N > 2$, $\varepsilon \neq 0$ is a parameter and f satisfies $f(x,0) = f(x,1) = 0$, $f(x,t) > 0$ for any $t \notin \{0;1\}$. Here, for fixed $0 < \lambda \leq \Lambda$, the Pucci's extremal operators are defined, as in [6], by:

$$\mathcal{M}^-_{\lambda,\Lambda}(A) = \lambda\sum_{e_i>0} e_i + \Lambda\sum_{e_i<0} e_i \quad \text{and} \quad \mathcal{M}^+_{\lambda,\Lambda}(A) = \Lambda\sum_{e_i>0} e_i + \lambda\sum_{e_i<0} e_i,$$

where A is a matrix of order N that belongs to $\mathcal{S}_N$, the space of all real symmetric $N\times N$ matrices, and $e_i = e_i(A)$, $i = 1, 2, \ldots, N$, are the eigenvalues of the matrix A. We remark that all our results here obtained are valid for the equation (P_ε), if we replace $\mathcal{M}^+_{\lambda,\Lambda}$ by $\mathcal{M}^-_{\lambda,\Lambda}$, but in order to simplify our presentation, in this study we only consider the operator $\mathcal{M}^+_{\lambda,\Lambda}$. Notice that Pucci defined this kind operators in the sixties, see [29]. Let us recall, from [6], the following definitions

Definition 1.1 Let Θ be a open subset of $\mathbb{R}^N$. We say that a continuous function u in Θ is a viscosity subsolution (resp. viscosity supersolution) of (P_ε) in Θ, when the following condition holds: if $x_0 \in \Theta$, $\varphi \in \mathcal{C}^2(\Theta)$ and $u - \varphi$ has a local maximum at x_0 then

$$-\varepsilon^2\mathcal{M}^+_{\lambda,\Lambda}(D^2\varphi)(x_0) \leq f(x_0, u(x_0))$$

(resp. if $u - \varphi$ has a local minimum at x_0 then $-\varepsilon^2\mathcal{M}^+_{\lambda,\Lambda}(D^2\varphi)(x_0) \geq f(x_0, u(x_0))$). We say that u is a viscosity solution of (P_ε) when it is subsolution and supersolution.

We say that $-\varepsilon^2\mathcal{M}^+_{\lambda,\Lambda}(D^2u)(x) \leq$ (resp. $\geq, =$) $f(x,u(x))$ in the viscosity sense in Θ whenever u is a subsolution (resp. supersolution, solution) of (P_ε) in Θ.

For the Laplacian operator, that is the case when $\lambda = \Lambda = 1$, problems with a nonnegative nonlinearity having a zero at a positive value were first considered in [25], and there was proved the existence of two solutions through topological degree arguments in the subcritical case. Existence and behavior of a solution below the zero of f has been studied in many works, see for instance [19] and references therein, and it can be proved that this solution converges to 1 (the positive zero of the nonlinearity) as $\varepsilon \to 0$. Existence of a solution whose maximum is above 1 is more delicate and usually requires some hypotheses on the growth of f at infinity. In [22], was considered the p-Laplacian operator and allowed to f depended on the x variable, but only in the subcritical case, then two positive solutions were obtained to the asymptotical problem at the origin for ε small, and was proved that both solutions converge at least pointwise to 1 as $\varepsilon \to 0$. This behavior suggests that also for our problem, truncation procedures like those in [7,26,24] could be used in order to prove the existence of two solutions when are considered critical or supercritical nonlinearities. However, the pointwise convergence is not enough to guarantee a suitable control on the $L^\infty-$norm of the solutions. Assuming that Ω is convex and f is independent on the x variable, for the $p-$Laplacian operator in [23] was proved the existence of two positive solutions for small values of ε without imposing conditions on f at infinity.

In this study, without imposing convexity on the domain Ω and under some local hypotheses on f, we will show the existence of at least two positive solutions for ε small, even in the case when f depends on the x variable, but without restrictions on the growth of the nonlinearity at infinity. In this way, our result generalizes the preceding ones.

In order to state our first result, we introduce the following critical exponent:

$$p^+ := \frac{\tilde{N}}{\tilde{N}-2} \quad \text{with } \tilde{N} = \frac{\lambda}{\Lambda}(N-1)+1.$$

In the case $\lambda = \Lambda = 1$, this exponent corresponds to the result of [20], where non existence of positive solution holds for

$$\Delta u + u^p \leq 0,$$

if $p \leq p^+$. An extension of these results to Pucci's operators was done in [11], and a generalize to more general operators, in [4,15,16,2]. Here, in the proof of the Liouville type theorem (see Theorem 1.2 below), we use some ideas given in [16] related with a generalization of the arguments in [11].

Now, we are in position to state precisely our assumptions on f:

(F_1) $f : \overline{\Omega} \times [0, +\infty) \to [0, +\infty)$ is a continuous function and $f(x, \cdot)$ is locally Lipschitz in $(0, \infty)$ for all $x \in \overline{\Omega}$, $f(x, 0) = f(x, 1) = 0$ and $f(x, t) > 0$ for $t \notin \{0; 1\}$.

(F_2) $\liminf_{t\to 0^+} \dfrac{f(x,t)}{t} = 1$ uniformly for $x \in \overline{\Omega}$.

(F_3) There exist a continuous function $a : \overline{\Omega} \to (0, \infty)$ and $\sigma \in (1, \tilde{N}/(\tilde{N}-2))$ such that

$$\lim_{t\to 1} \frac{f(x,t)}{|t-1|^\sigma} = a(x).$$

(F_4) There exist $k > 0$ and $T > 1$ such that the map $t \mapsto f(x, t) + kt$ is increasing for $t \in [0, T]$ and $x \in \Omega$.

A simple model of a function that verifies our assumptions is $f(t) = t|1-t|^\sigma e^t$, while a more complex model is $f(x, t) = (|x|t^p + \log(t+1))e^t|1-t|^\sigma$, with $p > 1$.

Our result is the following

Theorem 1.1 *Assume that Ω is a bounded smooth domain. Then, under the hypotheses (F_1) through (F_4), there exists $\varepsilon^* > 0$ such that the problem (P_ε) has at least two viscosity positive solutions $u_{1,\varepsilon}, u_{2,\varepsilon}$, for $0 < \varepsilon < \varepsilon^*$.*

Moreover, these solutions satisfy $\|u_{1,\varepsilon}\|_\infty \to 1^-$ and $\|u_{2,\varepsilon}\|_\infty \to 1^+$, as $\varepsilon \to 0$.

An important tool used in the proof of the preceding results is the following Liouville type theorem for a nonnegative function with zeros.

Theorem 1.2 *Let* $f : [0, \infty) \to [0, \infty)$ *be a continuous function satisfying the following three assumptions:*

(f_1) $f(t) = 0$ *if* $t = 0$ *or* $t = 1$, *and* $f(t) > 0$ *if* $t \neq 1$, $t > 0$.
(f_2) *There exist constants* $\gamma > 0$ *and* $\sigma \in (1, \tilde{N}/(\tilde{N} - 2))$ *such that* $f(t) \geq \gamma(t-1)^{\sigma}$, *for* $t > 1$.
(f_3) *There exists a constant* $\overline{b} > 0$ *such that* $\liminf_{t \longrightarrow 0^+} \frac{f(t)}{t} \geq \overline{b}$.

Then any bounded solution of the problem

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2 w) \geq f(w) \text{ in } \mathbb{R}^N, \\ w \geq 0, \end{cases} \tag{1.1}$$

is either the constant function $w \equiv 0$, *or else* $w \equiv 1$.

We strongly believe that our method can be extended to different operators for which this sort of Liouville type theorem holds, some already quoted above, and such that a Gidas-Spruck type theorem also has been obtained.

Remark 1.1 As our main motivation is dealing with supercritical models at infinity, two key step in our proof are: truncation arguments and a Liouville type result, which allow us to show that the solutions of the truncate problem are effectively the solutions of the original problem for ε small enough. However, note that little modifications in the proofs of our theorems above we allow us to obtain similar results in the case that f growth like t^p at infinity, with $p \in (1, \tilde{N}/(\tilde{N} - 2))$, but without need of using truncation arguments and some Liouville type result, see Propositions 3.3 and 3.4. A simple model in this situation is $f(t) = t^q|1 - t|^{\sigma}, 0 < q \leq 1$.

2 Nonlinear Liouville type theorem

Proposition 2.1 *Let* w *be a viscosity solution of the inequation*

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2 w) \geq f(w) \;\; in \;\; \mathbb{R}^N,$$

where f *is a continuous nonnegative function. Then either* $\inf_{\mathbb{R}^N} w = -\infty$, *or* $\inf_{\mathbb{R}^N} w$ *is a zero of* f.

Proof Let $U(r) = \inf_{|x|=r} w(x)$. Suppose by contradiction that $\inf_{\mathbb{R}^N} w = M \in \mathbb{R}$ with $f(M) > 0$. Let $M_0(r) = \inf_{|x| \leq r} w(x)$. Then we must have that $M_0(r) \to M^+$ as $r \to +\infty$. By the continuity of f, for a suitably large r_0 and some $\alpha > 0$, one has $f(w) \geq \alpha > 0$ provided $M \leq w \leq M_0(r_0)$.

We claim that $U(r)$ is strictly decreasing for $r > r_0$. Indeed, if not, should be r_1 and r_2 satisfying $r_0 < r_1 < r_2$ such that $U(r_1) \leq U(r_2)$, that is, w should have a minimum in $\{x : |x| < r_2\}$. Since in this case w satisfies

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2 w) \geq 0 \text{ in } B_{r_2}, \\ w \geq U(r_2) \qquad \text{on } \partial B_{r_2}, \end{cases}$$

it results by the Strong Maximum Principle for viscosity solutions (see [6, Proposition 4.9]) that either $w > U(r_2)$ in B_{r_2} or $w \equiv U(r_2)$ in B_{r_2}; the first possibility contradicts $U(r_1) \leq$

$U(r_2)$; however, by the definition of M_0, one has that if $w \equiv U(r_2)$ in B_{r_2} then $M_0(r_0) = M_0(r_2) = U(r_2)$, and as a consequence $-\mathcal{M}^+_{\lambda,\Lambda}(D^2 w) \geq \alpha > 0$ in B_{r_2}, which is impossible for a constant function: this proves the claim that $U(r)$ is strictly decreasing for $r > r_0$.

Now let

$$v(x) = \frac{1}{2\lambda N}|x|^2,$$

that is a radial solution of

$$\begin{cases} -\mathcal{M}^-_{\lambda,\Lambda}(D^2 v) = -1 \text{ in } \mathbb{R}^N, \\ v \geq 0,\ v(0) = 0. \end{cases}$$

Consider $W = w + \delta v$, with $\delta > 0$. Since $\lim_{|x| \longrightarrow \infty} v(x) = +\infty$, then W has a global minimum. Since $U(r)$ is strictly decreasing, we may choose δ sufficiently small so that this minimum lies at some point x_0, with $|x_0| > r_0$ and $w(x_0) < M_0(r_0)$. Hence, $f(w(x_0)) \geq \alpha > 0$. Now, choosing $\delta < \alpha/2$, from the definition of a viscosity supersolution, taking $-\delta v$ as a test function we obtain

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2(-\delta v(x_0))) = \delta \mathcal{M}^-(D^2 v(x_0)) = \delta \geq f(w(x_0)) \geq \alpha.$$

But this is a contradiction because $\delta < \alpha/2$. This completes the proof. □

Proof of Theorem 1.2. Let u be a bounded solution of (1.1). By Proposition 2.1 we have that the minimum of u must be a zero of f.

We initially suppose that $\inf u = 0$ and let $\eta : [0, +\infty) \to \mathbb{R}$ such that $0 \leq \eta(r) \leq 1$, $\eta \in \mathcal{C}^\infty$, η nonincreasing, $\eta(r) = 1$ if $0 \leq r \leq 1/2$ and $\eta(r) = 0$ if $r \geq 1$. It is obvious that there exists $C > 0$ such that

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2 \eta(|x|)) \leq C.$$

Define now $\xi(x) = m(R/2)\eta(|x|/R)$, where $m(r) := \min_{|x| \leq r} u(x)$. Then by scaling property of $\mathcal{M}^+_{\lambda,\Lambda}$ we have

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2 \xi(x)) \leq \frac{Cm(R/2)}{R^2}.$$

In addition, $\xi(x) = 0 \leq u(x)$ if $|x| > R$ and $\xi(x) = m(R/2) \leq u(x)$ if $|x| \leq R/2$. Thus there exists a global minimum of $u(x) - \xi(x)$ achieved in a point x_R with $|x_R| < R$. Note that $u(x_R) - \xi(x_R) \leq 0$ and so $u(x_R) \leq \xi(x_R) \leq m(R/2)$.

By using a similar device as those in [13], if we define $\varphi(x) := \xi(x) - \xi(x_R) + u(x_R)$ we obtain that φ is a test function for u at x_R and thus

$$f(u(x_R)) \leq -\mathcal{M}^+_{\lambda,\Lambda}(D^2 \varphi(x_R)) = -\mathcal{M}^+_{\lambda,\Lambda}(D^2 \xi(x_R)) \leq \frac{Cm(R/2)}{R^2}.$$

So, since $0 < u(x_R) \leq m(R/2)$, by (f_1) and (f_3) there exists $R_0 > 0$ large enough so that

$$\frac{\overline{b}}{2} u(x_R) < f(u(x_R)) \leq \frac{Cm(R/2)}{R^2},$$

for any $R > R_0$. This implies that

$$\frac{\overline{b}}{2} m(R) \leq \frac{Cm(R/2)}{R^2} < \frac{cm(R)}{R^2},$$

which is impossible if $m(R) \neq 0$ for all $R > R_0$. Then there is some $R > 0$ sufficiently large so that $m(R) = 0$. Hence, since u is a viscosity supersolution of $-\mathcal{M}^+_{\lambda,\Lambda}(D^2 u) \geq 0$ in

$B_T(0)$ by applying the Strong Maximum Principle in [6], we have that $u \equiv 0$ in $B_T(0)$ for any $T > R$, *i.e.* $u \equiv 0$ in $\mathbb{R}^N$.

Finally, in the case that $\inf u = 1$ in $\mathbb{R}^N$, setting $u_0 = u - 1$ we see that u_0 is a viscosity solution of

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2 u_0) \geq f(u_0 + 1) \text{ in } \mathbb{R}^N, \\ \quad u_0 \geq 0, \end{cases}$$

which satisfies $m_{u_0}(R) := \min_{|x| \leq R} u_0(x) \to 0$ as $R \to \infty$. Arguing as in the previous case and using (f_2) we obtain that

$$\gamma u_0(x_R)^\sigma \leq f(u_0(x_R) + 1) \leq \frac{C m_{u_0}(R)}{R^2},$$

which implies

$$m_{u_0}(R) R^{\frac{2}{\sigma-1}} \leq c,$$

i.e.

$$m_{u_0}(R) R^{\tilde{N}-2} \leq c R^{\tilde{N}-2-\frac{2}{\sigma-1}}.$$

But inequality above is a contradiction, because $m_{u_0}(R) R^{\tilde{N}-2}$ is increasing (see [13, Corollary 3.1]). □

3 Proof of Theorem 1.1

Note that by hypothesis (F_3) there exist $R > 1$ and $\gamma_1 > 0$ such that $f(x,t) \geq \gamma_1 |t-1|^\sigma$ for $t \in [1, R]$. Without loss of generality we may assume that $R \leq T$ from hypothesis (F_4). Then we truncate f as follows

$$f_R(x,t) = \begin{cases} f(x, t^+), \ t \leq R, \\ \frac{f(x,R)}{R^\sigma} t^\sigma, \ t \geq R, \end{cases}$$

where $t^+ = \max\{0, t\}$. Also, without loss of generality, we may assume that

$$\liminf_{t \to 0+} \frac{f_R(x,t)}{t} \geq 1 \text{ uniformly for } x \in \overline{\Omega}.$$

With this definition, f_R has a power growth at infinity with exponent less than $\tilde{N}/(\tilde{N}-2)$ and also satisfies the following properties:

$$f_R(x,t) \geq \gamma_2 |t-1|^\sigma \quad \text{for } t \geq 1$$

if $\gamma_2 = \min\left\{\gamma_1, \inf_{x \in \Omega} \frac{f(x,R)}{R^\sigma}\right\} > 0$, and

$$\text{the map } t \mapsto f_R(x,t) + kt \text{ is increasing for } t \in [0, +\infty], \tag{3.1}$$

where k is as in hypothesis (F_4).

We consider then the auxiliary problem

$$(Q_{\varepsilon,\tau}) \qquad \begin{cases} -\varepsilon^2 \mathcal{M}^+_{\lambda,\Lambda}(D^2 u) = f_R(x,u) + \varepsilon^2 \tau u^+ & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where τ is a nonnegative parameter.

We remark that since $f_R \geq 0$, by the Strong Maximum Principle in [6] one has the non-trivial solutions of the problem $(Q_{\varepsilon,\tau})$ are positive. Moreover, considering from now on μ_1^+ as the first positive eigenvalue of $-\mathcal{M}^+_{\lambda,\Lambda}$ in Ω (see [5, Proposition 1.1]), one gets that the problem $(Q_{\varepsilon,\tau})$ has no positive solution for $\tau > \mu_1^+$.

Our first step will be to derive some a priori estimates for solutions of $(Q_{\varepsilon,\tau})$.

Lemma 3.1 *Under hypotheses* (F_1) *and* (F_2), *we have given* $\widetilde{\varepsilon} > 0$, *there exists a constant* $D_{\widetilde{\varepsilon}}$ *such that, if* u *is a viscosity solution of problem* $(Q_{\varepsilon,\tau})$ *with* $0 < \varepsilon < \widetilde{\varepsilon}$ *and* $\tau \geq 0$ *then*

$$\|u\|_\infty \leq D_{\widetilde{\varepsilon}};$$

and therefore there is a positive constant C_ε *such that*

$$\|u\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})} \leq C_\varepsilon.$$

Proof Suppose, for sake of contradiction, that there exists a sequence $\{(u_n, \varepsilon_n, \tau_n)\}_{n\in\mathbb{N}}$ with u_n being a positive viscosity solution of $(Q_{\varepsilon_n,\tau_n})$, such that $S_n := \max_{\overline{\Omega}} u_n = u_n(x_n) \to \infty$ as $n \to \infty$, where $\{x_n\} \subset \Omega$ is a sequence of points where the maximum is attained. We remark that since we are not supposing $\tau > 0$ at this point, this sequence may not be bounded away from the boundary.

Let now $\delta_n = \text{dist}(x_n, \partial\Omega)$ and define $w_n(y) = S_n^{-1} u_n(A_n y + x_n)$, where A_n will be fixed later. Hence w_n satisfies

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2 w_n(y)) = \frac{A_n^2}{\varepsilon_n^2 S_n} f_R(A_n y + x_n, S_n w_n(y)) + \tau_n A_n^2 w_n(y) \text{ in } \Omega_n = A_n^{-1}(\Omega - x_n) \tag{3.2}$$

and $w_n(0) = \max w_n = 1$.

We choose $A_n^2 = \varepsilon_n^2 S_n^{1-\sigma} f(x_n, R)^{-1} R^\sigma$. Since $S_n \to \infty, 0 < \varepsilon_n < \widetilde{\varepsilon}$ and $\tau_n \leq \mu_1^+$ (because no positive solution of $(Q_{\varepsilon,\tau})$ exists for $\tau > \mu_1^+$), we conclude that $A_n \to 0$ and $\tau_n A_n^2 \to 0$. Then, the right hand side of (3.2) becomes

$$\frac{R^\sigma f_R(A_n y + x_n, S_n w_n)}{f(x_n, R) S_n^\sigma} + o(1),$$

and by the continuity of f and the definition of f_R, it is bounded. By regularity results (see [32] for instance), we have that, up to subsequence, $w_n \to w$ in compact sets of $\mathbb{R}^N$ or $\mathbb{R}^N_+$, according to whether the limit of δ_n/A_n is infinity or not, that is, Ω_n tends to $\mathbb{R}^N$ or to a half space.

Finally, taking the limit in (3.2), we have that w satisfies, in the viscosity sense, either

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 w) + w^\sigma = 0 \text{ in } \mathbb{R}^N$$

or

$$\begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2 w) + w^\sigma = 0 \text{ in } \mathbb{R}^N, \\ w = 0 \qquad\qquad\qquad\qquad \text{on } \partial\mathbb{R}^N_+. \end{cases}$$

But this contradicts the Liouville type theorem in [13, Theorem 4.1] in the case of $\mathbb{R}^N$ and the corresponding in [30, Theorem 1.5] in the case of the half space, because

$$1 < \sigma < \frac{\widetilde{N}}{\widetilde{N} - 2} < \frac{\widetilde{(N-1)}}{\widetilde{(N-1)} - 2} = \frac{\widetilde{N} - \frac{\Lambda}{\lambda}}{\widetilde{N} - 2 - \frac{\Lambda}{\lambda}}. \tag{3.3}$$

This contradiction proves that $\|u\|_\infty \leq C$ for any solution of problem $(Q_{\varepsilon,\tau})$ with $\varepsilon < \widetilde{\varepsilon}$ and $\tau \geq 0$. Finally using the $\mathcal{C}^{1,\alpha}$ estimates we obtain that there is a positive constant C_ε such that

$$\|u\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})} \leq C_\varepsilon,$$

for some $\alpha \in (0, 1)$ (see [32]). □

Now we look for a family of supersolutions of $(Q_{\varepsilon,\tau})$. For this purpose, we consider to the function ψ being the viscosity solution of

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2\psi) = 1 & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases}$$

and $A := \|\psi\|_\infty$. For the existence of such ψ see, for example, [31].

Lemma 3.2 *Under hypothesis (F_3), for any $\varepsilon > 0$ there exist $\tau^*_\varepsilon, \delta_\varepsilon > 0$ such that $v_\xi = 1+\xi+\frac{\delta_\varepsilon}{4A}\psi$ is a supersolution for $(Q_{\varepsilon,\tau})$ for any $\xi \in [-\delta_\varepsilon, \delta_\varepsilon/2]$, and $\tau \in [0, \tau^*_\varepsilon)$. Moreover, we may choose δ_ε as a nonincreasing function of ε^{-1}.*

Proof Fixed $\varepsilon > 0$, by the hypothesis (F_3) we have that

$$\lim_{t\to 1} \frac{f_R(x,t)}{\varepsilon^2|t-1|} = 0,$$

and then there exists $\delta > 0$ such that $\varepsilon^{-2} f_R(x,t) < \frac{|t-1|}{8A} < \frac{\delta}{8A}$ for $|t-1| \leq \delta$ and any $x \in \Omega$. Since this estimate still holds for lower values of ε, we deduce that δ may be chosen as a nonincreasing function of ε^{-1}.

If $\tau^* > 0$ is such that $\tau u < \frac{\delta}{8A}$ for $\tau \in [0, \tau^*),\ u \in (0, 1+\delta]$, then

$$\varepsilon^{-2} f_R(x,u) + \tau u < \frac{\delta}{4A} \quad \text{for } \tau \in [0,\tau^*),\ u \in [1-\delta, 1+\delta] \quad \text{and} \quad x \in \Omega.$$

If we define $v_\xi = 1+\xi+\frac{\delta}{4A}\psi$, we have that $v_\xi \in [1-\delta, 1+\delta]$ provided $\tau \in [0,\tau^*)$, $\xi \in [-\delta, \delta/2]$ and then

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2 v_\xi) = \frac{\delta}{4A} > \varepsilon^{-2} f_R(x, v_\xi) + \tau v_\xi,$$

which proves that v_ξ is a supersolution. □

Now we prove the existence of a first solution for $(Q_{\varepsilon,\tau})$ via the sub- and supersolutions method. For this we need hypothesis (F_4). For simplicity, for μ^+_1, the first eigenvalue of $-\mathcal{M}^+_{\lambda,\Lambda}$ in Ω, we denote from now on by φ^+_1 its first eigenfunction associated, which is positive in Ω, see [5, Proposition 1.1] or [30]. Without loss of generality, we can consider $\|\varphi^+_1\|_\infty = 1$.

Proposition 3.3 *If hypotheses $(F_1)-(F_4)$ hold, then problem $(Q_{\varepsilon,\tau})$ has a positive solution $u_{1,\varepsilon,\tau} < 1$ for $0 < \varepsilon < (1/\mu^+_1)^{1/2}$ and $0 \leq \tau < \tau^*_\varepsilon$. Moreover, the following property holds: given $0 < \overline{\varepsilon} < (1/\mu^+_1)^{1/2}$ there exists $\rho > 0$ such that $\rho\varphi^+_1 \leq u_{1,\varepsilon,\tau} < 1$ for any $0 < \varepsilon < \overline{\varepsilon}$ and $\tau \in [0, \tau^*_\varepsilon)$.*

Proof In a standard way, using (F_2), we may find a $\rho > 0$ (as small as desired) such that $f_R(x,t) > \varepsilon^2\mu^+_1 t$ for any $t \in (0,\rho)$, and $0 < \varepsilon < \overline{\varepsilon} < (1/\mu^+_1)^{1/2}$; then $\rho\varphi^+_1$ is a subsolution to problem $(Q_{\varepsilon,\tau})$ for any $\tau \geq 0$ and $0 < \varepsilon < \overline{\varepsilon}$.

For $\tau \in [0, \tau_\varepsilon^*)$, we have the supersolution $v_{-\delta_\varepsilon} < 1$ from Lemma 3.2; since δ_ε is not increasing in ε^{-1}, we may choose ρ such that $\rho\varphi_1^+ < v_{-\delta_\varepsilon/2}$ for any $0 < \varepsilon < \overline{\varepsilon}$. Denote now by X the Banach space of $\mathcal{C}^{1,\alpha}$-functions on $\overline{\Omega}$ which are 0 on $\partial\Omega$, endowed with the usual $\mathcal{C}^{1,\alpha}$-norm. Also, we will write $u \ll v$ to say that $u < v$ in Ω and $\frac{\partial u}{\partial \nu} > \frac{\partial v}{\partial \nu}$ on $\partial\Omega$, where ν denotes the unitary outward normal vector to $\partial\Omega$. Let k be as in 3.1 and $K_\tau : X \to X$ be defined as follows: $K_\tau v = u$, where u is the unique solution of the Dirichlet problem

$$\begin{cases} -\varepsilon^2 \mathcal{M}^+_{\lambda,\Lambda}(D^2 u) + ku = f_R(x, v) + (k + \varepsilon^2\tau)v & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

the mapping K_τ so defined is compact (see [17] for instance). Setting $D := \{u \in X : \rho\varphi_1^+ \le u \le v_{-\delta_\varepsilon/2}\}$, we have that $K_\tau : D \to D$ is an increasing map. Thus, using the monotone iteration method (see [3] for instance), we obtain a solution $u_{1,\varepsilon,\tau}$ with the claimed properties. □

Now, we work with $\tau > 0$ and we show that a second solution exists: we will apply a topological degree argument, adapting a result obtained by de Figueiredo and Lions in [18], see also [23] for the p-Laplacian case.

Proposition 3.4 *Assume hypotheses as Proposition 3.3. If* $0 < \varepsilon < (1/\mu_1^+)^{1/2}$ *and* $\tau_0 \in (0, \tau_\varepsilon^*)$, *then* (Q_{ε,τ_0}) *has a second positive solution* u_{2,ε,τ_0}. *Moreover* $\left\|u_{2,\varepsilon,\tau_0}\right\|_\infty > 1$.

Proof Let us fix $0 < \varepsilon < (1/\mu_1^+)^{1/2}$ and arguing as in the proof of Proposition 3.3, we consider the bounded open set

$$\mathcal{O} = \left\{u \in X : \|u\|_X < C_\varepsilon + B_\varepsilon + 1, u \gg \rho\varphi_1^+\right\},$$

where $C_\varepsilon,\ B_\varepsilon > 0$ will be chosen below (see in (3.4) and (3.6), respectively) and $\rho > 0$ is as in the proof of Proposition 3.3, so that $\rho\varphi_1^+ < 1$ and it is a strict subsolution for all problems $(Q_{\varepsilon,\tau})$, $\tau \ge 0$ (in particular $\varepsilon^2\mu_1^+\rho\varphi_1^+ < f_R(x, \rho\varphi_1^+)$).

We need that $0 \notin (I - K_\tau)(\partial\mathcal{O})$ (*i.e.* no solution of $(Q_{\varepsilon,\tau})$ lies on $\partial\mathcal{O}$), so that the degree $\deg(I - K_\tau, \mathcal{O}, 0)$ will be well defined and independent on τ. To obtain this we consider C_ε as in Lemma 3.1, so that

$$\|u\|_X \le C_\varepsilon \tag{3.4}$$

for all possible solutions of $(Q_{\varepsilon,\tau})$ with $\tau \ge 0$.

Then, we claim that any solution u of $(Q_{\varepsilon,\tau})$ such that $u \ge \rho\varphi_1^+$ in Ω satisfies $u \gg \rho\varphi_1^+$ (and then it is not on $\partial\mathcal{O}$). Actually, we have

$$\begin{cases} -\varepsilon^2 \mathcal{M}^+_{\lambda,\Lambda}(D^2 u) + k\,u = f_R(x, u) + (k + \varepsilon^2\tau)u & \text{in } \Omega, \\ -\varepsilon^2 \mathcal{M}^+_{\lambda,\Lambda}(D^2(\rho\varphi_1^+)) + k\rho\varphi_1^+ \le \varepsilon^2\mu_1^+\rho\varphi_1^+ + (k + \varepsilon^2\tau)\rho\varphi_1^+ & \text{in } \Omega, \end{cases} \tag{3.5}$$

by hypothesis (F_4), and since $u \ge \rho\varphi_1^+$, we have

$$f_R(x, u) + (k + \varepsilon^2\tau)u \ge f_R(x, \rho\varphi_1^+) + (k + \varepsilon^2\tau)\rho\varphi_1^+,$$

and then a strict inequality holds between the (continuous) right hand sides of 3.5. Thus, the claim is proved.

By the above computations, we obtain that

$$\deg(I - K_\tau, \mathcal{O}, 0) = 0 \quad \text{for any } \tau > 0,$$

since $(Q_{\varepsilon,\tau})$ has no solutions for $\tau > \mu_1^+$.

At this point we fix $\tau = \tau_0$, we consider the supersolution $\phi := v_{\xi=0} > 1$ from Lemma 3.2, and we assume that no solution of (Q_{ε,τ_0}) touches it, otherwise such a solution would satisfy the claim and the proposition would be true. Using the $\mathcal{C}^{1,\alpha}$ estimate in [32] we obtain that we may choose the constant $B_\varepsilon > 0$ such that

$$\|K_\tau v\|_X \leq B_\varepsilon, \qquad \forall\, v \in X : 0 \leq v \leq \phi; \tag{3.6}$$

we consider the open subset of $\mathcal{O}$

$$\mathcal{O}' = \{u \in \mathcal{O} :\ u < \phi \text{ in } \Omega\}$$

and we claim that $\deg(I - K_{\tau_0}, \mathcal{O}', 0) = 1$.

Observe that K_{τ_0} maps $\overline{\mathcal{O}'}$ into $\overline{\mathcal{O}'}$. Indeed, if $v \in \overline{\mathcal{O}'}$, then $\left\|K_{\tau_0} v\right\|_X \leq B_\varepsilon$ by 3.6, and if we consider $u = K_{\tau_0} v$ we have

$$\begin{cases} -\varepsilon^2 \mathcal{M}^+_{\lambda,\Lambda}(D^2\phi) + k\,\phi \geq f_R(x,\phi) + (k + \varepsilon^2\tau_0)\phi, \\ -\varepsilon^2 \mathcal{M}^+_{\lambda,\Lambda}(D^2 u) + u = f_R(x,v) + (k + \varepsilon^2\tau_0)v, \\ -\varepsilon^2 \mathcal{M}^+_{\lambda,\Lambda}(D^2(\rho\varphi_1^+)) + k\,\rho\varphi_1^+ \leq \varepsilon^2\mu_1^+\rho\varphi_1^+ + (k + \varepsilon^2\tau_0)\rho\varphi_1^+, \end{cases}$$

then, since $\rho\varphi_1^+ \leq v \leq \phi$, the comparison principle implies that $\rho\varphi_1^+ \leq K_{\tau_0} v \leq \phi$.

Now, let $u_0 \in \mathcal{O}'$ and consider the constant mapping $C : \overline{\mathcal{O}'} \to \overline{\mathcal{O}'}$ defined by $C(u) = u_0$. So, one obtains that $I - \mu K_{\tau_0}(v) - (1-\mu)u_0$, $\mu \in [0,1]$, is a homotopy between $I - K_{\tau_0}$ and $I - C$ in $\overline{\mathcal{O}'}$ without zeros on $\partial\mathcal{O}'$: in fact, if $v \in \partial\mathcal{O}'$ then (since $\mathcal{O}'$ is convex) $\mu K_{\tau_0}(v) + (1-\mu)u_0 \in \mathcal{O}'$ for $\mu \neq 1$, and then it is different from v, while for $\mu = 1$ we have $v \neq K_{\tau_0}(v)$ since we are assuming that no solution touches ϕ.

Hence $\deg(I - K_{\tau_0}, \mathcal{O}', 0) = \deg(I - C, \mathcal{O}', 0) = 1$, as we claimed.

Then, applying the excision property, it follows that $\deg(I - K_{\tau_0}, \mathcal{O} \setminus \overline{\mathcal{O}'}, 0) = -1$, so (Q_{ε,τ_0}) has a solution $u_2 \in \mathcal{O} \setminus \overline{\mathcal{O}'}$; in particular, $u_2(x_0) > \phi(x_0) > 1$ in some point $x_0 \in \Omega$, since otherwise it would be on $\partial\mathcal{O}'$, and then u_2 is distinct from u_{1,ε,τ_0} from Proposition 3.3. □

Now, we will obtain a solution for $(Q_{\varepsilon,0})$ as the limit of the solutions obtained in the previous Proposition.

Lemma 3.5 *Assume hypotheses as propositions 3.3-3.4. Then, given* $0 < \varepsilon < (1/\mu_1^+)^{1/2}$, *there exists a solution* $u_{2,\varepsilon,0}$ *for the problem* $(Q_{\varepsilon,0})$, *which satisfies* $\left\|u_{2,\varepsilon,0}\right\|_\infty \geq 1$.

Proof Given $0 < \varepsilon < (1/\mu_1^+)^{1/2}$ we will consider a sequence $\tau_n \to 0$ and we will focus on the solution $u_n := u_{2,\varepsilon,\tau_n}$ from Proposition 3.4, so that we know that $\|u_n\|_\infty > 1$.

By Lemma 3.1, we have a uniform bound for $\|u_n\|_{\mathcal{C}^{1,\alpha}}$ for some $\alpha \in (0,1)$. Then, up to a subsequence, $u_n \to u$ in $\mathcal{C}^1$, where u is a nonnegative viscosity solution of $(Q_{\varepsilon,0})$.

From $\|u_n\|_\infty > 1$ we obtain $\|u\|_\infty \geq 1$. Thus u is nontrivial and then positive. □

The following Lemma is based in a result due to Quaas and Sirakov (see [30, Theorem 3.2]) which relates the existence of a nonnegative solution in $\mathbb{R}^N_+$ with the existence of a positive solution in $\mathbb{R}^{N-1}$. This result will allow us to show that the maximum of the solutions are below of R, the parameter of truncation, for $\varepsilon \neq 0$ small enough.

Lemma 3.6 *The solutions* $u_{2,\varepsilon,0}$ *from Lemma 3.5 satisfy* $\left\|u_{2,\varepsilon,0}\right\|_\infty \to 1$ *as* $\varepsilon \to 0$. *In particular, there exists* $\varepsilon^* > 0$ *such that if* $0 < \varepsilon < \varepsilon^*$ *then* $\left\|u_{2,\varepsilon,0}\right\|_\infty \leq R$.

Proof Given $\eta > 1$, suppose by contradiction that there exists a sequence $\varepsilon_n \to 0^+$ such that the corresponding solutions $u_n := u_{2,\varepsilon_n,0}$ satisfy $\|u_n\| > \eta$, in particular there exists a sequence $x_n \in \Omega$ such that $d_n := dist(x_n, \partial\Omega)$ and $u_n(x_n) = \|u_n\|_\infty > \eta$.

Letting $w_n(x) = u_n(x_n + \varepsilon_n x)$ we see that w_n satisfies

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2 w_n)(x) = f_R(x_n + \varepsilon_n x, w_n) \text{ in } B(0, d_n\varepsilon_n^{-1})$$

and $w_n(0) = u_n(x_n)$.

As in the proof of Lemma 3.1, we obtain (since w_n is bounded in L^∞) also a uniform bound in the $\mathcal{C}^{1,\alpha}$ norm in compact sets, for some $\alpha \in (0, 1)$. Then, up to a subsequence, $w_n \to w$ in the $\mathcal{C}^1$ norm in compact sets and $x_n \to x_0$ in $\overline{\Omega}$, where now w is a $\mathcal{C}^1$ function defined in $\mathbb{R}^N$ or in a half space. Thus, w is a $\mathcal{C}$-viscosity solution of the problem

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2 w) = f_R(x_0, w), \\ w \geq 0. \end{cases}$$

in $\mathbb{R}^N$ or in an half space. If such w solves the problem in $\mathbb{R}^N$, then according to Theorem 1.2 we conclude that either $w \equiv 0$ or $w \equiv 1$. On the other hand, if such w solves the problem above in an half space, then from Theorem [30, Theorem 3.2], (3.3) and again Theorem 1.2, we conclude the same result for w.

In any case, this contradicts the fact that $w_n(0) = u_n(x_n) > \eta > 1$, and then the lemma is proved. □

Proof of the Theorem 1.1. The first solution is $u_{1,\varepsilon,0}$, which is the solution obtained in Proposition 3.3 for the problem $(Q_{\varepsilon,0})$. Indeed, note that $u_{1,\varepsilon,0}$ verifies $\|u_{1,\varepsilon,0}\|_\infty < 1$. Besides, by hypothesis (F_1) and (F_2), if t_ε is the largest real such that $f(x,t) > \varepsilon^2\mu_1^+ t$ for $t \in (0, t_\varepsilon)$, uniformly for $x \in \overline{\Omega}$, then $t_\varepsilon \to 1^-$ as $\varepsilon \to 0^+$. Since no positive solution of (P_ε) may exist below t_ε, we deduce that $\|u_{1,\varepsilon,0}\| \to 1^-$ as $\varepsilon \to 0^+$.

On the other hand, the second solution corresponds to $u_{2,\varepsilon,0}$, the solution to the problem $(Q_{\varepsilon,0})$ given in Lemma 3.5, which verifies $\|u_{2,\varepsilon,0}\|_\infty \geq 1$. Besides, by Lemma 3.6, $\|u_{2,\varepsilon,0}\|_\infty \leq R$ for $\varepsilon > 0$ small, where R is the parameter of truncation, and $\|u_{2,\varepsilon,0}\|_\infty \to 1^+$ as $\varepsilon \to 0^+$. □

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